Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education

PME39 Hobart, Australia

Hobart, Australia

July 13-18, 2015

Volume 2

Research Reports

Abt - Gin

Editors: Kim Beswick, Tracey Muir, & Jill Fielding-Wells

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Cite as: Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.) (2015). *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3). Hobart, Australia: PME.

The Proceedings are also available on-line at http://www.igpme.org

ISSN 0771-100X
ISBN 978-1-326-66413-8

Cover Design and Logo: Helen Chick
Printing: UniPrint, University of Tasmania

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THE ZONE OF PROXIMAL DEVELOPMENT AND THE AFFORDANCES OF THE MATHEMATICAL TOOLS

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I performed a meta-analysis of 52 fraction-related problem-solving activities drawn from the literature in order to investigate if the physical properties of mathematical tools play a role in the emergence of the Zone of Proximal Development (ZPD) as children interact with mathematical tools to solve a task. The results demonstrate that the physical properties and perceived affordances of mathematical tools act as mediators between children's physical actions and their mathematical problem solving and that this is recursively related to their perception of the tool and of the task. The findings also suggest that the ZPD emerges as children participate in collective interactions with mathematical tools that involve the use of guidance provided by the physical properties of the tools in the process of solving problems.

INTRODUCTION

Among all the topics in the elementary mathematics curriculum, fractions are the most mathematically complex and the most cognitively challenging (Steffe, 2003). Martin and Schwartz (2005) contend that, in order to understand fractions, a child needs to develop new interpretations of whole numbers. Re-interpretation is a complex process and is particularly difficult through thinking alone. Therefore it is crucial for students to have opportunities to engage in wide-ranging experiences while learning fractions. An environment that incorporates diverse categories of mathematical tools can help students to better understand fractions (Steffe, 2003). Mathematical tools are objects that can be handled by children in a sensory manner and that foster mathematical thinking (Swan & Marshall, 2010); examples include fraction circles, paper and the micro-world of TIMA. Although mathematical tools play an important role in the process of learning fractions, their level of usefulness varies in accordance with each child's perception of them.

Research shows that the physical properties of mathematical tools have both strengths and limitations. A tool that is useful for illustrating one task for one child is not necessarily useful for illustrating another task and/or for assisting another child. For example, Cramer and Wyberg's (2009) study shows the strengths of the fraction chart in improving students' interpretation of the part-whole construct for fractions and its limitations in supporting the estimation of fractions. The literature on the uses of tools in learning fractions considers questions related to the physical properties of mathematical tools that remain unanswered. Examples of these questions include "What is it about pattern blocks that did not support students' thinking on fraction order tasks?" (Cramer & Wyberg, 2009, p. 14) and "Does this sensory character [of the tools] itself make manipulatives helpful?" (Clements & McMillen, 1996, p. 270).

Focusing on the Zone of Proximal Development (ZPD), I analysed the children's problem-solving process as they interacted with the mathematical tools. The ZPD is described by Vygotsky as "the distance between the actual developmental level (independent problem solving) and the level of potential development (problem solving under adult guidance or in collaboration with more capable peers)" (Vygotsky, 1978, p. 69). In the field of mathematics education, the more knowledgeable others are often conceptualised as adults and peers. In my study, I propose to extend the concept of the "more knowledgeable other" to include the physical properties and affordances of the mathematical tools (Abtahi, 2014). Gibson (1977) uses the term "affordance" to refer to the characteristics of the tool that contribute to the kinds of interaction that occur. Similarly, I use the term "perception" to refer to the aspects of a child's thinking that contribute to the kinds of interaction that happen. In this study, I investigated the role of the physical properties and affordances of the tools in the possible emergence of the ZPD. My research question is: Does the ZPD emerge from guidance that is provided by the physical properties of the mathematical tools to assist children in solving fractional problems?

THEORETICAL FRAMEWORK: DIALECTIC MEDIATED INTERACTION

From a Vygotskian perspective, I consider a child's interaction with the mathematical tools to be both mediated and dialectic. Instead of acting directly in the social world, all of our actions are mediated by "tools" and "signs" (Vygotsky, 1978). Examples of tools include nails and paper. Examples of signs include language and various counting systems. Mathematical tools mediate between the internal process of mathematical learning and external physical actions. A child's interaction with mathematical tools is also dialectic. The child's external actions modify the mathematical tools and his or her mathematical perception is modified by the interactions with the tools. By modification, I refer to making changes like, for example, sliding the beads in an abacus or perceiving that in fraction circles two halves cover an entire disk. Useful modifications of the mathematical perceptions of the child may promote learning. Vygotsky (1978) argued that a fundamental feature of learning is that it creates the ZPD because "learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with *people* in his environment" (p. 67). Vygotsky's notion of learning referred to the emergence of the ZPD as the child interacted with others. My aim was to examine the possible emergence of the ZPD as the child interacted with the mathematical tools. To do this, I employed Roth and Radford's (2010) conceptualisation of the ZPD.

Roth and Radford's (2010) view of the ZPD rests on a "non-individualistic conception of the participants" (p. 301) in which the question of the more knowledgeable other arises from the collaborative interaction of the participants. By "participants" I refer to the parties that take part in the interaction. Roth and Radford (2010) argue that one of the most crucial aspects of the ZPD is "the emergence of a new form of collective consciousness, something that cannot be achieved if we act in solitary fashion" (p. 306). With respect to the ZPD, they conceptualise "knowing" as "the possibilities that become available to the participants for thinking, reflecting, arguing, and acting in a certain historically contingent cultural practice" (p. 301) which, in the case of my study, pertain to the solving of fraction-related tasks. Further, they use the notion of language and of other semiotic resources to explain how participants position themselves in the ZPD and "tune to others in conceptual and affective layers to collectively reach interactional achievement" (p. 307). In this study, I analysed 52 episodes of children's interactions with the tools, drawn from the literature to see who were the participants and to see what resources the children used as they interacted with the tool to solve a problem. I specifically focused on who/what was the more knowledgeable other in these engagements.

DATA COLLECTION AND ANALYSIS

Based on a search of databases, I selected 57 fraction-related studies. I reviewed the references of the articles and tracked the citations from study to study. I then selected 27 frequently cited studies. I proceeded to screen each report to determine if it met the criteria of: (a) being a qualitative study; (b) using mathematical tools; and (c) providing a detailed report on the children's activities. Given the nature of this study, I specifically looked for thorough descriptions of the children's interactions with the artefact pertaining to, for instance, the tasks, the artefacts and the inclusion of parts of the transcripts. I identified 21 studies, including work by Ball, Clements, Cramer, Hackenberg, Kieren, Mack, Olive, Pirie, Sáenz-Ludlow, Steffe, Steiner, Tzur and Wyberg. Within these studies, I identified 52 problem-solving activities and a total of 25 different mathematical tools including pattern blocks; fraction charts; fraction kits (Kerien & Pirie, 1994); fraction circles; computer micro-worlds (e.g., TIMA sticks, fraction bars); chips; paper for folding activities; markers and white boards; pencils and paper; cubes; and discrete quantities (e.g., bags of spices).

I read each study to understand the context. I then extracted the age/grades of the children; the descriptions of mathematical tasks and of the mathematical tools; any related figures or drawings; the researchers' explanation of the modifications to the tools (i.e., what the children did); and the related excerpts from the transcripts (i.e., what children said). I repeated the same protocol for all 52 activities. My subsequent readings focused on what the children did and placed a particular emphasis on the children's interactions and use of resources. To investigate the possibility of the emergence of the ZPD, I focused on how the participants positioned themselves in their interactions to work on the fractions problems. The question that guided this level of readings was "Now, who/what is the more knowledgeable other?"

Over the course of the analysis I was aware of the fact that the children were interacting with the mathematical tools in the presence of a teacher, of other children or of the researchers. Therefore the participants in these interactions were not always just the children and the tools. Nevertheless, my focus in this study was on the moments when the children were thinking, reflecting and acting interactively with the mathematical tools. These interactions might have been triggered by questions/suggestions from others.

RESULTS

The analysis of the data showed that children took part in the interactions with the mathematical tools, in which the participants were the children and the tools. Children used the resources provided by the physical properties of the mathematical tools to solve fractions problems. More specifically, they used the physical properties of the tools to perceive their affordances and then they used the perceived affordances as a guide to solve the mathematical task at hand. In other words, the children used resources for interactional achievements that could not have been attained in a solitary fashion and without interaction with the mathematical tools. Furthermore, the data analysis showed that, in the dialectic process of the children's interaction with the mathematical tools, the perceived affordances of the tools play a mediating role between the physical properties of the tools and the children's acquisition of knowledge of the mathematical concept. The ways in which the children perceived the affordances of the tools were recursively related to their mathematical knowing. This implies that, if the children's knowing of the mathematical concepts was incomplete and/or was not necessarily useful, the children had difficulties perceiving the mathematical affordances of the tools.

In the rest of this section, I provide examples drawn from the literature to illustrate the ways in which the children in the study participated in the interaction with the mathematical tools and used the affordances of the tools as resources in problem solving.

1. Useful perception of the fractional concepts

The first example is drawn from Cramer and Wyberg's (2009) study. The task was to model 3/4 with pattern blocks. Cramer and Wyberg (2009) noted that "The student uses the yellow hexagon as the unit and shows [3/4] using brown trapezoids" (p. 12). In this interaction, the participants were the child and the pieces of pattern blocks. The child perceived the guidance provided by the pattern blocks (i.e., the colours, the shapes and the interrelationships among the different shapes and sizes) to solve the task. As the child explained, "I used the yellow hexagon as the whole and four of the brown trapezoids would make one whole and so I put three to make 3/4" (Cramer & Wyberg, 2009, p. 12). In the continuation of the same task, the same child was asked to show 3/4 using another unit. Cramer and Wyberg (2009) observed that, in this action, "He had some difficulty" (p. 12). Drawing on his perception of the affordances of the tools and of the mathematical concept the child attempted to use the guidance provided by the physical properties of the tools to modify the arrangement of the tools. He tried to fit different shapes onto one another:

> He first tries to use the red trapezoid as the unit, placing green triangles on it; he tries the orange square as the unit with the purple pieces; then tries the blue rhombus with the purple triangles (p. 12).

Cramer and Wyberg (2009) further noted that “He has a difficult time fitting the purples into the blues at first but then is able” (p. 12). The child’s perception of the task and of the fractional concept was clear: he needed to choose a shape as a whole and find a different shape that would fit onto the whole four times. Yet, the affordances of the tools in relation to this new task were not easily perceived. In his new interaction with the pattern blocks, he was positioned in such a way that he needed to receive more guidance from the tools. I suggest that, at this moment, the more knowledgeable participant was the physical properties of the pattern blocks and that the guidance was provided by the ways in which the shapes and sizes were designed. The child used the physical properties of the blocks to randomly choose different shapes and, after a few trials, he managed to solve the task. In both of these dialectic interactions, the participants were the child (with his or her own perception of the task and of the tools) and the pieces of pattern blocks. Sean’s knowing – his thinking, reflecting and acting – was made possible by the ways in which he interacted with the blocks. The guidance provided by the physical properties of the pattern blocks (a non-semiotic resource) assisted him in solving the problem.

The next example is drawn from Ball’s (1995) study in which Sean was asked to use a marker and a white board to find 3/4 of a dozen crayons. Sean perceived the affordances of the white board (possibly a hard writeable surface) to dialectically modify it. Sean’s perception of the mathematical concept led him to draw sticks on the board and to group them into teams: “He drew 12 sticks to represent the 12 crayons, and marked off groups of four crayons” (Ball, 1995, p. 356) (Figure 1a). Sean perceived the new affordances of the drawing (i.e., the ways in which the groups of sticks were presented) to dialectically modify his mathematical thinking, as he indicated through the following statement:

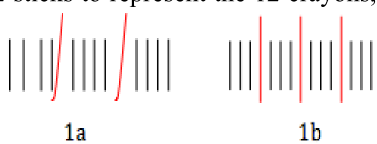

Figure 1:

> Well, I um counted these and I got, I went 1, 2, 3, 4 and I put a line down. So it’s … then I went 1, 2, 3, 4 and I put another line down and I add them up and it’s 8, and I put another line 1, 2, 3, 4. And that was 12… (Ball, 1995, p. 356)

In this interaction, the participants are Sean and his own drawing. Sean used the guidance provided by the physical properties of the tool to think, reflect and act; he “…looking at his own drawing immediately changed his mind” (Ball, 1995, p. 356). As he stated, “A quarter wouldn’t be that … because um, because that’s a third. There’s only three groups. There’s supposed to be four groups” (Ball, 1995, p. 356). I suggest that, at this moment, the more knowledgeable other was the physical property of the drawing. Sean’s perception of the affordances of the drawing led him to a new way of thinking, reflecting and acting. His new knowledge was made possible by his interaction with his drawing which guided him to dialectally re-modify the tool. He “drew new lines to mark off four groups of three crayons” (Ball, 1995, p. 356) (Figure 1b) and proceeded to solve the task:

> Because it's three fourths, that's what I said, it's three fourths so three crayons is a fourth, so three and (pointing at each group as he spoke) that's a fourth, that's a fourth and that's a fourth, so that's three fourths (p. 357).

Sean's knowing became possible through his participation in a collective interaction with the tools. It was through his participation with the tools that he was able to use the non-semiotic recourses provided by the drawing to solve the task.

2. Misperception of the fractional concepts

The following two examples show how children's un-clear perception of the fractional concepts prevented them from perceiving the affordances of the tools and solving the tasks. The first example is drawn from Mack's (2001) study, in which Lee was asked to find 1/4 of 4/5 of a chocolate cake. Lee modified the mathematical tools of paper and a pen. She "…drew a circle. Partitioned it into five equal-sized parts by drawing five radii one at a time. Put a dot on the one part" (Figure 2). Then she, drawing on her own perception of the fractional concept, perceived the affordance of the newly modified tool to explain that the one slice with the dot showed 1/4 of 4/5. The participants in this interaction are Lee and her drawing. Lee perceived the affordances of the tools, possibly the ways in which the circle was divided into five sections, to explain the following:

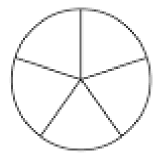

Figure 2:

> That one (points to one unmarked piece)! That's one fifth. 'Cause there's five of these (pieces in the whole cake), five fifths, and I gave one to him of these four there (indicated the four unmarked pieces) […] You said one fourth, so I need fourths. I need four pieces, and it's already cut into four, so that's four fourths.

The participants in the interaction were Lee herself and her newly modified tool with the drawing. Her unclear perception of the mathematical concept prevented her from using the resources provided by the tool to solve the task.

The final example is drawn from Olive and Vomvoridi's (2006) study. Tim used the micro-world of fraction bars to set up his own problem. He used the affordances of the tool (the computer actions of copying, pasting, moving, etc.) to create a rectangle and proceeded to partition "an on-screen unit bar into four equal parts. He then […] partitioned the first part on the left into three equal parts" (Olive & Vomvoridi, 2006, p. 25). In this interaction, the participants were Tim and the micro-world of fraction bars and the task was to find the fractional amount of the "the small piece below the original bar" (Olive & Vomvoridi, 2006, p. 25) (Figure 3). Tim had an unclear perception of the fractional concept. As Olive and Vomvoridi (2006) noted:

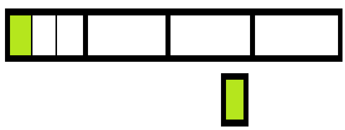

Figure 3: the Fraction Bars

> Tim's concept of a unit fraction was based primarily on the number of parts that were present (regardless of size) and did not take into consideration the part-to-whole relation of one part to a referent whole (p. 25).

Tim's misperception of the mathematical concept prevented him from perceiving the affordances provided by the fraction bars. The affordances might have been the interrelation between the sizes of the pieces with the whole. He concluded that the little piece shows the fraction of one sixth:

> Interviewer: Okay. So any idea what fraction this is of the unit bar [pointing to the small 1/12-piece that Tim pulled out of the bar]?
>
> Tim: One seventh? […] Cause there's seven in all and we pulled one out.
>
> Interviewer: Show me the seven that you have.
>
> Tim: [Pointing to each of the parts in his partitioned bar] one, two, three, four, five, six. Oh six! One sixth (Olive & Vomvoridi, 2006, p. 25)

At the time of working out the fractional amount of the little piece, the participants in the interaction were Tim and the micro-world of the fraction bars. Yet, because of the mathematical misperception, Tim could not perceive the useful affordance of the faction bar and therefore he could not reach an interactional achievement by collaborating with the tool. Although in this interaction one might consider the physical properties of the fraction bars to be the more knowledgeable other, Tim's mathematical development was not at the level (i.e., at the level of potential development) necessary to perceive this fact and to reflect and act accordingly.

DISCUSSION

The general issue that I brought up here is whether the zone of proximal development emerges as children interact with mathematical tools. Based on the evidence that I have outlined in this paper, I believe that it does. My claim about the mediating role played by the affordances of the tools, as exemplified in the cases mentioned above, suggests a possible extension to our common interpretation of Vygotsky's ZPD. Specifically, it suggests the inclusion of the non-semiotic recourses provided by the physical properties and of the affordances of the mathematical tools as "the more knowledgeable other" as it was under the guidance provided by the tools that children solved problems.

How then do I see a tool as a more knowledgeable other? My view draws on the essential property of the tools: tools (including mathematical tools) are culturally and historically based (Wertsch & Rupert, 1993) and their design is socially originated by us – humans. By the social origin of the design, I do not mean that tools are designed at once by a single act of an isolated person. On the contrary, I am referring to the historical evolution of tools as they have mediated our actions over time. The point that I am raising here is that the more knowledgeable other-ness of the tools is based on and is originated from the collection of the perceptions of many others who have historically designed, used and modified the tools.

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DIAGRAM CONSTRUCTION AND PERFORMANCE IN ADVANCED MATHEMATICS

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We report a study in which 100 students at the beginning of an undergraduate real analysis course were asked to construct diagrams to represent four general mathematical statements about functions. We present four theoretical criteria for analysing such diagrams and illustrate the range of student-produced diagrams; we then present an analysis showing that performance in the diagram-construction task was significantly related to subsequent performance in the course.

INTRODUCTION

There has been considerable research on students' use of mathematical diagrams (Presmeg, 2006). Some has sought to clarify relationships between mental imagery, external representations, and successful reasoning (Duval, 1999). Some has classified students as visualisers or otherwise (Presmeg, 1986; Stylianou & Silver, 2004), and some has investigated whether students can interpret graphical information representing real-world situations (Leinhardt, Zaslavsky, & Stein, 1990; Robert & Speer, 2001). Our work asks whether students *can* draw suitable diagrams rather than whether they are inclined to do so. Specifically, we asked students to construct diagrams to represent abstract statements from real analysis.

Real analysis lends itself to graphical representations because it involves real-valued functions and their properties. However, while students often see diagrams, they are less often asked to construct them. There is evidence that mathematicians and successful students can draw relevant diagrams and use them to construct mathematical arguments (Gibson, 1998; Stylianou & Silver, 2004), but studies at the undergraduate level are typically small-scale and focused on spontaneously-produced diagrams. Research is largely silent on the issue of whether a typical student can produce such diagrams, and thus on whether there is a systematic relationship between this skill and mathematical performance. This report addresses this gap by reporting a study that asked students to draw diagrams for four statements:

A: f is bounded on the set X if and only if $\exists M > 0$ s.t. $\forall x \in X$, $|f(x)| \leq M$.

B: Suppose that $f : [a, b] \rightarrow R$ is continuous and that y_0 is between $f(a)$ and $f(b)$. Then $\exists x_0 \in (a,b)$ s.t. $f(x_0) = y_0$.

C: If f is continuous at $x = a$ and $f(a) > 0$ then $\exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow f(x) > 0$.

D: Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) and that $f(a) = f(b)$. Then $\exists c \in (a, b)$ s.t. $f'(c) = 0$.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 9-16. Hobart, Australia: PME.

Our first aim was simply to investigate the extent to which students embarking upon a real analysis course were able to produce diagrams to represent such statements. Our second aim was to find out whether ability to draw diagrams like these is systematically related to performance in the course.

THEORETICAL BACKGROUND

In our study, students were asked to construct diagrams to represent statements written in a typical combination of words and symbols. They were thus required to translate between representation systems (Goldin, 1998), a process Duval (1999) calls *conversion*. Students are required to perform many such conversions during their mathematical education, and ability to do this is seen as evidence of mathematical understanding: both policy documents (NCTM, 2000) and research-related arguments (Janvier, 1987) stress its importance in flexible mathematical problem solving.

Undergraduate mathematics can also involve diagrams, and the intention is often that a diagram be interpreted as *generic* – as representing a whole class of functions, say. An individual might draw a diagram to facilitate proof construction (Gibson, 1998) via *semantic reasoning* (Alcock & Inglis, 2008; Goldin, 1998; Weber & Alcock, 2009), and both mathematicians and undergraduate students can and do use diagrams to understand statements (Gibson, 1998) and to explore relationships (Stylianou & Silver, 2004; Weber & Alcock, 2009). Diagrams arguably have particular utility for such purposes, because they allow simultaneous external representation of multiple aspects of a problem (Pantazaria, Gagatsis, & Elia, 2009). They can thus facilitate imagined variation of one or more of these aspects (Tall, 1995), recognition of relationships that may not be obvious from a problem statement (Pólya, 1957), and the correct set-up of equations necessary to solve a problem (Bremigan, 2005).

The extent to which a diagram is useful might vary, however. This observation is key to our study because we are interested in judging the value of diagrams produced in response to a direct request. In this paper we use four criteria to capture each diagram's possible value in supporting further semantic reasoning.

Our first criterion is *correctness*. If a diagram does not correctly represent the relationships under consideration, this shows that the person who produced it does not understand the statement or was not (in this instance) able to convert between representation systems appropriately. Either way, an incorrect diagram will not reliably lead to productive and correct further reasoning.

Our second criterion is *genericity*. For a diagram to function as a generic example, it should be neither too trivial nor too complicated, and analogy with other instances should be readily achieved (Rowland, 2002). In our context, a diagram might be too trivial if it incorporates function properties that oversimplify the situation: a function might be drawn as constant or monotonic or always positive, for instance (Haciomeroglu, Aspinwall, & Presmeg, 2010). A diagram might be too complicated if it includes potentially distracting irrelevant features such as multiple axis crossings or asymptotes. A diagram that is too simple might suggest invalid inferences, and one that is too complicated might impede focus on key properties.

Our third criterion for judging diagrams is quality of *labelling*. Incorrect labelling can result in misrepresentation, and a more subtle possibility is that some mathematical objects might not be explicitly labelled. This could be important because experts might be more consistent than novices in producing fully-labelled diagrams (Stylianou & Silver, 2004) and because quality of labelling might be a factor in enabling translation from a diagram-based insight to a formal argument, a process that can be difficult (Alcock & Weber, 2010; Weber & Alcock, 2009).

Our fourth criterion emerged during data analysis, so it is described in the Method section below. The Method section also describes data we collected in order to investigate any relationship between diagram construction and performance in the analysis course. Theoretically there could be such a relationship: ability to use and convert between a variety of mathematical representations might support understanding and semantic reasoning. On the other hand, performance in courses like real analysis is traditionally measured via formal work with definitions, theorems and proofs, and a student could learn to do such work without attending to diagrams.

METHOD

Task design

All participants were asked to draw a diagram to represent each of the four statements listed in the introduction; the order of presentation was randomised so that participants saw different versions of the task. We selected the statements from the real analysis course, using the following criteria. First, we wanted all terminology and symbols to be familiar to the students, so that ability to construct diagrams would not be confounded with ability to interpret the components of the statement. Second, we wanted statements which would be accessible but which the participants had not studied before, so that they would not try to 'remember' an appropriate diagram (for this reason we did not flag any of the statements as a definition or theorem). Third, we wanted statements for which participants would not be tripped up by inattention to the subtler points of calculus or analysis. For example, differentiability is key to statement D (Rolle's Theorem), but attention to this was unlikely to be problematic because students tend to think about differentiable functions; completeness of the real numbers is key to the statement B (Intermediate Value Theorem), but this was unlikely to cause problems under typical naïve conceptions of continuity.

Because of these criteria, we did not expect the participants to have trouble in literal reading of the statements. Nevertheless, we wanted to exclude the possibility that any apparent difficulties in diagram construction resulted from an inability to read the statements. We thus also asked the participants to write out in words exactly what each statement said. Except for occasional minor awkwardness in English expression, there was no evidence that any participant had difficulty reading the statements. To establish that this writing did not, in itself, improve diagram construction, we asked half of the participants to complete the drawing task first and half to complete the writing task first (the order in which the statements appeared in the writing task was randomised too). Results of this manipulation are reported later.

Participants and administration

A total of 100 students took part; 75 were in the second year of a single-honours mathematics degree and 25 were in the third year of a joint-honours mathematics degree. All had high pre-university mathematical attainment, all had studied two-semester courses in calculus and linear algebra, and all spent 50-100% of their study time on mathematics. The task was administered at the beginning of the first lecture in the real analysis course. Participants were given a booklet and asked to fill in a cover sheet stating that they understood that their responses would also be used for research and asking them to provide their ID number if they gave permission for the researchers to link this to information from the university database (all students gave this information). Drawing-first and writing-first versions of the task were interleaved so that students sitting next to each other did not receive the same type of task first. The participants were given 10 minutes to complete whichever task was first in their booklet, which asked them not to turn over until told to do so. When told to turn over, they then had the same amount of time to complete their second task.

Data analysis

Before analysing the student-produced diagrams, we collected diagrams for each statement from three mathematics lecturers (one author of this paper and two with recent calculus lecturing experience). This confirmed that expert diagrams were broadly similar. It also prompted us to introduce our further criterion for judging diagrams, for the following reason. Figure 1 shows two expert diagrams for statement C. Both are accurate, generic and fully labelled: as required, [illegible]. Also, however, outside the region where $|x-a|<\delta$, the function *does* take on values that are less than zero. This goes beyond the literal statement to indicate a sense of what is mathematically important about the claim.

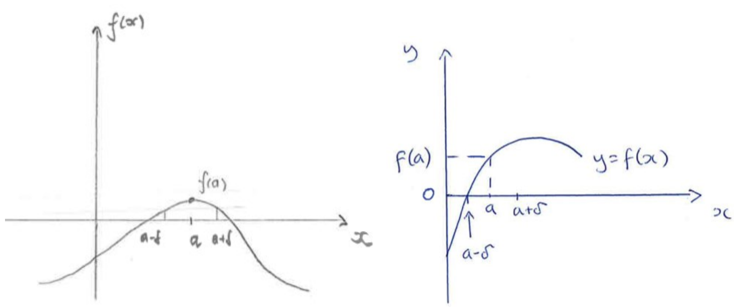

Figure 1: Expert diagrams for statement C.

We thus awarded each student-constructed diagram a score of 0, 1 or 2 for correctness (to allow for partially correct answers) and 0 or 1 for each of genericity, labelling and this new criterion, which we termed *completion*. This gave us a score out of 5 for each statement and an overall score out of 20 for each participant.

While scoring, we had reason to believe that seven students had misunderstood the task instructions (most had written instead of drawing; one had apparently begun trying to prove the statements); a further three were repeating the course. These ten students were excluded, leaving 90 participants for the descriptive analyses. For the remaining participants we collected prior performance scores from their earlier calculus course (as percentages); we considered calculus to be the most relevant as preparation for our task. We used the students' eventual real analysis final examination scores in two ways, looking at both raw score and a standardised score which excluded points from question parts that involved drawing diagrams. Of the 90 students who completed the drawing task, ten did not take the analysis examination and for one a calculus mark was not available. Thus the analytical results are based on a total of 79 participants.

RESULTS

Descriptive results: student-produced diagrams

Scores were low: the mean out of 20 was 7.0 (standard deviation 5.19) and, of the 90 participants, 14 scored zero. Statement B (the Intermediate Value Theorem) appeared easiest, with the highest mean score of 2.5. Figure 2 illustrates the types of errors and misinterpretations that can arise by showing two low-scoring student-produced diagrams (more diagrams will be shown in the presentation if this report is accepted).

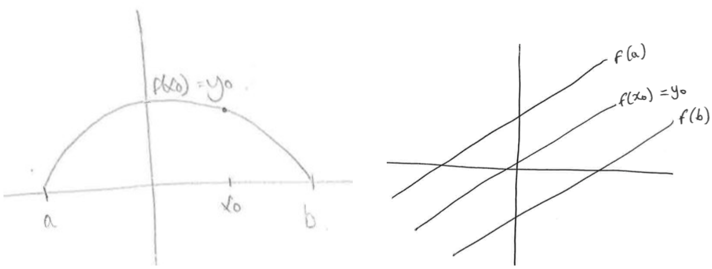

Figure 2: Low-scoring diagrams for statement B.

For statement D (Rolle's Theorem) the mean score was 2.1; for statement A (boundedness definition), 1.6, and for statement C (lemma), 0.8. Figure 3 shows low-scoring participant-produced diagrams for statement C. Very few participants were able to correctly represent the meaning of this statement – delta was rarely labelled in any way – and none captured the completion aspect.

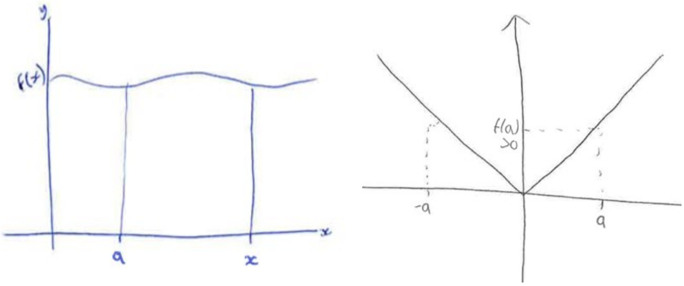

Figure 3: Low-scoring diagrams for statement C.

Analytical results: drawing scores and performance

As noted in the Method section, half of the students were asked first to draw and half were asked first to write out the statement in words. Diagram construction scores for the writing-first group (n=39; m=8.21, s.d.=4.97) were slightly higher than those for the drawing-first group (n=40; m=7.08, s.d.=5.30), but this difference was not statistically significant (t=0.98, p=0.33) so it is not used in further analyses.

Two linear models were considered, the first using the raw real analysis examination score as the dependent variable, and the second using the amended real analysis examination score as the dependent variable. In both cases, independent variables were the participants' calculus score, drawing-task score, year of study and interaction terms between year of study and calculus and drawing-task scores. In both models, all of the interaction terms and also year of study were found to be non-significant and were thus excluded. Both calculus score and drawing-task score were found to be statistically significant in both models, as shown in Table 1.

	Raw analysis exam score			Amended analysis exam score		
Variable	B	SE B	β	B	SE B	β
Calculus	0.48	0.11	0.42*	0.43	0.10	0.39*
Drawing	1.57	0.36	0.40*	1.53	0.36	0.41*
R^2	0.49			0.47		

Table 1: Summary of regression analysis for variables predicting raw and amended analysis examination scores; *p < .005.

In both cases, the estimated coefficients indicate that each additional 1% scored in calculus is associated with approximately an additional 0.5% in real analysis. More interestingly, each additional point out of 20 scored in the drawing task is associated with an additional 1.5% in real analysis. The standardised coefficients indicate that the predictive power of the drawing task score is on a par with that of prior attainment in calculus, even when performance in real analysis is measured exclusively via standard formal work.

DISCUSSION

The low scores on our drawing task indicate that constructing diagrams was not easy for participants. This could be considered unsurprising given that these students had no specific training in constructing diagrams for statements of this type, but it provides evidence regarding whether we can expect students at this level to make good use of diagrams in semantic reasoning. If students cannot produce such diagrams when specifically asked to, it seems unlikely that they would use them effectively as a natural part of reasoning. Of course, our study does not provide information on whether students can correctly *interpret* diagrams provided by others. Interpretation might be considerably easier than construction, and further research would be required to investigate whether this skill is related to academic success.

The relationship between drawing-task score and examination performance indicates that skill in producing diagrams might be an important factor in successful learning of advanced mathematics. It should be interpreted with caution, because all of the participants were enrolled in one course; this study does not enable us to tell whether this skill would be useful in *any* real analysis course, or whether features of the teaching simply made it useful in *this* course. It certainly does not provide evidence that this diagram construction skill is useful across the curriculum; it could be that it is of benefit in real analysis but not, say, in abstract algebra. Nevertheless, our findings provide reason to investigate diagram construction at this level more broadly, perhaps as one of a number of distinct mathematical skills that might benefit students in different mathematical domains.

Finally, examining expert-produced diagrams and scoring student-produced diagrams prompted us to articulate a clearer theoretical conceptualisation of what constitutes a good diagram. But student-produced diagrams also provide a valuable window into individual comprehension and mathematical reasoning, and we plan to report further on this issue in future work.

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CO-ACTION AND DYNAMIC GEOMETRY KNOWLEDGE

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To understand how interacting with dynamic geometry environment (DGE) shapes learners' geometric knowledge, we draw on the theory of instrumental genesis (Rabardel & Beguin, 2005) and the notion of co-action (Hegedus & Moreno-Armella, 2010) to understand learners' interaction with DGE. We analysed data of six teachers engaging in an online synchronous dynamic geometry environment in two semesters. Our analysis shows that the co-action between the teachers and the environment helped the teachers appropriate the dragging feature of DGE, which shaped their understanding of geometrical relations, particularly dependencies. This informs the broader question of how and what mathematical knowledge learners construct using certain technologies.

Understanding geometry is important in itself and in understanding other areas of mathematics. It contributes to logical and deductive reasoning about spatial objects and relationships. Geometry provides visual representations alongside the analytical representation of a mathematical concept (Goldenberg, 1988; Piez & Voxman, 1997). Pairing learning geometry with technological tools of Web 2.0 can allow learners to investigate collaboratively geometrical objects, properties, and relations and develop a flexible understanding of geometry. Though teaching with technology is recommended (Common Core State Standards Initiative, 2010, p. 7), meta-analytic studies indicate that teaching with technology cannot guarantee positive learning outcomes (Kaput & Thompson, 1994; Wenglinsky, 1998). Careful investigations are required to understand the appropriation of technology and how it shapes mathematics learning. To contribute to understanding this, we describe the influence of learners' appropriation of online, dynamic geometry environment on their geometric understanding. Dynamic geometry environments (DGEs) react to the users' actions through engineered infrastructure that corresponds to the theory of geometry. This reaction can inform users' actions and shape their thinking. This paper responds to the question: How does learners' appropriation of an online, collaborative dynamic geometry environment and its co-active functionality shape their geometrical understanding?

REVIEW OF LITERATURE AND THEORETICAL PERSPECTIVE

Researchers have investigated the use of digital technologies. Some investigate how teachers learn different geometric and algebraic topics of a DGE and changes in their knowledge (Hohenwarter, Hohenwarter, & Lavicza, 2009). Also with DGEs, others studied cognitive processes linked to different types of dragging and the use of dragging with trace (Arzarello, Olivero, Paola, & Robutti, 2002). Using the theory of instrumental genesis, others have investigated the complex and slow process students

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experience to transform graphing calculators from tools to mathematical instruments (Guin & Trouche, 1998). Others have also examined learners instrumental appropriation of digital technologies (Hegedus & Moreno-Armella, 2010; Rabardel & Beguin, 2005). However, given that digital tools such as DGEs can be used in collaborative environments, work needs to be done to understand how collaborating with each other and digital tools learners shape their development of geometric thinking.

To understand learners' appropriation of technological artefacts or tools, we draw on a Vygotskian perspective about goal-directed, instrument-mediated action and activity. Instrumental genesis (Rabardel & Beguin, 2005) posits that users' (teachers', students', or learners' in general) activity directed toward an object (material, mental, or semiotic) such as a task is mediated by tools, which may be material devices or semiotic constructs. To appropriate a tool, users develop their own knowledge of how to use it, a utilisation scheme. This scheme, along with the tool, forms the instrument. Rabardel and Beguin (2005) emphasise that the instrument is not just the tool, but "a mixed entity, born of both the user and the object: the instrument is a composite entity made up of a tool component and a scheme component." (p. 442). Therefore, an instrument is a two-fold entity, part artefactual and part psychological.

The transformation of a tool into an instrument occurs through two dialectical processes that account for potential changes in the instrument and in the users thorough instrumentalisation and instrumentation, respectively. In instrumentation, the structure and functionality of tools shape how the learner uses the tool, which result in shaping the learner's thinking. "Instrumentation concerns the emergence and development of utilization and instrumented action schemes" (Rabardel & Beguin, 2005, p. 444). In instrumentalisation, the learner's interactions with a tool also shapes the tool and how it is used, so that "the learner enriches the artefact properties" (Rabardel & Beguin, 2005, p. 444).

Particular infrastructural properties of DGEs give rise to a unique component of instrumentation. Hegedus and Moreno-Amrella (2010) theorize that DGEs' capability of responding to users movement of base points or hotspots establishes a dialectical co-active relationship. As users drag (click, hold, and slide) a base point or hotspot of a geometric figure, the DGE redraws and updates information on the screen, preserving all constructed mathematical relations among objects of the figure. In redrawing, the DGE creates a family of not only visually but also mathematically similar figures. Users may attend to the reaction of the DGE and experience underlying mathematical relations among objects, such as dependencies.

METHODS

Data come from a project that integrates a cyberlearning environment with digital tools for collaborative geometrical explorations—Virtual Math Teams with GeoGebra (VMTwG). For this paper, we analysed work from two online professional development courses for middle and high school teachers that occurred in the fall semester of 2013 and 2014. In small teams, these New Jersey teachers engaged in learning dynamic geometry through collaborating to solve tasks.

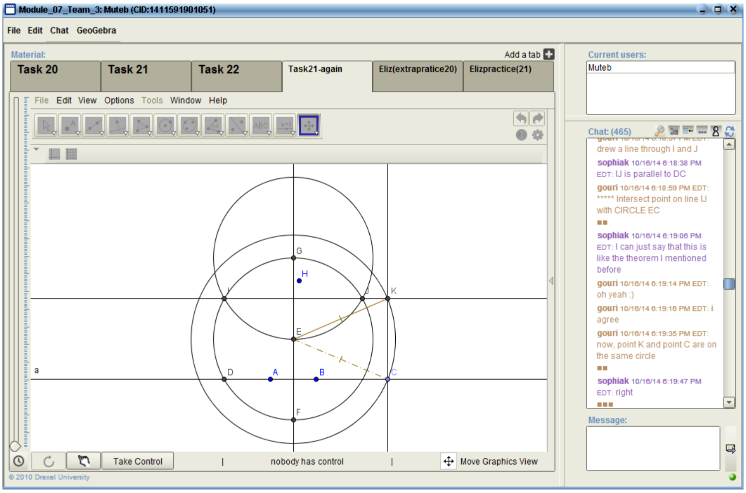

Figure 1: Team 3's construction of a perpendicular line that passes through an arbitrary point

VMTwG, a product of a collaborative research project among investigators at Rutgers University and Drexel University, contains support for chat rooms with collaborative tools for mathematical explorations, including a multi-user, dynamic version of GeoGebra, where team members can define objects and drag hotspots around on their screens (see Figure 1). VMTwG records users' chat postings and GeoGebra actions. The research team designed dynamic-geometry tasks that encourage participants to discuss and collaboratively manipulate and construct dynamic-geometry objects, notice dependencies and other relations among the objects, make conjectures, and build justifications.

We analyse the work of teachers in Team 1 from 2013, four teachers, and Team 3 from 2014, two teachers. Before this course, none of the six teachers had any experience with dynamic geometry. They met in VMTwG synchronously for two hours twice a week. We selected these teams because they demonstrate conspicuously how teams attended to the environment's reaction to their actions. We use the discursive and inscriptive data generated from their work on four different tasks. Team 1 worked to examine different types of triangles and construct them, then to re-examine previously examined triangles to discover dependencies involved in their construction. In a later session in the course, Team 1 examined and then constructed perpendicular bisectors. Team 3 discussed the construction of equilateral triangle and then construct one. Team members also constructed a perpendicular line that passes through an arbitrary point in a later session in the course.

Using conventional content analysis (Hsieh & Shannon, 2005), we analysed the teachers' data to understand their process of instrument appropriation and implications of that appropriation. We also used the construct of co-action to understand when, why, and how do teachers interact with hotspots; what feedback do they perceive and what they do with this feedback; and how does it shape their subsequent actions.

RESULTS

Our analysis focuses on understanding how the teachers' appropriation of VMTwG shapes their geometrical understanding. Our results show how through co-action teachers and VMTwG interact and how this interaction leads to shaping their understanding of affordances of dragging and the dynamic-geometry relation, dependency.

We also found that the teachers implemented their understanding of dependencies and dragging to solve geometrical problems. We will present the work of Team 1 on two tasks then the work of Team 3 on other two tasks in VMTwG.

Team 1:

During the first collaborative session with simple constructions, Team 1's members quickly understood dependency in dynamic geometry. In its second session, Team 1 worked to identify and construct different types of triangles and then to re-examine previously-examined triangles to discover dependencies involved in their construction. In the first collaborative session, the teachers examined different triangles. The vertices of the first triangle, ABC, were constructed as independent objects, so the team discussed it briefly. The second figure is an isosceles triangle, DEF. The lengths of DE and DF are equal. Point F is constrained to a hidden circle with radius DE. Points D and E are independent objects. Here is an excerpt from Team 1's discussion about the second figure:

386	Ceder:	so in the second one, f is dependent on g
387	Ceder:	I mean d
388	Ceder:	not g
389	Bhupinder_k:	E on D as well
390	Sunny blaze:	so ED and FD are dependent on angle D?
391	Bhupinder_k:	I think F depends on both E and D
392	Ceder:	f doesn't look dependent on anything now...am I missing something?
393	Ceder:	[after dragging F] ok, what am I missing? F can move independently, but when E is moved, F moves, so that makes which one dependent?
394	Bhupinder_k:	when you move F, ED stays fixed
395	Ceder:	right, so F is free to move anywhere
396	Ceder:	but not when E is moved

397 Ceder so F is sometimes dependent?

The team discusses dependencies among points, segments and angles. In lines 386 to 388, Ceder states that F is dependent on D then dismisses her assertion in line 392. Then, Sunny blaze states her understanding in a form of questions: “so ED and FD are dependent on angle D?” (Line 390). This highlights the struggle the teachers had to identify the dependency when the points are partially constrained. At line 397, Ceder asks whether point “F is sometimes dependent”. Though they had already seen and, a week before, constructed dependent objects in their first collaborative session, they struggled with a new and more complex situation. The concept of dependency is key for developing utilization schemes that allow learners to identify and build relationships in geometric constructions.

In a latter task, the team uses the concept of dependency to identify relations among objects. The task presents two circles constructed using the same radius, AB. Their points of intersections, C and D, were connected to create a perpendicular bisector to radius AB. In the session’s chat log, one teacher states that points C, D, and E are dependent on A and B. Another teacher states that the two circles share the same radius and that dragging the centre of one circle affects the size of the other, which makes the circles dependent on the centres. Through co-action, teachers appropriated the concept of dependency and used it to understand constructions in this task.

In these two tasks, the teachers attended to the reactions of the environment to their actions, which enabled them to appropriate the concept of dependency. Engaging with tasks where dependencies are key relations among geometrical objects was an important step. These tasks triggered a discussion about how to use the notion of dependency to create valid constructions. The teachers’ discussion was an important step that enabled them to understand how to apply their new concept, dependency. Next, they tested their understanding with another construction. After developing and testing their understanding of dependency, they applied their understanding in another task and identified dependencies among new sets of geometrical objects.

Team 3:

This Team of teachers worked on constructing an equilateral triangle (Task 8) and then constructing a perpendicular line that passes through an arbitrary point (Task 21) in a later session in the course. They worked on appropriating the dragging affordance of VMTwG in the first session, which was evident in the second session when they were constructing perpendicular line that passes through an arbitrary point.

Task 8 asks teachers to drag an equilateral triangle and then to discuss what they notice about the given figure and then construct a similar one in GeoGebra. Before this session, the teachers were asked to drag and notice relationships among basic geometrical objects to become aware of co-active relations between their actions and reactions of the VMTwG environment. As the following excerpt from Team 3’s chat log shows, the teachers, Gouri and Sophiak, felt the need to revisit their understanding of dragging after being instructed to create an equilateral triangle in Task 8.

26	Sophiak:	It seems that point C is fixed but pts A&B are not. I am thinking somehow A&B were used to create the circles which is why the make the circles bigger or smaller.
27	Sophiak:	How about you try to explore now?
28	Gouri:	ok I'll continue on with #2 [the second instruction in Task 8] as well
29	Sophiak:	No, I would like to create the objects as well. I think it is valuable if we both explore
30	Gouri:	C does seem fixed/constrained
31	Gouri:	sure - how about I do it and then you do it as well after?
32	Sophiak:	Sounds good. Please type what you do.
33	Gouri:	So far I created 2 circles
34	Gouri:	and overlapped the D point as the raius point for E
35	Gouri:	one more try
36	Gouri:	ok - I deleted the other circle because i dont need it
37	Gouri:	I somehow thought i could create all 3 points, abc through two circles
38	Sophiak:	How did you create F?
39	Gouri:	I added a point
40	Gouri:	then the polygon tool for the triangle
41	Sophiak:	Did you want to explore your picture to see if it behaves the same way as the original?
42	Gouri:	ok
43	Gouri:	[after dragging the pre-constructed figure for few minutes] I noticed that it's the points that make the circle dynamic
44	Gouri:	and not the circle (in black) itself

The teachers started by stating their noticings of the construction. In line 26, Sophiak mentions that point C is fixed (intersection point of the two circles) and points A and B are not and states that points A and B are used to construct the two circles. She states that since dragging points A and B affects the circles, they are used in constructing the circles. It indicates how Sophiak views the relationship between dependency and construction and how she is starting to identify the hotspots of the figure. The second team member, Gouri, successfully creates a similar figure to the task's figure. She states after dragging in lines 43 and 44 that "the points that make the circle dynamic and not the circle (in black) itself". These comments suggest that Gouri was concerned with what is being dragged in a dynamic geometry environment and what makes it dynamic.

This event shows that the teachers are distinguishing between different types of dragging. The co-action between the teachers and the environment helped the teachers develop an understanding of the dragging in DGE. This shows how teachers appropriate the environment through developing their understanding of dragging and dependencies.

Teachers' understanding of dragging different type of objects, hotspots and other objects, in DGE helped them appropriate the environment, which influenced the type of knowledge that teachers developed later in the course. Their work on Task 21 illustrates this. In the preceding task of Task 21, Task 20, the teachers constructed a line perpendicular to a given line. This team of teachers was unable to solve Task 21 in the first attempt and was asked to try again after revisiting Task 20.

The teachers met again and successfully solved Task 21. They used some insights from Task 20 to construct perpendicular lines multiple times. They started by constructing a line AB and an arbitrary point C (see Figure 1). Then using the technique from Task 20 (constructing circles with a common radius, mark their intersection points, then connect them), they constructed a line EF perpendicular to AB and dragged points A and B to test the construction. On that line, they marked point G and, employing the Task 20's technique, used it and point E to construct line IJ perpendicular to EF, which makes IJ parallel to AB. After that, they constructed circle EC and marked the intersection point of this circle with line IJ, point K. They dragged point C to test the behaviour of the construction. Finally, they constructed line KC, which is perpendicular to AB and passes through the arbitrary point C.

Proving that KC is perpendicular to AB is beyond the scope of this paper; however, it can be done easily using triangle congruency. The teachers collectively constructed their final solution. After each step of their construction, they dragged points A, B, and C to make sure that at each stage their construction maintained properties they intended. Their appropriation of dragging–what to drag, how to drag, and what to expect–was dominant in their problem solving of Task 21.

DISCUSSION

We introduced teams of teachers to a collaborative, online, dynamic geometry environment, VMTwG, in a semester-long professional development course. They interacted to notice variances and invariances of objects and relations in pre-constructed figures or figures that they constructed and to solve open-ended geometry problems. Our analysis of their interaction in two iterations of the course allowed us to understand how they appropriate the tools of VMTwG and how their appropriation shapes their geometrical knowledge. Team 1 worked on understanding dependencies among geometrical objects through dragging. Team 3 paid special attention to the characteristics of the objects they dragged. Their interactions indicate that they perceived the significance of dragging hotspots of a construction (Hegedus & Moreno-Armella, 2010). Co-action helped the teachers of both teams identify hotspots and use them to test their constructions and become aware of dependencies.

As mentioned above, teachers' appropriation of dragging and understanding of dependencies among geometrical objects were evident in their subsequent problem solving. Teachers' collaboration and the co-active nature of the environment helped the teachers in their appropriation of dragging and dependency. Integrating these two aspects, collaboration and co-action of DGEs, is an important feature of tasks designed for this type of environments.

Further research is needed to understand what other aspects of DGEs are important in the instrumentation process. Research that investigates how the appropriation of different aspects or tools of DGEs might influence learners' knowledge is also needed. Additionally, research is needed to investigate how teachers' understanding shapes how they integrate DGEs into their teaching practice.

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SUPPORTING RURAL AND REMOTE MATHEMATICS TEACHERS: RE-CONCEPTUALISING PROFESSIONAL DEVELOPMENT

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The purpose of this research was to understand how the context of regional conferences and lesson study provided opportunity for teachers in rural and remote areas to develop their ability to deliberately plan for whole-class mathematical discussions. Based on Smith and Stein's (2011) work on orchestrating class discussions and Murata's (2011) lesson study, teachers from multiple rural school districts developed collaborative working groups to develop their understandings of facilitating whole-class discussions. Results indicated that the structured formats of professional development for teachers in rural schools targeted the diverse needs of the populations. Findings provide perspective for structuring professional development in a multi-faceted approach for schools in rural settings.

STATEMENT OF PURPOSE

Recent findings in mathematics education highlight the importance of focusing on students' mathematical constructions for knowing how to support future learning and development (Norton & McCloskey, 2008). Engaging students in whole-class mathematical discussions is one means of developing understandings of what students know and in what areas misconceptions remain. Yet, teachers often struggle creating rich discursive environments due to the complex and dynamic nature of learning contexts (Smith & Stein, 2011). Training and professional development that is not embedded and on-going is less likely to have a meaningful impact or change in teachers' praxis as opposed to professional support that is recurring, in the context, and focused. Likewise, in rural and remote areas, providing opportunities for teachers to network and engage with others is not always a viable option. Therefore, there was a need to plan and implement a professional development structure that would provide opportunities for same grade level collaboration, but would also support ongoing professional growth within schools where there may be only one teacher of a specific grade. This model included regional conferences and a modified lesson study process (Murata, 2011) based on the regional conference work.

As a result, this study seeks to answer the following research question: How does a multi-tiered professional development and support model for teachers in rural and remote areas impact their perceived effectiveness?

THEORETICAL FRAMEWORK

Bauch (2001) indicated that the types of support provided to urban schools cannot necessarily be applied to rural settings, as they are fundamentally different with respect to available resources, community structure, and school configurations. Yet, rural

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 25-32. Hobart, Australia: PME.

districts are increasingly undergoing a "suburbanization" of their communities due to external influences related to state and national policy (Howley, Howley, Hendrickson, Belcher, & Howley, 2012; Yettick, Baker, Wickersham & Hupfeld, 2014). This means that rural and remote schools are expected to have the same outcomes as urban schools despite fundamental differences. We theoretically frame this study from the perspective of supporting teachers in rural and remote areas differs from supporting those in more populated areas (Bauch, 2001). As such, we assume a perspective that focuses on understanding the needs of teachers in rural areas and overcoming barriers to support their development.

RELATED LITERATURE

The following review of literature describes the importance of focusing on whole-class discussions to improve students' understanding of mathematics; the need to provide embedded, on-going, and multi-faceted professional develop for educators followed by literature on providing professional development in rural areas.

Whole-Class Discussions

Participating in focused mathematical discussions is fundamental to developing students' conceptual understanding of mathematics (Smith & Stein, 2011). Whole-class discussions require students to actively listen to other students' arguments, appropriately critique these arguments, and construct their own logical arguments based on mathematical evidence. This process of creating, interpreting, and defending ideas that explains *why* helps provide insight into the underlying mathematics (Lannin, Ellis, Elliot, & Zbiek, 2011). This means students are continually furthering their understandings of skills and procedures, important connections within and between other mathematical concepts, and similarities and differences between other students' ideas or representations. Through such interactions students are able to clarify their own thinking, make sense of the mathematics being discussed in nuanced ways, and become more engaged in their own learning (Blanton, 2002).

For students, participating in rich discursive interactions further develops essential mathematical behaviors that can support their learning across grade levels and content areas (Weiland, Hudson, & Amador, 2014). Too often, discussions are teacher-centered and do not promote rich and engaging situations for students to develop conceptual understanding (Kuhn, 2005; Roberts & Billings, 2009). Yet, deliberate interactions within a sociomathematical contexts have been also been found to help teachers notice and understand students' mathematical thinking (Jacobs, Lamb, & Phillip, 2010). Developing productive mathematical behaviors and habits of mind should be a central theme in mathematics classrooms at all levels. This means deliberately creating experiences and incorporating tasks with a high cognitive demand so that whole-class discussions can naturally arise. As such, it is imperative that teachers are knowledgeable about the pedagogical moves needed to help students develop their ability to construct mathematical arguments (Boerst, Sleep, Ball, & Bass, 2011; Zbiek, Martin, & Schielack, 2012) and thus create meaningful discursive interactions.

Multi-Faceted Professional Development

Effectively supporting teachers requires on-going and classroom embedded professional development (Desimone, 2009). This also means providing a multitude of experiences that are different both in design and intent. While workshops and conferences can be an effective means of initially presenting and exploring nuances of content and pedagogy, “one-time” experiences have little lasting effect. However, professional development that collaboratively focuses on teacher knowledge can be fostered through the process of lesson study, which can lead to improvement in instructional practice (Murata, 2011). During the lesson study process, a small group of participants begin by setting a goal for the lesson study process and then working collaboratively to plan a lesson to teach in one of their classrooms. To plan the lesson, participants study research materials, curriculum guides, and other artefacts to decide on appropriate instructional methods and content for a mathematics lesson. Next, a teacher in the group teaches the lesson to his or her students while the other participants observe the lesson. Following the lesson, the participants meet together to reflect on the lesson and to revise the lesson for teaching in another classroom. Once the lesson is revised, the modified version is taught in another teacher’s classroom. This student-focused process affords participants the opportunity to notice different aspects of the lesson (Murata, 2011). However, this process can be near impossible to effectively implement in areas with few teachers in a school or long distances between neighbouring schools.

Supporting Rural Teachers

Supporting teachers with professional growth can be a daunting task, especially in rural and remote areas where populations are limited, distances from one school to the next are immense, and geography inhibits travel. Accessing additional resources to support student learning is not merely a matter of covering great distances; often geographical barriers, such as mountains or bodies of water that are not easily overcome separate educators and make collaborating difficult. Distance technologies can assist in overcoming some of these barriers, but many rural schools do not have the bandwidth capabilities to support high speed internet, or other technological advances needed for distance communication. Thus, a three-faceted barrier exists in many parts of the world that impede the access to equitable professional growth for teachers: distance and geographical barriers, few professionals within a given location, and insufficient technology infrastructures. As such, rural and remote schools are unintentionally restricted in accessing equitable human and instructional resources and supports for their teachers and students. This complicates the implementation of new initiatives and meaningful professional growth and development of teachers. As a result, we were interested in designing a professional development model that would work to sustain teacher growth despite these barriers.

METHODOLOGY

This research focuses on the use of a multi-faceted professional development model for rural and remote teachers aimed at increasing teachers understanding of, and ability to, orchestrate whole-class discussions. Data include responses from post-conference questionnaires as well as responses from semi-structured interviews and lesson study sessions.

Participants and Context

The professional development model was designed so that participants would travel to attend regional conferences twice annually. The professional development team providing the support travelled to five different rural locations and conducted the same conference in each of the locations. Collectively, these conferences supported approximately 350 teachers who would travel from distances up to 300 km, with presenters traveling up to 600 km for each conference. The conferences for the teachers in their regions were approximately six months apart; in between the conferences, teachers from multiple grade levels were supported through a modified version of lesson study using videos recorded from their classrooms.

The regional conferences were two-day events that took place on Friday evenings and Saturdays to reduce the amount of time teachers would need to be away from school and to allow more rural teachers to attend; finding multiple substitute teachers for any given rural district during the school day is problematic. Typically, teachers in the rural areas would have time to drive after school for the two-day events. Conferences focused on both content and pedagogy for focused grade levels. The first regional conference focused on algebraic thinking and the second focused on orchestrating productive mathematical discussions (Smith & Stein, 2011).

The format for the lesson study process that occurred at the schools in between the conferences was based on Murata's (2011) conceptual description of lesson study. In the modified version, teachers from a given geographical location would meet together to discuss video-taped lessons. With traditional lesson study, the teachers would observe each other teach, but a lack of personnel to cover classes in rural areas, made this impossible. As a result, a group of teachers, typically four, would meet together and plan a lesson for one of the teachers to teach. Then, one teacher would video record him or herself teaching the lesson and then would meet together with the lesson study group to review the lesson, reflect on the teaching, and discuss modifications to the lesson. The teachers repeated these cycles approximately three times in between each of the regional conferences.

Data were collected from multiple lesson study groups and multiple groups within the conferences; however, for the purpose of this paper, we focus solely on the teachers' perceived outcomes from these experiences. These data include written responses about their experiences and learning as a result of their involvement in this process. Teachers completed a written form at the end of their involvement in the project. The written form included multiple likert-scale items focused on their ability to apply their learning in their classroom as well as open-ended prompts about their experience, learning, and noticing.

Data Analysis

To analyse data, we gathered written responses from participants; of the nearly 350 participants 305 submitted completed forms. All quantitative data were analysed, based on the individual locations, for means with a total possible of five as a descriptive measure of participants' perceptions. Recognizing that differences in rural locations and teachers' needs dictated differences in their perceptions, we calculated means for individual conference locations, but focused on the overall trends in the data as opposed to specific statistical analysis. The intent of the analysis was to gain a description about their perceptions. In addition to the teachers' perceptions using the likert-scale items, we analysed their open-ended responses, both in the questionnaires and in the transcribed interviews and lesson study sessions, using constant comparative methods (Corbin & Strauss, 2008). The two researchers independently coded all responses and then met to compare codes and discuss themes across the data set.

STATEMENT OF RESULTS

Findings from this study indicate: 1) teachers in rural areas considered the model to support their professional growth and the learning of their students, 2) teachers felt confident that they could apply their learning in their classrooms, 3) teachers noted an increased understanding of how to facilitate whole-class discussions as a result of participating in the modified lesson study process.

Support for Rural Teacher Professional Growth and Student Learning

The rural teachers noted that the professional development model would have a long-term impact on themselves as a professional educators and their participation would have a long-term impact on their students. They appreciated the opportunity to collaborate with others who taught in the same, or similar, grade level. Table 1 shows the means, out of five, from the participant groups for each of the five locations:

Location	Long-Term Impact as a Professional Educator	Long-Term Impact on Students
Region 1 (n=66)	4.397	4.283
Region 2 (n=75)	4.338	4.329
Region 3 (n=98)	4.626	4.510
Region 4 (n=14)	4.308	4.154
Region 5 (n=52)	4.472	4.444

Table 1. Perception of impact on professional practices.

Following the professional development, the teachers commented on the effects of the professional support, "This was a great opportunity to learn and talk to fellow educators," and "I loved all of the breakout sessions geared directly towards my grade level and my needs."

Often rural teachers do not have opportunities to work with teachers of their own grade level, so participants valued the experience. One teacher wrote, "One group of teachers 'adopted' a lone teacher from [another] rural district for the day, stating that she 'was their sister for the day.'" Teachers from rural and remote areas appreciated opportunities to work and collaborate with others, opportunities they stated they often do not have.

Classroom Application

One of the most common themes in the data set was the prevalence of comments from the teachers about the extent to which they would apply what they had learned to their classroom. Table 2 provides descriptive data about their views.

Location	Will Share Learning with Others at School	Ability to Apply Learning in the Classroom	Ability to Implement Ideas Immediately
Region 1 (n=66)	4.349	4.286	4.667
Region 2 (n=75)	4.308	4.218	4.519
Region 3 (n=98)	4.396	4.426	4.653
Region 4 (n=14)	3.923	4.154	4.385
Region 5 (n=52)	4.093	4.148	4.556

Table 2. Ability to apply learning in classroom

Not all teachers were specific about the exact components they would take back to their classrooms, but they expressed the ease of applicability between the professional support and their classrooms. One teacher commented, "This was a very worthwhile conference. The ideas and direction are very applicable for teachers. This is much needed to give teachers confidence with new teaching demands" and "I can use this information on Monday. This was the missing link to professional development I have received in the past. Great insights to improve instruction."

Collaborative Development

At the various conferences, the main sessions focused on how to anticipate and monitor student thinking, and then select appropriate student work as initial stages of facilitating a whole-class discussion (Smith & Stein, 2011). During the conferences, the researchers modelled effective techniques for orchestrating whole-class discussions and provided time for participants to collaboratively develop the initial stages of facilitating a whole-class discussion using rich tasks. This allowed for participants to experience the process to the intended outcomes. A common response shared by one teacher focused on the benefits of seeing this modelled at the conference and the immediate application to the classroom, "I greatly appreciated all of the tasks that were brought for us to work with. Brainstorming together and discussing the tasks deepened my understanding for implementation." While the conferences provided an opportunity to understand the process of facilitating an effective whole-class discussion but the follow-up lesson study project provided for more in-depth work within the teachers' classrooms.

One teacher commented that as a result of participating in the modified lesson study process she had a much clearer understanding of how to engage students in whole-class discussions. She stated:

- I just know that when we started out, we were relying mostly on our same old [techniques]. So without that support of other teachers, I'm not sure… it's scary. I mean, you never really know if what you're doing is correct. And so to have other… to sit and watch a video of you teaching in a classroom and have teachers say, 'Oh, I love that,' or 'could you have asked it this way and maybe that would've changed the way things would've gone?' That support has been crucial.

Another teacher commented that the process was "really enlightening" because:

- You see things that you don't even realize are going on. And that was very helpful. I think I ask great questions when I'm doing it. But when you watch it you can see where you could become a better, where I could create better questions to pose.

Thus the teachers were able to recognize the benefits of this modified model and working with others.

DISCUSSION OF RESULTS

The teachers in this project considered the professional development to support their efforts as educators in rural areas. As noted in the research, addressing the needs of rural teachers is challenging and the processes used for teachers in more urban areas are not always applicable (Bauch, 2001). The video modification to lesson study afforded opportunities for teachers to observe others through technological means when they otherwise had not had similar opportunities. While this method deviated from traditional lesson study (Murata, 2011), the process closely mirrored traditional lesson study and met the needs of teachers. Likewise, the regional conferences provided opportunities for teachers to meet with educators of similar grades form other areas. This provided opportunities to discuss mathematical discourse (Smith & Stein, 2011) with professionals teaching similar grades; this often is not possible in small rural schools. In sum, results indicate that this model proved beneficial for rural teachers, based on their perceptions of applicability of effective teaching practices.

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BEDOUIN ETHNOMATHEMATICS – HOW INTEGRATING CULTURAL ELEMENTS INTO MATHEMATICS CLASSROOMS IMPACTS MOTIVATION, SELF-ESTEEM, AND ACHIEVEMENT

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Our study attempted to address young Bedouin students' persistent difficulties with mathematics by integrating ethnomathematics into a standard curriculum. First, we conducted extensive interviews with 30 Bedouin elders to identify the mathematical elements of their daily lives – particularly traditional units of length and weight. We then combined these with the standard curriculum to make an integrated 30 hour 7th Grade teaching unit that was implemented in two Bedouin schools. Comparisons between the experimental group (75) and the control group (70) showed that studying the integrated curriculum improved the students' self-perception and motivation, but did not affect achievements in school tests. The experiment had an extra social impact, changing students' attitudes to their own culture and the tribe's older generation.

RATIONALE

For decades, mathematics education has served as a "critical filter" for entering mathematics-related well-paid occupations that promote social mobility. Today, admission to prestigious university faculties, such as engineering, computer science or medicine, requires the applicant to show substantial knowledge and high scores in mathematics. Israel's Bedouin community is dramatically underrepresented in these occupations. This underrepresentation can be traced in part to a lack of the necessary mathematical knowledge, since the Bedouin students exhibit the lowest scores in national tests, as well as low levels of motivation and a lack of self-esteem (Ministry of Education, 2013).

The Bedouin tribespeople on which this study focusses live in Israel's southern region. Over the past few decades they have been undergoing a process of transition, shifting from their traditional way of life as nomads to the life of a semi-modern, sedentary society. This change is influencing their social structure, their cultural and economic situation, and the patterns according to which they live their lives.

Efforts to improve achievement through special programs for the Bedouins, such as the integration of technology in their schools (Amit, Fried, & Abu-Najah, 2007) have yielded marginal results at best. The challenge, therefore, remains to develop students' self-esteem and increase their motivation to learn by finding more interesting and effective ways to convey mathematical content. With this goal in mind, we combined standard mathematics education with a cultural ethnomathematics program and tested the impact of this integration on the motivation, self-perception and achievements of Bedouin junior-high school students.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 33-42. Hobart, Australia: PME.

BACKGROUND ON ETHNOMATHEMATICS

Researchers and educators have been proclaiming the importance of integrating cultural values into mathematics education for several decades. As D'Ambrosio (1999) claims, education must instil cultural respect and take cultural values into account to an extent far greater than what is offered in a regular curriculum. He points out that the lack of such integration is particularly harsh in mathematics education, which often has no connection to the world children experience (D'Ambrosio, 2002). To address this problem, Adam, Alangui, and Barton (2003) suggested that cultural aspects must be integrated into students' learning environment in a holistic manner that includes the epistemology of the mathematics, its content, the classroom culture and the approach to learning mathematics.

According to Gilmer (1990), learning mathematics without a cultural context can be a factor in lower mathematical achievements amongst students. Conversely, when students are exposed to different mathematical cultures, they discover that they have useful knowledge beyond traditional mathematics; this strengthens their self-confidence and makes them more willing to learn. Powell and Frankenstein (1997) found that Ethno-mathematics can help students solve more complex problems and Lipka et al. (2012) found that teaching mathematics by means of cultural elements changed attitudes towards math, increased mathematical understanding and significantly improved students' test scores. Based on the potential positive impact shown in the research literature, we decided to take an ethnomathematical approach to improving all aspects of our young Bedouins' mathematics learning experience.

METHODOLOGY

The aim of the study was first to identify those Bedouin cultural elements that have mathematical potential, specifically traditional Bedouin units for measuring length and weight, and then to integrate them into the formal curriculum of the Ministry of Education. Finally, the modified learning unit was implemented in two 7th grade Bedouin classrooms and its cognitive and affective impact was assessed.

Research questions

- What traditional units for measuring length and weights can be found among the Bedouins and what is their source?
- To what extent does the integration of ethno-mathematical elements into mathematics education influence students' motivation to study mathematics, their self-esteem and their achievements?

Stages of the study:

The study was conducted in four stages: 1. Exploring and identifying the Bedouin ethnomatematical elements. 2. Designing an integrative teaching unit based on a combination of the standard national school curriculum and the ethno findings from stage one. 3. Implementing the teaching unit with the Bedouin 7th graders (6 hours per week for 5 weeks. 4. Systematically studying the influence of the new unit on the students' achievements, self-esteem and motivation.

Population and data collection:

The population for the first stage was approximately 35 older Bedouins from a well-known and prestigious tribe in Southern Israel. Data for Research Question 1 were gathered through videotaped personal interviews and conversations with a variety of adult members of the tribe - elders, sheiks, sons of sheiks, and women. Elderly male tribesmen were happy to talk, tell stories and be videotaped, but elderly women refused to be videotaped (agreeing only to showing their hands and body, without faces) because of the "bad luck that the video brings". Some of younger Bedouin men (in their 40-50s) were suspicious at first, and reluctant to share traditional knowledge despite the fact their interviewer was a Bedouin from the same tribe. In time, however, they became more cooperative and helpful.

The analysis of the interviews was qualitative. Videos were looked at again and again, with the aid of two Arab speaking mathematics teachers and one linguistics teacher. Traditional measures of weight and length were first extracted separately by each individual analyst, and then refined through common discussion until a consensus was reached about what the measures, their literal meaning, and their equivalent in universal, "standard" measures. Examples of these are given in the results section.

The population for the implementation and impact testing stages consisted of 145 7th Graders from four classes – two from each of the tribe's two schools. In each school, one class served as a test group and the other as a control group, so that altogether the test groups included 75 students and the control groups included 70. The background and the level of all four classes was the same, and the students in the classes were randomly divided. The test classes learned according to the integrated ethnomathematics program (see below), and the control classes learned only according to the official Ministry of Education program. Both groups had the same amount of teaching time and were taught by qualified, experienced teachers.

Data for Research Question 2, regarding the new program's influence, were gathered using "tailor-made," anonymous questionnaires, which were administrated pre- and post-intervention. These questionnaires were divided into 22 statements, some of which addressed students' motivation and some of which addressed their self-esteem (e.g., "How good are you in mathematics", "If we were to link between Bedouins and mathematics, would you put more effort in your studies?" Many rejected this latter statement at the beginning.). The students were asked to rate their level of agreement with each statement on a scale of 1-5.

To determine the reliability of the data from the questionnaires, we ran a Cronbach's alpha internal reliability test. Results were: Pre-experiment motivation: $\alpha = 0.796$, Post-experiment motivation: $\alpha = 0.860$, Pre-experiment self-perception: $\alpha = 0.777$, Post-experiment self-perception: $\alpha = 0.945$. (Note: although the reliability of the tests from different periods was slightly different, it was still relatively high overall.) To further track the process and progress of the new program, the teachers recorded personal interviews with students from both groups and wrote a reflective teachers' journal (the process is beyond the scope of this paper.)

RESULTS: TRADITIONAL UNITS OF LENGTH AND WEIGHT:

The interviews yielded more than 35 traditional units of length and weight. These were usually anchored in the tribal Bedouin tradition, and were once part of the Bedouins' everyday nomadic life. As the examples below show (four examples out of 35), most of units, but not all, were associated with a specific literal meaning:

1. Concept: مقرط العصا Read: M'krat ala'sa - Literal meaning: stick throwing distance. This term is one of the most common amongst the Bedouins, especially amongst the older generation. To understand this concept, it is important to clarify that most Bedouins make their living by herding sheep, goats, camels, or other animals. The man in charge of the herd would generally hold a stick (80-150 cm) with which to lead or direct the flock (see Figure 1). The term "Makart Alasa" is more of an expression (not the real distance of throwing) and means the range from 3 kilometers to 4 kilometers to the side that the stick points. In fact, it is a vector with a magnitude and a direction.

Figure 1: Traditional stick held by Bedouin shepherd

2. Concept: شوط Read: Shoot - Literal meaning: the distance a horse rider can cover at a run in one burst without stopping. This is one of the more common measures today, and was designed for measuring particularly long distances. When we asked how far it was, one interviewee told us that it was the distance between two (ancient) towns - Lod and Ramle – or approximately 18 km.

3. Concept: قربة Read: Kerbh - Literal meaning: vessel for carrying water or milk
The kerbh (see Figure 2) is a vessel made of goatskin for keeping milk in the tent or cooling water. This unit of measurement was used mainly for the sale of milk or its products, though some interviewees claimed that it was also a unit for weighing water when it was brought from the well for drinking, especially if more than one kerbh was brought up. One kerbh is worth 30kg.

Figure 2: Traditional unit of weight, kerbh

4. Concept: الاوقية, وقية Read: Wakeh - Literal meaning: none. This is the most basic Bedouin unit of weight, and it is still used in many tribes today, measured with a deep plate. Some claim that there are four wakeh in a kilogram, so that it is worth 250 grams.

The ethnomathematics teaching unit, implementation and influence

The experimental teaching unit employed in this study, (about 30 hours in 5 weeks) addressed the topic "units of measurement," and included exercises for measuring length and weight in both universal "standard" units and the traditional Bedouin units identified in stage one. Although the traditional measuring units were taken from what once was the everyday life of Bedouins, most of the students were not acquainted with them, and an introductory chapter, including tribal stories, was therefore designed and implemented.

The students in the experimental group learned the universal units (i.e. metric system) and the traditional units simultaneously. The students experienced actual use of the two types of units in the classrooms, and at home with their families. The exercise below (Figure 3) is taken from the very beginning of the integrated unit; it asks the students to conduct measurements using both universal (kilograms) and traditional units (for example, the "wakeh," see above) and thus develop a sense of both kinds.

الوزن بالكيلو غرام	الوزن بالوقيه	الموزون	الرقم
	16	وزن بطيخة كبيرة	1
60		وزن مربي الصف	2
1000		وزن سيارة متوسطة	3
	200	وزن كيس قمح	4
	40	وزن خروف رضيع	5
3000		وزن بالة قش	6
	3	وزن حاسوب شخصي	7
1/4		وزن حقنة ذهب	8

Figure 3: Measurement with traditional and standard units, classroom exercise.

Influence on self-esteem, motivation and achievement

The data show that in the experimental group's statistical test, which included 75 observations (per variable), there were significant differences for both variables between the pre and the post test. In the experimental group (N=75), both motivation and self-perception were significantly higher after the teaching unit's implementation, and the change was more pronounced in girls than in boys. In the control group (N=70), no significant differences were found between the pre- and post-tests (see Figure 4). It is worth noting that the pre intervention score for motivation was more or less the same for the test group and the control group, but that the initial self-esteem in the control group tested much lower. The teachers attributed this to the fact that not being chosen for the experiment had harmed these children's self-esteem.

Results for achievements were obtained from regular school tests before and after the intervention. The headmasters of the schools were forbidden to give exact scores, but both of them indicated separately that there were no significant differences in test scores between the experiment class and the control class at any time. The teachers confirmed this as well.

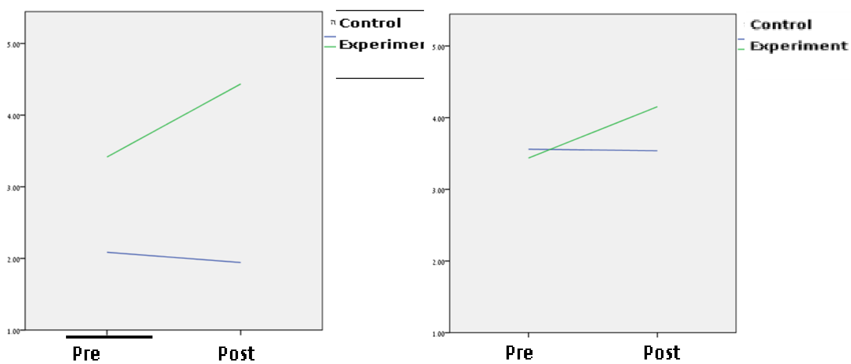

Figure 4: Self-esteem (left) and motivation (right)

DISCUSSION

This study arose from the need to find a way to help underprivileged students from the Bedouin tribes improve their mathematical achievements, motivation and self-confidence, after other attempts to do so had failed to produce results. Our experimental integration of ethno-mathematical elements into the teaching unit improved the motivation and the self-perception of the students who participated, particularly those of the girls, which supports the claims of researchers like D'Ambrosio (2000) and Fasheh (1982). On the other hand, the experiment's failure to improve the students' achievements contrasts with the findings of Lipka et al. (2012), where a significant improvement in national exams was found amongst Yup'ik students who studied ethno-integrated materials. This could be due to the fact that the Alaskan students studied the modified unit for a longer period, and that the rise in motivation over time translated into higher achievements, whereas in the relatively short time span of our own study the motivation had not yet matured to that extent.

Our study carries several additional benefits. First, this is the first time that all of these traditional Bedouin units of measurement have been systematically collected and categorised, and doing so will help preserve the remnants of a culture that may soon disappear. For the students, discovering that mathematics could be found everywhere all around them - particularly in the desert - was a thrilling experience. Moreover, they discovered that it was the older members of their tribe, those who do not drive cars or use cellular phones, who are, nevertheless, in possession of all this mathematical knowledge. The study unit led the students to ask their elders about the mathematics of their culture and helped to raise that generation somewhat in the estimation of their descendants. Lastly, the current study is limited in scope but promising. If this integrated study unit were expanded and extended in the long term, more Bedouin students might take a greater interest in mathematics as their motivation and self-esteem rose, improving their achievements in school and through them their chances of higher education and the socio-economic advantages that go with it.

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CONSTRUCTING AND CONSOLIDATING MATHEMATICAL KNOWLEDGE IN THE GEOGEBRA ENVIRONMENT BY A PAIR OF STUDENTS

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Understanding how students construct and consolidate abstract mathematical knowledge is a central aim of research in mathematics education. Abstraction in Context (AiC) is a theoretical-methodological framework for studying the processes involved in constructing abstract mathematical knowledge as they unfold in different contexts. This research uses the AiC framework to examine the processes used by two seventh-grade students working on a sequence of three tasks to construct and consolidate the Pythagorean Theorem and its expansions in the GeoGebra environment. The findings indicate that the pair constructed the majority of the expected knowledge elements. The two students also consolidated some of the components that were built.

INTRODUCTION

Understanding how students construct and consolidate abstract mathematical knowledge is a central aim of research in mathematics education (Dreyfus, 2012). Some researchers have tried to understand how students construct knowledge, especially deep, abstract mathematical knowledge such as concepts and strategies in learning situations. These researchers aimed at describing and understanding processes of knowledge construction and the conditions under which these processes (fail to) happen.

Abstraction occurs in different contexts, among them mathematical, social, curricular and learning-environmental. Abstraction in Context (AiC) is a theoretical framework for describing processes of abstraction in different contexts (Dreyfus, Hershkowitz, & Schwarz, 2001). These researchers defined abstraction as a process of vertically reorganizing previous mathematical constructs into a new structure. The AiC framework postulates that the genesis of abstraction passes through a three-stage process: the need for a new construct; the emergence of the new construct; and the consolidation of that construct.

The emergence of a new construct is described and analysed by the RBC model: recognising (R), building-with (B) and constructing (C). Recognising refers to the learner's realization that a previous knowledge construct is relevant for the situation at hand. Building-with involves the combination of recognised constructs in order to achieve a localised goal, such as the actualisation of a strategy, a justification or a

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, p. 40. Hobart, Australia: PME

solution to a problem. Constructing consists of assembling and integrating previous constructs by vertical mathematisation to produce a new construct.

Consolidation is a never-ending process through which students become aware of a construct, with the use of the construct becoming immediate and self-evident. Students' confidence in using the construct increases, and students demonstrate more flexibility in using the construct (Dreyfus & Tsamir, 2004). Consolidation of a construct is likely to occur whenever a construct that emerged in one activity is built-with in further activities. Hence, consolidation connects successive constructing processes and is closely related to the design of sequences of activities that enable it.

Various studies have used AiC to investigate learning processes in different contexts. Kidron and Dreyfus (2010) studied L's justification of bifurcations in a dynamic system, and specifically how instrumentation led to constructing actions and how the roles of the learner and a computer algebra system (CAS) intertwine during the process of constructing a justification. They showed that certain patterns of epistemic actions were facilitated by the CAS context. Dreyfus et al. (2001) used the AiC framework in the context of collaborative peer interaction and identified types of social interaction that support processes of abstraction.

Dreyfus and Tsamir (2004) conceptualised and studied the consolidation of students' constructs within the AiC framework. They developed an empirically based theoretical analysis of consolidation that emerges from a sequence of interviews with a talented student on the topic of infinite sets. They showed that consolidation can be identified by psychological and cognitive characteristics of self-evidence, confidence, immediacy, flexibility and awareness. They also found three modes of thinking conducive to consolidation: problem solving, reflective activity and an intermediate mode.

The current study aims at tracing processes of constructing and consolidating abstract mathematical knowledge in two seventh-grade students who solved a sequence of three tasks about the Pythagorean Theorem and its expansions in the GeoGebra environment.

METHOD

Salma and Sahar, two seventh-grade students from the same class, participated in the study. Their teacher attested to their high mathematical achievements. The participants and their parents gave their consent to participate in the study.

Three exploratory tasks were designed for the study. The first task dealt with the Pythagorean Theorem and its proof. The other tasks were based on the "What if not?" strategy (Brown & Walter, 1993). The second task dealt with the relations between areas of squares built on the sides of an obtuse/acute triangle, while the third task dealt with the relations between areas of regular polygons built on the sides of any triangle. GeoGebra was selected as the technological tool due to its dynamic nature and ease in use. An appropriate GeoGebra applet was built for each task.

In each of the three tasks the students were asked to propose a hypothesis regarding the mathematical situation and then to experiment with GeoGebra to verify or refute their hypothesis. Finally, they explained / proved the constructed mathematical concept / relation. Each task lasted about 45-55 min. and was recorded and transcribed verbatim.

In the a priori analysis for the expected construction and consolidation processes, the main knowledge elements and their sub-elements were assumed for all tasks. Figure 1 presents the a priori analysis of the connections between the knowledge elements subsequently described (E1 & E2 in the first task, E3-E6 in the second, and E7-E11 in the third task). An operational definition was developed for each element to guide the analysis of the students' abstraction activity. Due to space constraints we now offer only a few definitions.

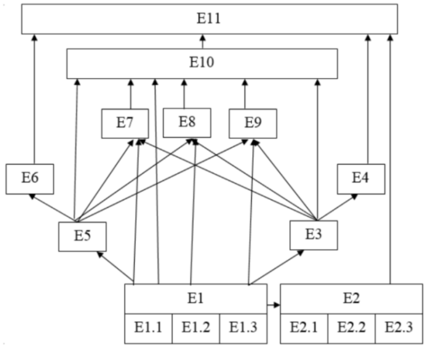

Figure 2: The connections between assumed knowledge elements

E1: Articulating the Pythagorean Theorem, with the following sub elements:

- E1.1: A right angle triangle whose sides are 3, 4, and 5 satisfies the Pythagorean Theorem.
- E1.2: Other right angle triangles satisfy the Pythagorean Theorem.
- E1.3: Generalisation: All right angle triangles satisfy the Pythagorean Theorem.

E2: Proof of Pythagorean Theorem, with the following sub-elements:

- E2.1: Recognising two squares as congruent.
- E2.2: The area is an additive magnitude.
- E2.3: The relationship between the areas of two congruent squares can be expressed as two equivalent algebraic expressions.

E3: The relations between areas of squares built on the edges of an obtuse triangle.

E4: The justification of E3.

E7: The relations between areas of equilateral triangles built on the sides of any triangle.

FINDINGS

The pair successfully constructed all the knowledge elements related to the Pythagorean Theorem and its expansions, with the exception of C1.1 and C11, which were partially built. Moreover, the pair consolidated some of the constructs that were built in the sequence of three tasks. This consolidation occurred in different situations: (a) when the pair tried to explain a certain constructed element (e.g., when explaining C3); (b) when the pair used a constructed element in further R and B actions to construct another element (e.g., using C1.1 to construct C1.2); and (c) when the pair thought reflexively about a constructed element (e.g., when the pair expressed their confidence in the correctness of C1.3).

The processes of constructing and consolidating knowledge in this study occurred in the context of the given sequence of tasks and the social and technological contexts. We demonstrate the analysis of three episodes: (i) the final constructing of C1.3 and the pair's confidence in its correctness; (ii) the beginning of constructing C3 and consolidating C1; and (iii) the construction of sub-element of C7 and consolidation of C3.

Episode 1: The final construction of C1.3 (first task)

1.		Inter.	So, what do you conclude?
2.	B1.2	Salma	The sum of the areas of the medium and small squares is equal to the area of the large square.
3.		Inter.	How can we refer to these squares other than as small, medium and large?
4.	R legs, hypo.	Salma	It is the same… we can call them legs and hypotenuse.
5.		Inter.	So, how can you express the relations?
6.	C1.3	Salma	[writing the answer] The sum of the areas of the small square built on the leg BC and the medium square built on the leg DC is equal to the area of the large square built on the hypotenuse CD.
7.		Sahar	Excellent, let's answer the next question.
8.		Salma	OK
9.		Sahar	[Reading the question] Are you sure that the relations you found in question 4 are satisfied in any right angle triangle? Explain!
10.		Salma	Sure.
11.		Sahar	Why? Explain!
12.	R1	Salma	Because the triangle will not be a right angle triangle if these are not equal.
13.		Sahar	Yes.
14.		Salma	Write it down. You are cleverer than me, right?
15.	R1.2	Sahar	[Writing] Yes, because we have observed many cases, right?
16.		Salma	No, wrong, give me the pencil…
17.		Inter.	You are saying that…
18.	R1.2	Sahar	Because we have observed many cases and the triangle was a right angle triangle.

19.	R1	Salma	We saw that we could not have any two numbers, three numbers, where the sum of these two areas, the small and the medium, is equal to the large one. That is, if the area of the two squares is not equal to the area of third square, the triangle will be acute or obtuse… that's why it's sure, sure.
20.		Sahar	OK, OK

At the beginning of constructing C1.3, the pair thought they could build a right angle triangle from any three sides. However, after working with the applet, they realised they could not do so. They also realised that to be a right angle triangle, the triangle has to satisfy the Pythagorean Theorem [turn 1]. In this construction process (generalization of the Pythagorean Theorem), Salma led the construction actions (R, B and C). At the beginning, she expressed C1.3 inaccurately [2]. By recognising the legs and the hypotenuse [4], Salma improved the construction of C1.3 [6]. Sahar agreed with Salma in all her actions.

The pair of students was confident about the correctness of C1.3 [10, 13]. While discussing the correctness of C1.3, they thought reflexively about C1.3 and consolidated it. Sahar was confident because she tried many cases [15], and Salma was confident because she began to construct the inverse theorem [12]. Statements 12 and 19 show that Salma inadvertently constructed the inverse theorem. Thus, the knowledge elements from the second task began to build the first one.

Episode 2: Beginning of construction of C3 and consolidation of C1 (second task)

The pair collected data (areas of squares built on edges of obtuse triangle) in three cases:

1.		Sahar	We have to find the relations.
2.		Salma	16, 9.994, 48.72 [areas of squares built on sides of triangle, fig.2a]
3.		Sahar	Salma… find the relationship, I do not know how!
4.	R1 B1	Salma	9 plus 4, 9 plus 4 [trying Pythagorean Theorem in the obtuse triangle, fig. 2b]
5.		Sahar	13
6.	B1	Salma	16 plus 7 [trying Pythagorean Theorem in obtuse triangle, fig. 2c]
7.		Sahar	What?!
8.		Salma	16 plus 7, 33, 23 increase it little
9.	B1	Sahar	Plus 11, it *is not as the previous relationship*, I am sure.
10.		Salma	You're right, it is not.

In Episode 2, the students tried to see whether "the equivalent relationship" found previously (C1: Pythagorean Theorem) holds even for an obtuse triangle. Thus, by building-with C1 (towards C3) they saw that the relations in the given situation are different from C1 [8, 9]. In the construction actions, Salma led for the most part. She recognised C1 [4] and built-with it the new relations [4, 6, 7, 9]. At the end of this episode, they both agreed that C1 would not hold in the case of obtuse triangles. It is important to note that in further discussions, the pair found it difficult to recognise "less than\more than" relations and that recognition of these relations was suggested by the researcher.

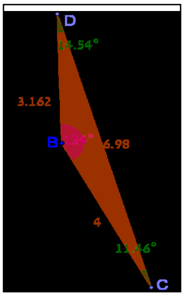

Figure 2a

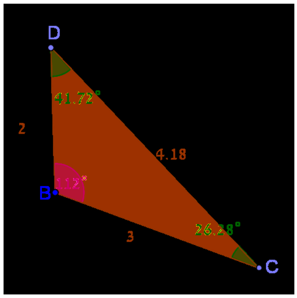

Figure 2b

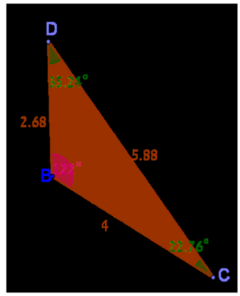

Figure 2c

Figure 2: The three cases the students worked on in Episode 2

Episode 3: Constructing C7.2 and consolidating C3

1.	R	Salma	This triangle is obtuse, write!
2.		Sahar	Wait! Before we found that in obtuse triangles the sum of the areas of the medium and small squares is less than the area of the large square.
3.	B3	Salma	Yes, yes [checking]. It's not the same…Hhhhhh…no, sorry, it's the same, the same… we said that it is less than. Let's try another obtuse triangle 6, 6… it's also the same… right? [Fig. 3a].
4.		Sahar	Because you did not change the length of the sides, you changed the angles.
5.		Inter.	You can change the length of the angles from here [the applet]
6.	B3	Sahar	9+10= 19, if we increase it by 2, the sum will be 21. . . so it's right [Fig. 3b].
7.		Salma	Sure it's right.
8.		Sahar	We can say 20.
9.		Salma	Write: "less than"… Do you want to try more cases? Sure it's "less than".
10.		Sahar	I will write it down. In the case of an obtuse triangle… mmm… What did we say before?
11.		Salma	"Less than" relationship.

12.	$C_{7.1}$	Sahar	The sum…, but we have to explain our "aim".
13.		Salma	"Less than" relationship! Write it down. After that we'll explain.
14.		Sahar	The relationship is…
15.		Salma	The relationship is "less than"… the sum of the areas of the two squares built on the edges comprising the obtuse angle …
16.		Sahar	…built on the edges comprising the obtuse angle is less than the area of the square built on the edge opposite to the obtuse angle.

Figure 3a

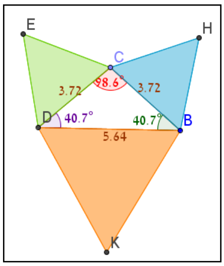

Figure 3b

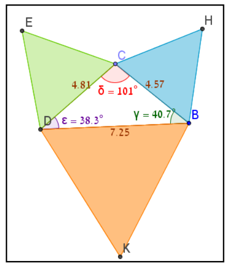

Figure 3a & 3b: The two cases explored by the pair in Episode 3

Episode 3 shows the process of constructing the sub-element C7 ($C_{7.1}$, the case of an obtuse triangle). The other two elements (the case of a right/acute triangle) were constructed similarly. The episode also demonstrates the process of consolidating C3. The two students constructed collaboratively: Salma recognised the obtuse triangle [1] and Sahar recognised C3 [2]. Then, the pair built-with these constructs [3, 6] to generate the new construct and expressed it collaboratively [15, 16]. The consolidation of C3 occurred when the pair recognised and built-with this construct to generate the new one (C7). They accomplished this with relative ease and immediacy. This indicates that they consolidated the previous construct.

DISCUSSION

The study traced the processes of construction and consolidation of the Pythagorean Theorem and its expansions by a pair of students who worked on a sequence of three tasks in the GeoGebra environment. The findings indicate that the pair constructed the majority of the knowledge elements and consolidated some of the constructed elements. The construction and consolidation processes occurred in a technological, social and task context.

The pair's working processes included exploration of different cases, generalization and explanation/proof. GeoGebra supported the learning during the exploration phase (for similar findings, see Becta, 2003). For example, in the process of constructing C1, the pair explored different cases by manipulating the triangle in the GeoGebra applet. The pair selected "representative" cases such as polygons with different types of side lengths: large/small numbers, fractions and integers. Exploring these different cases in GeoGebra enabled the students to generalise the relations (Dikovic, 2009).

Furthermore, our findings show that the pair consolidated some of the constructed elements. The consolidation occurred *during* a task and in the *following* tasks. Consolidating during a task occurred in three cases: (1) when the pair thought reflexively about a constructed element while explaining (e.g., while trying to justify C3), similar to Dreyfus and Tsamir (2004); (2) while they expressed their confidence in the correctness of a construct (e.g., expressing their confidence in the correctness of C1.3); and (3) while they used a constructed element in R and B actions in order to build a new element (e.g., when the pair used C1.1 in order to build C1.2), as in Tabach et al. (2006).

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JAPANESE FIRST GRADER'S CONCEPT FORMATION OF GEOMETRIC FIGURES: FOCUSING ON VIEWPOINT CHANGES WHILE IDENTIFYING FIGURES

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The purpose of this study is to describe how first graders change their viewpoints in identifying geometric figures through instructional tasks. The author conducted questionnaire for first graders (n=69) who have not yet formally studied the definition of triangles. The questionnaire consists of four parts; (1) classifying shapes, (2) identifying triangles and providing explanations, (3) engaging in instructional tasks, (4) re-identifying triangles and providing explanations. The students are divided into three groups according to instructional tasks as follows, answering the number of sides and vertices of triangles, drawing figures, reading definition of triangles. The results indicate that different changes in students' viewpoint took place, such as from visual viewpoints to descriptive ones.

INTRODUCTION

One of the aims of school mathematics is raising from experience-based to scientific concept. At the initial stage of concept formation of geometric figures, it is necessary to know how students form conception of geometric figures through learning mathematical viewpoints.

Over the past few years in Japan, few studies have been made on early elementary school children from the perspective of mathematics education (Matsuo, 2010). Some results of these research indicated difficulties of classification and identification (Masuda, 2011; Isobe et al., 2002). For example, students have constrains because of prototype schemes and they could see the parts not the whole of shape, example in the case of triangles students could identify "pointy" characteristics. This research reported students' understanding of geometric figures and characteristics of their understandings. In order to raise students' mathematical viewpoints of geometrical figures, we must consider what kinds of educational approach we should take and what kinds of activities lead to change their viewpoints. For these reasons, this paper focus on students' change of viewpoints under previously-planned activities regarding geometric figures, in order to shed light on how these students modify or change their ideas after engaging in instructional tasks.

THEORETICAL BACKGROUNDS

Characteristics of Pre-recognitive level

van Hiele (1986) introduced five levels of geometric thought, the visual level, the descriptive level, the theoretical level, formal logic and the nature of logical laws. However Clements and Battista (1992) pointed out the existence of pre-recognitive level before the visual level, in which young children could perceive shapes but could not recognise them. Regarding pre-cognitive level, some characteristics have been reported: Children cannot generalise shapes because of developmental reason grounded in a misconception influenced by prototype example (e.g., Clements, 1998; Magara & Fushimi, 1981; Shiraisi & Ota, 1986), children cannot grasp whole figures because of influence from visual characteristics like "pointy" and "skinny" (e.g., Clements et al., 1999; Tokyo Educational Research Institution, 1959; Isobe et al., 2002), there are different appearances in the same shape because of anisotropy of space (Katsui, 1971).

According to the view that the van Hiele levels, seen as types of reasoning, develop simultaneously (Clements et al.， 2001)， in the early years, students at the pre-recognitive level develop simultaneously three different types of reasoning (namely visual reasoning, descriptive-analytic reasoning, abstract-relational reasoning), being visual reasoning the dominant one. The author mainly describes first graders' condition of visual reasoning which they identify shapes according to their appearance as visual wholes, and descriptive- analytic reasoning which they describe parts and properties of shapes.

Five phases of understanding of geometric figures

How could students change from the condition that visual reasoning is dominant to the condition that descriptive-analytic reasoning is dominant? In order to analyse the changing situation, two aspects of students' figural concepts, image representation and linguistic representation are focused. It is because students form individual conception of geometric figures during classes, for example image representation is formed by recognising figures written on a textbook, blackboard and notebook, on the other hand linguistic representation is formed by understanding mathematical terms, definitions, properties and propositions.

Kawasaki (2007) makes "aspect models for understanding of figural concepts" (Figure 1), which is based on the relation between the two aspects, there are five phases of understanding of figural concepts from elementary school to junior high school as follows:

Phase 1: Figural concepts are recognised through visual image of shapes. Linguistic representation is not used consciously.

Phase 2: Figural concepts are recognised through image of prototype. Terms of Figural concepts are used consciously however image affects recognition stronger than terms.

Phase 3: Figural concepts are analysed by components of shapes. Students could express their attribute using components linguistically.

Phase 4: Necessity of definition of geometric figures are recognised. Students are aware of definition and properties.

Phase 5: Students become aware that figural concept could be described logically using linguistic expression.

According to Japanese Course of Study, first graders belong to mainly Phase 1 and 2. For students in these Phases, it is important to consider effective way to introduce linguistic representation. The function of linguistic representation is giving students objective mathematical concepts. Therefore this paper focuses on mathematical terms written in the explanation and analyses how they change those terms such as sides and corners. Through revealing one cross section of changing aspect, we describe how students develop descriptive-analytic reasoning being visual reasoning dominant.

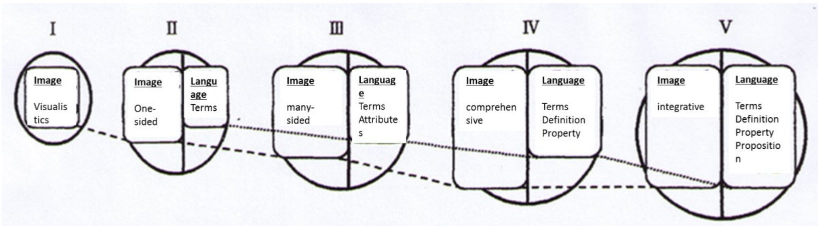


Figure 1: Aspect models for understanding of figural concepts (Kawasaki, 2007)

METHOD

Subjects

The participants in this survey were 69 first graders from a Japanese public elementary school. The survey was conducted in January 2012, until then they have experienced building and classifying various shapes using concrete objects in their circumstances, however mathematical terms like definition, name, side and vertex were not learned.

Three classes were chosen in Grade 1 and each class was examined by different instructional tasks included in the questionnaire. Instructional tasks in group A (n=23) is answering the number of sides and vertices of triangles, drawing figures, reading definition of triangles; group B (n=21) is answering the number of sides and vertices of triangles, drawing figures; and group C (n=25) is reading definition of triangles.

Questionnaire

Classroom teachers implemented the questionnaire and read questions aloud to help students to understand the questions. The questionnaire consists of four items; (1) classifying shapes (Figures 1 & 2), (2) identifying triangles and explanation (Figure 3), (3) instructional tasks, (4) re-identifying triangles and explanation (Figure 4).

Instructional tasks have two factors, operational and linguistic. Operational tasks are answering the number of sides and corners of a triangle, drawing a shape enclosed by four straight lines, linguistic task is reading the definition of triangles. Both of them are referred to in the Grade 2 textbook. The students in each group do the given activities as follows;

Group A: counting the number of sides and vertices, drawing a shape enclosed by four straight lines, reading the definition of triangles (operational and linguistic)

Group B: counting the number of sides and vertices, drawing a shape enclosed by four straight lines (operational)

Group C: reading the definition of triangles (linguistic)

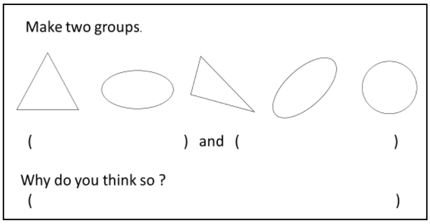


Figure 2: Classifying circles and triangles (Q1)

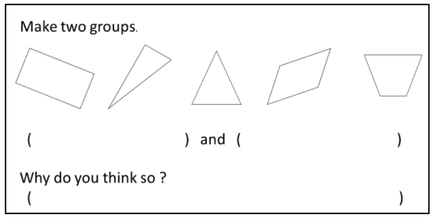


Figure 3: Classifying triangles and quadrilaterals (Q2)

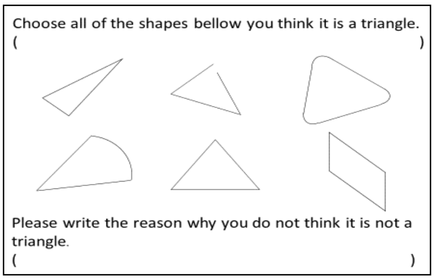


Figure 4: Identifying triangles before activities (Q3)

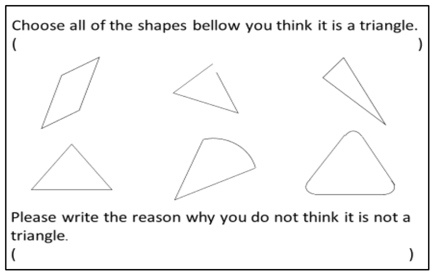


Figure 5: Identifying triangles after activities (Q4)

RESULTS AND DISCUSSION

Table 1 shows the mean and standard deviation for Question 3 in each class. There are statistically differences between group A and group C ($p<.04$), group B and group C ($p<.01$). Note: a maximum score is six.

	Group A(n=23)	Group B(n=21)	Group C(n=25)
Mean	4.26	4.38	3.76
S.D	0.56	0.64	0.77

Table 1: Mean and Standard Deviation of Identification (Q3)

We see from Table 2 that there is different tendency among three groups in each shape, for examples, group A has higher percentage at rounded triangle whereas group B has higher percentage at sector. It seemed that what classroom teacher taught influence this results, so that this paper does not compare among three groups, but describe the characteristics of the students whose visual reasoning is dominant and then focus on changing viewpoints of each student.

	Quadrilateral	Unclosed triangle	Scalene triangle	Equilateral triangle	Sector	Rounded triangle
A(n=23)	100%	74%	48%	78%	61%	65%
B(n=21)	100%	81%	19%	100%	100%	38%
C(n=25)	96%	40%	44%	88%	68%	40%

Table 2: The percentage of correct answer in six shapes (Q3)

Characteristics of students who cannot classify shapes

At first, 31 of 69 students who cannot classify circles and triangles, triangles and quadrilaterals are selected in order to clarify the characteristics of their viewpoint in identifying triangles.

Table 3 summarises the results of comparison between the students who cannot classify and who can classify in the case of quadrilateral, unclosed triangle, sector and rounded triangle. Table 3 indicated the students who cannot classify have tendency to make a judgement without certain reasons why it is not a triangle even if they judge correctly. For example, most of the students judge a quadrilateral is not a triangle but 58% of "cannot classify" cannot write their reasons. By contrast, "can classify" have tendency to make judgements with their reason such as name and component in any case of shapes.

Category 1: judging triangles

Category 2: without reasons or inadequate reasons (Because it is not a triangle)

Category 3: name of the shape (Because it is a quadrilateral)

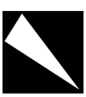
Category 4: components of the shape (Because the corner is rounded)

Category 5: concrete objects or impression of the shape (Because it is look like a pizza)

Category	Quadrilateral		Unclosed triangle		Sector		Rounded triangle	
	Cannot classify (n=31)	Can classify (n=38)	Cannot classify (n=31)	Can classify (n=38)	Cannot classify (n=31)	Can classify (n=38)	Cannot classify (n=31)	Can classify (n=38)
Judging triangles	3%	0%	39%	34%	32%	18%	42%	61%
Without reasons	58%	23%	35%	11%	59%	18%	39%	8%
Name	29%	53%	16%	29%	0%	3%	10%	11%
Components	3%	13%	7%	26%	6%	18%	6%	16%
Concrete objects	7%	8%	3%	0%	3%	42%	3%	15%

Table 3: Comparison between "cannot classify" and "can classify"

In addition, some characteristics of visual reasoning influenced by students' own image representation are;



Reason why this is not a triangle: It is a long triangle. The left hand side is long. Too thin and long. The upper part is long. Look like a paper airplane. Two sides are long. It is slant. It does not look like rice ball.

Reason why this is not a triangle: The underpart is not straight. The underpart of triangle is round. This is triangle but underpart is curved. (23students out of 69 expressed curved line "underpart")

According to these descriptions, students have some misconception like "the pointy corner should be situated on top", "triangles are not long". These reasons are evidence of influence of prototype examples, however we can see some signs of mathematical viewpoints such as "under part is …", "left hand side is …". Even though they could not use the math terms consciously, they could see the shapes not only from their appearance but also from the viewpoint of sides and corners.

The aspects of changing viewpoints depending on the tasks

The 38 students who can classify triangles and quadrilaterals are selected. First students who judge the rounded triangle is triangles are focused (23 students shown as 61% shaded in Table 3). Table 4 shows how they change their judgment after teh activities. It is indicated that five students out of six in group A, three students out of six in group C do not change their judgments, in contrast all students in group B change their ideas correctly, especially five students out of eleven use component.

	Judging triangles	Without reasons	Name	Components	Concrete objects
A(n=6)	5	0	0	0	1
B(n=11)	0	1	3	5	2
C(n=6)	3	0	1	2	0

Table 4: Changing situation after learning activity (Rounded triangle)

Now, we take a close look at changing viewpoints of geometric figures. In the case of sector, students mainly use two mathematical viewpoints, corners and sides, when they judge if it is triangles or not. Those who write the reason including components and concrete objects are selected (A: 7 students, B: 18 students, C: 10 students).

The terms in Table 5 indicate some examples as follows, for sides; curved line, not straight line, for corners; rounded corner down side is rounded, imitative word "KAKU (crooked)", for concrete objects: look like a pizza, a cake, a corn. As Table 5 indicates, four students in group A have a viewpoint "sides", eight students in group B have a viewpoint "corners", eight students in group C have a viewpoints "sides" after simple tasks. Considering these phenomena it is possible to say that the words "straight line" in triangle's definition affects the reasoning in group C, whereas the students in group B was not affected by the words "straight line" in the question "How many straight lines are there in triangles?". Rather the word "corner" in the question "How many corners are there in a triangle?" affects them. Regarding group A, it seems quite probable that the students could not focus on certain viewpoints because they might have been confused by definition and operational activities.

Comparing between group of "cannot classify" and "can classify", it is clear that group of "can classify" in group A and B change their reasons more than group of "cannot classify". However group C has different tendency as six students in "cannot classify" change their reason. One of the reasons for this phenomena might be that it is easier for them to capture the word “straight line” in the definition they read.

	Cannot classify			Can classify		
	Before	After	Number of students	Before	After	Number of students
A(n=7)				judge triangles	sides	1
				sides	sides	2
				inadequate	sides	1
				sides	inadequate	1
				concrete object	concrete object	2
B(n=18)	sides	sides	1	concrete object	corners	6
	sides	corners	1	concrete object	sides	3
				concrete object	concrete object	4
				sides	concrete object	2
				sides	corners	1
	nothing	sides	1	Judge triangle	sides	1
	sides	concrete object	1	inadequate	sides	1
C(n=10)	inadequate	sides	1	sides	sides	1
	sides	sides	1	sides	inadequate	1
	nothing	sides	1			
	inadequate	sides	1			

Table 5: Changing viewpoint of identification (Sector)

CONCLUSION

Most of the student used visual reasoning to identify triangles, and were noticeable influenced by prototype examples, however, we can see signs of mathematical points of view, such as “left hand side is ...”. We could say that they can change their reason after engaging in instructional tasks because of these conditions.

To sum up the results, depending on the nature of the instructional task students were engaged with, each group exhibited different tendencies in viewpoint change. For example, a few changes were noticeable in Group A, change to “corners” took place in Group B, and change to “sides” occurred in Group C. And also, it seems that the nature of the task has certain degree of influence in the categories “cannot classify” and “can classify”. For example, Group C has more students who “cannot classify” than the others.

These results indicate that students identify shapes using mathematical viewpoints under proper instructional tasks, even in the level of visual reasoning dominant, which means they cannot classify shapes because of their appearance. In order to create effective practices and lessons , we should consider three components as follows; (1)

Sign of mathematical viewpoint based on their experiences: What kinds of informal and mathematical terms students used before learning mathematical viewpoints, (2) Types of learning activity especially to foster linguistic aspect, (3) Level of understanding of geometric figures.

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MERLO: A NEW TOOL AND A NEW CHALLENGE IN MATHEMATICS TEACHING AND LEARNING

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The core element of this paper is an innovative tool for teaching and learning, based on equivalence of meaning across different kinds of representation and called MERLO (Meaning Equivalence Reusable Learning Objects). After presenting a general MERLO approach, we focus on its application in mathematics education and, in particular, in the Italian institutional context of secondary school. In this paper we describe a first level of experimentation regarding the use of MERLO in teachers' education, as part of an ongoing research. The lens of Meta-Didactical Transposition allows an analysis of the process of teachers' professional development with MERLO.

INTRODUCTION

MERLO (Meaning Equivalence Reusable Learning Objects) is a didactical and methodological tool developed and tested since the 1990s by Uri Shafrir and Masha Etkind at Ontario Institute for Studies in Education (OISE) of University of Toronto, and Ryerson University in Toronto, Canada (Shafrir & Etkind, 2010). It is a very adaptable tool, suitable for several subjects and uses: we can mention, for example, the use of MERLO that Masha Etkind is doing in architecture as an assessment tool to check students' deep understanding of concepts.

The aim of our research is the application of MERLO in mathematics education, linking the MERLO approach with some elements of the Italian institutional context. For this reason, the choice of the research context is that of a Master, held in the University of Turin, for in-service mathematics teachers who will become educators for other teachers.

In the first part of this paper we present the theoretical framework for MERLO approach. Then we focus on our experience concerning the use of MERLO in teachers' education, inside the Master context. A brief description of the Meta-Didactical Transposition model allows us to use it as lens for the analysis of the process of teachers' professional development with MERLO. The paper ends with a final discussion and some proposals for further research that we intend to develop.

MERLO APPROACH: THEORETICAL FRAMEWORK

MERLO (Arzarello, Kenett, Robutti, Shafrir, to be submitted; Etkind, Kenett, Shafrir, 2010) is a database, that is a sorted and organized collection of MERLO activities covering relevant concepts within a discipline, through multi-semiotic representations in multiple sign systems. Each element of the database is a structured MERLO activity,

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 57-64. Hobart, Australia: PME.

that includes one target statement TS (it encodes different features of an important concept) and four other statements, linked to the target statement by two sorting criteria: shared or not shared *meaning equivalence* with the target statement, shared or not shared *surface similarity* with the target statement.

The term *meaning equivalence* designates a commonality of meaning across several representations. The term *surface similarity* means that representations "look similar": they are similar only in appearance, sharing the same sign system, but not the meaning.

Based on these two sorting criteria it is possible to create four types of statements, called Q1, Q2, Q3, Q4, depending on the fact that they share or not share equivalence of meaning and/or surface similarity with the target statement. The four types of statements are shown in the figure below.

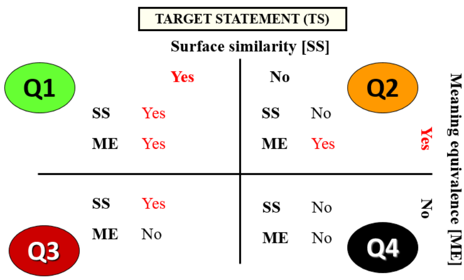


Figure 1: Types of statements, linked to a TS in a MERLO activity

A typical MERLO activity contains five statements: a target statement plus four additional statements of type Q2, Q3 and Q4; they can be in a variable number, provided that at least one Q2 statement is present, in addition to the TS. We avoid the inclusion of Q1 statements, because they make the activity too easy, for their equivalence both in appearance and in meaning with TS.

In the version of MERLO activity for students, obviously, the type of each statement is not revealed. The students are required to recognize the statements in multiple representations that share equivalence of meaning and to write the concept that they had in mind when making the decisions. In this way MERLO activity combines multiple-choice (recognition) and short answer (production). It gives a feedback on students with two main scores: recognition score and production score. This feedback is useful to the teacher for getting information about the level of understanding (the so called "deep understanding") of their students on a particular conceptual knowledge.

Regular use of MERLO activities enhances a particular way of thinking, named "conceptual thinking" (Etkind & Shafrir, 2013), requiring that learners explore patterns of equivalence-of-meaning among ideas, relations and underlying issues. Learners' attention is focused on conceptual contents and on meanings, through comparative analysis of multi-semiotic representations of conceptual situations.

MERLO approach to parsing and analysing concepts is applicable to various subjects for recognizing, representing, organizing, exploring and manipulating knowledge. It is particularly recommended in mathematics, where the ability to shift from one to another representation of the same object and the coordination of multiple representations in more than one semiotic register are fundamental competences, in order to access the underlying meaning and to understand mathematics (Duval, 2006). MERLO approach is in line with national (INVALSI, in Italy) and international (PISA, TIMSS) assessment tests, where the ability of shifting between representations is abundantly evaluated. Moreover, it is also a didactical tool for avoiding or overcoming the so called "duplication obstacle" (Duval, 1983). This kind of obstacle leads students to the consideration of two representations of the same mathematical object as two different mathematical objects, but also, conversely, it may represent students' inability in grasping two different meanings of a mathematical object in only one representation. The duplication obstacle described by Duval is a real source of difficulty in mathematical learning (Fishbein, 1987) and it may be the cause of failure in mathematics at school (examples in Ascari, 2012).

RESEARCH CONTEXT: MASTER FOR IN-SERVICE TEACHERS

The Master for Mathematics Teachers' Educators is an educational program of two years, held in the University of Turin to the Department of Mathematics and directed to Italian in-service secondary school teachers (30 enrolled teachers), who will become teachers' educators.

We think that the choice of this research context is the most appropriate for the application of MERLO in mathematics education and in teaching and learning in Italian secondary school: a fundamental aspect is teachers' education. For this reason, a training process of two years was implemented inside the Master context, in order to make teachers aware of this new didactical tool.

During the first year (2013/2014) teachers were involved in the following training phases:

Phase 1. Translation of MERLO activities, produced in other countries (Russia) and solution of them.

Phase 2. Construction of new MERLO activities in geometry, inside Italian school context and curriculum.

Phase 3. Solution of MERLO activities produced at phase 2.

The experimentation inside the Master context is going on during the second year (2014/2015) with a small group of seven voluntary teachers as teachers-researchers, and with the whole group as learners and practitioners. Teachers in this phase are involved in the design of new MERLO activities linked to INVALSI tests and m@t.abel activities.

INVALSI is a National Institute for Evaluation of Instruction and Education System in Italy and every year it addresses INVALSI tests to monitor the level of Italian students' learning and to compare it with other European realities.

m@t.abel is a national project, started in 2006 and promoted by the Ministry of Education, Universities and Research, the National Agency for School Development and Autonomy (ANSAS - INDIRE), the National Associations for Mathematics and Statistics (UMI - SIS). It points at the renovation and improvement of mathematics teaching and learning and it is aimed at mathematics teachers in secondary school levels. The contents are related to four basic Standards that are part of the curricula of many countries around the world, as well as in OECD-PISA and INVALSI tests: numbers, geometry, relations and functions, data and forecasts.

The choice to link MERLO activities with the national assessment INVALSI and the national project m@t.abel is not accidental, but has the aim to root MERLO approach in the Italian institutional dimension. The institutional dimension is important because the teachers' professional development is contextualised inside and constrained by the institutions, such as national curriculum, the Ministry of Education, policy makers, textbooks, national assessment and so on.

Teachers involved in the experimentation, starting from some INVALSI tests of the previous years and from some m@t.abel activities, produced several MERLO activities. Here we present two examples from INVALSI questions.

The first example (Figure 2) is designed for lower secondary school and it is about numbers: aim of the activity is to test students' comprehension of the concept of fraction. With this objective in mind, teachers' design process started from the choice of a graphical representation of a fraction, made of small squares. Using it as target statement, teachers created four other statements: two Q2 statements that share equivalence of meaning with TS but do not share surface similarity, because they are represented in a different semiotic system; a Q3 statement, linked with TS by surface similarity, sharing the same semiotic system, but not the meaning; a Q4 statement that does not share neither equivalence of meaning, nor surface similarity with TS.

The second example (Figure 3) is designed for upper secondary school and requires recognition of relations and functions in different semiotic systems. This MERLO activity is linked to a real life context and concerns two offers for ski-lifts. Teachers' design process started from a natural language description of the two offers, chosen as target statement. Then teachers created four other statements: three Q2 statements that share equivalence of meaning, but do not share surface similarity with TS, representing the same two offers in a different way (Cartesian graph, table and formal language); and a Q4 statement that does not share neither equivalence of meaning, nor surface similarity with TS.

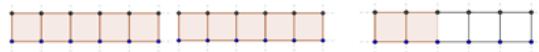
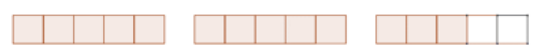
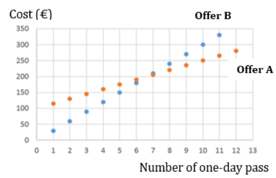
<table>
<tr><td></td><td>TS</td><td>Q2</td></tr>
<tr><td>At least two out of these five statements – but possibly more than two – share equivalence-of-meaning.
1. Mark all statements – but only those – that share equivalence-of-meaning
2. Write down briefly the reasons that guided you in making these decisions</td><td>A[ ]
</td><td>B[ ]
$\frac{12}{5}$</td></tr>
<tr><td>Q2</td><td>Q3</td><td>Q4</td></tr>
<tr><td>C[ ]
$2+\frac{2}{5}$</td><td>D[ ]
</td><td>E [ ]
$\frac{12+3}{5+3}$</td></tr>
</table>

Figure 2: first example of MERLO activity

<table>
<tr><td colspan="3">Mario is going on holiday at a ski area. To take advantage of ski lifts (chair lifts, cable cars, ...), he can choose between two offers, A and B, both valid for the entire winter season.</td></tr>
<tr><td></td><td>TS</td><td>Q2</td></tr>
<tr><td>At least two out of these five statements – but possibly more than two – share equivalence-of-meaning.
1. Mark all statements – but only those – that share equivalence-of-meaning
2. Write down briefly the reasons that guided you in making these decisions</td><td>A[ ]
Offer A: initial fixed cost 100 € plus 15 € per day (for every day you use ski lifts).

Offer B: 30 € per day, with no initial cost.</td><td>B[ ]

</td></tr>
<tr><td>Q2</td><td>Q2</td><td>Q4</td></tr>
<tr><td>C[ ]
<table>
<tr><th>Number of days in which you use ski lifts</th><th>Offer A (cost in €)</th><th>Offer B (cost in €)</th></tr>
<tr><td>1</td><td>115</td><td>30</td></tr>
<tr><td>2</td><td>130</td><td>60</td></tr>
<tr><td>3</td><td>145</td><td>90</td></tr>
<tr><td>4</td><td>160</td><td>120</td></tr>
<tr><td>5</td><td>175</td><td>150</td></tr>
<tr><td>6</td><td>190</td><td>180</td></tr>
</table>
</td><td>D[ ]
Offer A: $c = 100 + 15g$

Offer B: $c = 30g$

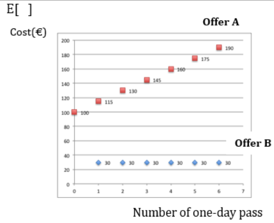
g number of one-day pass
c cost (€)</td><td>E[ ]

</td></tr>
</table>

Figure 3: second example of MERLO activity

The task for students is to recognize the equivalence of meaning across several kinds of representation of the same mathematical object (fraction in the first example, function in the second example). Students have to mark in the first example the three statements in position A, B, C (in the case shown in Figure 2) and in the second example the four statements in position A, B, C, D (in the case shown in Figure 3) because they share equivalence of meaning. We remind that the position of statements is changeable and their type (Q2, Q3 or Q4) is not revealed to students.

A further request is to explain the reasons for the choice, request that promotes argumentative expertise. The feedback received by teachers is not only that of a closed-answer test, but also an argumentative open-answer. Teachers can decide to use MERLO activities as a tool for final assessment or for formative assessment, in order to support a mathematical discussion in class. MERLO activities are based on the ability to read and interpret several kinds of representation, to see the same mathematical object represented in different sign systems and to be able to recognize it, even if there is a surface similarity but not equivalence of meaning with another object. We think all these aspects are fundamental in mathematics teaching and learning.

META-DIDACTICAL TRANSPOSITION AS LENS FOR ANALYSIS

The Meta-Didactical Transposition model (Aldon et al., 2013; Arzarello et al., 2014; Clark-Wilson et al., 2014) is useful and appropriate as a lens for the analysis of teachers' professional development process, in the research context just described: indeed, this theoretical model has been conceived to take into account the complexity arising from the intertwining of the processes involved during a teacher education program.

The theoretical background for the Meta-Didactical Transposition model is derived from Chevallard's Anthropological Theory of Didactics (Chevallard, 1985, 1992). In particular the model refers to the notions of didactical transposition and praxeology. This is the starting point for an expansion, which focuses mainly on "meta" aspects.

Through the Meta-Didactical Transposition model we can analyse teachers' professional development from a dynamic point of view, highlighting the interactions between the two communities involved in the teachers' education process (the community of researchers and the community of teachers) and observing their initial praxeologies and how they evolve over time, giving birth to new shared praxeologies.

During collaboration between the two communities some components that are internal for researchers become internal also for teachers, like the MERLO theoretical framework. However, there is not only a shift of theoretical knowledge from researchers to teachers, because each community adds something new, with the aim of arriving at a new praxeology shared by both of them, which we could call "MERLO pedagogy". At the moment of our research and experimentation, the shared praxeology is related to the task design of MERLO activities and to their possible use with students. Teachers, during meetings and working together with researchers, arrived to some methodological choices, necessary to have coherence in the design of new MERLO activities. We can mention the choice that all statements must be correct, that is a particular choice in respect to other traditional tests. The sharing of methodological choices in the design process led to the creation of MERLO activities by teachers, such as those presented in the previous section.

FINAL DISCUSSION AND PROPOSALS FOR FURTHER RESEARCH

MERLO activities are rooted on the construct of Meaning Equivalence, that is equivalence of meaning through different kinds of representation: the task for students is to recognize commonality of meaning in several sign systems. The experience with teachers in the research context of the Master highlighted the complexity of Meaning Equivalence construct and the delicacy of some choices during the design process of MERLO activities, because the kind of knowledge that will be tested on students depends from these choices.

The analysis of the examples produced by teachers, the discussions among them during meetings and the next reflection of researchers, led to introduce a sort of “empirical distance” between statements and concepts. For example, if it is simple to recognize a Q2 statement as equivalent with the target statement TS, it is more difficult recognizing what it means that a statement is “closer” to TS than another one, even if both are equivalent to TS.

We think that this sort of “distance” is fundamental in MERLO and so the idea for future research is to explore it in empirical way. In this regard, a possible task is to construct a Boundary of Meaning Map, that is a map where statements have to be placed inside or outside certain boundaries, associated with the meaning of a particular target statement TS, and into three different levels more or less close to the boundary.

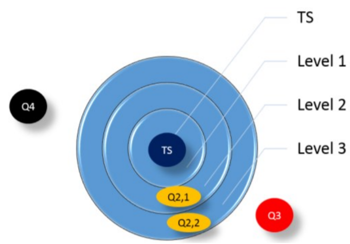


Figure 4: Boundary of Meaning Map

The same task can be addressed to different people, both teachers and students. The analysis of the collected data could give important information to the researchers, because different boundaries of meaning of the same concept might emerge and several gaps between students and teachers could be identified. The awareness of the existence of these gaps is the first step for the development of future research, which aims at didactical and pedagogical interventions to bridge them.

Acknowledgement

This work is part of a wider research project, which involves also Ron Kenett and Uri Shafrir: we want to thank them for the cooperation.

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A PROOF-OF-CONCEPT VIRTUAL LEARNING ENVIRONMENT FOR PROFESSIONAL LEARNING OF TEACHERS OF MATHEMATICS: STUDENTS' THINKING ABOUT DECIMALS

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This study details the trial of a proof-of-concept virtual learning environment (POC-VLE) for professional learning of teachers of mathematics, focussing on student thinking about the well-researched topic of decimal notation. The POC-VLE was situated in a virtual mathematics classroom where the teacher participant observed scripted interactions between teacher and student avatars. It was found that exploration of the POC-VLE was effective for enabling teacher participants to determine the thinking of an avatar student with 'whole number thinking'; demonstrating their understanding by answering items in the same way as the avatar student. Teacher participants predicted that other teachers of mathematics would have a similar experience and noted potential for using this VLE for professional learning.

BACKGROUND

This paper explores the feasibility of using a virtual learning environment (VLE) for professional learning for teachers of mathematics. The motivation for considering the development of a VLE was that members of the research team had provided face-to-face professional learning for teachers of mathematics over several decades and were cognisant of the need for more teachers of mathematics to be able to easily access research-based professional learning regardless of geographical location. Increasing access to high-speed broadband in Australia provided us with the opportunity to rethink delivery of professional learning. Hence the research team investigated the feasibility of a virtual learning environment that could be accessed by teachers of mathematics, to improve their pedagogical content knowledge (PCK). The ability to identify student thinking (including misconceptions) is an important element in the PCK framework of Chick, Baker, Pham and Cheng (2006).

The use of online interactive media for education is still a relatively new area of research (Hamid, Chang, Waycott & Kurnia, 2011). Early indications are that the use of online interactive media for education is a complex phenomenon that needs to consider many issues and conceptualisation and development of an online platform which addresses the issues is a challenge which bears further examination.

There are various types of online interactive media and these are referred to as Virtual Learning Environments (VLEs) by Mueller and Strohmeier (2011). Grossman (2010) described two types of VLEs used for prospective teachers. The first type are virtual field experiences where prospective teachers access websites to view videos of exemplary teaching or classrooms in real-time. The second type involves virtual

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 65-72. Hobart, Australia: PME.

classrooms which "can provide a sheltered opportunity for prospective teachers to engage in targeted practice of clinical skills" (Grossman 2010, p. 2). SimSchool, an example of this type of VLE (see Meletiou & Mavrou, 2013; Gibson & Kruse 2011), addresses the development of a teacher's general pedagogical knowledge rather than focus on pedagogical content knowledge for the teaching of mathematics. In addition, there are VLEs where participants take the persona of avatars, such as in second life (see for example Muir, Allen, Rayner, & Cleland, 2013).

The material for this proof-of-concept virtual learning environment (POC-VLE) was based on the *Teaching and Learning about Decimals CD* **(**Steinle, Stacey, & Chambers, 2006). During face-to-face professional learning we have used this CD to illustrate students' thinking to deepen teachers' PCK and to highlight to teachers the importance of paying attention to the reasons behind student answers. This CD was itself based on a decade of research into decimal misconceptions involving over 3000 students (from Grades 4 to 10) and nearly 10 000 tests. The misconception about decimal notation that is used in the POC-VLE is *whole number thinking* (see Steinle & Stacey, 2003a) where a student treats the digits to the right of the decimal point as another whole number. While this thinking can be simply explained (in this case a student would incorrectly think that 2.345 is larger than 2.8 as 345 is larger than 8) it is more complex for teachers to recognise this thinking from observing a student's responses (written or spoken), which are sometimes correct and sometimes incorrect.

While this POC-VLE is focussed on one misconception, Steinle and Stacey (2003a) describe ten possible misconceptions, grouped by the general behaviour of the students when comparing pairs of decimal numbers. The *longer-is-larger* group of misconceptions includes students who believe (for various reasons) that a decimal number with more digits is a larger number than one with fewer digits. Hence, these students would incorrectly choose 4.63 as a larger number than 4.8, yet correctly choose 5.73 as larger than 5.6. *Whole number thinking* is the most common reason for this behaviour, but there are others. In contrast, the *shorter-is-larger* group of misconceptions is characterised by students who believe (again for various reasons) that a decimal number with fewer digits is a larger number than one with more digits. Hence, these students would correctly choose 4.8 as a larger number than 4.63, yet incorrectly choose 5.6 as larger than 5.73. For example a student with *denominator focussed thinking* would choose 5.6 as larger than 5.73 as the first number involves tenths and the second involves hundredths, and 'tenths are bigger than hundredths'.

Steinle and Stacey (2003b) estimated that approximately 7 in 10 students would at some stage experience one of the longer-is-larger misconceptions (of which *whole number thinking* is by far the most common) while they are in Grades 4 to 6. For students in secondary school (Grades 7 to 10), the corresponding figure is 2 in 10. While our long term goal is to develop a VLE that assists teachers to identify specific misconceptions in their own classroom, it is because of the high incidence of *whole number thinking,* that it was chosen as the focus for the POC-VLE.

Through our face-to-face professional learning, we have observed a shift in paradigm where teachers move from believing that all incorrect answers are due to careless errors, to a view that an incorrect answer could be 'the tip of the iceberg'; incorrect answers might be the result of a deeply held misconception. A student with a misconception can provide predictable correct and incorrect responses based on a logical consequence of their thinking and teachers need to be able to recognise what this might look like in student work.

A virtual classroom can provide a 'textbook case' of a real-life situation, in that it provides a clear example of a type of situation so that the user is not distracted by extraneous details. This is an interim step before the user deals with a real, 'messy' classroom. Gibson and Kruse (2011, para 11) note that "… even a simple model of reality can benefit learning. For example, it shears away unnecessary details while simplifying a real system. Models allow us to hold — in our hands and minds — some aspects of a system that cannot otherwise be experienced".

In contrast to these VLEs, our POC-VLE focusses on deepening a teacher's PCK through providing opportunities to identify an avatar student's thinking in a virtual mathematics classroom. We are not aware of any other VLE where the participant (in-service or pre-service teacher) is an observer in a virtual mathematics classroom of the scripted interactions between teacher and student avatars.

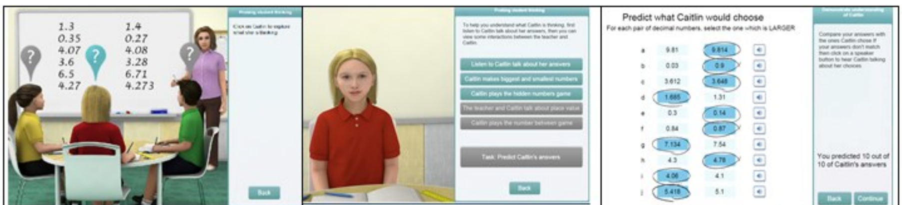
THE PROOF-OF-CONCEPT VIRTUAL LEARNING ENVIRONMENT

We wanted to investigate whether the complexity of the classroom can be encapsulated adequately in a VLE in a way that is both engaging and deepens teachers' PCK. Our first step was to conceptualise and create a proof-of-concept VLE, which is discussed here. The usability of the POC-VLE was an important consideration for time-poor teachers, but more importantly, usability would impact the ability of participants to engage with the ideas about student understanding.

We wanted the participant to actively explore the virtual classroom, acting like a detective to discover how an avatar student was thinking about decimal numbers, in the same way that they might in their own classrooms with real students. Participants had to infer student thinking about decimals from analysing interactions in the POC-VLE classroom, through observing and analysing responses to tasks by a student avatar (Caitlin) and through hearing her articulate her reasoning. Among other features, the POC-VLE enabled participants to: observe interactions between teacher and student avatars; observe a student avatar completing tasks and listen to them 'thinking out loud' to provide insight into the reasons for their responses; and demonstrate knowledge of one misconception by predicting the responses of a student avatar with this misconception.

To develop the POC-VLE the research team conceptualised the 'look and feel' and interactive features of the VLE and created story boards and scripts to translate some aspects of the *Teaching and Learning about Decimals* CD into a suitable format for the VASTPARK programmers. Figure 1(a) shows a screenshot of the end of the first mini-movie where avatar students chose the larger number from pairs of decimals. While the student responses for the first three pairs are the same (all correct), the responses for the fourth pair of numbers vary, indicating that further exploration is required. This sets the scene to engage the participant; they now need to be a detective to solve the mystery. The participant has reached a decision point and must choose an avatar student to explore their thinking. In the POC-VLE, the only option is Caitlin. In Figure 1(b) the options for further exploring Caitlin's thinking are shown. Each link provides a different mini-movie, showing different activities and interactions, providing further insight into Caitlin's thinking. For example in *Caitlin makes biggest and smallest numbers*, the avatar teacher and Caitlin discuss making decimal numbers (3._ _) with cards containing the digits 0-9 and Caitlin discusses her answers. Once the participant completes all links in (b) they can then choose to *Predict Caitlin's answers*. Figure 1(c) shows the result of matching Caitlin's responses precisely, which demonstrates the participant understandings of *whole number thinking*. The participant can listen to Caitlin explain her reasoning by selecting the radio button next to an item. From our experience in delivering face-to-face professional learning, the model presented in the POC-VLE, where teachers examine student responses (both written and verbal), promotes a paradigm shift for teachers. Teachers place greater emphasis on student reasoning, rather than focussing only on correctness of answers. Correct answers can sometimes be given for the wrong reasons. For example, when Caitlin circled 0.9 as larger than 0.03 (which is correct), she has compared two whole numbers, 9 and 3, rather than two decimal numbers.

The research question was: What are participants' experiences of using the POC-VLE, and in particular, can exploration of the POC-VLE help develop understanding of *whole number thinking*?



(a) Decision point at end of first mini-movie | (b) Options for probing Caitlin's thinking | (c) Predicting Caitlin's answers

Figure 1: Screenshots from POC-VLE

METHODOLOGY

Eight participants were involved in trialling the POC-VLE. The participants were selected based on their expertise: mathematics coordinators in schools, experienced users of technology in schools, and leadership at a system level (responsibility for organising professional learning for teachers within their educational system). The participants were: four secondary school mathematics coordinators (C1, C2, C3, C4);

one lecturer in technology education (T1); one mathematics teacher with extensive experience in teaching technology (T2); and two education systems perople (S1, S2).

Two focus groups of four participants were held. Participants completed Questionnaire 1 (background information) prior to the trial and Questionnaire 2 (usability) directly following the trial. To determine if the design features of the POC-VLE were memorable, Questionnaire 3, (screen-shot stimulated recall) was completed after a 30 minute break. This was followed by an audio-recorded focus group discussion with semi-structured questions.

In this paper we report the results of Questionnaire 2, with supporting comments from focus group discussions. In Questionnaire 2 there was one specific question which asked participants to comment on how well they were able to understand *whole number thinking* after exploring the POC-VLE and also how well a teacher new to teaching mathematics would be able to understand this type of student thinking. The responses to the items in Questionnaire 2 as well as the focus groups discussions were collated and are reported below.

We will discuss pedagogical and usability aspects of the POC-VLE. Pedagogical aspects relate to the ability of the POC-VLE to engage participants in learning about an avatar student's understanding of decimals. Usability aspects related to the "look and feel" of the POC-VLE. Navigation had to be intuitive and easy, otherwise participants would be frustrated. We were keen to get participant feedback on pedagogical and usability aspects to inform future development of the full virtual learning environment.

RESULTS AND DISCUSSION

There was overwhelming support for the POC-VLE. In particular, all eight participants rated the following three aspects as 'helpful' or 'very helpful' on a five point Likert scale: *watching a mini-movie to set the scene*; *watching and listening as the avatar teacher and Caitlin interact*; and *using the radio buttons (to hear Caitlin describe her thinking) in order to understand her thinking*. Not only did they rate these aspects highly from their own perspective, they also rated them similarly from their perception of the perspective of a teacher who is new to teaching mathematics.

The following pedagogical and usability aspects were important considerations for the researchers to ensure that any later iterations of the POC-VLE appealed to a wide audience, not just maths coordinators, technology enthusiasts or systems people.

Pedagogical: Ability to explore student thinking

In focus group discussions participants agreed that working through the POC-VLE enabled them to understand *whole number thinking*, and they predicted that other teachers would also have the same outcome. There were a range of positive comments about identifying *whole number thinking*, for example: "… did a good job in highlighting this misconception" (T1) and "Gets teachers to realise ways in which students can think" (C1). In addition, C1 noted that the ability to observe interactions between an avatar teacher and avatar student would "…remind all teachers [about] the importance of getting students to explain their responses", thus the POC-VLE promoted additional teaching messages, not just an understanding of *whole number thinking*. Participants also suggested that the POC-VLE might help other teachers to understand that incorrect thinking may not necessarily be the result of careless errors, but could be due to misconceptions. For example, S1 wrote "Students can have consistent, logical (in their mind) incorrect ways of thinking".

Pedagogical: The right tasks/questions are needed to diagnose student thinking

The POC-VLE had a range of tasks embedded in it, with the purpose of showing how to probe Caitlin's thinking. This modelled the need for teachers to ask a range of questions in order to understand student thinking. There was agreement that the POC-VLE not only helped to illuminate Caitlin's thinking, but also provided motivation for participants to reflect on their questioning practices in classrooms. For example, C1 commented that "Hearing dialogue between teacher and student [avatars] ….gives ideas on how to go about questioning …" and C2 agreed that the POC-VLE helped show how "…to model the type of questions teachers can ask students". The POC-VLE promoted careful questioning and task selection. It provided a model for teachers, promoting the need to target questions and to carefully analyse reasons behind a student's responses in order to uncover any mathematical misconceptions.

Usability: Aesthetics, navigation and control

Aesthetics of the POC-VLE, including design, animation, voice pitch and look of the classroom, was discussed in the focus group. There were a range of comments showing the diversity of views about aesthetics, for example, C4 noted "No problems with the "look" as in animated figures. The classroom was nothing like our classroom….The virtual classroom was possibly too idealised". In contrast, C2 thought that the look mimicked a classroom "It was very user friendly. The screens were clear and it looks like a classroom with teacher and students. It made sense for this look and was appropriate". The diversity shows that one 'look' may not immediately appeal to all, but the participants overcame any aesthetic issues quickly, noted by C3 who wrote "Starts off feeling a little "odd". The characters were static, i.e., no lip movement. After a few minutes it became more familiar".

The participants in this study generally agreed that the POC-VLE was intuitive and easy to use. For example, C2 indicated that the POC-VLE was "very easy to follow. The options were large and well located and there was not any confusion as to what to do next. Re-watching, pausing and moving on was easy". Participants felt that they were in control of their experience, which is particularly important for online professional learning which may be undertaken by out-of-field teachers who may not feel confident with the mathematical content.

Usability: Self-paced learning and the ability to revisit activities

The POC-VLE provides self-paced learning where participants can readily revisit activities, hence having control over the sequence and the pace. This contrasts with face-to-face professional learning where the presenter controls both. Participants reported a positive experience with the self-paced nature of the POC-VLE. For example, T2 wrote that "It was good to be able to go back if you needed to revisit anything, or go faster if it was repetitive." In addition, C4 valued the ability "…to pause, take notes, handle interruptions… repeat/explore thinking". For teachers unfamiliar with decimal misconceptions, of which *whole number thinking* is only one, there is the opportunity to revisit the classroom interactions and avatar student's responses to confirm understanding of the misconception. Future iterations of the VLE will include additional misconceptions in the one virtual classroom.

Overall, the POC-VLE provided good opportunities for participants to both identify student thinking and reflect on the way that incorrect answers may be the result of more than careless errors. The approaches used to highlight student thinking motivated participants to reflect on their approaches to questioning in class and the ability to revisit activities was seen to be valuable.

CONCLUSION AND FUTURE DIRECTIONS

A VLE for professional learning has the potential to enable large numbers of teachers of mathematics to engage with research findings about teaching and learning of school mathematics to deepen PCK, including knowledge of student understanding. The positive responses (albeit by a small number of participants) to the POC-VLE suggested that this goal could be achieved. The participants reported that the POC-VLE enabled them to understand *whole number thinking* and that it was easy to use. The participants were very favourable about the POC-VLE and its potential to be used in professional learning for teachers of mathematics; anticipating the next version which will incorporate additional student avatars (illustrating different mathematical thinking) and teaching exemplars.

The perceived usefulness of the POC-VLE for teacher professional learning was encapsulated by this comment by C4: "Excellent. I would give this to new teachers and teachers new to the teaching of year 7 maths. In fact, as a coordinator [of mathematics staff], I would insist on doing this … if you were new to the year level team".

Participants suggested a range of ways that the POC-VLE might be used with groups of teachers; it could be worked through individually, or used in mathematics staff meetings at school. They recognised wide applicability of the POC-VLE and there was anticipation for the next version, with more features, including teaching exemplars. Overall, the study found that there was potential for such a VLE to be used to provide meaningful professional learning for teachers of mathematics and to develop understanding of student mathematical thinking.

Acknowledgements

We would like to thank the participants and other members of the research team (Kaye Stacey, Robyn Pierce and Ken Clarke). This project was funded by The University of Melbourne and the Institute for a Broadband Enabled Society (IBES).

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THE READER AND THE WRITER PERSPECTIVES OR THE SUBTLETIES OF SYMBOLIC LITERACY

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This paper draws on a larger study (Bardini, 2003) where epistemology is envisaged as a complementary tool for didactic analyses of students' use and understanding of algebra. We will focus on one particular epistemological idea explored in our work namely 'the author and the reader' perspectives (Serfati, 2005), here referred to as 'the reader and the writer' perspectives. We will describe the potentials offered by such an epistemological lens when it comes to better understanding what underlies the construct of algebraic expressions and highlighting the multiple facets that constitute what we may call 'symbolic literacy'. As a practical implication, we will show how the epistemological framework provides a fine-grained tool for selecting and devising appropriate tasks aimed at assessing students' symbolic literacy.

INTRODUCTION

Mathematics derives much of its power from the use of symbols (Arcavi, 2005), but research at secondary level has shown that their conciseness and abstraction can be a barrier to learning (MacGregor & Stacey 1997; Pierce, Stacey & Bardini 2010). Since symbols form the basis of mathematical language, mathematical fluency, like fluency in any language, requires proficiency with symbols, which we call 'symbolic literacy'. Under the notion of symbolic literacy lies the notion of 'symbol sense' described by Arcavi (1994, 2005), which includes among other components the ability to manipulate as well as 'read through' symbolic expressions. For the purpose of the present paper, we have privileged the term 'literacy' in order to better convey the idea of mathematics as a language of discourse (Usiskin, 2012), that can take place in oral or written form, that one can either decipher ('read') or produce ('write'). As Usiskin notes:

> Mathematics is, among its many other attributes, a language of discourse. It is both a written language and a spoken language, for – particularly in school mathematics – we have words for virtually all the symbols. Familiarity with this language is a precursor to all understanding. (Usiskin, 2012, p.4)

The dual feature of symbolic literacy, namely the ability to both read and produce mathematical expressions, has previously been explored by Rubenstein and Thompson (2001), who analysed high school students' difficulties with symbols. In their work, the authors classified such difficulties as verbalisation challenges (translating symbols into spoken language), reading challenges (understanding the concepts represented by the symbols), and writing difficulties (producing symbols), while noting that combinations of these three groups of difficulties frequently occur together. Rubenstein and Thompson cite Usiskin (1996) who asserts that "if a student does not know how to read mathematics out loud, it is difficult to register the mathematics".

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In what follows we will show how epistemology complements and potentially enrich these didactical ideas. We will further describe in particular the two aforementioned theoretical approaches that come into play when exploring a mathematical expression: the analytical approach –carried out by the author/writer– and the synthetic approach –carried out by the reader. Our aim is to examine these epistemological ideas through a didactical lens. A set of mathematical tasks were designed to illustrate the theoretical principles; we suggest these may constitute the ground for assessing the different facets of symbolic literacy at different school levels.

THE READER PERSPECTIVE

Historical and epistemological background

What better source to look for the exact interpretation of a symbol than having its own 'inventor' describe it? This is specifically what Widmann did in the 1526 edition of his arithmetic work *Behennde unnd hüpsch Rechnüg auff allen Kauffmanschafften*, where the signs '+' and '–' (slightly more elongated than nowadays) first appeared in print. Because this was the first printed occurrence of such symbols, Widmann provided the reader with a series of symbolic expressions and their 'translations'. Thus, next to the expression '3+30' the reader could find the explanation: "Add the number 30 to the number 3". Similarly, the expression '4 – 17' should be interpreted, according to Widmann, as the following instruction: "Take away the number 17 from the number 4" (for a facsimile of the original work see Cajori 1928, Dover edition 1993, p. 130). Such interpretations for '+' and '–' were later on extended to mathematical expressions that also comprised signs for the unknown, such as in Clavius *Algebra* (1608) where the expression '$1x – 7$' (Clavius used a specific cossic sign distinct from our Latin letter x) had to be interpreted as: "From the unknown value, take away the number 7". As Serfati (2005) summarises, in both cases the author's goal was to

> provide the reader with a symbolic representation for an elementary *instruction*, that is a rule for carrying out an action or an operation (here the subtraction) between two quantities, should they be known numbers or unknown magnitudes. (Serfati, 2005, p.73 –our translation)

The rhetorical interpretation of the basic operations can be further applied to expressions that involve more than one operation. Thus, in front of e.g. '$(2 + x) \times 3.5$' one should, once an order has been decreed to the different operations in order to avoid all ambiguity, interpret: "Add the number 2 to the unknown number with sign x. Then multiply the result by the number 3.5". In other words, interpreting compound expressions comes down to executing a series of elementary instructions, in a very specific, prescribed order.

Didactical considerations

Figure 1 represents a task (to rather be considered as an instance of a potential set of tasks) that illustrates the theoretical idea of interpreting algebraic expressions as following/applying a series of instructions. This sort of task is not uncommon in algebra school textbooks, and can sometimes be found in the introductory parts related to the notion of function.

The following series of instructions constitutes a computational algorithm for the expression $\sqrt{2x-5}+3$. "Let x be a number, multiply it by 2, take away 5 from the result, take the square root of the result, then add 3 to the result." Write an algorithm for each of the following expressions: $[5(2+x)]^2$; $\sqrt{3+\frac{1}{x}}+2$; $[2(-x+3)]^2$

Figure 1: Task 1 – Interpreting mathematical expressions as series of instructions

As mentioned previously (and this applies to all tasks presented in this paper), this task should not be considered as 'closed'. Not only can it be adapted to different school levels (by modifying the complexity of the mathematical expressions – we will come back to this notion later on), but can also be presented in different ways to best fit the profile of the students of a same class. Let us now discuss some points regarding the algebraic expressions and their 'translation' into algorithms.

First, it is interesting to note that whenever there is a sign for the unknown, the first instruction of the algorithm is one that relates to the unknown, regardless of its position in the expression. This is demonstrated in the worked example but also holds for, e.g., the first given expression, where, in left to right reading, the unknown is not visually first. Even though the order of the operations carried out by the reader requires the addition to be considered first (cf. previous section), it does not impose any hierarchy for the two elements that it constitutes: the number with sign '2' and the unknown number of sign '*x*' are to be considered, from this point of view, equal. However this task, by its design, implicitly attributes a primacy to the unknown sign, theoretically nonexistent (since the addition is commutative). This phenomenon is all the more overt in the second given expression, where the sign of the unknown constitutes the denominator of a fraction, itself visually the second component of the addition. While the primacy of the unknown in the first case was more for clarity purposes (one could argue that we could have equally said "Take 2, now add an unknown number *x*,..."), in the later case it turns out to be a necessity for the algorithmic description of the expression. One may well question this necessity (arguing we could have alternatively said "Take the reciprocal of an unknown number *x*..."), but this is only because of the very specific status of the fraction (the reciprocal). When considering rational expressions with a more complex fraction than '1/*x*' (even in the simple case of e.g. $3+\frac{10}{2x-1}$), the structure of the fraction demands the unknown be addressed first.

As mentioned previously, applying this type of task in a classroom setting requires consideration of the complexity of the expressions involved in order to best suit the students' level(s). From what has been said above, it is easy to see that the complexity of a given expression is tightly related to the number of unknowns involved and its 'display', and less to the number of operations involved (for detail see Bardini, 2003, pp. 108-110). Indeed, an expression such $\sqrt{(4x+5)^2-2}$, with five operations, seems much easier to describe in terms of a sequence of instructions than e.g. $\frac{c+1}{(b+2)(a-1)}$, with also five operations. In the latter case, the difficulty is less due to the fact that there are multiple unknowns involved, rather it is because the unknowns appear both in the numerator and the denominator of the fraction, which requires the blocks of instructions to be considered 'in parallel', hierarchically equivalent. A possible description might be: "Let *b* be a number, add 2, let *a* be a number, subtract 1, multiply the first result with the second, let *c* be a number, add one, divide this result by the last result found". Having to temporarily 'store' the intermediate results makes the appropriate algorithmic description less evident than when the operations 'flow' linearly.

Finally, some comments about a possible variation of such task. Another – and maybe more common – version of Task 1 is when the sequence of instructions, rather than being written in full, are described diagrammatically, as in Figure 2 for $[2(-x+3)]^2$.

$$x \longrightarrow \qquad \longrightarrow \qquad \longrightarrow \qquad \longrightarrow f(x) = [2(-x+3)]^2$$

Figure 2: Variation of Task 1 – diagrammatic representation of a series of instructions

Although this variation does reflect the idea of the interpretation of an expression through a series of instructions (reader perspective) and may seem more 'accessible' when considering its application in a classroom setting, this approach presents serious limitations to the expressions that may be involved. More precisely, it excludes key categories of expressions possible in Task 1. First there are expressions with fractions that cannot be translated in terms of reciprocal. In this approach (Figure 2), every intermediate result must be 'active in the given instruction' thus excluding any compound expression in the denominator. More generally speaking, a fraction where unknowns (one or multiple) are present in both numerator and denominator is not possible. The limits go beyond the case of rational expressions. It is easy to see that for the same reasons, expressions such e.g. $5x(3x+1)$ cannot be part of the task. Unlike Task 1, it is not possible to carry out instructions 'in parallel' to later regroup. We can conclude by saying that this approach to Task 1 is only applicable to 'simple' expressions and contrary to the first version of Task 1, the complexity of the tasks will be highly linked to the number of operations involved.

THE WRITER PERSPECTIVE

Historical and epistemological background

In the previous sections we saw that expressing rhetorically the interpretation of a symbolic writing means translating it into a sequence of instructions prescribed in a very specific order. In fact, when deciphering an algebraic expression, the reader begins with the most 'internal' operation (or node if we consider expression trees) and progressively reconstructs the hierarchy, through a synthetic process. Thus the expression $[5(2+x)]^2$ may be rhetorically interpreted by the reader as "Add the number 2 to the unknown number with sign x. Multiply the result by the number 5. Square the last result". But such rhetoric description turns out to translate in the very opposite order the conceptual aim of author of the expression. If we consider the previous example, while the reader begins by interpreting the most internal operation (addition), the ultimate ambition of the author is to represent a square (which base is itself the result of a multiplication of two factors, etc.), the operation that conceptually structures the expression. In other words, while the reader of an expression tackles it by the most internal 'operations', the first combinatorially speaking (the root of the expression tree), what guides the author is the meaning of the expression, last described in the series of instructions. Let us note here that the double ordination that describes the decomposition and rebuilding processes conveys theoretical steps that, in practice, are neither entirely synthetic nor purely analytical. As Serfati notes, the actual reading of a symbolic writing is in fact a mix between analysis and synthesis, where "the synthesis comes after to consolidate the progress of the analysis" (Serfati, 2005, p.115). Similarly, in order to have a full picture of the expression he/she is producing, the author needs to consider the 'inner' operators at the same time he/she considers those conceptually more important.

Didactical considerations

Equally important to assessing students' ability in 'interpreting' symbolic writing is to give opportunities that place the student in the position of the author of an expression. This is usually done through activities where students are required to mathematically formulate a problem, given in a specific context (mathematical or not). Our idea here was to reproduce as faithfully as possible the theoretical idea of what it might be like to think like an 'author' of an expression, without necessarily having to set the task in a particular context (which adds further potential difficulties to be taken into account and hinders some aspects of 'symbolic literacy' not commonly considered).

Because we want to avoid framing the task as a 'formulation' problem, we have in some way to provide the mathematical expression that the student is asked to symbolically represent. One option is to describe in plain language the algebraic expression. Such description must however differ from the sequence of instructions as explored in the previous section since, as we saw, these better convey the reader perspective. If we want to place the student as an author, we should provide him/her with descriptions that are faithful to this approach, and ask the student to complete the ultimate step, that is translate it into symbols. Figure 3 is an example of such task, not uncommon in early algebra textbooks.

Write the following sentences as algebraic expressions: "The double of the square of *a*", "The sum of the square of 5 and double of *a*", "The difference between 3 and the product of 5 and *x*", "The square of the sum of 7 and *x*".

Figure 3: Task 2 – Translating into symbols expressions in plain language

Despite the fact that the expressions are given in plain language, they do convey the author perspective: the structuring sign (or operation) of the expression, the main one conceptually speaking, is alluded to at the very beginning of the sentence (e.g. "The sum" in the 2nd sentence).

This task presents some flexibility when considering its application at different school levels, however the very design limits considerably the expressions one may consider. Figure 3 presents the rhetorical description of expressions involving at most three operations, yet the reading of such sentences is not necessarily straightforward (e.g. the 2nd sentence). If we want students to be able to engage with this task, the complexity of the expressions (here tightly linked to the number of operations involved) should be kept to a minimum, at the risk of shifting the task genuine purpose.

Finally, we can suppose that the solution of the task (and potentially the difficulty in solving it) will reveal another aspect described in the epistemological analysis. While the rhetorical description of each expression begins with the most internal operation (the 'root' of the expression), in order to symbolically translate each sentence, one has to consider at the same time the '2nd most internal' operation, especially when the expression gets more complex. In "The difference between 3 and the product of 5 and *x*", the author's aim is to represent a difference. However, in order to define the terms of the subtraction one has got to also consider the product. Thus although the task has been designed to place the student in the author's position, the solving process involves constantly shifting back and forth between analytical and synthetic approaches, as noted in the epistemological analysis.

READING THE AUTHOR'S MIND – CONCLUSION AND DISCUSSION

Meaning making of mathematical symbols is certainly a very complex and multi-facetted issue that has drawn the attention of mathematics educators around the world for decades. Regardless of the framework adopted (process vs. object in Sfard 1991, process vs. concept in Tall et al. 2001, 'symbol sense' in Arcavi 1994, to name a few) there seems to be a consensus, namely the importance developing students' ability to recognise the conceptual meaning of mathematical expressions and thus being able to go beyond its 'surface structure' (Skemp, 1982), at risk of producing incoherent symbol arrangements.

In light of what has been discussed so far, we reframe this issue in the following terms: to what extent is a student, in front of a mathematical expression, thus theoretically placed as the 'reader', able to detach him/herself from the related order of interpretation and recognise the 'structure' of the given expression, thus adopting the author's point of view? Figure 4 presents an extract of a possible task that illustrates this .

Link each of the following expressions to the sentence that best describes it. If you select 'other(s)', please specify.

Expressions $A:\left(\frac{a}{3}\right)^2+\left(\frac{b}{3}\right)^2, B:\frac{a^2+b^2}{3}, C:\frac{(a+b)^2}{3}, D:\frac{a^2}{3}+\frac{b^2}{3}$

<u>Sentences</u> **1:** The third of the sum of the squares of *a* and *b* **2:** The sum of the squares of the thirds of *a* and *b* **3:** The sum of the thirds of the squares of *a* and *b* **4:** The third of the square of the sum of *a* and *b* **5:** Other(s)……………………………………………..

Figure 4: Task 3 – Blending the reader and the writer perspectives

Note that what is at stake here is not merely about recognising the structure of an expression – other scenarios might be more appropriated since the translation in plain language can constitute a major obstacle for students. Rather it is about going further in the investigation of what constitutes 'symbolic literacy', by examining the double position reader-writer within the same task. Can a student, *a priori* in a reader perspective, play the role of the author? Is he/she able to overcome the usual linear reading and bring out the structure of the given expression? Ultimately, does the student adopt a unique, privileged position?

Probably this task (and maybe more evidently than others presented in this paper) may appear rather 'artificial' in the sense that it might not be common in usual a classroom settings. But this task, as well as all others, does however unravel in other terms and hopefully in greater detail features of some of the key components of 'symbolic literacy' described elsewhere in the literature.

We believe it is not by chance that 'being engaged/involved' with mathematics is often translated, in different languages, by 'active' verbs such as *doing*. When it comes to mathematical symbols, *doing* mathematics certainly relates to (the very important) skill of manipulating symbols. But as the literature suggests, equally important to being able to manipulate symbols is to extract their meaning. As we tried to show in this paper, even in its more 'passive' way, namely 'reading', this does mean being fully active, making what Usiskin promotes when talking about 'reading out loud' mathematics truly active, as the reader does take the ownership of the expression as well.

Acknowledgements

I wish to thank Robyn Pierce and Jill Vincent for their invaluable inputs and careful review of this paper.

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BELIEVING WHAT WE PRACTICE: DOES SELF-ASSESSMENT COUNT?

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The constructivist research perspective (which adopts an interpretive stance) implies that meaning is co-constructed by the actors. In educational research, it is often argued that teachers' self-reporting on their practices is less valid than an observer's (researcher's) point of view. In this paper, I first argue that the practitioner's own interpretation of her practice is as valid as an outsider's interpretation. Then, I analyse classroom practices as described historically, and outline the results of a study where teachers' beliefs about their practice mirrored the criteria about best practices that were described in those documents. Finally, I provide recommendations for a research paradigm that adopts a more balanced approach to what counts as teachers' knowledge about their beliefs and practices.

INTRODUCTION

Mathematics education reform movements in New Zealand, Australia, Europe, the UK, Canada, and the United States have been underway for a number of years (e.g., Australian Education Council, 1990; Ministry of Education of New Zealand, 1992; Ministry of Education of the United Kingdom, 1959; Office for Standards in Education (OFSTED), 1994). New educational initiatives have been introduced by governments and learning institutions around the world with the implicit assumption that they would impact the practice of teachers (Brown, Hanley, Darby, & Calder, 2007). It is often argued that there is little evidence of their effects on teacher practice (Brown et al., 2007; Mullis, Martin, Foy, & Arora, 2012). This paper investigates some of our assumptions about interpretivist research, provides an historical view of what the mathematics education discourse has been, and describes a qualitative study where participating teachers' voices were heard.

Whose reality counts?

A constructivist research paradigm is interpretivist. It assumes that reality, as we know it, is constructed intersubjectively through the meanings and understandings that we develop socially and experientially. "Knowledge is a dynamic product of the interactive work of the mind made manifest in social practices and institutions" (Paul, 2005, p. 46). It includes an "*enacted*, or *constructed*, reality, composed of the interpretive, meaning-making, sense-ascribing … activities that produce meaningfulness and order in human life" (Lincoln, 2005, p. 61). Since we cannot separate ourselves from what we know, *truth* cannot be grounded in an objective reality. The axiological implication is that researchers must make "extraordinary efforts to reveal, or uncover, beliefs and values that guide and generate individual and

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group constructions" (Paul, 2005, p. 36).

It is also argued that beliefs about practice must be attended to in research studies that concentrate on teacher development (Barkatsas & Malone, 2005; Kagan, 1992; Pajares, 1992; Swan, 2006). Qualitative researchers are tasked with designing and enacting studies with trustworthiness and catalytic validity. According to Guba and Lincoln (2005) *catalytic validity* describes a research process that is able to prompt action on the part of research participants. Kemmis (2006) argued that transforming realities "requires truth-telling both with respect to the truths that arise from our work (our findings) and the methods by which we arrive at them. It requires that we critically evaluate how we have done our work – whether our findings are justified by our methods" (pp. 474-475).

Why then is it often assumed that another's (researcher's) interpretation of classroom practice is more objective, and therefore more valid than a participating teacher's interpretation? Many agree that it is time that practitioner-derived knowledge is acknowledged (Appelbaum & Davila, 2007; Hanley, Darby, Calder, & Brown, 2005). Rather than a focus on purely observed evidence, I argue that practitioner-derived knowledge must be considered trustworthy and relevant in research (Lerman, 1994; Smyth, 1989).

Teachers' self-reported practice, particularly as reported to colleagues in the *public sphere* about one's practice should be considered valid. In research methodology, this implies that particular texts, including what might be considered *indirect* observations such as reflective journals should count as a good representation of what those participating teachers *know*. Focus group methodology is strongly supported. Kemmis (2006) argued that it is important to establish "public sphere in which people realise and enact their communicative freedom", and that we must "open communicative spaces in which 'the way things are' is open to question and exploration" (p. 474). If this can be accomplished in a research setting, then surely, the teacher's own *public* declarations about her practice can count as the *truth*.

What has been the discourse?

In 1980, the National Council of Teachers of Mathematics in the United States published *An Agenda for Action*. This document invited conversation about what should happen in a mathematics classroom that is organised around problem solving. When compared to more recent documents, parallels can be found in the messages being sent to mathematics teachers about the nature of a classroom culture that includes problem solving. Hiebert et al. (1997) described an international collaboration (including studies from the US and from South Africa) with the goal of working towards a common definition of what it means to *understand* mathematics, and what is essential for teaching/facilitating students' understanding. Two current documents describing norms and teaching practices seem to commit to the same discourse (see Table 1). Due to the limited space, I have only reported on three of six practices.

Mathematics Teaching Practices (NCTM, 2014)	***Setting up Positive Norms in Math Class* (Boaler, 2014)**	***Making Sense* (Hiebert et al., 1997)**	**An Agenda for Action 1980s (NCTM, 1980)**
Implement tasks that promote reasoning and problem solving.	Depth is more important than speed.	Make mathematics problematic. Ideas and methods are valued. Students choose and share their methods.	The mathematics curriculum should be organised around problem solving. Problems should be presented in more natural settings.
Use and connect mathematical representations	Math is about connections and communicating.	Meaning for tools must be constructed by each user. Tools are used with purpose – to solve problems.	Mathematics programs must include the use of…visualisation. Assist the student to communicate about problems in a variety of modes, e.g., models, and schematic diagrams.
Facilitate meaningful mathematical discourse	Questions are really important.	Share essential information. Correctness resides in mathematical argument. Tools are used for recording, communicating, and thinking.	Assist the student to read and understand problems. Programs must include methods of gathering, organising, and interpreting information… and communicating results. Students…question, experiment, estimate, explore, and suggest explanations.

Table 1: Parallels Between Historic Current Teaching Practice Documents

After 35 years, it can be argued that the messages have shifted somewhat with respect to choice of vocabulary. However, it can also be argued that these messages share strong similarities.

The culture of mathematics is difficult to change. As Kegan and Lahey (2007) described: “[p]eople often form big assumptions early in life and then seldom, if ever, examine them…But only by bringing them into the light can people finally challenge their deepest beliefs and recognise why they’re engaging in seemingly contradictory behavior” (p. 50). Kegan and Lahey called these narratives that impede our growth and learning *subconscious competing commitments*, and described how it is important to help participants challenge those subconscious competing beliefs. In the following paragraphs, I summarise a study designed with professional development sense-making activities for secondary mathematics teachers where participants' assumptions were challenged, teachers took action, and where there was evidence of trustworthiness and catalytic validity.

INVESTIGATING AN EXAMPLE

The study's theoretical framework

The following is an example of a qualitative study that respected teacher voice, and where, what the teachers said about their practices in focus groups and in reflective journals equally counted as evidence of their practices. The conceptual framework of this study linked together theoretical constructs shared by the literature on empowerment. It drew on work that supports critical reflection as a process for providing an empowering space for those in educational organisations (Fisher, 2003; Smyth, 1989), including Schön's (1983) work on reflective practice, and Freire's (2000) critical pedagogy.

Through a qualitative professional learning methodology, participating teachers and the researcher worked together to design tasks that provided a problem-based and more *student-centered* environments. In order to do so, classroom norms and assumptions that might be considered most typical of our secondary mathematics classrooms were challenged. During the phases of this research project, teachers participated in meaning-making activities (facilitated by the researcher), in focus group sessions where they examined and discussed their practices, reflective journaling questions, and the design and implementation of tasks.

Desimone (2009) argued that "[p]rofessional development is a key to reforms in teaching and learning, making it essential that we use best practice to measure its effects" (p. 192). She proposed a core conceptual framework for studying the effects of professional development on teachers and students suggesting that the core features should include active learning, duration, and collective participation. These ideas were consistent with the implementation of this research study.

The study's methodology

The study involved a group of seven secondary mathematics teachers. They comprised a mathematics department in a small high school with grades from 8 to 12 that enrolled approximately 700 students. There were five male teachers and two female teachers. The school was in a rural area outside a larger city centre with a greater population of approximately 55000 (which included the rural area). Interpretive approaches rely heavily on naturalistic methods (such as after school professional learning meetings and reflective writing). At the beginning of the study, teachers completed an online survey of beliefs and practices that reflected various stances and pedagogies: traditional, reform, and ethno- and critical mathematics. This allowed teachers to investigate and self-evaluate with respect to how much they ascribed to the beliefs and practices of these paradigms for teaching mathematics. We articulated our beliefs together in focus group sessions, and then the teachers were asked to reflect on their answers through journaling exercises. All teachers took action by implementing a lesson, and then by collaborating together to share their lessons and reflect on their practice. Teachers had the opportunity to reformulate their philosophies through their actions, make more sense of their beliefs and practices, and change their consciousness to know better *who they are* (Fay, 1987) as educators. Their last journal reflection asked them to consider the processes they experienced. Near the end of the study, the following questions were asked to summarise what had occurred. Samples of the journaling questions were as follows:

> What, if anything, have I gained from writing reflections? What, if anything, have I gained from participating in reflective discussions with colleagues? How has our group benefited, or not, from the work we have done together. Do I believe that what I have learned in this study will affect my practice? Why or why not? (Journaling Questions)

Essential aspects of the study were the exploration of the participants' beliefs and practices, and a need for action and collaboration on the part of the teachers.

The study's results and analysis

The views described below were expressed in the teachers' Focus Group sessions and in their Reflective Journals near the end of the study. As teachers reflected and applied their new pedagogical perspectives, a number of themes emerged. Table 2 shows a sampling of the teachers' reflections based on the three practices on which I report in this paper.

So, what did teachers say they did?

Mathematics Teaching Practices	Examples of *Teachers Quotes* from Journals and from Focus Group Sessions (Baron, 2011)
Implement tasks that promote reasoning and problem solving	The majority of students enjoyed the lesson because it varied from the "typical" lesson when I am the "speaker" and they are the "listener" (Teacher A) I just went to "lean and mean" constructivism. We read the task, and that's all I gave them. (Teacher E)
Use and connect mathematical representations	I really liked the conversations about the room. They related the relationships between the graphs and equations. (Teacher F) They said they liked it because it was hands-on. (Teacher B)
Facilitate meaningful mathematical discourse	We started with going through the vocabulary and used technology. It gave a picture and it had the definition when you clicked on the word. (Teacher C)

	I read through the problem before starting and then asked for students to paraphrase it. (Teacher D)

Table 2: Parallels between what teachers expressed and teaching practices

Taking action and communicating with colleagues in the *public sphere*

The teachers in the study acted on their beliefs, and implemented a lesson that they considered to be a risk-taking experience. They then shared their lesson with the group, and reported on their successes and difficulties. This process of communicative action and collaboration was critical in building trust within the group.

> The lesson sharing was really beneficial…It allows me to realise that I am not alone in my struggle to change and develop new methods of instruction. It is reassuring to know that even teachers with way more teaching experience than I have encounter the same constraints and obstacles. (Teacher A)

As Kemmis and McTaggart (2005) predicted, the participants' communicative action opened the "communicative space…[and built]…solidarity between the people who open their understanding to one another" (p. 576). The participants expressed that the processes in the study had a positive impact on them, and most felt that they knew themselves better as teachers by the end of the study.

> It has given me the chance to figure out who I am as a teacher, and what I want to do, or how I want to teach. (Teacher B)

DISCUSSION AND IMPLICATIONS FOR RESEARCH

I first argued that, as interpretivist educational researchers, we have not yet freed ourselves of positivist assumptions about how we conduct research, and we have still allowed the argument to be made that another person's interpretation of a teacher's practice is more valid than a teacher's interpretation. I then sought to find the "practice" messages that have made up the discourse on mathematics education classroom practice over the past 35 years. I wondered how 'new' the new messages are, and then I provided the results of a study that showed teachers ascribing to the recommended practices. I conclude with the argument that mathematics educational researchers can and should be more loyal to the interpretivist stance, and that, perhaps, by including teachers voices as valid 'teacher change' data, we might find more positive results of teachers implementing the practices described in curriculum documents. When focusing on teachers' voices in our research, we may yet find that teachers' beliefs and practices have in fact shifted.

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LANGUAGE AS A RESOURCE: MULTIPLE LANGUAGES, DISCOURSES AND VOICES IN MATHEMATICS CLASSROOMS

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Research on the learning and teaching of mathematics in contexts of language diversity often draws on the notion of language as a resource. This idea has been valuable in understanding aspects of learning and teaching mathematics in such contexts, although research has for the most part focused on students' home languages and code-switching as the principal resources of interest. I use Bakhtin's concept of heteroglossia to develop a more elaborated and specific understanding of language as a resource, leading to the idea that multiplicity is a key feature. I draw on an earlier mathematics classroom ethnography to illustrate these ideas.

INTRODUCTION

There is now an established body of work looking at the learning and teaching of mathematics in contexts of language diversity (e.g., Barwell, 2009; Moschkovich, 2010). This work has been conducted in a range of settings including bilingual mathematics classrooms (e.g., Moschkovich, 2009), second language mathematics classrooms (e.g., Barwell, 2005; 2014), and multilingual mathematics classrooms (e.g., Adler, 2001; Setati, 2005). A common orientation in all this research is to see the use of more than one language as productive, rather than as a problem, and to challenge deficit perspectives that see bilingual or multilingual learners as less capable of learning mathematics (e.g., Moschkovich, 2002; Planas & Setati-Phakeng, 2014). In much of this research, this orientation includes a conceptualisation of language as a resource. In this research report, I look at how this conceptualisation has been developed in the existing literature. Drawing on theoretical ideas derived from Bakhtin's theory of language, I then develop an expanded and more specific notion of language as a resource. To illustrate these ideas, I present an analysis of language resources in use in a language diverse mathematics classroom in Canada.

LANGUAGE AS A RESOURCE IN MATHEMATICS CLASSROOMS

The notion of language as a resource first appears in research on the learning and teaching of mathematics in contexts of language diversity in the 1990s. Adler's work in multilingual South Africa (written up in 2001), in particular, makes explicit use of this idea, but it can also be seen, at least in passing, in work by Moschkovich (e.g., 1996) and Setati (e.g., 1998) among others. Adler (2000) discusses the concept of resource at length, although her focus is not exclusively on language. Drawing on both Bernstein, and Lave and Wenger, she argues that mathematics classrooms are sites of hybridity, in which everyday and academic mathematical practices intersect to produce distinctive school mathematical practices. To interpret this hybridity, she proposes a framework of different kinds of resource available in mathematics classrooms as a basis for this hybridisation. She proposes four forms of resource: (1) basic resources: material (e.g., school buildings) and human (e.g., class sizes); (2) material resources: technologies (e.g., calculators), school mathematics materials (e.g., textbooks), mathematical objects (e.g., number lines), and everyday objects (e.g., money); (3) social and cultural resources: language (e.g., code-switching) and time (e.g., the school time-table); (4) other resources: human (e.g., teachers' knowledge; collegiality) (see Adler, 2000, pp. 212-213).

These resources are, according to Adler, transparent: that is, they can be both visible and invisible—highlighted when relevant, yet able to be used without explicit attention. It is important to note two things about Adler's framework. First, language is just one component within a broader conceptualisation of resources—although the other components of the framework arguably all have a language dimension. Second, however, the language component of the framework is somewhat narrow: it includes a specific attention to languages spoken by students and teachers, the practice of code-switching (i.e., switching between these different languages), and a more diffuse concern with the use of classroom talk to facilitate mathematical learning.

The concept of resource has been taken up in more recent work by a number of researchers. Planas and Setati (2009), for example, conceptualise language as, in Adler's terms, a social and cultural resource. This conceptualisation reflects Adler's, in that it retains the same focus on languages used by learners and teachers, on code-switching, and on the more diffuse concern with using classroom talk to promote mathematical thinking and learning. Hence, for example, in their analysis of data from Catalonia, Planas and Setati note:

> These learners bring multiple competencies to the mathematics classroom – while they may have difficulties with one of the languages they may use the other language as a resource. A student who is missing English vocabulary, for instance, may be competent in describing mathematical processes and presenting mathematically sound arguments in Spanish. (p. 40)

A similar perspective is apparent in related work by Planas and Civil (2013), Planas (2014) and Planas and Setati-Phakeng (2014). Moschkovich (2007) has also emphasised code-switching as a specific feature of language as a resource.

There is more to classroom interaction than the choice of language, however. Moschkovich (2009), for example, traces in detail how a pair of Spanish-English bilingual students propose multiple interpretations of a graph as they attempt to complete their task. The teacher is able to build on these interpretations to support the students to understand the conventional mathematical way to read the graph. Moschkovich points out how the teacher does not see the students' different interpretations as 'wrong'. In my own work, I have shown how primary school students in the UK draw on generic features and narrative accounts to make sense of arithmetic word problems (e.g., Barwell, 2005). It is, therefore, apparent that students draw on various aspects of language to make sense of and participate in mathematics. There are, then, various candidates for resources beyond code-switching or students' home languages. There is thus a need for a more refined conceptualisation of language as a resource, in order to better understand how language can and does support the learning and teaching of mathematics in a wide range of contexts of language diversity.

THEORETICAL PERSPECTIVE: HETEROGLOSSIA

To develop the notion of language as a resource in the context of language diversity, I draw on Bakhtin's (1981) notion of heteroglossia. Bakhtin's work offers a complex theory in which language is understood as dialogic and as situated in time and space. By dialogic, Bakhtin is arguing that meaning in language arises from the relations between aspects of language, such as words, styles or national languages, rather than from these aspects themselves. In seeing language as situated in time and space, Bakhtin highlights how any utterance is in relation with preceding utterances, both from the immediate interactional context, as well as reaching back through the history of language. Ways of using language always have a history and meaning is derived from this history of use. Moreover, this history is not simply of abstract words; it is the history of people using words—of their voices. In Bakhtin's theory, voice is a crucial dimension of meaning. He suggests that when we use words, they are "half someone else's": they carry the echoes of previous voices using these words. When students first use a word like hypotenuse, for example, their use carries the voice of their teacher as much as their own.

Bakhtin (1981) also noticed that language use is caught up in a kind of tension between forces of standardisation on the one hand, and forces of diversification on the other. Bakhtin used the metaphor of centripetal and centrifugal forces to describe this tension. The constant diversification of language in use (always counterbalanced by a tendency to standardisation) is referred to as 'heteroglossia', a word introduced by Bakhtin's translators. It is often defined simply as 'social diversity of speech types' (p. 263). A more careful reading, however, suggests that the English term 'heteroglossia', in fact captures three related forms of diversity (each having a different word in Russian; see Busch, 2014, p. 24, whose terms I follow).

Multidiscursivity refers to speech types related to forms of social organisation, such as the language of professions, families, or particular kinds of activity, as well as the language of particular times in history. Mathematical discourses are part of multidiscursivity and are themselves diverse, since they include the discourses of different mathematical domains, different mathematical communities, different mathematical eras (antiquity, renaissance, contemporary) and different levels of mathematics (school, university, research). According to Busch (2014), "each of these spheres develops relatively stable types of speech genres and topics" (p. 24).

Multivoicedness derives from Bakhtin's attention to voice and, in particular, the idea that any utterance expresses the intentions not just of the speaker, but of those who previously used similar words. In mathematics classrooms, for example, students must learn to use words introduced by their teacher or by a textbook. When they do so, however, their utterances reflect multiple voices—their own, as they attempt to talk about mathematics, but also their teacher's, the textbook, the curriculum, etc.

Finally, *linguistic diversity* operates at broader scale of time and space and refers to diversity in relation to more traditionally understood distinctions between languages and dialects. Of course, these distinctions are somewhat fuzzy and it is important to understand that all three forms of heteroglossia are present all the time, even within a single utterance.

This characterisation of diversity in language use clarifies the nature of different forms of language resource. Looking back at the research discussed in the previous section, it is apparent that most attention has been given to resources relating to linguistic diversity. Code-switching, for example, arises from linguistic diversity and can be used in any discursive context and with multiple voices. Moschkovich's (2009) work on multiple interpretations, meanwhile, can be seen as relating to multivoicedness. Her examination of the interplay between students' interpretations (i.e., their mathematical intentions or voices) and those of the teacher contribute to the development of mathematical thinking. My own work (Barwell, 2005), particularly on the role of genre in second language learners' engagement with arithmetic word problems, relates to multidiscursivity. The students in my research were able to use the features of a common mathematics classroom genre (i.e., word problems) to participate in school mathematics.

AN ILLUSTRATIVE EXAMPLE

From 2008-2012, I conducted an ethnographic study of mathematics learning in different second language settings in Canada. In this report, I refer to one of these settings, located in an Anglophone school in the province of Quebec. The Grade 5-6 class was established for students identified by the school as falling behind in both English and mathematics. I visited the class regularly throughout the 2009-2010 academic year. During that time, enrolment in the class fluctuated but never went over 9 students. For most of the year, all of the students in the class were Cree, one of the original peoples of Canada. The students spoke Cree as a first language and English as a second language, though with a range of proficiency levels. I made audio recordings of whole-class interaction and some small-group work, including my own work with groups of students, as well as interviews with students and the teacher. I took notes and collected samples of students' work and photographs of other artefacts, such as posters or work written on the blackboard. After each visit, I wrote a brief report summarising my observations.

During the ethnography, a key focus that emerged for me was the nature and role of centripetal and centrifugal language forces in shaping mathematics learning and teaching in different contexts of language diversity. For the class described above, I developed a way to methodically analyse the data in order to identify situations in which the tension between centripetal and centrifugal forces was particularly salient.

As a result of this analysis, I identified three such situations: (1) the students' use of Cree during in mathematics; (2) mathematical word problems; and (3) students' mathematical explanations (Barwell, 2014). These three situations were not the only ones in which language tensions arose (indeed, in Bakhtin's theory, these tensions are always present). They were, however, three situations in which these tensions were highly salient.

For this research report, I consider these three situations from the point of view of the language resources the students used in their participation in mathematics.

The students' use of Cree in mathematics

There was a notice on the wall reminding the students to use English; but there was also a notice welcoming them in Cree. While the students' use of the home language happened more frequently outside of formal class time, I sometimes observed the use of Cree in mathematics lessons, as in the following observation, previously reported in Barwell (2014, p. 917: TA is the teacher):

> [Alex] calls TA over and asks 'what does this mean?' TA says 'the width'. Trevor starts the worksheet but doesn't understand. TA asks Alex to explain to him. Alex gets up and goes to Trevor's side of the desk. He explains in Cree. TA says from the other side of the room 'in English, explain in English' but Alex is already in full flow. TA tries again but Alex has finished the explanation. TA comments on it being 'too late now'. This exchange was interesting – it suggests that between themselves, Cree is the preferred language for explaining things.

This situation is different from the kind of code-switching reported in previous research, in which the home language is widely used (e.g., Setati, 2005; Moschkovich, 2002). These students' home language is highly marginalised and spoken by a relatively small population. This use of Cree is an example of heteroglossia as language diversity. From a Bakhtinian perspective, the resource here is not Cree per se; rather, it is the language diversity that is the resource. The students draw on both Cree and English in different ways according to the situation and, in particular, according to whom they are speaking. The two languages are in dialogic relation.

Mathematical word problems

As in many mathematics programs, the students regularly encountered word problems. In Quebec, such problems include more complex, text-rich tasks known as situational problems. On one occasion the students worked on a problem that included a lengthy text explaining the history of a local tulip festival. It then states:

> You are a gardener hired to plant tulip bulbs for the Canadian Tulip Festival in May. You decided to arrange the flowers in a V for Victory format. You decide to use a pattern to make your design. Here is the design you started. [The problem includes a diagram showing a pattern of increasing squares labelled in a cycle of three colours] How many purple, yellow, and pink tulips do you need to complete the design? Show all your work.

I worked on this task with small groups of students. They all struggled to interpret the text, not least because they were unfamiliar with the tulip festival and, indeed, tulips. I recorded one encounter in my notes:

> Ben moved first, drawing in rows of tulip bulbs in the boxes shown in the diagram. He did 5×5 in the first empty box and then moved on to the next box. Curtis looked at what he was doing and then did something similar. At some point, Curtis came up with a solution, fairly quickly. He just wrote three numbers at the bottom of the answer box. I didn't understand his solution but explained that he needed to explain how he worked it out. He wrote a sentence along the lines of 'I added the tulips' – something quite general. So I said he needed to be more precise, to explain what calculation he did […] he had little trouble solving the problem, and that most of the time was spent on writing it down in an 'acceptable' way.

The students' challenge here seems to be as much about how to interpret the problem text and how to produce the kind of written response that such texts demand.

Their challenge can be understood in terms of the heteroglossia of multidiscursivity. Ben and Curtis are able to make use of the diagram presented in the problem, a form of mathematical diagram embedded within the genre of a form of word problem. As such, the diagram can be seen as a resource for the students' mathematical thinking. The broader word problem structure, however, is not a resource they draw on. My role is therefore one of mediating the text in support of the students. The students' struggle with this kind of problem can be understood as a lack of available resources: the students need to be introduced to a broader repertoire of, for example, problem genres, in order to have necessary resources necessary to tackle such problems.

Students' mathematical explanations

Students were often asked to explain their thinking but often struggled to formulate suitable responses. In one case, for example, the teacher asked for examples of perimeter (partly reported in Barwell, 2014, p. 919):

> TA asks for examples. Alex points to the edge of his desk, saying 'this'. TA draws a (somewhat crooked) square on the blackboard and labels one side 5cm. The students seem to recognise the example, saying 'they're all the same' and suggesting how to work out the perimeter. Kevin says: times/ add it/ add it/ add those centimetre things/ 5 centimetres.
>
> Curtis: because
>
> Kevin: there's four sides.
>
> TA writes the formula on the blackboard and goes through it with them for the square. Again, the students seem to be following, certainly more than yesterday. I notice Jenny is vocalising, though her words are not taken up.
>
> Jenny: because/ length/ width

The students here make use of deictic terms (this, they, it, those things). They also clearly build on each other's contributions. This kind of joint production with heavy use of deixis was fairly typical. This pattern of language use can be understood as the heteroglossia of multivoicedness. The students make use of each other's contributions, as well as those of the teacher. The use of deixis makes sense in the context and students are able to take each other's contributions as resources with which to build mathematical thinking. Again, it is the multiplicity of dialogically related voices that acts as a resource, rather than any particular individual voice.

CONCLUSIONS

The notion of resources has been valuable in developing an understanding of the learning and teaching of mathematics in contexts of language diversity, although often researchers focus mainly on code-switching. As a step towards a more developed conceptualisation of language as a resource for mathematics learning, I have drawn on the three forms of heteroglossia proposed by Busch (2014), based on Bakhtin (1981). That is, language as a resource can be looked at in terms of the resource of language diversity, the resource of multidiscursivity, and the resource of multivoicedness. This approach suggests that it is multiplicity that acts as a resource in each case, rather than specific languages, discourses or voices. This perspective reflects the hybridity of mathematics classrooms. Multiplicity supports rich dialogic relations between languages, discourses and voices, which in turn support mathematical thinking and learning in contexts of language diversity.

Acknowledgments

I am grateful to the school and students for their generous participation. The project referred to in this paper was funded by the Social Science and Humanities Research Council of Canada, grant number 410-2008-0544. I also thank David Pimm for conversations about resources.

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HOW DO SECONDARY SCHOOL STUDENTS MAKE USE OF DIFFERENT REPRESENTATION FORMATS IN HEURISTIC WORKED EXAMPLES? AN ANALYSIS OF EYE MOVEMENTS

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Heuristic worked examples are an effective way to foster students' mathematical argumentation skills. This study explores how secondary school students make use of different types of representations when working with such kinds of heuristic worked examples. Eye tracking data showed that the students spent more time on pictures than on symbolic or textual representations, and more time on symbolic than on textual representations. Furthermore, the students tried to integrate information from the different types of representations by alternating between consecutive parts of different representation formats.

THEORETICAL FRAMEWORK

Mathematical argumentation is a key feature of academic mathematics. In a broader sense, it includes activities such as examining assumptions, conjecturing, and understanding and setting up formal proofs. At the university level, mathematics is typically introduced following a rigorous axiomatic structure. Albeit to a less rigorous degree, argumentation and proof is also implemented in curricula and educational standards for all levels of primary and secondary school mathematics (e.g. Kultusministerkonferenz, 2004; National Council of Teachers of Mathematics, 2000). Yet, many studies have shown that even at the end of secondary education, students struggle with constructing mathematical proofs and with developing rigorous argumentation. In a large study in England and Wales with almost 2,500 high-attaining grade 10 students, Healy and Hoyles (1998) investigated proof evaluation, the ability to construct proofs, and views on the role of proofs. The results revealed that even these high-attaining students had problems with proofs. Reiss, Klieme, and Heinze (2001) largely replicated these results with students in the upper secondary level in Germany. Other studies in Germany revealed similar results also for younger students of grades 7 and 8 (Reiss, Hellmich, & Reiss, 2002), and grade 9 (Ufer, Heinze, & Reiss, 2008). Thus these results can be found in different countries and different age groups.

One approach to foster argumentation skills is the use of heuristic worked examples (e.g. Reiss & Renkl, 2002) and their effectiveness could be shown for school students (e.g. Heinze, Reiss, & Groß, 2006) as well as for university students (e.g. Hilbert, Renkl, Kessler, & Reiss, 2008). While worked examples provide the learner with a direct solution to a given problem, heuristic worked examples include less direct solution steps. The solution process is presented in a realistic, not necessarily straight way and can also include explorative or misleading steps (Hilbert et al., 2008). Often

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a fictitious person explains his or her procedure. In particular, heuristic worked examples to foster argumentation skills present heuristic strategies that are beneficial for finding an argumentation or key ideas for generating a proof. Heuristic strategies, like generating systematic examples, are general techniques that can help to better understand a problem or to progress its solution (Schoenfeld, 1985). The structure of such heuristic worked examples is based on the process model by Boero (1999), which describes how experts develop proofs. In particular, the first four phases of the model are essential for heuristic worked examples to foster argumentation skills: *production of a conjecture*, *formulation of the statement according to shared textual conventions*, *exploration of the content*, and *selection and enchaining of coherent, theoretical arguments into a deductive chain*.

The present study addresses the question of how secondary school students make use of different types of representations included in a heuristic worked example to foster argumentation skills. In the context of argumentation, the role of different representations is of particular relevance. The reason is that as a final result, a mathematical proof needs to be represented in a formal way by means of mathematical symbols. However, to develop a mathematical proof, non-symbolic representations such as pictures or diagrams can be much more helpful. Even experts often do not start out with symbolic notations to develop a proof, but draw pictures or generate several concrete examples before they are eventually able to write down a proof in a formal way. From a psychological perspective, the reason seems to be that mathematical knowledge is not always stored in a symbolic representation format, but also in visual representation formats in the human's mind. For the learning process, theories on multimedia learning state that integrating textual and visual information is very effective (e.g. Schnotz, 2005). In the context of mathematical argumentation, pictures seem to be beneficial when they represent key ideas of the argument or the proof, and when they can easily be translated into the formal symbolic system.

Accordingly, the heuristic worked example that we used in this study included information in three different representation formats. *Text* was used to describe the ideas and conclusions of a fictitious person during the solution process. The person explained what she thought, what she wanted to try to do next, and what new insights she got from the previous step. That means that also the used heuristic strategies were presented in a textual format. To visualize these strategies, *pictures* were integrated in the example. Thus the pictures should help to understand and illustrate some of the main ideas of the solution process. Mathematical *symbols* were used to formalize the verbal or pictorial conclusions and to come to the sought proof. In the heuristic worked example used in the study, the following representations were alternated in most of the cases: text and picture, and text and symbols. With regards to content, these consecutive representations belonged together. Figure 1 shows a part of the heuristic worked example. All of the three types of representation formats and the alternating structure can be seen. The task of the example was, *"Prove: The sum of three consecutive numbers is always divisible by three"*.

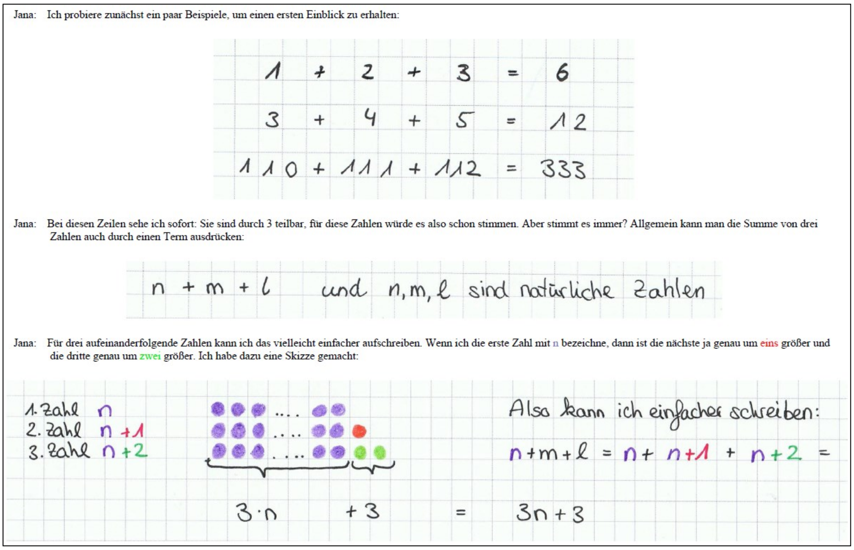

Figure 1: Part of the heuristic worked example used in the study with the three types of representations *text*, *picture*, and *symbol* with an alternating structure.

In the first line of Figure 1, the fictitious person says, *"First, I try out some examples to get a first impression"*. In the following symbolic part, these examples are shown in a concrete way. After that the person says, *"In these lines I see immediately: They are divisible by 3, accordingly for this numbers it would be true so far. But is it always true? Generally, one can express the sum of three numbers by a term"*. Afterwards this term is written down with the help of symbols. Then the person says, *"For three consecutive numbers I maybe can write that down more easily. When I call the first number n, then the next one is exactly larger by one, and the third one is exactly larger by two. I have drawn a picture for that"*. In the last section of Figure 1, this picture together with some symbolic and verbal conclusions can be seen.

The method of eye tracking was used in the present study to assess the participants' distribution of attention on different parts of the heuristic worked example. The use of eye tracking has several advantages, as it allows an exact measurement of the fixation time that individuals spend on specific areas of the presented content. An increasing number of previous studies has shown that eye tracking can be used successfully to investigate strategy use on mathematical tasks. For example, Inglis and Alcock (2012) used this technique in the field of mathematical proof reading. The authors showed that novices (i.e. students) spent more time on formulas (compared to the non-formula, that means textual parts of the proofs used in the study) than did experts (i.e. mathematicians). Furthermore the experts shifted their attention back and forth between the lines of the proof in order to look for between-line warrants more often than the novices. A study by Beitlich et al. (2014) made use of eye tracking to analyse the use of different representation formats in mathematical proofs. More precisely, the authors wanted to know whether and how adults with high expertise in mathematics looked at a picture given with a mathematical proof while reading the proof to comprehend it. They found that the participants mostly spent more time on the text parts of the proofs than on the picture. Furthermore the participants alternated between the text and the picture during reading the proofs. In view of these results it is assumable that eye tracking will be a feasible method to analyse how secondary school students make use of different types of representations in a heuristic worked example.

RESEARCH QUESTIONS AND HYPOTHESES

Although there is some evidence that the use of heuristic worked examples can enhance students' mathematical argumentation skills, little is known about whether students actually make use of the advantages that heuristic worked examples offer. This is particularly true for the use of different representations within a heuristic worked example. Accordingly, we investigated whether students at the secondary school level focus their attention on all different representation formats included in a heuristic worked example to foster argumentation skills or whether they take into account some representation formats more than others. This question is of practical relevance, as it can help to improve the design of heuristic worked examples for example in textbooks. We hypothesized that the students would spend more time on mathematical symbols than on regular text, because similar to the novices in the study of Inglis and Alcock (2012), the participants of our study were not very experienced in argumentation. We further expected that the pictures would be fixated longer than the symbols and consequently they would be fixated the longest, because as explained above, pictures are seen to be helpful for generating argumentation and even experts often use this kind of non-symbolic representations first in order to get a formal proof.

We also investigated whether the students would try to integrate information from different representation types. According to cognitive psychological theories on multimedia learning, the understanding of a text, that contains different representation types can be fostered by integrating the information from these different types (e.g. Schnotz, 2005). We assumed that the participants would actually try to integrate the information, because the heuristic worked example was designed in a way that suggests alternating between representations, especially between consecutive parts of different representation formats as these parts mostly belonged together directly with regards to content.

METHODOLOGY

The participants were 26 students (16 female) from grades 10 and 11 of the academic track of a German secondary school. Their mean age was 16 years (*SD* = 0.85).

The students sat in front of a computer screen, which was connected to a binocular remote contact free eye tracking device. The participants were instructed to read the heuristic worked example on the screen so that they would be able to answer questions about it afterwards. They were told that this is an example how argumentation tasks can be solved and that they should try to comprehend the given approach of a fictitious student. There was no time limit and the students could go from one page to the next by pressing a key by themselves, but they could not go back to read the previous pages again. All in all, the heuristic worked example consisted of four pages.

After calibration, the experiment started and the participants saw the first page of the heuristic worked example. On average, the students spent approximately four minutes (M = 4:06 min; SD = 1:19) on the whole heuristic worked example. When they had read the whole example, they had to answer six multiple-choice-questions by clicking on the correct answer on the screen. The questions required only recognizing parts of the example without asking for comprehension. For example the students had to answer the question which picture the fictitious person used for the first exploration of the task by choosing the correct answer out of four given pictures. There were two questions about pictorial parts of the heuristic worked example, two questions about symbolic parts, and two questions that asked for contents of different representations.

RESULTS

To analyse the eye movements, we defined three kinds of areas of interest (AOIs) for each form of representation: *text*, *picture*, and *symbol*. The AOIs were fitted around the respective parts of the heuristic worked example. The task was not taken into account. To compare fixation times on text, pictures, and symbols, we decided to divide the fixation times (in ms) for the three kinds of AOIs by the size (in pixel; px) of the respective AOIs to account for the different sizes of the AOIs (sums of AOI sizes: *text* = 815,359 px, *picture* = 197,543 px, *symbol* = 1,149,897 px).

On average, the participants spent most time on pictures (0.077 ms/px, SD = .045), followed by symbols (0.059 ms/px, SD = .027), and text (0.036 ms/px, SD = .015). These differences were statistically significant according to a one-way analysis of variance (ANOVA) with repeated measures, $F(1.27, 31.85) = 26.34$, $p = .00$, $\eta^2 = .53$. Post-hoc pairwise comparisons revealed significant differences between all of the three kinds of AOIs, (all $p < .05$). This result is in line with our hypothesis.

The outcome of the analysis of the eye tracking data seems to be related to the results of the multiple-choice-questions. The questions about the pictures used in the heuristic worked example were answered correctly by 98% of the participants, and the questions about symbolic parts by 67% of the students. That means that the parts that were fixated longer (according to eye tracking data) were better recognized (according to data from the questions).

To answer the question whether the students tried to integrate information from different representation types, we counted how often the participants alternated between two AOIs of different representation formats. The first transition from one AOI to another was not counted as this was in line with the natural reading behaviour. Also the transitions from an AOI of one representation type to an AOI of the same type of representation were not counted, as we were not interested in these transitions. For the analysis, we used sequence charts which show the order and the duration of fixations of the AOIs.

We found that every participant made on average 16.6 (*SD* = 9.9) transitions between areas of different representation types in the whole heuristic worked example. Even though there was a relatively high variance, this means that they actually alternated between AOIs of different representation formats. That suggests that they tried to integrate the information of the representations which is in line with our hypothesis.

Having a closer look at the sequence charts revealed that most of the transitions occurred between consecutive AOIs of different types of representations. This is also in line with our hypothesis and possibly indicates that the students were aware that with regards to content, the representations that were presented closely together belonged together.

In addition, the multiple-choice-questions about the contents of different representations were answered correctly by 63% of the participants. This could indicate that though the students tried to integrate information from different representation types, not all of them were successful.

DISCUSSION

The aim of our study was to better understand how secondary school students make use of different representation formats that are used in heuristic worked examples to enhance mathematical argumentation skills.

We found that the students spent most time on the pictures of the heuristic worked example, followed by symbols and text. This is in line with our hypothesis, but not in line with other studies. In the study by Beitlich et al. (2014) the participants spent more time on the text of mathematical proofs compared to pictures accompanying the text. One difference between this study and our study is that in the study by Beitlich et al. (2014) the picture had a less important role whereas in our study the pictures illustrated some key ideas of and strategies for the solution process and thus played an important role. Furthermore the authors used formal proofs and showed them to expert mathematicians whereas in our study we used a heuristic worked example with different types of representations to foster novices' argumentation skills. In view of these crucial differences, the different results are not surprising.

Another finding indicated that the participants at least tried to integrate information from different representation types by alternating between relevant areas. As we expected, the students especially alternated between consecutive AOIs of different representation formats. In the study by Beitlich et al. (2014) the experienced participants tried to integrate the information given in the text and the picture, too. In the study by Inglis and Alcock (2012) the experts did also show an alternating reading behaviour, but the novices alternated less between consecutive lines of mathematical proofs. One conclusion might be that heuristic worked examples can foster eligible reading strategies that are applied by experts in the field of mathematical argumentation and proof. However, based on the existing data of the present study, it is not possible to make a quantitative evaluation whether the amount of the transitions between different types of representations is high or low. For this kind of conclusion, additional data is necessary.

Analysing eye movements was a feasible method to get access to secondary school students' use of different representation formats in heuristic worked examples. If only the answers to the multiple-choice-questions would have been analysed, it would not have been possible to draw detailed conclusions. For example, it could have been only seen that questions about pictures, which asked for recognizing elements used in the heuristic worked example, were answered more correctly than questions about the recognition of symbols. However, it would not have been possible to suppose that this might be according to the fact that the pictures were fixated longer than the mathematical symbols. On the other hand, data from the questions enriched the eye tracking data regarding to the second research question. Hence, a combination of these methods seems to be beneficial.

To generalize the present results, it is necessary to replicate the study with a larger sample size and different heuristic worked examples. Furthermore it would be useful to conduct similar studies (also in different countries) with different age groups or with participants with different amounts of expertise in the field of mathematical argumentation to compare these groups. It would also be interesting whether high and low performing school students differ in their use of different representation formats in heuristic worked examples to foster argumentation skills, particularly as existing research suggests that they differ in the degree they benefit from heuristic worked examples (e.g. Heinze et al., 2006). In addition it should be noted that it would be possible to analyse the eye tracking data in a different way. For example instead of accounting for the different sizes of the AOIs, it could be meaningful to account for the different content of information. That would require conducting preliminary studies.

As an implication for educational practice, the results suggest that the use of different representation formats in heuristic worked examples to foster argumentation skills could be efficient and this kind of studies can help to improve the design of heuristic worked examples.

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IDENTITY AS AN EMBEDDER-OF-NUMERACY: A CROSS CASE ANALYSIS OF FOUR TEACHERS

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Numeracy needs to be developed across the curriculum. However, if teachers are to effectively embed numeracy into the subjects they teach, they need to be supported to develop this capacity. Using an adaption of Valsiner's zone theory, a cross case analysis of four teachers is presented. The findings suggest that assisting teachers to broaden their personal conception of numeracy and providing opportunities for them to develop appropriate pedagogical content knowledge may enhance their capacity to exploit numeracy learning opportunities across the curriculum.

INTRODUCTION AND BACKGROUND

Proficiency in literacy, numeracy, and problem-solving in technology-rich environments - competencies that the Organisation for Economic Co-operation and Development (OECD) describes as the *key information-processing skills* - has an effect on an individual's economic and social well-being (OECD, 2013). Although numeracy encompasses much more than mathematics (OECD, 2013), an individual cannot be numerate without sound mathematical knowledge. Gal (2013) has argued that mathematics education, in school and other settings, should focus on how individuals can be assisted to develop the capacity to act in a numerate way. For schools, he suggests that this means rethinking the tasks, pedagogy, and assessment used. However, while mathematics education has a role to play in developing students' numeracy capabilities (or mathematical literacy, as it is sometimes called), numeracy needs to be developed in a range of contexts and, for students at school, this means in their other subjects (Steen, 2001).

One way of promoting numeracy learning beyond the mathematics classroom involves taking an *embedded* approach by encouraging all teachers to exploit the numeracy learning opportunities that exist *across the curriculum* (e.g., ACARA, 2014). However, for this approach to be successful, teachers need to be able to effectively embed numeracy into the subjects they teach; in other words, identify opportunities within curriculum documents and design tasks that support both discipline and numeracy learning. In this paper, some findings from a study that aims to identify how teachers can be supported to develop this capacity are reported.

Teacher identity, specifically identity as an embedder-of-numeracy (hereafter referred to as *EoN Identity*) was used as the lens to enable a focus to be placed on factors, both cognitive and non-cognitive, that are likely to have most impact on a teacher's capacity to embed numeracy into the subjects they teach. A conceptual framework for EoN Identity was developed (Bennison, 2014a) and an adaptation of Valsiner's (1997) zone theory has been used as the theoretical framework for describing and analysing each participant's EoN Identity (e.g., Bennison, 2014c). Building on this previous work, a

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preliminary cross-case analysis of four teachers is presented. This analysis was informed by the following research question:

In what ways can teachers be supported to develop the capacity to embed numeracy into the subjects they teach?

THEORETICAL FRAMEWORK

The EoN Identity framework (Bennison, 2014a) can be employed to assist in the design of empirical studies because it provides a focus for data collection. However, the framework has limited use for analysing data collected in such studies because it is difficult to conceptualise how the characteristics identified in the framework interact to produce a particular EoN Identity. On the other hand, the framework is consistent with a sociocultural view of learning and readily aligns with the adaptation of Valsiner's (1997) zone theory used by Goos (2013) to understand teaching learning.

Valsiner (1997) drew a distinction between learning that was possible and learning that actually occurred and conceptualised this as the interaction between an individual's Zone of Proximal Development (ZPD), Zone of Free Movement (ZFM), and Zone of Promoted Action (ZPA). He defined the ZPD as an individual's current state of development, constituted by the knowledge and past experiences that an individual brings to any situation; the ZPA was defined as actions that were being promoted by others; and the ZFM as actions that were permitted within the environment. He argued that the ZFM and ZPA worked together in a ZFM/ZPA complex to structure development. Thus, learning will only occur if the individual has the capacity (ZPD) and is permitted within the environment (ZFM) to act in the way promoted (ZPA).

Goos (2013) viewed the zones from the perspective of teacher-as-learner. For her, the ZPD represented the ways in which a teacher could develop under the influence of teaching actions that were being promoted (ZPA) within the teacher's professional context (ZFM). Her approach involved mapping the characteristics known to influence teachers' use of technology onto their ZPD, ZFM, and ZPA. Therefore, applying this approach to the current study entailed mapping the characteristics within the EoN Identity framework onto a teacher's ZPD, ZFM, and ZPA (see Figure 1). For example, opportunities to learn about embedding numeracy across the curriculum (e.g., professional development activities) were included in a teacher's ZPA (see Bennison & Goos, 2013 for a description of this mapping process).

While Shulman (1987) suggested that seven types of knowledge were needed for teaching, only three are included in the ZPD; mathematical content knowledge (MCK), pedagogical content knowledge (PCK), and curriculum knowledge (CK). In the EoN Identity framework, CK was defined as the knowledge needed to identify numeracy learning demands and opportunities across the curriculum, PCK as the knowledge needed for designing activities to exploit these, and MCK as the associated mathematical knowledge (Bennison, 2014a).

Valsiner's zones	Characteristics of EoN Identity

Zone of Proximal Development (ZPD)	Mathematics content knowledge (MCK) Pedagogical content knowledge (PCK) Curriculum knowledge (CK) Beliefs about numeracy Confidence with numeracy
Zone of Free Movement (ZFM)	Support from colleagues and administrators Curriculum requirements Characteristics of students
Zone of Promoted Action (ZPA)	Professional development Participation in research projects Informal interactions with colleagues

Figure 1: Valsiner's zones and characteristics of EoN Identity

RESEARCH DESIGN AND METHODOLOGY

The study (2013 – 2014) reported on in this paper involved eight teachers in two schools in Australia and was conducted within the context of a larger project (hereafter referred to as the *Numeracy Project*). The four teachers whose case studies are presented in this paper were from one of the schools.

Data was collected, during four visits to the school, through lesson observations and interviews with the teachers. Lesson observations focused on the tasks used and how these tasks provided opportunities for students to develop the five dimensions of the numeracy model developed by Goos, Geiger, and Dole (2014): *context*, *mathematical knowledge*, *tools*, and *dispositions* which are embedded in a *critical orientation*. The subsequent interviews were about planning and implementation of tasks, student and teacher learning as well as teacher reflections on the lesson. Each teacher also participated in a scoping interview that sought information about background, beliefs about numeracy, school context, and past opportunities to learn about numeracy across the curriculum.

Interview transcripts were annotated to identify comments related to characteristics within each teacher's ZPD, ZFM, and ZPA. For example, comments a teacher made about access to resources contributed to their ZFM. However, the characteristics of the ZPD are internal and must be inferred from the actions of the teacher in conjunction with their comments. Therefore, assessment of PCK was based on past opportunities a teacher had to learn about embedding numeracy into subjects they teach and analysis of their classroom practice using Goos et al.'s (2014) numeracy model. This analysis enabled each of the zones to be "filled in" and a narrative constructed in which each teacher's EoN Identity was described in terms of their ZPD, ZFM, and ZPA, enabling identification of factors that that contribute significantly to the teacher's EoN Identity.

FINDINGS

The teacher's shared Zone of Free Movement

As the teacher's professional context constitutes the ZFM (Goos, 2013), this section situates the research within Australia, the Numeracy Project, and the school. Firstly, numeracy is identified in the Australian Curriculum (ACARA, 2014) as a general capability to be developed in all subject areas. However, the use of the National Assessment Program – Literacy and Numeracy (NAPLAN) as a measure of school performance and accountability places pressure on schools to improve NAPLAN results, which influences school organisation, curriculum, and pedagogy (Hardy, 2014). Secondly, the teachers had previously agreed to participate in the Numeracy Project (2012 - 2014), where the potential of a professional development approach based on Goos et al.'s (2014) numeracy model was being investigated. Finally, the school was a large metropolitan school where school NAPLAN results were substantially below the Australian schools' average. Junior classes (Grades 8 and 9) were organised in *POD groups*, where one teacher taught English and history and another teacher took mathematics and science, with these teachers located to the same multidisciplinary staffroom. There were four 70-minute lessons in a school day and three lessons per week for each of the subjects mentioned above. While the school had a laptop hire scheme, the teachers reported limited uptake by students.

The teachers

The four teachers were Michael, Michelle, Karen and Martin (pseudonyms). In this section, the case of Michael is presented to illustrate how a case study was developed for each teacher. This is followed by summaries of the cases of the other teachers.

Michael was a mid-career science teacher. He completed his pre-service teacher education about eight years ago, completing curriculum subjects in physical education and mathematics but no subjects that specifically addressed numeracy across the curriculum. Since he began teaching, Michael had not had any professional development related to numeracy other than his involvement in the Numeracy Project. He agreed to participate in the project because Michelle, who shared his Grade 9 POD group, was a participant. When POD classes were introduced at the school two years ago, Michael was given a Grade 9 POD class for mathematics and science. He claimed that he had the appropriate science content knowledge, having completed an Applied Science degree, but found managing practical work difficult. In this paper, the focus was on Michael's EoN Identity in science.

Michael saw numeracy as basic school mathematics, describing it as:

> a form of mathematics that has been taught in a maths class somewhere along the line, maybe more primary school or early, like [Grade] 8 or 9. So I think they are the same thing … the basics of mathematics that every student should know.

While he saw a relationship between *mathematics* and science ("science does have a fair bit of *maths* involved in it"), Michael conceded that he didn't focus on developing these skills as his main focus was *covering the content* ("we have a curriculum that we have to meet"). Michael reported that he found it difficult to keep students engaged for the duration of lessons, especially if he had mathematics and science in consecutive sessions, and that behaviour management issues influenced his classroom practice, as making lessons more student-centred would not enable him to "get to that goal at the end". A high level of student absenteeism presented an additional challenge for Michael, as if students missed a lesson they were "missing a whole concept".

This example of Michael's classroom practice comes from a unit on ecology that focussed on the impact of rabbits on native animals. Michael told students that two areas of land were studied over a five-year period. While both had bandicoots, dingoes, and wallabies, a small number of rabbits were introduced to one of the areas at the beginning of the study. Michael provided students with the feeding habits of the animals and asked them to predict the effect of rabbits on the native animal populations. He then presented data from the study and led a discussion about what to consider when displaying the data graphically. After giving students time to graph the data, a limited discussion about potential reasons for the observed population changes occurred. Michael's goals for this lesson were for students to *display* the data graphically and to *interpret* the data. He would have liked to focus on the latter goal but limited access to laptops meant that most of the lesson was devoted to drawing the graphs by hand. While the lesson provided a context (understanding the impact of introduced species) for the use of mathematical knowledge (translating data from tabular to graphical form) and tools (using tables and graphs to mediate thinking about the situation), the opportunity for students to apply a critical orientation was limited (due to lack of time) and there was no opportunity for students to develop positive dispositions towards using mathematics in the situation.

Michael's ZPD seemed to be limited by his personal conception of numeracy which focused on mathematical skills and limited PCK that resulted from the lack of opportunities he had to learn about embedding numeracy into the science curriculum. His ZFM appeared to be mainly constituted by elements that impeded his development of an EoN Identity. The need to cover the content, lack of access to resources, and the behaviour management issues he experienced combined to limit his capacity to fully exploit numeracy learning opportunities in science. The only element within his ZPA that would assist him to make the most of these opportunities was the Numeracy Project, where his participation was less than enthusiastic.

Michelle had been teaching for just less than ten years. After completing a Bachelor of Arts, majoring in Geography, she worked for a while then returned to university to complete a Graduate Diploma in Education. Michelle taught history and English but the focus in this paper is on her EoN Identity in history. While she appeared to have adequate MCK, her opportunities to develop the requisite PCK for embedding numeracy in history had been limited. She believed that numeracy was needed in everyday life but her personal conception of numeracy seemed to be mainly focussed on mathematical knowledge and context. Classroom observations suggested that she was able to identify numeracy learning opportunities in history (e.g., the use of budgeting to help students to understand what life was like in Australia in 1901) but she did not fully exploit the potential of this activity.

Michelle's ZPD seemed to lack the rich personal conception of numeracy and PCK that would facilitate her developing a strong EoN Identity. Her ZFM included the views of her colleagues, who saw numeracy as the domain of the mathematics department, and the limited availability of technology. Michelle was an enthusiastic participant in the Numeracy Project and actively sought to develop her PCK through her own reading; thus, her ZPA promoted embedding numeracy in history.

Karen was a recently graduated science teacher with no formal preparation to embed numeracy in science. She was keen however, to develop this capacity and sought to do so through her participation in the Numeracy Project, mentoring from more experienced colleagues, and her own reading. While Karen believed there was a place for numeracy in science, her personal conception of numeracy seemed limited to mathematical knowledge and context. Classroom observations revealed that Karen was able to identify numeracy learning opportunities in the science curriculum (e.g., the use of a scaled geological timeline) however she did not fully exploit these.

Within her ZPD, Karen was in the process of developing PCK, had a narrow personal conception of numeracy but believed that numeracy was a part of science. Karen's ZPA was promising, with the presence of several actions that support embedding numeracy in science. Her ZFM allowed her to utilise numeracy learning opportunities in science (new curriculum, supportive colleagues) but she felt constrained in how she implemented tasks by lack of access to appropriate technology and student attitudes towards school (see Bennison, 2014b).

Martin, an experienced history teacher with over thirty years of experience, shared a POD group with Karen. His teaching areas were physical education and history. Numeracy across the curriculum had not been part of his pre-service teacher education nor had he participated in any numeracy-related professional development, possibly resulting in inadequate PCK. While he believed that numeracy was part of everyday life and he wanted to utilise numeracy learning opportunities in history, Martin expressed lack of confidence with embedding numeracy in history that he attributed to his lack of formal mathematics education. During classroom observations, he demonstrated that he was able to identify numeracy learning opportunities (e.g., using data to help students understand the impact of the Industrial Revolution) but increased attention to the inherent mathematical knowledge would have enriched the tasks he used.

Martin's beliefs appeared to support embedding numeracy into history. However, his ZPD seemed to lack appropriate MCK and PCK and included a narrow personal conception of numeracy. Within his ZFM, the new curriculum presented challenges because of limited chances to interact with other history teachers. Martin's only exposure to professional development that promoted embedding numeracy across the curriculum (ZPA) was the Numeracy Project, but his engagement with this project appeared to have been limited (see Bennison, 2014c).

Discussion

Michael, Karen, Martin, and Michelle had different disciplinary backgrounds and levels of experience. Not surprisingly, differences emerged in their ZPDs. but there were also similarities. All teachers identified numeracy learning opportunities (demonstrating CK); however, none fully exploited these to develop all dimensions of Goos et al.'s (2014) numeracy model. This may have been due to inadequate PCK, as there had been limited opportunities for any of the teachers to develop this type of knowledge, or the teachers' narrow personal conceptions of numeracy that restricted their ability to "see" the full extent of numeracy in the activities used.

Although the four teachers were at the same school and, on the surface, appeared to have the same professional context, their individual ZFMs differed, sometimes as a result of how a teacher interpreted his/her individual context. For example, all the teachers were implementing the new curriculum that gave them permission to embed numeracy across the curriculum. On the other hand, the arrangement of classes into POD groups, with the resultant allocation of teachers to staffrooms, presented problems for Martin (limited opportunities to interact with other history teachers), whereas this arrangement presented an opportunity for Michelle to reorganise the time between history and English to achieve her goals in both subjects. Participation in the Numeracy Project was part of all the teachers' ZPA; however, while Martin and Michael were indifferent towards the project, Karen and Michelle were enthusiastic participants who engaged in other activities to develop their PCK.

CONCLUDING REMARKS

These findings suggest that assisting teachers to broaden their personal conception of numeracy and develop appropriate PCK (both part of the ZPD) may enable teachers to embed numeracy across the curriculum. Although based on the cross case analysis of only four teachers, the findings do illustrate how comparison of teachers' ZPDs, ZFMs, and ZPAs enables suggestions to be made about how to support teachers to strengthen their EoN Identity. However, teacher learning will only occur if actions that promote embedding numeracy across the curriculum are permitted in the teachers' professional context. Therefore, further work is needed to examine how the ZFM/ZPA complex can be mapped onto each teacher's ZPD to direct development. This may assist in deciding whether the focus for assistance should be individual teachers, groups of teachers of the same discipline, or the whole school community.

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AFFORDANCES OF MATHEMATICS TEXTBOOKS: A VYGOTSKIAN PERSPECTIVE

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I use a Vygotskian perspective to argue for the importance of mathematics textbooks in the learning of mathematics. I argue that a typical mathematics textbook contains, inter alia, implicit mediators such as worked examples, exercises, theorems, proof, definitions and multiple representations, all of which provide different forms of access to mathematical objects and all of which permit some usage of mathematical signs. This usage of signs, be it logical or not, is essential to learning. Also, in a typical textbook, scientific concepts as presented in definitions, theorems and proofs, are interwoven with the (relatively) everyday concepts privileged in diagrams and graphs, worked examples and exercises. This enables movement between the abstract and concrete, a crucial aspect of meaningful learning.

INTRODUCTION

In this paper, I wish to examine how mathematics textbooks may be used for learning mathematics in a high school or early undergraduate context. According to the Merriam-Webster online dictionary a textbook is "a book about a particular subject that is used in the study of that subject especially in a school". Nowadays, a textbook may be printed on paper or it may be online. Importantly, and for the purposes of this paper, a typical mathematics textbook is structured according to pedagogical principles (for example, the mathematical notions in each chapter build on those of the previous chapters) and mathematics content is mediated through expository text, definitions, theorems, proofs, multiple representations, worked examples, exercises and answers. My focus is on learners' appropriation of abstract mathematical objects such as functions, calculus and various geometric objects. As such my arguments may not be relevant to primary school mathematics learning.

Within a socio-cultural context a textbook may be regarded as a cultural artefact; it is a depository of a particular portion of mathematical knowledge written and structured with pedagogical intent. Examples of textbooks used in undergraduate mathematics courses are the precalculus textbook by Sullivan (2012) and the calculus textbook by Thomas, Weir and Hass (2010). Textbook use is motivated by a basic sociocultural assumption: knowledge is not constructed from scratch by each succeeding generation. Rather we build on the knowledge of our predecessors (much of which is contained in textbooks) and colleagues to construct our own mathematical knowledge and possibly to create new knowledge. Of course, mathematics textbooks vary in quality and trustworthiness. Indeed some so-called textbooks are very poorly written and riddled with mathematical and other errors. I am excluding such 'textbooks' from this discussion.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 113-120. Hobart, Australia: PME.

Although there are many other rich and exciting resources which can be used in the learning of mathematics, such as the YouTube videos of the Khan Academy, TED lessons, etc., the structure and pedagogical and subject matter coherence of a textbook distinguish it from these other resources. Indeed the sole use of disparate internet resources, unless within a highly structured learning programme, may promote fractured and fragmented knowledge which lacks pedagogical coherence. Furthermore, and as will be discussed below, a mathematics textbook has certain unique characteristics which may be exploited in the learning and teaching of mathematics.

Many mathematics educators acknowledge that textbooks have the potential to be an accessible and powerful resource in mathematics classrooms (Rezat, 2008). However this was not always so: indeed there is still a lingering view of textbooks as constraining and controlling (Apple & Junck, 1990, cited in Ball & Cohen, 1996). In this regard Ball and Cohen (1996, p. 6) argue that "educators often disparage textbooks, and many reform-oriented teachers repudiate them, announcing disdainfully that they do not use texts". In my own institution, several mathematics educators regard the use of a textbook with much suspicion. For them its use by the teacher in the classroom signifies a lack of teacher autonomy, knowledge and creativity. Nonetheless, this type of negative attitude to the use of a textbook has, in the last decade, lost some momentum and, in the mathematics education academy, there is generally more of a recognition of the positive role that curriculum materials such as textbooks may play in mathematics education (Stein, Remillard, & Smith, 2007).

In this paper I want to develop a way of understanding, in theoretical terms, the possible affordances of a textbook for a learner. In particular, I am looking at the textbook in the context of its possible use by a learner in a standard mathematics learning environment. In such an environment learners engage with mathematical knowledge both during lessons and at home. In school, the teacher uses the textbook to a greater or lesser extent as a resource for examples, expositions, homework exercises, etc. At home, the mathematics textbook is a potentially powerful resource for learning. It may be used by the learner to revisit what was done in class or to anticipate what will be taught in class. This use may be voluntary or prescribed by the teacher in the form of exercises and reading.

THEORIZING THE TEXTBOOK

This study uses the Vygotskian notion of mediation by physical tools and signs (see Figure 1) as its central theoretical tenet. In particular it theorizes the primary role of a mathematics textbook as a mediator. Mediation is a concept that is fundamental to Vygotskian theory. Vygotsky uses it to provide a theoretical explanation of the link between the socially and historically constituted world (which includes bodies of knowledge such as that which we call mathematics) and the individual's higher mental functions (for example, abstraction and generalisation). "The higher mental functions he (Vygotsky) argues, are irreducible to their primitive antecedents; they do not simply grow from the elementary functions as if the latter contained them in embryo. To appreciate the qualitative transformation that engenders the mature mind, we must look outside the head, for the higher mental functions are distinguished by their *mediation* by external means" (Bakhurst, 2007, p. 53, my italics).

According to Wertsch (2007), mediation may be explicit or implicit. With explicit mediation, the tool of mediation is introduced intentionally into an activity by an interested party. Its purpose is to mediate or support an individual's or group's activities. With explicit mediation, the mediator is a material object. For example, a thermometer used to measure temperature, a knot in a rope used to remind oneself of something, and so on. In my study, the textbook may be regarded as an explicit mediator, mediating between mathematical knowledge (the 'object' in the Vygotskian triad) and the learner (the 'subject') in the Vygotskian triad. See Figure 1.

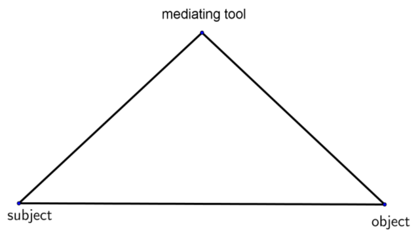

Figure 1: Vygotskian triad

In contrast, implicit forms of mediation are often covert: "Implicit mediation typically does not need to be artificially or intentionally introduced into ongoing action. Instead, it is already part of an ongoing communicative stream" (Wertsch, 2007, p. 180). Wertsch categorises both social and inner speech as implicit mediators. In this sense then, a teacher (who uses social speech) may be an implicit mediator, mediating between learner and mathematical textbook, in the course of teaching.

In addition, and in the mathematics education domain, I classify the different formal elements of mathematics such as theorems, worked examples, definitions, graphs, exercises with or without full solutions, etc. as implicit mediators (see Figure 2). These elements are mediators in that they provide epistemological paths, some more direct than others, to mathematical objects. They also have a dual nature in that they constitute mathematical knowledge in and of themselves. For example, consider the mathematical object, 'sine function'. This object is never accessed directly by the learner; rather it is accessed through various mediators such as its definition, graphical representations, diagrams, worked examples, exercises, theorems, and so on. For instance, in an exercise in which the learner has to prove some trigonometric identity (say, $\sin^2 x + \cos^2 x = 1$), the mathematical identity is a fact in itself. It also, however, requires the learner to work with definitions of sine and cosine (the definitions themselves serving as mediators between terms, 'sine' and 'cosine, and mathematical objects, 'sine' and 'cosine' respectively); so the exercise involving 'proof' has a mediating role. I return to the role of mathematical mediators in shaping various mathematical activities, such as abstracting, later in this paper.

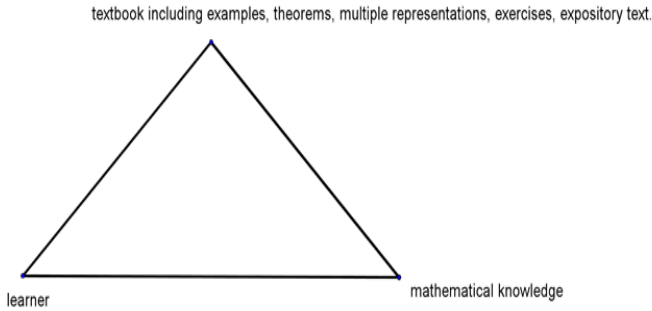

Figure 2: Textbook as mediator

A VYGOTSKIAN VIEW OF LEARNING

In order to look at the affordances of the textbook, it is necessary to briefly elaborate a theoretical view around the appropriation of abstract knowledge, such as mathematics. According to Vygotsky and as elaborated by Berger (2005) to the mathematical domain, appropriation of abstract knowledge (in this case mathematical knowledge) is a slow process involving the construction and renovation of conceptual objects through, inter alia, complex-thinking and pseudo-conceptual use of mathematical signs (such as words, symbols, diagrams).

With complex thinking, the usage of the sign is based on real but not logical connections between different aspects of the object, or between different objects. These connections may involve non-logical activities such as template-matching, associations, imitation, manipulations, etc. An example of complex thinking using association is as follows: On first encountering the derivative, $f'(x)$, of a function $f(x)$, many learners associate the properties of $f'(x)$ with the properties of $f(x)$. Accordingly, these learners assume that since $f(x)$ is continuous, so is $f'(x)$. Clearly this is not logical; indeed it is mathematically incorrect.

With pseudoconceptual thinking (itself a form of complex thinking), the learner uses mathematical signs correctly as if they understand the relevant mathematical object even if they do not fully understand it. The learner uses the signs to construct a mathematical object. This object may have some properties and structures commensurate with the properties and structures of the mathematical objects as defined by the mathematical community; or it may be an embryonic or deformed version of the object. No matter the ‘accuracy’, a pseudoconceptual usage of signs allows new learners to communicate with more knowledgeable others and to engage with textbooks. That is, the pseudoconceptual usage of a set of signs (in exercises, definitions, theorems, problem–solving, and so on) allows the learner to use mathematical signs in ways that are commensurate with their use by members of the mathematical community even before fully ‘understanding’ the mathematical object. It is this usage and communication which shapes learning and the construction or renovation of abstract objects. The hope is that through appropriate use and social interventions (be it through textbook use or teacher intervention), the pseudoconcept will get transformed into a concept.

Another aspect of Vygotskian theory that is relevant to an understanding of the mathematics textbook role in learning, is the relationship between scientific concepts and spontaneous or everyday concepts. Scientific concepts are those that form a “coherent, logical hierarchical system” (Daniels, 2007, p. 311). These concepts have a high degree of generality, for example, theorems, definitions. They are usually introduced by a teacher at school, and are key components of a textbook. In contrast, everyday concepts are those that develop in the context of “immediate, social, practical activity” (Daniels, 2007, p. 311). In the context of a textbook, I suggest that (relatively) everyday concepts are often used in the description of those concepts suggested by particular examples or visual representations. I elaborate on this point later. Most importantly, Vygotsky saw the development of these two types of concepts as interdependent: “The formation of concepts develops simultaneously from two directions: from the direction of the general and the particular” (Vygotsky, 1987, p. 163). Through this interdependence the scientific concepts get imbued with the richness of everyday contexts and meanings and the general and the abstract become visible in the everyday concepts.

AFFORDANCES OF A MATHEMATICS TEXTBOOK

In this section I discuss how the different implicit mediators in a mathematics textbook (for example, exercises, theorems) afford access, be it embryonic, partial or full, to mathematical objects. I also show that the juxtaposition of these different elements allows for an interweaving of scientific concepts with (relatively) everyday concepts.

I use Sfard’s (2008) description of mathematics as a discourse characterised by its narratives, routines, visual mediators and words to categorize the implicit mediators. Mediators of narrative are those that define or describe or justify the existence of mathematical objects, e.g. theorems, definitions and proofs; mediators of routines are those that refer to activities with these objects, e.g. worked examples, exercises; visual mediators are those that offer alternate representations of mathematical objects, e.g. symbols, expository text, graphs, diagrams. These three forms of mediators all employ ‘mathematical’ words and thus, for the purpose of this article, I do not distinguish ‘words’ as a separate category of mediator.

A mathematics textbook may be written with a deductive or inductive approach. Most textbooks adopt a mixture of both approaches. With the deductive approach the relevant section of the textbook gives “appropriate definitions or concepts, which are then exemplified and followed by exercises for students to practice”(Ensor et al., 2002, p. 22). In Vygotskian terms, this corresponds to going from the abstract to the concrete, or from the scientific (eg definitions) to the everyday (eg real-world examples using mathematics). With an inductive approach, the textbook "introduces a topic by engaging students in a range of activities that can be regarded as instances of the concepts which students are to master. Activities lead to definitions and from this point opportunities may be provided for students to practice" (Ensor et al., 2002, p. 23). In Vygotskian terms, this corresponds to going from the concrete to the abstract, or from the everyday (eg examples of particular instances in which the mathematics is applied), to the scientific (eg definitions).

In either case, and for purposes of this paper, each chapter of a mathematics textbook contains definitions, theorems and proofs; worked examples illustrating the concepts; exercises comprising both routine and non-standard exercises (with some answers at the back of the book) and multiple representations of the mathematical ideas in the form of expository text explaining or suggesting the mathematical concept, diagrams, graphs and so on. Each of these different mediators affords access to more specific or to more general and abstract instantiations of the relevant maths object. In the to-ing and fro-ing between these mediators, the everyday moves towards the scientific and the scientific moves towards the everyday, thereby enriching the understanding of the concepts.

More specifically, **worked examples** allow, inter alia, for imitation of 'new' mathematical signs. That is, the student through copying and mindfully adapting the solutions of worked examples in the textbook to other similar examples, is able to engage with the new mathematics object before full understanding. It is this imitative use of signs, probably using complex thinking, which gives the student initial access to the new mathematical object (its properties and characteristics).

In contrast, **exercises** afford the learner the opportunity to construct their own solutions to problems to which full solutions are not given. These constructions may be informed by complex or pseudo-conceptual thinking and may involve imitation, association, template-matching and so on. Of course, they may also involve, wholly or partially, conceptual thinking. As indicated above, appropriation of socially-sanctioned knowledge happens as a result of the construction of appropriate concepts; we are not born with prior knowledge of various socially-conceived concepts, nor can we just absorb them. Rather we need to construct them for ourselves. Part of this constructive activity is the using of the signs (words, symbols, diagrams) to communicate with the textbook or others, to a better or less degree, about the object. Being able to check the final answer against a back-of-the-book answer provides the social feedback which helps shape the learner's understanding of the mathematics object under scrutiny.

Definitions, theorems, proofs and other theoretically-orientated mediators may be regarded as scientific concepts in that they constitute the underlying structure and connections within the logical, hierarchical system of mathematics. For Vygotsky, the purpose of school education is to appropriate scientific concepts such as these. (These correspond to Sfard's 'narratives'.) However they are also implicit mediators: scientific concepts' "relationship to objects is mediated through other concepts" (Daniels, 2007, p. 311) which may themselves be scientific concepts. Bakhurst (2007, p. 70) argues that because "scientific concepts are verbally articulated, theoretically embedded, and tightly related to many other concepts, they seem abstract, general, and remote from concrete experience. But appreciation of such concepts, properly integrated into a system of knowledge, actually facilitates the understanding of objects in their particularity". Likewise in mathematics an understanding of definitions, theorems and proofs (sometimes glossed over by students and teachers) while essential to an understanding of the general and the abstract are also essential to an understanding of the particular.

Vygotsky did not speak of **multiple representations** of concepts, per se. However these representations, for example, graphs, diagrams or expositions involving applications or history of the mathematics object may be regarded as particular instances of an object. Many of these mediators, such as diagrams, evoke an understanding of the mathematical objects more rooted in the everyday, than do verbal and symbolic definitions. For example, a diagram showing how the slope of tangent to a curve may be used to describe the derivative of the curve at that point, is closer to the everyday than an examination of the formal symbolic definition of the derivative. It is through the interaction of these different forms of mediators (in this case diagram and definition) that the everyday concepts interweave with the scientific concepts, thus imbuing the object with meaning and locating it within a system of interconnected objects (scientific concepts).

CONCLUDING ARGUMENT

In this paper I have used a Vygotskian perspective to argue for the importance of a typical mathematics textbook in the learning of mathematics. I have shown how a typical textbook has certain features which are crucial for the learning of maths. These include various implicit mediators (such as worked examples, exercises, theorems, proof and definitions and multiple representations) all of which provide access to mathematical objects at different points and all of which permit some usage of mathematical signs. In Vygotskian terms, this usage of signs, be it logical or not, is essential to learning. Furthermore in the archetypal textbook, scientific concepts as represented in definitions, theorems and proofs are interwoven with the (relatively) everyday concepts privileged in diagrams and graphs (multiple representations), worked examples and exercises. This enables movement between the abstract and concrete, a crucial aspect of meaningful learning. Most importantly, in a typical mathematics textbook, the elements of mathematics are used with pedagogical anticipation (of what is to come) and pedagogical retrospection (of what went before). On-line resources, unless they are organised into a learning program, lack this pedagogical coherence and consistency.

In closing, this paper presents important possibilities for a learners' use of a textbook, and most importantly a theoretical framework for understanding these possibilities.

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INFERRING PRE-SERVICE TEACHERS' BELIEFS FROM THEIR COMMENTARY ON KNOWLEDGE ITEMS

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This paper explores the entailment of teachers' beliefs in pedagogical content knowledge for mathematics teaching. It reports interview data from one pre-service teacher discussing his responses to a multiple-choice item designed to assess an aspect of pedagogical content knowledge. The responses were analysed in two stages as the basis for inferring beliefs that appeared to underlie them. The results suggest that the pre-service teacher's choices relied heavily on his beliefs, and problematise the distinction between beliefs and pedagogical content knowledge.

INTRODUCTION

Beswick and Goos (2012) reported that primary pre-service teachers (PSTs) found it much easier to endorse progressive and student centred belief statements about the nature of mathematics, and mathematics teaching and learning than to provide appropriate answers to multiple choice items designed to measure their mathematical content knowledge or pedagogical content knowledge (PCK). They suggested that items that presented more complex scenarios that draw on PCK and imply a need for the respondent to take a position on the relative merits of different responses might provide a more effective way than Likert scale items to uncover the teachers' beliefs. Deducing beliefs from PCK item responses is consistent with the conceptualisation of teachers' knowledge as encompassing beliefs and confidence (Beswick, Callingham, & Watson, 2012). The essential equivalence of knowledge and beliefs has also been argued by Beswick (2011).

The study reported here explored the possibility of using PCK items to infer teachers' beliefs by examining interview data in which PSTs talked about their thinking in relation to a multiple choice question aimed at accessing their PCK. It addresses the research questions, "To what extent are beliefs entailed in PCK?" and "How effective might discussion of PCK scenarios be for uncovering the beliefs of PSTs?"

TEACHER BELIEFS AND THEIR MEASUREMENT

The influence of teachers' beliefs on their practice is well established. There is also broad consensus that teachers act rationally, with apparent inconsistencies between the beliefs they articulate and their classroom practices attributed either to incomplete understanding of the teachers' beliefs system on the part of the researcher, and/or the interaction of multiple beliefs of varying centrality in a given context (Leatham, 2006). Measures of teachers' beliefs have included questionnaires allowing the larger sample sizes needed to test theory and relationships among variables, and have typically comprised Likert type items (e.g., Beswick & Goos, 2012).

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Ambrose, Clement, Philipp and Chauvot (2004) attempted to overcome the shortcomings of Likert scale questionnaires – that they identified as difficulties in understanding how respondents have interpreted items, the fact that the relative importance to the respondents of various beliefs is not clear, and that statements are responded to without reference to a context – by devising an open-response survey that presented scenarios for teachers to consider along with questions for them to respond to. The responses were coded according to rubrics to obtain measures of seven pre-defined beliefs about mathematics and mathematics learning. Although the scenarios used were specific, the beliefs about which the responses to them were used as evidence were quite broad (e.g., "Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures" (p. 65)).

Some of the scenarios used by Ambrose et al. (2004) involved examples of teaching intended to provide opportunities for respondents to take a stance in relation to specific pedagogical actions (e.g., the degree of directiveness of the teachers' guidance). Others could, with different prompts, also have been used to make inferences about the respondents' PCK (e.g., ranking fraction problems according to their anticipated difficulty for students). In contrast the items in this study were designed to measure the participants' PCK, and the beliefs inferred from responses were emergent from the data rather than predetermined. This means that the items used in this study (1) were appropriate for exploring the entailment of beliefs and PCK, and (2) the beliefs identified are closely related to the contexts described in the items.

Pre-service teachers' beliefs

PSTs tend to begin initial teacher education programs with strongly held and quite traditional beliefs about mathematics teaching and learning that they have acquired as a result of their own experiences of learning mathematics at school (Van Es & Conroy, 2009). According to Philipp et al. (2007), many PSTs regard mathematics as a collection of rules and procedures – an orientation that aligns with Ernest's (1989) instrumentalist view or the static perspective described by Felbrich, Müller and Blömeke (2008). In addition, they may believe that mathematics learning amounts to learning the procedures (Philipp et al., 2007). Many studies have reported interventions that have appeared to achieve greater alignment between PSTs' beliefs and the student-centred beliefs that typically underpin teacher education programs (e.g., Philip et al., 2007). These are accompanied by appropriate circumspection about the longevity of the apparent influence.

PEDAGOGICAL CONTENT KNOWLEDGE

PCK, introduced by Shulman (1987), has been conceptualised and elaborated by mathematics educators in a range of ways. For example, Ball and colleagues (e.g., Ball, Thames, & Phelps, 2008) described it in relation to mathematical knowledge used in teaching. For them, PCK comprises knowledge of content and students, knowledge of content and curriculum, and knowledge of content and teaching. In their knowledge quartet, Rowland, Huckstep and Thwaites (2005) took a more dynamic view of knowledge used in teaching, whereas Chick, (2007) considered aspects of PCK falling into one of three categories forming a continuum and reflecting the relative prominence of content and pedagogical knowledge. These categories were knowledge that is Clearly PCK, Content Knowledge in a Pedagogical Context, and Pedagogical Knowledge in a Content Context. Within each, a range of overlapping and interdependent subcategories were identified. In this study aspects of Chick's (2007) framework, were used to operationalise PCK in terms of knowledge of "(1) analysing /anticipating/diagnosing student thinking, (2) constructing/choosing tasks/tools for teaching, (3) knowledge of representations, and (4) explaining mathematical concepts" (Beswick & Callingham, 2011), and hence to devise items designed to provide insight into PSTs' PCK.

THE STUDY

The study reported here was part of a larger study involving seven Australian universities and aimed at developing measures of PSTs' MCK, PCK and aspects of their beliefs with a view to establishing an evidence base for PST education. The principle data collection was conducted using an online multiple-choice and Likert scale item questionnaire. In addition to the questionnaire, individual interviews were conducted with small numbers of PSTs at some of the participating universities. It is part of one of these interviews that forms the basis of this paper.

Instrument

The semi-structured interviews asked PSTs about their backgrounds, what came to mind at the mention of mathematics, and what they thought were the most important things for teachers to know and believe in order to be effective. They were also asked to discuss three PCK items and three beliefs items from the questionnaire. The relevant interview question for this study concerned one of the PCK items answered by both primary and secondary PST interviewees. It is shown in Figure 1. After being shown the item participants were asked:

1. Which choice do you consider to be the most helpful?
2. Why do you consider (your choice) to be the most helpful one?
3. Are there any circumstances in which it would not be the most helpful?
4. Please comment on the other options. Are some more helpful than others? Why? Under what circumstances?
5. What other models might you use?

Participants and data analysis

This paper draws on the responses of the one PST – Geoff (pseudonym) – enrolled in a 2-year Master of Teaching program preparing to teach secondary mathematics. Prior to undertaking teacher education, Geoff had completed a Graduate Diploma in Agricultural Science. Geoff was in the first year of the M. Teach program. His responses illustrate the potential use of PCK items to elicit beliefs and were typical of the responses of the interviewed PSTs.

The interview transcripts were read firstly with a view to identifying statements of beliefs. These were extracted, sometimes with re-wording for clarity, and used as the basis for inferring more general beliefs that seemed likely to underpin the specific belief statements.

A teacher sets the following proportional reasoning task for an upper primary class:

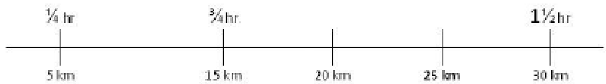
Bill and Ben were out on a Sunday morning bike ride. After three quarters of an hour they passed a sign that showed they had ridden 15 kilometres since they left home and that they still had 25 kilometres to reach their destination. How long will it take them to get there?

Which of the following representations is most helpful for the teacher to develop the students' understanding of proportional reasoning in solving this problem?

☐ Cross multiplying

Time (hr)	Distance (km)
¾	15
x	25

☐ Double number line



☐ Ratio table

Time	¾ hr	¼ hr	1 hr		
Distance	15 km	5 km		1 km	25 km

☐ Find the unit rate:

Riding 15 km in ¾ hr is equivalent to riding 1 km in ¾ ÷ 15 hr.

Figure 1: PCK item discussed by interviewees.

RESULTS

Geoff identified the double number line as the most helpful representation. The transcript of his responses to each of the questions about the item, are provided in Table 1. Table 1 also shows the results of the initial analysis in the form of beliefs statements apparent from the transcript. Geoff preferred the double line representation because it made the solution obvious for all students. In commenting on other possible models he also mentioned that the double line allowed the problem to be simplified, made it concrete, and would be enhanced by using physical number lines such as wooden rulers. He referred to the inability of primary school students to think abstractly or to engage in high level thought when justifying his choice of the double number line, and mentioned the need for these abilities in commenting on the cross multiplication and unit rate methods. Potential to confuse students was regarded as a disadvantage of cross-multiplication, but the fact that it was quick was referred to as positive feature of that approach. On the basis of these beliefs (evident from transcript shown in Table 1), the following six beliefs are suggested as underlying Geoff's thinking in relation to choosing representations:

1. Most upper primary aged students cannot think abstractly.
2. Being quick is a positive characteristic of a solution method.
3. Solutions that all students can understand are best.
4. It is important / helpful to simplify and make concrete, mathematics for students.
5. It is important to make mathematics problems concrete for students.
6. It is important not to confuse students.

Question	Response (transcript excerpt)	Beliefs (from transcript)
Why?	This is an upper primary school kid so visual representation is something that is very important to children of that age – they can't ... the majority are not at age yet where they are able to use abstract thinking to work out in their head this is ... to multiply this with that to give me 'how many km per hour'... so you can use the double line – you can even use a ruler on a big whiteboard; you could use a ruler for something concrete ... ¼, 2/4, ¾, a whole... you could visually represent it. It's easy to do and easy to show that... So I would use the double line. I guess it is possible that we do have a child in ... it would be unusual if not ... if you're talking about an upper primary kid, most of them wouldn't be able to do any abstract thinking. That would be able to understand the cross multiplying. However, having said that, the cross multiplying formula – the way it's done – would be very quick.	Visual representations are important for upper primary aged students. The majority of upper primary aged students aren't able to think abstractly. Working out a problem like this in one's head using cross multiplication is abstract thinking. Cross multiplication is quick.
When not most helpful?	No, can't see circumstances when this would not be applicable – would not be the most helpful. The reason being is that even for the most advanced student – just because it's obvious, they'd still get it.	The double number line makes the solution obvious.
Comment: ratio table	Is a visual representation – a good way of ... similar to the Double number line, but is confusing because the time given at the top is not progressive.	The ratio table is confusing because the times are not ordered by duration.
Comment: cross multiplication	The cross multiplying one... that is quick , and if I was to be confronted with this problem, that is how I would naturally do that ... that way of doing it. Because it is quick. It does rely a bit on abstract thinking ... in the method.	Understanding cross multiplication requires abstract thinking.

Comment: unit rate	That is pretty high maths and thinking there – it would be good for the very, very good students but the upper primaries, I have doubts whether they could get that. Because it's too... for the high end student, it's abstract.... involves applying and producing a formula from the information that's given to you.	Upper primary students probably couldn't understand the unit rate method. Unit rate method involves high level mathematical thinking. The unit rate method is abstract. The unit rate method involves applying and producing a formula from the information provided.
What other models?	You could possibly ... probably not another model. Probably I would use the double line, but make it more concrete, as in use pieces of wood or rulers... you could use one ruler on top of another ... you could simplify further and make it more concrete.	A concrete representation of a number line such as ruler or piece of wood (or two rulers or pieces of wood) would be even better than a drawn number line. Double line method allows the problem to be simplified. Double number line method allows problem to be made concrete.

Table 1: Analysis of Geoff's interview transcript

DISCUSSION

Geoff was concerned to make problems understandable and accessible to students, to make their solution as simple as possible, and to avoid confusion. To this end he valued representations that he described as concrete. The influence of Geoff's own experiences of learning mathematics and of observing mathematics teaching was evident from his attraction to cross-multiplication method despite believing it was too difficult for upper primary students because, "that is how I would naturally do that".

The extent to which the beliefs evident from Geoff's interview were consistent with messages received in his teacher education classes is not clear but the connections made with his personal preferences are consistent with Van Es and Conroy's (2009) claim that PSTs bring with them pre-existing beliefs about mathematics teaching and learning. There was also evidence that some of his beliefs, for example, his reference to quick methods and the value of simplifying problems for students, could be considered traditional as described by Van Es and Conroy (2009). Interestingly, Geoff made no specific reference to anything that he had learned as part of his M. Teach course.

CONCLUSION

This study sought to explore the extent to which beliefs are entailed in PCK, and the potential effectiveness for uncovering beliefs of discussions of PCK scenarios. The responses to the PCK item presented here did seem effective in inferring this PST's beliefs. A further step, that was beyond the scope of this study but that would help to validate this claim, would be to present the participants with the set of beliefs ascribed to them for their comment.

Examination of the transcript revealed that it was comprised almost entirely of belief statements, often stated without equivocation as if knowledge. This is consistent with conceptions of beliefs and knowledge as essentially equivalent (Beswick, 2011) and the inclusion of beliefs (including about one's own ability or efficacy – i.e., confidence) in a rich construct of knowledge (Beswick et al., 2012). The PCK expressed by this PST in choosing the option that the questionnaire designers considered best (the double number line) was not justified in terms of objective evidence but rather personal experience and anecdote. For him, PCK was founded on personal beliefs. That is, Geoff's beliefs constituted knowledge for him. It is often emphasised that beliefs are necessarily inferred from words or actions of individuals (Pajares, 1992) but less often acknowledged that constructs more commonly called knowledge are similarly inferred. For example, when a respondent provides or chooses a correct or approved answer an inference is made about that individual's knowledge.

This study provides further evidence for the blurring of possible lines between beliefs and knowledge. One implication could be that, rather than being a part of what is being attempted in PST (or any) education, beliefs change is in fact the whole task.

Acknowledgments

This research was supported by Australian Learning and Teaching Council Priority Projects grant no. PP10-1638. The contributions of Rosemary Callingham, Helen Chick, Julie Clark, Merrilyn Goos, Barry Kissane, Pep Serow, and Steve Thornton are acknowledged.

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THE MEANING OF RATIO: PROSPECTIVE MATHEMATICS TEACHERS' KNOWLEDGE ABOUT THE TEACHING AND LEARNING OF PROPORTIONAL REASONING

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This study shows the role of prospective teachers' mathematical knowledge in identifying students' understanding related to the idea of ratio as a component of proportional reasoning. Ninety two prospective teachers analysed primary school students' answers and answered questions probing the teachers' awareness of mathematics and of teaching strategies. We found evidence that prospective teachers with stronger mathematical knowledge were, in general, better able to suggest insightful strategies to support or extend the primary students' conceptual progression. However, there was one group of prospective teachers, strong at mathematics, who seemed to believe that students' answers were simply "right or wrong" and who were ineffective at suggesting strategies.

INTRODUCTION

The development of proportional reasoning is an important topic in primary and secondary school curricula that involves several interrelated cognitive processes ranging from qualitative thinking to multiplicative reasoning. Recent studies have shown that the teaching and learning of the ideas of ratio and proportion involved in the development of proportional reasoning are not easy tasks for teachers (Livy & Vale, 2011; Pitta-Pantazi & Christou, 2011). The knowledge needed to teach these concepts has an important role in the development of teachers' professional competences such as the recognition of students' mathematical understanding (the skill of noticing students' mathematical thinking).

The research presented here with primary school prospective teachers is embedded in two lines of research: studies focused on the teachers' skill of noticing students' mathematical thinking and teachers' knowledge and, studies focused on the development of proportional reasoning, particularly, on the idea of ratio. We have chosen it since previous research has shown the difficulties of the teaching and learning of this concept.

THEORICAL BACKGROUND

Recent research indicates that being able to identify relevant aspects of teaching and learning situations and interpret them to take instructional decisions (Mason, 2002) is an important teaching skill (professional noticing). These studies have also provided contexts and tasks for development in teacher education programs.

The skill of noticing students' mathematical thinking and the role of teachers'

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 129-136. Hobart, Australia: PME.

mathematics knowledge

A particular focus is the skill of noticing students' mathematical thinking (Coles, Fernández, & Brown, 2013; Jacobs, Lamb, & Philipp, 2010), understood as recognising evidence of student understanding, to take instructional decisions. Previous research has shown that identifying the relevant mathematical elements of problems (mathematical knowledge) plays an important role in recognising evidence of students' mathematical thinking in different mathematical domains. Fernández, Llinares, and Valls (2011; 2012) indicated that in the domain of proportionality discriminating between proportional and non-proportional situations was a key element in the development of prospective mathematics teachers' abilities to identify evidence of different levels of students' proportional reasoning. Bartell, Webel, Bowen, and Dyson (2013) examined the role mathematical content knowledge plays in prospective teachers' ability to recognise evidence of children's conceptual understanding. Magiera, van den Kieboom, and Moyer (2013) showed that prospective teachers demonstrated a limited ability to recognise and interpret the overall algebraic thinking exhibited by students in the context of one-to-one interviews. Sánchez-Matamoros, Fernández, and Llinares (2014) indicated that a key element in the development of prospective teachers' noticing of students' mathematical thinking in the domain of derivates was prospective teachers' progressive understanding of the mathematical elements that students use to solve problems (in the domain of the derivative). Prospective teachers' ability to identify the key mathematical elements needed to understand the concepts (key developmental understanding (KDU, Simon, 2006)) plays an important role in recognising the characteristics of students' understanding and also taking instructional decisions.

The development of proportional reasoning

According to Lamon (2007), proportional reasoning integrates different components, namely the meanings of the mathematical concepts (rational number interpretations; ratio, part-whole, measure, quotient and operator) and the ways of reasoning with these mathematical concepts (unitising process, reasoning up and down, quantities and covariance and relational thinking). Pitta-Pantazi and Christou (2011), based on Lamon's characterisation, added the ability to solve missing-value proportional problems, and the ability to discriminate proportional and non-proportional situations. These authors consider that the tasks of determining whether the contexts are proportional or non-proportional and solving missing-value problems can provide relevant information about the development proportional reasoning. We are going to focus on the ratio component understood as a relationship between two quantities that is a comparative index (Carraher, 1996; Freudenthal, 1983). In the following example, "In a new building, flats are sold with three different floor areas: 7.5 meters by 11.4 meters, 4.55 meters by 5.08 meters and 18.5 meters by 24.5 meters. Which one is more similar to a square?", the ratio is the mathematical concept useful to solve it. The ratio in the first flat is 7.5/11.4, in the second 4.55/5.08, and in the third 18.5/24.5. The squarest one, therefore, is the second because the ratio is closer to 1, and to be square the ratio has to be 1.

In this sense, understanding the meaning of ratio could be considered a KDU in the development of proportional reasoning in primary school students. Taking into account these aspects, our research questions are:

- How do prospective teachers understand the idea of ratio and what influences that understanding in recognising students' understanding?
- Which instructional decisions do prospective teachers take after the recognition of students' understanding?

PARTICIPANTS AND THE TASK

The participants were 92 prospective primary school teachers who were studying the third year of the degree at the University of Alicante (Spain) to become a primary school teacher. They had attended a course focused on numerical sense (first year) and one focused on geometrical sense (second year). In the third year, they were attending a course about the teaching and learning of mathematics in primary school and one of the units was about proportional reasoning. Data were collected after this unit.

Participants solved a task consisting of 12 primary school problems related to Lamon's 12 components of proportional reasoning (see above) and three primary school students' answers to each problem that showed different characteristics of students' understanding of each component. Prospective teachers had to answer four questions: the first one related to the learning objective of the primary school problem (Question a); and the second related to the recognition of students' mathematical understanding (Question b). The others related to the instructional decisions prospective teachers take to respond on the basis of students' understanding, supporting (Question c) and extending (Question d). In Figure 1, the primary school problem related to the ratio component, the three primary school students' answers, and the four questions are shown.

In this problem, students had to compare three ratios and look for the one closest to 1 to know which floor area is the squarest. The three primary students' answers show different characteristics of their understanding: the first writes the ratios and interprets that the floor area whose ratio is closer to 1 will be the squarer; the second writes the ratios between the sides but then provides a justification based on additive relations (there exists less difference, so it will be squarer because the sides are more equal); the third uses an additive strategy making the subtractions between the floor areas' sides. In trying to look for the smallest difference, this student chooses the difference closer to 0.

ANALYSIS

Data are the answers provided by prospective teachers to the four questions above. Three researchers categorised them, analysing the answers to each question individually. Validity and reliability were established by comparing sets of independent results, citing specific examples, clarifying the coding schemes, and re-coding the data until 100% of agreement was achieved.

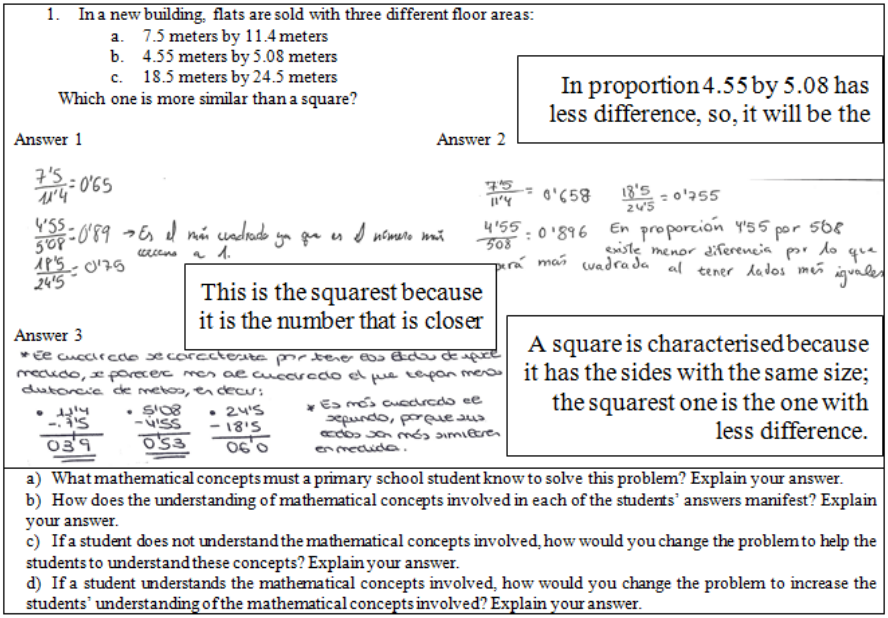

Figure 1. Prospective teachers' task related to the ratio component

With regard to the mathematical content that prospective teachers considered (Question a) that was involved in the primary school problem, we identified two categories: prospective teachers who recognised the idea of ratio as a measure (that is prospective teachers who recognised the squarest floor area as the one where the ratio between the sides is closer to 1), and prospective teachers who did not recognise this idea. We jointly analysed Questions a and b in order to check if, although the prospective teacher had not written the mathematical content of the task in Question a, they had used it to recognise students' understanding (Question b).

For prospective teachers who recognised students' understanding (Question b), we identified three categories. Prospective teachers who:

- provided general comments based on the correctness of the answer (*Answer 1: this student solves the task right because he understands and uses the mathematical concepts required correctly. Answer 2: this student solves the task right because he understands and uses the mathematical concepts required correctly. Answer 3: this student doesn't solve the task right because he doesn't use the idea of fraction as a ratio correctly, so he does a subtraction*);
- based their comments on a simple description of the students' answers (*Answer 1: he divides the sizes of the three lofts right and then points out that the one*

closer to 1 is squarer, so the answer is right. He uses the ratio and proportion concepts right. Answer 2: he divides the sizes of the three lofts and thinks that 4.55×5.08 is squarer because there exists less difference between these numbers and he knows that a square has equal sides. So, he uses the definition of a square. He divides the numbers but then he doesn't use them. Answer 3: he subtracts the measures of the three lofts and the one with the smaller difference will be the right one);

- recognised evidence of students' understanding (*Answer 1: the student solves the task right because he understands that it is a problem of proportionality and does the comparison ratio choosing as a result the one which is closer to 1. Answer 2: the student understands that he has to do a comparison ratio, but he doesn't understand that the right one is the one that is close to 1. He considers that the correct one is the one where the difference between numerator and denominator is smaller. Answer 3: the student doesn't understand that he has to do a comparison ratio, so, he does a subtraction. In this case, he considers that the right answer is the loft where the sides are more similar, that is, 5.08 and 4.55 because in a square the sides are equal*).

Finally, we analysed Questions c and d obtaining the categories shown in the results section.

RESULTS

Table 1 shows the categories of prospective teachers' answers that we obtained after the analysis. Results show that only prospective teachers who had identified the idea of ratio as a measure were able to recognise students' understanding (CI group). This result is also evidenced by the fact that all prospective teachers who had not identified the idea of ratio as a measure were unable to recognise students' understanding. They described or provided general comments (OD and OG groups). Therefore, identifying the key mathematical element (ratios as measure) helped prospective teachers to recognise evidence of students' understanding.

Identify the idea of ratio as a measure (33 PT)	Recognise students' understanding (CI)	14
	Describe the students' answers (CD)	13
	Provide general comments (CG)	6
Do not identify the idea of ratio as a measure (58 PT)	Describe the students' answers (OD)	24
	Provide general comments (OG)	32

Table 1: Categories of prospective teachers' answers

However, not all the prospective teachers who had identified the idea of ratios as a measure were able to recognise evidence of students' understanding. Some of them described the students' answers (CD group) indicating that those prospective teachers had difficulties in recognising students' understanding and others only provided general comments based on the correctness of the answer (CG group). This last group of prospective teachers leads us to think that although prospective teachers could talk about the mathematical concept in the problem they did not use this to talk about the students' understanding. They could have the belief that a student answer only could be "right or wrong".

Instructional decisions

Table 2 shows the problem modifications provided by prospective teachers to help students who did not understand the concept of ratio as a measure. The majority of prospective teachers proposed problem modifications based on the numbers or the ratios. What is significant is that the number of nonsense answers or blank answers increases in prospective teachers who had not identified the key mathematical content (ratio as a measure) and they only described students' answers or provided general comments.

Question c	CI	CD	CG	OD	OG
Use integer numbers / use smaller numbers /make differences between numbers bigger	7	7	2	12	13
Integer ratios	1	1		3	5
Use pictures / manipulatives	7	1		6	9
Change the context	1		1	3	
Blank answers / nonsense answer / same level	3	5	3	11	12

Table 2: Problem modifications to help students who do not understand the concept

Table 3 shows the problem modifications provided by prospective teachers to improve students' understanding. What is significant is that there were more prospective teachers who proposed nonsense answers or blank answers than for Question c. Therefore, it was more difficult for prospective teachers to modify the problem to improve students' understanding than to help students who do not understand the concept. We can observe also that the number of nonsense answers or blank answers increases in prospective teachers who had not identified the key mathematical concept (ratio as a measure) and they only described the answers or provided general comments.

Question d	CI	CD	CG	OD	OG
Use bigger numbers / use numbers with more decimals	2			3	7
Similar ratios / ratio bigger than 1 / same ratios	5	4	1	5	7
Change the context // percentages	3	1		3	2
Blank answers// nonsense answer// same level	5	8	4	15	19

Table 3: Problem modifications to improve students' understanding

DISCUSSION AND CONCLUSIONS

Identifying the meaning of ratio as measure as a KDU of proportional reasoning, led prospective teachers to recognise evidence of students' understanding and provide more variety of school problem modification to help students who do not understand the concept or to improve students' understanding. However, some prospective teachers, even though they had recognised students' understanding (CI), had difficulties in providing a task modification (particularly, the modification of the task related to the improvement of students' understanding).

Furthermore, there were prospective teachers who had identified the key mathematical content but they only described students' answers or provided general comments about students' understanding (CD and CG). This result seems to indicate that although some prospective teachers could talk about the mathematical concept in the problem they had difficulties in recognising evidence of students' understanding. One possible explanation of this fact is that they did not know how to use it to recognise students' understanding (that it is part of the mathematical knowledge for teaching). On the other hand, the fact that some prospective teachers provided general comments even though they had identified the key mathematical content also could indicate that these prospective teachers have the belief that a student answer is just "right or wrong". This feels like the dualism category in Perry´s Development Scheme (Copes, 1982, p. 38), the prospective teachers have not the richness or multiple or relativistic perspectives to be able to offer anything other than their one ´right answer. In other words, it is not enough simply to recognise the mathematical concepts in the problems. It may be that teacher education courses also need to work on moving some prospective teachers away from this "right/wrong" perspective, in order for them to be effective in responding to their students. Of course what we are talking about here is relatively profound changes in world-view and this is no easy task for students or teacher, and certainly more complex than learning some new mathematics. But we feel that unless teacher education is sensitive to the issue of prospective teachers seeing mathematics in dualistic terms, then we may not be equipping new teachers with the skills they need to recognise and therefore develop student understanding.

These results provide relevant information to prospective teacher training courses because our instrument can offer opportunities to work on the key mathematical concept to develop the professional competences of the teachers along with their own personal development.

Acknowledgements

The study reported here has been financed in part by the Project I+D para grupos de investigación emergentes GV/2014/075 of the Consellería de Educación, Cultura y Deporte de la Generalitat Valenciana and in part by the Project I+D+i EDU2011-27288 del Ministerio de Ciencia e Innovación, España.

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APPLYING GROWTH MIXTURE MODELING TO LONGITUDINALLY INVESTIGATING THE EFFECT OF MATHEMATICS CURRICULUM ON STUDENTS' LEARNING

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This study investigated the effect of the Connected Mathematics Project (CMP) (a middle school mathematics reformed curriculum) on students' learning through the use of longitudinal latent class modelling. Curriculum researchers often employ both classical statistical methods (e.g., repeated measures) and hierarchical linear modelling (HLM) to examine the longitudinal effect of a curriculum on students' learning. However, such methods do not allow for the examination of heterogeneity of the study population in terms of longitudinal growth trajectories. This study reported that students did exhibit different growth patterns. Growth mixture modelling (GMM) allows researchers to estimate the longitudinal effect of curriculum on students' learning by identifying latent (hidden) classes of growth trajectories.

PURPOSE

For many decades, mathematics curriculum has been considered a primary lever for educational reform (NCTM, 1989; NRC, 2004). In fact, the school mathematics curriculum has been, and remains a central focus in efforts to improve student learning. While curriculum has long been a subject of scholarly inquiry in mathematics education (e.g., Clapp, 1924; Davis, 1962), only recently have researchers started to investigate, longitudinally, the effect of curriculum on students' learning (Cai, Moyer, Wang, Hwang, Nie, & Garber, 2013; Cai, Wang, Moyer, & Nie, 2011; Grouws, Tarr, Chávez, Sears, Soria, & Taylan, 2013; Harwell, Medhanie, Post, Norman, & Dupuis, 2011).

This study was to investigate the effect of the Connected Mathematics Project (CMP) on students' learning through the use of longitudinal latent class modelling. Curriculum researchers often employ both classical statistical methods (e.g., repeated measures) and hierarchical linear modelling (HLM) to examine the longitudinal effect of a curriculum on students' learning. However, such methods do not allow for the examination of heterogeneity of the study population in terms of longitudinal growth trajectories, which often happens to be the case. In fact, it is quite possible that students exhibit different growth patterns. Growth mixture modelling (GMM) allows researchers to estimate the longitudinal effect of curriculum on students' learning by identifying latent (hidden) classes of growth trajectories. The Connected Mathematics Project (hereafter called CMP) is a complete middle school curriculum, which was

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 137-144. Hobart, Australia: PME.

developed with the support of NSF. We refer to all other more traditional types of curriculum as non-CMP. Of particular interest are the effects of the CMP vs. non-CMP curricula on students' learning growth trajectories as well as possible effects of gender and time-varying instructional covariates on the trajectories. In this study, we do not assume homogeneity of the students' growth patterns about the effects of the treatment and background variables on those patterns.

BACKGROUND AND DESIGN OF THE STUDY

The research reported here is a part of a large research project, *Longitudinal Investigation of the Effect of Curriculum on Algebra Learning* (LieCal). The LieCal Project was designed according to the National Research Council's recommendations (NRC, 2004). NRC recommended that curriculum studies should begin by identifying the program components that are different in each of the curricula (Cai, Nie, & Moyer, 2010). Studies must also carefully examine implementation components including whether all students in a school are taught using the program, how faithfully each of the curricula are taught, and how differently the assessments included in the instructional materials are used (Moyer, Cai, Nie, & Wang, 2011). Finally, the studies must measure the impact of the curricula on students' learning. In measuring student outcomes, studies must attend to the curricular validity of measures that are sensitive to the program's stated goals (Cai et al., 2011).

The LieCal Project used a quasi-experimental design with statistical controls to examine longitudinally the relationship between students' learning and their curricular experiences. The LieCal Project was first conducted in 14 middle schools in an urban school district serving a diverse student population in the United States (See Cai et al., 2011 for more information about the LieCal Project).

RESEARCH QUESTIONS

In this paper, we will specifically answer the following three research questions.

Research Question 1: *Are there latent classes of growth trajectories for student performance across the three middle school years on various assessment tasks?*

Research Question 2: *What are the treatment effects (CMP vs. non CMP) on the students' initial status (Fall of 6th grade) and their rate of growth on six different learning outcomes across the three middle school years, controlling for time-varying instructional covariates?*

Research Question 3: *Is there dependency between the classification of students into latent classes of growth trajectories on targeted outcomes and their gender?*

METHOD

Learning Outcome Measures

There were six learning outcome measures of interest in this study, namely: open-ended problem solving, equation solving (Component E), and four measures based on Mayer's (1987) cognitive components of word-problem solving (Component A:

translation, Component B: integration, Component C: planning, and Component D: execution). We used a combination of multiple-choice items and constructed-response items to measure students' procedural knowledge and routine problem-solving skills as well as high-level thinking skills. Table 1 shows the middle school data for this study.

Data Sources	Fall, 05	Spring, 06	Fall, 06	Spring, 07	Fall, 07	Spring, 08
State tests on both math and reading	All 6th graders		All 7th graders		All 8th graders	
Test on equation solving and the four components of Mayer's model	6th grade students (32 items)	6th grade students (32 items)		7th grade students (32 items)		8th grade students (32 items)
Test with open-ended tasks	6th grade students (6 items)	6th grade students (5 items)		7th grade students (5 items)		8th grade students (5 items)

Table 1: Middle school data source and time of data collection

Statistical Methods and Data Analysis

The *first two research questions* were addressed by using growth mixture modelling (GMM; e.g., Muthén, 2004). The GMM analysis included five time-varying covariates corresponding to the instructional variables discussed above, namely (a) teachers' conceptual emphasis in the classroom, (b) teachers' procedural emphasis in the classroom, (c) level of cognitive demand of instructional tasks as implemented in the classroom, (d) level of cognitive demand of instructional tasks as set up in the classroom, and (e) level of cognitive demand of algebraic homework tasks. GMM allows one to detect latent classes of different growth patterns that vary in initial status and rate of growth. Furthermore, GMM makes it possible to study relationships between parameters of latent classes (initial status and growth trajectories) and other variables, such as treatment conditions and different class invariant or class varying covariate effects, and to predict a distal outcome based on background characteristics and class membership. The *third research question* was addressed using the chi-square test for dependency between two categorical variables—in this case, (a) categories based on the students' classification into latent classes of growth trajectories, and (b) categories of student gender or ethnicity. This procedure allowed us to examine the frequency distribution of gender and ethnic groups across the latent classes being identified, as well as to test for underrepresentation or overrepresentation of gender and ethnic groups in latent classes of interest. (e.g., Muthén & Muthén, 2000; Muthén & Shedden, 1999).

SUMMARY OF RESULTS

Testing for Data Fit

Prior to searching for latent classes of growth trajectories for student performance on targeted outcomes, the growth models of interest were tested for data fit based on a commonly used chi-square test statistic in combination with several other goodness-of-fit indexes. The results are summarized in Table 2. The results in Table 2 indicate that there is an adequate data fit for the growth models used in this study to address the first and second research questions (Hu & Bentler, 1999).

Variable (Tasks)	Outcome χ^2	df	CFI	TLI	SRMR	RMSEA	90% CI for RMESA LL	UL
Open-Ended	527.86	52	.939	.918	.052	.065	.060	.070
Component A	128.70	52	.976	.968	.025	.026	.021	.032
Component B	81.38	52	.988	.984	.018	.016	.009	.023
Component C	169.60	52	.969	.958	.034	.032	.027	.038
Component D	131.55	52	.982	.976	.021	.027	.021	.032
Component E	81.14	52	.993	.990	.021	.016	.009	.023

Note. Limits of the 90% CI for RMSEA: LL = lower limit and UL = upper limit. Component A: translation, Component B: integration, Component C: planning, and Component D: execution. Component E: Equation Solving

Table 2: Goodness-of-fit indices for longitudinal growth models (2005-2008) on open-ended and multiple-choice component (A, B, C, D, E) tasks

Growth Mixture Models (Research Questions 1 and 2)

Three latent classes were retained in the GMM for growth trajectories of student performance on open-ended tasks and four latent classes were retained in the GMM for growth trajectories of student performance on each of the five multiple-choice components (A, B, C, D, and E). The estimates of treatment effects (CMP vs. non-CMP) on initial status and growth rate are provided in Table 3. A statistically significant CMP effect on growth rate indicates that the CMP and non-CMP students differed in their average rate of growth on the outcome variable across the three middle school years. The data for treatment was coded so that a positive effect favours the CMP students and a negative effect favours the non-CMP students. On open-ended tasks, within each latent class the CMP and non-CMP students had an equal start (fall 2005) and did not differ in growth rate across the three-year period of time (2005-2008).

Outcome/ Latent class[a]	Trend of growth	CMP effect initial status	CMP effect growth rate	Comments
OET Class 1 Class 2 Class 3	Moderate increase Close to flat Moderate increase	6.815 17.396 -1.109	-3.250 8.639 2.859	In all three classes, CMP and non-CMP have an equal start (fall 2005) and do not differ in growth rate across the three years.

A Class 1	Moderate increase	-26.138*	6.430**	CMP starts lower, grow faster.
Class 2	Slight decrease	10.168	-13.752	Equal start, similar in growth rate.
Class 3	Moderate increase	-20.673*	0.059	CMP starts lower, similar in growth rate.
Class 4	Close to flat	-47.550*	18.148**	CMP starts lower and grows faster.
B Class 1	Slight increase	-17.959	8.607	Equal start, no difference in growth rate.
Class 2	Close to flat	-11.802	39.119***	CMP starts lower and grows faster.
Class 3	Close to flat	2.833	-31.091***	Equal start, non-CMP grows faster.
Class 4	Sharp increase	3.564	1.422	Equal start, similar in growth rate.
C Class 1	Slight increase	12.330	29.449***	Equal start, CMP grows faster.
Class 2	Slight increase	-70.675***	-51.136***	Non-CMP starts lower and grows faster.
Class 3	Slight decrease	22.189	-16.432	Equal start, similar in growth rate.
Class 4	Slight increase	52.504***	16.296***	Equal start, CMP grows faster.
D Class 1	Slight increase	-81.100***	60.927***	CMP starts lower and grows faster.
Class 2	Moderate increase	-1.764	5.844*	Equal start, CMP grows faster.
Class 3	Slight increase	52.258*	26.282***	CMP starts higher and grows faster.
Class 4	Close to flat	15.796	-67.437***	Equal start, non-CMP grows faster.
E Class 1	Sharp increase	-51.267*	29.055	CMP starts lower, similar in growth rate.
Class 2	Sharp increase	-22.878**	6.207**	CMP starts lower and grow faster.
Class 3	Slight increase	-21.048***	4.967*	CMP starts lower and grow faster.
Class 4	Slight increase	0.580	-7.106	Equal start, similar in growth rate.

Note. **OET**: Scale score on open-ended tasks; **A, B**, **C, D**, **E**: Scale score on multiple-choice component A, B, C, D, E tasks, respectively.

p < .05. ** *p* < .01. *** *p* < .001; (statistically significant effects are given in bold).

Table 3: Unstandardized estimates of CMP effects on initial status and growth rate in latent classes of growth trajectories on targeted outcomes across years

With respect to the five multiple-choice components (A, B, C, D, E), the CMP effects on growth rate show that the CMP students grew faster than their non-CMP counterparts in half (10 out of 20) of the latent classes. In contrast, the non-CMP students grew faster in only three latent classes, namely class 3 for component B, class 2 for component C, and class 4 for component D. The growth trends for these three latent classes showing a non-CMP advantage were mostly "close to flat" across the three years. However, the growth trends for eight of the latent classes showing a CMP advantage were slightly increasing, moderately increasing, or sharply increasing. Of particular note are the five classes for which there was a statistically significant negative CMP treatment effect on initial status, but a positive CMP treatment effect on growth rate (classes 1 and 4 for component A, class 1 for component D, and classes 2 and 3 for component E).

Latent Classes and Gender (Research Question 3)

The third research question in this study addressed the representation across latent classes of students of different genders. Table 4 shows the frequency distributions for gender across latent classes of growth trajectories for the six targeted learning outcomes. The chi-square test for association between two categorical variables was used to test for possible dependence between the gender of the students and their classification into latent classes. Overrepresentation (or underrepresentation) of males or females in a specific latent class is indicated by a statistically significant positive (or negative) standardized residual for the respective case.

Outcome/Latent class	Trend of growth	Male(n = 982)	Female(n = 1083)
OET Class 1	Moderate increase	34.9%	36.6%
Class 2	Close to flat	4.2%	3.1%
Class 3	Moderate increase	60.9%	60.3%
A Class 1	Moderate increase	43.0%	47.7%
Class 2	Slight decrease	8.0%	5.4%
Class 3	Moderate increase	45.6%	43.4%
Class 4	Close to flat	3.4%	3.4%
B Class 1	Slight increase	25.6% (**U**)	32.9% (**O**)
Class 2	Close to flat	35.1%	32.3%
Class 3	Close to flat	30.5%	26.5%
Class 4	Sharp increase	8.8%	8.3%
C Class 1	Slight increase	34.4%	31.3%
Class 2	Slight increase	17.6%	18.5%
Class 3	Slight decrease	2.9%	2.8%
Class 4	Slight increase	45.1%	47.5%
D Class 1	Slight increase	29.6%	32.3%
Class 2	Moderate increase	17.3%	15.7%
Class 3	Slight increase	30.5%	30.3%
Class 4	Close to flat	22.5%	21.7%
E Class 1	Sharp increase	3.0%	4.7%

Class 2	Sharp increase	26.7%	29.1%
Class 3	Slight increase	63.6%	60.9%
Class 4	Slight increase	6.7%	5.3%

Table 4: Distribution of gender by latent classes of growth trajectories on targeted outcomes across three years

As shown in Table 4, dependence appears to exist only for multiple-choice component B. For component B tasks, males are underrepresented and females are overrepresented in latent class 1. There are no other cases of over- (or under-) representation of gender groups in latent classes. That means that only in multiple-choice component B, female students showed significantly higher growth than the male students. On any other learning outcome measures, male and female students have shown similar growth over the three middle school years.

SIGNIFICANCE OF THE STUDY

This study is to investigate the effect of the Connected Mathematics Project (CMP) (a middle school mathematics reformed curriculum) on students' learning through the use of longitudinal latent class modelling. Classical statistical methods (e.g., repeated measures) and hierarchical linear modelling (HLM) are usually used to examine the longitudinal effect of a curriculum on students' learning. However, such methods do not allow for the examination of heterogeneity of the study population in terms of longitudinal growth trajectories. The findings from this study showed that students did exhibit different growth patterns. Growth mixture modelling (GMM) allows researchers to estimate the longitudinal effect of curriculum on students' learning by identifying latent (hidden) classes of growth trajectories. This is significant not only because of the identification of the different classes of growth patterns, but also methodologically showed the importance of the examination of heterogeneity of the study population in terms of longitudinal growth trajectories.

Acknowledgement

The research reported here is part of a large LieCal Project, which is supported by grants from the National Science Foundation (ESI-0454739 and DRL-1008536). Any opinions expressed herein are those of the authors and do not necessarily represent the views of the National Science Foundation.

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EXPLORING PRIMARY TEACHERS' KNOWLEDGE FOR TEACHING MATHEMATICS

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Teachers' knowledge and pedagogical content knowledge of mathematics are important issues in the primary school. Much of this knowledge is tacit, and teachers take for granted their everyday practice. Focus groups of very experienced primary teachers were presented with scenarios about teaching mathematics, and asked to reflect on their own experiences. Outcomes from this process revealed a rich source of professional knowledge and awareness of both students' thinking and ways in which this can be developed.

INTRODUCTION

There is global interest in the knowledge that teachers draw on when teaching mathematics. It is known that teachers are a key determinant in students' outcomes (Hattie, 2008) and improving the quality of teaching is regarded as an important factor in improving students' mathematical outcomes (Ball, Lubienski, & Mewborn, 2001). Shulman (1978) first attempted to explicate the complexity of teachers' knowledge. In particular he used the term "pedagogical content knowledge" (PCK) to label the blend of content and pedagogical knowledge that is unique to a subject such as mathematics.

Since then there have been a number of studies that have addressed mathematical knowledge specifically in classrooms at different levels of schooling. In 1988, Carpenter and his associates used a framework of teachers' knowledge of distinctions between problem types, general knowledge of students' strategies, and capacity to predict the performance of specific students to obtain a measure of first grade teachers' knowledge in the contest of addition and subtraction (Carpenter, Fennema, Peterson, & Carey, 1988). Ball and her colleagues (Ball, Thames, & Phelps, 2008), also working with elementary school teachers, provided a fine-grained model of Mathematical Knowledge for Teaching (MKT) that included six components in two domains: Subject matter knowledge comprising Common Content Knowledge, Specialised Content Knowledge and Horizon Content Knowledge; and Pedagogical Content Knowledge including Knowledge of Content and Students, Knowledge of Content and Teaching, and Knowledge of Content and Curriculum. Baumert and colleagues (2010) worked with teachers in German Grade 10 mathematics classrooms. They defined teachers' knowledge in terms of PCK comprising three dimensions: tasks, assessing teachers' capacity to identify a range of solution paths, students, recognising students' misconceptions, and instruction, involving representations and explanations of mathematical concepts. All of these studies aimed to measure teachers' knowledge with the aim of linking this directly to students' outcomes.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 145-152. Hobart, Australia: PME.

Taking a different approach, Chick, Pham and Baker (2006) used classroom observations to identify some of the processes used by primary teachers. They suggested three categories of PCK: pedagogical knowledge in a content context, content knowledge in a pedagogical context, and one called "clearly PCK" in which the content and pedagogical knowledge were inseparable. Using video-taped observations, Rowland, Huckstep and Thwaites (2005) developed a "Knowledge Quartet" to describe beginning primary teachers' mathematical knowledge in teaching. The quartet consisted of foundation, transformation, connection, and contingency units, and was used to describe mathematics teaching development.

Although collectively these studies cover most of the compulsory years of teaching, there has been little attempt to consider how mathematical knowledge for teaching shifts and changes across the years of schooling, or with different topics within mathematics. Primary teachers, especially in Australia and New Zealand where this study was conducted, often teach different grades from year to year. Although the mathematical content becomes more complex, and the pedagogy in some instances becomes more formal, how teachers adapt their PCK to different situations, and particularly various age groups has been ignored.

CONTEXT OF THE STUDY

This study was the initial phase of a larger project that aims to map teachers' knowledge for teaching mathematics and English. It was undertaken in two states of Australia, Tasmania and Victoria, and in New Zealand. The two countries have different organisational structures for education, and curricula that have some similarities but also marked differences in approach. For example, the NZ curriculum is well supported with examples of pedagogical approaches, based on a constructivist paradigm (see Ministry of Education, 2007)) whereas the National Curriculum – Mathematics (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2013), taught in both Victoria and Tasmania, has no national pedagogical approach. Support is localised at the state level.

In the initial phase teachers who were regarded as experts in mathematics teaching were invited to participate in a local focus group to discuss their teaching of mathematics. The transcripts from three groups of primary mathematics teachers provided the data for this report. From this background and the original research aims, the following research question was posed:

How do teachers draw on their PCK as they adapt to teaching different students?

METHOD

Groups of highly experienced primary teachers in three locations were identified through local networks, including professional associations. The groups were deliberately small so that the conversation could be recorded. Each group was facilitated by a researcher, and all conversations were audio-recorded. A summary of the participants is provided in Table 1.

Location	No. Male	No. Female	Total
Tasmania	2	5	7
Victoria	1	2	3
New Zealand	0	3	3

Table 1: Summary of participants.

A series of scenarios was presented to the group as a stimulus for discussion. Most of the scenarios came from the researchers' prior experiences, and had provoked discussion among the research team. An example of a stimulus item is shown in Figure 1.

Children are writing their own subtraction problems. One child writes

$4 - 8 =$

What response would you make to this child?

Figure 1: Subtraction stimulus item.

In addition two additional open questions were posed at the end of the discussions. These questions focussed on creative approaches to teaching mathematics (the "zing" question) and what teachers would do if they were asked to teach some aspect of mathematics out of their immediate knowledge base (the "scary" question). The same protocol was used for each group, but because of time constraints not every group received every stimulus item.

The audio-recordings were transcribed. After careful reading, responses were identified that included information about both upper and lower primary grades. These responses were then clustered into themes that reflected some of Chick et al.'s (2006) categories.

FINDINGS

Students' misconceptions

These very experienced teachers were able to suggest many possibilities for apparent students' misconceptions. For example, for the subtraction scenario shown in Figure 1, all groups to which this scenario was presented recognised this as a common activity in lower primary school. They suggested possible issues such as left-right confusion, not recognising that order is important in subtraction, and difficulties because of the different meanings of subtraction and how these are expressed in words. They also recognised, however, that there might be no misconception with one teacher saying:

> I think we immediately start with an expectation that there's something wrong here, but in fact, I've had this experience with high functioning children, children who are actually exploring something that they've discovered. (MU001)

A student who did not understand non-commutativity of subtraction by upper primary, however, would be of concern with comments such as

> So they do come through the ranks … and they slip through the cracks—some would be in high school, upper primary— somewhere it all comes out kaboom, so they don't understand what all of this is about with no relationships at all. They can answer a few things because they've memorised certain answers. It makes no sense to them. (MU005)

The strategies that were suggested for intervening varied according to the diagnosis and these depended to some extent on the age group. For example, a Grade 1 child who appeared to understand the idea of negative numbers might be asked to represent a similar problem, or offered a number line or asked "What's interesting about it?" (MU001). For students in upper primary with a lack of understanding of basic number concepts there was general agreement with the comment "Hands on, hands on, hands on" (MP005).

Concrete materials/models

All groups emphasised the importance of using appropriate representations including concrete models. Whether in lower or upper grades concrete materials were perceived as useful as one teacher stated "But it's really important, even up in year 5 and 6 now, to not cut out using the concrete material" (VP01). There was also recognition that by upper primary school some students were not willing to use concrete materials. One teacher who taught a variety of low attaining students in withdrawal classes said "they were terrified initially that they might get it wrong, or to use equipment, or to draw a picture" (MP028). Her approach was to use blank books and allow the students to experiment with their own representations in order to develop confidence.

In two groups there was considerable discussion about measurement concepts and the importance of using a hands-on approach to addressing the common confusion between area and perimeter. Some common activities used in many schools were criticised, for example

> I mean students might be moving, but there's not a lot of higher order thinking going on where, you know, they might be doing busy work, and an example is, finding the perimeter of a basketball court. It takes an awful lot of time, but they don't actually get a lot of learning going from there, other than walking around the court with a, you know, long ruler and measuring it in a different way (MP029).

These highly experienced teachers recognised that activity for the sake of it was not helpful, but that carefully structured concrete experiences could be valuable at any age. For example, in response to a stimulus about using informal units, popsticks, to measure the length of a desk, the same teacher who raised the issue of busy work stated "I did a similar thing with year 7, 8 kids just the other day but we were looking at area coverage...we had paper napkins around the place so they used that" (MP029). This teacher recognised that the ways in which students developed understanding of measurement concepts was the same although the attribute differed.

Mathematical language

The complexity of mathematical language was raised as an issue in some way by all groups. One teacher suggested that “maths does have a set of procedures, and a way of recording and a language … like what does that plus sign mean” (MP028). In relation to the subtraction stimulus (Figure 1),

> There’s that language issue too because minus and take away, there’s a number of ways we describe that action, and sometimes take away is like take away food, I mean it’s got multiple examples of how we use that language other than the standard format. (MU001)

A stimulus about creating a square with a loop of string led to the following exchange when asked what kind of knowledge the activity might reveal:

> MU002 Beginning, you know, at their level of geometric thinking, with the language they’re using.
>
> MP004 I think language they’re using, so if they’re using the words like angles or corners …
>
> MU001 How you’re defining it? What is the property it represents? For this level, it’s kind of 1 [referring to Van Hiele levels], isn’t it really?

In another group, the area/perimeter discussion led to a comment “So I try to draw out all the language first and then try to define it more exactly, or specifically, and then start [building] that vocabulary.” (MV02). This comment did, however, lead to some dissent, especially from the one high school teacher in the group who stated “I’d probably go the other way around it. I’d probably avoid all language because it’s usually used so poorly that I leave language to the end, rather than the beginning” (MV03). This high school teacher went on to outline her approach to geometry using hands-on materials such as cut-out shapes and modelling clay to promote discussion that eventually would lead to formal mathematical language use. Despite the different emphasis in relation to language, the approaches to teaching were very similar across the grade levels.

Identifying student thinking

Regardless of grade level, there was general agreement that unless a teacher understood what a student was thinking intervention was not possible. Talking to students, class discussion, and appropriate questioning were all mentioned as important pedagogical strategies. One teacher made use of video:

> I’ll give them a task and I’ll deliberately leave the room so they can actually start talking because sometimes me being there hinders what they want to talk about. And I tell them I’m videoing it, and we’ll talk about it afterwards, and we’ll look at the video afterwards, and talk about what they’ve talked about. But that’s just one way to get them started. (MP029)

There was also concern about the issue of students only wanting to know whether they were right or not. One teacher described the distress of one boy when she asked him to explain why he thought his answer was correct:

> And I had one boy burst into tears the other day because I said, “What’s your thinking? Explain to me why you think that’s the answer.” “But am I right?” And I said, “Why do you think you’re right, explain to me.” And he just burst into tears and said, “I just want to know if I’m right.” (MP028)

Another teacher indicated that she often launched her lesson with a problem:

> You put the problem out there for the children. And you let them all explore it, and they’re all exploring the same. The same problem. And then after about, maybe after five minutes, you stop them all. And then you get the children’s strategies, and they all discuss, and they all talk. You haven’t been doing the talking, you’ve just given them the problem and then they do all the talking, and then you let them go back to the problem. (MV01)

The importance of classroom dialogue was universally recognised across all the years of schooling. The value of allowing students to talk through a problem was frequently mentioned with typical comments like “Yeah, that child might be able to start to explain it to you, his strategy, and then say, Oh! and self-correct, and you’re thinking that’s terrific” (MV01).

Discussion

Whether they were discussing students in the early years of primary school or in the high school years, these highly experienced teachers had broadly similar approaches to teaching. They demonstrated many of the qualities that Chick et al. (2006) referred to as “Clearly PCK” in which the mathematical and pedagogical knowledge was intertwined. The discussion about using paper napkins as informal units for measuring area is one such example.

The approach to teaching that these teachers espoused could broadly be described as a social-constructivist approach, in which students were presented with situations to explore mathematical ideas through discussion with both their peers and the teacher. Throughout the focus groups the teachers involved demonstrated a deep knowledge of school mathematics and how students learned mathematics. They appeared to have Ma’s (1999) Profound Understanding of Fundamental Mathematics, recognising the interconnections among the mathematical ideas, but also anticipating students’ problems.

Although sensitive to students’ ages, and the difficulties that some students might have in using approaches to learning that might be deemed “babyish”, rather than compromise their pedagogy these teachers sought ways to engage their students that were age-appropriate, such as using video, or allowing students to struggle with a problem. Some of these tactics could be described as Pedagogical Knowledge in a Content Context (Chick et al., 2006) drawing on a knowledge of the mathematics content to recognise mathematically productive attempts and discussions to help students’ understanding. They also understood that activity itself was not necessarily productive.

It seemed that in terms of their knowledge for teaching mathematics, the very experienced teachers in this study had deep understanding of the mathematics classroom. During the conversations in all groups there was mention of particular mathematics education researchers and theories, suggesting that their knowledge had been gained not only from practice but also from theory. Although information about qualifications was not formally requested, many of the teachers indicated that they either had, or were working on, post-graduate qualifications, or that they had undertaken extensive professional learning programs. The importance of bringing together theory and practice in the development of PCK or mathematical knowledge for teaching intuitively makes sense. How these two aspects interact, however, warrants further formal investigation.

It should be noted that the teachers involved in this study were deliberately chosen for their expertise, and could not be considered a typical sample. The next phase of this project is to observe a range of teachers from diverse backgrounds and experience to attempt to map teachers' knowledge in a more detailed way. In particular, although the approaches to teaching described by these teachers appeared similar across the years of schooling, what these are like in practice is still to be explored.

Acknowledgment

This study was funded by the Australian Research Council Grant No. DP130103144. The contributions of Kim Beswick, Helen Chick, Ian Hay and Tom Nicholson to this study are gratefully acknowledged.

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MATHEMATICS TEACHERS' KNOWING AS REFLECTIVE AWARENESS

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Teachers need to be equipped with appropriate ways of knowing to teach mathematics with deep understanding and help students to think mathematically. This paper focuses on reflective awareness [RA] as a central aspect of mathematics teachers' knowing for teaching. It reports on a study that investigated how RA was fostered in a group of elementary school mathematics teachers' learning and how it shaped their thinking and teaching. Findings indicted that questioning and creating pedagogical models are important to support RA; there is an important relationship between RA and teacher learning and RA and teaching mathematics; RA is central to teachers' development of inquiry stance, knowledge of mathematics, and knowledge of mathematics pedagogy.

INTRODUCTION

Mathematics teachers' knowledge specific to teaching mathematics has received significant attention in mathematics education research in recent years. However, given the complexity of this knowledge ongoing consideration of it is important if we are to help teachers to improve their practice. This paper contributes to our understanding of this knowledge with particular focus on teacher knowing as *reflective awareness* [RA]. It reports on a study of elementary teachers involved in a *professional development approach* [PD] aimed at transforming their teaching of mathematics to an inquiry-oriented perspective. Specific attention was on identifying aspects of the PD that supported RA in their learning and how RA shaped their thinking and teaching.

LITERATURE REVIEW

Ball, Thames and Phelps's (2008) category-based perspective of mathematical knowledge for teaching [MKT] has provided a basis for recent studies to investigate MKT in general and in relation to specific mathematics topics or concepts. However, this perspective of MKT does not provide a complete picture of what is necessary to teach mathematics with depth and understanding. Other perspectives of MKT have been suggested that provide a broader or alternative ways of making sense of it. For example, Ruthven (2011) identified four perspectives to MKT: *subject knowledge differentiated* (deals with categories), *subject knowledge situated* (deals with material and social context); *subject knowledge interactivated* (deals with mathematical (re)contextualizing and (re)construction in the classroom), and *subject knowledge mathematised* (deals with mathematical modes of inquiry). Others have considered teachers' mathematical knowledge as a way of being and acting, an orientation towards mathematics (e.g., embodying modes of mathematical enquiry) (Watson, 2008),

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 153-160. Hobart, Australia: PME.

pedagogical content knowing (i.e., knowing-to-act in particular teaching contexts) (Cochran, DeRuiter, & King, 1993), and a participatory attitude toward mathematics (Davis & Renert, 2009). These latter ways of viewing mathematics teachers' knowledge are related to the perspective that "teachers must act mathematically in order to enact mathematics with their students" Ruthven (2011, p. 91), which suggest that their ways of being, acting and knowing are important to their practice. Teachers who lack appropriate ways of knowing are unlikely to be equipped with appropriate knowledge to teach mathematics with deep understanding or to help students to think mathematically. These ways of knowing or thinking should include problem-solving thinking and mathematical thinking (Mason, Burton, & Stacey, 2010; Schoenfeld, 1992) and inquiry thinking (Dewey, 1933). Central to these ways of thinking is RA.

THEORETICAL PERSPECTIVE

Reflective awareness [RA] should be a central aspect of mathematics teachers knowing. The notion of reflection (e.g., Schon, 1983) is common in mathematics teacher education as a process of self-understanding and growth. RA is a unique aspect of this process with broader pedagogical implications regarding what teachers should know. RA has been used with some variations in the literature. For example, it has been associated with enabling professionals to perform tasks in their particular disciplines and to communicate their thinking, rationales, and judgments as they do so (Shulman, 1986) and with a deeper kind of observation that consists of suspending, redirecting, and letting go (Senge, Scharmer, Jaworski & Flowers, 2005). Studies on mathematics teachers have considered "awareness" in the context of noticing (i.e., what teachers are aware of or attend to, e.g., Sherin, Jacobs, & Philipp, 2010). Many of these studies deal with helping teachers to notice but do not explicitly address RA.

Theoretically, RA is being linked to works such as Mason (1998) and Dewey (1933). Mason noted: "The key notions underlying real teaching are the structure of attention and the nature of awareness" (p. 244). "To become an expert it is necessary to develop and articulate awareness of your awarenesses-in-action; to become a teacher in the full … , it is necessary to become aware of your awareness of those awarenesses-in-action" (p. 255). He identified three forms of awareness: *awareness-in-action*; *awareness-in-discipline* (awareness of awareness-in-action) and *awareness-in-counsel* (awareness of awareness-in-discipline). Such awareness of awareness of awareness involves a reflective process or some form of *reflective thinking*. From Dewey's (1933) perspective, reflective thinking is "central to all learning experiences enabling us to act in a deliberate and intentional fashion … [to] convert action that is merely … blind and impulsive into intelligent action" (p. 212). It begins when one encounters "a state of doubt, hesitation, perplexity, mental difficulty" (p. 12) to be cleared up by inquiry.

Such notions of awareness and reflective thinking contribute to the perspective of RA used in this study as involving a state of curiosity/wonderment/puzzlement that results in action through inquiry/questioning to resolve this state in order to know and grow.

It is not simply *seeing* something (instrumental awareness) but being able to see a puzzling situation in that something and to act on it. Instrumental awareness involves *seeing* what one knows while RA involves *seeing* something that is or could be different from what one already knows and results in questioning/inquiry to understand it. Teachers with knowledge of RA are curious about what is happening in the classroom and ask questions to understand, to check their own thinking and their students' thinking, and to consider alternative interpretations of an event or behavior.

RESEARCH METHOD

The **participants** were 14 grades 1 to 6 teachers from the same elementary school. They participated in a *self-directed professional development approach* [PD] aimed at transforming their teaching to an inquiry-oriented perspective. To fulfill their school's requirement of a professional growth plan, they formed a mathematics study group to work on bringing their teaching more in line with curriculum expectations involving a constructivist/inquiry perspective. My role was to provide support by responding to their needs rather than imposing direction. Thus, the PD was self-directed in that the teachers decided on what to do and how to do it. In the first year of the PD, considered here as part of a larger project, the group met in their school every three weeks for one and a half to two hours, plus three school-PD days, time to observe *experimental* lessons, and occasional lunch-break meetings to plan and reflect on these lessons.

The PD was consistent with current perspectives of effective teacher learning. For example, it followed a socially/culturally situated process of knowledge construction involving collaboration, discourse, reflection, inquiry and application. It also involved continuous interactive support over a substantial period of time, focused on specific educational content under guidance of an expert adopting a hands-off role, and revolved around artifacts that helped to foster a sense of ownership with teachers (e.g., Borko, 2004). As a *community of inquiry* (Wells, 1999), it included *dialogical inquiry,* that is, "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (Wells, 1999, p. 122). Chapman (2013) discusses the inquiry orientation of the PD. The focus here is on the features of the PD that were significant to the teachers' learning and use of RA.

Data collection for the larger project focused on two aspects of PD: the way it evolved for the teachers and the way it impacted their learning and practice. This included: (1) field notes and audio recording of PD sessions involving their discussions of, e.g., what to do and how, when and why to do it; their plans, observations and evaluation of their research lessons; and their students' work. (2) Samples of teaching artefacts (e.g., research lesson plans) and participants' notes (e.g., observations of video lessons) during the sessions. (3) Several classroom observations of each teacher. (4) Three open-ended group interviews and one with each teacher to probe their thinking about the PD, their learning about discourse and inquiry, and the impact on their teaching.

Data analysis for this study was guided by the research questions: What aspects of the PD supported RA in the teachers' learning? How did RA shape their thinking and teaching? Codes were developed based on the theoretical perspective of RA and used to identify the features of the PD that supported their RA and aspects of their thinking/ actions during the PD and their teaching that were characteristic of RA. The coded information was categorised in different ways that included: (1) their questions/prompts that were RA-oriented; (2) what they attended to in students' responses during discourse; (3) their intentions related to RA; and (4) their knowledge of RA. Themes emerging from these categories were used to draw conclusions regarding their learning of RA and use of RA in their teaching. Verification procedures included elimination of initial themes based on disconfirming evidence and *member checks* with the teachers.

FINDINGS

This summary of the findings focuses on two features of the PD that emerged as significant in supporting RA in the teachers' learning (i.e., *questioning and creating pedagogical models)* and, as an example, one teacher's use of RA in her teaching.

Questioning consisted of two categories: self-based and meaning-based questioning.

Self-based questioning involves posing questions that enable one to think about and talk about oneself. In the context of RA, it is triggered by curiosity or puzzlement about one's own or others' thinking or actions. The teachers did not initially demonstrate self -based questioning and had to be prompted to do so early in the PD. For example, after observing a video math lesson to gain understanding of inquiry-based communication in teaching mathematics, they focused more on the types of questions the teacher asked the students, recorded what they considered to be meaningful and shared what they recorded. However, this process occurred in a space external to them. There was passive acceptance of what they shared, e.g., seeing commonalities as validation of what they noticed and differences as new information of what they did not notice. They were prompted to be curious about their own and each other's thinking (e.g., by asking why-type questions to each other regarding what they recorded to help them to understand their thinking and teaching). While this shift in approach started slowly, with practice based on their understanding of it, their self-questions started to take on new forms that reflected their growing curiosity of, and interest in exploring, their thinking and teaching in relation to the video lessons. This enabled them to engage in RA to understand themselves and each other regarding how they made sense of aspects of their teaching and the changes they should pursue based on the video-lessons study.

Meaning-based questioning involves posing questions that require one to make sense of something in which one is interested. In the context of RA, it is triggered by curiosity or puzzlement about something one wants to act on. The teachers had no problem engaging in meaning-based questioning since this was central to their self-directed PD in defining how it evolved. Key questions that shaped the PD in its first year included: What does communication look like in an inquiry lesson? What attributes make up an inquiry lesson? What is a meaningful model of inquiry teaching for us? Such questions supported RA that included creating and testing hypotheses emerging from their learning through the self-based questioning and video lessons.

Creating pedagogical models (i.e., general approaches to guide their learning and teaching) also emerged as central to the teachers' engagement in RA. For example, they puzzled with, posed questions of, and inquired into the structure of the models they were interested in. Key models they created included: (1) an *inquiry-based teaching model* for their teaching of mathematics. They engaged in RA to understand how to integrate the model in their thinking and teaching. RA helped them to frame the model in relation to the students (i.e., learners/learning) and not themselves (i.e., teacher/teaching). Components of the model included engaging students in learning mathematics through free exploration, focused exploration, discussions, predictions, comparison, applications, evaluation, reflection, and extension of the mathematics concept being taught. (2) A *lesson observation model* to prompt their observations of experimental lessons. Their engagement in RA allowed them to represent the items in the model as questions that indicated what they were curious about regarding students' thinking about the mathematics concepts and their engagement in the mathematical activities as a basis to learn from it to improve their practice. (3) A *problem-solving inquiry model* to guide their teaching of problem solving. They engaged in RA to under -stand what they should do before, during, and after a student is engaged in solving a mathematics problem for which the solution method is not known in advance.

RA in their teaching: All of the teachers made significant changes to their teaching. However, the focus here is on the relationship of RA to their teaching. This is illustrated using the case of one of the teachers, Lena, who taught grade 3. She was one of the teachers whose teaching reflected RA consistently. Her teaching shifted to engaging students in inquiry-oriented discourse on an ongoing basis. For example, her questioning shifted to mirror what she learned from the PD. She explained:

> I have changed my question techniques after our work in the study group and the questions that we've come up with based on what we know that will promote good conversation. So I do use the techniques like "what have you noticed" … So my kids now know you have to explain the why, … how you make sense of this.

One of Lena's goals for questioning was her own learning. She was now curious about students' thinking and wanted to learn from it. As she explained:

> It's a little bit selfish, but I want to learn something. So I want to be aha'd! and surprised. … I almost get a rush … it's a weird thing, but a high when they teach me something. I'm not afraid to take risk, so I put myself out there to see what I can learn.

So instead of requiring students to see and accept her approach to solving a problem or way of thinking as being the most efficient or meaningful, Lena now tried to learn from them and engaged them in discourse about what they noticed about their approaches, when and why to use them, and the importance to use what made sense to them.

Lena's planning of her lessons also shifted as she started to think deeply about questioning and tried to imagine possible scenarios.

> So if I put this [question] out there, what direction could it go? And if it went that direction, what would I do, and if it went that direction, how would I help them?

Lena engaged in *self-based questioning* during whole-class discourse, for example,

> I'm always attending to: have I met their needs and where do I need to take this now? What do I need to do next with it? Does that make sense? I'm thinking: so what they said; can I come up with a question based on that to promote more thinking and discussion?

While her learning was important, the key goal of Lena's questioning was her students' learning. She encouraged them to be curious and to ask questions, which were central to their whole-class discourse. She explained:

> What sets the direction for it [discourse] now is the math questions that the kids are asking, because they were given freedom to say, tell me what you want to learn. … So what is important for it [discourse] is the interest of the kids and questions that they have.

When students wanted to know what type of questions was good to ask, she told them:

> It should be something you want to learn. Something that you might have seen or heard and you wonder about. ... So that's how we left it: wonder, curiosity, what if.

Lena also challenged students' thinking with *meaning-based* and *self-based questions*, such as: Why is it an even number? How do you know a number is even? Why are you taking away? What is the reasoning around that? What made your mind go that way? She often posed *self-based questions* that allowed students to think about what they knew based on prior knowledge or their experiences and what they wanted to know. She started lessons on new concepts by "always trying to find out what they are bringing to the lesson, before just bringing what I think in to know." For example, "what do you want to know about patterns? Or, what have you noticed about patterns? Or can you tell me about patterns in your world?" She encouraged students to think about what made sense for themselves. "Ask yourself in your head, did that make sense?" She encouraged them to "see the math" in their lives, to think about their problem-solving processes or strategies, and to reflect on their mathematical learning experience such as affective aspects of their problem-solving experience.

In general, Lena's questioning approach embodied her knowledge and use of RA. Questions asked by her and her students had a personal component of acting on a curiosity or perplexity that was resolved through *dialogic inquiry* or investigation of real-world experiences or mathematical tasks.

CONCLUSION

MTK is more complex than discrete categories of content and pedagogical content knowledge when considered from a broader perspective of what teachers should know to teach mathematics. Teachers need to learn to think in different ways that support mathematical thinking and meaningful mathematics pedagogy. This study suggests that there is an important relationship between RA and teacher learning and RA and teaching. RA was central to the teachers' development of inquiry stance, knowledge of mathematics (e.g., through students' thinking) and knowledge of mathematics pedagogy. They developed understanding of RA through the PD that impacted their teaching in meaningful ways. Self-based questioning, meaning-based questioning and creating pedagogical models were central to developing this understanding. Lena held knowledge of RA and used it to help her students to enhance their learning of mathematics. The shift in her thinking and teaching resulting from the PD showed depth in her RA. This shift was directed to her own learning and her students' learning. Her teaching approach included self-based and meaning-based questioning for herself and her students. Her case shows that RA is central to a teacher's ability to shape events in the classroom by being aware of and questioning the phenomena around which the discourse of the classroom is organised. Her RA was important to promote curiosity and questioning in students and to help them to develop their RA.

RA, then, is an important component of teachers' knowledge and knowing and should explicitly be treated as such in teacher education. Teachers should learn of the importance of RA in their own and students' learning and how to engage in RA and engage their students in it. They should learn to treat students' thinking not just as a source of information, but a means for them to engage in RA and to engage the students in RA. Future work should include investigating RA in students' learning of mathematics, prospective mathematics teachers' learning, and practicing mathematics teachers' thinking and classroom behaviors during mathematical activities to understand RA from a practice-based perspective for different levels of school.

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GEOMETRIC CONCEPTS OF TWO-DIMENSIONAL SHAPES BY PRIMARY SCHOOL TEACHERS IN TAIWAN

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This is a case study investigating whether primary school teachers in Taiwan have accurate geometric concepts for teaching the properties of two-dimensional (2-D) shapes to students. We present results from classroom observations and interviews of two Year 6 teachers. The purpose of the study is not only to reveal the possible misconceptions or errors that teachers might have and need to be improved, but also clarify the correct concepts and discuss the good geometric content knowledge for teaching shapes. The teacher training authorities in Taiwan should develop a systematic resource about 2-D shapes for teachers. Then the teachers can review and fully comprehend the geometric or mathematical content knowledge before teaching each concept.

INTRODUCTION

Studies have shown that students' geometric misconceptions and low academic achievement in geometry may be due to teachers' insufficient or incorrect knowledge (Portnoy, Grundmeier, & Graham, 2006; van Hiele, 1999). Thus, students' misconceptions of 2-D shapes might be because of teachers' having insufficient geometric content knowledge. In other words, if school teachers do not have clear and accurate geometric concepts for teaching, their students might also have incorrect concepts or misconceptions about these geometric concepts.

In Taiwan, the case study by Hwang (1995) found that pre-service primary school teachers lacked of mathematical content knowledge about reasoning and comprehending the mathematical concepts. Several research projects which focused on the geometric misconceptions of the students in the middle grades (e.g., Kao, 2002; Hsueh, 2002) showed that Taiwanese primary students had low success rates (from 11.6% to 57.4%) in recognising basic 2-D shapes. Shieh (2003) also found Year 6 students had over 15 misconceptions about quadrilaterals.

In the other countries, studies have shown teachers and students easily get confused about 2-D shapes, such as the properties of quadrilaterals (Usiskin, 2008; Leung, 2008). Thus, this study aims to investigate whether in-service primary school teachers fully comprehend the geometric conceptions of 2-D shapes and have good mathematical content (including geometric content) knowledge as the report of Teacher Education and Development Study in Mathematics (TEDS-M) (2012); or whether they show confusion and have misconceptions as the findings from Hwang (1995), Usiskin (2008) and Leung (2008) indicate.

METHODOLOGY

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 161-168. Hobart, Australia: PME.

This paper reports the result on just one aspect from a large study of Taiwanese teaching of geometry (see Chiang, 2012) which videotaped up to ten geometry lessons on one topic from each of ten Taiwanese teachers in 2008/2009. In this study, I report on two teachers called T1 and T2, who make an interesting pair for comparison and represent the extremes of the sample of ten teachers, with T1 displaying very strong content knowledge relative to the whole group and T2 displaying relatively weak knowledge. In addition, this case study mainly investigates whether the teachers fully comprehended the geometric concepts of 2-D shapes that they taught and whether they provided accurate knowledge and explanations to the students in their classes. Thus, in-depth observation and interview were the main methods for obtaining the data.

Participants

T1 and T2 were volunteers for the study who were fully qualified and were currently teaching the geometric topic, The Properties of 2-D shapes, to Year 6 students from the same public school. They had both been primary school teachers for at least 5 years, so had already established themselves in the profession. They also represented the diversity of the in-serviced teachers' backgrounds in Taiwan. Table 1 shows the background information about T1 and T2.

Attributes	T1	T2
Gender	Female	Female
Age	28	34
Years of teaching	5	10
Years teaching Year 6	3	2
Training path	Teachers' college	Graduate school of education
Major in mathematics	✓	×
Masters' degree	×	MEd
Number of lessons for topic	8	6
Textbook	Nan-Yi	Nan-Yi

Table 1: Details of the Case Study Teachers T1 & T2

Data collection and analysis

In the classroom observations, each geometric concept or problem solution given by T1 and T2 was judged and recorded immediately in the classroom observation framework (see detail in Table 2). Later the videos of observed lessons were reviewed to check the findings from the lesson observation frameworks. Concepts and answers in the category of further discussion were discussed and allocated to one of the other categories. This meaningful information was organised into tables so that the data could readily be compared. Secondly, after finishing this topic teaching, both teachers were interviewed by one question: "Are there any difficult concepts or questions for you to teach this topic? Why?" The interview audiotapes were inserted into the transcripts and then sorted into tables by the various emerging themes of the critical episodes. All data were established in Mandarin and translated to English. Table 2 shows the definitions of the categories in the classroom observation framework.

Category	Definition
Accurate	The teacher provides clear and correct explanation for the concept/answer.
Inaccurate	The teacher provides an incorrect explanation of the concept or makes an error for the answer.
Partially accurate	The teacher mainly provides clear and correct explanation for the concept/answer but with some small error(s).
Limited	The teacher provides the correct description of the concept or answer (e.g., from the textbook) but without any explanation or elaboration.
Conflicted	The teacher provides a correct concept/answer that is different from the one in the textbook or teachers' guide
Further Discussion	The researcher cannot judge the teacher's instruction immediately in the observed lesson and will need to review or discuss with the other mathematical experts later.

Table 2: The Definitions of the Categories in the Classroom Observation Framework

RESULTS

Responses to the interview question

The teachers were asked about any difficult concepts or questions for teaching 2-D shapes. Both classed this topic as a review of material that students had learnt and known about, but T1 pointed to weaknesses that students might have for learning the definitions of the quadrilaterals, whereas T2 did not consider the errors or misconceptions the students may have. T1 mentioned the Venn diagram is appropriate for clearing the misconceptions and strengthening the students' memories of the similar definitions of quadrilaterals. By contrast, T2 thought the topic was "relaxed" [without complex calculations] for the students. Table 3 shows their responses.

Teacher	Interview response
T1	This [topic] is a review because they [students] had learnt these before. I know they realize these concepts but did not pay attention to memorise them …The students I taught before always got trouble on these [definitions of quadrilaterals]. For examples, why is square also a kind of parallelogram? …their misconceptions, like the definitions between square and rectangle are quite similar but they are different shapes…I think the circles [the Venn diagram] can help them to clear these misconceptions and help them [students] to memorise the concepts in the topic.
T2	The concepts of shapes have been taught before, now they are just like the review. They [students] had a similar chapter in Year 5 but it did not compare the properties with each other…It is more relaxed so they [the students] rarely asked me why. They have knowledge of each shape so they won't have many questions on it.

Table 3: The Replies for the Interview Question by T1 & T2

Results of the classroom observations

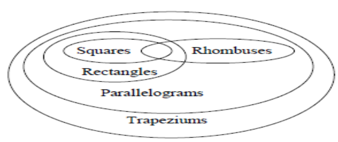
Both T1 and T2 taught the topic by closely following the textbook. Table 4 summarises the concepts the teachers taught, and ratings of accuracy on GCK for T1 and T2.

Geometric concept	T1	T2
Parallel lines	Accurate	Inaccurate
Perpendicular lines	Accurate	Limited
The definition of quadrilaterals	Accurate	Limited
The definition and properties of rectangles	Accurate	Limited
The definition and properties of squares	Accurate	Limited
The definition and properties of parallelograms	Accurate	Limited
The definition and properties of rhombuses	Accurate	Limited
The definition and properties of trapeziums	Conflicted	Limited
The inclusive relationship of quadrilaterals (e.g., a square is a kind of rhombus)	Partially Accurate	Limited
The definition and properties of triangles	Accurate	Limited

Table 4: Detail of Accuracy by T1 and T2 from the classroom observations

Generally T1 provided a logical procedure in illustrating and explaining each concept. The only imperfections were in accidentally using a different definition of trapeziums from the one in the textbook and in being unable to correctly position the rhombuses in a Venn diagram (an extension not shown in the textbook or teachers' guide). There are two definitions of trapezium in textbooks throughout the world: the "inclusive definition" (a quadrilateral with **at least** one pair of parallel sides) and the "exclusive definition" (a quadrilateral with **exactly** one pair of parallel sides) (Usiskin, 2008). The Nai-Yi textbook used the exclusive definition (只有一雙對邊平行的四邊形叫梯形), so that a rectangle or parallelogram (with two pairs of parallel sides) is not a trapezium. However, T1 clarified the definition of trapeziums with the students by the inclusive definition (**只要有**一雙對邊平行的四邊形便叫梯形). She mentioned that parallelograms, rectangles, and squares are kinds of trapeziums without noticing that the other definition was used in the textbook. Then a student asked where rhombuses should be put in the Venn diagram. After a short discussion, T1 said, "It overlaps somewhere…" Thus, T1 put the domain of rhombuses overlapping the regions of rectangles and parallelograms but still incorrectly, as shown in Figures 1 (a) and (b).

(a)



(b)

Figure 1: (a) T1'2 second Venn diagram and writing in Chinese (b) Copy of the Venn diagram and writing from (a) translated to English

By the end of this lesson, T1 clearly knew the diagram was still incorrect so she erased the hand-drawn Venn diagram, and said, “All right, don’t look at rhombuses because you [the students] will be confused ...”. In the next lesson, T1 moved to the next concept without further discussion. In contrast, T2 showed inaccurate and limited concepts in her teaching. She tended to teach each geometric concept by rote and to write down the descriptions for the definitions and properties of different quadrilaterals and triangles without further explanations. Thus, the majority of teaching concepts by T2 were rated as limited.

Comparison of T1 and T2 teaching the concept of parallel lines

T2 showed inaccurate concept of parallel lines because she did not correctly drew the parallel lines on the blackboard, and modified the students’ mistakes in solving the question of drawing a parallel line in the textbook as shown in the following figure 2.

Figure 2*:* Textbook geometric problem: “Please draw a line through B parallel with line M” (Nai-Yi Year 6 (first semester) textbook, 2008, p. 44)

Firstly, T2 drew the parallel lines by eye with the set-square and ask the students what the definition of parallel lines is and how to solve this question. One of the students (S1) came to the blackboard and quickly took a set-square and positioned it to go through B as he copied T2’s action, then drawing a line through B which S1 judged by eye to be parallel. Manipulations are shown in Figure 3.

(a) (b)

Figure 3: S1’s manipulations for drawing a parallel line with line M

When S1 finished the drawing, the other student (S2) claimed S1 was incorrect because the line (segment) he drew was not the same length as the line (segment) M. Then S1 quickly drew a longer line which seemed to be the same length as line M. T2 did not dispute this action, or discuss the misconception with either S1 or S2, and quickly moved to the next question. The teaching behavior above led to the concept of parallel lines being rated as “inaccurate”. In contrast, before solving this question, T1 intended to disrupt the students’ misconception that parallel lines must be horizontal lines. Figures 4 (a)-(g) show how T1 represented the concept of parallel lines and its drawing by the third volunteer student (S3) who was the first to do the correct drawing.

(a) (b) (c) (d) (e) (f) (g)

Figure 4: Screen shots of T1 (a)~(c) and S1 (d)~(g) drawing parallel lines

In this episode, she also clarified several potential misconceptions about parallel lines, including that parallel lines are horizontal lines, that two parallel lines must be equal length, and that there many lines parallel to a given line. Figured 5(a)-(d) show how T1 constructed the third different length of parallel line.

(a) (b) (c) (d)

Figure 5: Screen shots of T1 (a)-(d) for constructing of three parallel lines.

From these episodes, it is suggested that T1 who had stronger mathematical background understood the definition and properties of these basic geometric elements, whereas T2 did not.

CONCLUSIONS & IMPLICATIONS

The results in this study support previous findings that primary school teachers have poor performance with basic geometric knowledge (Jones, Mooney, & Harries, 2002). The result is consistent with the finding of Hwang (1995), that pre-service primary school teachers lack mathematical content knowledge for reasoning about mathematical concepts, and shows the teachers' confusions about the different definitions of quadrilaterals that have been reported (Usiskin, 2008; Leung, 2008).

In addition, this study is consistent with other findings that the teachers who have stronger mathematical backgrounds have better mathematical content knowledge for teaching (e.g., Hill, 2007). T2, who did not have mathematical background, considered this topic was simple for the students in the interview. She provided the students with accurate defintions and properties of the quadrilaterals and triangles by rote, and wrote answers for each question on the blackboard in the classroom. However, when the application of the parallel-line question appeared (see Figure 2), T2 and the students obviously were confused and could not solve it. In contrast, T1 provided a logical procedure in illustrating each concept without hastily posting the solutions for the students. She spent more time elaborating and extending the concept of parallel lines and the relationships among the 2-D shapes for her students. Although T1 failed to show the correct position of rhombuses in the Venn diagram, she and her students cooperatively challenged this question, and had the marvelous finding that "It overlaps somewhere". This episode represents the good teaching of mathematics or geometry, which is to promote learners' mathematical or geometric thinking by failures or being stuck (Mason, Burton, & Stacey). Providing accurate or standard answers to students is not enough for teaching the properties of 2-D shapes.

Moreover, the research group in National Academy for Educational Research (NAER) in Taiwan has developed a systematic resource about 2-D shapes for teachers, such as the Venn diagram of quadrilaterals is shown in this official website. If it can include the common errors and misconceptions that students at different grades might have, teachers can be alerted to look for these and know how to clarify them by activities, tasks or strategies. For example, in Australia, the educational authority Department of Education and Early Childhood Development of the state of Victoria systematically built for teachers the Mathematics Developmental Continuum P-10 (first year of school to Year 10 level) and included a section on concepts of 2-D shapes, named "Changing conceptions of shapes" (Stacey et al., 2006). It shows the basic educational aims, disciplines, phases and progression points for this topic, but also clearly demonstrates the activities, teaching strategies, workable solving tasks, and the misconceptions or mistakes that students at different ages might have. The teachers can clearly follow the instruction for reviewing all geometric concepts before teaching, and reduce inaccurate instruction for the students.

To sum up, it is recommended that all teachers carefully review geometric concepts before teaching, no matter how "easy" they think they are. Teacher education authorities should emphasise primary school teachers' geometric content knowledge as a basis for developing other professional knowledge, in particular for the concepts of 2-D shapes, and their possible misconceptions or errors.

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TEXTBOOK SIGNATURES: AN EXPLORATORY STUDY OF THE NOTION OF GRADIENT IN GERMANY, SINGAPORE AND SOUTH KOREA

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University of Auckland | Arizona State University | University of Duisburg-Essen

This study focuses on how textbook signatures can be formulated to represent similarities and differences in mathematics textbooks across different countries. In this paper, we examine the teaching of gradient (Grade 8) in Germany, Singapore, and South Korea by characterising the textbooks in terms of contextual (educational factors), content, and instructional variables. Findings suggest an alignment between these variables and the respective curriculum emphases.

BACKGROUND OF STUDY

Textbooks have an important role in shaping what and how mathematics is taught in schools (Charalambous, Delaney, Hsu, & Mesa, 2010; Mayer, Sims, & Tajika, 1995; Stigler, Fuson, Ham, & Kim, 1986). An analysis of textbooks is one way to compare learning opportunities in mathematics among different countries, because it is a medium that most probably influences the teaching and learning of mathematics. The aim of this study is to help us understand how different educational systems and cultures support future learning in mathematics. We assume that an analysis of textbooks can show us the similarities and differences in Germany, South Korea, and Singapore regarding the following questions:

1. Which conceptualisations of the concept of gradient are introduced and in which order?
2. Which representations are used for the concept of gradient?
3. What levels of cognitive demands are required by the tasks in the textbooks?
4. Are there any unique characteristics of the textbook for each country?

The concept of gradient was chosen as a case to study how textbooks from different countries introduce the same topic. It is considered to be a fundamental topic at secondary level because of its multiple facets of definition and different modes of representation, which make it a difficult concept to learn and teach. This concept is also very important for the learning and teaching calculus, especially when dealing with functions and their graphs.

THEORETICAL CONSIDERATIONS

Textbook Signatures

Textbook analyses have attracted the attention of mathematics educators in the past three decades, but this development in research on textbooks has been unbalanced (Fan, Zhu, & Miao, 2013). In particular, Fan et al. (2013) argue that it is important to examine the relationships between textbooks and their educational contexts, in order to understand the differences between textbooks. To compare textbooks across different countries, Charalambous et al. (2010) proposed that textbooks within the same country may have a "textbook signature" or "uniform distinctive patterns" (p. 146). This notion of a textbook signature may provide a way to represent, and relate textbook variables to a wider educational context. In this paper, we propose and demonstrate how textbook signatures of three countries—Germany, Singapore, and South Korea—can be represented and analysed in context. Building on the content and instructional variables proposed by Huntley (2008), we integrated a horizontal and vertical analysis of the mathematical content (Charalambous et al., 2010) into our study. Table 1 shows the variables which we consider in our comparison of the three textbooks in this paper.

Variables	*Description*
Context variables	Education system; academic year; time allocated for mathematics; number of textbooks; publishing process; and curriculum emphasis.
Content variables	Conceptualisations of gradient (e.g., geometric ratio; algebraic ratio; physical property; functional property; parametric coefficient; trigonometric conception; calculus conception; real-world situation; determining property; behaviour indicator; and linear constant) Definitions; rules; representation; explanations; examples; exercises.
Instructional variables	Representations of gradient Cognitive demand of tasks, exercises, and examples.

Table 1: Proposed framework to analyse textbooks.

Conceptualisations of Gradient

Gradient (i.e. also called slope, steepness etc.) is an important topic in mathematics because it can be the basis for learning more advanced mathematical knowledge. However, research has demonstrated that students have difficulties in understanding the concepts of gradient (Stump, 2001; Walter & Gerson, 2007). For example, Stump (2001) states that students mainly think of gradient as an angle, a formula, rise over run, or steepness. She also found that students struggled to make connections between various representations of these concepts, particularly the connection between rate of change and gradient. Walter and Gerson (2007) contend that the emphasis of "rise-over-run" concept has contributed to the students' difficulties in making connections among slope, line position, and rate of change. In addition, Stump (1999) highlights that secondary teachers have a limited understanding of gradient concepts because they focus mainly on the geometric ratio, and procedural aspects of gradient. Building on these studies, Moore-Russo, Conner, and Rugg (2011) suggest that there are 11 conceptualisations of gradient as shown in Table 1 (content variables).

METHODOLOGY

Context of Countries

All of the selected countries have achieved above-average results in international mathematics student assessment programmes, such as PISA 2012. However, whereas Germany ranked 16^{th}, South Korea and Singapore achieved higher scores in mathematics, the 2^{th} and 5^{th} best score respectively among all participating countries and economies.

These three countries represent a good spectrum of educational contexts. In contrast to South Korea and Singapore, Germany does not have a centralised educational system. All textbooks from the three countries are written by private publishers and are subject to approval by the local authority before publishing. There are only two secondary textbooks (currently) available in Singapore in comparison to multiple mathematics textbooks in South Korea and Germany. Whereas secondary education is compulsory in Germany (grade 5 to 12/13 in grammar schools) and South Korea (grade 7 to 9), only primary education (grade 1 to 6) is compulsory in Singapore.

German mathematics curriculum focuses mainly on modelling, problem solving, argumentation and reasoning, and communication. South Korean mathematics curriculum emphasises conceptual understanding, and aims to develop mathematical attitudes, thinking and communication skills in a creative way; whereas the Singapore curriculum focuses on problem solving, and stresses conceptual understanding, skills proficiency, mathematical processes, attitudes, and metacognition.

Textbook Selection

Three secondary mathematics textbooks from Germany (*Elemente der Mathematik 8*), Singapore (Discovering Mathematics-2nd Edition), and South Korea (8^{th} grade *Kum Sung* math textbook) were chosen for this study. The German textbook, published by Schroedel Publishing Company, represents the traditional German grammar school curriculum. It is approved for use in many states, and exerts a strong influence over other later textbooks in the market. The textbook from Singapore is one of the two approved textbooks for the current 2013 syllabus, and has been widely adopted by many schools. Similarly, the textbook published by Kum Sung Publishing Company is one of the popular textbooks in South Korea, and is based on the 9^{th} revised Korean mathematics curriculum.

Data Analysis

To analyse the mathematics textbooks, we reviewed curriculum documents and textbooks, which cover the concepts of gradient. The selected textbooks were then coded by the respective researcher for three variables: conceptualisations of gradient (Moore-Russo et al., 2011), representations of gradient (pictorial, numerical, graphical, and symbolic); and cognitive demands of tasks, exercises and examples (Smith & Stein, 1998).

Lastly, we created the textbook signature for each country by representing how these three variables co-occurred on each page (see Figures 1, 2, and 3). Distinct conceptualisations of gradient on each page was listed (See Figure 1); while each aspect for the other two variables was listed and accompanied by its own histogram, which represents the frequency of occurrence across all the pages in that textbook. The numbers of the left indicate the frequency, while the numbers at the bottom show the sequence of pages. Comparing the textbook signatures across different countries provided a way to see the similarities and differences, both visually and numerically. These patterns were then analysed and accounted for within the textbook, before they were discussed in light of the wider educational context.

RESULTS AND DISCUSSION

Conceptualisations of Gradient

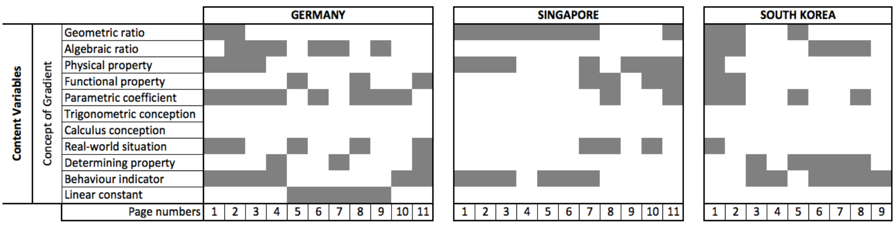

Figure 3: Conceptualisation of gradient in the textbooks.

Referring to Figure 1, the German textbook focuses on different conceptualisations of gradient simultaneously across the 11 pages. In the first few pages, gradient is introduced as geometric ratio; physical property; behaviour indicator; algebraic ratio; real-world situation and parametric coefficient, by referring to the steepness of a street and mathematising it using the gradient triangle and proportional functions. The last four conceptualisations are represented throughout the whole chapter, whilst functional property, determining property and linear constant are introduced later in the chapter. In Singapore, gradient is introduced primarily as a geometric ratio, by making references to the steepness of a line (physical and behavioural) without using the algebraic ratio notion. As seen in Figure 3, these conceptualisations give way to others such as functional and parametric coefficients when connections are made to the notion of equation of a straight line. Gradient is also introduced as real-world situations in some of the tasks.

In South Korea, the concept of gradient is first introduced in relation to the steepness of a staircase, before being compared with determining gradients in graphical representations using geometric ratio. Based on this initial introduction, the concept of gradient is formally defined as algebraic ratio with simple comments on functional property and parametric coefficient. Later, the main focus of the notion of gradient changes to the determining property and behaviour indicator, in connection to the equation of a straight line.

The different emphases in the conceptualisations reflect, to a large extent, the curriculum focus in each country. Firstly, all three countries do not feature both trigonometric and calculus conceptions because these are more advanced notions that are not required when gradient is first introduced. Secondly, the more varied conceptualisation of gradient in Germany seems to reflect its emphasis on modelling, which demands different notions to be represented simultaneously. In contrast, there seems to be a more structured and distinct transition in the conceptualisations for both Singapore and South Korea, which may reflect the curriculum focus on gradient as part of a larger unit on straight-line graphs.

Representations of Gradient

As seen in Figure 2, all three textbooks focus predominantly on the graphical and symbolic representations of gradient. However, the German textbook features more pictorial representations of gradient than the other two countries. For example, no pictorial representation of gradient is used in the Singapore textbook, and numerical representations are only introduced in conjunction with graphical and symbolic ones, when straight line graphs are used in the context of the task. Similarly, in South Korea, there are only a few numerical representations, and they are only used to indicate the algebraic ratio, rather than to encourage students to find the patterns of constant rate of change for the functional property of gradient. Graphical and symbolic representations appear together because of the emphasis on the translations between graphical and symbolic representations.

There are two noteworthy aspects with regard to the use of representations across the three countries: the density and its association with the conceptualisations. First, the German textbook seems to have a “denser” representation patterns per page, as compared to the other two countries. This may be associated with the type of tasks and conceptualisations emphasised in the textbooks. For example, in Germany, there seems to be a relationship between the real-world conception of gradient and the use of pictorial representations. This may be due to the use of pictures to help students make sense of the problems that are set in a real-life context.

On the other hand, the other two countries have similar representation patterns that are less dense, and are more focused on the graphical and symbolic representations. This is the case because both focus on gradient of straight lines as the main context, instead of modelling the real life situations. Nevertheless, while the South Korean textbook seems to emphasise more on algebraic representations because of its attention to the algebraic conceptualisation; the Singapore textbook has less symbolic representations because gradients are determined mostly through the geometric ratio without making reference to the algebraic conceptualisation.

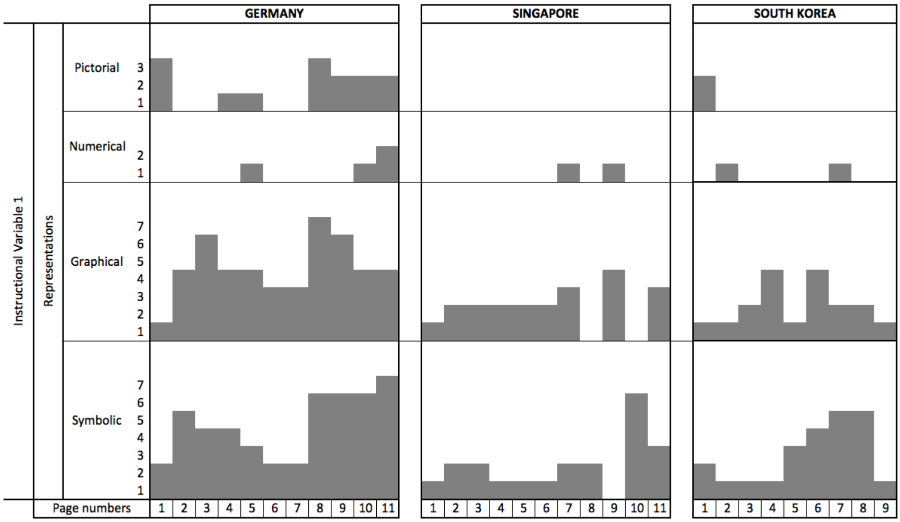

Figure 4: Representations of gradients used in the textbooks.

Cognitive Demand of Tasks

As seen in Figure 3, although all four types of cognitive demand problems are present in the German textbook, most can be classified as procedure without connection (70%). This is followed by procedures with connection (19%), while tasks of doing mathematics (6%) and memorisation (5%) are present to a much lesser extent. Similarly, in Singapore, almost all the examples are computational in nature, and hence they are classified as procedures without connections (56%). The higher-demand questions (36%) are found at the end of the section, and the layout of the questions seems to follow a sequence from less-cognitively demanding to more demanding ones. In contrast, tasks in the South Korean textbook are classified mainly as procedure with connection (50%), followed by procedures without connection (36%), doing mathematics (10%), and memorisation (4%). This really stands out amongst the three textbooks because almost 60% of the tasks in the South Korean textbook are of high-level cognitive demands.

All three textbooks appear to focus less on "memorisation" tasks, but at the same time, they include only a few "doing mathematics" tasks. In Germany, given the high representation density and the varied conceptualisation of gradients, it seems surprising that a huge majority of the questions are classified as procedures without connections. Even though the German examples require complex mathematical thinking, and connections between the multiple representations for mathematical understanding, the tasks for the students are rather algorithmic and not highly demanding (e.g. drawing a graph for a given equation or determining equation for a given graph). In contrast, the cognitive demands of the tasks in Singapore seem to reflect its curriculum approach of moving from simpler to more difficult problems in the analysed chapter. Likewise, the high use of "procedures with connection" type tasks in South Korea is aligned with its curriculum emphasis on understanding mathematical concepts, and its key focus on graphical-symbolic translations.

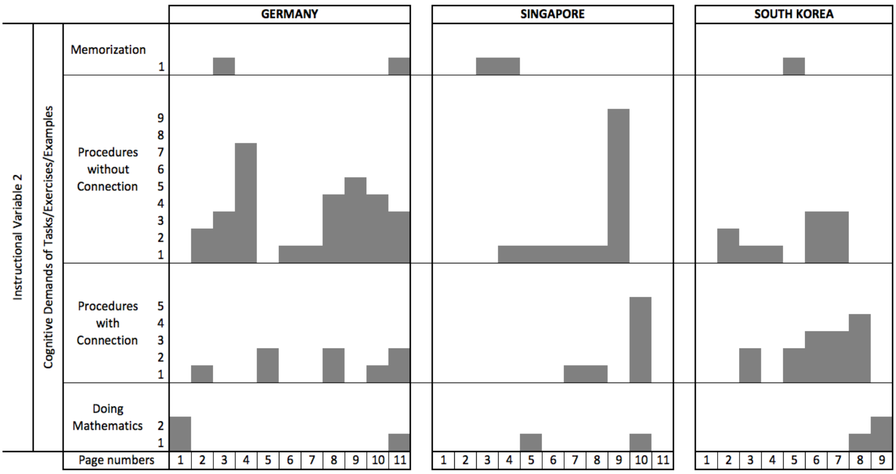

Figure 5: Cognitive demand of tasks in the textbooks.

CONCLUDING REMARKS

This exploratory study suggests that the textbook signature of each country is unique. More importantly, the patterns suggest an alignment between the three variables, and the curriculum emphasis of each country. The textbook signature can also provide a visual representation of the strengths, and areas for improvement in the design of textbooks. For instance, all three countries may benefit from including more "doing mathematics" tasks in order to reflect the focus on modelling, problem solving, and conceptual understanding. However, two main limitations hinder us from drawing further implications. Firstly, the representation of the textbooks in Germany and South Korea can be improved if a more representative sample can be drawn. Secondly, the reliability of the coding can be improved by translating the analysed textbooks, and clarifying our coding procedures by taking into consideration the difference in context (e.g., what we consider as "procedures with connections" may be different in each country). Notwithstanding these limitations, this study highlights how textbook signatures can be used to examine textbooks across different countries. Further investigation on formulating and representing these signatures will be a fruitful area for future research.

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COMPARATIVE RESEARCH IN MATHEMATICS EDUCATION: BOUNDARY CROSSING AND BOUNDARY CREATION

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This paper addresses comparative research in mathematics education from the perspective of boundary crossing and argues that all research is intrinsically comparative and, as such, continually engages in the useful and productive activity of constructing and reconstructing boundaries. Recognition of the significance of acts of comparison in both boundary crossing and boundary construction foregrounds comparison as a key tool in the essential act of boundary deconstruction. International comparative research in mathematics education provides the examples illustrative of the points being made. Researchers in mathematics education must consider what boundaries they invoke in their comparisons and to examine critically the form of boundary crossing implicit in their particular comparative activity.

COMPARATIVE RESEARCH AND BOUNDARIES

It is the assertion of this paper that it is the business of research to continually engage in the useful and productive activity of constructing boundaries. It is also true that some of the least useful and most harmful boundaries are also products of research. I would argue further that there is a fundamental redundancy to the expression "Comparative Research," since comparison is implicit in all research. Nonetheless, this paper will continue to employ the expression "Comparative Research" to refer to those research designs for which the focus is on specific, differentiated objects, communities or systems about which an act of comparison is to be undertaken. By contrast, a longitudinal study of evolving practice in a single mathematics classroom would not conventionally be thought of as a comparative study, yet the comparison between current and recent practice is continual in such a design. Acts of research comparison necessarily construct boundaries that distinguish between the objects, groups, communities, settings or systems that are compared. These boundaries are important. Without them, our acts of comparison are meaningless. As a consequence, boundary construction is an inevitable entailment of all research activity.

RESEARCH AS COMPARISON: THE RIGHT TO COMPARE

An earlier paper (Clarke et al., 2012) posited the Validity-Comparability Compromise as a central consideration in cross-cultural research in mathematics education. Commensurability was interpreted as the right to compare. This right to compare cannot be assumed, but is contingent on our capacity to legitimise both the act of comparison and the categories through which this act is performed. It was argued that any value that might be derived from international comparisons of curricula or classroom practice is critically contingent on how the research design addresses the competing priorities of validity and comparability.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 177-184. Hobart, Australia: PME.

This paper examines the nature of the boundaries constructed through our acts of comparison, the status that might be accorded to those boundaries, and our responsibilities as researchers to acknowledge our role in boundary construction. Further, I argue that sensitivity to the entailments of our comparative acts can assist us in the deconstruction of those boundaries created by our research. Such deconstruction would then better equip us to celebrate the useful work performed by those boundaries, while sensitising us to possible dangers, such as unwarranted extrapolation or generalisation, reification, segregation, stagnation or sanctification.

COMPARATIVE RESEARCH AS BOUNDARY CROSSING

If all research involves comparison, and all comparisons invoke or create boundaries, then my further proposition is that all research, and Comparative Research in particular, involves acts of boundary crossing. It is useful at this point to consider the proliferation of boundary-related terms pervading educational literature at the moment: boundary crossing, boundary object, boundary interactions, boundary practices, and boundary zones (see Akkerman & Bakker (2011) for a useful discussion). Underlying all these terms is an inevitable uncertainty about what the term "boundary" actually refers to; inevitable, because its use and referent will vary from study to study. Boundaries separate the entities to be compared. They are constructions, built of language through discourse. However, we respond to boundaries in different ways. Sometimes the boundary appears as a natural feature, like a river, separating one habitat from another; sometimes, as an artefact, like a wall, constructed to enclose or to separate; and, sometimes, as the principles by which the members of a club or society are distinguished from non-members. Given such variation, the nature of boundary crossing itself must take different forms. The remainder of this paper addresses possible different approaches to boundary crossing and attempts to illustrate its points with examples relevant to mathematics education. The question that structures this discussion is "How do you cross a boundary?" This question directs attention to the nature of the particular boundary and consequently to the assumptions and consequences of research in mathematics education. Each method of boundary crossing comes with its own caveat.

METHODS OF BOUNDARY CROSSING

One way to cross a boundary is to abolish it.

> The insertion of cultural artifacts into human actions was revolutionary in that the basic unit of analysis now overcame the split between the Cartesian individual and the untouchable societal structure (Engeström, 2001, p. 134).

In this instance, the boundary between the individual and the physical world was abolished as a matter of theoretical dictate. In the field of research, the redefinition of metrics can significantly reconstruct boundaries. As a case in point, between the 2000 and 2003 administrations of PISA, Australia moved from "low equity" to "high equity" status without apparent change in practice, but through "slight variation in the way 'equity' was measured in PISA" (Gorur, 2014). In such cases, boundaries are re-drawn without additional evidence and a school system may cross from one grouping to another as a matter of legislation, rather than any change in either practice or outcome. Political examples of such boundary crossing by proclamation are extremely common. Every act of boundary crossing can be associated with at least one potential danger, represented in this paper as a caveat.

CAVEAT: the abolition of boundaries can deny the recognition of diversity.

Each abolished boundary assigns an integrity or connectedness to otherwise distinguished entities (students, teachers, school systems, or task types) as members of a unified aggregate that conceals diversity. These concealed diversities may disempower the communities now integrated and may deny the researcher both explanatory alternatives and possibilities for advocacy of action. A particularly obvious example is the national aggregation of student achievement scores across category distinctions of ethnicity or socio-economic status that, once dissolved, no longer offer avenues for researcher comparison, explanation, advocacy or political action (eg. Berliner, 2001; see also, Clarke, 2003).

Another way to cross a boundary is to demolish it.

The distinction between abolition and demolition for me is one of theoretical dictate vs empirical demonstration. Theory or simply accepted wisdom (entrenched belief) may treat a boundary as well-established in that it distinguishes in a useful way two categories of occurrence or situational domains that are conceptually distinct. However, if empirical evidence pertinent to the characteristics held to distinguish the bounded domains is not consistent with the posited difference, then the boundary must be considered demolished (or at least destabilised) on evidential grounds. This destabilizing of boundaries can be highly productive. The lack of evidence of difference, where difference might be expected, should lead us to interrogate the original assumptions on which that difference was posited.

As a case in point, PISA student achievement performance is commonly invoked as suggesting curricular or pedagogical difference. Research in classrooms in Korea and Finland problematise any simplistic clustering of Korean and Finnish school systems as pedagogically similar. The inability of PISA scores to distinguish between Korea and Finland, therefore demolishes a putative boundary that would have those two school systems in distinct domains. The comparability of Korea and Finland in this one respect suggests that the dissolution of boundaries is highly specific and cannot be simplistically generalised. It does, however, suggest the particularly useful question: “For what other educational attributes might Korea and Finland be considered to reside in the same domain?”

CAVEAT: The demolition of a boundary is specifically contingent on the nature of the empirical warrants. Boundaries are situated constructions of prescribed conceptual tenure.

Yet another way to cross a boundary is to build a bridge.

What is the work of a bridge? A bridge conveys individuals, groups, ideas or artefacts between domains. It does not interact with the boundary, but passes over it. The assumption that a construct such as mathematics achievement can be defined in a commensurable fashion across two school systems builds a bridge between those school systems. Emergent empirical differences reify the boundary without necessarily interacting with or interrogating it. The assumption that mathematical performance is commensurable across national or curricular boundaries is dependent upon assumptions of curricular comparability with respect to mathematics. However, we know from comparative analyses of mathematics curricula that different school systems do not organise their mathematics content in the same way (see Figure 1). Figures 1 and 2 show the results of comparative analyses of the Australian, Chinese and Finnish national mathematics curricula for the years of compulsory schooling. The categories employed in both analyses are adapted from the work of Porter and his colleagues (Porter & Smithson, 2001; and see Xu, Kang & Clarke, 2011).

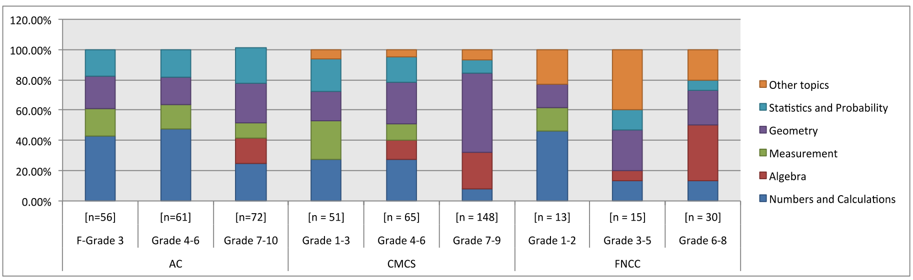

Figure 1. Comparison of Australian, Chinese and Finnish Mathematics Curricula by Content Category

As can be seen from Figure 1, both the content and its sequencing differ significantly between the three countries. As importantly, the types of mathematical performances (levels of cognitive demand) specified in the three curricula also differ significantly (see Figure 2). Figures 1 and 2 demonstrate profound differences in not only the nature of the mathematics considered essential in each school system but in the types of student performances promoted in relation to this content. PISA compares levels of student achievement, products of curricula that are different in structure and in aspiration. The measurement of student mathematical achievement on international tests such as PISA or TIMSS constructs a bridge between the mathematics curricula in the participating countries that affords comparison with respect to the performances attributable to students benefiting from the various curricula.

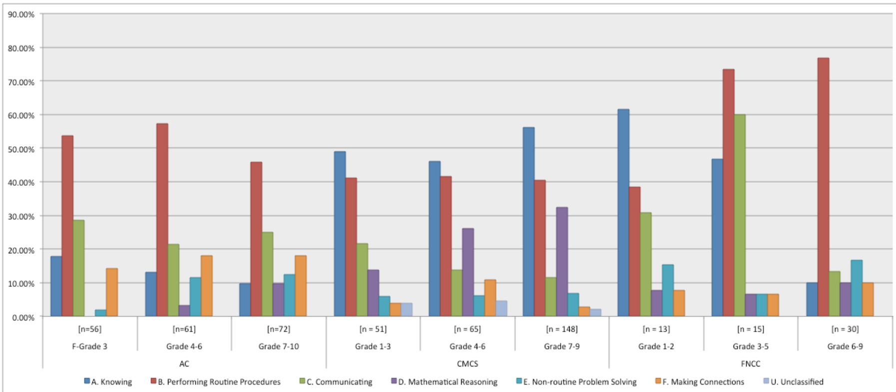

Figure 2. Comparison of Australian, Chinese and Finnish Mathematics Curricula by Performance Type (Cognitive Demand)

The measured performance stands as the single surrogate for the varied aspirations of the different curricula. The institution of international testing provides the bridge for this form of border crossing and reifies through the international acclamation of its findings the boundaries its acts of comparison have simultaneously surmounted and invoked.

> *CAVEAT: Bridges can institutionalise both difference and the defining boundary, differentiating what is being connected.*

The paradox of simultaneously differentiating what is being connected through the act of comparison is at the core of the activity of research comparison. We must scrutinise the legitimacy of the act of comparison because its consequences can consolidate the boundary it appears to transcend: both constructing and concealing difference (Clarke, 2013).

A fourth way is to find objects to which the boundary is permeable.

A truly impermeable boundary would prevent all possibility of comparison. Another way to say the same thing is that there would be no objects pertaining to one domain that had meaning within the other domain and nothing, therefore, that could serve as the basis for comparison. In one form of contemporary boundary-speak, this means there would be no possibility of a "boundary object" (Star & Griesemer, 1989).

To provide a contemporary context for this form of boundary crossing, I would like to situate the discussion around the acronym "STEM" (Science, Technology, Engineering and Mathematics). We have become so accustomed to the subject grouping for which STEM is the acronym, that it is difficult to recognise that STEM could be the name for a fairly monumental category error. One approach is to consider the nature of the truth claims characteristic of each discipline and the authorities to which these might appeal:

Science – empirical consistency; Technology – tool utility; Engineering – built viability; and, Mathematics – logical coherence. These are fundamental differences between STEM disciplines. If STEM, as a unitary aggregate or assemblage of component domains, is to be of value in educational (or other) settings, then we need a mechanism to enable boundary crossing between the STEM disciplines. In this fourth approach to boundary crossing, we examine those constructs to which the boundary walls of the STEM disciplines seem most permeable. What we need to identify are constructs that demonstrably do explanatory or at least classificatory work in more than one domain within STEM.

Take "Evidence" as a construct having currency in each of the STEM disciplines. What qualifies as evidence in the domain of mathematics may be differently conceived than in science. Yet the function of evidence remains arguably the same in each domain: the validation of truth claims. Research seeking to compare phenomena across the STEM disciplines can do useful work by addressing how constructs such as Evidence are employed. How are these constructs transformed in their passage between STEM cells? Do we find conservation of function accompanied by transformation of form?

> *CAVEAT: How are these objects transformed in their passage through the 'permeable' boundary? Does conservation of function but transformation of form maintain object identity and consequently comparability?*

The empirically-driven opacity (impermeability) of the boundary undermines the legitimacy of the very comparison that is rendering it more opaque. I suggest that the status of our "boundary object" as "boundary object" is critically dependent on the balance between sufficient similarity to support comparison and sufficient difference to sustain the boundary.

A fifth way to cross a boundary is to federalise the collective of bounded regions into a structured unity.

STEM also provides an example amenable to our fifth method of boundary crossing. If we consider STEM to be a confederation of states subject to the same legislative and constitutional principles, but independently organised for many practical purposes, then boundary crossing is achieved through the identification or articulation of those constitutional (and constituting) principles. Not only does this approach constitute a form of boundary crossing by transcending intercellular STEM boundaries, but it also holds the capacity to regulate the process of boundary crossing by legislating which responsibilities are shared and which are the specific province of each domain. For example, is Evidence universally invoked, but Proof restricted to the domain of Mathematics? The mechanism whereby such principles of intellectual trafficking are laid down will reflect the relative agency and voice given to the constituent entities in the federated states of STEM. Dominance of any particular voice (eg Science) in determining the principles of exchange (eg the standards for evidence-based practice) would constitute an act of colonization.

> *CAVEAT: Federation is a commendable aspiration provided it does not become colonization. Who speaks for each bounded region?*

Again, we find echoes of the concerns expressed by Clarke et al. (2012), elaborating the proposition: “Comparison must not be unilateral” (Stengers, 2011).

A sixth way to cross a boundary is to accept responsibility for its construction (and deconstruction).

Each act of comparison simultaneously achieves the researcher’s creation of the domains that are the subject of comparison and the boundary by which the domains are defined and distinguished. Each research report solicits the reader’s complicity in these acts of construction and distinction. As already discussed, the activity of comparison may be predicated on a presumption of difference that provided the warrant for comparison, but the consequences of the comparative activity may provide evidence that could either consolidate or destabilise the boundary on which the legitimacy of the comparison was predicated.

From this perspective, boundaries must be seen as fragile entities, ephemeral, continually changing and immensely useful. Whatever ideological commitments we might all feel to inclusivity, our practice as researchers acts to divide, to create boundaries. We do this most visibly in Comparative Research, where our acts of comparison are foregrounded, as are the domains across which we compare. As has been argued, these acts of comparison have the inevitable outcome of constructing boundaries. Our obligation as researchers is to acknowledge this activity and engage simultaneously in both the construction and the deconstruction of these boundaries. In this way, by accepting our role in boundary construction, we position ourselves across (on both sides of) the boundary, not only able to make comparison but also to examine the implications of that comparison for the boundary it presumes. This examination requires the deconstruction of the boundary, providing insight into its utility, its fluidity and what I have called its conceptual tenure.

SUMMATIVE DISCUSSION

Boundaries are constructions, built of language through discourse. They are inevitably purposeful and can be both useful and affirming. They must also be fluid, in the sense that they must always be subject to contention, to destabilisation, and, consequently, open to deconstruction and reconstruction.

In this paper, I have foregrounded the role of comparison (in Comparative Research in mathematics education, and in research in general) in creating and crossing boundaries. Viewing the various activities of Comparative Research from the perspective of boundary crossing sensitises us to the role research plays in creating boundaries and to the implications for our research of both the possible nature of these boundaries and of the process of boundary crossing that is also intrinsic to our research activity.

International comparative research in mathematics education can both create and destabilise boundaries in ways that enhance or impede our ability to benefit from the practices of mathematics classrooms and school systems elsewhere. The boundaries we construct should clarify our understandings, not impede their application. Equally, our destabilisation of existing boundaries should result from our demonstration that some boundaries do no useful work, but rather inhibit our consideration of alternative ways to conceptualise our discipline, our pedagogy, and even our research.

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STUDENTS AND FINANCIAL LITERACY: WHAT DO MIDDLE SCHOOL STUDENTS KNOW? WHAT DO TEACHERS WANT THEM TO KNOW?

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Given the international recognition of the importance of financial literacy, the generally unsatisfactory results of international financial literacy tests for youth and adults, and the overlaps between financial literacy and mathematical thinking, the need to better understand how and why it is taught is crucial. In this paper, we provide a beginning research contribution to financial education by examining the perspectives of three different stakeholders: teachers, business volunteer instructors, and middle school students to explore what students know about financial concepts and what teachers would like them to understand about financial matters.

INTRODUCTION AND BACKGROUND

The OECD has recognized financial education as an important factor in developing financial stability, economic development, and individual financial empowerment and well-being. In particular, financial choices among young people are expected to be more challenging than in past generations with more complex products. Increased life expectancies and decreased welfare and occupational benefits will increase exposure to financial risks making financial literacy skills essential (OECD, 2014). While there is no single definition of financial literacy or capability, most include the acquisition and use of knowledge and skills for effectively managing one's financial resources and making informed financial decisions (Mandell, 2008; OECD, 2014; Social & Enterprise Development Innovations [SEDI] 2008; Task Force on Financial Literacy in Canada, 2010;). From simple daily spending and budgeting, to saving for major life events, or choosing banking products, these decisions have a profound impact on financial well-being and inclusion (SEDI, 2008).

Given this importance, the OECD identified the need for reliable data that can inform financial education strategies by providing benchmark measures of financial literacy levels. Through the Programme for International Student Assessment (PISA), the OECD created the 2012 Financial Literacy Assessment, which was completed by 29,000 fifteen year olds in 18 countries and economies and included questions about money and transactions, planning and managing finances, balancing risk and reward, and general character and features of the financial world. The results, released in July 2014, show about 15% of students scored below the baseline Level 2 and 10% scored at proficient Level 5 with wide variations within each OECD country. Notable is Shanghai-China, where 41% of students scored as proficient compared to the OECD average of 7.9%. In general, financial literacy in OECD countries (13 participated) was

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 185-192. Hobart, Australia: PME.

strongly correlated with scores in mathematics (.83) and reading (.79). This relationship was weaker in countries with established financial literacy programs and professional development for teachers where students perform better on financial literacy than their scores in mathematics and reading (OECD, 2014).

In Canada, the Federal Minister of Finance declared financial literacy an essential skill and authorized a federal task force to study the issue (Financial Consumer Agency of Canada, 2009). In 2011, a Financial Literacy Action Group was formed as a coalition of seven organizations that work to improve the financial literacy of Canadians. Similar concerns and efforts have been underway in the United States resulting in the formation of a President's Advisory Council and significant efforts by organizations to advance financial literacy. The US-based Jump$tart Coalition was founded in 1995 by organizations that share an interest in advancing financial literacy among K-12 students. It has grown to include more than 180 national partners and 48 affiliated state coalitions.

Despite widespread initiatives to improve financial literacy skills, study after study show a poor grasp of financial concepts among young people and adults. Less than 18% of U.S. baby boomers could correctly answer a question regarding interest compounding over two years (Lusardi, 2012) and the U.S. National Foundation for Credit Counseling, NFCC, reports that 48% of adults express concern about insufficient savings for retirement (NFCC, 2011). Credit card debt is on the rise and people do not fully understand the harmful consequences of making only minimum monthly payments nor the benefits of modest increases over these small payments (Soll, Keeney & Larrick, 2013).

In Canada, the British Columbia Securities Commission (BCSC) commissioned, in 2011, the first comprehensive Canadian benchmark study on youth financial skills (ages 17-20) which found marginally better scores among those students who recall taking a "very comprehensive" high school course, while those whose courses were "somewhat" or "not very" comprehensive scored no differently than those who did not take a course at all. Sixty-six percent reported learning what they know about personal finance from their family, followed distantly by 10% who attributed their knowledge to a course in high school (BCSC, 2011). Similar results were reported in the 2008 U.S. Jump$tart survey report: "We have long noted with dismay that students who take a high school course in personal finance tend to do no better on our exam [biennial since 1997] than those who do not" (Mandell, 2008, p.5). The report continues that such results are "a great disappointment" to those who support high school courses in personal finance and "it points to the need for better materials and teacher training" (Mandell, 2008, p.5). Families have also been found to be the most important influences in the acquisition of financial skills in Canada, US and Australia, with youth's understandings about money tied to family backgrounds and financial circumstances (BCSC, 2011; LaChance & Choquette-Bernier, 2004; Lusardi, 2012; Lusardi, Mitchell & Curto, 2010; Sawatzki, 2014). As a consequence, those with less-than-financially-savvy parents are at a significant disadvantage which points to a societal fairness justification for effective youth financial education programs and "without attention to such issues, financial literacy education is reduced to replicating inequities and contributes to the continued marginalization of already vulnerable populations..." (Pinto, 2012, p.113).

When considering mathematics, aspects of financial literacy can also be considered the use of mathematical thinking in a particular context – a financial context. In fact, exploring the understandings middle school children have about money enabled Sawatzki (2014) to develop, trial and refine a financial literacy intervention that found merit in using financial dilemmas to engage students in everyday applications of mathematics that connect social and mathematical thinking, and that require them to seek out and consider multiple alternative options. This approach was proposed as a method to prepare students to be active and critical problem-solvers and make informed financial decisions in the future. Sawatzki (2014) also highlighted the need for quality, research-based professional learning opportunities and materials to guide teacher's approach to financial literacy education.

Given the international recognition of the importance of financial literacy, the generally unsatisfactory results of international financial literacy tests for youth and adults, and the overlaps between financial literacy and mathematical thinking the need to better understand how and why financial literacy is taught is crucial. There is little research on the pedagogical practices of developing middle school students' financial literacy (Sawatzki, 2014 is one exception). This paper provides a beginning contribution to the research in this area. We examine what middle school students know about financial concepts and what teachers would like their students to understand about financial matters.

METHODOLOGY

A mixed methods (quantitative and qualitative) approach was used to gather data from students and teachers, by the first author, in the process of developing new financial literacy materials for middle school students. This Canadian joint research included a national financial education non-profit organization, a financial institution, the University of British Columbia Interdisciplinary Studies program and a government-sponsored research program. The materials were developed for the national non-profit organization for delivery by business volunteers from the financial services industry in Canadian middle school classrooms.

Participants

There were three groups of participants in this study, all from British Columbia, Canada: teachers, volunteer instructors, and students. The teachers, 23 in total, had each requested and hosted volunteers to present the existing financial literacy program in their Grades 6 to 8 classrooms in the year preceding the study. The volunteer instructors, 10 in total, were from the financial services industry and had experience teaching the existing financial literacy program. The students, 45 in total, were in Grade 7 aged 11-12 (n=26) or Grade 8 aged 13-14 (n=19). The Grade 7 students were from a diverse urban school with 50% ESL (English as a Second Language) students, while Grade 8 students were from a French Immersion class in a suburban high school with 14% ESL students. Each school performed with average academic achievement scores for the province.

Data Collection

The classroom teachers responded to an online questionnaire about their reasons for requesting the financial literacy program, and their priorities on topics and teaching methods. The volunteer instructors were interviewed through face-to-face semi-structured interviews on their experiences with teaching the existing financial literacy program, and reasons for choosing to volunteer. The students completed a pre- and post-survey of financial background knowledge and the intervention materials were delivered in their classrooms.

Data Analysis

The data collected from the online survey of the classroom teachers were analysed for frequency of responses using the reporting and graphing capabilities of the online software, and the qualitative responses were summarized for common themes. The volunteer instructor interviews were recorded with handwritten notes, which were then entered into a database and summarized for common themes. The student pre- and post-surveys were completed by the students on paper and later entered into a database for analysis of response frequencies and common themes.

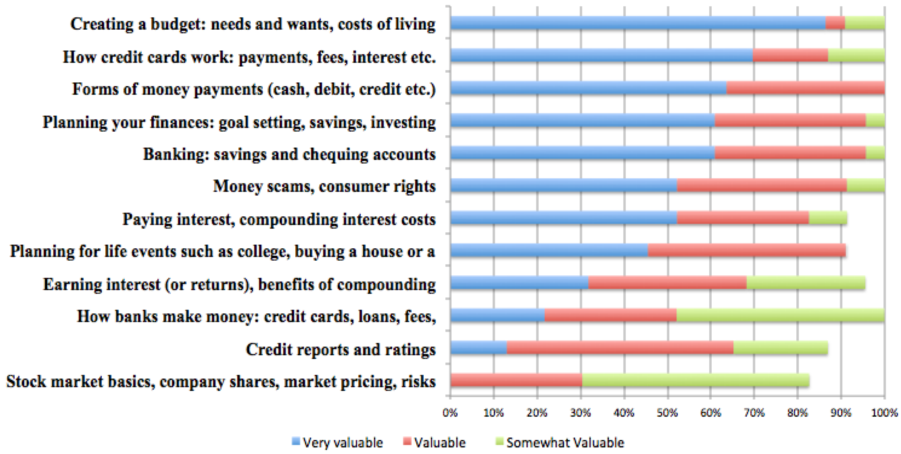
RESULTS

Teachers' perspectives on financial literacy

The primary reason that teachers had requested and hosted the existing financial literacy program over the prior year was that they felt it was important for students to increase their financial knowledge (82%). Almost half had hosted the program in the past and were pleased with the results and half had heard about the program and thought they would try it. Over one third requested the program because they felt that instructors from the business community were more knowledgeable about financial topics. Most of the teachers (78%) had integrated financial concepts into their math or social studies curriculum, 40% had used websites to teach financial skills, a few (17%) had used books, videos and sharing of financial news articles.

Teachers reported that using an instructor with group exercises was very effective (87%) for delivering financial education in the classroom, followed closely by just a live instructor, and more distantly by computer, board games and tablet apps. Several teachers stated that exercises or games that linked mathematics to real-life such as banking, saving money, interest rates, and setting up decision-making exercises or a small business of some sort, were particularly effective for teaching financial skills.

Teachers note that many of their middle school students have very little experience with earning and saving money, though they can have very different knowledge levels based on their socio-economic backgrounds. Some from less privileged backgrounds were said to "know almost nothing about formal money systems and act suspicious and contemptuous of banks." Teachers rated how valuable they felt the following financial concepts were to their students:



Teachers in this study affirmed the importance of using an instructor with group exercises that linked mathematics to real-life financial activities to be a very effective approach for their students. The use of volunteer instructors from the financial services community had proven to be effective with these teachers, or had been recommended from their colleagues, with more than a third of teachers considering the business volunteers to be more knowledgeable about financial topics.

Financial services industry volunteers as instructors

All the program instructors were volunteers with experience in the financial services industry at banks, credit unions, brokerage firms, or in corporate finance and had taught the financial education program an average of three times during the last two years. Most were motivated by a desire to help young people learn valuable money management skills and avoid financial problems by sharing their expertise, and by the interest and enthusiasm of the students. Sample comments included: "In grade 8, I got allowances and I wish I had started saving a lot sooner. I am motivated by news of credit card debt levels, and bad advice given to seniors." "The course makes a difference with kids and can impact their lives." Others felt that "depending on their parents, [financial knowledge] kids may or may not be learning appropriately." "Because this is not stuff they [students] learn traditionally. I learned it from my mom." "Kids like to talk about how they work with money, their experiences." And "I walk away feeling really good about what kids have learned. I love it."

Program elements that they found particularly engaging and effective were: real-life problems with hands-on activities such as creating a financial product advertisement, a comparison shopping exercise, or developing a budget for a trip or event. Areas the volunteer instructors thought were important to cover included: cost of buying fast food ($100/month for 20 years), credit cards, fees and interest, banking information such as loans, types of accounts, bank's profit models. Several volunteers mentioned encountering sensitivity with ranges of socio-economic backgrounds of students with exercises such as classifying needs and wants, or during a shopping decision making activity. One instructor reported her/his experience with a Grade 7 class where only 1 or 2 students understood credit and debit cards, while in a second class, at a private school, Grade 8 students had credit cards on their parent's account.

In their professional lives, the volunteers observe adults struggling with financial literacy issues and believe that sharing their expertise could positively impact the students' financial behaviours and future well-being. They are a motivated and important resource for the teachers in this study who value the real-life perspective and knowledge they bring to the classroom.

Students' perspectives and understanding of financial literacy

In the Grade 8 class, 84% of the students had a bank account though only 32% had ever used their debit cards to make a purchase. Most students (79%) said they had learned about finance in the past, 68% of them listed math class, and half said they learned about interest rates. Additional topics mentioned by one or two students each included: exponential or compound growth, mortgages, sales taxes, debt and investment and counting money. In the Grade 7 class, 62% said they had learned about finance in math and in a provincially required course called Personal Planning, which had a theme called "What's Money got to do with it?" and one student said she learned about "careers and how money can't buy you happiness." Most of the mathematics themes reported by Grade 7 students focused on counting money and making change though one student mentioned saving and spending and another reported computing percentages as being tied to developing financial skills.

Over 73% of the students reported that their parents had taught them about money and more than half the time these lessons were about saving money and spending wisely, such as not buying cheap toys or items they won't use, and waiting for sales. A handful of students mentioned learning about the value of money, that it "doesn't grow on trees" and that "it takes hard work." A few students mentioned parental lessons that dealt with online banking, setting up a child stock account, foreign exchange, supply and demand, difference between credit and debit cards, and how to organize three savings pots: for donating, saving and spending.

When asked to describe their "money personalities" in terms of "I like to save, spend or earn money" most students (78%) stated they like saving money, half also included either earning or spending, distributed equally. About 50% of the students described being careful about spending money by saving up for purchases they really wanted, or spending at a slower rate than saving. Perspectives included: "I'm sort of mixed between saving and spending struggling [with] what I want but don't need." And "I like to earn and save money, whenever I spend it I feel like I could've saved it for something better." Several students admitted they "like to spend money a lot."

When asked in the pre-survey the main reasons for putting your money in a bank vs. a piggy bank, almost all students identified safety as a main reason; only 25% listed earning interest. While 90% of Grade 8 students knew what 1% of $100 was, less than half could correctly compute 20% of $1000. The intervention covered information on credit and debit cards, including interest rates, and was ranked most useful by 95% of students. Money problem solving role-play exercises were rated as interesting and enjoyable by 95% of students.

The results indicate that middle school students are developing understandings about finances and have internalized lessons from their parents and teachers and begun to form savings and spending behaviours, curious about and sometimes struggling with how to manage money appropriately. The students were especially motivated during the financial problem solving role-play exercises, confirming the experiences of the teachers and instructors. This research found considerable variation between the financial lessons different students were taught at home and also in the student's experience and ability to work with financial math concepts such as interest rates.

CONCLUSIONS

In this paper, we provide a beginning contribution to the research in financial education for middle school students. Examining the perspectives of three different stakeholders: teachers, business volunteer instructors, and students, reveals that teachers saw a need to increase their student's financial knowledge in a range of areas and selected a program delivered by business volunteers to bring a real-world perspective and specialized expertise to their classrooms. The volunteer instructors were motivated to share their knowledge and positively impact the students' financial behaviours and the students shared their widely varied understandings about and experiences with finances and were engaged and attentive, though challenged with financial mathematics concepts such as computing interest. For these middle school students, parental guidance on financial habits and behaviours and the corresponding socio-economic influences were apparent and seem to play an important role in the student's self-described values around saving and spending. This observation is consistent with the research on parental influence cited earlier in this paper, including the finding that financial literacy is strongly and significantly correlated with parent's education, (in particular the mother's) (Lusardi, 2012). This research also supports Sawatzki's (2014) emphasis on the important role that student's understandings and values around finances serve in the development of effective interventions.

This study involved a short intervention in the important area of financial literacy and provides a snapshot of three stakeholder's perspectives. Further research could seek to bring the worlds of mathematical thinking (literacy) and financial literacy together to further explore students' values, behaviours, and understandings of finance and the mathematical understandings they use and those they will need in order to make capable financial decisions in the future.

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EVALUATING PRE-SERVICE TEACHER NUMERACY – BUILD BRIDGES RATHER THAN ROADBLOCKS

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Numeracy is an important skill for many university students. In particular, it is being focused upon for pre-service teachers through the development and administration of skills tests that address numeracy and literacy. These tests are high-stakes as they have the capacity to stop pre-service teachers from completing their qualifications or registering with the required teaching authority. The aim of this theoretical research report is to explore the potential impacts of a high-stakes numeracy test and to discuss whether these impacts have unexpected consequences for pre-service teachers, pre-service teacher educators, and the community.

INTRODUCTION

Numeracy is an essential component in the development of lifelong skills, such as qualitative and critical thinking skills (Office of the Chief Scientist, 2014a, p. 23), and a strong focus of STEM education and careers (Office of the Chief Scientist, 2014b). Brady (2014) identified numeracy as an essential component in a range of courses at university. However, she stated that students' lack of numeracy could hamper their engagement with their courses. The importance of the development of numeracy, particularly quantitative and critical thinking skills (Kemp & Hogan, 2000), should be a priority in education. In pre-service teacher education, the numeracy requirement is often formalised.

Teacher education programs in Australia need to demonstrate how their pre-service teachers develop mathematical discipline knowledge (Australian Institute for Teaching and School Leadership [AITSL], 2011). In the United Kingdom (UK), the Teachers' Standards (Department for Education, 2011) requires teachers to have sufficient understanding of the subjects they will teach to enable them to produce experiences that will engage their students. This reflects the National Council for Mathematics Teachers [NCTM] (n.d.) consideration of the need for teachers to generate discourse in mathematics education. Situated alongside this emphasis is the testing of pre-service teacher numeracy. However, rather than working in simpatico, these two may diverge and the resultant conflict may hinder the development of numeracy in pre-service teachers, especially at the level required to engage their students in mathematical experiences that incorporate discussion and discourse.

Numeracy

This paper positions numeracy as mathematics skills, mathematical competency, and disposition towards mathematics (Cooke, 2015). This is not a new way of viewing numeracy – the Australian Association of Mathematics Teachers [AAMT] stated that

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 193-200. Hobart, Australia: PME.

the disposition to use mathematics is crucial to numeracy in 1997. However, not all of the components of this position are shared in the tests developed to assess pre-service teacher numeracy.

PRE-SERVICE TEACHER NUMERACY

Approaches to assessing pre-service teacher numeracy

While the UK has had a test for pre-service teachers in place for several years, Australia will introduce a numeracy test for pre-service teachers in 2015. A review of the information available for both numeracy tests, from the websites of the relevant organisations in Australia and the UK, is outlined in the following paragraphs. It should be noted that, presumably due to the longer timeframe in which the UK numeracy test for pre-service teachers has been enacted, more detailed information on the UK numeracy test is available. In the UK, students who are intending to complete pre-service teacher courses must pass the professional skills tests before they can start their pre-service teacher course (Department for Education [DfE], n.d. a). The professional skills test numeracy component incorporates a mental arithmetic test given aurally and a written data and arithmetic test provided online with on-screen questions and answers (Department for Education [DfE], n.d. b). Up to three attempts can be made to pass the numeracy test, but information on what happens after a third failed attempt was not found on the Department for Education (DfE) website.

The Australian Institute for Teaching and School Leadership [AITSL] (n.d. a) states that the pre-service teachers need to have literacy and numeracy skills in the top 30% of the population. Pre-service teacher education programs are required to develop processes to that enable their students have sufficient numeracy and literacy to engage with their course and to teach, as well as to ensure their students are in the top 30% of the population prior to graduation. In addition, tests of literacy and numeracy will be introduced in 2015. Although these tests will be the focus of determining the literacy and numeracy of pre-service teachers, skills not easily measured in tests will be expected to be assessed as part of the teacher education program (AITSL, n.d. b). Examples of skills outside of the tests that were provided by AITSL (n.d. b) include elements incorporating communication, such as speaking and listening.

There are two key concerns regarding these approaches to testing numeracy and these concerns could be roadblocks to the development of numeracy. The first relates to both the Australian and UK numeracy tests – due to the risk of not being able to either enrol in the pre-service teacher course (as with the UK test) or able to teach (as with the Australian test), it will be a high-stakes test. The second involves the practice documents provided for the UK numeracy test – although it is beneficial for the students to be able to see what the test questions involve, the practice paper provides "show me" and "further help" information (DfE, n.d. b) that are procedural. However, a focus on procedural knowledge is contrary to the mathematical knowledge required for teaching mathematics (Cooke & Sparrow, 2012) as it does not address why a process would be use, the understanding behind a solution, or how different understandings fit together (Booth, 2011).

Potential impacts of a high-stakes test for pre-service teachers

Several impacts may result from high-stakes tests that could be roadblocks to the development of pre-service teacher numeracy. Beilock (2008) found that high pressure test situations could impact on achievement in the tests. However, she found that students who relied on their working memory had results that were impacted more than students who had lower demands on their working memory. This could reflect the findings from Ashcraft and Krause (2007), where students who where anxious increased their speed to the detriment of accuracy. Beilock also found that those with lower working memory demands could successfully select short cuts in both low pressure and high pressure test situations. The capacity to appropriately select short cuts could indicate that they have an understanding that reflects Booth's (2011) description of conceptual knowledge – understanding behind the solution that can fit together different ways of solving a problem.

Cooke, Cavanagh, Hurst, & Sparrow (2011) investigated mathematics anxiety reported by pre-service teachers when thinking about using mathematics in a group situation, when thinking about completing a mathematics assessment or test, and when thinking about teaching mathematics in a classroom situation. Cooke et al. found three of the five statements that were easiest to affirm were the same in all situations – one was being aware of previous failures. The other two statements that were easiest to affirm were also the same – being scared of making a mistake – when thinking of completing a mathematics assessment or test and when thinking about teaching mathematics in a classroom situation. An awareness of failures may indicate that the pre-service teacher is within a negative loop of experiences. Metje, Frank, and Croft (2007) proposed that these types of experiences could build on each other, creating failure cycles. Once in a failure cycle, attitudes towards mathematics could become more negative. This may be why Cooke et al. found that students were most likely to affirm that they were aware of previous failures, regardless of the situation they were considering. This indicates potential consequences of a high-stakes numeracy test, both in terms of previous experiences impacting on the test and on the test impacting on future experiences. It also suggests the provision of experiences beyond what are often negative remembrances from school is essential (Cooke & Sparrow, 2012). These new experiences may enable students to build bridges that overcome negative recollections and move towards the development of numeracy needed to teach.

Having to pass a high-stakes numeracy test may be seen by pre-service teachers as a negative experience that contributes to attitudes and anxiety. Núñez-Peña, Suárez-Pellicioni, and Bono (2013) found that student performance at university could be negatively impacted by negative attitudes towards mathematics and mathematics anxiety. Unfortunately, if mathematics anxiety developed, Ashcraft and Krause (2007) proposed that it would be exacerbated by "cultural attitudes that undermine math achievement—for example, that math is hard, one either is or is not good at math, regardless of how hard one works" (p. 247). The negative experiences of the pre-service teacher may also feed into their teaching once they graduate. This has the capacity to then flow into the mathematical experiences they create. Ashcraft and Krause (2007) proposed that young students at risk of developing mathematics anxiety may be more likely to do so if working with a teacher who is not supportive. It may be that a teacher who has negative attitudes towards mathematics, is not certain of their mathematics ability or their ability to teach mathematics, or has anxiety regarding mathematics may not have the content knowledge or pedagogy to be supportive. Certainly, Beilock, Gunderson, Ramirez, and Levine (2009) found that female elementary teacher mathematics anxiety negatively impacted on the mathematical achievements of their female students. This could be due to the use of fewer teaching strategies (Swars, Daane, & Giesen, 2007) or a focus on using textbooks and worksheets (Choppin, 2011). These impacts could also be contributing to the reduced interest in STEM careers (Erickson & Heit, 2013).

Potential impacts of a focus on procedural knowledge of mathematics for pre-service teachers

A focus on following one set way to solve a mathematical problem could limit the mathematical knowledge of pre-service teachers and generate a roadblock to the development of numeracy. Raghubar, Barnes, and Hect (2010) proposed that mathematical competence incorporates conceptual understanding, problem solving, and knowledge of procedures, and processing of new and relevant information. Erickson and Heit (2013) added in metacognitive judgement, particularly the capacity to accurately judge capabilities to reduce overconfidence. Likewise, Kemp and Hogan (2000) stated "numerate behaviour involves a blend mathematical, contextual and strategic knowledge" (p. 11). All of these capabilities would be needed to be able to discuss mathematics and to share mathematical experiences. A teacher with these capabilities would build numeracy bridges for their children to travel. The strength required of these bridges is evident in the National Council of Mathematics Teachers [NCTM] (n.d.) proposition that discourse and discussion are important elements of mathematical experiences. The NCTM (n.d.) describes discourse as the written and oral engagements used by students and teachers to represent, argue, and communicate their thoughts. The role of the teacher is to "orchestrate and promote discourse" (NCTM, n.d., para. 1) to encourage the students to develop appropriate mathematical understandings. Cirillo (2013a) adds "students learn mathematics best when they are given the opportunities to speak about mathematics using the language of mathematics" (p. 1).

Pre-service teachers need to be able to recognise and use mathematics within any subject they are teaching and to be able to engage the students in discussing the mathematics involved in any task, regardless of the subject in which it is embedded (Kemp & Hogan, 2000). This capability to engage in discussions with mathematics requires more than procedural knowledge. When discussion is used in the mathematics learning experience, Cirillo (2013b) it expands the opportunities for mathematical exploration and understanding. Mathematical discussion also has the potential to move the mathematical authority from the teacher only to the participants within the class (Cirillo, 2013b). These considerations make it imperative that pre-service teachers are given the opportunity to experience mathematics beyond set procedures and to truly develop numeracy. A document provided by the organisation administering a test of numeracy that focuses on procedural knowledge and understanding would imply to the user that this focus is what numeracy and mathematical knowledge should contain.

DEVELOPING NUMERACY

A focus on numeracy

Developing numeracy should incorporate a focus on mathematics skills, mathematical competency, and disposition towards mathematics. Ananiadou and Claro (2009) saw competence as involving the application of skills but proposed that attitudes could impact on competence. This reflects the AAMT's (1997) statement that "a person's disposition to use mathematics is also critical in numeracy" (p. 14). Disposition to use mathematics is expanded by the clarification that it "includes personal confidence, comfort and willingness to 'have-a-go'" (AAMT, p. 14). Disposition is an important consideration for numeracy (Cooke, 2015), particularly as having a skill does not mean that the skill will be used (Dottin, 2009). Richardt (2002) stressed that it was disposition that moved abilities into actions – from skills that are possessed to actually choosing to use those skills. In doing so, he referenced Dewey's (as cited in Richardt, 2002) reflection that the desire to use knowledge builds into a person's disposition towards using that knowledge.

Importance of a focus on numeracy for pre-service teachers

As Metje et al. (2007) proposed for lecturers, providing experiences that create positive cycles for pre-service teachers should be a focus of pre-service teacher education. This would become crucial if pre-service teachers were exposed to high-stakes tests of their numeracy. The two risks associated with tests for numeracy, namely the possibility of developing mathematics anxiety and negative attitudes towards maths or entrenching a procedural view of mathematics, could impact on both the mathematical opportunities for the pre-service teacher and for their future students. An example of this continuing impact can be seen through Ashcraft and Krause's (2007) assertion that mathematics anxiety "leads to a global avoidance pattern—whenever possible" (p. 247), in other words, an avoidance of mathematics related activities, which would include preparation and enactment of mathematical experiences in a classroom (Choppin, 2011; Swars et al., 2007).

To be able to teach mathematics, teachers and pre-service teachers need mathematical knowledge (Beswick, Watson, & Brown 2006). Mathematical knowledge should incorporate procedural, conceptual, and strategic knowledge and understanding, problems solving capabilities, metacognitive judgement, and an appropriate disposition towards mathematics (Cooke, 2015; Erickson & Heit, 2013; Kemp &

Hogan, 2000; Raghubar et al., 2010). In addition, confidence with mathematics and beliefs about mathematics that lead the development of appropriate mathematical experiences are required (Beswick, Callingham, & Watson, 2012). Using high-stakes tests of numeracy has the capacity to undermine the breadth and depth and type of knowledge and dispositions needed to ensure pre-service teachers are prepared to fully engage in mathematical experiences with their future students (NCTM, n.d.). The incorporation of tests of numeracy would necessitate pre-service teacher education courses developing experiences that counteract unhelpful numeracy perceptions that may result from the tests.

CONCLUSION

Pre-service teachers do need to be numerate and the community needs to be reassured that teachers of the future will have the skills needed to teach their children. The issue presented in this theoretical research report is that tests of pre-service teacher numeracy may have the capacity to create roadblocks to the development of numeracy, both for the pre-service teacher and, potentially, their future students. If this were to occur, these tests would create detrimental ripples that spread through the community. The introduction of tests of numeracy (and literacy) should be closely monitored to determine the short and long term impacts. If numeracy tests are found to negatively impact on pre-service teacher numeracy, pre-service teacher education programs will need to provide the building blocks through which their pre-service teachers can build bridges to help their develop numeracy.

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CRUCIAL EVENTS IN PRE-SERVICE PRIMARY TEACHERS' MATHEMATICAL EXPERIENCE

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Relationship with mathematics is a crucial variable in the professional development of pre-service primary teachers: it can largely affect pre-service teachers' reactions to educational prompts and also their future didactical choices in classroom. Many researches showed that pre-service primary teachers have often developed a negative relationship with mathematics during their mathematical experience as students. This paper adopts a narrative methodology to investigate about the origins and development of these relationships. Specifically, it investigates how pre-service primary teachers tell the events that they recognize as crucial for the development of their relationship with mathematics. Results indicate the relevance of success/failure experiences, and the key-role of the teacher therein.

INTRODUCTION AND THEORETICAL BACKGROUND

Research in mathematics teacher education has highlighted the influence of prior experiences with mathematics as students on pre-service primary teachers' professional development. Studies carried out in different countries show that pre-service primary teachers have often lived negative experiences with mathematics and have consequently developed a negative relationship with mathematics (Lutovac & Kaasila, 2014). These negative experiences with mathematics can generate uncertainty, low perceived self-efficacy as future teacher, and produce "deep seated beliefs [that] often run counter to contemporary research on what constitutes good practice" (Liljedahl et al., 2007, p. 320). On the other hand, a reflection about the causes of the difficulties met in the school experiences with math can be the germ for the *math-redemption phenomenon*, i.e. pre-service teachers' desire to reconstruct their personal relationship with math and to avoid the mistakes imputed to their past mathematics teachers (Coppola, Di Martino, Mollo, Pacelli, & Sabena, 2013).

Both for research and for practice in teacher education programs, it appears therefore significant to investigate about the origin and development of the pre-service teachers' different relationships with math.

A narrative approach seems to be particularly suitable for this purpose (Kaasila, 2007). The goal of the narrative approach is to get the narrator to describe stories in which aspects that he/she considers significant come to the fore. As Kaasila underlines, through a narrative approach we can focus not only on pre-service teachers'

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experiences, but also on how they describe them. In particular, Connelly and Clindinin (1990, p. 2) state that:

> The main claim for the use of narrative in educational research is that humans are storytelling organisms who, individually and socially, lead storied lives. The study of narrative, therefore, is the study of the ways humans experience the world. This general notion translates into the view that education is the construction and reconstruction of personal and social stories; teachers and learners are storytellers and characters in their own and other's stories.

The narrative study conducted by Di Martino and Zan (2010) about school students' relationship with mathematics shows that students identify some specific events as crucial for the development of their relationship with mathematics. Moreover, they are often characterized by the description of *ruptures*, i.e. these events constitute what Bruner (1990) calls *turning points*. As Bruner observes, when an important rupture occurs in the plot of a personal story, the narrator usually recalls and describes with most details and emotional transport a specific event (or some events) related to this rupture. In the context of mathematics education, Drake (2006) carried out a very interesting narrative inquiry focused on turning points in mathematical experiences of six primary teachers. The case study conducted by Drake confirms that turning points are an inestimable source of information for the interpretation of teachers' prior experience with mathematics.

Within this framework, as part of a long-standing Italian research project focused on primary pre-service teachers (Di Martino & Sabena, 2010; Coppola, Di Martino, Pacelli & Sabena, 2012), we carried out a narrative study focused on crucial events for the development of pre-service primary teachers' relationship with mathematics. The goal of the study was to identify recurrent crucial events (and factors involved in these events) in the narrations of future primary teachers, being steered by the following research question:

How do crucial events affect future primary teachers' development of the relationship with mathematics?

METHODOLOGY

Population and procedure

The study involved 145 future primary school teachers enrolled at the University degree for primary school teachers of two different universities: one in the South and one in the North of Italy.

The choice of the research instruments is always not *neutral*, reflecting researchers' values, assumption and beliefs. In particular, there is a variety of different ways of collecting narrative data. We decided to collect autobiographical writing to give respondents *space and time* for thinking what and how narrate. We proposed the following prompt: "*Narrate an episode in your school experience as student that you consider significant for the development of your relationship with mathematics. If possible, describe the details that you remember and the emotions felt. Explain why* *you consider the episode significant*".

Respondents were asked to write their narration anonymously, in order to prevent any conditioning aimed at gratifying the reader or at describing a better image of himself (Connelly and Clandinin, 1990, call this phenomenon "the Hollywood plot"). They were asked to provide a nickname, to allow us to combine their narratives to other possible investigations. We will use such nicknames in discussing the results.

The approach to the data

Narrative research is embedded in an interpretive framework: through the collection and analysis of narrative productions, researchers seek to understand, produce sense and interpret the world in terms of its actors and starting from narrators' words (Bell, 2002). Lieblich, Tuval-Mashiach, and Zilber (1998) identify two main independent dimensions in the analysis of narratives, *categorical versus holistic* and *content versus form*:

> The first dimension refers to the unit of analysis, whether an utterance or section abstracted from a complete text or the narrative as a whole. […] The second dimension, that is, the distinction between the content and form of a story, refers to the traditional dichotomy made in literary reading of texts (*ibid.*, p. 12)

We are aware that in many cases the distinctions introduced are not so clear-cut: as Lieblich, Tuval-Mashiach, and Zilber underline, a purely categorical or holistic approach is not possible practically. Moreover, combining the different dimensions permits to grasp a deeper understanding of the collected narratives.

Concerning the first dimension, using an investigator triangulation method, we started with a holistic analysis to identify the narratives that include one or more episodes recognized by the narrator as crucial in the development of his/her personal relationship with math. Then we carried on with a categorical approach in order to recognise factors that are recurrent in the description of the episodes and more in general in the collected narratives.

Also regarding the content/form dimension we developed the analysis through a multiple approach: our attention was mainly focused on content, but we considered particularly significant also the structure of the plot and the occurrence of specific phrases in the narratives.

RESULTS AND DISCUSSION

A first quantitative data analysis indicates that the 39% of the sample does not report an episode as requested in the assignment, but a description of the personal development of the relationship with mathematics during the entire school period. Narrators explain the reasons for not reporting a specific episode; there are two main cases: the relationship with math is perceived by the narrator as stable during school years, without the occurrence of any events that have modified this trend (Margherita writes: "*I have always had a good relationship with math (...) it seems to me that an episode in which this relationship has improved or worsened has never happened*");

the relationship with math is recognized by the narrator as determined by a certain period rather than by a single episode (Nina88 writes: "*I can't remember a specific episode, but an entire school period that changed my relationship with math*"). In this latter case, in analogy with the terminology introduced by Bruner (1990) for a single event, we use the expression *turning period*.

School transitions appear to be the main perceived causes for a turning period (Anonymous: "*Passing from primary school to middle school, it is as if the solid link between me and math was suddenly and magically broken*"): different teachers, topics, practices and also mathematical success criteria often – for better or worse – provoke a *crisis* that can determine a change in the personal relationship with math (Sissi: "*When I arrived to the Lyceum, I was 'traumatized' (...) from that moment on I have had a difficult relationship with mathematics*"; Austin: "*In primary school I had a good approach with math (...) At middle school there was an overturning of the situation, the enthusiasm for the discipline had been reset (...) Fortunately, when I arrived at secondary school there was 'the big turn'*"). This confirms the topicality of the "transition problem", well-known (at least) in western educational systems:

> Students move, in mathematics, from one type of institution with its characteristic culture to another type with another culture, which produces marked discontinuities in the transition process (…) mathematics is perceived and treated so differently at the different levels that one can hardly speak of the same subject, even if it carries the same name throughout the system (Niss, 2003, p. 117).

Most of the time the figure of the teacher is recognized as crucial in the turning periods, even assuming epic traits in the narratives (Hakuna Matata: "*The encounter with these teachers represents* my significant episode"). In the "positive" cases, the teacher is seen as a mentor, sometimes as the unique factor that determines the development of a good relationship with math (Bubby: "*There wasn't an episode that determine my view of math, but a teacher that, through his teaching, has determined the rebirth of my passion for math*"). Conversely, in the "negative" cases, it happens that the teacher is seen as a sort of cruel and detached "persecutor" (Killylilly: *"In middle school, teacher was the reason for my hate for math: she explained, if you were able to understand well, if you were not able to understand she didn't help you (...) I was terrified in classroom*").

The analysis of the narratives that describe a specific episode (the remaining 61% of our sample) offers several interesting causes to reflection. Future teachers report at least one of the following three reasons for the identification of an episode as significant: i) the episode has caused a change of beliefs; ii) the episode has determined a change in the personal relationship with math (*turning points* for the development of the relationship with math); iii) the episode recalls significant and unexpected emotions.

We will analyse more in depth the case ii), but we want to underline significant common aspects. All the three cases are related to a *rupture*, and the events are often narrated as vivid although some of them occurred many years ago (Lobianco: "*Primary school, third year, fourth day of school, I remember it as it happened yesterday*"). Moreover, the narrator often reports that recalling the episode still elicits strong emotions (Benedetta, describing an episode happened when she was in grade 2: "*It was autumn, I remember this detail because I was wearing my favourite jersey (...) I remember with fear those minutes. When I close my eyes and think over that episode, I can feel my heart beating faster*").

Analysing the school period the specific episodes referred to, it emerges that, even though the primary school experience is obviously the less recent one, yet the 38% of the narrated episodes refers just to this school period (row 1 in Table 1).

	Primary	Middle	High	University
Narrated episodes	38%	17%	41%	4%
Change of beliefs	33%	20%	27%	20%
Turning points	26%	19%	53%	2%
Emotions	57%	17%	27%	0%

Table 1: School period of the narrated episodes (according to the different typologies)

As researchers and teacher educators, we underline the importance of developing a reflection about the data related to primary period with future primary teachers.

The 57% of the narrated episodes recalling significant and unexpected emotions are placed in the primary school period (row 4 of the table). Furthermore, the 26% of the turning points for the development of the relationship with mathematics is placed in the same period (row 3). In particular, reading the narratives, we highlight the occurrence of terms that characterise strong emotional states such as *very happy, delight, love,* but also *terror, hate, frustration*: for better or for worst, early school experiences with math are strongly charged with emotions. For instance, describing the episode related to a test on multiplication tables, Ale92 writes: "*I was a very anxious child, I was scared (although I knew that there wasn't any punishment), and when I was not able to understand something I used to cry. I was scared to disappoint my parents and the teacher (...) surely my anxiety was triggered by the fact that all my classmates were able to do it well and I wasn't*". In her narration it appears clearly that the strong emotions during mathematics activities are linked to social aspects, which thus influence the development of the negative relationship with math. The social relevance of math is probably one of the reasons because mathematics elicits so strong emotions particularly in primary school: the fear to disappoint parents or teachers, and possibly to get discredit from the classmates, can be very strong.

The social relevance of math seems to determine an interesting peculiarity of mathematics that emerges from the narratives: mathematics has the force to provoke strong opposite feelings and perceptions within the same person (idg: "*Since primary* *school, math was the unique subject able to make me satisfied, confident and in the meantime it was able to make me feel incompetent*").

Analysing the *form* of all the narratives that include turning points, we can observe that they are characterized by the occurrences of words such as "*always*", "*never*" and of the expression "*from that moment on*". This data analysis suggests that a turning point in early school years may prematurely determine the student's relationship with mathematics. Furthermore, in case of "negative" turning points (i.e. those episodes that determine the development of a negative relationship) all the subsequent educational choices are affected, even with the outcome of avoiding mathematics as much as possible (Valentina: "*Resulting by this experience* [primary school experience], *I tried to do mathematics the least possible in the following schools, moreover my negative relationship with mathematics affected my decision concerning high school*"). This "avoiding strategy" sometimes prevents the students to pursue some personal drives and can provoke regrets later (Francy: "*Having a second thoughts, I would do the High School of Science: I regret that, at the time, I hadn't the force of make this decision*").

The analysis of turning points

Analysing the *content* of the narratives that include turning points, it emerges that sometimes the turning points are determined by the introduction of some specific topic that represents an insurmountable obstacle (also related to an unexpected failure or decrease in perceived competence) or it is considered meaningless. A typical example of this is the introduction of the letters in algebra (Carmen: "*In grade 11, letters took the place of numbers (...) mathematics become increasingly distant and obscure. I was sure I was never been able to be successful*").

Success, failure and perceived competence represent recurrent factors in turning points. The majority of future primary teachers had not a smooth experience with mathematics, therefore many of them recognize as turning point a school episode of success or failure in mathematics, which determines a strong emotional state (Giu, describing a written exam in grade 6 where she got a very bad mark: "*This episode will be always present in my mind, because it was the mark more humiliating of my school experience. I would have wanted to die from shame!*"), or a significant change in the perceived competence (Meli recognizes her turning point in her first successful written exam in grade 9: "*In Middle school I always had low marks in mathematics, I believed that I wasn't talented at math, therefore I studied it badly and reluctantly (...) at last, after three years, I had overcame a stumbling block, had cancelled the belief that I would never have been able to success in math. From that moment on, I have nurtured my interest for math, and I find a great pleasure in doing it*").

Again, most of the time the teacher strongly affects the consequences of a turning point event: the same event, be it a success or a failure, can have negative or positive development consequently to the teacher's actions. For instance, Nike993 writes that "*My worst experience comes from middle school experiences when, in front of a failure, the negative reactions of the teacher determined the beginning of my hate* *towards mathematics*". On the contrary, Fede V. recalls an oral test concerning geometry in grade 10. She had great difficulty: "*In that moment, when I wasn't able to conclude these problems, I felt terrible about myself, an incompetent*". But the teacher did not scold her, and demonstrated instead to consider her difficulties, by underlining that she should not be afraid of making mistakes. This was particularly felt as supporting: "*From that moment on, I began to improve in math and to become fond of geometry*".

Although the consideration of classmates and parents also affect the reactions to and the consequences of success/failure events, the teacher is reported to be the main undisputed factor in the development of turning points. In particular, the importance that the teacher trusts the students' capacities emerges (Franpolla: "*During the High School my relationship with math changed thanks to my teacher. She believed in me and she allowed me to recover the confidence in my math abilities*"). Concerning this aspect, the story narrated by Magiusa is paradigmatic, in the negative: the perception of the teacher's *surrender* represents her turning point for the development of her negative relationship with mathematics: "*the teacher took cognizance of my* white flag*, factually legitimating it and compromising thus any possibility for having interest in the subject"*.

CONCLUSIONS

The relationship with mathematics that future primary teachers have developed during their experiences as students is often strongly negative. The risk is affecting the way pre-service teachers use professional development opportunities and also their future didactical choices when they will be teachers. Studying these relationships, their dynamics and developments seems to be important both as researchers and as teacher educators.

We strongly believe that in order to study these aspects it is crucial to focus on "the ways humans experience the world" (Connelly & Clandinin, 1990, p. 2): "listening" the voice of future teachers through a narrative inquiry to understand their purposes, reasons and actions.

This methodological choice is also an educational choice: asking future teachers to tell about their math story may also represent the early impulse for an in depth reflection about own past experiences and reasons that have affected the development of their personal relationship with math. It is interesting to notice that, in some cases, this impulse also represents a sort of math-therapy: at the end of her narrative (about a love-hate relationship with math), PisoloTo writes: "*I want to underline that telling my story with mathematics helped me a lot…I've never done it before*".

Moreover the methodological choice has influenced the quantity and quality of the data collected. Focusing in detail on the episodes reported as turning points, we were able to identify which factors were perceived as crucial in the development of these events. If it is true that turning points are mainly related to specific episodes of success/failure in mathematics, it is also true that teachers are often the principal actor of the narrated story: more importantly, most of the times, he/she also strongly affects "the end of the story".

Reflecting on their own experiences and confronting themselves with these results can be useful to future primary teachers to recover their personal relationship with mathematics, and also to become writers of "happy end" stories when they will be again in the classroom.

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"NOT TO LOSE THE CHAIN IN LEARNING MATHEMATICS": EXPERT TEACHING WITH VARIATION IN SHANGHAI

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This paper reports on an expert teacher's ideas and practice of teaching with variation that underpin her guidance of junior teachers in lesson design study in a research project in Shanghai, China. The data we analysed included the teacher's lesson plan, teaching references, classroom materials, together with the video of the lesson and its transcript. Using the framework of teaching with variation, we identified four types of variation: task variation, example variation, calculation method variation, and exercises variation. The findings help towards a deeper understanding of the complexity of teaching expertise valued in the Chinese mathematics classroom.

RATIONALE AND STUDY BACKGROUD

With an ongoing research focus on identifying effective ways of enhancing mathematics teaching within our teacher professional development (TPD) project in Shanghai (SH), China (see Ding et al., 2014), this paper focuses on the pedagogical ideas and practices of an expert teacher of mathematics in the local school context of Shanghai. Our rationale is that, as Li and Kaiser (2012) point out, understanding the conception and nature of teacher expertise in mathematics instruction remains quite limited. Our study aims to contribute better understanding the complexity of teaching expertise valued in the Chinese classroom context (Li, Huang, & Yang, 2011).

It has been known for some time that pedagogical approaches that limit learners to rote learning and that accentuate instrumental understanding have relatively poor long term effects, with learners not being able to apply knowledge to new situations nor in their everyday life as citizens (e.g., Skemp, 1976). To understand the complexity of teaching expertise valued in the Chinese classroom context, we focus on two aspects of the practice of an SH expert teacher of mathematics: one is the relationship between the teacher's leading teaching role and students' active learning role; the other is the relationship between conceptual and procedural knowledge in mathematics and in students' learning .

Teaching with variation (briefly called 'TwithV' in this paper) has long been widely practiced by mathematics teachers in China. Perhaps as a consequence, different notions of variation have been identified as characterizing the features of TwithV. For instance, Gu (1981) showed aspects of figural variation in teaching and learning mathematical concepts, developing independent thinking skills in problem solving and in establishing knowledge systems in geometry. Huang, Mok, and Leung (2006) identified classroom practice in SH in terms of implicit variation; that is, where the changes from the origins to their variations "have to be discerned by abstract and logical analysis by learners ...so that the conditions or strategies for applying relevant knowledge are implicit and not obvious" (p.265). Sun (2011) characterised the variation of problems in terms of "one problem multiple solution" and "one problem multiple changes" (p.65). Li, Peng, and Song (2011) identify that teaching algebra with variation involves "aspects of orientation of variation, types of variation, levels of variation, and variation exploration" (p.546).

To date, researchers have been largely engaged in tackling two key questions in studying TwithV in Chinese mathematics teaching. The first question is 'why an emphasis on variation should be made in mathematics teaching and learning'; the other question is 'how to design such variation for the effective teaching and learning of mathematics'. To the first question, Sun (2007) points out that TwithV enables students to appreciate the abstract nature of 'invariable in variation" of mathematical laws and to develop insight into the mathematical system by the idea of "applying the invariant concepts to the varied situations" (p.16, translated by the first author). The second question remains a challenge in mathematics education research and is our focus in this paper. We use the work of Gu (1981, 1994, 2014) as the theoretical framework within which we address our research question: how does an expert teacher help Grade 2 students to establish the internal relationship of new concepts and methods with previous ones by TwithV in a lesson on division with remainder?

THEORETICAL FRAMEWORK

In the '*Qingpu experiment study*' (a project led by Gu, in collaboration with a number of teachers and researchers, from 1977 to 1994 that focused on improving the effectiveness of teaching and learning of mathematics in Qingpu district, SH), Gu (1994) found that the most effective teachers were able to deliberately arrange what we might call multiple layers of teaching and learning. Here, the multiple layers refer to the *Xun Xu Jian Jin* principle of Confucius; that is, to make progress by following foundational principles such as the development of understanding from shallow to deep, the subject content from easy to difficult, the learning from simple to complicated, and the practice from single to complex tasks.

Based on this, Gu *et al*. (2004) identify and illustrate two forms of TwithV, namely *conceptual variation* (*CV*) and *procedural variation* (*PV*). Within *CV*, there are two means of variations: (1) concept variation (e.g., varying connotation of a concept); (2) non-concept variation (e.g., giving counterexamples). Thus, *CV* emphasizes understanding concepts from multiple perspectives. In contrast, *PV* highlights a hierarchical system in unfolding mathematics activities (e.g., different steps to arrive at a solution or different strategies to solve problems). In this paper we aim to develop a deeper insight into the use of *PV* by an SH expert teacher.

Here we note that in his most recent writing, Gu (2014) further explains that it is *PV* that plays a key role as *Pu Dian* (铺垫); that is, in setting up a proper distance between previous and new knowledge in students' learning. Akin to the notion of 'scaffolding', *Pu Dian* means to build up one or several layers so as to enable learners to complete tasks that they cannot complete independently. In this paper we aim, in particular, to develop a deeper understanding of how *PV* creates the 'proper distance' for all students in learning in the Chinese classroom context.

METHOD

Our lesson design study is being conducted through a school-based TPD in a local laboratory school located in the western suburb of SH. The overall approach is a form of the Action Education (AE) model developed by Gu and Wang (2003) (for more on our use of the AE model, see Ding et al., 2014). During the process of supporting one of the case teachers to redesign and re-implement her lesson plan according to TwithV (for the details of our study cycle see Ding et al., 2014), we noted that in her teaching, expert teacher Mei constantly addressed the idea "not to lose the chain in mathematics learning". As Mei gave an open lesson on the same mathematics topic and was video-recorded for both her school and her school district key junior teachers (those teachers considered as potentially effective young teachers by their schools) as part of the school-based TPD activity in 2010, it became interesting and possible for us to examine Mei's idea in her own teaching practice. The term 'expert teacher' in our study recognizes that Mei is not only an effective teacher in subject teaching, but that she also plays the multiple roles that are described by Yang (2014, p.271-2).

The data we present in this paper includes Mei's own lesson plan, teaching references and learning materials of the lesson (e.g., the textbook, worksheet), and the lesson video and transcript. Mei has over 30 years teaching experiences in elementary mathematics teaching in her school district. She has taken the leadership of the in-service elementary mathematics teachers TPD program at her school district level since 2009. Her school is a public school, and the school size in 2010 was about 2500 students (from grade 1 to grade 9), 56 classes and about 200 teachers (all subjects).

The class in this lesson was Grade 2 (students age 7-8 years old). The length of the lesson was 35 minutes. There were 44 students in the class. 25 of them were boys and the rest were girls. According to the school's regular learning assessment in mathematics, more than 80% of the students in this class were excellent at mathematics, 15% of them were good at mathematics and the last 5% also passed in all school tests. This means that there was not a student who was really weak in mathematics in this class. The lesson topic was division with remainder, which has remained one of the key and the most difficult topics in the SH reformed elementary mathematics curriculum.

The data was analysed through three main stages: (1) Mei's lesson plan, the textbook and teaching references and the video transcripts were carefully studied and key codes for analysing the lesson were developed as follows: the lesson structure (e.g., introduction activity, the main activity and exercise activity), the learning goals of the lesson (e.g., the key points and the difficult points of the lesson), teaching strategies (e.g., questioning, using concrete materials such as drawing or pictures, hands on experiments, and use of multiplication table), classroom interactions (e.g., teacher-whole class, teacher-individual, students in pairs). (2) We also developed codes to analyse the teaching tasks (e.g. solving problems, division operational procedure) and teaching strategies of the TwithV largely according to Gu et al. (2004). (3) We also referred to what Mei analysed of her lesson according to her instructional intention of TwithV that Gu has not yet sufficiently explained (e.g., the variation of tasks to tackle the individual differences in the class).

DATA ANALYSIS

We analysed Mei's TwithV in the observed lesson according to two key points of learning goals in Mei's lesson plan (see the left column in Table 1). Noticeably, Mei considered that the difficult point of students' learning was to correctly use the method of 'trying quotient' by the multiplication table (briefly called MT in this paper) in the operation of division with remainder. In the first place, we use Table 1 to outline the main lesson structure that focused on the two key points and the difficult point of learning. Then we focus on analysing of Mei's TwithV in relation to these points.

1. The observed lesson

Lesson structure & learning goals	Key teaching tasks	Examples of the task outcome
1. Learning goal in teaching: To know the new concept of "division with remainder" and to develop an understanding of the fact that 'a remainder is always smaller than a divisor'.	*Task 1*. A problem of sharing 12 peaches by 3 monkeys. *Task 2*. (1) Sharing 13 peaches by 3 monkeys. (2) Sharing 14 peaches by 3 monkeys. (3) Sharing 15 peaches by 3 monkeys. *Task 3*. (1) Sharing 17 strawberries by 4 friends. (2) Sharing 17 strawberries by 6 friends.	*Task 1*. 12÷3=4 *Task 2*. (1) 13÷3=4……1 (2) 14÷3=4……2 (3) 15÷3=5 or 15÷3=4……3? *Task 3*. (1) 17÷4=3……5 (2) 17÷6=2……5
2. Learning goal: To learn to correctly calculate the division with remainder when both divisor and remainder are one digital.	Task 4. Sharing 11 oranges by 4 friends, with the help of a picture. Task 5. Representing thinking method of how to operate '11÷3=?' (to use students' hands to respectively represent the quotient and remainder).	Task 4. 11÷4=2……3 Task 5. 11÷3=3……2 Task 6. 11÷5=2……1 Task 7. 11÷6=1……5

	Task 6. Exchanging ideas and representation with neighbour student of the operation of '11÷5=?'. Task 7. Without a picture, explaining the method of the operation of '11÷6=?'.	
3. Learning goal: To correctly use the method of 'trying quotient'.	*Task 8*. 31÷5=( ) …… ( ). To think: 31－ _ =_.	31÷5=(6) …… (1). To think: 31－30 =1.

Table 1: The lesson structure, learning goals and key teaching tasks

2. Teaching with variation

(1) The introduction of the concept of division with remainder

Teaching episode one: Task variation to generate conflict and interest in learning new concept.

In *Task 1* (see Table 1), we note that students already learned to use the division method to solve a problem in a situation involving the concepts such as 'equal' and 'sharing'. They were also able to correctly use the MT to get the quotient 4 in the division; that is, the previously learned procedural operation of division in relation with the MT, together with the concepts like 'equal', 'sharing', 'dividend', 'divisor' and 'quotient', are the "anchoring part of knowledge" (Gu et al., 2004, p.325) for students to be able actively to explore new knowledge/problem. The interaction between Mei and the class below shows that Mei deliberately helped students to establish such knowledge anchor for the new learning in *Task 1*.

Teacher (T): (asked the class) What do the numbers 12, 3, and 4 respectively mean?

Student1 (S1): (one student was invited to give his answer.) 12 means 12 peaches. 3 means 3 monkeys. 4 means each monkey got 4 peaches.

T: Very good. But how did you get the quotient 4? Why did you think so? (another student was invited to give the answer.)

S2: I used the statement (a brief way used by the class to mean the multiplication table), that is, three four is twelve (3×4=12).

T: Very good. Why did you think of this statement? What did you refer to?

S2: Because the divisor is 3, I therefore thought about the statement of 3. And, the dividend is 12, so I thought that three four is twelve.

The interaction above also shows how Mei helped students to develop relational understanding (Skemp, 1976) by making connections between division and multiplication explicit. Emphasizing that 12 ÷ 4 = 3 because 3 times 4 is 12 can be seen both as an essential part of a relational understanding of division and multiplication and as an essential part of the *PV*. By starting with a division task that students were already familiar with, Mei also deliberately enable students to develop their autonomy and to generate new interest in learning.

In *Task 2* (see Table 1), Mei carefully varied the dividend (added one more peach to the 12 peaches in *Task 1*), while the 3 monkeys (the divisor) and the question of sharing were kept unvaried. Such a task variation recognized students' early learning experience and enabled them to develop learning autonomy in new problem situation. Noticeably, Mei also used a set of same questions to facilitate such autonomy during the process of solving *Task 2* (1). For instance, "*what does 13 mean here? What do 3, 4 mean then? What does 1 mean? Why did you not divide this one peach?*".

Teaching episode two: Example variation to deepen understanding of new concept

To define the connotation of the concept and further understand the concept of 'remainder' and its relationship with divisor, Mei deliberately applied the non-concept variation of the *CV* (Gu *et al*., 2004, p. 318) in *Task 2*(3) (see Table 1). Firstly, Mei challenged students by a non-concept example '15÷3=4……3'. By comparing it with '15÷3=5', students were able to discern the fact that "a remainder should be smaller than a divisor".

Next, in *Task 3* (see Table 1), Mei purposefully requested students to explain why the remainder 5 is incorrect in '17÷4=3……5', while it is correct in '17÷6=2……5'. It appears that the teacher's leading role in varying examples and questions is necessary here as it is not natural for young students to make it explicit of their thinking process of the fact that 'a remainder is ALWAYS smaller than a divisor' automatically establish on their own.

(2) Develop mathematical thinking through the calculations

Teaching episode three: Calculation method variation to experience the process of mathematisation.

During the next four tasks (see Table 1), Mei deliberately varied the calculation methods (from the concrete (e.g., *Task 1*), the semi-concrete (*Task 4&5*), the semi-abstract (*Task 6*), to the abstract (*Task 7*); Gu et al. 2004, p. 330) to enable students to gradually experience the process of mathematisation. In such a process of accumulation, Mei constantly used the same set of questions such as "*what is the quotient? What is the remainder? What statement [of the MT] do you think? What do you think firstly [dividend or divisor]? How do you find this statement? How did you get the remainder? etc.*". Mei considered that the teacher's leading role is important to make individual students' implicit inner thinking process explicit and to enable different students at various levels of understanding to communicate of their thoughts and to learn from each other in the class.

(3) Improve calculation skill by using the method of 'trying quotient' by the MT

Teaching episode four: Exercises variation to improve mental calculation skill.

Mei also set up multiple layers of classroom exercises. In this paper, we focus on an analysis on one of the exercises, Task 8 (see Table 1). Here, Mei considered that the form "31－ _ =_" was essential to enable those students who had difficulty in making a direct shift from their early learned procedure of division without remainder to the newly-learned procedure of division with remainder. The "31－ _ =_" form can be considered as a stepping stone set up by Mei in establishing the 'proper distance' for those students in active learning. By thinking of "31－ _ =_", what became visible to these students was the intricate relationship amongst the dividend (31), the outcome of the right statement of the multiplication table (here, it meant 30 as 5×6=30), and the remainder (1 for 31-30=1).

DISCUSSION AND CONCLUSION

In this paper, we identified four types of variation underlying the expert teacher Mei's teaching idea to 'not to lose the chain in learning mathematics'. These are task variation, example variation, calculation method variation and exercises variation. In the first place, the example variation is the *CV* (Gu et al., 2004) for deepening students' understanding of the concept of 'remainder in division' and the fact 'a remainder is *always* smaller than a divisor'. Next, the task, calculation method and exercises variation consists of the multiple layers of *PV* (Gu et al., 2004).

We identified three layers that Mei carefully set up to create the 'proper distances' in learning for understanding the internal relationship of a new concept and method with previous ones through these three types of variation in *PV*. The first layer was to use task variation (e.g., see the variation from *Task 1* to *Task 2*) to enable students to develop learning autonomy in using the same method of the MT in trying the quotient in division with remainder. The second layer was to use the calculation method variation through four similar tasks (from *Task 4* to *Task 7*) to enable students not only to make a shift from physical objects to arithmetic forms, but also to make their individual inner thinking process explicit to their classmates in the class. The third layer was to set up the exercise variation to tackle students' learning difficulty in making a shift from their early learned procedure of division without remainder to the newly learned procedure of division with remainder.

Findings of our study highlight that the teacher's leading role is essential not only in engaging students in effective learning, but also developing their learning autonomy and motivation. As illustrated by the data analysis above, Mei not only used the four types of variation to create the proper learning distances for students to develop understanding of the new concept, fact and calculation procedure, but also skilfully used questioning strategies to create various kinds of classroom learning space (e.g., learning by individuals, learning between students, and learning between the teacher and the whole class). This use of four types of variation relates to Li *et al*. (2011) analysis of three broad categories of teaching expertise in the mathematics classroom context in China, namely teacher knowledge for teaching, mathematics-specific instruction and student-oriented approaches (p.190). As such, this paper is offered as a contribution towards developing a deeper understanding of the complexity of teaching expertise valued in the Chinese mathematics classroom. An important aspect for the teacher is 'not to lose the chain in learning mathematics'.

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A CASE STUDY OF TEACHER QUESTIONING STRATEGIES IN MATHEMATICS CLASSROOMS IN CHINA AND AUSTRALIA

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This study was designed to extend the understanding of teachers' questioning practices in classrooms through a fine-grained analysis of mathematics lessons taught by two competent junior secondary teachers separately from mainland China and Australia. A comprehensive coding system was developed to analyse what kinds of verbal questions were initiated by the teachers to elicit mathematical information and in what ways the teachers took students' verbal contributions into consideration so as to promote the construction and acquisition of mathematical knowledge. The process of coding teacher questions in cross-cultural settings has provided particular insights and these will be summarised and discussed.

INTRODUCTION

Question asking is one of the most common strategies used by teachers in their classroom instructional. Many attempts have been made to categorise teacher questions in classroom practices and to present teachers' skilful questioning strategies, which highlight the context where the questions are asked, the appropriate use of different types of questions, the learning opportunities created in the sequences of teacher-student interactions (Hiebert & Wearne, 1993; Boaler & Brodie, 2004; Kawanaka & Stigler, 1999). Meanwhile, instead of focusing only on the isolated "good questions", researchers are more aware of the significance of "good questioning" practices in mathematics classrooms, which involves the use of good questions as part of good questioning practices by teachers (Aizikovitsh-Udi, Clarke, & Star, 2013).

Nonetheless, there is still a lack of investigation on teacher questioning strategies by taking full account of both the descriptions of the types of teacher questions and the analysis of the teacher's strategies in terms of arranging these different types of questions together to fulfil pedagogical purposes. For one thing, some of the studies mentioned above (i.e., Sahin & Kulm, 2008) focused mainly on several particular types of teacher questions without providing a complete description of those mathematical questions asked by the teacher in the classroom practices (Stolk, 2013). For another, some studies mainly focused on the macro analysis of the sequences of teacher-student interactions, without a fine-grained exploration of what types of questions constitute these sequences (i.e., Franke et al., 2007). This is significant particularly when considering that a fine-grained analysis of teaching practices is necessary to reveal the complexity of mathematics teaching and to generate usable knowledge for teaching (Kazemi, 2008). Besides, few studies in mathematics classrooms have considered the distinctions between *initiation questions* which are those questions asked by teachers for initiating purposes (such as to start conversation or discussion), and *follow-up*

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 217-224. Hobart, Australia: PME.

questions which refer to those questions asked for following-up purposes (e.g., in response to students' answers to teachers' previous questions) (Oliveira, 2010). This distinction is significant for the researchers and practitioners with regard to reflecting on and improving mathematics teaching practices (Franke et al., 2009; Peterson & Leatham, 2009).

With an attempt to bridge the gaps mentioned above, this study intended to develop a comprehensive framework with regard to teacher questioning and thereby to analyse what kinds of verbal questions and prompts were initiated by the teachers to elicit mathematical information and in what ways the teachers took students' verbal contributions into consideration so as to facilitate students' construction, acquisition and articulation of mathematical knowledge.

METHODOLOGY

The case study design allows the exploration of complex phenomena within their contexts (Baxter & Jack, 2008) and the detailed case studies of teaching practices cases could also help researchers and practitioners to interpret and critically reflect on a teacher's actions and interactions in the classroom, and to consider the different courses of actions open to the teacher (Smith & Friel, 2008, pp. 2; Stein, Smith, Henningsen, & Silver, 2009, pp.147). Furthermore, the depth of teacher questioning strategies could be revealed more clearly through exploring the cases from different cultural settings (Kawanaka & Stigler, 1999; Koizumi, 2013).

While the IRF (Initiation-Response-Follow up) structure (Cazden, 2001) has been criticised as limiting the potential of teacher-pupil dialogue in promoting pupils' conceptual learning in mathematics classrooms (i.e., Kyriacou & Issitt, 2007), more and more researchers have pointed out the effects of the IRF structure depend on the way in which the teacher implements this structure in classrooms. For example, Nathan and Kim (2009) and Franke, et al. (2009) both claimed that the IRF structure could be used to facilitate students' articulation of mathematical thinking and to engage students in sophisticated reasoning.

Considering the above analysis, a cross-cultural lens was adopted in the present study and the IRF structure was taken into account when analysing teachers' discourse process of how the teacher initiates questions and builds up on student responses.

SETTING AND PARTICIPANTS

Data were from video-recorded observations of two junior secondary mathematics teachers' lessons. The participants are separately from the city of Nantong in China and Melbourne in Australia. Both teachers are recognised as competent according to local criteria and a whole unit of consecutive lessons were recorded for each. The study presents the analysis of the first three lessons by the Chinese teacher and the first two lessons by the Australian teacher. The details of the lessons are in Table 1.

Teacher	Year	Lesson content	Time

AUS	9	***Lesson1*** Re-introducing the Pythagoras' Theorem and its applications in triangles	103mins
		Lesson2 Application of the Pythagoras' Theorem in the Cartesian plane and in real life situations	36mins
CHN	8	***Lesson1*** An introduction to quadratic functions	45mins
		Lesson2 Investigating the graph of $y=ax^2$	45mins
		Lesson3 Investigating the graph $y=a(x\text{-}h)^2+k$	45mins

Table 2: Lesson Topics Delivered by the Two Participating Teachers

DATA ANALYSIS

The term "question" refers to what the teacher says to elicit students' verbal responses related to mathematical content. Three types of occasions were identified when the teachers interacted with students by using questions. Interactions in which the students reply to teacher questions and then the teacher does not have verbal response are categorised as Question-Answer (Q&A) pairs. In addition, two types of IRF sequences were identified: (1) IRF (single) in which the teacher asks a question and then gives closed follow-up moves (e.g., evaluation) so as to accomplish the current discussion, and (2) IRF (multiple) in which the teacher asks a question and then gives open follow-up moves (e.g., clarification or elaboration) to continue the current discussion.

When analysing teacher questions, a distinction was highlighted between initiation questions and follow-up questions. Initiation questions are those questions asked by teachers for initiating purposes, such as to start conversation or discussion. By contrast, follow-up questions are those questions asked, for example in response to students' answers or contributions to teachers' previous questions. In this study, the Q&A pair and IRF (single) sequence contain teacher initiation questions and the IRF (multiple) sequence includes teacher initiation and teacher follow-up questions.

A coding system was developed to categorise the initiation questions and follow-up questions. Instead of inventing the name of each category in advance, those questions documented in our data were analysed first and then attempts were made to provide names to describe these different kinds of questions. The development of the coding system in this study was informed by some previous researchers (Boaler & Brodie, 2004; Hunkins, 1995; Hiebert & Wearne, 1993; Oliveira, 2010). The coding systems are presented in Table 2 and Table 3 in which the examples are shown in italics.

Category	Description & Example
Understanding check	Questions used to check whether students can follow the teacher. "Is everyone OK with how I get from the 2nd line to the 3rd line?"
Evaluation	Questioning requiring students' comments. "Now let's look at these two descriptions, which one do you prefer to agree with?"
Review	Questions used to elicit the previously learnt or mentioned

	mathematics knowledge. "Now what do I know about squares in their area?"
Information extraction	Questions requiring students to identify and select information from text descriptions, graphs, tables, or diagrams. "What is (b), what's the mathematical word for what (b) is asking you to find?"
Link/ application	Questions requiring students to provide examples or application of mathematical knowledge. "Could you list some examples?"
Result/ product	Questions requiring results of mathematical operations or the final answer of the problem solving. "What is the square root of 80?"
Strategy/ procedure	Questions used to elicit the procedures or strategies of problem solving. "How can we solve this problem?"
Explanation	Questions requiring students to provide explanations "How would it be interpreted from the perspective of a function?"
Progress Monitoring	Questions requiring regulation of the process of reasoning or problem solving. "But have you answered the question?"
Comparison	Questions requiring the comparison. "Is this different from the previous questions?"
Reflection	Questions requiring the reflection after mathematical activities. "What mathematics have we already used in solving triangles?"
Variation	Questions requiring students to consider the variations of mathematical tasks. "So what if I got a hundred and twenty seven in that answer?"
Generation	The teacher requires students to generate a problem, or components of a problem, to fit given constraints. "I need *a* and *b*, so give me numbers."

Table 3: Sub-categories for initiation questions

Category	Description & Example
Clarification	Questions requiring a student to show more details about his/her answers or solutions. "How did you get this 16?"
Justification	Questions requiring students to justify their answers. "Why did you choose this method to solve this problem?"
Reformulation request	Questions requiring students to reformulate his or her answer, especially when the teacher asks a question to a whole class and a couple of answers are given by the students. "Say that again."
Elaboration	Questions requiring for additional information especially when the students fail to fully achieve the teacher's goals. "In other words, the green line becomes the what?"
Extension	Questions used to extend the topics under discussion to other situations or to connect the knowledge under discussion with the previous

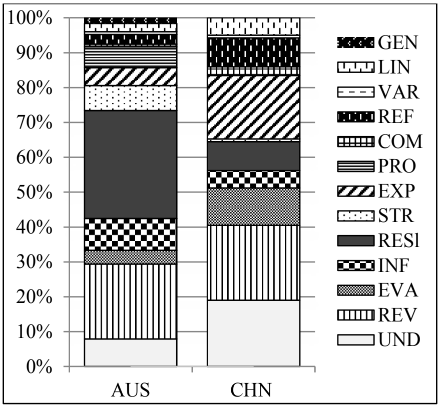
	knowledge. "Would this work with other numbers?"
Supplement	Questions used to request for supplement. "Did anyone do this problem in a different way?"
Cueing	Questions used to direct students to focus on key elements or aspects of situation in order to enable problem-solving. "What is the problem asking you to find?"
Refocusing	Questions used to guides students to refocus on the key points, especially when students are off the right track. "But what was the question, if this was a textbook question, what would it look like?"
Repeat/ rephrase	The teacher repeats or rephrases the question
Agreement request	Questions used to check whether the rest of the class agrees with the student who gives the answer. "So would you agree that the height of this one is going to be a hundred and forty nine?"

Table 4: Sub-categories for follow-up questions

FINDINGS

The coding systems presented above were applied to analyse the selected lessons taught by the two teachers. In total, 252 initiation questions and 157 follow-up questions were asked by the Australian teacher in two lessons of 139 minutes altogether, and 121 initiation questions and 116 questions by the Chinese teacher in three lessons of 135minutes altogether. On the average, the Australian teacher asked around 2.9 (409/139) questions per minute, whereas the Chinese teacher raised approximately 1.8 (237/135) questions per minute. Compared with the Chinese teacher, one more question was asked in per by his Australian counterpart. This reflects that the Australian teacher spent more time on interacting with students than the Chinese teacher. Nevertheless, the Chinese teacher tended to ask follow-up questions more frequently after raising the initiation questions. For almost every initiation question, the Chinese teacher used one (116/121) follow-up question. By contrast, the Australian teacher asked only 0.6 (157/252) follow-up questions for every initiation question. In summary, the Australian teacher asked more questions altogether, but the Chinese teacher was more likely to ask follow-up questions after students responded.

The detailed information in terms of the breakdown of imitation questions and follow-up questions is shown in Figures 1 and 2. Although a relatively broad range of question types was identified, it is obvious that both their initiation questions and follow-up questions consist predominantly of a couple of question types. For the Australian teacher, the majority of initiation questions comprises those requiring understanding check, review and result, whereas questions asked for understanding check, review and explanation constitute the main body of the initiation questions raised by the Chinese teacher. In terms of the follow-up questions, the Australian teachers mainly asked students for the purpose of clarification, justification, cueing and repetition. By contrast, the Chinese teacher employed follow-up questions with the intention of clarification, elaboration, cueing, and agreement request.



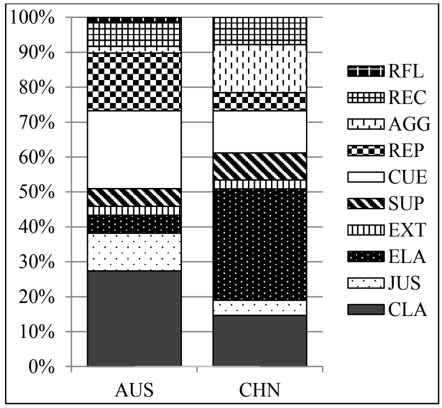


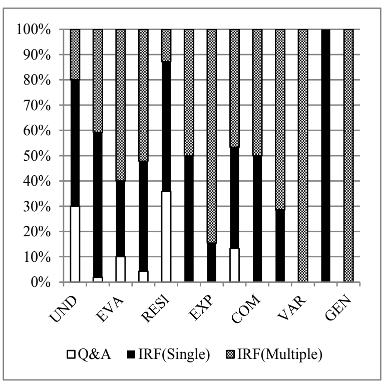
Figure 1: Initiation questions

Figure 2: Follow-up questions

The meaning of the abbreviations in Figure 1 and Figure 2 are listed below:

Initiation questions: UND=Understanding Check; REV=Review; EVA=Evaluation; INF=Information extraction; RSL=Result; STG=Strategy; EXP=Explanation; PRO=Progress monitoring; COM=Comparison; REF=Reflection; VAR=Variation; LIN=Link; GEN=Generation.

Follow-up questions: CLA=Clarification; JUS=Justification; ELA=Elaboration; EXT=Extension; SUP=Supplement; CUE=Cueing; REP=Repeat; AGG=Agreement request; REC=Refocusing; RFL=Reformulation request



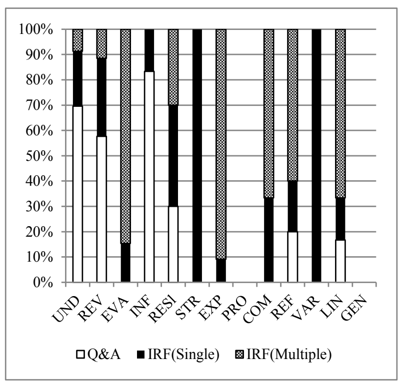


Figure 3: The AUS teacher

Figure 4: The CHN teacher

Note: Please refer to Figure 1 and 2 for the meaning of the abbreviations

In order to reveal what types of initiation questions were more frequently accompanied with follow-up questions, Figures 3 and 4 show the proportions of the three occasions (Q&A, IRF-single, and IRF-multiple) where the initiation questions were asked.

It suggests that after the two teachers asked questions for evaluation, explanation, and reflection, they both tended to build on students' responses with follow-up questions, which thereby resulted in longer sequences of teacher-student interaction. Apart from these three commonly used question types, the Australian teacher also asked questions for information extraction, variation and generation before the follow-up questions, whereas the Chinese teacher chose to ask more follow-up questions after the questions for comparison and link/application.

CONCLUSION

The coding system developed in this research covers and identifies a large range of question types used by the two participating teachers. By distinguishing the different roles of initiation questions and follow-up questions, this research has been able to reveal the complex nature of teacher questioning practices in mathematics classrooms. Furthermore, by analysing the IRF structure occurred in teacher-student interaction, this research also reveals the circumstances where the two teachers tended to continue the interaction by asking follow-up questions. This research provides researchers and practitioners with a new perspective to interpret and analyse teacher questioning practices in mathematics classrooms. It also has potential implications for teachers' professional development by allowing the teachers to employ the coding system when reflecting on their own practices and analysing other experienced teachers' practice.

It is unlikely that the analysis of two teachers could be claimed as sufficient for the understanding of the complex questioning practices in mathematics classrooms. Yet, it is doubtless that cases of this sort could help us to develop the theory and methods necessary to study questioning practices in greater depth (Nathan & Kim, 2009). This study is a part of the authors' work on teacher questioning and the work is still ongoing. The coding framework will be used to assist the exploration of more issues, such as students' opportunities created in the questioning sequences.

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LINKS BETWEEN MULTIPLICATIVE STRUCTURES AND THE DEVELOPMENT OF MULTIPLICATIVE THINKING

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This paper presents the findings of one aspect of a study that investigated Grade 3 students' development of multiplicative thinking. Of particular interest was the extent to which students could successfully perform on tasks relating to different semantic structures and the influence of each structure on their strategy choice. The findings suggest that Grade 3 students are capable of solving problems relating to the different semantic structures and many do so using multiplicative strategies.

INTRODUCTION

A growing concern among teachers in the middle school (Grades 6-8) is the increasing number of students who rely on additive thinking to solve proportional reasoning problems when multiplicative thinking is required, or cannot distinguish between when a task requires additively thinking or multiplicative thinking. This may be attributed to an emphasis in the junior elementary grades on multiplication as repeated addition, equal groups and arrays and an expectation on teachers to model multiplicative situations using repeated addition. This paper presents evidence to suggest that students in elementary grades need experiences with different multiplicative situations to make the transition from additive to multiplicative thinking.

THEORETICAL FRAMEWORK

Multiplicative thinking is the basis of proportional reasoning, and a necessary pre-requisite for understanding algebra, ratio and rate, interpreting statistical and probability situations, and understanding and reading scale (Lamon, 1993; Siemon, Breed, & Virgona, 2005; Singh, 2000). Lamon (1993) purports that to "understand the functional and scalar relationships inherent in a proportion" (p. 58) students need to comprehend the multiplicative nature of situations involving ratio and proportion, and see the need to make relative comparisons. This supports the view that the development of ratio and proportion concepts is embedded within the development of the multiplicative conceptual fields (Greer, 1988; Vergnaud, 1988).

Numerous studies indicate that students' difficulties in solving problems involving fractions, decimals, ratio and proportion are attributed to a reliance on additive reasoning when multiplicative reasoning is required (e.g., Hart, 1988; Singh, 2000; Van Doreen, De Bock, & Verschaffel, 2010). Hart (1988) found this was particularly evident for enlargement questions which involved ratios other than n: 1 or n: 2. For example, in solving the problem, "The dimensions of a rectangle are 6cm long and 8cm wide, which when stretched gave a width of 15cm. What is the length of the other side?" (Singh, 2000, p. 287) a student found the difference between 8 and 15 (7) and

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added this to 6 and said it was 13cm. It appears that because the multiplication relationship was not understood the student looked for any relationship that matched, which in this case was additive (Singh, 2000).

Specific studies argue that the development of multiplicative thinking is more conceptually demanding than additive thinking (e.g., Clark & Kamii, 1996; Steffe, 1994) due to the levels of abstraction required. Steffe (1994) describes the demands of multiplicative thinking in the following way:

> For a situation to be established as multiplicative, it is necessary at least to co-ordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit. (p. 19)

To do this requires a level of abstraction and inclusive relationships that are not required in additive thinking (Clark & Kamii, 1996). In additive thinking there is just one level of abstraction whereas multiplicative thinking requires a double, or nested, level of abstraction; one must be able to think of both the numbers of objects in each group and the number of groups simultaneously. Singh (2000) found that when students move from additive to multiplicative thinking with whole numbers, two important changes occur, the first being a shift from "operating with singleton units to coordinating composite units" (p. 273) and the second being a change in the meaning given to a number.

A consistent theme in the literature is that students need experiences with different multiplicative situations such as Rectangular Array/Area, Cartesian Product, Product of Measures, Multiplicative Comparisons, Rate, to support their development of multiplicative thinking (e.g., Anghileri, 1989; Greer, 1994; Lamon, 1993). Specific studies identified subtleties and differences inherent in the structures such as the role of the numbers, the mathematical structure, and the visual models, to highlight the different thinking processes involved (e.g., Greer, 1992;Vergnaud, 1988). For example, Allocation/Rate involves a many-to-one correspondence in which equal sets of objects are matched with a tally set. Sophian and Madriad, (2003) suggested in situations such as these the multiplier is a rate variable that stipulates the mapping relation between an individual target object and its multiple counterparts (e.g., four cookies per child), whereas the multiplicand specifies the number of target objects and hence the number of iterations of that mapping relation (e.g., 3 children, so 3 iterations of the 4 cookies). The Rectangular Array structure on the other hand, gives a visual representation of the mapping of two spaces into a third (Vergnaud, 1988) and assists students to develop a sense of the relationship between the numbers. It is argued that such a representation encourages students to develop their thinking about multiplication as a binary operation, and for making the mathematical property of commutativity, intuitively acceptable (Greer, 1992).

Multiplicative Comparison or Times-as-Many-as is considered to be a preliminary stage to ratio and has the advantage over the groups of aspect as it relates directly to the nature of multiplication (Anghileri, 1992; Greer, 1992). Nesher (1988) argued that multiplicative comparison problems are syntactically complicated. A characteristic of this kind of problem is the formulation of a mathematical function (y=f(x)), or as an expression: *x has n times-as-many-as y*. For example, “Dan has 5 marbles. Ruth has 4 times-as-many marbles as Dan. How many marbles does Ruth have?” (p. 22). Others argue that Cartesian Product is the most difficult (e.g., Nesher, 1988; Van Doreen et al., 2010). The reason is that is an implicit assumption not explicitly expressed in the text of the problem that must be taken into account when solving it. In the example, “Ruth has 4 skirts and 3 blouses. How many different combinations of skirts and blouses outfits can Ruth make?” (Nesher, p. 23) the assumption is that each skirt is cross-multiplied with each blouse to identify the number of outfits. The absence of this description can result in students considering additive combinations and not recognising it as a multiplicative situation. Greer (1992) maintained that while the distinctions between models of situations are important pedagogically, and provide an analytical framework for guiding research, one must be mindful that the way in which a situation is interpreted depends on a student’s perception of it. For example, mathematically identical problems from different semantic structures can induce dissimilar solution strategies (e.g., Anghileri, 1989; Greer, 1992). Greer (1988) also suggested the need to provide multi-step word problems, rather than single operation word problems, to push students to think more deeply about which operations to use and to move beyond superficial strategies. A common theme that emerged from the different studies was that multiples strategies were employed across the different problem types and that the size of the numbers and semantic structure influenced students’ strategy choice.

METHODOLOGY

This paper draws on one finding of a larger study of young children’s development of multiplicative thinking. The study involved Grade 3 students (aged eight and nine) in two elementary schools, one grade was part of a teaching experiment; the other was used as a control group (Comparison cohort). Thirteen students, representing a cross section of each grade, were selected according to their mathematical achievement. Four weeks after the teaching experiment the students were interviewed to gain insights into their understanding of and approaches to multiplicative problems.

The author developed a multiplication task-based interview, consisting of 15 tasks in the form of word problems across five semantic structures identified by Anghileri (1989) and Greer (1992): three Equal Groups tasks; four Allocation/Rate tasks; four Rectangular Array tasks; three Times-as-Many tasks; and one Cartesian Product task (a decision made following the trialling of the tasks). The Allocation/Rate tasks included two two-step tasks and two one-step tasks to gain a better sense of a student’s strategy choice. Each task consisted of three levels of difficulty (easy, medium, challenge). In some instances an extra challenge question was offered if the student appeared to find the challenge task relatively easy. The following examples of word problems for each semantic structure illustrate the contexts used and the different language demands required to interpret the problems.

Equal Groups: On the table are 8 boxes of crayons. There are 6 crayons in each box. How many crayons are there altogether?

Allocation/ Rate: How many wings does a butterfly have? How many antennae? How many wings and antennae would 9 butterflies have?

Rectangular Array: Here is a plan of my veranda (Outline of rectangle 9×7 and one 2 cm tile). How many tiles would cover the whole veranda?

Times-as-Many: The Phoenix scored 8 goals in a netball match. The Kestrals scored 16 times-as-many goals. How many goals did the Kestrals score?

Cartesian Product: At the ice-cream shop I can choose from 4 different flavours and 3 different size cones. How many different single flavoured ice-creams can I order?

Each interview was audio taped and took approximately 30 to 45 minutes, depending on the complexity of students' explanations. Problems were presented orally and students were encouraged to solve them mentally, however, paper and pencils were available for students to use at any time. Students were asked to explain their thinking, and if they could work the problem out a quicker way. Once a response and explanation was given, the student was asked to record a number sentence on paper. Their responses were recorded and any written responses retained.

Method of Analysis

The data were coded for two purposes, first to ascertain student performance and second to identify student approaches to multiplication tasks. As the researcher was interested in knowing both the approaches students used and components of the task that may influence their strategy choice, an extensive analysis was undertaken of each of these components (e.g., semantic structure, level of difficulty, number triples). While acknowledging that providing students with a choice contributed to the richness of the findings, it also added to the level of complexity both in the analysis and presentation of data. Students' strategies were coded according to the level of abstractness and degree of sophistication, informed by the categories of earlier studies (e.g., Mulligan & Mitchelmore, 1997; Sherin & Fuson, 2005). For the purpose of this paper, the term abstraction refers to a student's ability to solve a problem mentally without the use of any physical objects (including fingers), drawings or tally marks. The strategies chosen by the students were categorised in the following way. The first category Building Up is additive, whereas the others are considered multiplicative.

Building Up: Visualises the groups and the multiplication fact but relies on skip counting, or a combination of skip counting and doubling to calculate an answer.

Doubling/Halving: Derives solution using doubling or halving and estimation, attending to both the multiplier and multiplicand.

Multiplicative Calculation: Automatically recalls known multiplication facts, or derives easily known multiplication facts.

Holistic Thinking: Treats the numbers as wholes—partitions numbers using distributive property, chunking, and/or use of estimation.

RESULTS AND DISCUSSION

From the analysis of the data three findings were evident. First, a high percentage of accuracy was evident by students in both cohorts (Experimental cohort 98%, Comparison cohort 87%). Second, the high success rate (100% and 74% respectively) on the Times-as-Many tasks was unexpected, given this semantic structure is considered more difficult (Nesher, 1988) than the Equal Groups and Rectangular Array structures. Third, the high success rate on the Allocation/rate tasks (100% and 98%) suggests that Grade 3 students are capable of interpreting this semantic structure. Table 1 presents the frequency of correct responses by each cohort for each semantic structure across the different levels of difficulties. There were three tasks for both Equal Groups and Times-as-Many structures indicating a maximum of 39 correct responses for each, whereas Allocation/Rate and Rectangular Array structures each had four tasks indicating a maximum of 52 correct responses. The highlighted cells indicate tasks in which some students were unsuccessful. The final row of the table is an aggregate of correct responses of each cohort for each semantic structure.

	Experimental Cohort (n=13)					Comparison Cohort (n=13)				
Level of Diff	Easy	Med	Chall	ExChall	Total	Easy	Med	Chall	ExChall	Total
Equal Groups		7	21	11	39	4	21	13		38
Allocation/Rate		23	24	5	52	15	27	9		51
Rectangular Array		18	32	2	52	10	27	9	2	48
Times-as-Many		17	22		39	2	16	11		29
Cartesian Product	3	6	1		10	1	2	1		4
Total	3	71	100	18	192	32	93	43	2	170

Table 1: Frequency of correct responses across each semantic structure

A higher proportion of responses (61%) of the Experimental cohort across the fifteen tasks, were for the challenge and extra challenge levels of difficulty, than the Comparison cohort (26%). It is worth noting that some of these students who chose the challenging tasks and used sophisticated strategies were not higher performing students on the pre-test interviews.

	Experimental Cohort (n=13)		Comparison Cohort (n=13)	
	Additive	Multiplicative	Additive	Multiplicative
Equal Groups	20	19	28	10
Allocation/Rate	16	36	34	17
Rectangular Array	19	33	28	20
Times-as-Many	13	26	12	17
Cartesian Product	4	6	4	0
Total	72	120	106	64

Table 2: Frequency of strategy choice for each semantic structure for each cohort

Three findings are apparent from the data presented in Table 2. First, multiplicative strategies (doubling/halving, multiplicative calculation, holistic thinking) accounted for 63% (120 out of 192) responses of the Experimental cohort and 38% of the responses of the Comparison cohort. This suggests that students who consistently use these strategies are thinking multiplicatively rather than additively. It seems that offering students the opportunity to engage with a range of task types and number triples beyond what is commonly posed at this level prompts the use of multiplicative solution strategies. Second, a higher proportion of multiplicative strategies were evident for Allocation/rate (69%), Rectangular Array (63%) and Times-as-Many (66%) tasks for the Experimental cohort than the more familiar Equal Groups (49%) semantic structures. A similar pattern was evident for the Comparison cohort with a higher proportion of multiplicative strategies used for these structures (33%, 41%, 60% respectively) than for the Equal Groups (26%) sematic structure. These results are in stark contrast to what was anticipated, given the Allocation/rate and Times-as-Many semantic structures were less familiar to the students than Equal Groups and Rectangular Arrays. Six students solved the Cartesian Product task using a multiplicative strategy suggests that some students at this level can solve tasks of this nature with understanding. Third, the high proportion of multiplicative strategies overall for the Allocation/rate tasks was unexpected given the lack of familiarity and experience of the Comparison cohort with this structure and multi-step tasks. However, the use of the distributive property by some students for the challenge one-step and multi-step tasks is further evidence of the need to incorporate tasks such as these into regular classroom practice. The following responses for the Allocation/rate task, "How many wings and antennae would 9 butterflies have?" characterised the type of multiplicative thinking students used for the challenge tasks.

> Annie: I done [*sic*] 6 times 10 is 60 and took away 6 to get 54. So for the antennae I did 9 times 2 is 18 and for the wings I did 10 times 4 and took away 4 to get 36. Thirty-six and eighteen is fifty-four and that's the same as nine times six.
>
> Sean: I know 9 fours are 36 that's how many wings, and I halved it to get the feelers so that's 18. I added 36 and 18 to get 54.

Both students used known facts as a starting point. Annie used two solution strategies. First, she combined the wings and antennae and multiplied by 10 and subtracted 6 to compensate for rounding up to 10. Second, she used partial products and again rounded the 9 up to 10 and subtracted to compensate. Both methods reflect her understanding of the problem and use of multiplicative reasoning. Sean, who was in the Comparison cohort and had no experience with such tasks prior to the interview, used his knowledge of doubling and halving to find the number of antennae rather than doing two separate calculations. His solution strategy revealed his capacity to engage with two-step tasks such as this using a sophisticated strategy.

Fourth, Holistic Thinking was the preferred strategy for the Times-as-Many tasks (65% or 17 out of 26) of the Experimental cohort. Given that this aspect of multiplication is quite different from the other structures and some of the number triples were outside their range of experiences one might have expected students to choose a less sophisticated strategy. The following abridged excerpts from the interviews illustrate students' use of this strategy for Times-as-Many challenge task 13 "The Phoenix scored 8 goals in a netball match. The Kestrals scored 16 times as many goals. How many goals did the Kestrals score?"

> Miles: 128 goals. I split the 16 'cause I know 8 tens is 80 and 8 sixes are 48. Eighty and forty are 120 and 8 more is 128.

> Sharne: I can halve 16 to get 8. I know 8 eights are 64 and another 8 eights is 128, because 8 times 8 and 8 times 8 is the same as 16 times 8.

In this task, all three students partitioned the 16 into known facts as a starting point. Miles who was from the Comparison cohort partitioned the 16 into ten and six using his place value knowledge and operated on each separately. Sharne halved the multiplier and operated on each separately. Both students showed an understanding of the distributive property and used it to enable them to solve the problem mentally.

These four findings highlight the value of providing students with experiences of less familiar semantic structures, and reveal students' untapped mathematical capabilities. These findings suggest that enabling students to engage with complex tasks relating to different sematic structures prompts the use of more sophisticated strategies than may normally be the case. It could be argued that the complexity of the semantic structure and the number triples facilitated students' level of thinking.

CONCLUDING REMARKS

The findings of this study suggest that giving students opportunities to experience complex and less familiar semantic structures such as Times-as-Many and Cartesian Products, that require them to think more deeply, will encourage them to move beyond the need for models or a reliance on additive thinking to multiplicative thinking. The findings also indicate that not only can Grade 3 students engage with tasks across less familiar semantic structures such as Allocation/Rate and Times-as-Many, but do so using more sophisticated strategies that one might expect. As suggested by Greer (1988) providing multi-step word problems and less familiar situations push students to think more deeply about which operations to use and move beyond superficial strategies. Engaging students in a range of semantic structures also develops a deeper understanding of the nature of multiplication. This also indicates the importance for teachers of students as young as Grade 3 not to delay the development of multiplicative thinking by restricting students to the use of models that oversimplify multiplicative situations.

From the findings it is inconclusive as to whether the Cartesian Product semantic structure is too difficult for Grade 3 students. One might conjecture that with experience students may be capable of interpreting it using multiplicative thinking. Research into this conjecture would provide further debate as to whether this semantic structure supports young students' understanding of multiplication and of its place in the curriculum.

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PCK ABOUT USING MULTIPLE REPRESENTATIONS – AN ANALYSIS OF TASKS TEACHERS USE TO ASSESS STUDENTS' CONCEPTUAL UNDERSTANDING OF FRACTIONS

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Task-related research formats may afford highly practice-relevant insight into teachers' professional knowledge and views. This study explores what kinds of tasks in-service teachers think of when aiming to assess their students' conceptual understanding of fractions. In a top-down coding approach taking into account a sample of 87 teachers a particular focus was put on core aspects of fractions addressed by the teachers and on requirements regarding conversions of representations. Moreover, possible interrelations of such content domain-specific PCK with the teachers' pedagogical content views on teaching and learning mathematics were considered. The results indicate that most teachers focused only on a few core aspects of fractions and suggest interrelations with more global views.

INTRODUCTION

When mathematics teachers assess students' conceptual understanding, tasks play an important role. It may thus be assumed that the problems and questions that teachers choose to this end reflect their professional knowledge and views. In the case of conceptual understanding related to fractions, especially domain-specific aspects of pedagogical content knowledge (PCK) play a role, such as the emphasis teachers give to different aspects of fractions or PCK related to the use of representations and requirements connected to conversions of representations. Moreover, the problems and questions teachers choose can be expected to reflect also more general PCK: For instance, it is possible to spot whether teachers favor open format questions and thus are aware that such questions are more useful to find out what students think. These PCK components may also be interrelated with general views of the teachers: For instance, a cognitive constructivist orientation might support teachers to choose tasks which have a higher potential of showing many facets of a student's conceptual understanding. However, even though fractions are a key content area in the mathematics classroom, there are hardly any quantitative empirical studies with a specific focus as outlined above. Consequently, this study focuses on these aspects of professional knowledge and views.

THEORETICAL BACKGROUND

Among researchers and practitioners in mathematics education, it is widely acknowledged that fractions is one of the most problematic topics in school mathematics (e.g., Charalambous & Pitta-Pantazi, 2009; Niemi, 1996; Padberg, 2009). A main reason for learners' problems in understanding fractions was found to be the

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 233-240. Hobart, Australia: PME.

fact that the concept of fractions is very multi-faceted: Several authors (Ball, 1993; Charalambous & Pitta-Pantazi, 2009; Padberg, 2009) have described a number of different so-called core aspects encompassed in the concept of fractions. The list by Padberg (2009) includes the main aspects that are often emphasized: proportion (part-whole), measure, operator, ratio, quotient, solution of a linear equation, scale value (point on the number line) and quasi-cardinality. Essential for students to acquire an elaborated conceptual understanding of fractions is to see the concept of fractions from different perspectives and to integrate several of these core aspects in the sense of being able to interpret fractions according to different such aspects.

Moreover, different kinds of fraction representations typically emphasize different core aspects of the concept (e.g., Lamon, 2001). Hence, the development of a rich concept image is closely tied to competencies of making connections between different representations (Ainsworth, 2006; Dreher, Kuntze, & Winkel, 2014; Duval, 2006; Tall, 1988). As a framework for the idea of using multiple representations in the mathematics classroom we refer to Duval's (2006) theory of representation registers. He emphasized the predominant role of multiple representations of mathematical objects and their transformations for mathematical activities as well as for conceptual understanding. His theoretical framework centres around systems of representations that have rules for performing transformations of representations within the system without changing the mathematical object that is represented. He referred to these systems of representations by the notion of *representation registers* (Duval, 2006, p. 11). Moreover, by the term *conversion*, Duval (2006) referred to a transformation from one representation register to another. A transformation from the number line representation of a fraction to the pie chart representation would for instance be a conversion, since these representations belong to different registers, even though both of them are pictorial. Duval pointed out that conversions of representation are often crucial to gain insight into mathematical concepts and hence learners should be fostered in making connections and conversions between different representation registers. In particular, Duval argued that both directions of conversions of representations should be considered, since students often have problems in reversing conversions. Addressing both directions of conversions affords however insight into whether learners can carry out such conversions merely on a procedural level or whether they have a deeper conceptual understanding.

When accompanying the learning process of their students in the domain of fractions, teachers have to assess their conceptual understanding. For this purpose, they are very likely to use problems and questions which afford insight into their students' reasoning and thus the ways they understand fractions. For being able to assess the students' conceptual understanding and for detecting potential deficits or misconceptions, teachers need specific professional knowledge related to problems and questions they can use in such assessment situations. Against the background of the above reasoning, such tasks should focus on core aspects of the concept, in this case fractions, and they should encourage students to show whether they see several of these aspects.

Furthermore, conceptual understanding can be made visible by addressing different representation registers and conversions of representations, where in particular both directions should be taken into account. Beyond these characteristics of tasks tightly bound to the idea of using multiple representations, also other aspects of tasks may play a role for how deeply students' conceptual understanding of fractions can be assessed. For instance, open tasks (e.g., "Give at least three different pictorial representations of the fraction ¾") tend to afford deeper insight into learners' understanding than closed format tasks (e.g., Draw a number line and mark the fraction ¾"). Thus, not only domain-specific knowledge about what is required to understand the concept of fractions may be crystallized in the tasks teachers select to assess students' understanding, but also more global professional knowledge and views may play a role: In particular cognitive constructivist versus direct transmission views on teaching and learning which were found to influence students' achievement gains in mathematics (Staub & Stern, 2002), may also influence what kind of tasks teachers see as being appropriate to assess learners' conceptual understanding. A cognitive constructivist point of view may for instance better facilitate using open question formats to give the students an active role and freedom in presenting their knowledge instead of expecting them to reproduce a certain solution that was presented by the teacher, which would reflect rather a direct transmission point of view. As an indicator for global views on teaching and learning mathematics we hence included cognitive constructivist versus direct transmission views.

Assessing students' understanding depends of course not only on the tasks that are used, but also on the analysis of students' answers to these questions. However, the selection of tasks is crucial for the potential of the questions to uncover students' conceptual understanding. Hence, these tasks and their characteristics certainly merit attention and may afford practice-relevant insight into PCK teachers use for assessing their students' conceptual understanding of fractions. Since corresponding empirical research is however scarce, this study aims to provide a first exploratory insight.

RESEARCH INTEREST

According to the need for research pointed out in the previous section the study presented here aims to provide evidence for the following research questions:

What conclusions about teachers' domain-specific PCK about using multiple representations can be drawn from tasks they think of when aiming to assess students' conceptual understanding of fractions?

Is such content domain-specific PCK interrelated with the teachers' general pedagogical content views on teaching and learning mathematics?

SAMPLE AND METHODS

For answering these research questions a corresponding paper-pencil questionnaire was designed. This questionnaire was answered by a sample of 87 German in-service teachers (45 female, 41 male, 1 without data), where 64 were teaching at academic-track secondary schools and 23 were teaching at secondary schools for lower-achieving students. These participants had a mean age of 40.4 years (SD = 12.0) and they were teaching mathematics on average for 11.7 years (SD = 11.2). The teachers completed the questionnaire at their schools in the presence of the first author or a student research assistant and they were given as much time as they needed.

Corresponding to the research questions for this study two parts of the questionnaire were included in the evaluations. In the first part the participants were asked to write down tasks they would use to assess students' conceptual understanding of fractions:

Which questions/problems would you pose in order to find out if a student has understood what a fraction is? Please suggest tasks, imagining that you are creating a test at the end of the teaching unit "fractions" in year 6.

In order to analyse the teachers' answers regarding specific PCK about using multiple representations indicated by the suggested questions and problems for assessing students' conceptual understanding of fractions, each set of tasks was coded in a top-down approach with respect to the following criteria:

- Did the teacher state open/closed questions?
- Which core aspects of fractions (see Padberg, 2009) did the teacher address by the stated tasks?
- How many different representation registers are addressed by the tasks?
- How many different conversions of representations are addressed by the tasks?
- Do the tasks take account for both directions of a conversion of representations?

Accordingly, all answers by the participants were double-coded by the first author and a student research assistant with high inter-rater reliability, where Cohen's kappa was at least .86 for each coding criterion. Discrepancies were resolved through discussion in which an agreement could always be reached.

Scale	Sample item	# items	Cronbach's α
Cognitive constructivist view	Students should be allowed to come up with their own ways of solving problems before the teacher demonstrates how to solve them.	5	.76
Direct transmission view	Students learn mathematics best by attending to the teacher's explanations.	6	.74

Table 1: Scales regarding cognitive constructivist vs. direct-transmission views

The part of the questionnaire addressing cognitive constructivist and direct transmission views used items from the survey instrument by Staub and Stern (2002) which is based on scales by Peterson, Fennema, Carpenter, and Loef (1989). Table 1 shows sample items of the corresponding two four-point Likert scales.

RESULTS

Focusing on the first research question, we start with the results concerning the tasks teachers think of when aiming to assess students' conceptual understanding of fractions. In the following the findings regarding the analysis of the teachers' answers outlined in the previous section will be presented.

The first criterion that was coded concerned open/closed questions. The data displayed in Figure 1 show that almost all of the participating teachers stated closed questions, whereas less than half of them enclosed open questions in their set of tasks.

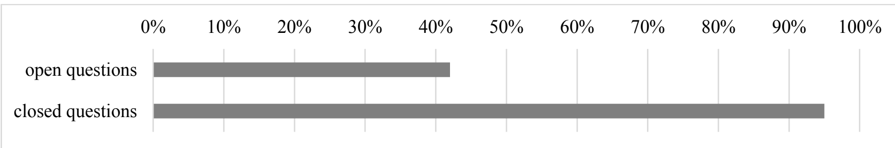

Figure 1: Percentage of teachers who stated closed/open questions

Secondly, each teacher's set of tasks was analysed with respect to which core aspects of fractions were addressed by the questions. The corresponding results presented in Figure 2 indicate that the tasks that were stated by the participants focused predominantly on fractions as being a proportion, whereas all of the other core aspects of fractions were addressed by less than a third of the teachers. Merely 16 % of the participants addressed more than two core aspects of fractions with the tasks that they suggested.

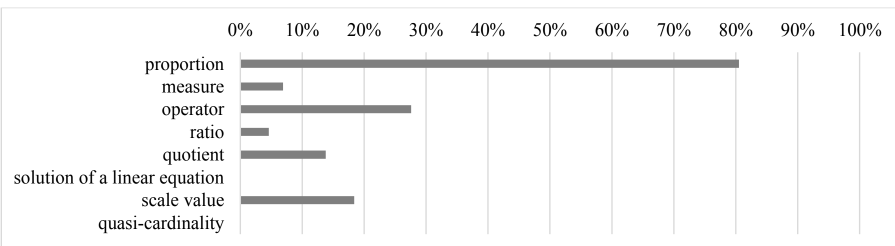

Figure 2: Percentage of teachers who addressed the core aspects of fractions

Analysing the sets of tasks suggested by the participants for assessing students' conceptual understanding of fractions regarding the number of representation registers and the number of conversion addressed yielded the data shown in Figure 3. Accordingly, more than half of the participating teachers focused on more than three different registers and about the same percentage requested more than two different conversions by the questions and problems they suggested. However, 79 % of the participating teachers never took into account both directions of a conversion of representations in their set of tasks.

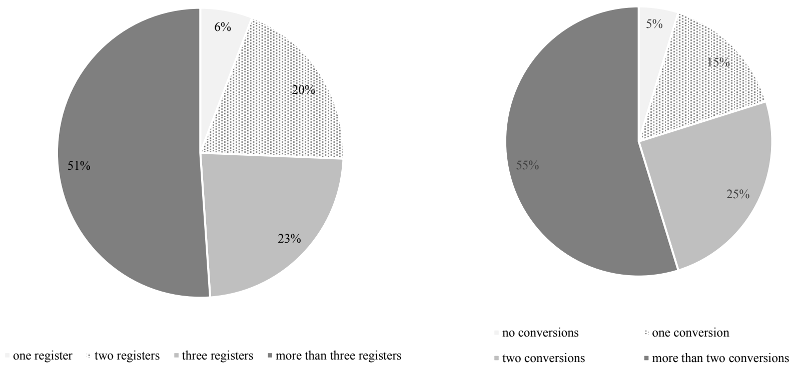

Figure 3: Number of different registers/conversions addressed by the teachers' tasks

For finding answers to the second research question we focused next on the teachers' general pedagogic content views on teaching and learning mathematics by taking a look at their cognitive constructivist and direct transmission views. Figure 4 shows the participants' means and standard errors regarding the corresponding two scales.

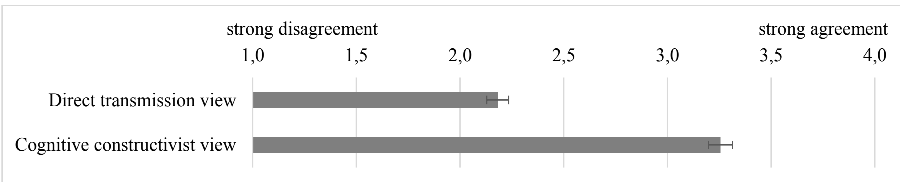

Figure 4: Direct transmission and cognitive constructivist views

This data indicate that on average the participating teachers agreed more with the cognitive constructivist view than with the direct transmission view. In order to explore whether such general pedagogical content views may play a role for the teachers' choice of tasks for assessing students' conceptual understanding of fractions, possible interrelations with the investigated task characteristics were focused upon. Corresponding analyses yielded the following: Those teachers who came up with open questions agreed on average more with the cognitive constructivist view ($t(84) = 2.29$, $p = .024$, $d = 0.50$) and less with the direct transmission view ($t(84) = 2.32$, $p = .023$, $d = 0.51$) than those teachers who stated closed questions only. Furthermore, those teachers whose set of tasks addressed at least once both directions of a conversion of representations approved more of the cognitive constructivist view ($t(75) = 2.23$, $p = .029$, $d = 0.61$) and less of the direct transmission view ($t(75) = 2.06$, $p = .042$, $d = 0.55$) than their colleagues.

DISCUSSION AND CONCLUSIONS

Research investigating teachers' competencies in assessing learners' conceptual understanding often focuses on the conclusions teachers draw from specific students' solutions (e.g., Leuders & Leuders, 2014). The first step, namely choosing problems and questions for assessing students' understanding, is frequently neglected. This choice is however decisive for how meaningful the students' solutions are for uncovering their thinking and conceptual understanding and thus it depends on corresponding PCK. The finding that less than half of the participating teachers would pose open questions in order to assess their students' conceptual understanding of fractions suggests for instance that the majority of these teachers have insufficient PCK regarding the potential of open tasks to reveal student thinking. The result that those teachers who came up with open tasks were on average more in favor of the cognitive constructivist point of view and agreed less with the direct transmission view indicates that not only knowledge, but also more global views may play a role when teachers choose tasks in order to assess their students' conceptual understanding.

From the perspective of the idea of using multiple representation in the mathematics classroom the findings of this study suggest that at least a majority of the teachers took into account several different representation registers and conversions of representations when thinking of tasks to assess learners' conceptual understanding of fractions. Considering the result that most participants addressed merely a single or at most two core aspects of fractions with their set of tasks, indicates however that the different representations of fractions which were taken into account were frequently not chosen in a way that they reflect a variety of different aspects and complement each other towards a multi-faceted concept image. Even though some aspects, such as seeing a fraction as a solution of a linear equation, may not be central for an appropriate conceptual understanding of fractions of a sixth grader, certainly more than just the part-whole aspect is needed. The operator aspect is for instance essential for understanding fraction multiplication (e.g., Padberg, 2009), but was addressed by merely 28 % of the participating teachers.

The findings of this study hence indicate specific needs for teacher education and professional development. Corresponding learning opportunities should not only focus on PCK about using multiple representations of fractions, but should combine such theoretical knowledge with working on specific fraction tasks. A key feature of such specific professional development could be the analysis of such tasks with respect to their potential to uncover students' conceptual understanding and making connections with domain-specific professional knowledge about different core aspects of fractions.

We would like to recall that the results of this study should be interpreted with care, since the sample, the restriction to the domain of fractions, as well as the design of the study constitute clear limitations with respect to the possibility to make broader generalizations. The findings hence call for further research which could combine the analysis of tasks teachers use to assess students' conceptual understanding with a focus on how these teachers interpret their students' solutions to these tasks. Future research studies should also take into account further content domains. Moreover, since the findings of this study suggest that also more global pedagogical content views play a role for how teachers assess learners' conceptual understanding, such interrelations should be investigated in greater depth.

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LEARNING TRAJECTORIES AS BOUNDARY OBJECTS IN PROFESSIONAL DEVELOPMENT SETTINGS

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In this paper, we present our work in designing boundary objects that translate research-based frameworks of students' mathematical thinking into useful tools for teachers. We share our adaptation of learning trajectories and discuss how it was taken up and used by teachers in a professional development setting.

INTRODUCTION

When researchers and teachers come together in professional development (PD) settings, they have the opportunity to exchange knowledge related to mathematics teaching and learning. Wenger (1998) conceptualizes these opportunities as boundary encounters. In such encounters, each community values different practices that are aligned with their goals and hence, they bring different knowledge to the PD setting. For example, knowledge in a teaching community may involve ways of promoting mathematics learning in complex contexts. For the research community, knowledge includes investigating students' thinking and its implications for teachers.

Akkerman and Baker (2011) argued that research on boundary encounters in educational settings can provide insight into the learning potential of these encounters. Yet, knowledge exchange across boundaries can be challenging because those involved in boundary encounters come from distinct communities with different identities and agendas (Wenger, 1998). In such contexts, it is exactly this challenge that can promote transformative learning for the communities involved. Artifacts seeking to bridge both communities, called boundary objects, can support shared meaning-making in boundary encounters.

In this report, we discuss the design of a boundary object used to create shared meaning across the teaching and research communities in mathematics PD. Our work focuses on designing PD to share research-based knowledge on children's mathematics learning trajectories and to learn from teachers about how this knowledge may be useful in their instruction. A critical feature of such PD is the design of boundary objects that carry both varied and similar meanings for researchers and teachers.

Boundary Objects

When individuals from different communities come together in boundary encounters, representations of knowledge that convey meanings across these different communities serve as a catalyst for working together. Star and Griesemer (1989) defined these representations as *boundary objects* or "objects which both inhabit several intersecting social worlds and satisfy the information requirements of each of them" (p. 393). Such

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objects are flexible enough to be adapted to different communities' needs while at the same time maintaining a common identity across communities. Star and Griesemer claim, "the creation and management of boundary objects is a key process in developing and maintaining coherence across intersecting social worlds" (p. 393).

Learning Trajectories

As representations of students' mathematics, learning trajectories (LTs) have gained prominence as a tool for educational reform. The United States' adoption of the Common Core State Standards for Mathematics (CCSSM) (CCSSI, 2010) which were created around "research-based learning progressions", has increased attention towards the use of LTs in teacher education and PD. While many LTs are created by the research community for assessment design (Confrey, 2012) and curriculum development (Clements, Wilson, & Sarama, 2004), Daro, Mosher, and Corcoran (2011) propose that learning trajectories be translated into "useable tools for teachers" (p. 57) to support teachers' growth and instruction.

While various definitions of LTs exist, they all recognize the importance of instruction in supporting students' movement through particular mathematical domains. Confrey and colleagues (2009) define LTs as "research-conjectured, empirically-supported descriptions of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representations, articulations, and reflection, towards increasingly complex concepts over time" (p. 347).

Studies of teacher learning of LTs point to the value of LTs as a framework for making instructional decisions and focusing on students' thinking (Clements, Sarama, Spitler, Lange, & Wolfe, 2011). Knowledge of LTs can deepen teachers' own mathematical knowledge (Wilson, Sztajn, Edgington, & Confrey, 2014), identify and describe student thinking with greater detail (Wickstrom, Baek, Barrett, Cullen, & Tobias, 2012), and help teachers foster rich mathematical discussions (Clements & Sarama, 2009).

In our work around LTs, we use a variety of artefacts from practice that share meaning across the research and teaching communities, such as students' written work and videos of clinical interviews with students. We also use a representation of the LTs themselves. In what follows, we describe our re-representation of four LTs that we designed to serve as a boundary object, how it was used in our PD, and what the teachers learned from its use.

METHODS

This study is part of a larger design experiment to examine teacher learning of students' LTs and student-centered instruction. Design experiments allow for the examination of both the processes of learning as well as the contexts of practice and result in theory about the nature of learning within the particular context (Cobb et al., 2003). A fundamental aspect of design research is the articulation of learning conjectures that drive the design of the learning environment.

Our initial conjecture guiding the design experiment was that the LTs themselves may be too narrow to be useful for teachers—they focus on a small slice of content, and teachers often need to consider their students' thinking across different mathematics topics. As such, we created a consolidated representation of four number and operations LTs (Clements & Sarama, 2009) as a boundary object that related the LTs to one another by focusing on: a) the student (rather than the mathematics) (Philipp, 2008) through *learner profiles*, and b) on key mathematical ideas from LTs which we call *markers*. We call this designed boundary object the LT Profile Table, and we conjectured that the markers and profiles would initially bridge the specific LT levels for teachers in a way that is more accessible and meaningful. As teachers learned more about the LTs, the levels would provide a more nuanced tool for understanding students' thinking. This paper addresses the question: *How do teachers integrate their understanding of markers, profiles, and LTs in their PD discussions?*

Learning Trajectory Based Instruction (LTBI)

Project LTBI is a multi-year PD project bringing elementary mathematics teachers and researchers together in ways that promote knowledge exchange about students' LTs among teachers and researchers. A goal of the project is to improve the practice of both the teaching and research communities. In its third year, the project partnered with one elementary school in a mid-sized suburban school district in the Southeastern United States. Researchers and twelve elementary grades teachers worked together for one year around number and operations LTs developed by Clements and Sarama (2009), as well as a conceptual model of instruction encouraging the use of LTs, open instructional tasks, and pedagogical practices to centralize students' mathematical thinking in instruction (Sztajn, Wilson, Confrey, & Edgington, 2012). The PD consisted of a total of 55 hours, beginning with a 30 hour intensive summer institute that took place over four days prior to the start of the 2012-2013 school year. This was the second iteration of the PD program. During the year, researchers and teachers met for six monthly meetings and the project concluded with a one-day meeting the following summer.

The LT Profile Table

Clements and Sarama (2009) present ten empirically supported LTs on various mathematical topics, describing how children understand and learn each topic. Based on their emphasis in the CCSSM, we chose four LTs as the content focus in our PD: 1) Quantity, Number, and Subitizing, 2) Counting, 3) Addition and Subtraction, and 4) Composition of Number and Place Value. Each LT includes levels of thinking that describes a typical path children follow in developing understanding of the particular topic as well as instructional tasks matched to each level.

Once we identified the LTs that would comprise the content focus of the PD, we created the learner profiles spanning across LTs, as shown in the rows in Table 1. These profiles acknowledged significant markers of students' development within each trajectory. For example, the Perceptual Child profile identified *cardinality* as an essential marker for the Counting LT. These markers were isolated as a way for teachers to recall larger milestones within each trajectory. The columns of the table include the specific levels of each LT. For the researchers, the table represented empirically-supported descriptions of student thinking that we were aiming to share. We hypothesised that, for teachers, the table represented a formalisation of intuitions and observations of their students' mathematical work.

We introduced each learner profile by engaging participants in professional learning tasks that explored one or more of the markers within each profile. For example, to introduce the Counting On Child profile, we showed teachers two clinical interviews of students solving a join-change unknown story problem where one child used a counting-on strategy and the other child used a direct modelling strategy, and asked teachers to note differences and similarities in the students' strategies. Through discussion and an examination of problem types, we formalized *counting on* from the Counting LT as well as *counting strategies* and problem types from the Addition and Subtraction LT. After introducing each profile, we then focused on each LT individually for the remainder of the PD.

Data Sources and Analysis

Data consists of video recordings of all PD meetings and transcripts of audio recordings of group discussions. In addition, the research team created a conjecture log and met regularly during the year to revise our initial conjectures. This log was used to document what we as researchers learned throughout the PD. In order to understand the meaning teachers created around the LT Profile Table, we examined the data for instances when teachers referred to specific learner profiles, markers, or used language from the LTs or the Profile Table to discuss students' mathematical thinking. Using constant comparison methods (Glaser & Strauss, 1967), we looked across all such instances to understand how teachers' used the markers, profiles, and LT levels.

LT Profile	QUANTITY	COUNT	ADD/SUB TR	PLACE VALUE
Percep tual Child	PERCEPTU AL SUBTIZING	CARDINALI TY	USE OF SMALL COLLECTI ONS	
	• Maker of small collections • Perceptual Subtizer to 4 then 5	• Reciter (10) • Corresponder • Counter Small Number • Counter to 10 • Producer to 5	• Small Number	
Early Counti ng	CONCEPTU AL SUBTIZING	FLEXIBLE NUMBER SEQUENCE	DIRECT MODELIN G	COMPO SING TO 10

Child	• Conceptual Subtizer to 5 then 10	• Counter and Producer to 10+ • Counter backward from 10 • Counter from N (N+1, N-1) • Skip Counter by 10s to 100 • Counter to 100	• Find result (joining, part-part-whole with direct modeling, counting all; take away using objects) • Make it N (adds on to make another number) • Find change (finds missing addend using objects) ◦ Join-to (count all)/separate from (count all)/match (count rest)	• Composer to 4, 5, 7 then 10 (knows number combinations, doubles to 10)
	EXTENDED SUBTIZING	COUNTING ON	COUNTING STRATEGIES	
Counting on Child	• Conceptual Subtizer to 20 (uses groups)	• Counter on using patterns (keeps track using numerical patterns) • Skip Counter (counts by fives and twos) • Counter of imagined items (counts mental images) • Counter on keeping track • Counter of quantitative units and place value (understands base 10 system, decompose a ten into ones when useful) • Counter to 200 (recognizes patterns)	• Counting strategies (join and part-part-whole) ◦ Using finger patterns/counting on/counting up to • Part-whole (flexibly solve all previous problem types, sometimes start unknown) • Number-in-number (keeps part and whole in mind simultaneously; uses counting strategies for start unknown) • Deriver (flexibly uses strategies and derived combinations	
	PLACE VALUE SUBTIZING	CONSERVATION	PROBLEM SOLVER	COMPOSING WITH TENS & ONES
Place Value Child	• Conceptual Subtizer with Place Value and Skip Counting (e.g. uses groups of twos, tens or fives)	• Number Conserver • Counter Forward and Back	• Problem solver (flexibly uses strategies and known combinations)	• Composer with tens and ones (understands 2 digit number as tens and ones; counts with dimes/pennies; regrouping)
	MULTIPLICATION SUBTIZING		MULTIDIGIT	
Multi Digit Child	• Conceptual Subtizer with Place Value and Multiplication		• Multidigit solved by incrementing or using tens and ones	

Table 1: The LT Profile Table (adapted from Clements & Sarama, 2009).

FINDINGS

We share two findings related to teachers' uses of the markers, profiles, and LT levels in their discussions during the LTBI PD. First, the markers and profiles gave teachers initial traction in focusing on students' mathematical thinking. Because the focus during the first three days of the PD was introducing teachers to the Learner Profiles, it was not surprising when teachers used the profile names to label and categorize students' mathematical thinking, attending to the markers within each profile. For example, on the second day of the PD, teachers worked in small groups to analyzing students' written work and described the student's thinking based on the profiles:

H3: Perceptual [child] can count to five, right? [G2 agreeing] So would this be the next level child? Because he's counting past five.

G2: So he can count to ten and he can recite up to ten. Perceptual.

H3: But he's producing.

K2: Then he would be a child that has got perceptual, but he's not really counting yet because he doesn't have…

G2: They've got to have…direct modelling is the key and cardinality is the key. That's why I would say he's got the cardinality of 10 and 3, so he's reaching over there.

H3: Ok, so he's beginning.

G2: Mm-hmm. He's beginning to be an early counting child. How about that?

In this discussion, the teachers used the markers of "cardinality" and "direct modeling" to distinguish between the Perceptual Child and the Early Counting Child profiles. G2's comment, "he's beginning to be," suggests that teachers sensed the progressive nature of the LT. In keeping with our conjecture, we believed the specificity of the LT levels would further support teachers' learning of the LT.

Second, though teachers used the LT levels later in the PD, these levels were not their primary tool for understanding students' thinking. Rather, teachers continued to use the profiles and markers as the PD progressed. In these discussions, teachers repeatedly categorized student thinking based on the profiles stating, "She was moving from Early Counting to Counting On," or "He was getting ready to be a Place Value child". Teachers did come to use the specific LT levels as evidence of how they categorized their students' thinking, yet they continued to focus on the markers and profiles. For instance, in the third monthly meeting, teachers discussed the results of several student interviews and how the LTs helped them think about who to interview and what questions to ask. One teacher commented:

> For the one child who had a hard time even managing 1:1 correspondence, he stayed down in the cardinality range. He couldn't even hold onto what he counted, so we just practiced counting out different objects. And for the other child, she was doing some of the joining and finding change and things like that. Some of it even seemed to be in her head, but also she was using her fingers and stuff. So I went back and thought, well, I don't know if she's able to skip count to 100 by tens, or if she's able to count backwards. So I just went back to see what she was able to do in the different parts [of the LTs] to see where she was moving into being a Counting On Child.

The teacher used a specific LT level (e.g., 1:1 correspondence) as evidence for a claim about the student's development of a marker (cardinality). She looked across LTs and used the other student's work on addition problems (e.g. finding change) as well as counting as evidence of the student's movement toward the Counting On profile.

DISCUSSION

In the LTBI PD, the LT Profile Table served as a boundary object between the research and teaching communities. The table was designed in such a way as to bring forth central ideas from each LT and also connect ideas across them. We conjectured that the markers and profiles would initially support teachers in learning the LTs but would be replaced by the LT. However, our findings indicate that teachers integrated the markers, profiles, and LT levels in their PD discussions. Teachers first relied on the markers to categorize student thinking into one of the profiles. Later, teachers began attending to specific LT levels to provide evidence for a particular learning profile classification or progress toward a marker. These results point to the potential benefits and constraints of decisions made when designing boundary objects and underscore the careful considerations researchers should make when translating research results for teachers.

Acknowledgements

This report is based upon work supported by the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this report are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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CHANGES IN EXPRESSION WHEN TRANSLATING ARITHMETIC WORD QUESTIONS

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Teaching and assessing mathematics in Indigenous languages which do not have developed school mathematics registers may require teachers to translate word questions from a source language into their own language. Differences may occur between the source and the translation due to semantic and syntactic requirements of the target languages and for other reasons during the translation process. We present examples of some translations by teachers of word problems from English into languages of Central Province, Papua New Guinea. Even simple statements can contain multiple changes in translation. Some statements may be difficult or impossible to translate while retaining a comparable mathematical difficulty.

LANGUAGE OF INSTRUCTION IN MATHEMATICS

There are powerful arguments for teaching mathematics in children's first languages. First languages connect with children's social and cultural identities. It is also easier for children to learn subject matter in a language in which they are fluent (Matang & Owens, 2014; Pinnock & Vijayakumar, 2009). Where students are trying to learn mathematics in a language in which they are not yet fluent, the outcome may be impaired learning. For example, the difficulty of word problems in mathematics can be exacerbated for those learning in an additional language (Jones, 1982; Lean, Clements & Del Campo, 1990).

However, there are many languages spoken by children that do not have developed mathematics registers for use in school. A mathematics register is the meanings, words and grammatical structures used in talking about, learning and doing mathematics (Halliday, 1978). These languages may not have a history of being used in school mathematics, there may not be text books or curricula, and teachers who speak these languages may not have received their own mathematics education in these languages. While the language to teach and learn mathematics can be developed in any language, the process of identifying or developing appropriate language requires both care and time (Meaney, Trinick & Fairhall, 2011). When there are many languages spoken in a single country, there may not be the resources to formally develop a mathematics register in all of them. In this case, individual teachers may be in the position of working out how to use the language. This may include translating material from another language.

Papua New Guinea has a population of around 7 million people. With close to 850 languages spoken, it is one of the most linguistically diverse regions of the world (Lewis, Simons, & Fennig, 2013). Papua New Guinea is also economically

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undeveloped, with schools being poorly resourced, often with little access to books, electricity or computer technology. The country inherited an English language education system from its time as an Australian colony. In the 1990s a reform of the education system occurred, introducing indigenous languages, known as Tok Ples, and culture into education, along with the provision of three years of elementary schooling (Klaus, 2003). A step model of language instruction was introduced, with instruction beginning in local languages, and a transition to English occurring in the third year of primary school, with the amount of English increasing over the next two years (Department of Education Papua New Guinea, 2004).

This included the development of a *Cultural Mathematics* program, a specific approach in Papua New Guinea that seeks to integrate local mathematical knowledge and practices, such as counting or measurement, into the school program, thus building on the home knowledge of the students (Department of Education Papua New Guinea, 2004). Much of the challenge of developing a teaching program in so many languages has had to be met by individual teachers at a village level. This was a huge shift in the role of the teacher in Papua New Guinea (Feeger, 1997).

Teaching mathematics in indigenous languages can require both in the moment decisions about language use and planned language use, such as translating mathematics questions from English. In both cases, teachers need to decide which terms to use and how to use them. In this paper we ask what is lost and what is gained in translations – what is the mathematical significance of syntactic and semantic changes in translated questions?

EQUIVALENCE OF TRANSLATED QUESTIONS

Within a single language there can be simpler or more complex phrasing of arguably equivalent questions. Making even small changes to the sentence structure of a mathematical question within a language can affect the difficulty of the question (De Corte & Verschaffel, 1987; Solano-Flores, 2010). In some contexts, there may be a case for ensuring that mathematical questions to be translated have a simple structure. Some guidelines for translatability include using short, simple sentences; using the active voice; and using the full noun in all contexts rather than pronouns (Ercikan,1998). However, it is not possible to completely separate the readability of a mathematical question from the mathematics of the question (Österholm & Bergqvist, 2012). At times, the inferential understanding of a passively and complexly phrased question may be part of students' mathematical literacy that we would like to assess. We would argue that this also applies to the complexity of questions posed orally. Teachers' own judgements of the mathematical difficulty of questions is, to some extent, also based on the language difficulty of the questions (Solano-Flores, 2010).

There can also be simpler, more complex, or just plain different phrasing of equivalent questions translated between languages. In some cases the differences are unavoidable due to grammatical imperatives of the languages. In other cases the differences may be structurally avoidable but the question may have the wrong tone or sound awkward in the target language. That is, the languages may have similar possibilities of expression, but one expression will sound more natural or normal in one language. Differences may also occur because the translator is primarily concentrating on retaining what they see as the mathematical content with less attention paid to the structure of the sentence, or content seen as not mathematically important. Finally, there are also cases where a translation is not really possible, as the question contains terms and concepts for which there is no equivalent in the target language. This is particularly the case when attempting translation into a language that does not have a developed school mathematics register.

One change that can have mathematical significance is a change in the level of abstraction or concreteness of a question. There are both advantages and disadvantages to processes of abstraction. Sfard (2008) claims that mathematical discourse in a large part serves to objectify "things" such as numbers through processes of abstraction, reification and alienation. This includes the elimination of the human subject in talking about numbers by using passive structures and nominalisation. Such "impersonal discursive forms are very effective in implying the extradiscursive existence of numbers," (p. 50) which facilitates communication about these mathematical objects. On the other hand, abstract objects such as bare numbers are more difficult to operate with than concrete objects whether real or imagined (Huttenlocher, Jordan, & Levine, 1994). Especially in the early years of mathematics education, there are many contexts in which it is preferable to anchor mathematical problems or situation to the concrete.

The linguistic practices associated with objectification in mathematics such as nominalisation are reasonably easy to perform in English and other Indo-European languages, but are not as available in all languages. Note that we need see the different grammatical properties of different languages as limitations only when we ourselves are limited in our goals to the reproduction of the established mathematical discourse in those languages. Barton (2009) claims that our mathematics have developed in line with the ways that our languages delineate our world, but also that there is potential in other, dissimilar languages to develop new logics and new mathematics. Nevertheless, there are many situations in which the primary aim is the reproduction of an established mathematical discourse, particularly in terms of providing access to the social and economic benefits associated with competency in school mathematics.

PAPUA NEW GUINEA CONTEXT

Levels of resourcing and the sheer number of languages mean that developing a formal mathematics register in all the languages that are used in schools in Papua New Guinea is not currently feasible. This paper reports on part of a three year study which is exploring how best to identify and use cultural mathematical proficiencies to assist young students to transition to school mathematics in Papua New Guinea. A week long professional development workshop is delivered to elementary school teachers. The workshop teaches how to bring local cultural and language into a teaching cycle that includes diagnostic assessment of children's mathematical knowledge and proficiencies. The assessment includes an interview to be conducted one-on-one with individual children. Interview results are also to be collected as part of our data collection on the success of the project. The interview is provided in English, but in a large proportion of cases needs to be conducted in indigenous Papua New Guinean languages, and the teachers need to make the translations themselves.

Time was spent working on these translations during a workshop in a village in Central Province, which surrounds the National Capital District which contains Port Moresby. Many villagers either work in the capital or rely on the income of family members who do. Motu, an Austronesian language, is spoken widely in the immediate region of the village, as is Polis Motu (also called Hiri Motu), a simplified lingua franca developed from but not mutually intelligible with Motu, and which is spoken across several provinces. Many villagers can also speak English and Tok Pisin. Other local languages which were used in the workshops were Hula and Koiari. Those participants who did not speak Motu nevertheless had a working comprehension of it. One of the two team members could speak Motu and Tok Pisin as well as English; the other team member shared only English with the participants. The workshop was predominately delivered in English with some discussion, particularly within small groups, in Motu.

TRANSLATING QUESTIONS

In this workshop, the language focus was on translating the questions in the children's interviews. The questions cover a range of early number, arithmetic, measurement and geometrical topics. Some of the arithmetic questions started with a statement about an initial number of objects, either real (stones used as counters which were displayed to the children) or imaginary (such as bananas), which was followed by another quantity to be added or subtracted. For example: One of the questions began with the statement "I have 5 ripe bananas". The adjective 'ripe' was used because some PNG languages have different words for eating bananas and cooking bananas, as well as names for many varieties of bananas and for different parts of the plant. In Tok Pisin, an eating banana is called 'ripe'. The translated sentences together with their literal meanings are presented below.

(1) Motu:	*ravao*	*mage-dia*	*ima*	*eini*		
	banana	ripe-FOC	#5	here		
(2) Hula:	*au*	*piku*	*mera*	*imaima*		
	I	banana	ripe	#5		
(3) Koiari:	*daike*	*uhi*	*bae*	*adahakibe*	*da-agenahuma*	
	I	banana	ripe	#5	I-have_them(edible)	
(4) Polis Motu:	*lau*	*be* *biku*		*mage-na*	*bona*	*5*
	I	FOC banana		ripe-FOC	together	#5
(5) Tok Pisin:	*mi*	*gat* *faipela*		*mao* *banana*		
	I	have #5		ripe banana		

The translations produced by the teachers do not necessarily represent standard sentences in each language. Tok Pisin (5) was the only language in which a word for word translation was provided. In standard Tok Pisin the adjective should follow the noun, but the Tok Pisin in this region shows this influence from English. The Motu translation (1), which was the dominant language in the workshop apart from English, uses *(h)eini* 'here', which is not in the English original, and does not contain an 'I have' construction. The Hula (2) and the Polis Motu (4) include the pronoun 'I' but do not have a possessive verb 'have'. The possession is indicated by the word order. Polis Motu, which is a creole based on Motu, introduces a grouping word *bona* 'together' which emphasises the bananas as a group as well as a focal particle *be*. The Koiari statement (3) has a possessive verb *da-agenahuma* which is specific to the possession of edible things. If the sentence was "I have 5 stones", the verb *da-agedahuma* 'I have them (inedible)' would have been used. Some of the features of the translations are necessary in the target languages. The specification of the edibility of the bananas, which is implicit in the English version's inclusion of the adjective "ripe", is required in Koiari as there is no neutral possessive verb. In Hula, it would be possible to include the verb *maparara* 'have'; however, because it is not necessary, it may be that including the verb would place an undue emphasis on the possessive aspect of the statement.

In the case of Motu, it would also be possible, and more correct to use a more literal translation using *dekeguai* 'have'. The Motu sentence (1) occurred between questions which began with similar statements that the same teacher translated more literally:

(6) **"Here are 9 stones."**

nadi	*tauratoi*	*toi*	*eini*
stone	#6	#3 (9)	here

(7) **"I have 5 stones."**

lau	*egu*	*nadi*	*ima*
I	POSSESSIVE(my)	stone	#5

Again with (7) a more correct Motu sentence would use *dekeguai* 'have', and it would be more common to use *taurahanita* (#8 + #1) for nine. As the two types of statements are both clearly possible in Motu, it is hard to say why the teacher who translated sentence (1) chose to do so in this way. It is also not clear whether he would then have been able to ask the question beginning in this way. The statements such as "Here are 5 stones" are designed to be accompanied by actual stones to use as concrete counters during the interview. Would it still make sense to say "Here are 5 bananas" in the absence of actual bananas? In Motu, it might. In English it makes sense to say "There are 5 bananas", referring to imaginary bananas, as an existential statement than a positional statement that some bananas are at some place.

The above examples shows how necessary and unnecessary changes can enter the translation of a very simple statement. There were other questions that presented more difficulties, or that required greater deviations from a literal translation.

(8) **"What are two numbers that add to 8?"**

edadia	*numera*	*rua*	*baita*	*haboudia-mu*
which	number	#2	FUTURE	put_them_together- PRESENT
taurahani	*be-davaria-mu.*			
#8	FOCUS-find-PRESENT			

(9) **"What are another two numbers that add to 8?"**

numera	*idaudia*	*rua*	*ma*	*ba*	*haboudia*
number	other	#2	FOCUS	make	put_them_together
taurahani	*baina*	*davaria*			
#8	FUTURE	find			

A literal translation of (8) would be something like "(Can you) find two numbers that (you can) put together (to make) 8?" and of (9) "(Can you) find two other numbers that put together (to make) 8?" The original English question has an impersonal structure that is typical of mathematical discourse. There is no human agent, the question asks about numbers that "add to 8", regardless of who is doing the adding, and in fact that add to 8 without anyone to add at all. The interviewee is being asked to identify and name the numbers. In the Motu translations, the interviewee is being asked to *find* the numbers and, at least in the first sentence, to put them together. The mood has changed, with the English question inquiring after numbers that are indicated to exist, to the Motu sentence indicating a hypothetical situation via future particles. While the English question inquires about pairs of numbers that have the property of adding to 8, the Motu question is asking for numbers with which the interviewee can make 8.

There were several questions in the interview which presented difficulties in trying to find a suitable translation at all. In one of these, the interviewer lays out stones in two colours in a repeating pattern, such as 2 black, 1 white ..., and asks "show me how you continue this pattern." When asked what the word for pattern in their language is, many of the teachers have a ready answer. However, this will often turn out to be a word that refers to designs such as those that are used in weaving. The word will generally not be able to be used about a number pattern. This is an example of a concept which is not present in many of the languages of Papua New Guinea. Similarly, another question asks the interviewee to explain "what area is". Many of the PNG languages do not appear to have a term equivalent to area as used in mathematics. Interviewees might offer a word that also translates as 'place', but does not refer to an area measure bale by a smaller area. There are ways to talk about and measure area in some contexts, but it is impossible to translate a question that asks for a definition when the term is not defined in that language.

CONCLUSION

Translating even the most straightforward of mathematical statements and questions from English into PNG languages presented challenges. Changes were introduced by the teachers into almost all of the sentences. The reasons included the grammatical and stylistic requirements of the target languages, decisions by the teachers about what parts of the sentences were mathematically significant, and the available vocabulary in the target languages. While some of the changes appear to be very minor, differences in the level of abstraction in the question may lead to the question being more or less difficult in the translated form than in the original form.

Teachers' decisions about terminology were affected by the extent of their own mathematical knowledge. In discussion it sometimes became clear that the teachers did not know how a concept explored in the early years of school would be developed in the later years of mathematics education. There also seemed to be a relationship between the topics not well understood by the teachers and topics for which local vernacular terms were not readily available, such as patterns and area. The translated sentences show differences in the degree of abstraction both as a result of teachers' choices in their translations and of requirements of the individual languages, such as the Koiari requirement that possession be distinguished between edible and non-edible items. The trend was to make questions more concrete and less abstract. It is clear that the results of such interviews conducted by children need to be interpreted with the awareness that even minor changes may alter the ease of children answering a particular question in one language comparted to another.

Acknowledgements

The project is funded through an Australian Development Research Award led by Kay Owens. The views expressed in the publication are those of the authors and not necessarily those of the Commonwealth of Australia which accepts no responsibility for any loss, damage or injury resulting from reliance on any of the information or views contained in this publication.

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BETWEEN MATHEMATICS AND TALMUD – THE CONSTRUCTION OF A HYBRID DISCOURSE IN AN ULTRA-OTHODOX CLASSROOM

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This paper examines the case of adult ultra-orthodox Jews studying algebra for the first time, in a pre-college course. First, the social context and Talmudic background of the students is presented. Then, we analyse how cultural resources from both the Talmudic studies, the main practice of ultra-orthodox culture, and the mathematics classroom culture, were used by students to construct a new hybrid discourse. We conclude by discussing how our analysis demonstrates some productive possibilities and potential problems for the students' mathematical learning.

INTRODUCTION

Culture has been of increasing interest to mathematics education researchers (ex. Sfard & Prusak, 2005). A particular focus has been placed on the difficulties of minority groups to participate in mathematical classroom discourse (ex. Cobb & Hodge, 2002). Less attention has been given to instances where the cultural background of students may also *enhance* productive participation in learning. Building on the linguistic anthropological notion of "hybrid discourse" (Lefstein & Snell, 2010) we analyse the discourse patterns in a classroom of ultra-orthodox adults who are being introduced to the concept of proof and the activity of mathematical proving (Stylianides, 2007). In this setting, students come from a background of highly argumentative learning (as practiced in Talmudic studies) together with almost no disciplinary knowledge in mathematics other than basic arithmetic. Our goal in this paper is to examine the intersection between mathematical classroom discourse and the cultural discourse students bring into class.

THEORETICAL BACKGROUND

In our analysis, we will adopt a *communicational* lens (Sfard, 2008), which views learning mathematics as participation in a certain type of discourse characterized by a particular set of *words*, *routines* and *narratives*. In particular, we pay attention to the *interactional routines* followed by participants (Empson, 2003; Heyd-Metzuyanim, 2013). Specifically, we look at who *initiates* an interaction, where does the *authority* for making mathematical claims lie (Veel, 1999), and in what ways key words are being used (Sfard, 2008).

On top of Sfard's framework, we wish to use the notion of hybrid discourse put forward by Lefstein and Snell (2010). Lefstein and Snell propose that discourses are dynamic processes rather than static entities. Therefor there are gaps between a discourse prototype and its realization in practice. These gaps - which stem from the individual's agency and creativity, and from the complexity of social situations - are filled by borrowing from other cultural resources, thereby constructing a hybrid discourse. In communicational terms, we will expect this hybridity to take place by a drift of words, routines, narratives, and even goals and purposes, from one discourse to the other. Yet before moving on to showing how such hybridity may take place between Talmudic and mathematical discourses, we must provide some context and background on the ultra-orthodox Talmudic culture.

The ultra-orthodox males in Israel ideally devote most of their time to Talmudic studies. In the elementary years, they receive some basic education in “secular” domains such as mathematics, English, and geography during a short period of the day. This secular education ends at the age of 12-13 after which males go into “Yeshivas”. There, only sacred texts are studied, mostly the books of the Talmud, which are often descriptions of the mythological scholars' disputes concerning older texts.

Some researchers, such as Schwarz (in press) and Segal (2011), have already drawn attention to the parallels between Talmudic learning practices and instructional practices that have been highly valued in reform efforts to turn classrooms into “discussion based” learning environments (Herbel-Eisenmann & Cirillo, 2009; Lampert, 1990). However, Schwarz and Segal also point to some attributes which are significantly different than practices prevalent in traditional pre academic mathematical classrooms. In what follows, we briefly outline these differences.

Learning purposes: Unlike pre academic mathematics, which is often studied as a preparation for more advanced mathematics courses or for reasons external to the discipline (such as admission to academic tracks), in Talmudic studies the act of learning is held up as a goal in itself .The ultra-orthodox religious values encourage engagement with the sacred texts as an activity worth of its own. In Blum-Kulka’s words:

> The religious obligation to study the (Talmudic) law is not goal-oriented, but concerns itself merely with process. … The ideal of *Torah li-shmah (*Torah study as an end unto itself.) underscores the perception that time spent on disagreement is of the same religious value as that expended on reaching an agreement. (Blum-Kulka 2002, p. 1576).

The structure of interactional routines: The basic and very common Talmud structure of interactional routines is that of two learners ('Havruta') who engage with a text without the constant guidance of an instructor. In contrast, the structure of the pre academic mathematics class is that of a teacher-led instruction, followed by some independent class work.

Routines for endorsement of narratives: Talmudic and mathematical discourses differ in the ways they determine a statement as “true”. In mathematics, two opposing proposition cannot be simultaneously declared as “true”. The “truth” of a statement is based on its coherence (or agreement) with all mathematical narratives that have been endorsed up to that point. Talmudic justification, on the other hand, involves reasoning between several, often equally plausible alternatives. As a consequence, one's Talmudic interpretation must be supported by evidence ("Re'aia"), but it doesn't necessarily refute other interpretations, as mathematical counter-examples do.

Authority structure: As mentioned above, Talmudic studies are often performed in pairs. While the teacher (Rabbi) is often present and has a voice in the discussion, he is not regarded as the ultimate arbitrator. In mastering the "Cultural preference for disagreement" (Blum-Kulka, ibid), students are given the authority to both disagree and put forward unique and creative arguments. In contrast, in mathematics classrooms (especially those described as “teacher-centred” or “traditional”), the authority to state what is true or false mainly lies with the teacher. Students in such classrooms are accustomed to rely on the teacher's authority and develop intricate methods of interpreting his stance, even when it is not stated explicitly. Though the “teacher as ultimate arbitrator” phenomenon has been fought against within reform attempts (ex. Lampert, 1990), it can be seen as an attribute of the pre-academic mathematical classroom discourse. In mathematics, there is often only one correct answer, and the teacher, being more knowledgeable than the students, mostly has access to it before they do.

In light of the above similarities and differences, we would like to closely analyse the hybrid discourse observed in our research, and to ask what discursive possibilities did it open or close for students’ participation and for the development of their mathematical discourse?

THE STUDY

The current study follows a class of pre-academic mathematics for adult ultra-orthodox males, at the ages of 18 to 30, preparing them for bachelor studies in business school. The course took place 6 hours a week (two days) during 13 months, from January 2013 to February 2014, and was taught by Nadav (the first author). At the period when the study was conducted, Nadav held a B.Sc. in mathematics & Computer-Science with only minimal formal training as a teacher. The course started from the most basic algebraic signs and methods, and ended with an exam equivalent to 3 points of 'Bagrut' exam (the basic level of the Israeli mathematics matriculation exam).

Seven students attended the lesson described below, out of ten who were enrolled in the course. During the recordings, a stationary video camera pointed at the teacher to allow a view of the board. No other cameras were in use, mainly for political reasons. The topic of ultra-orthodox integration in the modern Israeli society has been controversial in the past years. Therefore, we were careful not to steer objections to the documentation, neither by the students nor by the college management. The lesson was transcribed in Hebrew and later translated by the authors into English.

AN EPISODE OF ARGUMENTATION ABOUT MATHEMATICAL PROOF

The particular episode examined here took place in September 2013 after 8 months of instruction. It was chosen for close analysis because of the rich discussion that took place in it, and because it had seemed to contain evidence for the construction of a hybrid discourse from both ultra-orthodox and mathematics classroom cultures.

The session concerned the proving of the quadratic formula. Through that, the teacher planned to discuss what constitutes a proof in mathematics and what does not. First, he wrote on the board an example of a quadratic equation (x2 – 3x + 2=0) and, with the aid of the students, established that by either factoring or using the quadratic formula, one can find that its roots are 1 and 2, and to verify it using substitution. Then the following proceeded (all names are pseudonyms):

1.	T	... In this example the quadratic formula has given two solutions that are indeed true. You found it using substitution (Joshua: yes). Does it mean it is really true? Does it mean it always finds the solutions?
2.	Joshua	Not yet. Not yet.
3.	Abraham	If I see ten twenty of those.. then.. then what?
4.	T	Then?
5.	Abraham	It's sure to be true
6.	Joshua	If you see that it's infinite. If you see that it's some infinite process that makes it always substitute the…

The above excerpt shows that some students, and in particular Abraham, started out holding a naïve, or inductive idea of proof. The more a formula is empirically checked to provide true solutions (by substituting the symbols for real numbers), the surer one is that it is "true". In contrast, Joshua caught on to the teacher's questioning of the inductive claim and declared that such substitution would not suffice [2]. Instead, one should look for an "infinite process" that would always make it true [6]. Capitalizing on another student's comment that "there's always an exception", the teacher moved to explain that in mathematics, an example is not a proof, whereas a counter-example is. Therefore, even showing hundreds of examples where a formula is correct wouldn't prove it. At this point, Abraham responded: "so you need to understand its (the formula's) logic" showing that he was moving away from the inductive reasoning.

Having established that examples cannot be considered as proof, the teacher offered a procedure for proving the quadratic formula beyond doubt. He did not, however, present it as an established proof but rather as a hypothetical procedure that might be considered a proof if the students accepted it.

44.	T	What of the things that I have here (points to the general quadratic equation $ax^2+bx+c=0$ and the quadratic formula written on the board) can I substitute (and) where? *(Silence)* Just like I took x_{12}… I had one and two… I substituted it here (points to $x^2 – 3x + 2=0$)
45.	Avraham	That does not prove it, we said that
46.	T	It doesn't prove it. If I take these x one and two, for instance? (*points at the quadratic formula*) (*Silence*. *Teacher writes the following quadratic*

equation on the board, while saying it out loud)

$$a\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)^2+b\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)+c=0$$

47.	Avraham	(*As the teacher is writing*) It's better 'cause it shows you like that for any number… It proves 'cause it shows me that any number that I'll put here..
48.	T	Good

Here occurred the main transformation in Abraham's claims about what "proof" was. Instead of claiming that "10 or 20" correct trials would prove a formula, he now sees that what the teacher was suggesting would "show it for any number" [47]. Notably, this important transformation in Abraham's claims occurred through participating in a teacher-led IRE interaction, and the idea of such a proof was completely initiated by the teacher (the students at this stage of their studies had no way of coming up with such a proof on their own). However, what unfolded next was a change in the classroom's structure of interactional routines. Concurrently with the interaction between the teacher and other students who were seeking clarifications (mostly Yehuda, who was sitting at the front desk), an episode of argumentation between Joshua and Abraham (who were sitting in the back) enabled the latter to take ownership and practice authority over the teacher's idea. The next excerpt concentrates on the dialogue between Joshua and Abraham:

70.	T	Instead of substituting (with) the number 2 I'll substitute the expression with letters…. (Joshua: OK), All of it , and I'll see that at the end, all of this (points to the quadratic formula) becomes zero, so that will tell me that actually this always becomes zero. Do you agree that this is a way that really shows me this formula is always correct?
71.	Joshua	As long as there's no counter example, yeah
72.	*At this point Yehuda starts asking questions and conversing with the teacher.*	
73.	Abraham	There can't be a counter example. Any number that you put will work.
74.	Joshua	Why not? And if you put an incorrect number?
75.	Abraham	But it can't be. (Joshua: Why not?) 'Cause these letters told you that any number that you substitute for them, you'll get zero.
76.	Joshua	That's if it works. And what if it won't work?
77.	Abraham	But it can't be that it wouldn't work. What did he tell you? What's the evidence (Hebrew: re'aya) that he gave you?
78.	Joshua	He doesn't have any evidence yet
79.	Abraham	He does have evidence. That's the evidence he's giving you!
80.	Yehuda	No… That's not evidence
81.	Abraham	Cause these- these expressions actually tell you that any number that you put instead of the expression will give you zero

Continuing the argument while the teacher turned to answer Yehuda, Abraham tried as best as he could to convince Joshua. He said: "Here he's giving you an expression. That means that anything that you put instead of the expression will give you the same thing". Several more turns occurred between Joshua and Abraham, mainly stating the same challenge and response. However, Joshua remained unconvinced and the episode ended with the teacher moving to show a proof that was more intelligible to the students (the “completing the squares” proof).

ANALYSIS AND DISCUSSION

In analysing the above episode, we first wish to demonstrate it as a hybrid discourse with resources in the ultra-orthodox Talmudic culture and the pre-academic mathematics classroom culture. We then conclude in discussing some of the affordances and problems the hybrid discourse presented the students with, focusing on Joshua and Abraham.

Hybridity

The interactional routines consisted of students actively seeking clarifications both from the teacher, and more importantly, from each other (as in “what’s the evidence he’s giving you?” or “that’s what he’s telling you”). This happened mostly in the last segment, right after the structure of interactional routines had changed and Abraham and Joshua were discussing the problem between themselves in the back while the teacher was conversing with Yehuda in the front. In Havruta studies, the most common form of debate is that of using the text to prove one’s point and argue (by challenging or rebutting) against the Havruta peer’s claims. In this episode, we believe the same pattern occurred, with the difference being that here the teacher’s talk was serving as the “text” over which students were arguing.

Another hint for the growing dominancy of the Talmud discourse at that point of the lesson was the usage of the key word “re’aya” (evidence), a common notion in the Talmudic studies and debates. This term was inserted into the debate by Abraham and immediately taken up by Joshua. The teacher, being an outsider to the ultra-orthodox world, was not used to this word, neither in the daily nor the mathematical context.

The main key-word the teacher introduced was taken from the mathematical discourse: “counter-example”. This was used to introduce the routine of refuting a mathematical claim. Joshua willingly adopted this new key word, yet he used it in a way that was not intended by the teacher. He repeatedly claimed that Abraham (and thus the teacher) would only be correct "As long as there’s no counter example" [71]. In order to endorse the proof presented by the teacher, one had to realize why there could not be any counter-example in that case. In other words, one would have to accept the function that letters (or algebraic symbols) and algebraic manipulations had as generalization tools, and that using a letter instead of a number renders the search for numerical counter-examples to be unnecessary.

Discursive Constraints – the case of Joshua

Joshua did not make any signs of accepting this discursive rule of proving by using algebraic notation. The reason for that might be found in the drifting of Talmudic *authority structure* into classroom discourse. In contrast to Abraham, who at least partially relied on the traditional mathematics classroom authority structure (teacher as arbitrator of truth), Joshua interpreted the teacher's proposal as an object for debate. He was therefore unwilling to "suspend his disbelief" (Ben-Zvi & Sfard, 2007) enough to take the teacher's claim under serious consideration. Perhaps a different conception of the teacher's authority, less Talmudic and more mathematical, would have moved Joshua's focus at that point from interpersonal to intrapersonal activity in order to examine this new idea that the teacher and Abraham were suggesting.

Discursive Possibilities – the case of Abraham

Unlike to Joshua, we claim that the hybrid discourse seen in this episode provided Abraham unique opportunities for learning. Such opportunities would not be available neither in a traditional learning setting where the instruction is strictly teacher-led (in IRE fashion) nor in settings that are mostly student centered (as in small group problem solving). Abraham's mathematical claims developed first through an IRE interaction with the teacher, where he achieved the important realization that there was a deductive way of proving a formula, irrespective of the empirical trials of checking its truth value. However, the intensive "Havruta" episode with Joshua provided the opportunity to elaborate and restate his newly acquired narrative in ways that wouldn't have been possible had the conversation remained solely between Abraham and the teacher. Joshua was coming up with questions and challenges that the teacher, who was the one who offered the solution to begin with, was unlikely to present to Abraham.

CONCLUSION

Lest we be misunderstood, we wish to clarify that we are *not* making any claims about ultra-orthodox Jews' general propensity to neither succeed nor fail in mathematics. Neither are we making general claims about the affordances of a Talmudic background for the study of mathematics. Further research would be needed for making such claims. Rather, the case brought here shows an example of constructing a hybrid discourse from different cultures. The study further illustrates that cultural differences can both lever and hinder the learning of mathematics.

We believe this is an essential step in the derivation of pedagogical implementations. For instance, the insights gleaned from this study can assist teachers of the ultra-orthodox population by raising their awareness to the affordances and obstacles that the Talmudic background may provide. More generally, we believe that such examination of hybrid discourses in the mathematics classroom is a fruitful path for educators wishing to integrate students from diverse cultural backgrounds.

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TEACHERS' PERSPECTIVES USED TO EXPLAIN STUDENTS' RESPONSES IN PATTERN GENERALISATION

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The purpose of this paper is to explore teachers' perspectives to explain students' responses in pattern generalisation tasks. Individual interviews were done with 15 in-service school mathematics teachers who were asked to explain students' responses to five pattern generalisation tasks. Results of data analysis showed that teachers' perspectives used to explain student response in pattern generalisation fell into four categories: student lens, teacher lens, fluctuated between teacher lens and student lens, inability to explain student response. The findings also showed that teachers' perspectives used to explain students' responses were mediated by the pattern generalisation type. In particular, the teacher perspective adopting student lens increased with the increase in the generality demands of the task.

BACKGROUND

Pattern generalisation is a core area in mathematics characterized by requiring more strategic knowledge than content (mathematical) knowledge. Mathematics teachers by their training normally have more indirect exposure to the strategic knowledge of pattern generalisations than their students because of their opportunities to study topic like functions, sequences, and series. Thus one would expect that teachers and students would have different perspectives regarding solving pattern generalisation problems.

Few research studies explored teachers' perspectives in explaining students' responses in pattern generalisation. One direction was the study of teacher ability to explain student reasoning in pattern generalisation in terms of identifying the elements which constitute a complete explanation (El Mouhayar & Jurdak, 2013; El Mouhayar, 2014). For example, El Mouhayar and Jurdak (2013) showed that more than half of the in-service mathematics school teachers who participated in the study were unable to provide complete explanations for students' reasoning in pattern generalisation tasks. In particular, the findings reported that while teachers' explanations focused on constant-related counting elements, those explanations were lacking in terms of identifying variable-related counting elements. El Mouhayar and Jurdak (2013) also found out that teachers' ability to explain students' reasoning in far generalisation tasks (questions which are difficult to be solved by step-by-step drawing or counting) depend on their ability to explain students' reasoning in near generalisation tasks (questions which can be solved by step-by-step drawing or counting). Another direction in the literature focused on teachers' knowledge of pattern generalisation. For example, Rivera and Becker (2007) reported that prospective teachers showed the ability to successfully generalize patterns using different strategies. The findings reported that some prospective teachers formulated rules from the sequence of numbers that are listed in a pattern whereas others used relationships and cues that are established from the figural structure of a pattern.

RATIONALE OF THE STUDY

There are few research studies on teachers' explanations of students' responses to mathematical tasks; although teachers' explanations of student work is critical for the interaction between teacher and *individual* student. Teachers' explanations of student work are particularly important in pattern generalisation because of the strategic nature of reasoning in the latter. This study extends previous research on teachers' ability to explain student response in pattern generalisation in three directions. First, the present study aims at exploring the perspectives that the teachers use to analyse students' responses whereas previous research focused on exploring teacher ability to identify and explain student reasoning in pattern generalisation in terms of the elements of student response (El Mouhayar, 2014; El Mouhayar & Jurdak, 2013). Second, the present study aims at exploring the impact of pattern generalisation type on teachers' perspectives to explain students' responses whereas previous studies addressed the influence of pattern generalisation type on teacher ability to explain student reasoning (El Mouhayar, 2014). Third, the present study uses interviews about authentic student work with in-service mathematics teachers whereas previous research, dealing with teachers' knowledge to explain student reasoning in pattern generalisation, has used surveys consisting of authentic student work or of contrived illustrative models of students' responses taken from the literature.

RESEARCH QUESTIONS

- What perspectives do in-service mathematics teachers use to explain students' responses to pattern generalisation tasks?
- How is the use of teacher perspective to explain students' responses influenced by pattern generalisation type?

METHOD

Participants

Ninety one in-service school mathematics teachers from different grade levels were selected from 20 schools in Lebanon, particularly Beirut and its suburbs, to participate in a previous study (El Mouhayar, 2014). Of the ninety one participants, fifteen were selected to participate in the present study. The majority of those participants (60%) had five or more years of experience in teaching mathematics and 80% were females.

Instrument

In a previous study, teachers filled out an instrument, consisting of 10 items, in order to examine their ability to explain student response in pattern generalisation (El Mouhayar, 2014). A sample of students' responses were taken from a survey used in a previous study (Jurdak & El Mouhayar, 2014) involving 1232 Lebanese students from grades 4 to 11. The survey included four pattern generalisation tasks. Each of the items displayed the problem (a figural growing pattern showing the first four figural steps) and a student's response to: (1) immediate generalisation (predicting steps 5), (2) near generalisation (predicting step 9) or (3) far generalisation (predicting step 100 or step n). For each item, participants were asked to respond to the following question: "How did the student think to get the number of squares?". Participants filled out the questionnaire individually in the presence of the investigator. Filling out the questionnaire took around 90 minutes. Out of the ninety one teachers who filled the survey, fifteen were selected to participate in the present study. The selection of the fifteen participants was based on their different abilities to explain students' responses in pattern generalisation and on their experience in teaching in different grade levels.

Five out of the ten items from the original survey were selected for the present study. The items represented different reasoning approaches and strategies used by students to generalize immediate, near and far generalisation tasks. Participants were asked to re-explain students' responses for each of the five items and/or to clarify their written explanation about student response that is included in the survey. For example, participants were asked to explain student response for step n (Figure 1).

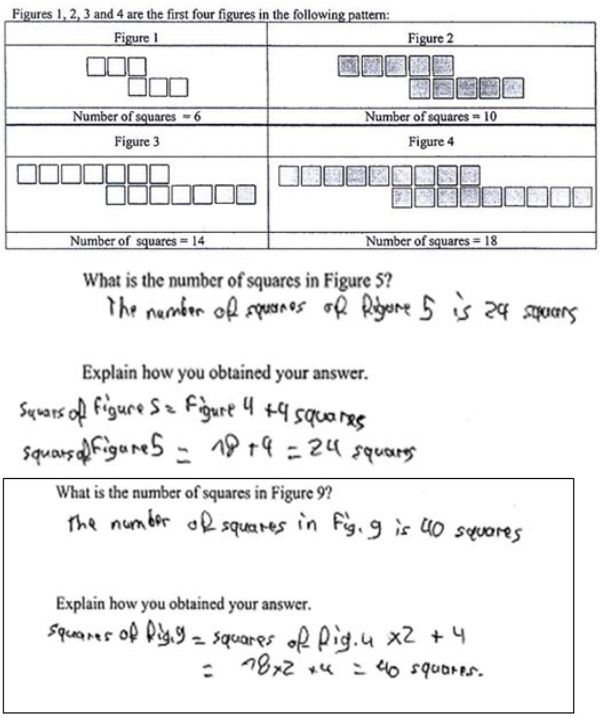
Figures 1, 2, 3 and 4 are the first four figures in the following pattern:

Figure 1	Figure 2
Number of squares = 6	Number of squares = 10
Figure 3	**Figure 4**
Number of squares = 14	Number of squares = 18

What is the number of squares in Figure 5?

The number of squares of figure 5 is 24 squares

Explain how you obtained your answer.

Squares of figure 5 = figure 4 +4 squares

squares of figure 5 = 18 + 4 = 24 squares

What is the number of squares in Figure 9?

the number of squares in Fig. 9 is 40 squares

Explain how you obtained your answer.

squares of fig. 9 = squares of fig. 4 x2 + 4

= 18x2 +4 = 40 squares.

Figure 1: Student response to a near generalisation task (step 9)

Data Collection and analysis

The researchers interviewed each of the participants individually for about 30 minutes. For each of the five items, the researchers showed the participants the item, containing the task and student response, in addition to the teacher written explanation from the survey. The researchers asked the participants to explain the students' responses. Follow up questions were asked by the researchers, when needed, in order to further clarify their explanation of students' responses.

Interviews were transcribed and the obtained transcriptions were subjected to a series of analyses. First, the researchers identified the elements and relationships that constituted a complete and coherent explanation of the students' responses for each of the 5 items in the questionnaire. Second, a constant comparative method of qualitative analysis (Glaser & Strauss, 1967) was applied to identify teachers' perspectives to explain students' responses. One researcher coded the data by one task at a time and several meetings between the two researchers followed where disagreements in identifying teachers' perspectives and in coding were discussed until a consensus was reached. Second, frequencies and percentages were determined for each of the categories concerning teachers' perspectives to explain students' responses. Third, a cross tabulation of teacher perspective to explain student response by pattern generalisation type was done to explore the possibility of significant differences in the variation of teacher perspective across generalisation type. Fourth, percentages of teachers' perspectives to explain students' responses across pattern generalisation types were presented by bar graphs in order to identify the highest frequency of teacher perspective within each generalisation type and trends of variation of teacher perspective across generalisation type.

FINDINGS

Perspectives used by teachers to analyse students' responses

Qualitative analysis resulted in four perspectives that the teachers used to analyse student responses in figural pattern generalisation. A perspective is the lens through which the teacher views and analyses student response. The four perspectives are:

- Student lens: The teacher viewed and analysed student response exclusively through the lens of the written solution provided by the student. For example, excerpt 1 (Table 1) shows how the teacher adopted student lens to analyse student response to a near generalisation task (step 9).
- Teacher lens: The teacher viewed and analysed student response exclusively through the lens of the solution that is provided by the teacher. For example, excerpt 2 (Table 1) shows how the teacher adopted the teacher lens to analyse student response to a near generalisation task (step 9).
- Teacher fluctuated between teacher lens and student lens: The teacher goes back and forth between using student lens and teacher lens.

- Teacher perspective to analyse student reasoning could not be identified by the researchers or teacher expressed his/her inability to explain student response

The findings showed that the majority of teachers adopted student lens (74%) followed by teacher lens (11%) or fluctuation of teacher and student lenses (9.6%).

Excerpt	Teacher perspective	Sample teacher explanation
1	Adopted student lens	Teacher: "*The student confused between the number of the figure and the number of squares within a figure. He [student] considered number 9 as if it is the number of squares in the figure, but it is not, it is the number of the figure. He [student] considered that since 9 can be written as 4*2+1 then the number of squares of figure 9 is the number of squares in figure 4*2+4 (the additional squares) = 40*." [italics added]
2	Adopted teacher lens	Teacher: "*The number of squares in figure 9 is related to the figure number*. For example, in figure 2 I have here 2×2 plus 2×3. So *the number of squares in figure n is 2n+2(n+1)* such that n is the number of the figure and it is equal to the number of columns. 2n since each column is formed of 2 squares. The number of squares on the sides in figure 2 is 3 and there are 2 sides so 2*3 which is 2(n+1) is the number of squares located on the sides. This is how I thought about it. Therefore, the answer that I got is 2n+2(n+1), which is equal to 4n+2. Here look what the student did; he said that the number of squares in figure 5 is equal to the number of squares in figure 4 + 4 squares; so 18+4 squares. As you read student's response, there isn't a convincing explanation to what he is doing since he [student] did not connect the number of squares to the figure number. *The student just counted the number of squares and he increased by 4 each time*, but he [student] does not know that the number of squares is related to the figure number. I did not understand what he [student] meant by his explanation since I believe that the number of squares should be related to the number of the figure." [italics added]

Table 1: Samples of teachers' explanations of student response

Excerpt 1 in Table 1 provides an example of a teacher adopting student lens to explain student response in a near generalisation task presented in Figure 1. The teacher adopted student lens since the teacher pointed out how the student considered that the number of squares of step 9 is the number of squares of step 8 plus 4. The teacher explained that the student recognized that 4 is the constant difference between the consecutive steps of the growing pattern. The teacher also pointed out that the student multiplied the number of squares of step 4 by 2 in order to find the number of squares in step 8. The teacher explained why the student multiplied the number of squares in figure 4 by 2 to get the number of squares of figure 8 by stating that the student confused between the number of squares in step 8 and the step number.

Excerpt 2 in table 1 provides an example of a teacher adopting teacher lens to explain student response in a near generalisation task presented in Figure 1. In excerpt 2, the teacher adopted exclusively the teacher solution as a reference to explain student response. The teacher solution was based on relating the figural step number to the growing parts of the pattern.

Influence of pattern generalisation type on teacher perspective to explain students' responses

A cross tabulation of the perspectives used by the teachers to explain students' responses by pattern generalisation type (immediate, near and far generalisations) was done. Results show that chi-squared was significant ($\chi 2$ (6) = 27.376, $p<0.00$) indicating that the perspectives that are used by the teacher to analyse students' responses were significantly influenced by the type of the pattern generalisation task. Additionally, a cross tabulation of teacher ability to correctly recognize student numerical and figural reasoning approach by pattern generalisation type (immediate, near and far generalisations) was done. Results show that chi-squared was significant ($\chi 2$ (4) = 30.93, p= 0.00) indicating that teacher ability to correctly recognize student reasoning approach was significantly influenced by the type of the pattern generalisation task.

The teacher's perspective in analysing students' responses was mediated by the type of pattern generalisation task. Figure 2 shows that teachers adopted a student lens most frequently in far generalisation tasks while they adopted a teacher lens most frequently in near generalisation tasks whereas they fluctuated between teacher and student perspectives most frequently within the immediate generalisation tasks. Figure 2 also shows that teachers' adoption of student lens increased across generalisation types from 16.7% in immediate generalisation to 25.9% in near generalisation to 57.4% in far generalisation, whereas, teachers' fluctuation between teacher and student lenses decreased from 75% in immediate generalisation to 12.5% in each of near and far generalisations.

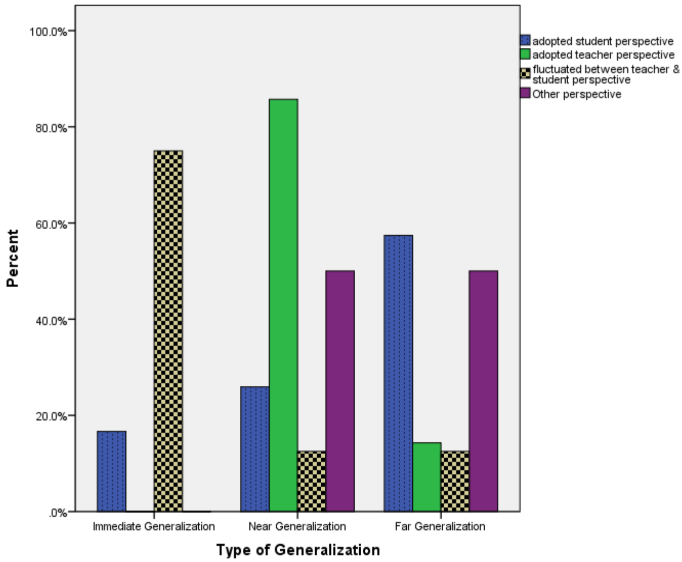

Figure 2: Bar graphs representing the percentages of perspectives used by teachers to analyse students' responses

DISCUSSION

On major finding in this study is that teachers predominantly used student lens to explain students' responses, however teachers' use of perspectives was mediated by pattern generalisation type. The frequency of different types of teachers' perspectives used to explain students' responses differed according to the generalisation type (immediate, near and far). The frequency of teachers adopting student lens increased with the increase in the generality demands of the task, whereas, the frequency of teachers fluctuating between teacher and student lenses decreased with the increase in the generality demands of the task. One plausible explanation for this finding may be due to the alignment between student and teacher knowledge of pattern generalisation in far generalisation tasks. According to the Lebanese curriculum, most of the tasks that are related to pattern-related topics involve variables and involve providing algebraic expressions to determine any term of a sequence or determine an output of a function presented in the form of a rule. On the other hand, teachers' knowledge of pattern generalisation is similarly affected by their exposure in their university education.

A second finding of this study is that the teachers used teacher's lens more often than student lens to explain student responses in near generalisation tasks. Again this may be explained in terms of possible discrepancy between student and teacher knowledge where teachers are less familiar with dealing with small figural step n. For example, El Mouhayar (2014) showed that teachers showed higher ability to explain students' responses for far generalisation tasks by using a larger amount of data (elements and relationships) found in student response compared to teachers' explanations for near generalisation tasks.

In conclusion, the area of teachers' explanations of student work is underrepresented in mathematics research, although the ability of teachers to explain their students' work is critical for them to understand their student thinking. This research is a step in this direction. It is hoped that more studies be carried out in this area in order to generate enough knowledge that may be incorporated in teacher education programs.

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TEACHING NUMERACY IN PRACTICE: INCREASING FAMILIARITY WITH MATHEMATICAL PROCESSES

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Education policy in Australia and elsewhere has emphasised the importance of numeracy as a necessary skill for living in contemporary society. While there is global agreement on the relevance of numeracy, there is limited support for teachers in enhancing student numeracy outcomes at the classroom level. A program of professional development set within a framework of building understanding about mathematical content and processes was implemented in one Australian school, and results suggest completing such a program builds teachers' familiarity with numeracy and increases their capacity for identifying and supporting numeracy-based learning opportunities.

INTRODUCTION

This paper reports on data gathered during a professional development session aimed at assisting secondary school teachers to identify the mathematical knowledge and processes inherent in their subject area. The program took place in a 1300-student, semi-rural comprehensive government secondary school in outer Sydney, Australia, where the author is also the Head of Mathematics. The session aimed to equip the participants with knowledge on identifying lesson activities that rely on numeracy, and increase their confidence in supporting students to develop the disposition required to be effective users of numeracy.

This paper considers selected responses that highlight the challenges faced by teachers in embedding numeracy within their classrooms due to limited awareness of the extent of numeracy-based skills and attitudes, and compounded by limited scope of materials that may support the teaching and learning of numeracy. The growth observed in the participants was greater than expected, as participants suggested lesson modifications and proposed teaching stimulus ideas in addition to identifying opportunities to extend numeracy learning.

This study is a side-project of a larger undertaking towards the author's doctoral thesis that will examine current practices of embedding numeracy within secondary schools, and the influence of teachers' mathematical confidence and attainment on their numeracy practices within the classroom.

BACKGROUND AND THEORETICAL FRAMEWORK

Numeracy was first introduced into the lexicon of educators and education stakeholders in the Crowther report in the United Kingdom (Crowther, 1959). In the decades since, there have been multiple descriptions of what numeracy means and variations in

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 273-280. Hobart, Australia: PME.

terminology - quantitative literacy, mathematical literacy, quantitative reasoning - but there is a common theme across all these descriptions: the disposition to choose, use and apply mathematical knowledge, skills and ideas in the context of everyday living.

In the decades since the Crowther report, the notion of numeracy has evolved from a term synonymous with the application of school mathematics, to a more abstract view of being the connection between mathematics skills taught at school and their real-life applications, although this debate is still ongoing (Park, 2010). In the Australian context, numeracy is the term of choice, and it has been provided a platform in education policy documents for nearly two decades. It has a high status in the national curriculum, described as an ultimate goal for schooling or a *general capability*: a critical component of being successful, confident, and creative individuals who are active and informed citizens (Australian Curriculum Assessment and Reporting Authority [ACARA], 2013). The general capabilities encompass:

> the knowledge, skills, behaviours and dispositions that, together with curriculum content in each learning area and the cross-curriculum priorities, will assist students to live and work successfully in the twenty-first century (ACARA, 2013)

Exploration of numeracy elements identified within the Australian Curriculum documents has largely been undertaken by the mathematics education community (Goos, Dole, & Geiger, 2012), and has established that the curriculum documents present a range of opportunities for teachers to develop numeracy skills in their students. However, despite the high status afforded to numeracy nationally, relatively few studies have been conducted on how numeracy in treated in Australian schools. The small number of studies that have been undertaken suggest that teachers have a narrow conception of numeracy and focus on numerical aspects of mathematics, overlooking non-numerical ideas (Callingham, Beswick, & Ferme, 2015), and the curriculum-identified numeracy elements are seen by teachers as an “add-on” rather than tasks that use mathematics skills to support the learning of their subject’s content (Ferme, 2014a; 2014b).

It could be argued that a narrow presentation of what numeracy is in school contexts may be reflected in international data on school numeracy performance. The Performance in School Achievement (PISA) tests (Organisation for Economic Co-Operation and Development [OECD], 2014) indicate that, in their mathematics lessons, students report less exposure to real-world applied problems that arise in everyday life or work and require the application of suitable mathematical knowledge, than exposure to applied problems where mathematics is a context in itself. PISA focuses more on assessing what 15 year-old students can do with their mathematical knowledge, than what mathematics they know and reports an international downwards trend, and Australian students are not immune to this.

Teachers play a key role in assisting students make links between mathematical ideas to develop their numeracy skills (Askew, Rhodes, Brown, Wiliam, & Johnson, 1997) but there is little research on how to engage non-mathematics teachers in learning and critical discussion about numeracy (Sowder, 2007). Hogan (2000) proposes that students need help in developing their mathematical, strategic and contextual knowledge to become fluent numeracy operators, but teachers need to be mindful of the context in which the students use this knowledge and develop their understanding accordingly, but should not style themselves as teachers of mathematics. Goos (2010, cited in Goos, et al., 2012) developed a model for numeracy that that acknowledges the contexts and it is this model used in the study described in this paper.

RESEARCH CONTEXT AND METHODOLOGY

The professional development session discussed in this paper was presented to staff as a session within the mandatory professional development period at the conclusion of the 2014 school year. Staff were required to select six, hour-long sessions from ten options presented by members of the school's executive leadership team.

Delivery and format of the session were planned with consideration of Hawley and Valli's (1999) principles for effective professional development. Considerable emphasis was placed on teacher involvement in a collaborative, school-based setting with student goals as the primary aim. Participants were presented with current, information-rich material that was based on current research, and although the session did not locate itself within the school's immediate goals for numeracy (improving results of standardised tests), those in attendance were able to take key ideas from the session and contribute to the school's long-term change process.

The format of the professional development lent itself to a multiple case-study model, which enabled the author to recognise the complexity of educational research and expose the subtleties that lie within school environments (Cohen, Manion, & Morrison, 2011). As a colleague to the participants and leader within the school, the author assumed a participant-observer role typical of phenomenological studies that examine a small number of cases (Atkinson & Hammersley, 1994; Cohen, et al., 2011). The researcher was involved in the process of examining the objects presented to the participants, but in this respect her interactions were limited to responding to participant questions as they arose, or facilitating further discussion on specific ideas resulting from participant dialogue through the use of Socratic questioning.

The 24 participants comprised representatives from the learning areas of Mathematics, Science, Humanities and Social Science, English, Physical Health and Development, Design and Technology, and Learning Intervention. The participants, who ranged in teaching experience from 4 weeks to more than 25 years, worked collaboratively in faculty focus groups resulting in collegial responses that would highlight the participants' agenda over that of the researcher (Cohen, et al., 2011).

The session was presented in three parts (see Figure 1). The first and third sections centred on examining pre-selected student learning activities relevant to each group's learning area and recording responses to a set of stimulus questions focusing on interpreting the activities' mathematical knowledge, skills and ideas; numeracy demands; and teaching strategies and opportunities for enhancing numeracy.

The stimulus activities were drawn from an online resource bank available to teachers for the purposes of assessment moderation (Board of Studies NSW [BOS], 2014). The activities were selected by the author based on their breadth and depth of mathematical content: it was intended to present activities to staff that would be identifiable as requiring mathematical knowledge or skills, but also containing opportunities for development of mathematical processes.

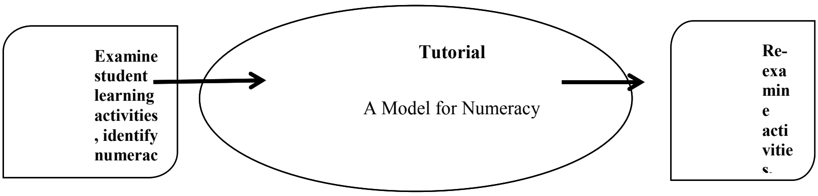

Figure 1: Session Design

The principle focus for the session was the tutorial that bridged the two examination activities. The tutorial presented Goos' (2012) model for numeracy, and described the kinds of processes that engage students in worthwhile mathematics (National Council of Teachers of Mathematics, 2000). Specific attention was given to making connections between mathematical concepts and other disciplines, and, in particular, the local curriculum's emphasis on develop understanding and fluency in mathematics through "inquiry, exploring and connecting mathematical concepts, choosing and applying problem-solving skills and mathematical techniques, communicating and reasoning" (BOS, 2012).

The author matched each learning activity with the relevant Australian Curriculum content descriptions and numeracy elements, but these links were not made explicit to the participants. The pre- and post-tutorial written responses, and the author's field notes for each group, were compared to the curriculum numeracy elements. A similar procedure was undertaken to record changes between pre- and post- tutorial responses that identified numeracy-focused teaching opportunities against a scaffold of Goos' model for numeracy.

RESULTS AND DISCUSSION

The professional development session emphasised developing an understanding of the range of mathematical knowledge, skills, ideas and processes relevant to secondary schooling, and the connections to other faculty areas those concepts may bridge under the auspices of numeracy. Teachers are required to identify the numeracy demands of their area and provide learning experiences that support numeracy application and development (ACARA, 2013), and participating in a professional development program that encouraged collaborative work to develop a sense of self as a teacher of mathematical processes (Sowder, 2007) appears to have contributed to addressing that requirement as discussed in this section.

Ten written responses were returned. One of the returned responses was not written against a specific student learning activity, but with respect to the numeracy demands of the faculty focus group's curriculum (Design and Technology). This response to the task arose after a brief discussion within the group that established reluctance to narrow their curriculum down to one learning activity, as even after a cursory examination of the supplied learning activities they felt that numeracy was "in everything [they] do and can't be separated", and reducing the number of numeracy demands they could consider for the purposes of the session they felt minimised the relevance of numeracy in their subject.

Identifying Mathematical Concepts

The stimulus questions asked participants to identify mathematical- and numeracy-based skills separately, as previous research has indicated that teachers often use the terms interchangeably, and may not be fully appraised of the differences between the two (Ferme, 2014b; Hogan, 2012). The ensuing verbal discussion within a number of groups reiterated this phenomenon, most notably with a group of Science specialists who were unwilling to proceed with the first task until they came to an agreement on the differences, which they identified as being context-dependent, reflecting Beswick's (2008) research linking numeracy proficiency and context. Similarly, another group listed "chronology and sequencing" as a mathematical skill required for their learning activity and "time lines" as a corresponding numeracy skill. That faculty focus group came to the agreement that these were separate concepts, referring to national standardised tests where construction of timelines are frequently included, but there is less or no emphasis on "questions that just ask students to arrange [events] in chronological order". These responses reiterate the notion that national testing regimes emphasise mathematics skill over context and have a strong influence on how numeracy is perceived (Hogan, 2012).

This narrow interpretation of numeracy was concurrently identified through cross-referencing the learning activities to the Australian Curriculum documents during the coding process. For example, one student learning activity, "Investigating the Local Environment", required students to source and label contemporary maps, draw diagrams of fauna and their habitat, describe historical changes to the built environment, and record climate data for a known locality. Pre-tutorial participant responses described generic mathematical ideas: mapping, scale, graphing, recording, and data interpretation, mirroring the corresponding Australian Curriculum numeracy elements for this activity: estimate and calculate; interpret maps and diagrams; estimate and measure with metric units, and; interpret data displays. These instances of a limited scope afforded to numeracy support Madison and Steen's (2008) research that suggests that the existing curriculum and assessment documents reflect the overly simplistic view of numeracy many people hold.

Post-tutorial participant responses to stimulus questions about the mathematical- and numeracy- based skills required in the activities yielded more specific descriptions of mathematical concepts. For instance, one group had listed graphing, measurement, reading scales, and number sequences in their pre-tutorial responses, but added patterns and trends, interpreting bivariate data, graphing relationships, and data collection and representation in their post-tutorial responses. This suggests that equipping teachers with specific knowledge about the range of school-based mathematical concepts goes some way to improving recognition of numeracy-based activities across learning areas.

Supporting Student Dispositions

The pre-tutorial participant responses recorded no suggestions for inclusion of specific mathematics language, despite the expectation for teachers to use correct mathematical language appropriate to their learning areas (ACARA, 2013). The tutorial, however, included specific reference to the importance of using accurate mathematical language and the importance of strong literacy skills to numerate behaviours (OECD, 2009). In post-tutorial responses, some faculty focus groups included literacy-based tasks as opportunities to enhance student numeracy outcomes. For example, for a Science activity that required students to plan a first-hand investigation and write a Science report, the focus group suggested that students also be encouraged to "write up a brochure to market/justify [the results]".

Using numeracy as a tool to support other learning areas was heavily emphasised during the tutorial by way of describing mathematical communication, problem solving, reasoning, understanding and fluency as being involved in the application of numeracy. Post-tutorial group responses such as the one exemplified above indicated that this focus increased participants' awareness of how mathematics can be used to make sense of ideas from different fields. When mathematical skills are used to make an abstract idea concrete, that subject becomes more accessible (Phillips, 2002), as reflected in a number of post-tutorial responses. One illustration of this was a group's proposal that students should compare the human impact of past and present mudslides on the local population in an activity that explored natural disasters.

Sowder (2007) notes that professional growth is marked by a change in teachers' instructional strategies and knowledge, and the greatest area of change recorded in the data was that of participants' propensity to modify learning activities to better support student dispositions for using numeracy. The responses from the focus groups demonstrated a quantitative increase in the incidence of attending to student numeracy learning needs, in particular scaffolding or modifying tasks to attend to the range of mathematical capacities in students. For instance, "use grid paper to sketch out materials for waste reduction [in woodwork]", "offer blank templates for drawing Venn Diagrams", and "model how the data can be manipulated" were suggested as opportunities to enhance student numeracy outcomes in the post-tutorial responses.

It should be noted that the groups may not have had enough time to complete their post-tutorial responses or think deeply about their practice, but these results provide starting points for further investigation.

CONCLUSION

Building professional capital by outfitting teachers with the requisite knowledge of mathematical processes and procedures that underpin numeracy has a distinct role in classroom learning (Callingham, et al., 2015). The responses provided by the participants in this study demonstrate that, even with brief exposure to specific knowledge about numeracy and strategies to enhance student outcomes, qualitative changes can be made in the way teachers think about their practice.

The data from this project suggest that presenting an accessible model for numeracy and verbalising the mathematical processes that underpin the application of mathematical knowledge and skills, improves teachers' development of numeracy-focused student learning activities. The knowledge gained from this project has the potential to inform further research on current practices of embedding numeracy within secondary schools and professional development of secondary school teachers.

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LEARNING ABOUT STUDENTS' MATHEMATICAL THINKING USING "KDU"

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The goal of this study is to examine how prospective teachers (PTs) learn to recognise the understanding of inclusive relations as a Key Development Understanding(KDU) (Simon, 2006) when secondary school students are learning about the classification of quadrilaterals. Our findings suggest that prospective mathematics teachers' learning was not uniform and, as a consequence, we characterised transitions in their learning. We provide a prospective teachers' hypothetical learning trajectory and discuss the role played by the identification of a KDU of the mathematic topic in describing prospective teachers' learning.

THEORETICAL BACKGROUND

Researchers have adopted several different approaches in order to determine how prospective teachers learn to identify evidence of students' mathematical understanding. The results of these studies have provided an insight into how prospective teachers learn to use mathematical knowledge in order to interpret students' mathematical thinking (Bartell, Webel, Bowen, & Dyson, 2013; Coles, Fernández, & Brown, 2013; Fernández, Llinares, & Valls, 2011; 2012; Jacobs, Lamb, & Philipp, 2010).

A consequence of such research is that teacher trainers have begun to design resources that support prospective teachers' learning about how students learn mathematical concepts and how their understanding develops (Sánchez-Matamoros, Fernández, & Llinares, 2014; Wilson, Mojica, & Confrey, 2013). One construct that is useful in thinking about the way in which prospective teachers learn about student learning is the learning trajectory. Sztajn and colleagues (Sztajn, Confrey, Wilson, & Edgington, 2012) have used the learning trajectory construct to refine the meanings given to mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). In particular, they have emphasised that teachers' knowledge of learning trajectories provides information about how learners progress from less to more sophisticated ways of thinking. They have also underlined the importance of selecting tasks that will facilitate student progress over the course of the learning trajectory, namely tasks that support students' cognitive development. In order for prospective teachers to learn how to help students progress through learning trajectory, it is necessary that they identify the critical transitions that are essential for students to progress. In this sense, the "key developmental understanding" (KDU) construct proposed by Simon (2006) can help

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 289-296. Hobart, Australia: PME.

prospective teachers identify crucial steps in students' conceptual development and propose tasks that will assist in such progression throughout the learning trajectory.

Although advances have been made in characterising learning trajectories in different mathematical domains (Clements & Sarama, 2009), less attention has been paid to how prospective teachers identify progression in secondary school students' understanding and how such identification can help them make instructional decisions. Our research examined how prospective teachers participating in a sequence of activities designed ad-hoc identified secondary school students' progression in their understanding of the classification of quadrilaterals, and how they used this information to select and generate tasks that could support students' transition towards a more sophisticated understanding.

Learning trajectory of the classification of quadrilaterals

Learning how to classify quadrilaterals causes difficulties for secondary school students. These difficulties are related to the role played by inclusion relations between quadrilaterals, as students recognise the various quadrilaterals by means of prototype examples without considering the inclusion relations associated with the classification processes (De Villiers, 1994; Fujita, 2012). Inclusive classifications result when the application of classifying criteria to a specific set creates subsets in which it is possible to establish an inclusion relation (hierarchical chain) among its elements. For example, in an inclusive classification of a set of parallelograms, the square can be considered a special type of rhombus; while in an exclusive classification (partition) the square and the rhombus belong to separate groups. Understanding the role inclusive relations play in the classification of quadrilaterals (Usiskin, & Griffin, 2008) has relevance to prospective teachers learning about students' mathematical understanding.

The development of secondary school students' understanding of inclusion relations can be understood as a learning trajectory. Multiple definitions of learning trajectory have been proposed in mathematics education (Clements & Sarama, 2009; Simon, 1995; Wilson, et al., 2013). In this study, a learning trajectory was defined as a hypothetical pathway along which students can progress in their learning. In the context of the classification of quadrilaterals, an understanding of inclusive classifications implies conceptual progress for students since it enables them to understand inclusive definitions (for example, that a square is a special type of rhombus). This understanding is, therefore, a key developmental understanding (KDU) in the learning trajectory for the classification of quadrilaterals.

Our research is located in the field of prospective teachers' learning related to identification of secondary school students' understanding employing learning trajectories. In particular, our research question is: how do prospective secondary school mathematics teachers learn to recognise the inclusive relations in the classification of quadrilaterals as a key developmental understanding?

PARTICIPANTS AND THE LEARNING ENVIRONMENT

The participants were 48 prospective secondary school teachers (PTs) enrolled in an initial training programme. The participants comprised mathematics and engineering graduates pursuing training to become secondary school mathematics teachers. They were enrolled in a subject focus on the interpretation of secondary school students' mathematical thinking.

One of the learning environments of this subject was focused on students' mathematical understanding about classifying and defining quadrilaterals. The learning environment consisted of six sessions each lasting two hours, and an online discussion in which prospective teachers participated for 10 days. The design of the learning environment incorporated a socio-cultural perspective (Wells, 2002) and considered four aspects: Experience, Information, Knowledge Building and Understanding. ''Experience'' is the prior knowledge that prospective teachers have built during their participation in learning and teaching situations. ''Information'' consists of our understanding (as a scientific community) of the quadrilateral classification processes (theoretical information) that we, as teacher educators, provide to prospective teachers (De Villiers, 1994; Usiskin, & Griffin, 2008). ''Knowledge building'' is related to how prospective teachers engage in meaning-making with others in an attempt to extend and transform their understanding of a student's mathematical thinking; finally, ''Understanding'' constitutes the interpretative framework in terms of which prospective teachers make sense of new situations, that is, what they mobilise to identify students' mathematical thinking. Figure 1 shows the design of the learning environment.

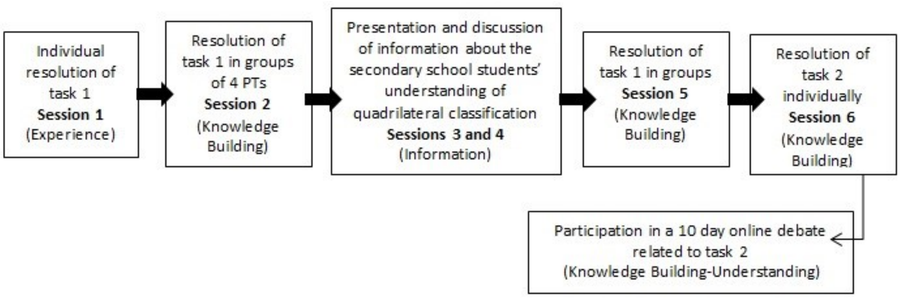


Figure 1. Design of the learning environment

Tasks

The two tasks (task 1 and task 2) consisted of two quadrilateral classification problems (ages 14-15), and six questions about teaching and learning in which prospective teachers have to anticipate possible secondary school students' answers to the problems reflecting different levels of understanding and make teaching decisions:

A1. Indicate exactly what Maria, a 3rd year secondary school student (aged 14-15), would have to do and say in each problem in order to demonstrate that she has achieved the learning objective assigned for the problem (Classify the quadrilaterals according to different criteria).

A2. Explain which aspects of Maria's answer to each problem make you think that she has understood the classification of quadrilaterals. Explain your answer.

B1. Indicate exactly what Pedro, another 3rd year secondary school student (aged 14-15), would have to do and say in each problem in order to demonstrate an understanding of certain elements of the classification of quadrilaterals while remaining unable to achieve the learning objective. Explain your answer.

B2. Explain which aspects of Pedro's answer to each problem makes you think that he has not achieved the intended learning objective. Explain your answer.

C. If you were the teacher of these students, how would you modify/extend the task in order to confirm that Maria has achieved the intended learning objective? Explain your answer. How would you modify/extend the task so that Pedro achieves the intended learning objective? Explain your answer.

The first four questions refer to the prospective teacher's ability to anticipate possible answers to the problems that reflect different levels of secondary school students' understanding of the process of classifying quadrilaterals. The last two questions (section C) are related to teaching decisions; in other words, to the decisions that the teachers should take in order to promote student progress on the learning trajectory. In this research report we are going to focus on the first four questions.

Textbook problems in task 1 and task 2 were different but they implied the classification of a set of quadrilaterals, applying different criteria. Figure 2 shows the two quadrilateral classification problems of task 2.

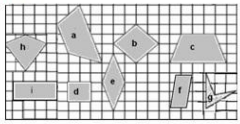
ANALYSIS

The data analysed in this study consisted of the written answers given by the prospective teachers in response to tasks 1 and 2, both individually and in groups (face-to-face and virtual). We identified the responses to the problems that prospective teachers considered as evidence of different levels of understanding of the classification process and the justifications provided (learning trajectory). Then, we identified how the previously discussed theoretical information changed what the PTs understood as evidence of secondary students' understanding of the classification process (taking into account how PTs considered the inclusive relations as a KDU).

The data was analysed by four researchers generating characteristics of the prospective teachers' answers. The initial characteristics were redefined as new data was added. Points of agreement and disagreement were discussed, with the aim of reaching a consensus on the inferences from the data by means of a process that looked for evidence that did or did not confirm the characteristics initially produced.

Problem 1

1.1. Classify the following quadrilaterals according to the congruence of all their sides and angles

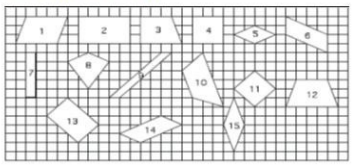


		All the angles are equal	
		YES	NO
All the sides are equal	YES		
	NO		

1.2. Answer the following questions:
A. Are there any quadrilaterals in which all the sides and angles are equal? Which ones?
B. Are there any quadrilaterals in which all the sides are equal but not all the angles are equal? Which ones?
C. Are there any quadrilaterals in which all the angles are equal but not all the sides are equal? Which ones?
D. Are there any quadrilaterals in which all the sides and angles are unequal? Which ones?

Problem 2

Which of the following quadrilaterals:



A) Are parallelograms? Explain your answer.

B) Are rhombuses? Explain your answer.

C) Are rectangles? Explain your answer.

D) Are squares? Explain your answer.

Figure 2. The two quadrilateral classification problems of task 2 adapted from Fujita's research (2012, p. 63)

RESULTS

Prospective teachers initiated the learning environment from three different standpoints. One group believed that understanding the classification of quadrilaterals was linked to defining prototypical figures, identifying all the properties that distinguished them from one another. This generated singleton subsets without relations between them (exclusive classification). The second group of prospective teachers held the understanding that the classification of quadrilaterals was linked to being able to perform classifications by forming non-singleton sets but without specifying any inclusive relationship between figures within a set. Lastly, a third group of prospective teachers linked understanding of the classification of quadrilaterals to students' abilities to establish relationships between some properties of quadrilaterals.

This made it possible to link understanding of the classification of quadrilaterals to the ability to recognise, for example, that squares can be considered a particular type of rhombus (inclusive classification).

By the end of the learning environment, we identified three changes in how prospective teachers identified inclusive relationships as a KDU for the classification of quadrilaterals (Figure 3). Some prospective teachers became aware that a student's ability to establish inclusion relations between the figures within a set constituted evidence of understanding the classification of quadrilaterals (changes 2 and 3, Figure 3), that is, these PTs were able to consider a square as a particular kind of rectangle. So these PTs began to use the understanding of inclusive relations as an indicator of students' conceptual understanding (KDU). However, even after completion of the tasks and discussions in small groups and a large group, one group of prospective teachers still did not recognise the role played by the understanding of inclusive relations as a key developmental understanding of the classification of quadrilaterals. These prospective teachers considered that understanding was evidenced by the ability to create partitions of the set of quadrilaterals assuming the existence of non-singleton sets, but without progressing towards recognition of the relationships between the properties that would generate some kind of inclusive classification (change 1, Figure 3). For these prospective teachers, the development of secondary school students' conceptual understanding was simply the visual recognition of the figures' properties without underscoring the relationship between them. In these cases, the prospective teachers relied on visual aspects and prototypical examples linked to the definitions that they knew in order to generate indicators of understanding, and were therefore unable to consider achieving an understanding of inclusive relations as a learning goal for students.

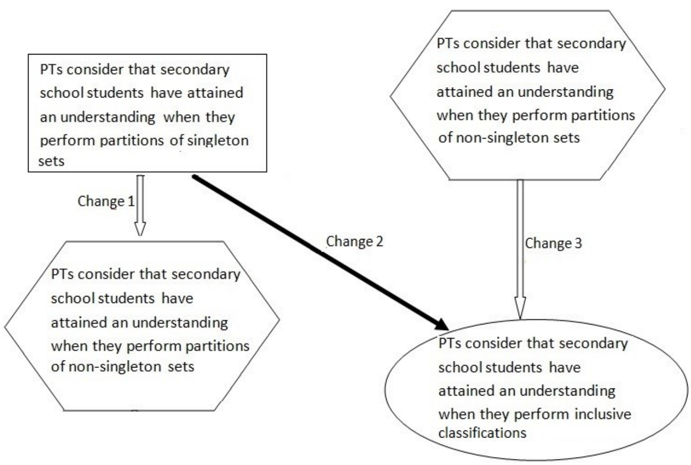


Figure 3. The changes identified that characterised prospective teachers' learning

The three changes that became apparent enabled us to identify transitions in prospective teachers' learning when they were learning about secondary school students' understanding of the process of classifying quadrilaterals. The results described indicate the characteristics of prospective teachers' hypothetical learning trajectory for the development of students' understanding. In this trajectory, some prospective teachers moved from considering that students' understanding was evidenced by their ability to generate partitions with non-singleton sets to recognising that inclusive relations constituted a key developmental understanding that determined the development of students' conceptual understanding (Figure 4).

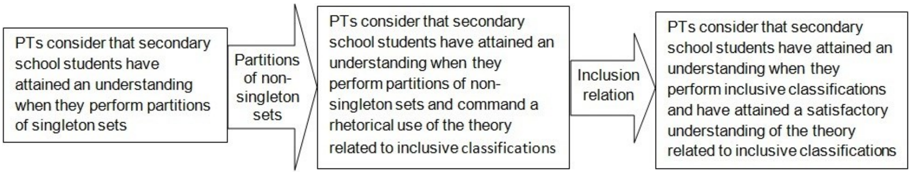


Figure 4. Prospective teachers' hypothetical learning trajectory for the classification of quadrilaterals

DISCUSSION

Our results suggest that recognition of inclusive classifications as a KDU enabled prospective teachers to modify what they considered evidence of students' understanding of the classification of quadrilaterals. Therefore, KDU construct (Simon, 2006) can help to shed light on prospective teachers' learning about students' mathematical thinking, in our case, in the extent to which the understanding of inclusive classifications was seen as an important element in explaining the development of secondary school students' conceptual understanding.

The study presented here is an example of how knowledge about students' mathematical thinking generated by research can be used in teacher training and provides information about how we can understand the prospective teacher' learning about students' mathematical thinking. Although it is not possible to generalise from the results, these offer evidence of how the use of knowledge about students' mathematical thinking in teacher training can help focus prospective teachers' learning on "key developmental understanding" related to student learning.

Acknowledgements. The study reported here has been financed in part by Ministerio de Educación y Ciencia, Dirección General de Investigación, Spain, under Grant no. EDU2011-27288 and in part by the Project I+D para grupos de investigación emergentes GV/2014/075 of the Conselleria de Educación, Cultura y Deporte de la Generalitat Valenciana.

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THE CLASSROOM DISCUSSION AND THE EXPLOITATION OF OPPORTUNITIES TO LEARN MATHEMATICS

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We analyse effects of a planned classroom discussion on the students' ability to improve their initial solutions on a similarity task. Data was collected from the preparation of the classroom discussion, the actual discussion, and from 14 and 15-year-old students' written responses to the task managed by the teacher. It was revealed that a systematically prepared classroom discussion became useful to create and exploit opportunities to learn mathematics related to similarity. However, results from one particular lesson showed that the exploitation of learning opportunities could easily be overrated, since learning was fragmented in the discussion and not all students completely took up the mathematical findings.

INTRODUCTION

Mathematics education research has placed much emphasis on the study of teaching practices that teachers can learn in order to improve the didactical effects of classroom discussions (Stein & Smith, 2011). Research on the potential of whole-class settings is a crucial resource for mathematics teaching, since teachers find it hard to lead quality classroom discussions and do not usually know what to focus on and how to do it (Smit, Van Eerde & Bakker, 2013). For the purpose of studying to what extent and how classroom discussions create opportunities to learn in secondary students, we present knowledge from a study with the following goals: (a) to characterize the instructional resources and strategies that can create opportunities to learn about similarity in a classroom discussion, and (b) to determine the effect of a classroom discussion on the student's ability to revise and improve their initial solutions on a similarity task.

THEORETICAL NOTIONS AND PERSPECTIVES

The notion of *opportunity to learn* has been extensively studied in the literature. Brewer and Stasz (1996) considered three aspects to define opportunities: curriculum content, instructional resources and instructional strategies. These aspects cannot be seen as something rigid, but as a connected structure, since "developing a fuller picture of classroom activities hinges on being able to identify interactions and overlap between curriculum, pedagogy and resources, and their effects on learning" (Brewer & Stasz, 1996, p. 3). Further, the quality of the opportunities to learn can be connected to the quality of the instruction received by students and their alignment with the classroom norms for justification and with what is assessed during the lessons.

In particular, opportunities to learn mathematics in a classroom discussion can be studied by focusing on the two connected aspects: (a) *instructional resources,* that is, the contents of mathematical knowledge, procedural and conceptual with respect to the

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 289-296. Hobart, Australia: PME.

available tools and tasks and expected solutions (Niss & Højgaard, 2011); and (b) *instructional strategies,* that is, the types of instrumental orchestration and discursive choices (Drijvers et al., 2010; Morera & Fortuny, 2012), and the actual discursive actions generated by the interaction processes of the mathematics classroom that potentially contribute to facilitate the students' learning.

Beyond being able to identify opportunities to learn, an additional issue is the task of studying how to connect opportunities with student achievement. We suggest that in order to decide whether a certain opportunity to learn is exploited, evidence of change in mathematical understanding is needed, in the context of utterances or written responses of students during classroom discussion. Consequently, the study of opportunities to learn requires a prior systematic analysis of instructional situations, which focuses on instrumental types of orchestration, discursive choices, actions in classroom discussions and progress in students' mathematical utterances and writing.

Preparing the classroom discussion and the selection of instructional resources

Teacher's preparation of classroom discussions is essential for promoting productive lessons. Stein and Smith (2011) identified and described several teaching practices that support mathematical discussions. Next to the discursive choices that need to be considered, the teacher also needs to take into account possible solutions or strategies. The potential of these contributions is highly dependent of the tasks and the resources that were provided to the students in advance as a preparation for the classroom discussion. Therefore, the selection of appropriate tasks is crucial for being able to elicit and build on students' thinking in a discussion. For instance, challenging open-ended tasks and cast in context that is realistic for students (Gravemeijer & Doorman, 1999) have the potential to evoke key concepts about the studied topic (e.g., similarity). The tasks also need to have an appropriate cognitive demand to provide starting points for a particular target group to work with main notions of the topic (e.g., shape and ratio); ideally, that work results in a rich variety of responses that offer starting points for classroom discussion. Furthermore, the selection of different tools, such as GeoGebra, and the orchestration of the discussion through these tools are important aspects to consider when preparing a classroom discussion.

Episodes and actions of a classroom discussion

The episodes of a classroom discussion are defined through the articulation of: the instrumental dimension, about the artefacts and the way in which these are used in class, and the discursive dimension, about the interactional patterns that help to understand the generic development of the episodes. In the instrumental dimension, six types of instrumental orchestration are considered: *exploring the artefact*, *explaining through the artefact*, *linking artefacts*, *discussing the artefact*, *discovering through the artefact* and *experiencing the instrument*. They are all inspired by the types of Drijvers et al. (2010), but have been generalised for instructional situations that do not necessary contain an intensive use of the technological artefacts.

The discursive dimension is framed in terms of stages of the discussion of a problem, which are organised according to an idealized development of the solution process: *situating the problem*, *presenting a solution*, *studying different solutions or explanatory strategies*, *studying particular or extreme cases*, *contrasting solutions*, *connecting with other situations*, *generalizing and conceptualizing*, and *reflecting on mathematical progress* (Morera & Fortuny, 2012). In addition, we interpret episodes as systems of actions that occur during the actual course of the discussion. Differently to how episodes are seen, actions are tied to the subject performing the action, student or teacher, and their role in the organisation of participation during discussion.

METHODS

Context and data

We created an environment with an instructional sequence of similarity problems to be implemented in a secondary classroom of Barcelona over a total of eight lessons. For this report we have selected the first task in the sequence (see Fig. 1), whose wording presents an open problem and whose solution is tied to proportional thinking. This problem is expected to create opportunities to learn, because it has more than one solution strategy and connections need to be made with underlying mathematical concepts. When solving the task some students can draw a figure by doubling all sides of the original polygon and they can obtain a figure with twice the perimeter. At the same time, others can draw a polygon that has twice the area, which will be only similar to the original when the ratio of all corresponding sides is $\sqrt{2}$.

Doubling figures!

Given the following letter from the alphabet, represent another letter that is twice the size. Below, briefly explain how you obtained it and compare it with the original.

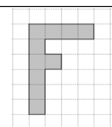


Figure 1: Formulation of the task

The whole sequence of tasks was discussed with three researchers and a secondary teacher, Laura. The teacher was used to participating with activities of our research group and was motivated to take part in research to improve her teaching and her students' learning. At the moment of the experiment she had eight years of teaching experience and was working in an urban school of a medium-high social area. In the first meeting the teacher explained that students had, in general, a positive attitude towards mathematics and a medium-high development of mathematical skills.

Data was collected in February 2014. The first author was present in all lessons, but did not intervene in the development of the activity. In the twenty-minute classroom discussion for the task in Figure 1, three video cameras recorded the participation of the students and the teacher. Sixteen students' written materials, produced during pair and individual work after classroom discussion, were collected. Answers were on the same document, although students were required to write down their responses in different colours, blue or black for pair work, and red for individual reflection.

Analysis

The teacher's preparation of the classroom discussion, the classroom recordings and created transcripts, and the students' responses to the task were analysed with a qualitative approach. Firstly, we searched for instructional resources and strategies that had the potential to create opportunities to learn mathematics about similarity during classroom discussion. In particular, we analysed the preparation of the discussion by the teacher, the teaching practices she selected to manage the discussion, the chosen artefacts and the criterion for the selection of interventions.

Actions within the episodes of the discussion were analysed to investigate relationships between discursive and instrumental choices and the possibilities to exploit intended opportunities to learn. We analysed effects of the classroom discussion on the students' ability to revise and improve their initial solutions on the similarity task. With the focus on the identified opportunities to learn, we compared the students' responses before and after the discussion in order to determine particular elements that were included in the students' individually written solutions. We looked for instances of students' exploitation of opportunities to learn by searching for explicit changes in the understanding of the mathematical procedures and contents that had been involved in actions during classroom discussion.

RESULTS

Some results of the study are presented as a storyline, by linking teacher's actions and students' learning processes. The storyline is illustrated with an excerpt of the classroom discussion and with selected students' responses to the similarity task.

Preparation of the instructional resources and the classroom discussion

A diagram to scaffold the students' mathematical activity was obtained due to the attempts by Laura to foresee the students' responses to the task before pair work (see Fig. 2). The diagram included possible difficulties and misconstructions (e.g., some students might forget to double the central part of the figure when trying to double the perimeter, or could make a drawing that did not have twice the size), two different strategies to solve the task (doubling perimeter or area), and questions to involve students in new mathematical challenges or to help them in solving the task (e.g., finding strategies to solve the problem in other ways).

The prepared guidelines for the experiment were thoroughly preserved by the teacher. Starting the lesson with a resolution in pairs let students to agree on an initial solution to the task. Later, different ways of understanding the wording could be compared and a definition of similarity based on the equality of homologous angles and the proportionality of corresponding sides could be introduced in classroom discussion. The students could add relevant elements from the discussion in their individually written reflections that improved the initial solutions.

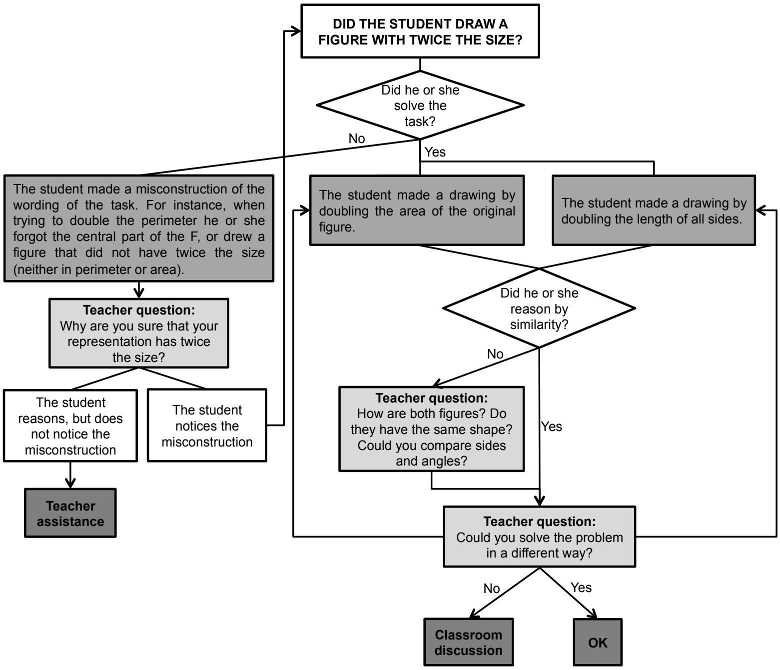


Figure 2: Lesson plan prepared by the teacher

Related to the selection and use of artefacts, Laura decided to combine a paper-and-pencil resolution of the task in pairs and the use of blackboard and GeoGebra during classroom discussion. Although GeoGebra was not essential to solve the task, the teacher said to have decided to use it in order to improve the visualization of the two interpretations of the wording: doubling perimeter and area.

Laura worked on the selection of particular situations in advance so as to organize the discussion. After having reviewed pair resolutions, she decided to start the discussion with drawings that doubled the length of all sides of the original polygon so that, later, they would discuss what happened when doubling the area. For instance, the teacher selected Martí to intervene in the classroom discussion since his solution, by doubling all sides, could be useful as a starting point of the lesson.

The teacher followed an ordered preparation of the instructional resources and the classroom discussion. She prepared an extensive diagram about the resolution of the task, preserved the teaching experiment thoroughly, selected and used different artefacts, and identified appropriate starting points for the discussion.

The exploitation of an opportunity to learn mathematics

The classroom discussion for the exemplified lesson was divided into six episodes that were characterized according to a type of instrumental orchestration and a discussion stage. We have selected the fifth episode of the discussion (*discovering through the artefact*, *generalizing and conceptualizing*). In the transcript below (turns are marked with T –Teacher or S –Student) we can observe that the teacher was guiding the discussion, although what was contributed could not all be predicted due to some student's unexpected interventions that changed the planned dynamics of the discussion. Víctor generalized the information by the artefacts and made a statement about the use of √2 to double the area by preserving similarity.

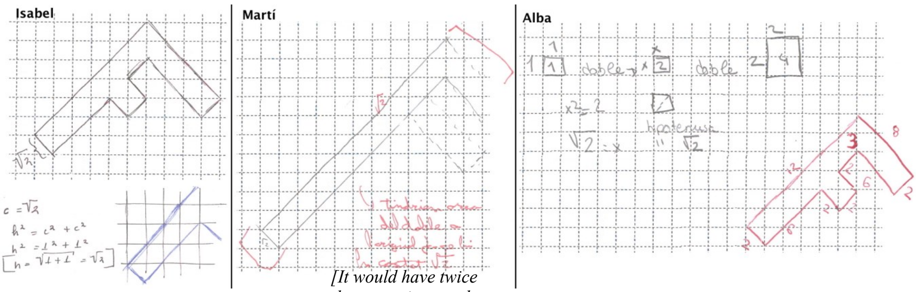
T - Laura:	The area is 40. So, you have not made twice the size in area, you have multiplied it by four, haven't you?
S - Víctor:	Yes, but then, using this grid, there is not any side that measures √2.
T - Laura:	You are telling us you would like to have √2. Why do you need it?
S - Víctor:	To double the area, but using this grid is not possible.
T - Laura:	Has everybody understood this statement about √2?
S - Group:	No.
T - Laura:	[to Víctor] Please, try to explain it better.
S - Víctor:	Because in the initial figure each side measures 1 and, here, each side measures 2 [referring to the figure with twice the perimeter], so its area is four times bigger. We would like to have a square whose sides had a measure of √2, because then the area would be 2.
T - Laura:	[drawing squares of different sides on the board] Okay, to draw a square with area 2 you need that all sides measure √2.

Víctor's intervention emerged in the discussion, without being prepared in advanced by the teacher. For this reason there was uncertainty about the opportunities to learn created in that classroom discussion, since any situation, planned or not, could generate mathematical learning to the whole group. Víctor's emphasis on the need to use √2 to make twice the size in area, created a procedural opportunity to learn, that of 'using √2 to double the area by preserving similarity'. The opportunity emerged due to the student's empirical justification, although teacher's follow-up questions and revoicing of this student's explanations were crucial to bringing about this opportunity to learn.

Focusing on the above opportunity to learn, we looked for instances of explicit changes in students' solutions, before and after classroom discussion, that suggested the creation of mathematical knowledge. Particularly, we examined whether students could double the area of the original figure by preserving similarity. We paid attention to whether the student drew a new figure correctly in diagonal by using the grid, wrote the term √2 on the paper, and identified √2 on the representation.

Our results reveal three types of different solutions about the use of √2 when the students solved the task after classroom discussion: '√2-similar rotation' (i.e., rotation of the initial figure to double the area by preserving similarity), '√2-non-similar rotation' (i.e., using the grid to rotate the figure, but doubling the area incorrectly or not identifying √2 properly), and 'solving the task without rotation'.

After the discussion twelve out of sixteen students drew a figure in diagonal, although only four out of the twelve solutions were completely correct due to preserve similarity and identify √2 correctly on the drawing (see Isabel's solution in Fig. 3). In addition, three out of the twelve responses also used the diagonals of the grid to rotate the figure, but its area was not twice the initial. For instance, Martí's solution (see Fig. 3), although not being complete, had a new drawing that was bigger than twice the area.



[It would have twice
the area, since each

Figure 3: Solutions of Isabel, Martí and Alba about the use of √2

Many students wrote down the term √2, which was obtained by applying the Pythagorean theorem, but five out of the twelve students with √2-responses did not identify the mathematical meaning of √2 correctly. For instance, Alba represented a square whose sides measured √2, but the drawing was too small and its area clearly measured less than one square unit. This student did not identify correctly the lengths of the sides on the rotated figure, since all of them had an integer measure instead of having a square root length (see Alba's solution in Fig. 3).

In brief, most students identified the need to rotate the initial figure to double the area and preserve similarity by using the grid. However, certain solutions were mathematically incorrect or incomplete, so that not all students took advantage of the opportunities to learn in the same way, although all had been involved in the discussion and the opportunities were created apparently the same for all of them.

FINAL DISCUSSION

Our first goal was to characterize resources and strategies that created opportunities to learn mathematics in classroom discussions. The study identified a systematically ordered preparation of the instructional resources and the discussion by the teacher, when managing a lesson of similarity. In particular, the diagram she prepared helped students to scaffold their mathematical activity so as to solve a similarity problem in an environment that combined work in pairs, classroom discussion and individual written solutions. This environment afforded to put in practice the instructional resources and the teaching strategies that helped students to exploit the opportunities to learn created in the discussion. Furthermore, the selection and use of technological artefacts such as GeoGebra in class became useful to promote the students' learning through the projection of simulations that showed the application of problem solving strategies.

The study of the second goal, which was about effects of classroom discussion on the students' ability to improve their initial solutions to the task, revealed that the opportunity to learn created in the discussion could be exploited by students afterwards. We have detected a positive effect of the teacher's interventions, mostly based on follow-up questions and revoicing of students' contributions, on the students' ability to comprehend and solve the task. However, individual responses after discussion showed that certain processes, such as identifying the meaning of $\sqrt{2}$ to double the area by preserving similarity, were not taken up completely. Further research is needed to study classroom discussions of other problems and more teachers to determine how to better prepare follow-up questions and activities to involve all students in discussions and make progress possible for all students.

Acknowledgements

Project EDU2011-23240 and Grant FPI BES-2012-053575, Spanish Ministry of Economy and Competitiveness; SGR2014-972, GIPEAM, Catalan Government. We greatly appreciate the comments made by Núria Planas (Universitat Autònoma de Barcelona) on previous drafts of this report.

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"IF IT DOESN'T HAVE AN APEX IT'S NOT A PYRAMID": ARGUMENTATION AS A BRIDGE TO MATHEMATICAL REASONING

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Argumentation offers potential for students to engage in deep scientific learning, and to be enculturated into the practices of science. The need to make a claim, provide evidence, and justify the claim using evidence, serves to deepen students' scientific reasoning. The research reported here introduces a model of argumentation to a class of Year 5 students through a geometry problem: "Can a pyramid have a scalene face?". Observations suggest that many of the benefits of classroom argumentation practices in science may be apparent in mathematics education.

INTRODUCTION AND LITERATURE REVIEW

Argumentation refers to the collaborative, discursive *process* that leads to the argument as a *product*. Argumentation has been introduced, with reported success, into school science (Jimenez-Aleixandre & Erduran, 2007). Students have opportunities to engage with scientific phenomena and seek to explain the phenomena with theories that they support using scientific reasoning. In this way, students are supported to engage in science as if they were practitioners themselves. By doing so, they are enculturated into science as a discipline, learning its language, conventions, practices and so forth. Could the same be done with school mathematics?

Colloquially, an argument is often presented with the purpose of 'winning'; that is, to convince someone of a statement or concept, or to have them take a specific action. To these ends, a variety of means may be adopted, including the employment of logic, fact, expert statement, or statistics, any of which may be purposely and selectively presented to support a particular viewpoint; even plays to emotion or values (Toulmin, Rieke, & Janik, 1984). *Scientific argumentation* has a goal of collaboratively exploring and resolving an issue in order to construct an explanation which best fits all available evidence and logic (Sampson & Clark, 2008). As such, scientific argumentation is considered "a social and collaborative process necessary to solve problems and advance knowledge" (Duschl & Osborne, 2002, p. 41). Such an approach values high quality evidence which is open to critique, accurate and valid. *Epistemic argumentation* (Siegel & Biro, 1997) identifies argumentative discourses as those which collectively seek the truth through critical reasoning and justification. Ideally, not only is the final claim agreed upon, but there is also consensus regarding the evidence and justification leading to the claim with the strength of an argument evaluated on epistemic criteria only (Lumer, 2010).

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 297-304. Hobart, Australia: PME.

Jimenez-Aleixandre and Erduran (2007) propose five primary benefits to the use of argumentation in the scientific classroom: Supporting access to the cognitive and metacognitive processes that characterise expert performance; supporting the development of communicative competencies; supporting the achievement of subject-specific literacy; supporting enculturation into the practices and the 'ways of knowing' of a discipline; and, supporting the development of reasoning based on rational criteria. Many of these benefits would come about through allowing students and the teacher to access the cognitive processes, specifically the thinking and reasoning, of each other.

Argumentation enables the exchange of opposing views, the articulation of conjectures, thoughts and understandings, and the opening of these to exploration and challenge (Leitão, 2000). This discourse may also incorporate representations, such as diagrams, models, graphs, or equations. Engaging in discourse enhances students' ability to appreciate alternative perspectives and approaches while enabling teacher insight into student's understandings and the potential to identify and challenge immature conceptions (Jimenez-Aleixandre & Erduran, 2007). The teacher is also able to model the cognitive and metacognitive processes of the practitioner through discursive interaction.

While potential benefits to using argumentation within the science classroom have been well documented, little research into Inquiry-Based Argument in mathematics education is apparent, and much of what exists more specifically relates to proof. Thus, the aim of the research reported here was to engage younger students in epistemic argumentation in mathematics learning and to observe whether potential benefits noted in science might also be applicable to mathematical thinking and reasoning.

THEORETICAL FRAMEWORK

Toulmin et al. (1984) provided an argument structure which has been widely adopted for the composition and decomposition of arguments, as well as identification of fallacious aspects of arguments. However, this structure has been criticised when used in research with children due to both the complexity of the structure and difficulty in identifying argument components (Erduran, 2007). One model that has been used with students in science education is Claim-Evidence-Reasoning (CER) (McNeill & Krajcik, 2012). The claim is the conclusion that addresses the original question, evidence is the data that support the claim, and the reasoning is the logic that enables the evidence to be used to establish and support the claim (McNeill & Krajcik, 2012).

For the purpose of science education, the reasoning should include the big idea or science concept that is the focus of the lesson. Including reasoning in the argument encourages students to consider and reflect on these scientific ideas, as well as improving their fluency with scientific terms and language (Jimenez-Aleixandre & Erduran, 2007). To extrapolate to mathematics, reasoning would incorporate the big ideas, principles, language and terminology of mathematics.

METHODOLOGY

The aim of the research reported here was to engage younger students in epistemic argumentation in mathematics learning and to observe whether potential benefits noted in science might also be applicable to mathematical thinking and reasoning.

This type of learning differs significantly from current educational practices and therefore conditions had to be explicitly created in which the instructional theory could be developed and tested. Design Research was chosen as the methodological approach as it is consistent with planning learning interventions and then systematically studying the learning reflectively and within its context (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Lesh, 2002). The research consisted of cycles of preparation and design, experimental teaching, and then analysis and reflection, which in turn led to the next teaching cycle. Design Research is built on the premise of making the research relevant to practice (Lesh, 2002) and this was an essential component of the research.

The findings reported here derive from a larger project in which students were first introduced to an Evidence Model (Figure 1, left) for use with a proportional reasoning inquiry problem (Fielding-Wells, Dole, & Makar, 2014). The second iteration, reported here, was the introduction to students of argument structure as an extension to the Evidence Model. This is referred to as the Argument Model (Figure 1, right). A third iteration extended students mathematical knowledge to critiquing and improving mathematical arguments based on statistical data generated by students (not yet reported).

The participants consisted of 27 children (aged 10-11 years) in a Year 5 class from a metropolitan government school in Queensland, Australia. The class had significant prior experience with Inquiry-Based Learning (IBL) and it was thought that this would prepare them for moving into argumentation through an existing focus on evidence. In engaging in IBL, these students had become accustomed to addressing ambiguous or ill-structured questions; however, the question posed in this instance, "*Can a pyramid have a scalene face?*", was more structured than usual in order to assist the students to focus on the evidence needed rather than complexity of the question context.

RESULTS

The students commenced the unit by reviewing the Evidence Model (Figure 1, left) and then engaging in discussion around the components of a conclusion. Between students' prior knowledge of expositions and guided questioning from the teacher, the Evidence Model was expanded into a model of Argumentation (Figure 1, right). The specific details of this development are reported elsewhere (Wells, 2014).

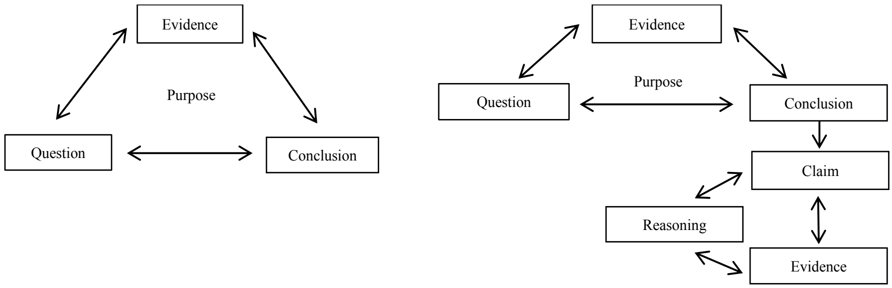

Figure 1: The evidence model (left, Fielding-Wells, 2010) and argumentation model (right, Wells, 2014).

Developing Evidence

The question, "*Can a pyramid have a scalene face*?", was originally posed by a student at the completion of a more traditional geometry lesson on pyramids. As the students were considered to have the necessary mathematical understanding to address the question, at least initially, they moved into groups to consider useful evidence they might generate. After a few minutes, each group shared their ideas. It was apparent that students could easily envisage evidence they needed to demonstrate a scalene-faced pyramid, but envisaging evidence for a converse claim caused difficulty; with students suggesting it could only be supported by a large quantity of 'failed' attempts. An advantage of students working first in groups and then sharing was that it enabled them to question each other's ideas and thus refine and improve their own plans.

Each group had decided to produce one or more of a net, model or diagram, but the mathematical evidence was more easily conceptualised than achieved. Under the Australian curriculum (ACARA, 2014), students would be expected to have deconstructed 3D shapes to determine their nets prior to Year 5. However, designing an irregular net or model was far more complex than simply folding an existing one. Most students drew what they conceived the net of a scalene-faced pyramid would look like and then attempted to fold it to make a model. The inherent difficulty afforded the opportunity for students to strengthen their understanding and appreciation of pyramids and their attributes. Ongoing errors, such as numbers of faces needed, provided inherent feedback, but one issue which caused greater difficulty was consideration of the lengths of the sides on each face.

A significant breakthrough came when Delmar realised that adjacent sides (on the net) had to be of equal length. As more students struggled with this problem, the teacher asked Delmar to share his discovery. Delmar drew a diagram on the board and indicated the edges as he talked.

Delmar: So you have to make those exactly the same (points to two adjacent triangle sides which will come together to form a common edge)

otherwise one side of the face won't reach up to the other. …
[the teacher uses this as a teachable moment to review the convention for marking sides to indicate equal length]

Teacher: Tell me something. Is this a general rule? I mean if I build any pyramid now that follows this rule, is it going to work? [draws a net in which every pair of adjoining sides are equal length]. Is that always going to make a pyramid?

Delmar: It should.

The students engaged with building pyramids: assisting each other as necessary. They quickly discovered that even when a net was constructed that met Delmar's criteria for adjacent edges being the same length, they still failed to form pyramids. When students began to struggle, the teacher instigated a class discussion to enable the students to support each other. A few students had successfully created a scalene-faced pyramid using three different but successful approaches. They shared their methods with the class, illustrated below in the students' own words; for example:

Sally: Well I drew a scalene triangle and I cut it out then I traced around it to make like one of these [a face] and so I cut it out and traced around it and kept going.

Lucy: I thought maybe you could get a pyramid with equilateral sides and then just adjust them to make it scalene and then see if that would work.

These examples enabled their classmates to find a way forward; afterwards, students were able to use one of these methods, or a method inspired by these methods, to construct a scalene-faced pyramid. They worked to develop accurate models and continuously assessed the quality of their models and nets. In this respect, the task itself served to provide feedback to students, rather than the teacher or peers.

Having made the pyramids, students needed to consider how the pyramids served to support the final claim that a pyramid could have a scalene face, and how they could best articulate and link the mathematical justification for making such a claim.

Developing Reasoning

Reasoning is essentially the connector between claim and evidence: it is that which justifies making an evidence-derived claim. Reasoning can serve to identify the mathematical principles that underlie a shift from evidence to claim. It is here that the need for deeper mathematical understanding becomes apparent. In the excerpt below, the teacher engaged the students in a discussion around reasoning; at later stages the teacher, and later the students, took on the role of audience and challenged class members who they felt had not provided adequate mathematical evidence. Once the students had discussed the importance of reasoning as a whole class, instances of reasoning became apparent in their small group discussions:

Teacher: And what will the model actually have to convince me that it has a scalene face? Samuel, can you think of a way that you could actually show me that it is a scalene face?

Samuel: Maybe show the lengths [of the sides of the faces].

Teacher: You also said that you wanted to show that it had an apex. Why would you want to show that your pyramid has an apex?

Kody: Every pyramid has an apex and if it doesn't then it means it's not a pyramid.

Teacher: So you're trying to convince me both that it has a scalene side *and* that it is a pyramid?

It was during these conversations that the students' surface knowledge of pyramids became apparent as they hurried to mathematics dictionaries to determine answers to questions they had not previously considered: Must the apex be centred over the base? How do you know which face is the base on a triangular-based pyramid? Must a pyramid have a regular base? Can a pyramid's base be any shape? In this way, students developed a richer understanding than they otherwise would have, as few of these issues were addressed in the regular coverage of curriculum content.

DISCUSSION AND CONCLUSION

This research aimed to engage younger students in epistemic argumentation in mathematics learning, and to observe salient features of argumentation practice, with particular focus on sharing of thinking and reasoning. While it was only possible to present a brief snapshot of the dialogue that took place, the process of developing evidence, making a claim based on the evidence, and the identification of the mathematics that makes it so (the reasoning) can begin to be appreciated. Jimenez-Aleixandre and Erduran's (2007) five benefits of argumentation on science education can be identified in this activity.

Supporting access to the cognitive and metacognitive processes that characterise expert performance: There is sometimes a perception in mathematics that it is a fixed body of facts, that there is nothing to discover. However, mathematicians wrestle with puzzles of mathematics regularly - albeit more complex ones. In this instance, students had the opportunity to pose a question of their own and apply the process of collective mathematical argumentation in order to support a response. In this way, students were empowered to create their own approaches and take multiple approaches to solving a problem. They were exposed to the cognitive processes of others as ongoing classroom discussions (in groups or whole class) enabled the thinking of others to guide and assist students with their own thinking [Lines 9-13].

Supporting the development of communicative competencies: There is a significant difference between students holding what they believe to be a 'correct' understanding and students articulating that understanding in such a way as to be understood by others and potentially have the understanding challenged (Sampson & Clark, 2008). In this unit, students repeatedly posited ideas, such as "It isn't a pyramid because the apex isn't over the centre of the base!", only to have those ideas challenged by peers.

Supporting the achievement of subject-specific literacy: As students engaged in discussions, many mathematical terms were initially used hesitantly or not at all [for example, Line 10]. By the end of the unit, students were demonstrating greater ease with mathematical language, possibly due to the repeated opportunities they had to express their ideas to each other and the teacher. For example, 'faces' and 'sides' were largely used interchangeably initially whereas they had taken on more accurate and specific meaning by the completion. Furthermore, in order for students to write mathematically, including the production of representations, students needed to be familiar with the symbolic code that permeates written mathematical knowledge. In this activity, students became familiar with ways to symbolically represent equality of length (side lengths), and angles and angle sizes.

Supporting enculturation into the practices and the 'ways of knowing' of a discipline: Mathematicians, and those who use mathematics to address problems, rarely do so as a sole participant. Even when they do, the results must typically be provided in a form that is able to be understood by others, explained sufficiently, checked and defended as necessary. Engaging students in such activities serves to introduce them to the discipline of mathematics authentically. Mathematics, like science, values evidence-based explanation, as distinct from emotive reasoning. Through argumentation, students were able to identify, under the guidance of teacher-practitioner as well as other students, what evidence and reasoning serve to satisfy others. In other words, they worked out what is considered acceptable 'ways of knowing' to the discipline.

Supporting the development of reasoning based on rational criteria: This is a central tenet of argumentation with the demonstration of mathematical underpinnings to support reasoning [Lines 14-23].

The excerpts included here give an incomplete picture of the full argumentation unit; however, even such small insight illustrates students at work with mathematical argument and suggests that the benefits may be significant. Certainly benefits similar to those identified by (Jimenez-Aleixandre & Erduran, 2007) are suggested.

The implications for mathematics teaching are of importance. If argumentation has potential for inculcating students into the discipline of mathematics, in such a way as to enculturate rather than provide a surface coverage, then it is at least worthy of further research. Specifically, research into context-rich problems would be of benefit. This study was limited in that the students were being introduced to argumentation for the first time, and so the context and the mathematics were kept relatively simple to encourage students to focus on aspects of the argument.

ACKNOWLEDGEMENT

The authors would like to acknowledge funding from an Australian Postgraduate Award and the Australian Research Council (DP120100690).

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ASSESSING A STATISTICAL INQUIRY

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As statistics education becomes more firmly embedded in the school curriculum and its value across the curriculum is recognised, attention moves from knowing procedures, such as calculating a mean or drawing a graph, to understanding the purpose of a statistical investigation in decision making in many disciplines. As students learn to complete the stages of an investigation, the question of meaningful assessment of the process arises. This paper considers models for carrying out a statistical inquiry and, based on a four-phase model, creates a developmental sequence that can be used for the assessment of outcomes from each of the four phases as well as for the complete inquiry. The developmental sequence is based on the SOLO model, focussing on the "observed" outcomes during the inquiry process.

INTRODUCTION

Contemporary assessment practices acknowledge the advantages of assessing student learning as they work their way through the learning process rather than relying solely on summative assessment conducted upon the completion of a learning sequence. Often termed as "assessment for learning" the evidence gathered is used to monitor student progress and guide the development of subsequent learning activities (Earl, 2003). The goal is to reveal the quality of students' understanding and thinking as well as specific content knowledge and skills development through the integration of assessment into the learning experience. One of the purposes of the shifting emphasis is to support conceptual development of ideas as well as procedural competence. This requires learning sequences to be sustained and ongoing. In statistics education, inquiries that engage students actively in the learning process provide the opportunity for this to be achievable (English & Watson, 2012).

Examples of how to assess the progressive learning outcomes of statistical inquiries are scant. For the most part, assessment reported in statistics education literature is based on statistical literacy rather than actually carrying out a statistical inquiry, involves only part of the practice of statistics, and suggests particular methods such as projects, portfolios, and use of computers (e.g., Bidgood, Hunt, & Jolliffe, 2010; Gal & Garfield, 1997). Research projects have focused on determining the progression of student understanding and application of statistical content or the ability to think and reason statistically, usually accomplished through the evaluation of individual items designed to target particular statistical concepts. This type of research has led to the development of hierarchies of learning that characterise student understanding at different levels, such as a statistical literacy framework (Watson, 2006). There are, however, few examples of research that involves the assessment of student understanding as they work systematically through a statistical inquiry.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 305-312. Hobart, Australia: PME.

TOWARDS A STATISTICAL INQUIRY FRAMEWORK

Fundamental to an inquiry-based pedagogy is the need for teachers not only to allow students to construct their own learning but also to support and scaffold that learning (Bell, Urhahne, Schanze, & Ploetzner, 2010; Makar, 2012). This is fostered through the application of inquiry frameworks that guide the implementation of a series of learning activities. Common to the many inquiry frameworks described in the literature are orientation and questioning processes in the beginning, followed by processes of investigation, and finalised with activities that demand students draw conclusions and evaluate findings (Bell et al.). These processes are exemplified in a traditional science inquiry that involves students working through a sequence of question, predict, experiment, model, and apply (White & Frederiksen, 1998).

Statistics education research offers some examples of frameworks for describing statistical thinking and reasoning that encompass the notions of inquiry. An extensive four-dimensional model proposed by Wild and Pfannkuch (1999) and elaborated on by Pfannkuch and Wild in 2004, is based on the way statisticians work and think statistically and can be applied to the way in which students engage in statistical investigations. It includes: Dimension 1: The investigative cycle, Dimension 2: Types of thinking; Dimension 3: The interrogative cycle; and Dimension 4: Dispositions.

Dimension 1 is related to the thinking processes employed when working through a statistical investigation. This involves posing a question, planning an investigation, collecting data, analysing data, and drawing conclusions. Dimension 2 is related to general and particularly statistical thinking. Wild and Pfannkuch (1999) posit that the types of thinking in this dimension are “the foundations on which statistical thinking rests” (p. 227). Dimension 3 adopts a cyclical process of data interrogation that involves thinking critically about the data in order to distil and encapsulate ideas and information. Dimension 4 includes the personal qualities, dispositions, and habits of mind employed when working with data. These dimensions underpin the way in which people should work statistically but are not all transferred easily to teaching and learning contexts.

More recently, Allmond, Wells, and Makar (2010) have provided guidance on how to develop learning experiences with a mathematical inquiry focus. Their framework encompasses the need to make connections to the context of a problem and to recognise the purpose of investigating a problem; to plan an investigation that provides the evidence needed to answer the problem; to draw on a range of mathematical concepts and skills to collect, represent, and interpret data; and to communicate and justify findings to an audience. Their model includes the following phases:

- Discover: Situating a question within a context and understanding its purpose.
- Devise: Planning an investigation.
- Develop: Engaging in mathematical reasoning.
- Defend: Communicating and justifying a conclusion.

The mathematical inquiry framework developed by Allmond and her colleagues is generic and can be applied to any mathematical investigation. It does not, however, provide specific guidance for designing or implementing a statistical inquiry.

Similar to other inquiry frameworks, the *Model of Statistical Investigation* (Figure 1) developed by Watson (2009) starts a statistical investigation with a question set in a context. The question sets the scene for an inquiry and draws in the context of the data. The Data Collection step provides the data that can be represented in a number of forms – numerical, pictorial or graphical. The data are often reduced using statistical calculations of measures of centre or graphical representations such as a box-and-whisker plot. These representations or measures are then used to make inferences about the data that answer the question posed in the initial stage of the inquiry. Part of the inference step is recognising the level of uncertainty associated with the conclusions drawn. The steps encompass the aspects of working statistically detailed in the *GAISE Report* curriculum framework (Franklin et al., 2007), particularly, "actively collecting, organising, summarising and interpreting data" (p. 63). In addition, Watson's model recognises the importance of context, the notion that not all conclusions can be made with the same level of confidence, and the underpinning idea that variation is fundamental to all statistical inquiries. This model provides a comprehensive view of what students should do as part of an inquiry.

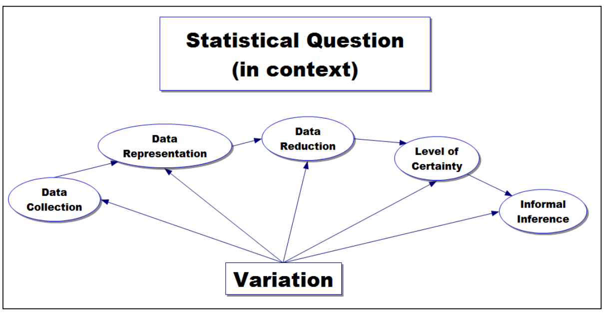

Figure 1. Model of Statistical Investigation (Adapted from Watson, 2009, p. 91).

Acknowledging the potential complexity of the model in Figure 1 for the classroom and in accordance with *GAISE*, an inquiry is summarised in four phases: (1) Pose questions, (2) Collect data, (3) Analyse data, (4) Make decisions.

These four phases constitute a statistical inquiry framework that is more applicable in statistics education than generic inquiry frameworks or those developed for science investigations. At the classroom level it has the potential to support teachers to plan a meaningful sequence of learning that can be communicated easily to students, who can keep track of their progress throughout the inquiry by relating activities undertaken at any stage of the inquiry to the phases of the framework. From a research perspective, it details specifically phases of a statistical inquiry that can be evaluated and interrogated individually to ascertain student understanding at particular phases of an inquiry. It encapsulates the notions of statistical thinking and reasoning highlighted by Pfannkuch and Wild (2004) and the process of inquiry advocated by Allmond et al. (2010).

IMPLEMENTING A STATISTICAL INQUIRY

It is essential for strategies developed for the assessment of a statistical inquiry to accommodate the way in which an inquiry is implemented. This process is not necessarily linear and entry into an inquiry may occur at any of the four phases depending on the background provided to students. This flexibility offers the opportunity to scaffold student learning and stage the development of the necessary skills and strategies for each of the phases before they are required to work through a complete inquiry, which includes working through the four phases sequentially. The advantage of using a staged approach is that skills developed during one phase may be consolidated with the skills developed from another phase (Watson & Fitzallen, 2010), thereby building students' capacity to initiate and work through the full inquiry process.

The implication for using a statistical inquiry either completely or partially is that teaching and learning and assessment practises need to accommodate both scenarios. When teachers and researchers use activities that focus on one of the statistical inquiry phases, they need to be aware of the elements of that phase so that they can provide the support needed for students to bring the ideas from each of the elements together to develop an understanding of the learning outcomes associated with that particular phase.

The way in which elements of understanding can underpin a concept is exemplified in a general developmental model of graph creation (Watson & Fitzallen, 2010), which includes three hierarchical sequences of learning development: the concept of graph, the ability to create or choose appropriate graphs, and informal decision making from graphs. In the context of graph creation and interpretation the second sequences constitutes two parallel sub-sequences: one for when more than one attribute is involved and another for when a large data set is used.

DESCRIBING LEVELS OF UNDERSTANDING

An essential component of the assessment process is having a structured sequence of the expected learning and its outcome. A useful model is the Structure of Observed Learning Outcomes (SOLO) of Biggs and Collis (1982). In assessing learning outcomes, such as related to a statistical inquiry, the focus is on what is observed during the process rather than responding to a test item at a later time.

A neo-Piagetian model, SOLO includes multiple modes of functioning, of which the concrete symbolic is of interest here because of the symbolic learning that takes place in schools based in empirical elements and concrete materials. Within the concrete symbolic mode, learning sequences can be identified in a hierarchy described as prestructural (P), unistructural (U), multistructural (M), and relational (R). In terms of the elements provided as part of the learning tasks, there may be none employed (P level); single elements may be used but are totally unrelated to each other (U-level); several separate elements may be employed in a sequence (M-level); or all of the elements may be combined in an integrated fashion showing their relationship to produce a conclusion (R-level). Once the result of a particular learning sequence is consolidated, it may in turn provide an element for a higher order sequence for which it is an essential ingredient. Watson and Fitzallen (2010) illustrated this for the concept of average and the development of graph understanding and its application to graph interpretation and decision making. Other characteristics of the U-M-R levels include the potential lack of recognition of conflict or identification of contradiction at the U-level, their recognition but lack of resolution at the M-level, and their resolution at the R-level should they arise. Conflict or contradictions arise when decisions made and ideas expressed by students are incorrect or there is a mis-match of ideas and information.

As a starting point for assessing learning outcomes from a complete statistical inquiry it is suggested that there are U-M-R sequences associated with each of the four phases of an inquiry. Each of these phases, when complete, provides an element for a U-M-R sequence that describes the development of understanding the practice of statistics. A general developmental sequence is seen in Figure 2. It applies either to one of the phases or to the complete inquiry.

	Successful Learning Outcome			
Relational level	Combines all elements in an integrated fashion to achieve the outcome; resolves any conflicts/contradictions recognised.			
Multistructural level	Links several elements in sequence; may recognise but not resolve conflict/contradictions.			
Unistructural level	Use single elements unlinked; does not recognise conflicts/contradictions.			
	Element 1	Element 2	Element 3	Element4*

*There may be more than 4 necessary elements.

Figure 2. General developmental sequence for a phase of a statistical inquiry.

The consolidation of each phase of the inquiry becomes a new Element to be employed in a subsequent phase or complete inquiry. Table 1 suggests the elements that are likely to be employed in the phases of an inquiry. Although acknowledging student achievement could occur at any of the levels of the developmental hierarchy, the relational level is the desired level of achievement for the targeted learning outcome.

Inquiry Phase	Elements
Pose questions	Context, Population, Type of measurement, Attributes
Collect data	Question, Type of data, Instruments, Sample Size, Variation

Analyse data	Question, Data, Graphical representation tools, Data reduction tools
Make decisions	Context, Question, Analysis, Uncertainty, Interpretation

Table 1. Elements required for the 4 phases of a statistical inquiry.

To exemplify the relationships in Figure 2 the inquiry phase of Pose questions is illustrated in Figure 3 with the Elements suggested in Table 1. The examples are from Year 6 students involved in posing and refining questions within the context of a claim that “athletes are improving over time” (English & Watson, 2012). Not all students chose the same sport or sporting event, measurements, or time frames. This resulted in many different examples at each of the developmental levels. Also varying among the students’ responses was the way in which they incorporated the different elements identified.

<table>
<tr><td></td><td colspan="4">Pose Questions (within a context) e.g., “A claim that Athletes are improving over time”</td></tr>
<tr><td>R-level</td><td colspan="4">All elements integrated; no conflict or contradiction; e.g., “Are the times of the gold medal 100m men sprinters in seconds generally improving over the years of each Olympic Games?”</td></tr>
<tr><td>M-level</td><td colspan="4">Several elements in sequence; recognise but not completely resolve of conflict or contradiction; e.g., “Are people who sprint 100m at Olympics improving over their career?”</td></tr>
<tr><td>U-level</td><td colspan="4">Single elements, unlinked, unrecognised conflict or contradiction; e.g., “In what age group do 100m men’s athletes win gold?”</td></tr>
<tr><td>Elements</td><td>Pick sporting event; e.g., Olympic games
(Context)</td><td>Specify athletes; e.g., men’s100m sprint
(Population)</td><td>Specify dates; e.g., 1896-2012
(Type of measurement)</td><td>Identify measurement; e.g., winner’s time
(Measurement criterion)</td></tr>
</table>

Figure 3. Examples of student responses across the developmental sequence for the *Pose questions* phase of an inquiry about athletics performance.

Next is a culminating sequence, which uses the outcomes of the four phases as elements for a complete statistical inquiry (Figure 4). This sequence recognises the necessity to integrate all phases of the inquiry, depending on the task set, and may represent thinking moving from the concrete symbolic to the formal mode of the SOLO model (Biggs & Collis, 1982).

	Complete statistical inquiry			
R-level	Integrates all elements; e.g., includes all 4 Elements accurately combined with uncertainty recognised in the Decision			
M-level	Links several elements in sequence; e.g., sets up Question with Analysis and Decision without Data or recognition of Uncertainty			
U-level	Single parts of the inquiry unlinked; e.g., only discusses Analysis			
Elements	Pose questions	Collect data	Analyse data	Make decisions

Figure 4. General developmental sequence for a complete statistical inquiry.

CONCLUSION

Assessment of the learning outcomes generated by students completing multi-staged statistical inquiries is complex. It needs to encompass evaluation of both the understanding of statistical content and the application of statistical processes. Presented in the previous sections is an example of how assessment can be integrated into the learning process when students conduct statistical inquiries. The identification of the phases of an inquiry and the characterisation of the elements that make up each of the phases, together with the application of SOLO (Biggs & Collis, 1982) to describe the progression of learning within each phase and across a complete inquiry, provide assessment constructs that have the potential to be used to not only determine students' level of achievement at various stages throughout a statistical inquiry but also support curriculum planning (Earl, 2003). As a teaching point, the developmental nature of the sequences sets up the importance of students reaching the R-level as a desired end point for each individual phase and complete inquiry. Supporting teachers to be cognisant of whether students have attained understanding at that level may also result in the utilisation of activities that provide the opportunity to achieve at that level. Specifying the statistical knowledge and skills associated with the elements of each phase signposts for teachers aspects of the statistics curriculum that need to be emphasised and highlighted. The way in which the outcomes from one phase are consolidated in subsequent phases has the potential to facilitate students' transitions across phases (Watson & Fitzallen, 2010). This also has the potential to support students to develop the skills and knowledge necessary to have the capacity to progress independently through a complete statistical inquiry, a focus of future research, which will involve investigating ways of translating the assessment constructs described in this paper into classroom assessment tools (English, Watson, & Fitzallen, 2015).

Acknowledgement

This research is supported by a Discovery Project Grant (DP120100158) from the Australian Research Council.

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PRE-SERVICE TEACHERS AND NUMERACY READINESS

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Although, in the Australian context, teachers are urged to address cross-curricular numeracy demands, whether pre-service teachers recognise and are prepared for this responsibility is rarely addressed. In this paper we report data from a pilot study in which we explored the numeracy capabilities of 151 pre-service teachers, their expectations of the numeracy demands they will face, and possible differences in the responses of primary and secondary pre-service teachers. Most in the sample considered themselves to be at least average at mathematics and were able to solve numeracy items considered suitable for 15 year-olds. However, more than half were uncertain, or did not believe, that there are mathematical demands on teachers in schools apart from what they teach their students.

INTRODUCTION

In the past decade the relationship between mathematics and numeracy has attracted increased attention in many countries. In the research literature a range of definitions of numeracy (sometimes termed quantitative literacy or mathematical literacy) is found. Yet it is clear that numeracy is not synonymous with mathematics, although it is recognised that mathematics underpins numeracy (National Numeracy Review Report Panel, 2008). The Programme for International Student Assessment [PISA], for example, measures mathematical literacy, not mathematics achievement, that is, "the emphasis is on mathematical knowledge put to functional use in a multitude of different contexts and a variety of ways that call for reflection and insight" (OECD, 1999, p.41). The Australian Association of Mathematics Teachers [AAMT] (c.1998) defined being numerate as being able "to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life" (p. 2). While these two definitions are not identical, both imply that it is the use of mathematical skills in context that characterises numeracy.

In the current Australian curriculum it is recognised that much of the development of explicit numeracy skills occurs in mathematics classrooms. Yet it is also acknowledged that "a commitment to numeracy development is an essential component of learning areas across the curriculum and [is] a responsibility for all teachers" (Australian Curriculum and Assessment Reporting Authority [ACARA], 2015).

Consistent with the AAMT and OECD definitions of numeracy (mathematical literacy), and ACARA's (2015) emphasis on numeracy across the curriculum, an instrument was developed which included items covering basic mathematical skills, as

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 313-320. Hobart, Australia: PME.

well as the mathematics needed to deal with everyday life, community participation, and the mathematical demands in the school as the workplace of teachers. The instrument was administered to pre-service teachers from one university in Australia.

In this paper, initial findings from the instrument are presented and the implications of the findings and directions for future research are discussed.

PREVIOUS RESEARCH

In a very recent Australian newspaper report it was claimed that “Typical student teachers have the maths ability of a 12-year-old child, leaving them ill-equipped to teach the subject — let alone even pass a Year 9 NAPLAN test” (Bita, 2014). Such headlines, with little critique in the ensuing discussion on the robustness of the research reported, only serves to fuel negative views about teachers and teacher education programs.

The mathematical content knowledge required for primary teaching has been widely researched, yet the best way to enhance this knowledge in pre-service programs remains open to debate (National Numeracy Review Report, 2008). Mathematical content knowledge is often assumed in the pre-service preparation of secondary mathematics teachers with emphasis placed on pedagogical approaches. “While there might be some debate about the extent and form of the mathematical knowledge required to teach effectively, there is clearly a considerable body of knowledge that is required by prospective teachers of mathematics” (National Numeracy Report Review, 2008, p. 68). According to Groves, Mousley and Forgasz (2006), however, “many pre-service teachers believe they are insufficiently prepared in terms of mathematics content, pedagogy and pedagogical content knowledge, but believe they are sufficiently prepared in terms of their knowledge of mathematics curriculum” (p. 204).

In the Australian context, teachers have been charged with the responsibility to address cross-curricular numeracy (and literacy) demands. More than a decade ago Thornton and Hogan (2004) argued: “If the role of school education is, at least in part, to equip the population with the knowledge, skills and strategies to be thoughtful, productive and critical members of society, then numeracy is everyone’s responsibility (pp. 318-319). Yet research is sparse on effective ways to prepare prospective teachers to meet this responsibility. In a recent study, White and Cranitch (2010) described the impact of a course, *Curriculum Literacies* (numeracy and literacy), encountered by final year secondary pre-service teachers. They reported that the effects varied by curriculum content area and that “literacy was generally found to be more relevant and more easily integrated than numeracy” (p. 1). Amongst its recommendations, the National Numeracy Report Review (2008) acknowledged that “[B]oth pre- and in-service teacher education should… recognise and prepare all teachers as teachers of numeracy, acknowledging that this may in some cases be ‘subject specific numeracy’” (p. xii). Little appears to be known about the views of pre-service teachers about the numeracy demands they face or their numeracy capabilities. We address these issues in this paper.

THE STUDY

Aims

The main aim of the study reported in this paper was to develop an instrument and trial it with pre-service teachers to gauge i) their views on the numeracy demands on Australian teachers and ii) their numeracy capabilities consistent with the AAMT's (c. 1998) and PISA (OECD, 1999) definitions of numeracy. Several research questions were associated with the trial of the instrument. Of interest in this paper are:

1. What are primary and secondary pre-service teachers' views on the utility of numeracy for teaching?
2. Do primary and secondary pre-service teachers differ in their performance on selected numeracy items and their confidence in providing correct responses?

Instrument

The instrument included biographical items (e.g., gender, whether course was for primary or secondary teaching), and items tapping views about and attitudes towards mathematics (e.g., importance of mathematics for teachers, levels of confidence, etc.) and the utility of numeracy skills for teaching.

As well as numerical items gauging basic mathematical skills, numeracy problems were set in the following contexts: everyday life, informed citizenry, and the workplace (the school), Concomitant with their answers to the numerical and numeracy items, participants were also required to indicate their level of self-efficacy for the responses they gave. Most of the umeraical items were in multiple-choice format; for others, participants had to provide answers and explain their responses. The numerical items were drawn from publicly available Australian Grade 9 NAPLAN (National Assesment Program for Literacy and Numeracy) tests and from the pool of released PISA items (with permission); a few items were developed by the researchers. The instrumnet was prepared for online completion using Qualtrics (www.qualtrics.com).

Instrument items to gauge teachers' views on the utility of numeracy for teaching

1. How good are you at mathematics? (5-point response format: 1=weak to 5=excellent).
2. Is it important for teachers to be good at mathematics? (Yes/No/Unsure). Please explain your answer.
3. Have you studied enough mathematics to be a competent teacher? (Yes/No/Unsure). Please explain your answer.
4. Are there mathematical demands on teachers in schools apart from what is taught to students? (Yes/No/Unsure). Please explain your answer.

Space constraints unfortunately prevent inclusion of the often quite thoughtful and informative explanations given.

Numerical items of interest in the present study (correct response underlined)

A. With the lid on, the mass of this box (drawn) is 232g. With the lid off, the mass of the box is 186g. What is the mass of the lid? [<u>46g</u>, 56g, 144g, 54g]

B. A set of traffic lights is red for half the time, orange for 1/10 of the time and green for the rest of the time. For what fraction of time is the set of traffic lights green? [1/3, <u>2/5</u>, 6/10, 10/12]?

C. Helen's office has a security alarm. To turn it off, Helen has to type her 4-digit code into this (drawn) keypad. Helen's code is 0051. Including Helen's code, how many 4-digit codes are possible? [Answer to be supplied by respondent: <u>10^4</u>]

D. Chris has just received a car driving licence and wants to buy a car. This table shows details of four cars at a local car dealer [A table with details is presented]. Chris wants a car that meets all the following conditions: the distance travelled is not higher than 120,000km; it was made in the year 2000 or later; the advertised price is not higher than 4500 zeds. Which car best suits Chris' needs? [Alpha, <u>Bolte</u>, Castel, Dezal]

To gauge the pre-service teachers' self-efficacy for their responses to each of the items listed above, they were asked if they believed the answer they gave to each item was correct [Yes, No, Unsure].

Data gathering

All enrolled pre-service teachers at one Australian university were invited to participate in the pilot study. The university offers undergraduate and graduate programs in teacher education. The university's guidelines for the recruitment of its students for research studies were adopted: advertisements were placed on selected Moodle sites with a link to the online instrument; lecturers in core units of study advertised the study in their classes; and posters and flyers were displayed within the buildings at the university campuses where the students were enrolled. A four week timeframe was allowed for the online instrument to be completed.

SAMPLE AND RESULTS

The sample comprised 237 students. Of these 23 (10%) opted out of the survey after answering only the first two or three items; these surveys were excluded from the analyses. Of the remaining 214 respondents who answered all or most of the items, 174 (81%) were female and 40 (19%) were male. Just over half, 119 (56%), were aged under 25. Of the rest, 53 (25%) were aged between 25 and 34, while 42 (20%) indicated they were older than 35. Most of the respondents, 164 (78 %) of the 211 who answered the question, had completed their secondary schooling in Australia.

Level of schooling qualification

The university from which the sample was drawn offers teacher education courses that would qualify teachers for early years [EY] teaching (birth to 8 years of age), primary (elementary) [P] teaching (grades Prep to 6), secondary [S] teaching (grades 7 to 12), as well as two cross-sectorial levels: EY-P (birth to grade 6) and P-S (grades P to 12).

The majority of respondents indicated that at the end of their studies they would be qualified to teach primary grades (80: 14 males, 66 females) or secondary grades (71: 21 males, 50 females). It is the responses from these 151 pre-service teachers that are the focus in this article. It should be noted that in Australia, primary teachers are generalists and secondary teachers are specialists. Of the 71 secondary pre-service teachers, only seven nominated mathematics as one of their two teaching specialisations.

Views on the utility of mathematics for teaching

1. *How good are you at mathematics?* Of the pre-service teachers, 11 (8%) considered themselves weak or below average at mathematics, 51 (37%) average, 66 (47.8%) good, and 10 (7.2%) excellent. A chi-square test revealed that there was no significant difference in the responses from the primary and secondary pre-service teachers.
2. *Is it important for teachers to be good at mathematics?* Overall, 106 (76.8%) said "Yes", it is important for teachers to be good at mathematics. A significantly higher proportion of primary (85.1%) than secondary (67.2%) pre-service teachers (χ^2= 6.88, df=2, p<.05) believed this to be the case.
3. *Have you studied enough mathematics to be a competent teacher?* 74 (53.6%) of the pre-service teachers said "Yes", 34 (24.6%) said "No", and 30 (21.7%) were "Unsure". While a higher proportion of primary (31.1%) than secondary (17.2%) pre-service teachers believed they had not studied enough mathematics to be a competent teacher, this difference was not statistically significantly different.
4. *Are there mathematical demands on teachers in schools apart from what is taught to students?* Overall, only 43.1% (59) pre-service teachers said "yes", the same proportion was "unsure", and 13.9% (19) said "no". A significantly higher proportion of primary (48.6%) than secondary (36.5%) pre-service teachers said "yes", with a higher proportion of secondary (22.2%) than primary (6.8%) pre-service teachers saying "no" (χ^2= 7.12, df=2, p<.05).

The responses to these four items revealed that 92% considered themselves at least "average" at mathematics, and a majority, 76.8%, believed that it was important for teachers to be good at mathematics. Interestingly, a higher proportion of the pre-service primary teachers (85.1%) – who will all have to teach mathematics – than the pre-service secondary teachers (67.2%) – who will need to deal with the numeracy demands in their discipline areas – agreed that it was important for teachers to be good at mathematics. Overall, however, 24.6% did not believe they had studied enough mathematics to be competent teachers. It was also clear that very many pre-service teachers were uncertain (43.1%) or did not believe (13.9%) that there are mathematical demands on teachers in schools apart from what is taught to students. Secondary pre-service teachers (22.2%) were significantly more certain that there were no such demands than were primary pre-service teachers (6.8%).

Responses to the four (A-D) numerical items

The frequencies and valid percentages of all primary and secondary pre-service teachers who provided correct responses to each of the four items, as well as the frequencies and valid percentages of those who believed they provided the correct responses, are shown in Table 1. Chi-square tests were conducted to determine if the frequency distributions of the responses of primary and secondary pre-service teachers differed. Statistically significantly different responses are underlined in Table 1.

	All		Primary		Secondary	
	Correct response	Confident correct	Correct response	Confident correct	Correct response	Confident correct
Item A	133: 96.4%	129: 93.5%	71: 95.9%	68: 91.9%	62: 96.9%	61: 95.3%
Item B	115: 86.5%	120: 92.3%	<u>67: 91.8%</u>	69: 95.8%	<u>48: 80.0%</u>	51: 87.9%
Item C	40: 36.7%	47: 37.0%	20: 35.7%	<u>19: 27.5%</u>	20: 37.7%	<u>28: 48.3%</u>
Item D	122: 96.1%	124: 99.2%	68: 98.6%	67: 98.5%	54 93.1%	57: 100.0%

Table 1: Frequencies and valid percentages of students with correct responses and those who were confident that their response was correct on the four numerical items

The data in Table 1 reveal that:

- The vast majority of primary and secondary pre-service teachers provided correct responses to Items A (mass of box lid) and D (car that met Chris' criteria for purchase) and that they were also very confident that the answers they provided were correct.
- A significantly higher proportion of primary (91.8%) than secondary (80.0%) pre-service teachers provided the correct answer to Item B (fraction of time the set of traffic lights was green); although not statistically significant, a higher proportion of primary than secondary pre-service teachers was confident that the answer provided was correct.
- Only about one third of primary and secondary pre-service teachers (35.7% and 37.7%) provided the correct answer to Item C (number of possible combinations of 4-digit codes on security panel); however, a significantly higher proportion of secondary (48.3%) than primary (27.5%) pre-service teachers was confident that the provided response was correct. Effectively the primary pre-service teachers under-estimated their performance while the secondary pre-service teachers over-estimated their performance.

In general, with over 85% providing correct responses, the performance for the whole group of pre-service teachers was good on three out of the four numerical items.

For both groups, the question about the number of possible 4-digit codes for the security pad (Item C), an item that can be considered to test "powers of 10" or "number of combinations" proved to be the most challenging. Why the secondary pre-service teachers were so much more confident than the primary pre-service teachers that their responses to this item were correct was an unexpected finding.

It was also unexpected that for the four numerical items the primary pre-service teachers generally performed at the same or higher level than the secondary pre-service teachers, and that a significantly higher proportion of primary (91.8%) than secondary (80.0%) pre-service teachers provided the correct response to Item B, a question involving addition and subtraction of fractions.

FINAL WORDS

It is noteworthy that the putative poor performance of pre-service teachers on comparatively simple mathematical tasks reported in Bita (2014) was not replicated in this pilot study. Most of the participants in this study, including those preparing to be primary teachers and who would thus have teaching mathematics as part of their load, were able to solve the four items which were taken from a NAPLAN Year 9 or PISA test. The somewhat higher percentage of prospective primary teachers who considered it to be important to be good at mathematics and their relatively strong performance on the numerical items, compared with those aiming to be secondary teachers, may well be a reflection of the small number (seven) in the latter group who counted mathematics as their subject of specialisation. Yet despite the small number of mathematics "specialists", as a group the prospective secondary teachers were generally more confident about their answer than were those preparing to be primary teachers. This issue warrants further scrutiny.

An important aim of the pilot study was to explore pre-service teachers' views on the utility of numeracy for teaching. That only just over half of the respondents in our sample believed that they had "studied enough mathematics to be a competent teacher" is clearly a matter of concern. Working towards a better appreciation of "mathematical demands on teachers in schools apart from what is taught in schools" – currently recognised by less than half the group – is, it seems, still essential. Careful scrutiny of the explanations for the answers given to the two items mentioned above may serve as useful starting points for constructive and manageable interventions during teacher education courses.

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CONCEPTUALISING TECHNOLOGY INTEGRATED MATHEMATICS TEACHING: THE STAMP KNOWLEDGE FRAMEWORK

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The Technological Pedagogical Content Knowledge (TPACK) framework is increasingly in use by educational technology researchers. The framework provides a generic description of the knowledge requirements for teachers using technology in all subjects. This paper describes the development of a mathematics specific version of the TPACK framework. We show how a particular conception of the knowledge required to teach mathematics can be integrated with the TPACK framework as the basis for understanding technology integrated mathematics teaching. The resulting framework provides a sharper lens than the generic TPACK framework alone and a better understanding of the knowledge required to use technology in teaching mathematics.

INTRODUCTION

Over the decades in which mathematics education researchers have worked to conceptualise Shulman's (1987) Pedagogical Content Knowledge (PCK) (e.g., Ball, Thames & Phelps, 2008; Chick, Baker, Pham, & Cheng, 2006; Rowland & Turner, 2007), Information and Communication Technology (ICT) has assumed an increasingly prominent place in students' learning. As a result, the knowledge required for ICT integration into teaching has received considerable attention from researchers who have stressed the importance of technical ICT knowledge, content knowledge, and pedagogical knowledge in teaching (e.g., Chee, Horani, & Daniel, 2005). In response, a conceptual framework that integrates technology, content, and pedagogical knowledge was proposed by Mishra and Koehler (2006), which they called "Technological Pedagogical Content Knowledge," (TPACK). The framework describes the knowledge required by teachers to use technology in teaching in effective ways. TPACK stems from the notion that technology integration benefits from an alignment of content, pedagogy, and technology knowledge. To integrate technology in their teaching practice, teachers need to be competent in all three domains. Only a few studies, however, have related the TPACK framework to particular subject matter content contexts (e.g., Guerrero, 2010; Jang & Chen, 2010).

In this paper, we explore the subject specific use of the TPACK framework in teaching technology integrated mathematics teaching to provide a model for further subject specific definitions and understandings of TPACK. The re-conceptualisation relies on the combination of two frameworks: TPACK and a mathematics specific conceptualisation of PCK, underpinned by the identification of complementary aspects of the two frameworks. The mathematics PCK framework used is Ball et al.'s (2008) description of Mathematical Knowledge for Teaching (MKT), chosen because of its

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 321-328. Hobart, Australia: PME.

influence in mathematics education, and the fact that it elaborates Shulman's (1987) concept of PCK adapted for the context of mathematics. The combined framework extends MKT by adding understandings arising from Mishra and Koehler's (2006) TPACK framework. At the same time, it adapts the TPACK framework by replacing content and pedagogical knowledge with specialised, mathematics specific conceptualisations of these aspects. The result is a new mathematics specific perspective. The paper is organised as follows. First, the development of the two existing frameworks, MKT and TPACK, are discussed separately. This is followed by a discussion of the implications of each framework for, and the development of, a conceptual framework for understanding a mathematics specific TPACK framework.

TEACHERS' KNOWLEDGE FOR TEACHING MATHEMATICS

In mathematics, PCK is regarded as essential (Ball et al., 2008; Park et al., 2011). Some have argued, however, that PCK detracts from the importance of teachers' knowledge of and about mathematics for being effective mathematics teachers (Ball et al., 2008). Thus, the description of PCK in a specialised subject of study, rather than as a generic concept, represents an important contribution to understanding the requirements for effective mathematics teaching (e.g., Ball et al., 2008; Chick, Baker, Pham, & Cheng, 2006; Rowland & Turner, 2007).

Rowland and Turner (2007), for example, identified four different knowledge categories required for teaching mathematics that they described as the knowledge quartet - namely foundation, transformation, connection and contingency. Theirs is a dynamic framework focused on the ways in which teachers use knowledge in the practice of mathematics teaching. Chick et al., (2006) considered the interaction of content and pedagogical knowledge for mathematics teaching in terms of a continuum.

Ball et al.'s (2008) notion of MKT emphasises the role of mathematical content knowledge. Their model comprises two major knowledge types: subject matter knowledge and PCK. Within subject matter knowledge they distinguish Common Content Knowledge (CCK), Specialised Content Knowledge (SCK) and Horizon Content Knowledge (HCK). CCK is necessary but not sufficient for teaching mathematics. It is the mathematical knowledge and skills used in settings other than teaching and used in a wide variety of situations. SCK is the mathematical knowledge and skill unique to teaching and not typically needed for purposes other than teaching mathematics. For example, recognising a wrong answer and being able to carry out a mathematical procedure is part of CCK, whereas recognising the mathematical steps that resulted in a student's error requires SCK in that it is mathematical knowledge not required by people other than teachers. Ball et al. (2008) divided Shulman's notion of PCK into Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC). Familiarity with common errors students are likely to make is an aspect of KCS. Addressing students' errors by selecting appropriate instructional methods is illustrative of KCT. Ball et al. provisionally placed KCC and HCK as third categories of pedagogical content and subject matter knowledge respectively. They described HCK as an awareness of how mathematical topics are related over the span of the mathematics curriculum and KCC as knowledge of the materials and programs that teachers use in their everyday work. Ball and colleagues' conceptualisation of MKT has been criticised, for example, for failing to distinguish clearly between content knowledge and PCK despite purporting to do so; content knowledge is included in each aspect of PCK even though subject matter knowledge is presented as a separate domain (Chick, 2011). Nevertheless, its emphasis on mathematical knowledge suited the purpose of developing a mathematics specific framework for technology integrated teaching.

TEACHERS' KNOWLEDGE FOR TECHNOLOGY INTEGRATED TEACHING

It has been argued that teachers' knowledge of ICT is not the only criterion for effectively using ICT in teaching; sound pedagogical and content knowledge are also critical to success (Chee et al., 2005). Shulman's (1987) notion of PCK is thus relevant but requires the additional dimension of technological knowledge. The incorporation of a technology component into PCK resulted in the development of the notion of "Technological Pedagogical Content Knowledge" (TPACK) (Mishra & Koehler, 2006). According to the TPACK framework, the combination of technology, pedagogy and content knowledge can reinforce each other to realise advantages afforded by technology in the teaching and learning process. The combinations of technology (TK), pedagogy (PK) and content (CK) result in four additional composite knowledge types, namely: Technological Content Knowledge (TCK), Technological Pedagogical Knowledge (TPK), Pedagogical Content Knowledge (PCK) and TPACK. The TPACK framework is presented in Figure 1 followed a definition of each knowledge type.

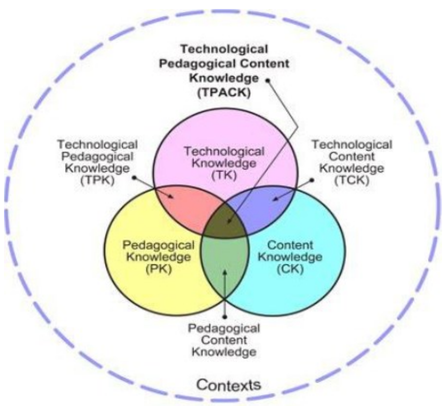

Figure 1: TPACK framework (source: Koehler & Mishra, 2009, p.63)

TK: Skills of teachers to use a particular technology. This could be using a particular software program and installing or removing it.

PK: Knowledge about process and practices of teaching. It includes, for example, students' learning styles, classroom management, students' evaluations and lesson planning.

CK: Knowledge of a subject matter to be taught. It demands understanding core principles, facts, theories, procedures and concepts of a particular subject matter.

PCK: Knowledge of how particular pedagogical approaches are suited to teaching particular content and vice versa.

TCK: Knowledge of how technology and content interact in effective teaching. It includes teachers' understanding of how subject matter can be changed by the use of technology.

TPK: Knowledge of how to use various technologies with different pedagogical approaches. It involves recognising and making use of the affordances of technologies and choosing pedagogical approaches that fit particular technologies and vice versa.

TPACK: The basis of effective teaching with the application of technology and requires an understanding of pedagogical techniques that use technologies in constructive ways to assist students to overcome difficulties and to learn content effectively.

Mishra and Koehler's (2006) knowledge types are based on a generic definition of content and hence of pedagogical knowledge and PCK. The re-conceptualisation of the TPACK framework for mathematics teaching that is presented in this paper is based on a definition of the mathematics content knowledge that teachers need that is more than simple knowledge of mathematics. Rather it is the SCK defined by Ball et al. (2008). This notion in turn influences the conceptualisation of mathematical pedagogical knowledge; it is not simply generic pedagogical knowledge applied to mathematics teaching but rather the knowledge of pedagogy needed to use specialised (mathematical) content knowledge effectively in teaching. This idea in turn has implications for PCK conceptualised as incorporating KCS and KCT as defined by Ball et al. (2008).

TECHNOLOGY INTEGRATED MATHEMATICS TEACHING

Adapting the TPACK framework to apply specifically to mathematics teaching requires understanding the three components (technology, pedagogy and content) from the perspective of mathematics teaching and the knowledge required to teach mathematics. The advantage of using Ball et al. (2008), rather than other conceptualisations of mathematics teacher knowledge, relates to its grounding in Shulman's (1987) notion of PCK which also informed the TPACK framework. The focus of this paper is on bringing together the TPACK framework and of Ball et al.'s (2008) MKT.

While the TPACK framework starts from the definition of PCK of Shulman (1987), we begin from the more detailed conceptualisation of MKT (Ball et al., 2008). In both cases, TK is seen as a component to be added. This is illustrated in Figure 2 with the left side of the figure showing the addition of TK to Shulman's notion of PCK, the right side showing its addition to Ball et al.'s (2008) MKT framework.

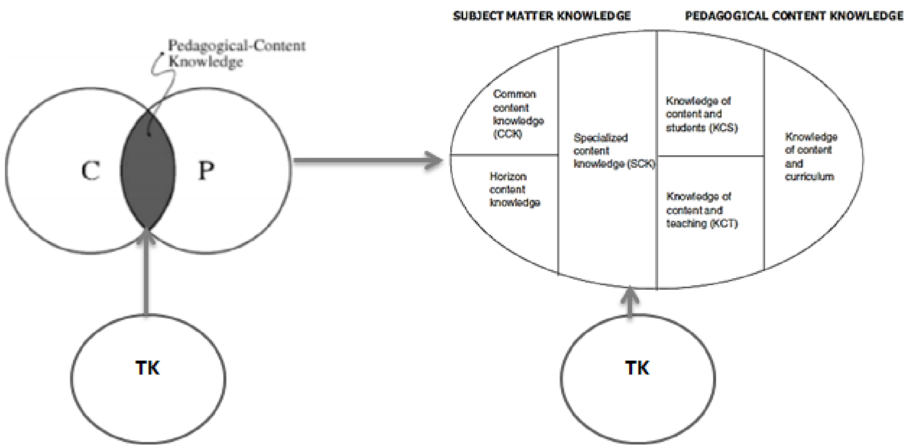

Figure 2. Explaining TPACK in mathematics teaching

Developing Specialised Technological and Mathematics Pedagogical (STAMP) Knowledge Framework

In the STAMP knowledge framework, Mishra and Koehler's (2006) Content Knowledge (CK) is redefined as Specialised Mathematics Knowledge (SMK) and Pedagogical Knowledge (PK) as Specialised Pedagogical Knowledge (SPK). Technological Knowledge (TK) remains as defined by Mishra and Koehler's (2006). SMK and SPK are defined together with each of the knowledge types that result from the intersection of the two frameworks. How each element of the STAMPK framework differs from the corresponding component of the TPACK framework is explained in the following section. The framework is illustrated in Figure 3.

Specialised Mathematics Knowledge (SMK)

SCK, as defined by Ball et al. (2008), is knowledge needed by teachers of mathematics in addition to CCK and HCK, but not by the general population or teachers of other subjects. For example, rather than simply knowing how to perform fraction calculations, teachers need to understand the multiple and subtly different meanings of fractions (e.g., as division, as parts of a whole, as points on a number line).

Specialised Pedagogical Knowledge (SPK)

Ball et al. (2008) did not define pedagogical knowledge but described PCK for mathematics teaching in terms of KCS, KCT, and KCC. Nevertheless, there are parallels with the PK used by Mishra and Koehler (2006) with generic assumptions. Mishra and Koehler (2006, p.1026) defined PK as "knowledge about techniques or methods to be used in the classroom; the nature of the target audience; and strategies for evaluating student understanding. It is about how students construct knowledge, acquire skills, and develop habits of mind and positive dispositions toward learning". From this definition one can see the importance of teaching methods (analogous to KCT), knowledge of students (analogous to KCS), and knowledge of the curriculum (analogous to KCC) which are defined by Ball et al.'s model pertinent to mathematics teaching and redefined in the STAMP knowledge framework by combing all KCT, KCS and KCC as SPK.

Technological Knowledge (TK)

This refers to skills required of teachers to use a particular technology. This could involve using a particular software program and installing or removing it.

Specialised Pedagogical Mathematics Knowledge (SPMK)

The intersection of SPK and SMK takes the place of PCK in Mishra and Koehler's (2006) TPACK framework. SPMK can be understood as the mathematics specific and specialised (as opposed to generic or everyday) knowledge for teaching mathematics. It includes knowledge of ways in which mathematical concepts can be represented, the affordances of particular mathematical problems, resources and the specific difficulties that students are likely to encounter in relation to particular mathematical concepts. For example, knowing the affordances of and appropriate uses of various representations of fractions (e.g., as areas, parts of collections, or points on number lines).

Specialised Technological Mathematics Knowledge (STMK)

The intersection of TK and SMK results in STMK. This is the knowledge required by teachers of mathematics in which the application of technology influences mathematical content. Teachers' selection of technology should fit with the special type of mathematics knowledge needed in teaching. STMK allows teachers to identify and use technology appropriately to facilitate the teaching of mathematical concept effectively. For example, the use of spreadsheets could transform the task of explaining the difference between a square and a rectangle to one of creating, changing and checking the properties of many figures that fit the definition of a rectangle and identifying that some of these are square.

Specialised Technological Pedagogical Knowledge (STPK)

The intersection of TK and SPK (with the definition given earlier) gives rise to STPK. This is the knowledge required by mathematics teachers in which teachers' mathematics specific pedagogical knowledge is influenced by the application of technology. For example, knowing that using dynamic graphing software to remove the tedium of creating scatter plots can enable students to access more sophisticated ideas about the relationships between variables than would be possible in the absence of technology.

Specialised Technological and Mathematics Pedagogical Knowledge (STAMPK)

Finally, the interplay of TK, SMK, and SPK gives rise to STAMPK. This is the unique knowledge for teaching mathematics with the application of technology. It is the integration of these three knowledge types that enables teachers to incorporate technology effectively into mathematics specific pedagogies in such a way that students are assisted to make meaning of the targeted mathematical ideas. It also includes understanding the instructional advantage of different instructional methods, specialised mathematics knowledge and technologies and combining these knowledge types in the classroom for effective learning of mathematics. The TPACK framework (Figure 1) is thus reconceptualised as the STAMPK framework for teaching mathematics with the application of technology as shown in Figure 3.

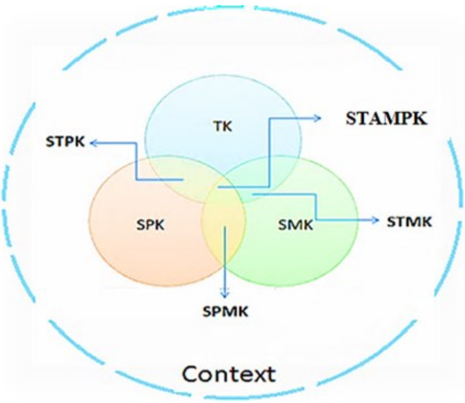

Figure 3. Knowledge required in teaching mathematics with technology

Teaching mathematics successfully with technology requires each component of knowledge described in the STAMPK framework as well as the knowledge types arising from their constructive combination. Their application will depend upon the particular context in which they are employed with such things as the availability of technologies, time, the nature of students, and course assessments.

CONCLUSION

The framework proposed provides an approach to specifying the TPACK framework for teaching mathematics. The resulting STAMPK framework interprets the TPACK framework in terms of Ball et al.'s (2008) influential model of Mathematics Knowledge for Teaching. The work was predicated on the belief that subject specific knowledge frameworks are of greater use than generic frameworks to both researchers and practitioners with interest in a specific subject. Ball et al.'s (2008) model was used because of its emphasis on mathematical knowledge and its origins in the seminal work of Shulman (1987), as well as its status in the field, however, other models of mathematics teachers' knowledge such as those of Chick et al. (2006) and Rowland

and Turner (2007) might also have been used. These would likely have yielded different but similarly useful insights into the knowledge demands of technology integrated mathematics teaching. Much work is needed to explore such options, weigh their relative merits, and tackle the ongoing challenge of operationalising conceptions of teacher knowledge within a technology rich environment. The STAMPK framework provides a starting point for one such line of inquiry.

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ON THE ABSENCE OF BASIC FLUENCY AND FLEXIBILITY IN NOVICES' GEOMETRY PROOFS

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We present an empirical study that sheds light on novice difficulties with geometry proofs from a rather different perspective – a perspective of fluency and flexibility, which are essential notions of creativity. We first motivate and illustrate the relevance of this perspective, and then display an empirical study of 8th graders' basic geometry proofs. The study's results reveal the absence of a suitable discipline of fluency and flexibility. The results illuminate novice behaviour characteristics with decomposition, and shed light on difficulties with figures of two interleaved triangles, as well as on difficulties with very basic manipulations with the givens.

INTRODUCTION

Fluency and flexibility, together with novelty, are three primary notions of creativity. *Fluency* involves the number of responses, or ideas generated in response to a prompt; *flexibility* involves shifts and alternations in the generated responses; and *novelty* involves the innovation, or uniqueness of (one or more of) the generated responses. Mathematics educators examine the appearances of these notions in both problem solving and problem posing (e.g., Silver, 1997; Torrance, 1988).

The notions of fluency and flexibility belong to a contemporary view of creativity. This view expands former views, which underlined the notion of novelty, and regarded creativity as "rare mental feats" and "occasional bursts of insight" (Silver, 1997). The embedment of the notions of fluency and flexibility in creativity offers a perspective that involves relevant elements "lighter" than innovation.

Embedding fluency and flexibility yields a variety of problem solving considerations with respect to creativity, including the recognition of distinct task features, the generation of different solution directions, and the development of alternative solutions. Novelty is examined with respect to original associations that yield progress in less usual ways.

Some studies examined creativity in geometry proofs. The proofs were not straightforward. They involved careful analysis and deduction, a range of geometrical structures and theorems, and invocations of various heuristics such as backward reasoning and auxiliary constructions (e.g., Kantowski, 1977; Levav-Waynberg & Leikin, 2012). Challenging and rich proof tasks are indeed relevant for examination with creativity notions. What about elementary proof tasks? They may seem irrelevant. But, are they so? Consider the following two tasks, in Figures 1 and 2.

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The tasks in Figures 1 and 2 are very simple proof tasks. We posed them to 8th graders, after they were well acquainted with basic triangle properties, and with the three fundamental triangle congruence theorems – Side-Side-Side (SSS), Side-Angle-Side (SAS), and Angle-Side-Angle (ASA). The tasks "call for" the use of these theorems.

Given: AE=AD, $\angle AEC = \angle ADB$

Prove: $\angle B = \angle C$

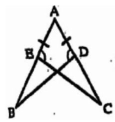

Fig 1. *An elementary fluency task.*

Given: KM=OP, PL=KN, ML=ON

Prove: $\Delta KMN \cong \Delta POL$

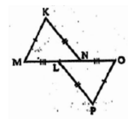

Fig 2. *An elementary flexibility task.*

In the task of Figure 1, problem solvers may demonstrate basic *fluency*. They may consider different directions – the relation between the triangles ADB and ACE (which are congruent), relations between angles, the relation between the two bottom, small triangles, and possibly additional relations. The fluency, of different directions, stems from different decompositions of the given structure. What did students do? Quite a few students attempted the relation between the two bottom, small triangles, realized a dead-end, and either gave up about the proof, or argued that these two triangles are congruent, "based" on the SAS theorem, or the ASA theorem, which they "forced" without any sound argumentation. They did not examine other directions.

In the task of figure 2, problem solvers may demonstrate basic *flexibility*. There are only two triangles in the figure. No information about angles is provided. It is rather clear that the SSS theorem should be invoked. But, not all the corresponding sides are marked equal. Two pairs of corresponding sides are marked equal, but the pair of the corresponding sides MN and LO are not known to be equal. Only their parts, ML and ON are marked equal. Yet, they share a common segment – LN. The flexibility required here is to capitalize on this feature, decomposed the line MO into three segments, separately recompose ML+LN and LN+NO, and show that MN=LO. (The proof of MN=LO may be obtained differently, by subtraction, yielding also a *fluency* aspect.) What did students do? Quite a few provided an unsound proof – they ignored the segment LN, and used the SSS theorem, with ML=NO as "corresponding sides".

We may try to examine the student responses in light of previous studies of novice difficulties with proofs and visualization. Studies of difficulties with proofs reveal a variety of erroneous perceptions of the nature of a proof (e.g., an example as a proof), of the form of a proof, of forms of argumentation, of the role of a proof, and more (Harel & Sowder, 2007). This, however, is not the case here. Students who provided unsound proofs indicated in interviews that they were aware of the fact that their proofs were improper. Some offered sound proofs for other proof tasks that we posed to them, and all of them offered a suitable, ordered proof for the task in Figure 3 below (which was the 1st task in a written questionnaire that was posed to them).

Given: $BC=CD$, $\angle C_1 = \angle C_2$

Prove: a. $\Delta ABC \cong \Delta ADC$ b. $AB=AD$

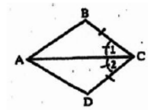

Fig 3. *An initial task, for examining proof acquaintance.*

Studies of difficulties with visualisation underline the difference between vision and visualization, and relate to *operative apprehension* (Duval, 1995, 1998), which involves directed figural processing. Duval, then Gal & Linchevski (2010), and others analysed novice difficulties also in light of Gestalt principles, which suggest that the natural perception of a form is considered a unitary global structure, which hampers the decomposition of a structure into sub-units. Duval pointed out the challenge of addressing the gap between operative apprehension and mathematical deduction. Gal and Linchevski offered a VPR model, composed of *organisation*, *recognition*, and *representation* components, for analysing geometry visualisation difficulties.

We may attempt to explain the difficulties described earlier, of our student responses, with the above perspectives of visualisation. But, do these perspectives unfold the whole picture? Our students demonstrated their ability to properly decompose a global structure into parts, or sub-units in their answers to the task in figure 3. They also showed suitable recognition (of corresponding sides) and deduction (using the suitable congruent theorem) in their solutions of this task. Yet, they still demonstrated difficulties in their solutions to the tasks in Figures 1 and 2. It seems that difficulties stemmed from elements additional to those described in the above visualisation perspectives. We believe that these additional elements derive from the absence of a suitable discipline of fluency and flexibility.

One may wonder: How is it that fluency and flexibility, which are naturally employed with challenging tasks, play a central role here, in such basic proof tasks? Our response to that is that the challenge is "in the eye of the beholder". Proof tasks are not "rote learning" tasks. They involve problem solving. As we have shown with the tasks of Figure 1 and Figure 2, even basic proof tasks require careful considerations, decomposition, re-composition, recognition of a suitable path to follow (possibly among several paths), manipulation of sub-units, and proper deduction. Competence with these elements involves challenge, even at the basic level.

Following the above, the objective of the study presented in this paper is to shed light on novices' corresponding fluency and flexibility. Our research question is: W*hat are the characteristics of novices' fluency and flexibility in elementary geometry proofs*?

In the next section, we describe the methodology of our study; in the section that follows, we present our primary findings; and in the last section we discuss these findings in light of previous studies in geometry and problem solving.

METHODOLOGY

Population

The study's population included 83 8th grade students, from three junior-high schools. They were all well acquainted with triangle terms, triangle properties, and triangle proofs. Their geometry learning started a year earlier, in 7th grade, in which they learned fundamental geometry terms and calculations, such as circumference, area, angle computations, segment subtraction, and more. In 8th grade they learnt and practiced a variety of triangle-congruence proofs, using the congruence theorems of SSS, SAS, and ASA. In both years of geometry studies they have seen and practiced many tasks that required structure decomposition, in calculations and proofs.

Tools

Our study's questionnaire involved 12 proof tasks, of which 8 were very elementary, as those presented in the Introduction. (Four additional tasks were a bit more involved.) These eight tasks required only a couple of deductive steps. In this paper we focus on these tasks.

The primary focus of the tasks was the employment of the heuristic of *decomposition*. One task also required the heuristic of *auxiliary construction*. Each task included a figure, which involved the composition of two triangles. Three of the compositions were of a *concatenation* form, where each of the triangles was "glued" (by a segment or a vertex) to the other; and five of the compositions were of an *interleaving* form, where parts of one triangle were interleaved with parts of the other. In some interleaving tasks the triangles shared an angle, in some they shared a side, and in all of these tasks sides were crossed. Previous studies in computer science (Ginat et al., 2013; Soloway, 1986) explicitly differentiated between concatenation and interleaving upon schema utilization. Interleaving is more complex, harder to handle, and increases cognitive load (Ginat et al., 2011; Sweller, 1988).

We also divided the eight proof tasks into several categories with respect to the relevance of fluency and flexibility in their solutions. Task 1 required no fluency and no flexibility. The focus of two tasks was fluency, the focus of three other tasks was flexibility, and the focus of two additional tasks was both fluency and flexibility. Next, we display the tasks and elaborate on them, together with the student solutions.

Process

The students were given 90 minutes to solve the written questionnaire. Some students requested time extension, and we let them work as long as they needed. Following their written solutions, about 15% of the students, who demonstrated different levels of competence, were interviewed about their solutions. We analysed the solutions in light of the notions of fluency and flexibility, as well as the perspectives mentioned in the Introduction section, and Schoenfeld's model of problem solving (1992).

RESULTS

In what follows we display the questionnaire tasks followed by characteristics of their solutions, together with some students' comments in interviews about their solutions. Task 1 of our questionnaire (shown earlier in Figure 3) involves a concatenation of two triangles, and requires simple decomposition, but no fluency and flexibility. It was answered properly by all the students.

Fluency

The tasks in Figures 4 and 5 below display interleaved triangles, and their focus is fluency. The triangles in Figure 4 share a side, and the triangles in Figure 5 share an angle. In both figures two sides cross one another. We name the crossing point **O**. In both tasks one may carry out various decompositions. An experienced problem solver will decompose the structure in Figure 4 into the two interleaved triangles ADB and ACB. A less experienced problem solver may start by examining the small triangles ADO and OCB. A similar phenomenon may occur in the task of Figure 5. In this task one may also examine all the angles, and reach a nice proof based solely on angles.

Given: AC=BD, BC=AD

Prove: $\angle C = \angle D$

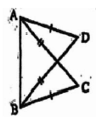

Fig 4. Task 4.

Given: AE=AD, $\angle AEC = \angle ADB$

Prove: $\angle B = \angle C$

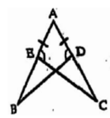

Fig 5. *Task 5 (the task of Fig. 1).*

Task 4 was properly solved by 59% of the students, and Task 5 – by 67%. In a following interview, the student B, who did not solve Task 4 said: "... I was unable to understand ... there are a few triangles for which I may show congruence ...". The student then added that the variety of possibilities inhibited his problem solving.

Another student, R, offered a proof in which he "showed" that ADO and OCB are congruent. He recognized the vertical angles around the crossing point O, and argued that ADO and OCB are congruent "based on the SAS theorem ... where the sides relevant in ADO are AD and AO, and the angle is O ...". However, the angle O is not between these two sides. In a following interview he indicated that this is "what he got", and that the employment of the SAS theorem here is improper. He added: "... I looked only at the small triangles ... it seemed like the rest was kind of blurred ...". Student H indicated that she attempted Task 5, did not advance, left the task, and later returned to it, and succeeded in the second attempt. The phenomenon of leaving a task and then returning to it later is typical in creative thinking.

Another student, V, indicated that Task 5 was easier for him than task 4, since the data included information about the angles, and this led him to immediately turn to the "large triangles'. His indication may explain the better success in task 5. A few students followed an original way that we did not expect. Although they knew that the focus of the questionnaire is congruence theorems, it was simpler for them to solely focus of angles. They showed that all the corresponding angles of the two triangles ADB and ACE are equal. They did not use the given equality AD=AE.

Flexibility

The first task in this category was the one discussed earlier in (Figure 2 of) the Introduction. Two additional tasks are displayed below: task 3 (of our questionnaire), in Figure 6, which involves concatenation, and Task 8, which involves interleaving. Both focus on flexibility. Task 3 requires the manipulation and capitalisation on the common part of two angles (similar to the task in Figure 2, which involves a common segment of two sides). Task 8 does not show triangles. One has to obtain them by adding an auxiliary construction, connecting the points B and C.

Given: NL=MN, KN=ON,

$\angle LNM = \angle KNO$

Prove: $\Delta KMN \cong \Delta OLN$

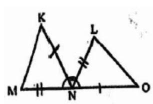

Fig 6. Task 3.

Given: AB=CD, and they are segments of the same line.

DE=AF, $\angle A = \angle D$.

Prove: $\angle F = \angle E$

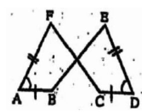

Fig 7. Task 8.

Task 3 was properly solved by 61% of the students, and task 8 – by 47%. In task 3, some students proved congruence by using the two obtuse angles, rather than performing the necessary angle subtractions.

In Task 8, students attempted diverse kinds of directions. The student A said: "... there are no triangles here ..."; R said: "... we didn't learn quadrilateral congruence ...". Some added two heights to the figure. Quite a few students added the necessary segment BC but did not capitalise on this auxiliary construction. The student M attempted SAS congruence by using the segments AB and CD, even though she added the segment BC herself. In a following interview she said: "... I can show congruence using SAS with a side, an angle, and a kind of a side ...", overlooking the relevant utilization of the segment she herself added!

Some students who properly solved Task 8 enjoyed this task. The student S indicated that he could not solve the task at first, so he left it, then returned to it, solved it, and felt satisfied. The student Y said: "... I like this kind of questions, which are challenging, though not hard ... I tried to think creatively ... there is something here that you do not see at first glance ... you need to think a bit ...". It seemed like some of the students referred to the requirements of this task with a sense, or a feeling of novelty.

Fluency and Flexibility

The last two tasks were slightly more challenging, and involved interleaved triangles, fluency and flexibility. We do not elaborate on these tasks due to lack of space. The first of them was solved by 43% of the students, and the other by only 36%. All in all, 53% of the students (44 students) solved between 1-4 tasks, 27% solved 5-6 tasks, and only 20% solved 7-8 tasks.

DISCUSSION

We showed in this study that fluency and flexibility, which are primary creativity notions, may be relevant for elaborating on elementary, novices' proofs. We demonstrated the employment of fluency and flexibility lenses in the Introduction, and continued in the previous, Results section. We discuss it further, below, addressing the research question posed in the end of the Introduction.

The proof tasks of our study required (among additional things) careful decomposition of given structures, and an examination of possible alternatives to follow. Suitable decomposition was a challenge for quite a few students. Some explicitly mentioned a vague "picture", upon looking at figures with interleaved compositions of two triangles. In a sense, what was necessary here was "to see the trees for the forest", rather than "see the forest for the trees". Some students could not explicitly recognize and separate parts of the whole; and some who did recognize and separate a part or two, did not examine other parts. The latter characteristic may be partly explained with Schoenfeld's characterization of novices' tendency to follow a single solution path, after little (or no) preliminary task analysis (1992).

But, there is more to that. It seems that a figure, with two interleaved triangles, was enough for increasing some novices' cognitive load to a point which inhibited progress. Crossing lines and more than one or two sub-structures yielded confusion. Some students were unable to conduct any operative progress, and others reached an unsuitable point where they "forced" an unsound proof. There was little or no fluency in their decomposition actions, and little, unsuitable fluency in their attempts to obtain a sound proof. We believe that an ordered practice of decomposition may help novices reduce cognitive load. Together with elaboration of fluency awareness, they may improve their control behaviour (Schoenfeld, 1992), perhaps significantly.

Novices should also be able to flexibly manipulate decomposed parts, and combine them in ways that yield progress. The proof tasks in our study required rather simple manipulations of decomposed parts with the givens. Yet, quite a few students were unable to turn to simple manipulations such as segment (or angle) addition or subtraction, even though they were well acquainted with such manipulations. Sometimes (as can be seen with task 8) they did attempt various manipulations, but were unable to direct them to their needs. At times, it even seemed that they were "overlooking" useful outcomes of their own manipulations. Marshall (1995) advocates the important role of flexibility, together with additional, fundamental problem solving elements. We believe that here too, as with fluency, practice and the elaboration of awareness may make a difference, in both control and beliefs. It is our hope that the perspective we offer may encourage tutors to underline the essential role of fluency and flexibility, already at the early teaching of proofs.

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