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Volume 4

Research Reports

Per - Zha

Editors: Kim Beswick, Tracey Muir, & Jill Fielding-Wells

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STUDENTS' PERCEPTIONS OF A GOOD TEACHER

Pamela Perger — University of Auckland
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No one can make someone learn, they can only help and guide them. So what is it that makes some people better at teaching than others? Ministers of Education, researchers, principals, teachers, parents and students all have their own view on what makes a good teacher. This paper presents the views of a class of Year 8 students (12 year olds) on what makes the perfect mathematics teacher. The characteristics the students identified show a mature understanding of which teacher characteristics support their learning of mathematics. Many of the characteristics the students identified align with the New Zealand Teachers Graduating Teacher Standards.

INTRODUCTION

It has long been recognised that the classroom teacher has a significant influence over student learning, (Anthony & Walshaw, 2007). Having spent at least eleven years in the schooling system, many people have a memory of at least one teacher whom they remember fondly (or otherwise). Based on their own experiences of school people develop a personal belief as to what makes an effective teacher. Hattie (2003) states that it is the teacher, after the student her/himself, who has the largest effect on student achievement. Developing effective teachers of mathematics continues to be a focus for anyone involved in mathematics education. There is a wealth of literature about good teaching but the debate about what characteristics make an effective teacher is ongoing (Anthony & Walshaw, 2007; Hattie, 2003; Murphy, Delli, & Edwards, 2004).

So what are the characteristics of an effective teacher identified in the literature? Hattie (2003) states that 'expert teachers':

- can identify essential representations of their subject,
- can guide learning through classroom interactions,
- can monitor learning and provide feedback,
- can attend to affective attributes, and
- can influence student outcomes' (p6).

Clarke and Clarke (2004) recognise that to be an effective teacher of mathematics you need to be able to:

- focus students on the important maths ideas,
- provide a clear structure to the learning along with purposeful tasks that will engage children,
- use a variety of materials, representations, and contexts,
- identify connections between mathematical tasks,
- provide opportunities for students to engage in mathematical thinking,
- engage children in learning communities,

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- maintain high but realistic mathematics expectations for all learners,
- encourage students to engage in mathematical reflection, and
- effectively use assessment as part of the teaching and learning process.

A current focus in education research is on building communities of inquiry that involve students in the process of learning. This process requires the student to take the responsibility for making sense of the mathematics they are learning. For students to be successful in this process of learning they require an effective teacher who is able to scaffold them into taking on this responsibility (Hunter, 2007). To be an effective mathematics teacher a teacher needs not only to be confident in their personal mathematics knowledge but also to have strong pedagogical knowledge, if they are to successfully engage students in the learning of mathematics (Anthony & Walshaw, 2007; Clarke & Clarke, 2004).

Researchers (Anthony & Walshaw, 2007; Clarke & Clarke, 2004; Hattie, 2003; Hunter, 2007; Murphy, Delli, & Edwards, 2004) all identify characteristics important in making an effective teacher. Murphy, Delli, and Edwards (2004) believe students begin developing their idea as to what an effective teacher is the day they start school. Students starting school already have an expectation of the teacher's role and what learning at school is 'supposed' to involve. McCullum, Hargreaves, and Gripps (2000) noted that students as young as six and seven years old demonstrated they could identify factors which impact on their learning. The factors the students identified included those associated with the role of the teacher. Brown & McIntyre (1993) found the twelve and thirteen year olds in their study recognised that an effective teacher needed to have a positive attitude if a teacher were to engage learners. The secondary school students Hill and Hawke (2000) interviewed identified that an effective teacher was one who respected their students, and encouraged/allowed students to work in pairs. Perger (2008) found that both higher and lower achievers (11 – 12 year olds) in a low socio economic area had the same expectations of their teacher. The characteristics identified in Perger's study fell into two categories; personal attributes of the teacher, and the learning environment the students expected the teacher to provide for them. The criteria identified by these students included someone to inspire them, someone who enjoyed teaching maths, someone who would interact with them, who was firm but not too strict, who would challenge them, give them hints not answers, and someone who would respect them.

Many of the characteristics described above are visible in the graduating teacher standards (New Zealand Teachers Council, 2014). The New Zealand Teacher Registration Board recognises seven standards a graduating teacher must meet to become a registered teacher. These seven standards are grouped into three categories; Professional Knowledge, Professional Practice, and Professional Values and Relationships. Standards One to Three (Professional Knowledge) state that graduating teachers need to: know what to teach, know about learners and how they learn, and understand how contextual factors influence teaching and learning. Standards Four and Five (Professional Practice) recognises that graduating teachers need to use professional knowledge to plan for a safe, high quality teaching and learning environment. Standards Six and Seven focus on positive relationships with learners and the learning community and commitment to the profession. These standards ensure teachers entering the profession understand the complex but crucial role they have in enabling all learners to achieve (New Zealand Teachers Council, 2014).

This study aims to identify what characteristics the students in the second author's class considered a requirement for an effective mathematics teacher.

METHOD

The 30 Year 8 students whose views are discussed in this paper attended an Intermediate School (Years 7 and 8) in a high socio economic area in Auckland, New Zealand. The second author had been their teacher for the past six months. She had developed a class culture where students were supportive of each other, expected to take responsibility for their own learning, and were prepared to take on a learning challenge. As part of a long-term project, the students had been working towards developing peer-tutoring skills with the first author. One of the tasks set at an early stage in this process was to brainstorm the characteristics of an effective mathematics teacher. Students worked in self-chosen groups of three to record their responses to the question '*What are the characteristics of an effective mathematics teacher?*' The use of small groups where participants can share their opinions often enables them to develop their ideas further than if they were involved in an individual interview situation (Flick, 2011). During the analysis of student responses the authors grouped characteristics the students had identified on the brainstorm sheets. Characteristics that were similar were collated into themes. During this process strong themes emerged. Once the themes were identified the occurrence of each response within each theme was noted as a percentage of the total responses (Figures 1 and 2).

RESULTS

Each brainstorm sheet contained between 7 and 21 characteristics (average number - 10 characteristics per sheet) the group of students considered a requirement for an effective teacher of mathematics. The students' responses were grouped into the four emerging themes: personal qualities, professional values / relationships, content knowledge of the teacher, and pedagogical knowledge of the teacher. Characteristics such as kind and caring were noted as personal qualities students considered a teacher required to be an effective teacher. Students also identified that effective teachers had to be fair to all students (Figure 1).

Personal Qualities	**Percentage of Total Responses**
Kind / Caring / Patient / Tolerant Honest Easy to approach Nice / Happy / Cheerful Creative Good looking Writes with both hands Sporty / Fun / Plays Games	26%

Professional Values / Relationships	**Percentage of Total Responses**
Treats every student equally – no favoritism Doesn't judge people, understand everyone is different with different strengths & weaknesses Respectful towards students rights	6%

Figure 1: Personal Qualities, Professional Values / Relationships of Effective Teachers

The students identified a range of responses that related to teacher knowledge. Eight of the ninety-six student responses identified actions / behaviours that clearly linked to teacher's personal knowledge of mathematics. Responses that described or listed teaching actions / behaviours the students considered supported their learning of mathematics were collated under the theme relating to teacher's pedagogical knowledge. This theme included responses regarding teacher knowledge and teacher expectations of students as well as specific teacher actions such as, breaking down the question so that student could understand what they were being asked to do (see Figure 2).

Knowledge of the Teacher - Content	Percentage of Total Responses
Knowledgeable Knows everything Knows their maths Has knowledge of the subjects Knows the answer	8%

Knowledge of the Teacher - Pedagogy	Percentage of Total Responses
Good at teaching Knows how to reflect on before Knows what is best for the students Understands the student Accepts everyone's ideas Knows your weaknesses and strengths / when you need help / goes at your speed Explains things (problem's meaning / vocabulary) Breaks down the question if you don't understand it Gives examples to work on Shows how to solve an example Relates it to things you can do / makes sure you understand Helps you when you are stuck Provides fun activities to get kids engaged Gives questions to work on at the end Links problems to things we know (context) Gets us to give it a go Helps you figure it out without telling you the answer Gives hints / Asks questions that make you think Doesn't tell you you are wrong, makes you work it out for yourself Makes you work on a problem first then explains and makes you do it again Even if you do something wrong the teacher will help you do the right thing.	60%

Figure 2: Knowledge of the Teacher – Content and Pedagogy

DISCUSSION

Student responses to the question *'What are the characteristics of an effective mathematics teacher?'* showed that the students in the second author's class were able to identify the characteristics of an effective mathematics teacher. The large majority of responses linked to the pedagogical knowledge / practices of a teacher. The students identified that an effective teacher would expect them to take responsibility for their learning (Clarke & Clarke, 2004; Hunter, 2007). The students recognised that they needed to do the thinking involved in solving problems, commenting that an effective teacher would ask questions that make you think and would make you work it out for yourself. The students identified that an effective teacher would be a guide (Clarke & Clarke, 2004; Hattie, 2003) helping the students to understand what the question was asking or explaining words they did not understand, explains things like what the problem means / what the words in the problem mean. Students also saw an effective teacher as someone who would help them make connections to what they already knew (Clarke & Clarke, 2004), relate it [the maths] to things you can do.

The students saw the role of an effective teacher as someone who challenged them (Perger, 2008) - gets us to give it a go; someone who makes them work on a problem first then explains if they need help. Students also saw the teacher as the person who challenged them to monitor their own learning, doesn't tell you you are wrong, makes you work it out for yourself, further reinforcing the idea that the students themselves are responsible for their own learning. An effective teacher was someone who provided activities that would engage them (Clarke & Clarke, 2004; Perger, 2008) - provides fun activities to get kids engaged and provide students with work that helped them consolidate their learning gives questions to work on at the end (Clarke & Clarke, 2004).

There was a strong feeling that an effective teacher would know your weaknesses and strengths and when you need help. Students expected that an effective teacher would monitor their learning (Hattie, 2003) and know what is best for them. As well as having a good knowledge of student potential, the students in Kim's class expected an effective teacher to know their maths (Anthony & Walshaw, 2007) implying the teacher would be able to actually know the answer to the mathematical problems they are asking their students to solve.

The Year 8 students in this study had the same expectation of being respected by the teacher as the secondary students did in Hill & Hawke's (2000) study. As well as not judging people and understanding [that] everyone is different the students believed that an effective teacher would treat every student equally, no favouritism. The students expected an effective teacher would be strict, but not too strict (Perger, 2008) and have reasonable rules and expectations so that all students knew what was expected of them.

From the characteristics of an effective teacher identified by the students in this study, four strong themes emerged. The interesting point about these themes was how similar they were to the New Zealand Graduating Standards (New Zealand Teachers Council, 2014). Of the four identified themes: personal qualities, professional values / relationships, content knowledge of the teacher, and pedagogical knowledge of the teacher, three link directly with the Graduating Teacher Standards. Graduating Teacher Standard One (Professional Knowledge) states that Graduating Teachers know what to teach. The Year 8 students in the class identified that an effective teacher needed to have knowledge of the subject: Standard Two (Professional Knowledge) states that Graduating Teachers know about learners and how they learn (pedagogy). Most of the characteristics identified by the students in this study fitted into this theme showing that these students had developed an understanding of what it takes to teach. The characteristics that made up the third theme matched the Graduating Teacher Standard 6 which states Graduating Teachers develop positive relationships with the learner. The comments on three brainstorm sheets (nine students) focused on the teacher showing students respect. The students recognised that an effective teacher was respectful towards students' rights.

An unexpected outcome of this study was, as the second author stated, seeing that students recognised and appreciated many of the 'teaching practices' I use. As a teacher having the effort I put into teaching recognised is very rewarding. The characteristics the students identified as required to make an effective teacher also strongly supported the classroom culture the teacher had strived to develop. The students' brainstorms not only demonstrated that students can identify the characteristics of an effective teacher but that they were able to recogniseyo practices that support their own learning.

References

Anthony, G., & Walshaw, M. (2007). *Effective pedagogy in mathematics/paangarau: Best evidence synthesis iteration [BES]*. Wellington: Ministry of Education.

Brown, S., & McIntyre, D. (1993). *Developing teachers and teaching: Making sense of teaching.* Buckingham: Open University Press.

Clarke, B. A., & Clarke, D. M. (2004). Patterns of participation in the mathematics classroom. In M. Hoines & A. Fuglestad (Eds.). *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 231-238). Bergen: PME

Flick, U. (2011). *Introducing research methodology: A beginner's guide to doing a research project.* London: Sage.

Hill, J., & Hawk, K. (2000). *Making a difference in the classroom: Effective practice in low decile multicultural schools.* Wellington: Ministry of Education. Retrieved from http://www.minedu.govt.nz

Hattie, J. (2003). *Teachers make a difference: What is the research evidence?* Paper presented at the Australian Council for Educational Research Annual Conference on Building Teacher Quality, Melbourne.

Hunter, R., K., (2007). *Teachers developing communities of mathematical inquiry*. New Zealand: Massey University.

McCullun, B., Hargreaves, E., & Gipps, C. (2000). *Learning: The pupils' voice*. Cambridge Journal of Education 30 (2), 275-289.

Murphy, P. K., Delli, L. M., & Edwards, M. N. (2004). The good teacher and good teaching: Comparing beliefs of second-grader students, preservice teachers and inservice teachers. *The Journal of Experimental Education, 72*(2), 69-92.

New Zealand Teachers Council. (2014). *Graduating teacher standards: Aotearoa New Zealand.* Retrieved from http://www.teacherscouncil.govt.nz/sites/default/files/gts-poster.pdf

Perger, P. (2008). Wanted – one great maths teacher. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds.). *Proc. 32nd Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 105-112). Morelia, Mexico: PME

A REFLECTION ON MATHEMATICS EDUCATION AND LANGUAGE DIVERSITY IN PME CONFERENCES

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In this paper we review research on mathematics education and language diversity in the PME community by exploring Research Reports published in the conference proceedings since its first conference in 1977. A total of 36 contributions on mathematics and language diversity have been indentified and examined. Although an increase in the representation of this area of study can be confirmed, progress towards emerging themes still remains poor in terms of disparities throughout different countries and continents, with some regions being more widely involved in this area. We provide evidence of how several of the main themes have been distributed in the period 1977-2014, and finish by reflecting on what could be done to more strongly address newer terrains in this area of study.

INTRODUCTION

Research on mathematics education and language diversity is fairly recent. It began with the exploration of the role of language and mathematics learning with a focus on bilingualism, bilingual learners and on bilingual classrooms and thereafter the focus moved towards multilingualism (Phakeng, *forthcoming*). In the PME community, the shift to multilingualism came in 1995 with a paper entitled, "Participatory, inquiry pedagogy, communicative competence and mathematical knowledge in multilingual classrooms: a vignette", authored by Adler almost after two decades into the PME conferences. This also marked the starting point of a shift to investigate the socio-political role of language in mathematics teaching and learning. This analysis presented in this paper explores the emergence and growth of research in this field in the PME community as showcased in PME conferences since inception in 1977. We do this exploration by working through the Research Reports published in the conference proceedings and focusing on the following questions:

- What research has been published in the PME proceedings in this area of study?
- What are the major themes that have emerged in this area?
- What are the gaps and silences?

The first Research Report focusing on language diversity was presented at the fifth annual conference of PME in 1981. This was two years after the first paper in an international journal focusing on this area of study appeared in 1979 in *Educational Studies in Mathematics* authored by Austin and Howson. In the thirty-eight editions of PME conferences so far, 36 Research Reports on language diversity have appeared in the proceedings, out of which 22 appeared in the last twelve editions between 2003 and 2014. Although these numbers give a clear indication of the growing body of research

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focusing on language diversity, it is important for us to reflect on whether this research is moving ahead in terms of progress and growth of this area in mathematics education research, whether we are asking new questions and addressing newer terrains in this domain and whether we have better understanding of the complexities of mathematics teaching and learning in contexts of language diversity.

Why this review and why now?

The world has become more multilingual over the years and multilingual classrooms are becoming even more multilingual. With a change in the language policies and practices in some countries, the nature of multilingualism is also changing. This notwithstanding, in the PME community, mathematics and language diversity has remained an area of study that has received limited focus. The low number of Research Reports published in the PME proceedings indicate not only a slow growth but also a relatively smaller group of researchers internationally working in this research domain. It is now almost 40 years since the advent of PME conferences and thus timely and relevant for us as a community to look back and review the advances made and the gaps that remain so that we can craft future directions that emerge from this journey.

THEORETICAL ORIENTATION

Debates on the role of language diversity on mathematics teaching and learning received a major boost in the mid-70s with the occurrence of two major international events that acknowledged the lack of research on the relationship between mathematics learning and teaching and use of languages which in many cases are non-home languages. These events were the second International Congress on Mathematical Education (ICME-2) held in the UK in 1972 and the International Symposium on 'Interactions between linguistics and mathematical education' held in Kenya in 1974. Historically, there was a widespread belief that bilingualism was a hindrance to language development, cognitive and intelligence growth and the ability to think (Reynold, 1928 as cited in Saunders, 1988). Only in the 60s did the effects of bilingualism on the intellectual functioning of children emerge and knowledge of more than one language was seen as an asset. However, there were criticisms over the validity of the claim that bilingualism was helpful (Macnamara, 1966). Studies that followed like that by Ianco-Worrall (1972) with Afrikaans-English bilinguals established the positive effects of knowing more languages by showing that the semantic differences of words were preferred over their phonetic qualities.

Barwell (2003) highlights the dominance of the English language in the research presented at PME in comparison to other languages. Furthermore, he argues that there is an indication of possible discrimination that mathematics education research faces both within the "community of researchers" as well as "in the practice of research" generated on the basis of language (Barwell, 2003, p. 37). Such a phenomenon could be one of the possible reasons behind lesser number of papers in PME on language diversity. Even if one argues that the issue of language diversity is not ignored in PME, it is clear that the presence of multiple languages is not always acknowledged and engaged with in much of the research in mathematics education even if that research was conducted in linguistically diverse classrooms. That may mean that we as researchers are getting only part of the story.

With respect to research in mathematics education as it has developed over time, Skovsmose (2011) argues and conjectures that "90% of research in mathematics education concentrates on the 10% of the most affluent classroom environments in the world, while 10% of the research addresses the remaining 90% of the classrooms" (p. 18). Skovsmose fears that "strong paradigmatic criteria might be operating within mathematics education research [that] has shaped and constructed the prototypical mathematics classroom [which also] dominates the research literature" (p. 18).

METHODOLOGY

To do this review we identified Research Reports in the PME proceedings that were focused on mathematics education and language diversity. We excluded presentations made in plenary lectures, research fora, short orals, posters, as well as discussion and working groups. Given the tradition of strict and rigorous reviewing criteria and process in PME conferences, it is reasonable to consider Research Reports as specialised quality representations of international research activity. Herein, we have ignored all those papers focussing on different aspects of language and communication in mathematics education or communicating mathematically or on the nature of mathematical language. We decided to exclude these papers because they do not have a specific focus on language diversity. Table 1 below gives details of the number of Research Reports published in PME proceedings since 1977.

Period		*Number of RRs published*	*Number of RRs published in 10 years*
1977-1986	1977-1981	1	3
	1982-1986	2	
1987-1996	1987-1991	1	4
	1992-1996	3	
1997-2006	1997-2001	8	17
	2002-2006	9	
2007-2014	2007-2011	11	12
	2012-2014	1	
	Total		*36*

Table 1. Decadewise number of Research Reports (RRs) published

Table 2 lists the dominant themes that were studied or focused on in the above contributions. In order to systematise this review, a framework has been developed by noting the central problem, research approach, arguments and level of education (school/ tertiary/vocational/out-of-school) of each study. It is complex to distinguish one single theme that a paper belongs to among others. This is why we have focused on the major topic in the wording of the central problem that the paper addressed rather than the issues that emerged therein. In order to make thematic groups of the RRs, we have created themes in ways that are sufficiently wide and do not consider unnecessary details according to our purposes of prioritising the major topic of the central problem. The final number of seven themes, as ordered by frequency in Table 2 below, was generated to assist the grouping of papers as well as the further analysis of work on mathematics education and language diversity in the context of the PME community.

Theme of the RR	*Number of RRs*
Teaching activity	11
Learner performance	7
Code-switching	6
Learner participation	6
Theoretical perspectives	2
Methodology	2
Policy	2
Total	*36*

Table 2. Themes and number of RRs

In what follows, we explore the extent to which the seven themes above have come to structure a visible, coherent and consistent theme on mathematics education and language diversity in PME Research Reports.

REVIEW OF THE RESEARCH: AN ANALYSIS

The themes that have received relative major focus since the inception of PME conferences are highly interconnected: ‘Teaching activity’, for instance, often appears in relation to how teachers support multilingual learners including language practices like ‘Code-switching’ (sometimes referred to as language-switching by authors). ‘Learner performance’ and ‘Learner participation’ of students who learn mathematics in language(s) other than their home or first language has remained another major concern among researchers in this area. This is not surprising because as Setati (2012) has argued, what lies at the core of research in this area of study is a need to address the uneven distribution of knowledge and success in mathematics. Some of the studies compare performance of learners who learn mathematics in their home or first language as against those who do not. Evans (2007) argues that poor performance is linked to the comprehension of the test language. It can be drawn from these studies that in order to enhance mathematical understanding, learners' language(s) need to match with the language(s) of the teacher and the textbook used. Works published elsewhere (Clarkson, 2007) have also suggested that competence in the home (or first) language and the language of instruction is instrumental in mathematics achievement.

While 'Theoretical perspectives' are attended to in all papers, tour analysis shows it as a less represented theme. Only a few reports elaborate on the role and use of theories in research on mathematics education and language diversity, along with awareness of issues in putting theories into frameworks for understanding practice. For example, a 2003 Research Report by Morales, Khisty, and Chval relates theories of discourse with a complementary multimodal perspective for the analysis of mathematics learning in multilingual contexts. That paper prioritises the focus on how the integration of certain theoretical perspectives is central to the study of learning in the multilingual mathematics classroom. Other authors draw on the explanation and application of more specific theories like the 'Pirie-Kieren theory', as described by Manu in his 1995 report on mathematical understanding in bilingual settings. What is missing or weak in some of these papers is the examination of the constraints of the different theories that are contemplated and often recontextualised from linguistics into mathematics education.

Although many of the examined papers pose important methodological questions, only two of them primarily address 'Methodology' as a major theme. Some of the authors tend to say that they have discussed methodological issues elsewhere. Barwell (2001) is the author who overtly discusses the need for and development of a methodology, in this case based on discursive psychology and conversational analysis to investigate data of what he refers to as "English as Additional Language" learners during mathematical interaction with English native speakers as they engage in solving word problems. This paper by Barwell also addresses learner participation but this theme comes across as a justification of why the explained methodology becomes valuable. It is interesting to note that two years later, in 2003, this same author contributes a second empirical paper that refines some of the initiated methodological concerns but now with a major focus on the study of learner participation. Still with respect to this author, we find in 2005 another paper that again foregrounds methodology. It summarises, however, a very different approach for the construction of a framework aimed at the comparison of PME research into multilingual mathematics education in diverse sociolinguistic settings.

More generally, papers documenting works from single countries (such as the UK, the US, Australia, or South Africa) show responsibility for producing significant impact at an international level when describing policy issues which may transcend particular country boundaries. Nevertheless, 'Policy' has been only presented as the major focus in two of the reports during these years of PME conferences. Despite the few papers focusing on policy, it can be argued that the shift toward socio-political approaches is as a result of policy issues being included among the relevant aspects in the analyses of some of the recent papers. What we have, therefore, is that political complexity tends to be commented on in these papers as something that importantly affects the conditions of mathematics teaching and learning, however, this is often done on the level of additional considerations and future research, or in ways that are not empirically situated. An exception is the paper by Civil in 2008, where the policy around the relationships between the learners' families and the school system is a clear variable in the analysis of interview data.

The research approach adopted in most Research Reports of empirical nature has commonly been small-scale qualitative studies (often in the form of case studies) as well as a few conceptual studies. Alongside the strength of rigorous qualitative research, quantitative approaches are equally necessary to further advance the domain of mathematics education and language diversity. Adoption of quantitative methodology as a research approach has remained minimal in the examined RRs with a slight exception of a Research Report by Clarkson (1984) entitled, "Language and mathematics in Papua New Guinea: A land of 720 languages". In this paper, Clarkson used statistical techniques to conclude that the extent of students' use of English did not significantly correlate with their mathematical tests.

Excluding the few reports that drew on data from research in out-of-school contexts and taking into account the total absence of Research Reports on vocational contexts of learning and language diversity, it can be seen that a majority of the reported studies were conducted at the school level up to the learner-age of 15 years and practically none at the tertiary level (post-school and university). Although the published Research Reports help us understand the complexities of mathematics education and language diversity at the school level, similar exposure to complexities of language at the university level remains elusive.

GAPS AND SILENCES

Almost after 10 years of the publication of the *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future*, it is even more relevant what was stated there: "In the real world, multilingualism is closer to the norm than monolingualism. However, much research reported at PME conferences is conducted in classrooms where many (or all) the students are multilingual and learning in a second or additional language, but usually no mention is even made of this" (Gates, 2006, p. 387). In our analysis of the Research Reports, we have updated the diagnosis by Gates and have detected enduring gaps and silences that have become even more visible.

Research Reports that focus on the complexities and challenges faced by immigrant learners who learn mathematics in a language that is not their own are largely conducted in developed countries in Europe and the USA. This remains the case almost 40 years since inception of PME conferences, despite the growing number of immigrant communities all over the world. What we also found interesting is the absence of Research Reports from developing countries that are multilingual such as

India and Pakistan as well as the Middle East and in the Eastern European region. The first Research Report on mathematics and language diversity published in the PME proceedings was authored by Dawe in 1981, "Bilingualism and reasoning in mathematics", and reported on a study conducted in Great Britain. While there were many papers that reported studies conducted in locations like Australia, Germany, Malaysia, Mozambique, Papua New Guinea, South Africa, Catalonia-Spain, UK, and US, papers from other countries in Africa, Asia, Europe, and Latin America are elusive in the PME proceedings.

Studies focused on pre-service and in-service teacher education and language development among teacher educators as support for pedagogical practices have not clearly emerged in the analysis. In the Research Report published so far there is no consideration of how language issues are dealt with in the professional knowledge that is being constructed by pre-service and in-service mathematics teachers. This means in particular that Research Reports that deal with teacher education and teaching activity have not yet taken on the significant contributions by Adler on the dilemmas that teachers have to face while teaching mathematics in contexts of language diversity. Although this research domain is growing, as a prominent research community, PME remains elusive of a clear and unanimous message of how to address the complexities of teaching and learning mathematics in contexts of language diversity. There are reports from small-scale studies that show the depth in a particular problem, but as argued earlier there are not many large-scale studies that can help us see the extent of the problem. Moreover, there have been no large-scale cross-country studies which might increase our understanding of commonalities in this research domain beyond boundaries.

LOOKING AHEAD BY LOOKING BACK

Not only is there a limited number of published work in this area of study but the themes explored are also limited. For example, the current trend of papers in the PME proceedings on this topic shows that they are not connected – they neither draw on previous research nor build a coherent body of knowledge. There has been a discrete and discontinuous nature of the flow of the knowledge build-up. In addition, most of the recent papers published in the recent PME proceedings are disjointed. Such a trend raises questions about how does the PME community build a body of knowledge. What have we learnt from the work and how have we addressed the gaps? Are recent papers citing other PME papers? Are we asking the same questions that have been explored in the past?

In the 2013 reconstruction of the PME research categories for authors to submit their papers for review, the category of studies in mathematics education and multilingualism was added. This is an internal small change in the review system that well represents the commitment of the International Committee in charge of the PME community toward the research domain that we have discussed in this report. These sorts of technical rearrangements are helpful in moving relatively new terrains ahead.

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References

Barwell, R. (2003). Patterns of attention in the interaction of a primary school mathematics student with English as an additional language. *Educational Studies in Mathematics*, *52*(1), 35-59.

Evans, S. (2007). Differential performance of items in mathematics assessment materials for 7-year-old pupils in English-medium and Welsh-medium versions. *Educational Studies in Mathematics*, *64*, 145-168.

Clarkson (2007). Australian Vietnamese students learning mathematics: High ability bilinguals and their use of their languages. *Educational Studies in Mathematics. 64,* 191-215.

Gates, P. (2006). The place of equity and social justice in the history of PME. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, present and future* (pp. 367-402). Rotterdam, The Netherlands: Sense Publishers.

Ianco-Worral, A. D. (1972). Bilingualism and cognitive development. *Child Development, 43*, 1390-1400.

Macnamara, J (1966). *Bilingualism and primary education*. Edinburgh, UK: Edinburgh University Press.

Phakeng, M. (*forthcoming*). Mathematics education and language diversity: Past, present and future. In A. Halai & P. Clarkson (Eds.). *Teaching and learning mathematics in multilingual classrooms: Issues for policy and practice*. Rotterdam: Sense Publishers.

Saunders, G. (1988). *Bilingual children: From birth to teens*. Philadelphia, PA: Multilingual Matters.

Setati, M. (2012) Mathematics in multilingual classrooms in South Africa: From understanding the problem to exploring possible solutions. In B. Herbel-Eisenmann, J. Choppin, D. Wagner & D. Pimm (Eds.), *Equity in discourse for mathematics education: Theories, practices and policies* (pp. 125-145). New York: Springer.

Skovsmose, O. (2011). *An invitation to critical mathematics education*. Rotterdam, Netherlands: Sense Publishers.

STATISTICAL LITERACY IN A PROFESSIONAL CONTEXT

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Many adults need to interpret workplace data as part of their employment. Their capacity to do so depends on their statistical literacy levels and their attitudes to statistics. To identify and develop the statistical literacy of adults with limited understanding it is important to know what approaches to professional learning might be effective. This paper presents an overview of an extended project that examined teachers' statistical literacy for dealing with student achievement data, and developed professional learning materials to support teachers in areas of identified difficulty. It highlights several issues associated with workplace statistical literacy.

INTRODUCTION AND CONTEXT

Statistical literacy, vital in today's data-driven world, involves the ability to interpret, evaluate, and communicate statistical information. Many professionals require knowledge of numeracy, statistics, and data presentation to make use of quantitative reports in a professional setting, and they also need a positive disposition towards the use of such data (e.g., Gal, 2002; Watson, 2006). The specific technical knowledge required will vary among and within professions, and so professionals may require targeted learning opportunities. This paper reports on a project that investigated teachers' professional statistical literacy and used this data to design instructional materials. Although teachers are the focus of the study, the study involves general principles that have the potential to be applied in other professional contexts.

In Australia a national push towards assessing students' literacy and numeracy understanding has resulted in schools and teachers receiving data that report on individual, class, school, state, and national outcomes. As in many countries, there has been contention about the value of such testing, which may impact on attitudes to the data. The preparation and distribution of reports occur at the state level, and, in Victoria where the study took place, many results are presented using boxplots (with whiskers to the 10th and 90th percentiles). The present study, then, is concerned with Victorian teachers interpreting data about "system reports of student achievement" (hereafter, SRSA), with the boxplot a common representation. Beyond this specific context and data representation type, however, the broader question of interpreting data—in whatever form or in whatever workplace—is a complex one. To investigate this, two frameworks were useful for the present study, and may be useful elsewhere.

FRAMEWORKS

The framework for professional statistical literacy shown in Figure 1 was used to interpret the statistical demands of the SRSA data presented to teachers, and underpinned the design of tasks to assess statistical literacy. Past research literature

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 17-25. Hobart, Australia: PME.

guided the framework's development. Curcio's 1987 study of graph comprehension highlighted the ideas of "reading the data" (to read literally the direct factual information on the graph), "reading between [or within] the data" (attend to two or more data points on the graph, often for comparisons), and "reading beyond the data" (extend, predict, and infer). More recent work of Shaughnessy, Garfield and Greer (1996) suggested an additional category, "reading behind the data," which attends to the context from which the data arise. Watson (2006) also emphasised the place of context in the interpretative process, with a three-tiered hierarchy building on basic terminology, through understanding of concepts in their context, to challenging and questioning statistical claims. The statistical knowledge base posited by Gal (2002, p. 10) also indicates the importance of knowing why data are needed, having familiarity with basic terms, and understanding how statistical conclusions are reached.

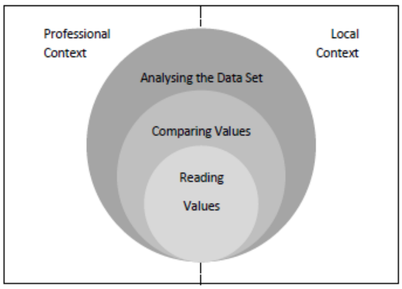

Figure 1: Framework for professional statistical literacy (Pierce & Chick, 2013)

Each of the earlier frameworks alone was insufficient for examining the statistical literacy required to interpret typical workplace data (Chick & Pierce, 2013). The framework in Figure 1 draws on all of them, however, indicating a hierarchy associated with increasingly sophisticated statistical knowledge at the same time as acknowledging the impact of context on interpretation. Figure 1 implies that reading values is simpler than comparing values, which is easier than analysing the data set holistically. The hierarchy suggests higher-level interpretation demands competencies from the lower level/s. Statistical skills are applied in a real-world context, with interpretation of data/statistical reports dependent on understanding these contexts as well as the technical aspects of the statistics. In the education workplace *professional context* would include understanding the source of test data (e.g., how the scores arise from testing and are used in reports), and factors specific to the professional's *local context* may also be needed to interpret data successfully (e.g., the socio-economic background of a school or knowing a child was ill on test day).

The second framework useful for investigating professional statistical literacy was the Theory of Planned Behaviour (TPB), (Ajzen, 1991). This looks at factors that either encourage and enable or, in contrast, create real/perceived barriers to change. The TPB suggests that three main groups of factors affect a person's intention to change behaviour, which in this case refers to engaging with statistical reports on SRSA data. These factors are attitudes (Do I think it will be professionally informative and worthwhile?), subjective norms (Do I think that others, whose opinion I value, think it is important?), and perceived behavioural controls (Do I see barriers that will make the new activity difficult for me?). In this study TPB was taken as a framework for examining teachers' disposition towards engagement with statistical reports.

THIS STUDY

This study first examined the attitudes and perceptions of school teachers towards SRSA (the statistical data that they received about students' literacy and numeracy performance), and then how well teachers understand and interpret the statistics of SRSA. Based on the frameworks above data collection instruments were designed and data collected in 2010 and 2011 as set out in Table 1.

Data collected	Sample and sample size
2010: Paper-based survey targeted demographics, access to SRSA, attitudes, and statistical literacy.	Cluster sample of 10 primary and 10 secondary government schools across Victoria, with 7 teachers plus principal or nominee from each school (n=150)
2011: Online survey using revised form of paper survey.	Random sample of 104 primary and secondary government schools (n=704)
2011: Face-to-face professional learning trialled, and evaluated.	2010 sample split into experimental (n=42) and control (n=31) groups.
2012: Online tutorials created, trialled, and evaluated	3 secondary and 6 primary schools not previously involved (n=86)
2013: "Using Assessment Data" tutorials online	Available from: http://usingassessmentdata.vcaa.vic.edu.au/index.aspx/

Table 1: Data collected during the project

Later stages of the project (2011-2013) used the teachers' statistical literacy results to develop professional learning experiences, first for face-to-face presentation, and then via on-line learning modules, as discussed later. Our main purpose was to consider teachers' statistical literacy, specifically, and factors influencing their capacity to interpret student assessment data, but there are implications for broader issues of workplace statistical literacy: what attitudinal factors affect it, what statistical knowledge is required, and what might enhance that statistical knowledge.

RESULTS ABOUT TEACHERS' STATISTICAL LITERACY

Results from items framed by the Theory of Planned Behaviour

Items investigating attitudes framed by TPB were included on both paper and online surveys from 2010-2011. The results showed that a majority of the teachers had a positive attitude towards the use of student achievement data (SRSA), as indicated by agreement or strong agreement with statements in Figure 2. A minority had negative views, perhaps reflecting negativity about the whole national student testing process. The results showed little impact of subjective norms (influence of peers), but did indicate that a significant minority perceived difficulties that would be barriers to their engagement with SRSA. More details about these results are found in Pierce and Chick (2011) and Pierce, Chick, and Gordon (2013).

Attitude (perception of the worth/value of the activity)

85% Student achievement data are something that my school's leadership team expect me to pay close attention to.

82% SRSA are useful for identifying topics in the curriculum that need attention.

80% SRSA are useful to inform whole school planning.

67% SRSA are helpful for grouping students according to ability.

58% SRSA are helpful for planning my lessons.

55% SRSA tell me things about my students that I had not realised.

Perceived behavioural controls (perception of factors influencing ability to engage)

15% I don't feel I can adequately interpret the SRSA I receive at our school.

29% SRSA take too long to interpret.

33% The amount of data presented in the SRSA I see is overwhelming.

Figure 2: Percentage of teachers who agreed with the given statements

Statistical knowledge: Results from Rasch modelling

Early work informed by the Figure 1 framework which investigated the statistical literacy demands of data representations being sent to teachers (Chick & Pierce, 2013) led to the design of items for the surveys that would assess teachers' capacity to interpret statistical data, including the boxplots that were typically used in their context. Teachers were presented with items using reports of the same format that they receive at their schools (see Figure 3). Items focused on the different statistical literacy levels: reading, comparing, and analysing. The teachers' responses on these items were subjected to Rasch modelling (see Pierce, Chick, Watson, Les, & Dalton, 2014) to rank the items' difficulties. Items that respondents found easiest involved:

- Reading values from a table; and
- Identification of school's weakest area from box plots or table (where the differences were sufficiently gross that interpretation was straightforward).

The items that teachers found hardest required:

- Justification of their choice of a box plot that matched a given histogram;
- Understanding what the "box" component of a boxplot represents;
- Recognising that the boxplots used in these particular reports have whiskers extending only to the 10^{th} and 90^{th} percentiles; and

- Correctly conceiving that boxplots indicate the distribution and density rather than the frequency of scores.

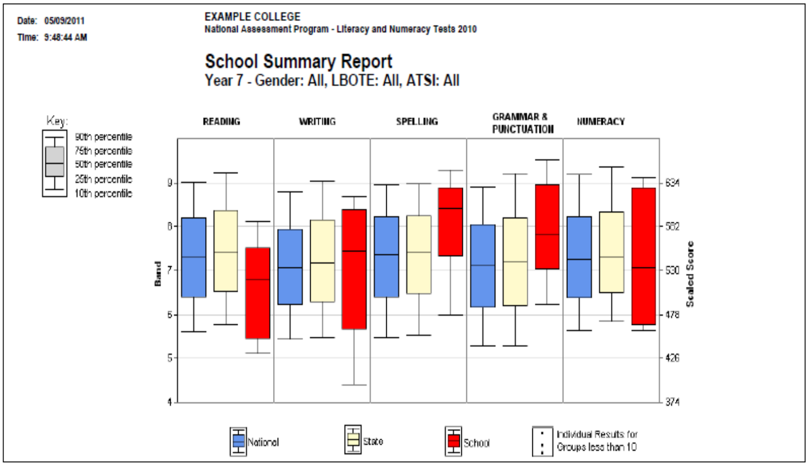

Figure 3: School summary report for a fictitious Example College

The teachers demonstrated appropriate basic statistical expertise, even considering their diverse backgrounds. Many teachers had little or no in-depth statistical training beyond interpreting simple histograms, and determining means, medians, and modes. Once the tasks became more complex, however, requiring an understanding of the nuances of the representation, many teachers' limited statistics backgrounds impacted on their capacity to correctly interpret the data. In particular, the density/frequency misconception arose often for boxplots, exacerbated because the visual impact of the representation tempts the inexperienced user's brain to interpret "more box" as "more data points." An added complexity here was the fact that the Victorian boxplots have non-standard whiskers, although a key *was* supplied to indicate what was depicted.

These results provided evidence that a professional learning program to develop teachers' statistical literacy should focus on alleviating misconceptions with regard to boxplots, assist teachers to analyse data in ways that will inform not only whole school planning but also classroom teaching, and embed a message that these reports provide information about students and classes and are not meaningless numbers.

Investigating workplace statistical literacy more generally

As discussed, these results apply to teachers and the SRSA data that they receive. For other professions in which adults are expected to interpret data, similar issues might be expected to arise. The TPB items require only straightforward adaptation for research in other contexts; the attitudes towards the use, usefulness, and interpretation of data are likely to reflect the workers' statistical backgrounds, their attitudes about the value of the data itself (and not just its presentation), and the context in which they work with the data (e.g., the teachers felt that the SRSA data came too late in the year for them to use it in adapting their teaching). In terms of investigating statistical knowledge and skills, the focus on boxplots in this study was very much associated with the Victorian teaching context; in another workplace context it seems reasonable to expect that, while boxplots may not be the representation-of-the-day, there might well be other representations that give rise to interpretation difficulties peculiar to those particular data approaches. Furthermore, it is likely that, like teachers, the workers' difficulties will be associated with the complexity of the statistical situation involved and limits on their background statistical knowledge.

DEVELOPING A PROFESSIONAL LEARNING PROGRAM

The professional learning program to help teachers develop statistical literacy went through two design phases: a face-to-face program first and then an on-line program for wider dissemination. The focus of the program was informed by the results of the statistical literacy surveys, specifically boxplot understanding and developing appropriate language for describing statistical results.

A key activity for the face-to-face program involved a set of 30 fictitious students representing an Australian Year 7 class, illustrated as cartoon characters on cards with their test scores. The data were constructed to produce boxplots exhibiting characteristics the teachers had found difficult to interpret correctly. Figure 3 above was produced from these data. Teachers first placed the individual cards on a scale to create a pictogram. Then a rough but conceptually-based boxplot was constructed by dividing the class into quarters, placing a rectangular card over the middle 50%, turning over the cards for the top and bottom 10% of students (to depict the outlying students not shown because the Victorian boxplots have whiskers to the 10th and 90th percentiles), and then placing appropriate strips of card to depict the whiskers. Finally, the student cards were removed, leaving the "abstract" boxplot. A number of audible "ah-ha" moments occurred as teachers realised what could and could not be inferred about the boxplot distribution. Their difficulties interpreting boxplots usually had their basis in not reading the key and misunderstanding the role of the percentiles. Many participants expressed surprise that longer whiskers did not equate to more students but rather to a greater spread of results. A consistent comment was how the use of students with names and faces brought the data to life and helped teachers think about such statistics providing information about their own students. This activity helped to link "abstract" data to "concrete" interpretations.

Other materials included ten printed statements describing boxplots. Participants judged the merits of the statements, discarding incorrect ones, and ranking the remainder in terms of their professional usefulness. As participants worked in groups, they clarified their thoughts regarding the wording of these statements, particularly in light of their recent insights gained through the building of a boxplot activity.

The following quote from one participant summarises teachers' feedback six weeks following the professional learning program.

> It was excellent. However, you tend to forget … stuff because you are only exposed to it for a short period of time. You need … a few more sessions to not only become confident with it … but then become so familiar … that it becomes second nature.

For these teachers their school data typically would be analysed once each year. The evaluation of the face-to-face professional learning program showed it was highly successful *on the day*, with teachers immediately and appropriately applying the principles learnt to their own school's data. However teachers were not able to recall this knowledge some weeks later. To address this, online "refresher" tutorials were developed. These new tutorials were trialled during 2013; they were designed to have a focus on a single issue per tutorial, with each taking only a few minutes to complete. This time the teachers' feedback focused almost exclusively on technical layout and computer concerns rather than issues of statistical literacy. Where possible the tutorials were made interactive, and the teachers responded positively to these drag-and-drop tasks and sections requiring them to enter answers to questions with immediate feedback. It seems that the hands-on face-to-face activities laid the foundations for understanding that later use of "just in time" support via the online resources could help sustain. The extent to which the online tutorials *alone* can provide appropriate professional learning has not been investigated yet.

CONCLUSIONS

The research showed that teachers were generally positive about interpreting and using student achievement data, but that there were limitations to their capacity to do so, due to misconceptions associated with the statistical representations being used. When statistical approaches summarise data—through calculating a single statistical value like the mean, or producing a depiction of a distribution as in the case of boxplots—there is compression of information, and so establishing the possibilities for what the original data set might be like can prove challenging. For these teachers, this was reflected in their difficulties with certain aspects of boxplots, exacerbated by the fact that most had little prior experience with this particular representation. In other professional environments similar issues are likely to arise: there are characteristic features of any data-summarising approach that need to be deeply understood in order to interpret data successfully. As soon as interpretation requires understanding of distributions then dealing with statistics starts to become complicated, and intuitions will need to be actively developed rather than assumed.

This means that, as in the professional learning sessions provided for the teachers, it is essential to identify and target key misconceptions in any training that is provided to develop professionals' statistical literacy. Moreover, if professionals are only consulting data intermittently there may well be a need not only for initial professional learning, but for accessible "just-in-time" follow up material that reviews key issues effectively. Increased confidence in their ability to interpret data is likely to reduce the barriers to engaging with data, and make data usage more effective.

Finally, it should be noted that although the statistical literacy framework is intended to treat statistical literacy as a broad construct, the teachers' situation had some very specific technical requirements (e.g., understanding boxplots). There is no guarantee that a professional's need to understand a particular statistical tool is necessarily generalizable. So, for example, while the project may have enhanced teachers' capacity to deal with boxplots and possibly increased some aspects of broader statistical literacy, it is not clear that it will have increased their understanding of other specific representations.

ACKNOWLEDGEMENTS

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References

Ajzen, I. (1991). The theory of planned behavior. *Organizational Behavior and Human Decision Processes, 50,* 197-211.

Chick, H. & Pierce, R. (2013). The statistical literacy needed to interpret school assessment data. *Mathematics Teacher Education Development, 15*(2), 5-26.

Curcio, F. (1987). Comprehension of mathematical relationships expressed in graphs. *Journal for Research in Mathematics Education, 18*, 382-393.

Gal, I. (2002). Adults' statistical literacy: Meanings, components, responsibilities. *International Statistical Review, 70*, 1-51.

Pierce, R., & Chick, H. (2011). Teachers' intentions to use national literacy and numeracy assessment data: A pilot study. *The Australian Educational Researcher*, *38*, 433-447.

Pierce, R. & Chick, H. (2013). Workplace statistical literacy for teachers: Interpreting box plots. *Mathematics Education Research Journal, 25*, 189-205.

Pierce, R., Chick, H. & Gordon, I. (2013). Teachers' perceptions of the factors influencing their engagement with statistical reports on student achievement data. *Australian Journal of Education, 57*, 237-255.

Pierce, R., Chick, H., Watson, J., Les, M., & Dalton, M. (2014). A statistical literacy hierarchy for interpreting educational system data. *Australian Journal of Education, 58*, 195-217.

Shaughnessy, J. M., Garfield, J., & Greer, B. (1996). Data handling. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics in education* (pp. 205-237). Dordrecht, The Netherlands: Kluwer.

Watson, J. M. (2006). *Statistical literacy at school.* Mahwah, NJ: Lawrence Erlbaum Associates.

A METHODOLOGY FOR THE DESIGN OF QUESTIONNAIRES TO EXPLORE RELEVANT ASPECTS OF DIDACTIC-MATHEMATICAL KNOWLEDGE OF TEACHERS

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This paper presents some of the results obtained in a study that explored prospective secondary teachers' Didactic-Mathematical Knowledge (DMK) regarding the derivative. The specific focus was on mathematical dimension of DMK. The study was carried out in three stages: 1) design of a questionnaire; 2) pilot application of this questionnaire so as to develop its final version; and 3) application of the final version to a sample of Mexican prospective secondary teachers. Both the design of the questionnaire and the responses of the prospective teachers reveal the complex set of mathematical practices, objects and processes that are brought into play when solving tasks on derivatives. The three mentioned stages are provided as a useful methodology for the design of instruments for assessing relevant aspects of teachers' knowledge.

INTRODUCTION

As we noted in a previous study presented at the 36th Conference of the International Group for the Psychology of Mathematics Education (Pino-Fan, Godino, Font & Castro, 2012), the last three decades have seen a growing interest in determining the knowledge which teachers require to teach specific topics such as the derivative. Our understanding of this knowledge has been advanced through the work of authors such as Shulman (1986), Ball (2000) and Hill, Ball and Schilling (2008). However, this research, along with that carried out by Fennema and Franke (1992), Llinares and Krainer (2006), Ponte and Chapman (2006), among others, has led to a multifaceted view of the way in which knowledge for teaching is constructed. Indeed, there is no universal agreement as to the theoretical framework that should be used to describe mathematics teachers' knowledge. Furthermore, although progress has been made in terms of defining the categories of knowledge that mathematics teachers need in order to teach effectively, a number of questions remain to be resolved. For example, how, or according to what criteria, might we measure the knowledge corresponding to each of these categories? Having identified such criteria, are they then useful for developing and promoting among prospective teachers the kinds of mathematical knowledge required by the different aspects of mathematics teaching? The research described in this paper seeks to address precisely these two questions.

In Pino-Fan et al. (2012) we presented the results obtained in the first part of a study that explored what we call the epistemic facet of prospective secondary teachers' didactic-mathematical knowledge about the derivative. The study as a whole involved three stages: 1) design of a questionnaire to evaluate teachers' knowledge, specifically as regards the epistemic facet of didactic-mathematical knowledge about the derivative; 2) analysis both of the results obtained in a pilot application of this questionnaire, as well as of the comments made by a number of experts who were consulted regarding the validity and reliability of the questionnaire; and 3) based on the information gathered in stage 2, development and application of a final version of the questionnaire, and analysis of the results obtained. This third stage also included interviews in order to obtain a more detailed understanding of prospective teachers' knowledge. The present paper provides an overview of these three stages. However, as stage 1 and part of stage 2 have already been addressed in Pino-Fan et al. (2012) the emphasis here is on stage 2 and, especially, stage 3. The results obtained provide partial answers to the two questions that motivated the research. Furthermore, the three mentioned stages are provided as a useful methodology for the design of instruments for assessing both the mathematical dimension and other dimensions of DMK.

THEORETICAL FRAMEWORK

The reference framework for this study is the Onto-Semiotic Approach (OSA) to mathematical knowledge and teaching; one that has been developed in several studies since 1994 (Godino, Batanero & Font, 2007). More specifically, we make use of a model developed by Godino (2009), and refined in different works (Pino-Fan, Godino & Font, 2013; Pino-Fan, Godino & Font, 2014), within the framework of the OSA, namely the Didactic-Mathematical Knowledge (DMK) model, which was designed to categorise and analyse the knowledge that teachers require to teach specific topics. The DMK model interprets and characterises a teacher's knowledge from three dimensions (Pino-Fan, et al., 2014): *mathematical dimension*, *didactical dimension* and *meta didactic-mathematical dimension*. Each of these dimensions considers subcategories of knowledge, which, in turn, also include theoretical and methodological tools that allow operationalising knowledge analysis regarding each subcategory. Furthermore, these dimensions, with their corresponding analysis tools, are involved in each of the phases proposed for the elaboration of Instructional Designs: preliminary study, design, implementation and evaluation. The relationships between dimensions and theoretical-methodological tools proposals by the DMK, with the contributions of other models of teachers' knowledge, can be found in Pino-Fan and Godino (2014).

In the case of the present paper, the topic is the derivative, and the focus is on mathematical dimension and one of the six facets included in the didactical dimension of DMK model, the epistemic facet. DMK's mathematical dimension makes reference to the knowledge that allows the teacher to solve the problem or mathematical activity that is to be implemented in the classroom and link it with mathematical objects that can later be found in the school mathematics curriculum. It includes two subcategories of knowledge: common content knowledge and extended content knowledge. The first subcategory, *common content knowledge*, is the knowledge of a specific mathematical object, which is considered as sufficient to solve problems and tasks proposed in the mathematics curriculum and in the textbooks of a certain educational level. The second subcategory, *extended knowledge*, refers to the knowledge that the teacher must have about mathematical notions that, taking the mathematical notions that are being studied at a certain time as a reference (for example, derivatives), come ahead in the curriculum of the educational level in question or in the next level (for example, integers in high school or the fundamental theorem of calculus in college). Extended content knowledge provides the teacher with the necessary mathematical foundations to suggest new mathematical challenges in the classroom, to link a certain mathematical object being studied with other mathematical notions and to guide students to the study of subsequent mathematical notions to the notion that is being studied (Pino-Fan, et al., 2014). For its part, the *epistemic facet*, one of the facets involved in the didactical dimension of DMK, refers to specialised knowledge of the mathematical dimension. The teacher must have a certain amount of mathematical knowledge "shaped" for teaching; that is to say, the teacher must be able to mobilise several representations of a mathematical object, to solve a task through different procedures, to link mathematical objects with other mathematical objects taught at a certain educational level or from previous or upcoming levels, to comprehend and mobilise the diversity of partial meanings for a single mathematical object, that are part of the holistic meaning for such object (Pino-Fan, Godino & Font, 2011), to provide several justifications and argumentations, and to identify the knowledge at play during the process of solving a mathematical task (Pino-Fan & Godino, 2014). For the *mathematical dimension* and the *epistemic facet*, two levels of analysis are proposed: 1) *mathematical and didactic practices*; in other words, a description of the actions performed to solve the proposed mathematical tasks, so as to contextualise content and promote learning; 2) *configuration of objects and processes*; that is, a description of the mathematical objects and processes involved in the mathematical practices under study, as well as those which emerge out of them.

METHOD

The research was based on a mixed methods approach (Johnson & Onwuegbuzie, 2004), since it was an exploratory study that examined both quantitative (level of accuracy of answers to questionnaire items: correct, partially correct and incorrect) and qualitative variables (type of cognitive configuration activated when solving the tasks set). The three stages of the research were as follows.

Stage 1: design of the questionnaire

In this first stage we began by creating a bank of tasks involving the derivative. These tasks were drawn from various studies on the teaching of calculus. In order to select the tasks that would be included in the questionnaire we considered three criteria: 1) the questionnaire needed to cover the different meanings of the derivative, taking as a reference the holistic view of the derivative that is set out in Pino-Fan, et al., (2011); 2) it had to capture movement between the different ways of representing both the function and its derivative; and 3) it had to reflect the type of didactic-mathematical knowledge that corresponds to the epistemic facet and mathematical dimension. This first stage is described in detail in Pino-Fan et al. (2012).

Stage 2 (part one): pilot application of the questionnaire

Having designed the initial version of the questionnaire, which included seven tasks that reflected the three criteria, we then selected an intentional sample of prospective secondary teachers, all of whom were students in the degree in mathematics teaching offered by the Autonomous University of Yucatan (UADY) in Mexico. This degree course comprises eight semesters, and the 53 prospective teachers who made up the sample were enrolled in semesters six and eight. These students had already taken courses related not only to mathematical analysis (differential and integral calculus, differential equations, etc.) but also to the teaching of mathematical subject areas (geometry, differential and integral calculus, conceptual development of calculus, etc.). The questionnaire was administered at the beginning of February 2011, and took two hours to complete. The results are reported in Pino-Fan et al. (2012).

Stage 2 (part two): expert triangulation

In order to ensure the reliability and validity of the questionnaire, it was subjected to a process of expert triangulation. By means of a survey, eight experts were asked to assess the relevance of the questionnaire items for evaluating each of the following aspects: 1) different meanings of the derivative; 2) representations activated both in the item statements and their possible solutions; and 3) the type of didactic-mathematical knowledge (corresponding to the epistemic facet and mathematical dimension) required to solve the tasks. In order to make any necessary improvements to the pilot version, the experts were also asked for their opinion regarding any important content that was missing from the questionnaire, as well as the wording and comprehensibility of items. It should be noted that we use the word "expert" to refer to researchers who have extensive trajectory in the research on the didactic of calculus and the teachers training. So, research professors from France, Mexico, Spain, Colombia and Portugal, participated in this study of the "expert triangulation".

Stage 3: application of the final version of the questionnaire

The final version of the questionnaire was administered to 49 prospective teachers at the beginning of February 2012. These prospective teachers constituted the total number of students who, at that time, were enrolled in semesters six and eight of the degree in mathematics teaching offered by the UADY. As in the pilot, the final version took two hours to complete, and the students to whom it was administered had, like their counterparts in stage 1 of the research (in 2011), already taken courses about mathematical analysis and how to teach calculus. One week after administration of the questionnaire, and once the results had been gathered, a series of semi-structured interviews were scheduled with the students. The aim of these interviews, which took into account the suggestions of the eight experts, was to explore in greater detail the knowledge and, specifically, the cognitive configurations used by the prospective teachers when solving the tasks set.

RESULTS AND DISCUSSION

Stage 2 (part two): expert triangulation

In general, the DMK-Derivative Questionnaire was viewed positively by the experts, as illustrated by the following two comments:

> I think the questionnaire is very thorough. It's worth noting that it measures not only the types of knowledge but also the way in which they have been acquired. I'm referring here to the difference between a highly mechanical or routine form of learning and one that is meaningful, where the derivative of a point has various representations (Expert E7).
>
> …I would like to point out that the proposed tasks are rich and varied, and I think they are capable of measuring prospective secondary teachers' didactic-mathematical knowledge about the derivative. I would also like to highlight the importance of the criteria defined in order to select the tasks: different meanings of the derivative; the use of different representations activated both in the item statements and the solutions to them; and the three components of prospective teachers' didactic-mathematical knowledge [common, extended and specialised content knowledge] (Expert E8).

The experts also made a number of observations regarding each of the tasks included in the questionnaire, notably the following:

> Some students may have a background in economics or business administration, where the derivative is associated with concepts such as marginality, and where it appears as the function of marginal utility... It would be useful to include a question that relates the derivative to economics… (Expert E5).
>
> …it is necessary to include activities related to modelling, such as optimisation problems or those involving the instantaneous rate of change. In my view it would also be important to include a problem in which verbal expressions play a more significant role … (Expert E1).

Along with other observations and suggestions these comments by experts E5 and E1 led us to include the tasks shown in Figure 1. One of the experts (E8) proposed the inclusion of five new tasks so as to explore in greater depth the variety of representations. However, we did not follow this suggestion, as we agreed with a comment made by expert E7:

> Rather than add new tasks I think it would be better to complement the questionnaire with interviews that could provide more detailed information about the students' responses. Of course, more tasks can always be added, but it is important that the questionnaire does not become too long.

9. In a company the total cost of producing q units is given by the function $C(q)=\frac{1}{3}q^3-12q^2+150q+2304$
   a) Find the functions that determine the mean total cost and the marginal cost.
   b) Determine the marginal cost and the mean total cost when producing 3 and 6 units.
10. The kinetic energy of an object is directly proportional to the square of its velocity and it has been shown experimentally that the proportionality constant is half its mass. What is the rate of change in kinetic energy with respect to velocity when $v=0\,{}^{mts}/_{seg}$? Justify your answer.
11. Is it possible to find two numbers such that their sum is 120 and the product of one number and the square of the other number is a maximum? If so, what are these numbers? Justify your answer.

Figure 1. Three new tasks included in the final version of the Questionnaire

Stage 3: application of the final version of the questionnaire

In order to evaluate the quantitative variable 'level of accuracy of the students' answers', each of the questionnaire items was scored as follows: 2 points if the answer was correct, 1 if it was partially correct and 0 if it was incorrect. Thus, the maximum possible score for a student who offered a correct solution to all the items of all the tasks was 36. Figure 2 shows the distribution of scores obtained by the 49 prospective teachers when responding to the final version of the questionnaire, and also indicates that the mean score was 13.8 (38.3% of the possible total score). It should be noted that of the 49 students who responded to the questionnaire, only 28 (57.1%) scored higher than the mean, and none of them reached a score of 24. This means that 42.9% of these prospective teachers scored below the mean. The tasks which they found most difficult were the three shown in Figure 1, which had a difficulty index of almost zero.

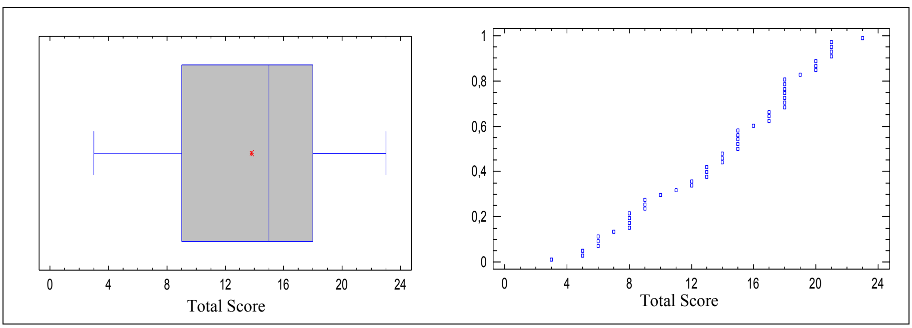

Figure 2: Distribution and mean of the scores obtained on the questionnaire

The qualitative variable of interest in the present study was the *type of cognitive configuration,* which refers to the systematic description of primary mathematical objects (linguistic elements, concepts, propositions, procedures and arguments) and of the processes and interactions between them. The analysis of the cognitive configurations involved in the students' solutions, together with the data obtained through the subsequent interviews, revealed that: 1) the prospective teachers did not make connections between the different partial meanings of the derivative; 2) when attempting to solve the tasks they found it difficult to make use not only of the elements considered to be part of mathematical dimension of DMK (primary mathematical objects, processes and their links) but also of the epistemic facet in general; and 3) their grasp of the epistemic facet of didactic-mathematical knowledge would not be sufficient to enable them to manage adequately their future students' learning about the derivative. For reasons of space we are unable here to present one of the tasks to illustrate the type of qualitative analysis carried out. However, examples of the type of analysis, as well as of the cognitive configurations observed in the prospective teachers' answers, can be found in Pino-Fan, et al., (2014).

FINAL REFLECTIONS

This paper has described aspects of a research project carried out over the last three years and whose purpose was to explore the epistemic facet of prospective teachers' didactic-mathematical knowledge (DMK) about the derivative. Originally one of our aims was to examine and foster this kind of knowledge so that prospective teachers would be better equipped to teach the derivative. However, we then encountered the difficulty of how to explore all the different aspects of the mathematical knowledge that is required to teach the derivative. Consequently, we restricted our study to the mathematical dimension and the epistemic facet of DMK among prospective teachers. The results of our research show that the variable and theoretical-methodological tool '*type of cognitive configuration*' activated in the prospective teachers' answers is useful for understanding the kind of didactic-mathematical knowledge they possess. Specifically it helps us understand their mathematical knowledge. This variable was analysed by means of a tool that we refer to as the 'configuration of primary mathematical objects and processes'; one which facilitates the analysis and categorisation of certain features of the epistemic facet of prospective teachers' didactic-mathematical knowledge. Likewise, the dimensions and sub-dimensions proposed by the DMK, as well as the theoretical-methodological tools proposed to operationalise these dimensions, allow an approach to the answers of the questions raised at the beginning. The design of the questionnaire used in this study, as well as the responses of prospective teachers to it, reveal the complex set of mathematical practices, objects and processes that are brought into play when solving tasks related to the derivative. Teachers need to become aware of this complexity during their training so that they will be able to develop and assess the mathematical competence of their future students.

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References

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, *51*, 241-247.

Fennema, E. & Franke, M. L. (1992). Teachers' knowledge and its impact. In A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 147-164). New York: Macmillan.

Godino, J.D. (2009). Categories for analysing the knowledge of mathematics teachers. *Unión, Revista Iberoamericana de Educación Matemática*, 20, 13-31.

Godino, J. D., Batanero, C. & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education*, *39*(1), 127-135.

Hill, H. C., Ball, D. L. & Schlling, S. G. (2008). Unpacking pedagogical content knowledge of students. *Journal for Research in Mathematics Education*, *39*, 372-400.

Johnson, R.B. & Onwuegbuzie, A. (2004). Mixed methods research: a research paradigm whose time has come. *Educational Research*, *33*(7), 14-26.

Llinares, S. & Krainer, K. (2006). Mathematics (student) teachers and teacher educators as learners. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 429-459). Rotterdam: Sense Publishers.

Pino-Fan, L., & Godino, J. D. (2014). Extended perspective of didactic-mathematical knowledge of the teacher. *Manuscript submitted for publication*. Available at http://docente.ulagos.cl/luispino/

Pino-Fan, L., Godino, J.D. & Font, V. (2011). The epistemic facet of mathematical and didactic knowledge about the derivative. *Educação Matematica Pesquisa*, *13*(1), 141-178.

Pino-Fan, L., Godino, J. D. & Font, V. (2013a). Design and implementation of an instrument to explore the epistemic facet of didactic-mathematical knowledge of future teachers on the derivative (First Part). *REVEMAT*, *8*(2), 1 – 49.

Pino-Fan, L., Godino, J.D. & Font, V. (2014). A proposal for the analysis of prospective teachers' mathematical practices on derivatives. *BOLEMA*, (In press).

Pino-Fan, L., Godino, J.D., Font, V. & Castro, W.F. (2012). Key epistemic features of mathematical knowledge for teaching the derivative. In Tso, T.Y. (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 297-304). Taipei, Taiwan: PME.

Ponte, J.P. & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 461-494). Rotterdam: Sense Publishers.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

THE THEORY OF REGISTERS OF SEMIOTIC REPRESENTATION AND THE ONTO-SEMIOTIC APPROACH TO MATHEMATICAL COGNITION AND INSTRUCTION: LINKING LOOKS FOR THE STUDY OF MATHEMATICAL UNDERSTANDING

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In this research report, we describe a comparative study between two theoretical approaches that allowed carrying out cognitive analysis from the subjects' performance: Theory of Register of Semiotic Representation and the Onto-Semiotic Approach of mathematical cognition and instruction. In order to carry out this study, we analysed the performance of a future high school teacher in a task related to the differentiability of the absolute value function. As a result of this study, the complementarities between these two theoretical perspectives, which might allow more complete and detailed analysis of the students' performance, are evidenced.

INTRODUCTION

One of the main concerns of the research community in Mathematics Education is determining which are the difficulties that learners face on their way to understanding, and therefore, learning, mathematical notions. This interest is reflected in the fact that one of the main focuses of research within our scientific discipline has been the features of the learner's cognitive activity. Currently, there are several theoretical positions that allow carrying out cognitive analysis (of students, prospective teachers or teachers) depending on what is desired to observe and which is the concerned mathematical notion (Duval, 2006; Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996; Godino, Batanero & Font, 2007). However, the complex nature of the subjects' learning phenomena has directed research groups to make efforts to revise and find possible complementarities between theoretical and methodological approaches that allow providing more detailed and precise explanations of such learning processes.

In this research report, we present a comparative study between two theoretical approaches, the Theory of Register of Semiotic Representation (TRSR) and the Onto-Semiotic Approach (OSA) of mathematical cognition and instruction, which allows carrying out cognitive analysis from the subjects' performance. In order to carry out this study, and following the proposed methodology for the works within the framework of the networking of theories, we analysed the performance of a future high school teacher in a task related to the differentiability of the absolute value function. As a result of this study, the complementarities between these two theoretical perspectives, which might allow more complete and detailed analysis of the students' performance, are evidenced.

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THEORETICAL FRAMEWORKS

Theory of Register of Semiotic Representation (TRSR)

In the context of cognitive psychology, the notion of representation plays an important role regarding the acquisition and the treatment of an individual's knowledge. As Duval (1995) points out: "There's no knowledge that can be mobilised by an individual without a representation activity" (p. 15).

The comprehension of the theory on registers of representation requires consideration of three key characteristics:

1. There are as many different semiotic representations of the same mathematical object, as semiotic registers utilised in mathematics.
2. Each different semiotic representation of the same mathematical object does not explicitly state the same properties of the object being represented; what is being explicitly stated is the content of the representation.
3. The content of semiotic representations must never be confused with the mathematical objects that these represent.

One of the specificities of semiotic representations consists of its dependence on an organised system of signs such as language, numerical writing, symbolic writing and Cartesian graphs. Consequently, all semiotic representations must be considered, primarily, based on the register where it was produced; then, based on what it explicitly does and what it cannot represent; secondly, based on what it explicitly does and what it cannot represent of the properties of the object of knowledge being analysed; and finally, based on the object itself to which it refers to.

Another of the essential specificities of the semiotic representations is the cognitive operation of conversion of the representations from one system into another; in other words, the transformation of semiotic representations into other semiotic representations. Duval (1995, p. 17) points it out as: "The notion of semiotic representation presupposes the consideration of different semiotic systems and a cognitive operation of conversion of the representations form one system into another". This conversion operation has been considered as a change of form: moving from a verbal statement into an algebraic operation, or draw the curve of a second-degree equation. These examples illustrate the change in the form that knowledge is represented.

It is important to point out that there are two fundamental cognitive activities within the TRSR: treatment and conversion. The activity of treatment consists of a transformation carried out in the same register, in other words, only one register is mobilised. The activity of conversion, on the other hand, consists of the mobilisation from one register into another, where the articulation of representation becomes fundamental. According to Duval (1995), the study of the activity of conversion makes it possible to comprehend the close relation between 'noesis' and 'semiosis', which is essential in intellectual learning. However, it must be taken into account that the operation of conversion brings some difficulties, including the fact that the representation of the source register does not have the same content as the destination register. Another difficulty lies in the treatment, which becomes complex by the use of the register of natural language and those registers that allow 'visualizing' (graphs, geometrical shapes, etc.).

Semiotic systems that allow studying the pairs 'representation, knowledge', must satisfy the three cognitive activities related to representation: 1) Constituting a trace or an assembling of traces that are identifiable as a representation of an object or thing; 2) Transforming representations according to the rules typical to the system in order to obtain other representations that might provide more knowledge to the initial representations; 3) Converting representations produced in a system of representation into another system, so that the latter allow making other meanings explicit to what is being represented. Not all semiotic systems allow these three cognitive activities. Semiotic systems that do allow said cognitive activities are what Duval (1995) calls *registers of semiotic representation*. These registers of semiotic representation constitute the degrees of freedom that a subject has to objectify an idea that is initially confusing, a beating feeling, taking advantage of information, or communicating with an interlocutor.

The Onto-Semiotic Approach of mathematical cognition and instruction

The Onto-Semiotic Approach (OSA) to cognition and mathematical instruction is a theoretical and methodological framework that has been developed since 1994 by Godino and colleagues (Godino, Batanero & Font, 2007; Font, Godino & Gallardo, 2013). The theoretical framework includes an epistemological model about mathematics, on anthropological and sociocultural bases, a cognitive model on semiotic bases from a pragmatic nature, and an instructional model coherent to the others mentioned above. There are then six facets or dimensions that are considered in OSA, for the study of the processes of teaching and learning, in relation to a specific mathematical content (Godino, Batanero & Font, 2007): epistemic, cognitive, affective, interactional, meditational and ecological. The cognitive facet refers to the development of personal meanings (learning of students).

Within the onto-semiotic approach, the notion of 'system of practices' plays an important role for the teaching and learning of mathematics. Godino & Batanero (1994) refer to the system of practices as "any performance or manifestation (linguistic or not) carried out by someone in order to solve mathematical problems, to communicate the solution to others, to validate the solution and to generalise it to other contexts and problems" (p. 334). Font, Godino & Gallardo (2013), point out that mathematical practices can be conceptualised as the combination of an operative practice, through which mathematical texts can be read and produced, and a discursive practice, which allows the reflection on operative practices. These practices can be carried out by one person (system of personal practices) or shared within an institution (system of institutional practices).

Within the OSA, certain pragmatism is adopted since mathematical objects are considered as entities that emerge from the systems of practices carried out in a field of problems (Godino & Batanero, 1994). Font, Godino & Gallardo (2013) put it this way: “Our ontological proposal originates from mathematical practices, and these become the basic context from which individuals gain experience and mathematical objects emerge from. Consequently, the object gains a status originated from the practices that precede it” (p. 104). Ostensive objects (symbols, graphs, etc.) and non-ostensive objects (concepts, propositions, etc.) intervene in mathematical practices, which we evoke while doing mathematics and are represented in a textual, oral, graphic, symbolic and even gestural way. New objects emerge from the systems of operative and discursive mathematical practices and these show their organisation and structure (Godino, Batanero & Font, 2007). If the systems of practices are shared within the core of an institution, then the emerging objects will be considered as ‘institutional objects’, while, on the other hand, if such systems correspond to one person, then these will be considered as ‘personal objects’. The emergence of a personal object is progressive during the history of a subject, as a consequence of experience and learning, while the emergence of an institutional object is progressive over time.

The notion of ‘system of practices’ is useful for a certain type of macro didactic analysis. For a ‘finer’ analysis of mathematical activity, the following typology of primary mathematical objects that intervene in the systems of practices, have been introduced in the OSA: 1) situations-problems (extra-mathematical applications, exercises,…); 2) linguistic elements (terms, expressions, notations, graphs,…) in diverse registers (written, oral, gestural,…); 3) concepts/definitions (introduced through definitions or descriptions: line, point, number, average, function, derivative,…); 4) propositions/properties (statements about concepts,…); 5) procedures (algorithms, operations, calculation techniques,…); and 6) arguments (statements used to validate or explain propositions and procedures, deductive or of another type,…). When an agent performs and evaluates a mathematical practice, a conglomerate formed by situation-problems, languages, concepts, propositions, procedures and arguments, is activated. These primary mathematical objects are connected with each other, forming intervening networks of objects, emerging from the systems of practices, which in OSA are known as configurations. These configurations can be socio-epistemic (networks of institutional objects) or cognitive (networks of personal objects).

METHODOLOGY

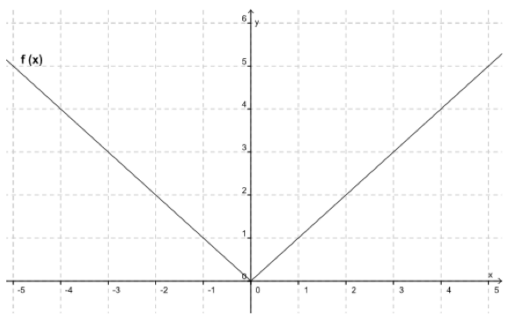
We use the proposed methodology for studies on networking of theories, which suggest for this type of studies, among other things, to select a problem or particular case and analyse this case or problem under the theoretical perspectives involved in the study. In our case, we select a task on the differentiability of the absolute value function and the solution provided by a student of university level to this task. This student, whom we refer to as Juliet, was enrolled in the final modules (eighth semester) of the degree in mathematics teaching offered by the Autonomous University of Yucatan (UADY) in Mexico. She had studied differential calculus in the first semester of their degree course, and had subsequently completed other modules related to mathematical analysis (integral calculus, vector calculus, differential equations, etc.). She had also studied subjects related to the teaching of mathematics. Both the task and the solution provided by Juliet can be found in the study of Pino-Fan (2014). Juliet's answer was chosen intentionally due to its cognitive complexity.

The task and the solution provided by Juliet

This task (Figure 1) has been studied in an investigation on teacher training (Pino-Fan, 2014).

Task

Examine the function $f(x) = |x|$ and its graph.



a) For what values of x is $f(x)$ derivable?

b) If possible, calculate $f'(2)$ and draw a graphic representation of your solution. If not possible, explain why.

c) If possible, calculate $f'(0)$ and draw a graphic representation of your solution. If not possible, explain why.

Figure 3: Task about derivability of the absolute value function

Juliet's solution, translated from Spanish to English language, is presented in Figure 2. The answer of Juliet was analysed from two perspectives (TRSR and OSA). From the point of view of the TSRS, the analysis focused on the identification and description of the semiotic registers of representation mobilised by Juliet, and the study of congruence between the activities of treatments or conversions/passages. From the point of view of the OSA, the mathematical practice of Juliet and the cognitive configuration (linguistic elements, concepts/definitions, properties/propositions, procedures and arguments) that mobilised as part of such a practice, were characterised.

a) In the first instance, the function $f(x) = |x|$ is not differentiable, because the graph has a peak at the point $x = 0$, If it is considered as a function of the type $f(x) = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x < 0 \end{cases}$, the function would be differentiable in the whole domain, in other words, $(-\infty, \infty)$.

b) As mentioned above, the function is not differentiable, therefore $f'(2)$ cannot be calculated and if it was differentiable, its graphic representation would be a point on the Cartesian plane.
Considering this type of function: $f(x) = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x < 0 \end{cases}$, the derivative would be $f'(x) = 1$, and then it would represent a point

c) As mentioned above, if we consider the function $f(x) = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x < 0 \end{cases}$ the derivative would be $f'(x) = 1$ and the graphic representation would be

Figure 4: Juliet's solution to the task

ANALYSIS FROM BOTH PERSPECTIVES

For reasons of space, we present in this research report a summary of analyses carried out from both approaches. The analysis carried out both with TRSR and OSA, show deficiencies in Juliet's mathematical activity, related to the lack of connection of the interpretations and treatments that she makes in the graphic and symbolic representations of the absolute value function. Through the lens of the TRSR it could be observed that the Juliet's answers for the items a), b) and c), show that she knows the definition of the absolute value function, and that she can express it in the symbolic register. Regarding the derivative function, however, she shows deficiencies, because although she answers that if the graphic of the function presents a corner or peak on x=0, then the function is not derivable, in her upcoming arguments some confusions are perceived regarding the domain and little detailed graphic of f '(2). She manages representing for the non-negative values of x, the derivative of f in symbols, but does not represent it graphically, perhaps indicating insufficient knowledge. Although cognitively, she had the symbolic and graphic registers, she does not succeed in the mathematical knowledge of the derivative function, which might be because it appears in an implicit way in the task. In conclusion, a disconnection between the graphic and symbolic registers in which Juliet stands to give her answers is observed. In this sense, we can conclude that Juliette does not carry out a cognitive operation of coordination and articulation between such registers.

With the lens of OSA we observe that Juliet begins her practice based on a visual justification to answer, although wrongly, subtask a), pointing out the existence of a 'peak' at the point of domain of the function x=0. From the beginning of her practice, we can observe that Juliet confuses the non-derivability (local) at a point of domain of the absolute value function with her misconception of non-derivability of the function (global). Later, Juliet writes the symbolic definition, by parts, of the absolute value function. We could say that, in a certain way, such definition is correct, however, she does not make crucial considerations, for example, that the point of domain of the function $x = 0$ belongs to both $f(x) = x$ and $f(x) = -x$. This fact leads her to a cognitive conflict that is shown in her sentence "*If it is considered as a function of the type... the function would be differentiable in the whole domain, in other words*, $(-\infty,\infty)$". This cognitive conflict generated from her visual interpretation of the graph of the function (the function is not derivable since it has a peak in $x = 0$) in contraposition to her interpretation of the symbolic definition, by parts, of the function (she considers that $f(x) = x$ exclusively for $x \geq 0$), is what leads her to give incorrect answers to the other subtasks.

CONCLUSION

The results of the comparison of analysis show that between these two theoretical perspectives there are complementarities that would allow performing more precise and 'finer' cognitive analysis, from the subjects' production. In this way, it is plausible to provide better explanations about the aspects that make it possible or impossible to comprehend mathematical notions. While the analysis from the OSA perspective focused on the subjects' mathematical practices, and mathematical objects, processes and their meanings, that emerge from such practices, the TRSR focused its analysis primarily on the registers of representation that the subject mobilises in his/her productions. In this way, the methodology proposed by TRSR can be considered as more 'global', in the sense that the subjects' cognitive activity is analysed without performing valuations from a mathematical point of view, as it is done with the tools of OSA. So, the OSA provides a level of analysis of the subject's cognitive activity that shows mathematical objects that are involved in the processes of treatment and conversion/passages between registers of semiotic representation. This level of analysis complements the analysis carried out using the tools of TRSR, because with the tools 'configuration of objects and processes' and 'semiotic function', the contents of representations become explicit and are utilised as part of such cognitive activity. It is clear that the registers of representation are implicitly involved in semiotic functions; however, these emphasise the mathematical content of the representation. However, it should be noted that within the OSA there is not systematisation for the analysis of linguistic elements. As a part of the methodology proposed by OSA, language signs – linguistic elements– can be identified, but these different languages could make reference both to register of semiotic representation and semiotic systems. TRSR makes a clear distinction between register of semiotic representation and semiotic system. Thus, the notion of register of semiotic representation of TRSR, complements and enriches the notion of linguistic elements of OSA, by making a very clear distinction between register and semiotic system, and systematising the analysis of such registers.

Finally, these complementarities between the TRSR and OSA show us guidelines for creating a methodology to perform cognitive analysis most 'comprehensive' and 'profound', which is the next step in our research. We are convinced that the relationship between notions of *mathematical objects* (as considered in OSA) and *semiotic representation* (as considered in TRSR), are essential for the analysis and characterisation of mathematical knowledge.

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References

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), Research in collegiate mathematics education II (pp. 1–32). Providence, RI: American Mathematical Society.

Duval, R. (1995). Sémiosis et pensée: registres sémiotiques et apprentissages intellectuels [Semiosis and human thought. Semiotic registers and intellectual learning]. Berne, Switzerland: Peter Lang.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. Educational Studies in Mathematics, 61(1-2), 103-131.

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. Educational Studies in Mathematics, 82(1), 97-124.

Godino, J. D., y Batanero, C. (1994). Significado institucional y personal de los objetos matemáticos [Institutional and personal meanings of mathematical objects]. Recherches en Didactique des Mathématiques, 14(3), 325-355.

Godino, J.D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. ZDM-The International Journal on Mathematics Education, 39(1), pp. 127-135.

Pino-Fan, L. (2014). Evaluación de la faceta epistémica del conocimiento didáctico-matemático de futuros profesores de bachillerato sobre la derivada [Assessment of the epistemic facet of didactic-mathematical knowledge of high school prospective teachers on derivative]. Granada: Universidad de Granada.

BILINGUAL MATHEMATICS TEACHERS AND LEARNERS: THE CHALLENGE OF ALTERNATIVE WORLDS

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In this report we explore language practices when learning and teaching mathematics in a classroom with bilingual learners and a bilingual teacher. We focus on data from the analysis of the whole class discussions in seven lessons. Up to eight episodes distributed in five of the lessons reveal the fragility of switching languages as a reiterated finding, particularly experienced by those learners whose dominant home language is not the language of instruction. Each time a student "dares" to switch languages there are more or less subtle reactions which affect the course and contents of the discussion. We conclude that individual positions and language practices are resources for research in that their study informs of interpretative and action frames that can be mobilised to explain shifts in the direction of mathematical interactions.

INTRODUCTION AND RATIONALE

For many years now, we have been examining (non-) participation trajectories of students in mathematics classrooms in our two geographically and linguistically different research contexts. In Civil and Planas (2004), we reported cases of students with diverse recognised statuses and "disadvantages" on the basis of socioeconomic issues, all of them with home languages different from the language of instruction and histories of recent immigration in their families. We have come to know that none of our respective school systems are properly supporting mathematics learners whose home languages are not dominant in the class. Our situated use of the terms (non-) participation draws on the idea of participation as a shifting collective activity of individuals oriented by identity work (Cobb & Hodge, 2011). Prior results from numerous lessons and interviews raise the value of this approach to participation as an activity that keeps shifting (Civil & Planas, 2012; Planas & Civil, 2013).

In this report we address individual positions and language practices as two of the interconnected reasons why students whose home language is not official in their school system may resist mathematical participation and challenge monolingual practices. The notions of positions (Wagner & Herbel-Eisenmann, 2009) and practices (Setati, 1998) put the emphasis on different aspects of mathematics teaching and learning (i.e., positions are more oriented to identity work, while practices are more oriented to action). Nevertheless, these two notions are resources for research in that their study leads to interpretative frames that can be mobilized to explain situations of mathematical (non-)participation in the classroom as well as shifts in the direction of mathematical interactions. To illustrate how we work with the notions of individual positions and language practices in the analysis of the bilingual mathematics classroom, we take an episode where a bilingual student code-switches in the middle

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of a mathematical explanation to the whole group and the bilingual teacher translates the switched term into the language of instruction.

FIGURED AND PARALLEL WORLDS

We claim that a double focus on positions by the learners and the teacher and language practices is necessary for understanding the way mathematics participation is performed and may shift in particular classroom interactions. Along with the study of how positions and practices are actually experienced by some participants, we also claim the relevance of examining how they are alternatively experienced, intended or imagined by some others. This view is informed by the constructs of "figured worlds" (Holland, Lachicotte, Skinner, & Cain, 1998) and "parallel worlds" (Jorgensen, Gates, & Roper, 2013). They are complementary constructs that overlap in many senses but come from the elaboration on different types of data.

The anthropological construct of "figured worlds" was introduced by Holland et al. (1998), on the basis of individual narrative accounts, to refer to sites where (future) identities are produced in ways that are not completely determined by the social and cultural dominant conditions. People modify and develop identities in figured worlds under the strong influence of their individual positions and the movements in the relationships with other people who are participants in those same worlds. It may happen that some of the practices coming up point to alternative forms of relationships and practices with little significance in other simultaneous intersecting worlds.

Similarly to Holland et al. (1998) but contextualised into mathematics education and on the basis of lesson accounts, Jorgensen et al. (2013) take a sociological perspective to frame conditions for participation and success in mathematics classrooms with working-class and culturally diverse students. These authors examine structural circumstances with an influence on the social practices taking place within the micro context of the classroom, some of them working to marginalize certain students while preserving the participation of others. What interests us from that work is the idea of the students and the teacher creating "parallel worlds which are structured quite differently inside and outside the classroom" (p. 221). Thus it makes sense to imagine forms of resistance in the mathematics classroom that challenge the status quo in ways that would not be possible or predictable in other contexts. This is a perspective that leaves room for identity work among participants toward the continuous creation and implementation of new forms of mathematical participation.

For students who are marginalized due to issues of status and home languages, participation in the mathematics classroom needs additional effort in comparison to other students whose home language is the language of instruction and whose social group represents or is closer to hegemonic knowledge. The metaphor of parallel worlds, which geometrically suggests worlds that never get in touch, is indeed an exaggeration as all contexts of social participation are connected. This metaphor, however, is useful in that it focuses on the development of different positions and practices in different scenarios. Also, it enables us to view classrooms as unique organizational alternatives where participants reconstruct (future) conditions from other worlds by means of particular positions and new improvised practices.

Various authors have examined innovative language practices in the multilingual mathematics classroom from the perspective of their contribution to the increase of mathematical participation. Code-switching, for example, has been documented as a tool for the benefit of Latina/o bilingual students in their learning of mathematics (Moschkovich, 2007). More generally, in the work with Latina/o bilingual students in the US, Gutiérrez (2002) enumerates a number of successful pedagogical practices like group work and relates them to the use of the students' home language. Gutiérrez states that practices of switching languages help to develop alternative worlds in the classroom by encouraging the mathematical participation of students from socially underrepresented group whose home language is not the language of instruction. This author, however, anticipates the issue of tensions and contradictions between counter-hegemonic bilingual practices and mainstream monolingual discourses.

Tensions coming from the dominance of monolingual practices and the needs for the learning of mathematics, for which the use of and the competence in the home language and the language of instruction are instrumental, point to the many challenges of teaching and learning mathematics in the bilingual classroom.

METHODS FOR INTERPRETING DATA

For the last fifteen years, we have conducted small-scale qualitative work in multilingual classrooms with lesson observations as well as individual and group interviews. We have focused on the obstacles and affordances to the participation in whole class discussion for learners whose home language is different from the language of instruction. Our analytical approach concentrates on the significance and impact of particular mathematical interactions from the perspective of how the participants involved seem to be interpreting these interactions, either individually or collectively, and with a critical eye on the role and use of the students' languages.

We argue that it is critical that we look at whole class interactions guided by the bilingual teacher, where the course of student mathematical participation tends to be dominated by those whose home language is the language of instruction. For this purpose, we have been examining moments of whole class interaction by means of episodes. To conform what we call an episode, we group consecutive turns of a lesson in which more than one language is used or referred to with respect to the resolution of a mathematical task. This procedure leads to episodes of various sizes, sometimes with several turns in between those for which we have detected explicit uses of more than one language. Moreover, we have often conducted interviews with teachers and students afterwards, to hear their interpretations and explanations of concrete events in a lesson (what happened in that part of the lesson, who did what, how and why); this method has allowed us to match diverse interpretations for one episode (as multiple positions, relationships and practices can be outlined) and from there to elaborate explanations that are consistent with what is prioritised in the analysis.

In an episode of a given lesson, for the identification of individual positions with respect to a language practice, we search for turns that introduce the use of or reference to that practice and the related positions by the teacher and by at least a learner whose home language is not the language of instruction. The episode becomes somehow the context of mathematical activity in which positions toward public language practices are manifested. In order to consider the emergence of parallel/figured worlds, we explore distinct and apparently opposite meanings between what is done through the actual practices in the episode and what is intended by means of other suggested or explicitly recommended practices. We refer to a pair of opposite language practices (with some distinct opposite meanings) when one of them promotes the use and value of the students' languages while the other promotes the exclusive use and value of the language of instruction. Consequently, we concentrate our study on classroom practices involving references to or uses of the students' languages during instances of whole class mathematical interaction. Although the episode is the primary source for the analysis, interviews are important in that they help to confirm some interpretations and may reveal insights which the videos and transcripts of the lesson could not show.

In what follows we focus on one mathematics classroom that the first author observed for seven lessons in Barcelona, Catalonia-Spain. The students were in their first year of secondary school (12 years old). This author also conducted audio-taped interviews with the teacher and some of the learners. All students and the teacher were bilingual because they could speak Spanish and Catalan, though they were not equally fluent in their two languages, particularly those from Latin American families whose parents did not speak Catalan. The teacher, with ten years of teaching experience, was dominant speaker of Catalan, the language of instruction in that part of the country.

POSITIONS AND PRACTICES AROUND CODE-SWITCHING

In the selected example, a tension comes from the experience of opposite practices around code-switching during the resolution of a task. The task in the lesson asks for rectangles with equal perimeter and area, without any mentioning of magnitudes or units of measure. The students had been working in small groups for about thirty minutes before the whole class discussion started. In two of the groups, respectively with two and three students of Latin American origin out of four, the conversation took place with frequent instances of code-switching between Spanish and Catalan, but all groups used only Catalan to produce their written reports of the task. At the beginning of whole class discussion, a student took the initiative to explain that, "squares whose sides measure four solve the problem because four times four is four squared". Amanda, a girl from Argentina with Spanish as her home language, reacted to it by sharing what had been mathematically discussed in her group when the same case had been considered. This is the starting point of the episode below.

00 Amanda: No pot ser perquè el perímetre és el *contorno*. No n'hi ha. [It's not possible because the perimeter is the boundary. There aren't.]

01 Teacher: Vols dir que és el *contorn*? [Do you mean it's the boundary?] *Contorn*, sí? [Boundary, eh?] I doncs? Per què? [So what? Why?]

02	Amanda:	Això… *contorn*. [This… boundary.]
03	Teacher:	D'acord. Què havies dit? [Okay. What did you say?]
04	Amanda:	(Silence)
05	Students:	El perímetre del rectangle... [The perimeter of the rectangle.]
06	Teacher:	[Looking at Amanda] Volies dir una cosa important! Endavant! [You wanted to say something important! Go ahead!]
07	Amanda:	Volia dir *contorn*. [I meant boundary.]
08	Teacher:	Sí, però dius que no hi ha solucions? [Yes, but you say that there are not solutions?]
09	Amanda:	Perquè l'àrea és una altra dimensió i no poden ser iguals. [Because the area is another dimension and they cannot be the same.]
10	Teacher:	Què vols dir amb una altra dimensió? [What do you mean by another dimension?]
11	Amanda:	Les àrees volen dir dues dimensions i els perímetres volen dir una dimensió. No es poden igualar els nombres sense tenir en compte això. [Areas stand for two dimensions and perimeters stand for one dimension. You cannot equal the numbers without taking this into account]

Amanda code-switches to Spanish in the first turn of the episode, and the teacher quickly translates the Spanish term for boundary, *contorno*, into Catalan, *contorn*, in the second turn. Italics, in this case, are used for the switched term, in its two linguistic forms. This situation does not necessarily point to resistance toward the language of instruction or to any other intentional strategy of contestation on the side of the student. It can be inferred that she is behaving as a bilingual person who uses her two languages, with different levels of intensity, as resources for communication and participation with other bilinguals. In this episode, Amanda's use of Spanish follows from her group work in which she has been alternating Catalan and Spanish to elaborate on the idea of the task not being solvable due to differences in the measured magnitudes. From the video of the lesson and field notes, it is clear how she positions herself as bilingual in the small group with two other peers from Colombia who are Spanish dominant and one Catalan dominant speaker. Her position toward the use of her two languages seems less strong in whole class discussion with the presence of all students and the teacher, but it is still there with the introduction of the term *contorno*.

The teacher reacts to Amanda's practice of code-switching with a practice of literal translation. Literal translation can be interpreted as either removing an error or revoicing a term, with different implications for the development of participation. We regard code-switching and literal translation as opposite practices in that they consider differently the value and the use of the students' languages in the course of the mathematical discussion. Although neither the student nor the teacher may be intending to impose, respectively, bilingual and monolingual practices, and they may not be consciously competing for specific language use, a tension in the direction of the discussion comes across when its emphasis on the mathematics is interrupted by an emphasis on the language. This shift in the direction of the discussion suggests tensions between the actual world of a classroom in a school system with a preferred language (i.e., what needs to be done with the language of instruction), and a figured alternative world with other languages in use (i.e., what can be done with the students' languages and why it is (not) done).

From our data in this classroom we could have chosen a different example of an episode with a similar tension resulting in an interruption in the mathematical participation by the learner who code switches. We have decided, instead, to select an example where the fragility of the alternative figured world is expressed in a more subtle way because, in the end and despite a brief silence, the student in question can finish her explanation. In the example, the teacher goes back to Amanda, reestablishes her participation and, by doing it, he contributes to letting her create a relevant learning opportunity around the inaccurate comparison of two numbers representing measures of different magnitudes. The situated effect of literal translation may not be mathematically severe for Amanda in the short term of this episode, but it can become severe for other students (or even for Amanda if it is a recurring practice) who may choose not to participate if they do not know or remember a word in the language of instruction. In the construction of figured present and future actions, the teacher's reaction to code-switching informs about what language use is preferred in this class.

Up to eight episodes distributed in five of the seven lessons reveal the fragility of switching languages as a finding of the analysis of whole class interaction in that classroom. A similar pattern takes place when a learner switches to Spanish for a word or sentence, and the teacher translates it to the language of instruction (six episodes including the example with Amanda) or asks other students if they know how to say it in Catalan (two episodes). Eight episodes are not representative of a frequent phenomenon but we cannot say that such fragility is rare. After the sixth lesson, in an interview the teacher was asked to talk about learners switching languages in his class. The episode with Amanda, which had happened that morning, served to initiate the conversation on this topic:

Author 1:	(...) Què ha passat amb la paraula per contorn? [What happened with the word for boundary?]
Teacher:	No res. Potser l'havia d'aprendre. [It's nothing. Maybe she had to learn it.]
Author 1:	El que deia, s'entenia. [What she was saying, it was understandable.]
Teacher:	Sí, esclar, s'entenia bé. [Yes, of course, it was well understandable.]
Author 1:	Llavors? [So?]
Teacher:	Expressar-se correctament és important. [To express oneself correctly is important.]

The teacher points to a normative lens when interpreting code-switching in the course of mathematical explanations: students have to learn to express themselves correctly and the teacher has a role in that. We do not question that the teacher used literal translation to help Amanda, and indeed we assume that his idea of students' expressing themselves correctly is based on a mathematically strong notion of correctness and far more complex than only considering language choice. What we miss, however, is a clear understanding of what he means by "correctly" in a bilingual context of mathematics teaching and learning, whether he would see it differently with monolingual students, and how tightly in his teaching activity he might be relating the use of the language of instruction to the recognition of mathematical correctness.

FINAL REMARKS AND FUTURE RESEARCH

Truxaw and Rojas (2014) stress that mathematical participation of learners who are not dominant in the language of instruction requires a position where one "dares to do it" (p. 26), that is, where one exercises sufficient power to make decisions about when and how to participate in the classroom discussion even if this implies that others are confronted with the use of a non official language. For the creation of worlds in the bilingual classroom that benefit all mathematics learners independently of their dominant language, positions like that of Amanda "daring" to publicly use her two languages are crucial, but also collective practices that reinforce such positions are necessary. In the example, Amanda draws on the language capital that serves her in other worlds of experience, but the creation of a bilingual world in the classroom is resisted in the ways that code-switching is contested. This is not a mere result of the language practices by the bilingual teacher and his modelling of monolingualism within the local context of the classroom, but something that requires the examination of the diversity of intersecting worlds (see Barwell, 2012, for a discussion on forces in competition from diverse worlds entering the multilingual mathematics classroom).

In this report we have explored the experience of a bilingual mathematics learner and a bilingual mathematics teacher in a classroom with an official language of teaching and learning. On the one hand, the strength of individual positions has offered an explanation for the introduction of bilingual practices and for the fragility of the suggested alternative worlds; on the other, the phenomenon of fragility around switching languages has been discussed from the perspective of access to recognized mathematical correctness. These are important findings for further exploration in the area of mathematics education research and language diversity with a focus on bilingual classrooms, bilingual teachers and bilingual learners. While the complexities and challenges faced by multilingual students in their mathematics learning have been increasingly studied in several countries, not many studies around bilingual students who are taught mathematics by bilingual teachers have emerged yet. This is a gap that needs to be addressed by researchers in this area of study.

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References

Barwell, R. (2012). Heteroglossia in multilingual mathematics classrooms: Gender, culture and diversity. In H. Forgasz & F. Rivera (Eds.), *Towards equity in mathematics education* (pp. 315-332). New York: Springer.

Cobb, P., & Hodge, L. L. (2011). Culture, identity, and equity in the mathematics classroom. In E. Yackel, K. Gravemeijer & A. Sfard (Eds.), *A journey in mathematics education research* (pp. 179-195). New York: Springer.

Civil, M., & Planas, N. (2004). Participation in the mathematics classroom: does every student have a voice? *For the Learning of Mathematics, 24*(1), 7-12.

Civil, M., & Planas, N. (2012). Whose language is it? Reflections on mathematics education and language diversity from two contexts. In S. Mukhopadhyay & W.-H. Roth (Eds.), *Alternative forms of knowing (in) mathematics: celebrations of diversity of mathematical practices* (pp. 71-89). Rotterdam, The Netherlands: Sense Publishers.

Gutiérrez, R. (2002). Beyond essentialism: the complexity of language in teaching mathematics to Latina/o students. *American Educational Research Journal*, *39*, 1047-1088.

Holland, D., Lachicotte, W., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Cambridge, MS: Harvard University Press

Jorgensen, R., Gates, P., & Roper, V. (2013). Structural exclusion through school mathematics: using Bourdieu to understand mathematics as a social practice. *Educational Studies in Mathematics, 87*, 1-19.

Moschkovich, J. N. (2007). Using two languages when learning mathematics. *Educational Studies in Mathematics, 64*, 121-144.

Planas, N., & Civil, M. (2013). Language-as-resource and language-as-political: tensions in the bilingual mathematics classroom. *Mathematics Education Research Journal, 25*(3), 367-381.

Setati, M. (1998). Code-switching in a senior primary class of second-language mathematics learners. *For the Learning of Mathematics, 18*(1), 34-40.

Truxaw, M. P., & Rojas, E. D. (2014). Challenges and affordances of learning mathematics in a second language. *Journal of Urban Mathematics Education, 7*(2), 21-30.

Wagner, D., & Herbel-Eisenmann, B. (2009). Re-mythologizing mathematics through attention to classroom positioning. *Educational Studies in Mathematics, 72*, 1-15.

SOLVING PROBLEMS AND MATHEMATICAL ACTIVITY THROUGH GIBSON'S CONCEPT OF AFFORDANCES

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In this paper I explore James Gibson's notion of affordance, a concept that is being increasingly studied in mathematics education research. In my explorations I intend to delve into epistemological aspects related to this concept and how these enable fruitful conceptualisations of mathematical problem-solving. In order to emphasise this epistemological exploration of affordances, I ground my study in the enactivist theory of cognition and supplement it with data extracts to illustrate what thinking in terms of affordances (at the epistemological level) can offer.

Recently, a number of mathematics education researchers have developed an interest in James Gibson's (e.g. 1986) concept of affordance as a way of discussing learning environments and how students interact with these, particularly in PME/PME-NA conferences (see e.g. Gresalfi, 2013; Brown, 2014). Most researchers make use of affordances by relating them to tasks solved or to the technological environments into which students are plunged (see e.g. WG-15 at CERME-8). However, something seems missing. The epistemological grounding of the notion of affordance, that is, regarding the generation of knowledge and the nature of the mathematical activity, has not been explored in depth in mathematics education research. And as it has often been argued in our community, delving into the epistemological roots of a concept or a theory can offer significant insights and distinctions for better understanding mathematical teaching-learning processes and enrich our research endeavours. In sum, we could gain from exploring Gibson's notion of affordance at an epistemological level, for example in order to strengthen our understanding of students' mathematical activity when solving problems (and our analysis of it).

This paper is partly theoretical, offering a perspective for conceptualizing students' mathematical solving processes, and partly practical, using data from my own studies to illustrate and make sense of the points highlighted. Hence, in this paper, I develop and push deeper the concept of affordance at an epistemological level, in order to enrich understandings of students' mathematical activity. I say "push deeper" because I do not use Gibson's concept of affordance as a rigid thing, a fixed once-and-for-all concept that is immutable and cannot grow. Rather, I use it as a springboard for exploring where it can lead, for delving into epistemological possibilities and dimensions for understanding students' mathematical activity when solving problems. To do so, I first discuss Gibson's concept of affordance and outline its meaning and possible outcomes for understanding students' solving processes. I then relate it to aspects of the enactivist theory of cognition (inspired e.g. by Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991; see Research Forum05 at PME-33) to ground theoretically the understandings put forth. Then I illustrate what an analysis in these terms, that is, of affordances taken at an epistemological level and related to enactivist thought, might mean by looking at data taken from a mental mathematics study on operations on functions (building on last year's PME-38 paper in Proulx, 2014). I then draw conclusions on its potential for understanding students' solving processes.

GIBSON'S CONCEPT OF AFFORDANCES

In his work, Gibson positioned himself strongly against dualism, that is, the separation of the environment and the organism, and argued for their inseparability. For this, he developed the discipline of ecological psychology, recognizing the co-evolution of animals and their environment, thus emphasising animal-environment reciprocity. This mutual reciprocity, complementarity, and inseparability of animal and environment became important in his work to the point where he argued for the presence of one in the other: "[…] to perceive the world is to coperceive oneself" (1986, p. 141); "Information about the self accompanies information about the environment, and the two are inseparable" (1986, p. 126). It is thus in this context that Gibson developed, and coined, the concept of affordances:

The affordances of the environment are what it offers the animal, what it provides or furnishes […] I mean by it something that refers to both the environment and the animal in a way that no existing term does. It implies the complementarity of the animal and the environment […]. If a terrestrial surface is nearly horizontal (instead of slanted), nearly flat (instead of convex or concave), and sufficiently extended (relative to the size of the animal) and if its substance is rigid (relative to the weight of the animal), then the surface affords support […]. Note that the four properties listed – horizontal, flat, extended, and rigid – would be physical properties of a surface if they were measured with the scales and standard units used in physics. *As an affordance of support for a species of animal, however, they have to be measured relative to the animal. They are unique for that animal. They are not just abstract physical properties*. (1986, p. 127, emphasis added)

Thus, an affordance could be said to be about *interaction*. It is in the interaction that the properties of objects arise and become the properties of objects, when the observer interacts with it. It is in interaction *with it* that a wall is rigid for humans, that air affords flying for birds, that a task affords algebra for a student or a teacher. Let me build on these examples. A sheet of paper or a tree leaf is rigid for an ant and affords solidity for walking on it. For a human like myself, it is malleable and can even be crushed if I walk on it. So is a tree leaf rigid in itself? Yes, in interaction with the ant. Is it malleable? Yes, in interaction with humans. Thus, rigidity and malleability are not properties of the object in an absolute manner, but only in relation, in interaction, with the ant or a human or any other species. The qualities of objects, their properties, emerge in the interaction with them (whether by physically interacting with them or by simply making sense of them, as I discuss below).

The concept of affordances that Gibson developed challenges traditional realist ontologies. For Gibson, properties and "truth" do not lie in the objects themselves, as realists would assert (see e.g. Vacher, 1998): the key to nature is not in nature. However it *is*, because these properties become properties through the interaction of an organism with the environment. Thus it does not offer a solipsistic view, for which constructivists have often been criticized, and neither does it offer a representationalist view, for which realists have also been criticized.

But, actually, an affordance is neither an objective property nor a subjective property; or it is both if you like. *An affordance cuts across the dichotomy of subjective-objective and helps us to understand its inadequacy*. It is equally a fact of the environment and a fact of behaviour. It is both physical and psychical, yet neither. An affordance points both ways, to the environment and to the observer. (Gibson, 1986, p. 129, my emphasis)

This perspective offers a particular way of conceptualising what a problem is in mathematics. For example, in "solve $6/x = 3/5$", Gibson's affordance theory suggests that there is no "algebra", "proportionality", or anything else inherent in the task because these properties arise in the interaction between the solver and the task. This is not to say that these properties are not "there" and appear suddenly "out of the blue", created from scratch. It is mainly that they arise in the interaction between the solver and task: these properties emerge from this interaction.

To better ground theoretically this epistemological view of affordances, based on interaction, I refer to aspects of the enactivist theory of cognition, which is especially concerned in mathematics education with issues of emergence and contingency of learners' mathematical activity in *interaction* with their environment. In particular, I focus on the distinction made between problem-posing and problem-solving to offer ways to deepen this conceptualisation of affordances at the epistemological level.

BRIDGING ASPECTS OF THE ENACTIVIST THEORY OF COGNITION

For Varela (1996), problem-solving implies that problems are already in the world, independent of us, waiting to be solved. Varela explains, on the contrary, that we pose them, specifying the problems we encounter through the meanings we make of the world in which we live. We do not choose problems that are out there in the world independent of our actions. Rather, we bring problems forth: "The most important ability of all living cognition is precisely, to a large extent, to pose the relevant questions that emerge at each moment of our life. They are not predefined but enacted, we bring them forth against a background." (p. 91). The problems that we encounter, the questions that we ask, are as much a part of us as they are a part of our environment: they emerge from our interaction with it.

In this perspective one cannot assume, as René de Cotret (1999) explains, that properties are present in the tasks and that these *causally determine* solvers' reactions. Even if each prompt is designed following specific intentions, which can play a role in how solvers pose problems (e.g. one often does not react to a square-root function as one does to a linear function), properties of the task, its affordances, emerge in the interaction of the solver with the task; they are affordances of this task for *this* solver. In that sense, following Simmt (2000), it is not tasks that are given to students, but mainly prompts that are taken up by students who themselves create tasks with. Prompts become tasks when students engage with them, when, as Varela would say, they pose them as problems. Students *make* the “wording” or the “prompt” a multiplication task, a ratio task, a function task, an algebra task, and so forth. It is in this sense that each prompt can be seen to have affordances that emerge in the interaction between solver and tasks.

For Maturana and Varela (1992), these affordances play the role of *triggers* in relation to the solver’s posing. Hence reactions to a prompt do not reside in either the solver or the prompt: they emerge from the solver’s interaction with the prompt, through posing it as a task. Strategies are triggered by the prompt’s affordances (for that solver), where issues explored in a prompt are those that resonate with the solver. This being said, the notion of emergence does not assert that strategies are new in the sense that they have never been encountered, but mostly that these strategies are generated for solving the task posed, created for or gave birth to, as Hannah Arendt would say, at the meeting of solver and prompt. Thus, the task posed is as much representative of the solver as it is of the prompt itself. The notion of affordances offers a way to make sense of solving processes, stressing the fundamental issue of interaction to analyse the emergence of students’ mathematical activity.

ANALYSING STRATEGIES THROUGH AFFORDANCES

Here, I re-visit the data presented in last year’s paper-38 (Proulx, 2014), but now through the angle of affordances and exploring what this can offer for data analysis through an epistemological lens and what meanings can be developed from it. The study focused on the nature of the mathematical activity that students brought forth when working on mental mathematics tasks, here on operations on functions in a graphical environment. This study took place in two Grade 11 classrooms where students had to operate mentally on functions in a graphical environment, that is, solving without paper-and-pencil or any other computational/material aids. E.g. a typical prompt consisted of showing two functions (without their algebraic expression) in the same graph and asking students to add or subtract them (see Fig.1).

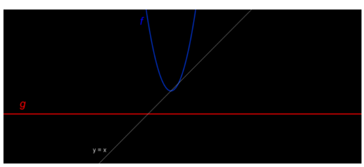

Figure 1: Example of a graphical prompt on operations on functions [$f(x) \pm g(x)$]

I offer here examples of three strategies put forth by students to solve the task and focus on using the concept of affordances in order to give meaning to these strategies (for a more complete analysis of the data, see last year’s paper in Proulx, 2014.)

Strategy 1. Algebraic/Parametric

Explanation of the strategy. Many students engaged in algebra to solve the prompts, even if these were proposed in a graphical context with no algebraic expressions. Students brought forth parameters from the algebraic expression (the ‘a’ and ‘b’ of the linear function $f(x)$=ax + b) to make sense of the graphs and add them. E.g. in the addition prompt (see Fig.2), many students explained that “both functions looked symmetrical, so the ‘a’ parameter of each line would cancel out, as well as the ‘b’ and thus give x=0”.

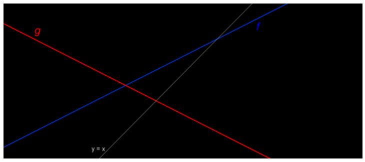

Figure 2: Addition of function graphical prompt

> In prompts where e.g. a linear function *f* would be added to a constant function, students would say that the “a” parameter of the function *f* does not change when added with a constant function that “does not have an ‘a’ parameter, so the function’s steepness stays the same and only the ‘b’ changes” giving a function parallel to *f* with a *y*-intercept at “b” instead of at 0.

Discussion in terms of affordances. Making sense of this strategy in terms of affordances, one can say that these students were able to draw out an algebraic context from the prompt, to pose the prompt as an algebraic task and develop a strategy to solve with/in that context. Literally, the algebra is not there in the prompt: it arises in the interaction of the solver with the prompt. The prompt afforded algebra *for these students*. Students made emerge the ‘a’ and ‘b’ parameters, posing the task in these terms and solving it in these same terms. But the task is not algebraic *per se*, even if an *a priori* analysis could state that this might be a possibility for solving it. Re-using Gibson’s above quotation, one can say that “as an affordance of [algebra] for [these students], they have to be measured relative to the [students]. They are unique for [these students]. They are not just abstract [attributes of the prompt].”

Strategy 2. Graphical/Geometric

Explanation of the strategy. When facing a function that was not linear (e.g. quadratic, square root, rational, hyperbolic), students generated particular ways of working with slope and parallelism. They assigned a constantly changing rate of change/slope to some nonlinear functions with which they were dealing (students used the expressions *slope* and *rate of change* interchangeably, hence the “/”). E.g. with the addition of a quadratic and a constant function (see Fig.1), students explained that the rate of change of the quadratic function was not affected by the addition of a constant function, because a constant function “did not have a variation” and thus the slope of the quadratic function: “will continue to vary in a constant way”. Thus the resulting function of their addition would have the “same rate of change as the quadratic function" but would be "translated down" in the graph since the constant function was "negative". For another prompt, one student said that for the square-root function "its rate of change is left untouched when i add the constant function, since it has no variation". Also, in cases where students faced more than one nonlinear function, some students began analysing functions in terms of "parallelism". E.g. in Figure 3 where the function *g* is to be found, some students expressed that "each function was parallel to the other" and that *g* had to be a constant function "for the curve to be translated down" and that it was "negative for bringing the curve lower".

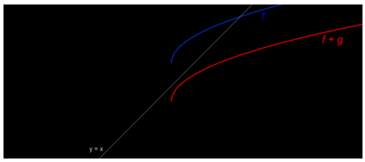

Figure 3: A prompt for which the parallelism strategy was used

Discussion in terms of affordances. Students brought forth a geometrical view of slope as a property not of the function, but of the curve present on the graph, talking about a geometric rate of change/slope (reminiscent of Zaslavsky, Hagit & Leron, 2002). By posing the prompt in geometrical terms, they generated geometric affordances and developed graphical/geometric strategies for solving it. One can then say that these prompts afforded geometry *for these students*. Through their geometrical slope, students brought forth the nonlinearity of nonlinear functions and developed ways of engaging with it. This geometrical path arose in the interaction of solver and the prompt, not as a property of the prompt, but as something that arose when interacting with it. Thus the same can be said as for the algebraic/parametric strategy, that is, that the geometry has to be measured, its presence has to be assessed relative to these students, because it is unique for these students. In fact, these geometric posings of the task are indeed unique, as they can even be mathematically questioned (e.g. the notion of parallelism of curves is a matter that is not settled in mathematics). By posing the task in geometrical terms, students made emerge affordances of the prompt for them and solved in relation to these affordances.

Strategy 3. Graphical/Numerical

Explanation of the strategy. For solving, some students referred to points in the graphs of functions (related to Even's, 1998, pointwise approach). Through those points, they generated exact and approximate answers (Kahane, 2003), which they combined to find the resulting function. E.g. in Figure 4 students had to find the function that resulted from the addition of *f* and *g*. In this case, they would bring forth specific points: (1) where *f* crosses the *x*-axis (*x*-intercept), giving an exact calculation, as the addition of the image for *f* (which is of length 0) with that for *g* results in an image for *f*+*g* that is the same as that for *g* (it has the same image for *g* to which 0 was added); (2) where both *f* and *g* intersect, giving an approximate calculation, as both images at *f* and *g* are the same, so the resulting image is double the value of the intersection point; but a precise location is impossible without knowing the exact location of the intersecting point in terms of precise length; (3) where *f* and *g* cross the *y*-axis (*y*-intercept), giving the same as in case (2); (4) where *g* cross the *x*-axis (*x*-intercept), giving the same as in case (1). In doing this, students mingled both exact and approximate answers to find points for the resulting function.

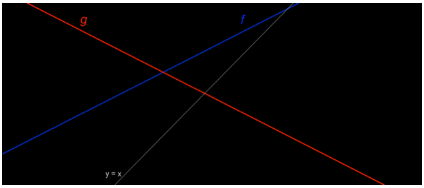

Figure 4: An addition of function prompt for which points were outlined

Discussion in terms of affordances. Students generated precise and approximate points to determine the resulting function. They were no longer in an algebraic context, but in a blend of numerical and graphical contexts, generating numbers/coordinates that had meaning for them in the graph. Using Gibson's affordances, these coordinates arose in the interaction of students and prompt, where this prompt afforded a numerical/pointwise interpretation *for these students*. Their numerical/pointwise posing made the prompt about points, illustrating how the prompt afforded points *for these students*, as they developed a strategy for solving it.

DISCUSSION OF FINDINGS AND FINAL REMARKS

These various entries into the same sort of prompts show how students posed their problems, making emerge affordances of the problems when interacting with them, that is, algebraic, parametric, geometric, point-wise, and so forth. Thus an algebraic posing of the functions produced an algebraic strategy; a graphical posing produced a graphical strategy; a numerical/pointwise posing produced a numerical/pointwise strategy. These affordances are to be seen relative to students and the prompts, as affordances *for these students* interacting with these prompts: they are not properties of the prompt in themselves, but are brought forth in the interaction with it when posing it as such. In this sense, the affordances (hence the task posed) are not independent of who the poser/solver is. Gibson's affordances underline that tasks do not possess attributes in themselves, but mainly in relation to a solver who interacts with it. Through the generation of affordances, students illustrate how they posed the task (it became a task about these affordances) and how they solved it (concerning these specific affordances). It can thus be seen as a double-emergent phenomenon, from the posing to the solving. By posing the task, students generate a context in which to solve it: here an algebra context, a graphical context, a point-by-point context. The task for them then becomes about this.

This being said, the concept of affordances can lead us much farther in this epistemological quest, because I have accounted for only one side of the story: that of the prompts. Because affordances and solver go hand in hand, affordances are said to be relative to the solver. But also, the solver can be said to be relative to the affordances themselves. If there are no affordances outside of the interaction with the solver, there can be no solver outside of the interaction with the affordances. This raises questions. Could this mean that the solver affords the task in return? That students emerge as solvers when they pose the task and make affordances; that is, that affordances make students emerge as solvers? Considering this would request stepping out of the subjectivity/objectivity duality, as Gibson emphasized. It is entering a terrain where subject and object co-define and mutually specify each other; an epistemological terrain where enactivism has offered promises for understanding students' mathematical meaning making (again see Research Forum05 at PME-33); it is a terrain worth exploring for mathematics education research.

References

Brown, J.P. (2014). Student strategies in enacting affordances. *Proceedings of joint PME-38 and PME-NA 36* (vol. 2, pp. 201-208). Vancouver, Canada: PME.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior*, *17*(1), 105-121.

Gibson, J.J. (1986). The ecologic*al approach to visual perception*. Hillsdale, NJ: Erlbaum.

Gresalfi, M.S. (2013). Technology in mathematics education: a discussion of affordances. *Proceedings of PME-NA 35* (pp. 16-28). Chicago, Illinois.

Kahane, J.-P. (2003). *Commission de réflexion sur l'enseignement des mathématiques — rapport d'étape sur le calcul*. Paris, Centre national de documentation pédagogique.

Maturana, H.R., & Varela, F.J. (1992). *The tree of knowledge*. Boston, MA: Shambhala.

Proulx, J., Simmt, E., & Towers, J. (2009). RF05: The enactivist theory of cognition and mathematics education research: Issues of the past, current questions and future directions. *Proceedings of PME-33* (vol. 1, pp. 249-278). Thessaloniki, Greece: PME.

Proulx, J. (2014). Mental mathematics and operations on functions. *Proceedings of joint PME-38 and PME-NA 36* (vol. 5, pp. 17-24).Vancouver, Canada: PME.

René de Cotret, S. (1999). Perspective bio-cognitive pour l'étude des relations didactiques. In *Le cognitif en didactique des mathématiques* (pp. 103-120). Montreal, Qc: PUM.

Simmt, E. (2000). *Mathematics knowing in action: A fully embodied interpretation*. PhD dissertation. University of Alberta, Edmonton, Canada.

Vacher, L-M. (1998). *La passion du réel*. Liber: Montreal.

Varela, F.J. (1996). *Invitation aux sciences cognitives*. Paris: Éditions du Seuil.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind*. MIT Press: Cambridge.

Zaslavsky, O., Hagit, S., & Leron, U. (2002). Being sloppy about slope: The effect of changing the scale. *Ed. Studies in Mathematics*, *49*(1), 119-140.

COGNITIVE STYLE, SPATIAL VISUALISATION AND PROBLEM SOLVING PERFORMANCE: PERSPECTIVES FROM GRADE 6 STUDENTS

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This study investigated the ways in which 807 Grade 6 students' encoded and processed mathematics information. Specifically, the study examined the relationship between the students' cognitive style, spatial visualisation and mathematics performance. Results revealed that students who employed spatial imagery information processing, in contrast to object imagery or verbal processing, tended to have higher spatial visualisation ability and higher mathematics problem-solving performance. In subsequent analysis, the instrument used to measure the cognitive style constructs was tested for content validity.

BACKGROUND AND CONTEXT

An individual's proficiency in solving mathematical tasks depends on a myriad of factors associated with cognitive ability and task design. Additionally, psychologists suggest that it is important to consider the ways in which an individual encodes and decodes cognitive tasks (Blazhenkova, Becker, Kozhevnikov, 2011). Problem solvers tend to have a preference for how they encode mathematical information, often referred to as cognitive style. Some individuals rely more on imagery to process cognitive tasks (referred to as visualizer) while others tend to use more verbal-analytical strategies (referred to as verbalizer).

Although there has been strong agreement regarding cognitive style on a dichotomous verbalizer-visualizer continuum, researchers have proposed additional categorisations. For instance, Krutetskii (1976) added an intermediate dimension that he referred to as harmonic cognitive style to suggest that some people tend to be both visualizers and verbalizers. Riding (1991) focused on this continuum in terms of Wholistic-Analytic and Imagery-Verbal as measured by the often-cited Cognitive Style Analysis (CSA) instrument. Recently, Blazhenkova & Kozhevnikov (2009) added a third dimension to the bipolar Visual-Verbal cognitive style. They distinguished among three dimensions of cognitive style, namely object imagers, spatial imagers and verbalizers. Object imagers prefer to use concrete, detailed and pictorial images of objects to interpret information. Spatial imagers represent spatial relations among objects schematically and perform complex spatial transformations. Verbalizers prefer to use verbal-analytical tools to solve cognitive tasks. This study is framed within this three-tier categorisation of cognitive style.

Given the observed relationship between cognitive style and mathematical performance in previous research, there is ongoing interest to understand the ways in

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which the former influence the latter or the interrelationship between them. For instance, Chrysostomou, Pitta-Pantazi, Tsingi, Cleanthous, and Christou (2013) observed that spatial imagery is related to number sense and algebraic reasoning. Similarly, Pitta-Pantazi and Christou (2010) observed that students with different levels of spatial imagery performed differently in problems involving spatial tasks such as arranging three-dimensional arrays of cubes. Anderson et al. (2008) found that both spatial imagery and verbal deductive cognitive style were important in geometry tasks unlike object imagery. In terms of word problems, Hegarty & Kozhevnikov (1999) pointed out that spatial imagery was positively correlated to success in finding the solution.

Previous studies (Hegarty & Kozhenikov, 1999; van Garderen, 2006) have provided evidence for the relationship among the three variables of interest, namely mathematics problem solving performance, spatial visualisation and cognitive style. However, these studies involved relatively small samples of students (33 boys and 66 students respectively). Thus, these studies are quite limited in terms of the nature and size of their samples. The current investigation involved 807 students and provides a more extensive base for investigation.

The following two research questions were posed:

- To what extent are cognitive styles (verbal information processing, spatial-imagery and object-imagery) related to spatial visualisation and performance on mathematics tasks?
- To what extent does the self-reported C-OSIVQ instrument provide valid measures of cognitive style?

The second research question emanated in the course of answering the first research question. As we compared the characteristics of verbalizers, object-imagers and spatial-imagers on the basis of the C-OSIVQ scales, we could observe the proximity of the content of the items in the spatial imagery dimension. This led us to question the extent to which the self-report C-OSIVQ instrument measure the three constructs in terms of content validity.

METHOD

Participants

Eight hundred and seven Grade 6 (aged 11-12) students (392 boys, 415 girls) from 8 Singapore schools (6 government and 2 government-aided) took part in the study.

Instrument 1: Measurement of cognitive style

The C-OSIVQ questionnaire (Blazhenkova, et al., 2011) consists of 3 sets of 15 items, each set corresponding to one particular type of cognitive style; namely object imagery, spatial imagery or verbal information processing. Participants rated each item on a 5-point Likert scale (1 = total disagreement; 5 = total agreement). The scores in each of the three sets were averaged to produce an object-imagery score, a spatial-imagery score and a verbal information processing score (see Discussion and Conclusion sections for sample items).

Instrument 2: Measurement of spatial visualisation

The Paper Folding Test (Ekstrom, French, & Harman, 1976), abbreviated as PFT, is one of the most commonly used instruments for measuring spatial visualisation ability. It also gives indication of students' use of schematic representations (Hegarty & Kozhevnikov, 1999). In this timed test, students are required to visualise the folding action of a square sheet of paper. A hole is then punched in one part of the fold and students are to identify the resultant design when the paper is reopened. The Paper Folding Test consists of 20 items. A correct item is given a score of 1 mark. The total score is calculated as follows: Number of items marked correctly minus one-fifth the number marked incorrectly. The minimum score is -4 and the maximum score is 20.

Instrument 3: Measurement of problem solving performance in mathematics

The mathematics performance of the students was measured through the Mathematics Processing Instrument (Lowrie, 2013), abbreviated as MPI. This instrument consists of 24 contextual items involving a combination of purely word problems and graphic-embedded tasks. It involves items from different areas of mathematics including numbers, measurement, statistics, probability, pre-algebra and spatial reasoning. Although the MPI is not a standard test of mathematics ability, the nature of the higher order tasks gives a measure of Grade 6 students' problem solving performance. A correct answer is given a score of 1 while an incorrect answer is marked as 0. Thus, the maximum score on this test is 24.

RESULTS

Cognitive style, spatial ability and performance in mathematics

Descriptive statistics for the five constructs are presented in Table 1.

Test	Minimum	Maximum	M	SD
PFT	-1.60	20	9.98	4.17
MPI	3	24	17.13	4.63
C-OSIVQ-Object	1.60	5.00	3.75	0.68
C-OSIVQ-Spatial	1.00	5.00	3.20	0.78
C-OSIVQ-Verbal	1.33	5.00	3.20	0.68

Table 1: Distribution characteristics of the instruments

There were significant correlations among the object, spatial and verbal dimensions of cognitive styles (see Table 2). Among the three dimensions, spatial imagery has the highest correlation to spatial visualisation ability as measured by the Paper Folding Test. The verbal dimension was not correlated to spatial visualisation ability as expected. Similarly, only the spatial imagery dimension of cognitive style was correlated to performance in the mathematics test. There was a significant correlation between spatial visualisation ability and mathematics performance.

Measure	C-OSIVQ Object	C-OSIVQ Spatial	C-OSIVQ Verbal	PFT	MPI
C-OSIVQ-Object	1.00	0.41**	0.57**	0.09**	0.04
C-OSIVQ-Spatial		1.00	0.29**	0.22**	0.11**
C-OSIVQ-Verbal			1.00	-0.00	0.06
PFT				1.00	0.36**
MPI					1.00

Note: ** $p < 0.01$

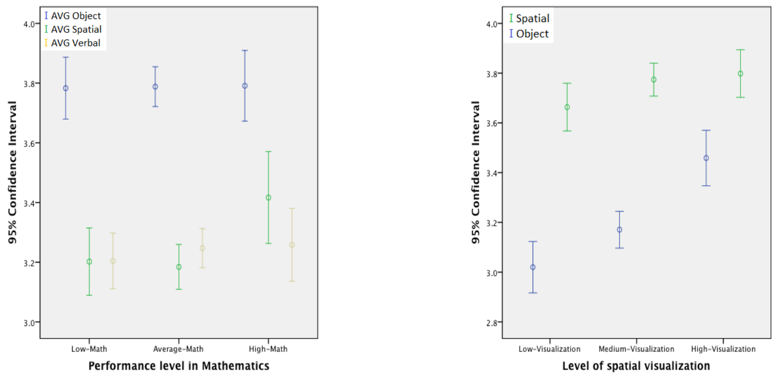
Table 2: Correlations among variables

Relation between the three dimensions of C-OSIVQ and MPI

The performance of the students on the MPI was split into three categories to determine whether there were variations in students' performance and the three dimensions of cognitive styles. The participants were classified as Low-Math (bottom 25% of the distribution, MPI score <14.75), High-Math (top 25% of the distribution, MPI score >21) and Average-Math (middle 50%, MPI score between 14.75 and 21). The 95% confidence interval for the three dimensions of the cognitive style is presented in Figure 1(a). The mean of the object scores were much higher than the two other categories of cognitive styles across performance level of students. This observation is in accord with Blazhenkova, et al. (2011) who maintained that object scale ratings tended to be higher than the other two scales. In addition, the verbal score of the high-performing students was higher than the two other groups. Further, ANOVA results ($F(2,739) = 4.84$, $p = 0.008$), suggested that there were significant differences among the three performance levels in the spatial imagery dimension of the C-OSIVQ between Low-Math and High-Math; and between Average-Math and High-Math. However, such differences were not exhibited for the other two dimensions of cognitive style.

Relation between Paper Folding Test, spatial imagery and object imagery scores

Table 2 revealed significant correlations between two of the three dimensions of cognitive style (namely spatial imagery and object imagery) and spatial visualisation. We attempted to study whether the level of spatial visualisation of the participants was related to the cognitive style (as we did for performance earlier in Figure 1(a)). Students with high spatial visualisation ability had high spatial imagery scores. Further, there are significant differences in spatial imagery scores between students who had low spatial visualisation ability compared to those who had high spatial visualisation ability (Figure 1(b)).



(a) (b)

Figure 1: Distribution of cognitive style according to (a) performance level in mathematics (b) level of spatial visualisation ability

Relation between spatial visualisation ability and performance on the MPI

Table 3 showed that Medium and High spatial visualizers had significant correlations with problem solving performance in mathematics, in contrast to Low spatial visualizers. Thus, mathematics performance is related to spatial visualisation ability.

Level of Spatial Visualisation Ability	Correlation coefficient	Significance (p-value)
Low	-0.029	0.685
Medium	0.186	0.000
High	0.206	0.004

Table 3: Correlation between spatial visualisation ability and performance on MPI

A re-analysis of the factor structure and content validity of C-OSIVQ

In order to measure the cognitive styles of the students using the C-OSIVQ instrument, the students were asked to rate a set of items on a Likert scale. Although the reliability of the instrument is reported to be high, it is critical to determine the extent to which these self-ratings provide adequate measure of unobservable behaviors involved in the encoding and processing of mathematical information. The C-OSIVQ questionnaire is based on three constructs assumed to characterize cognitive style, namely object imagery, spatial imagery and verbal image processing. For the present sample of Singapore elementary students, Cronbach's $\propto$ for the Object scale, Spatial scale and Verbal scale was 0.87, 0.88, and 0.84 respectively. The first two of these values are above the minimum coefficients (0.85) recommended by McKelvie's (1994) guidelines for judging psychometric properties of imagery scales. The test designers (Blazhenkova, et al., 2011) established the three dimensions of the C-OSIVQ by conducting Principal Component Analysis. We performed a similar analysis to determine how our sample compared with that reported by the test designers. The Kaiser-Mayer-Olkin measure verified the sampling adequacy for the analysis, KMO = 0.92. The initial Principal Component Analysis revealed nine components with eigenvalues greater than 1 but the scree plot suggested the retention of only three components. These 3 factors cumulatively accounted for 38.3% of the variance.

We rotated the solution orthogonally (using Varimax) as used in the parent version of the C-OSIVQ questionnaire (Blazhenkova & Kozhevnikov, 2009) to be able to interpret the factor structure. The first component consisted of nine spatial-imagery items while the second component consisted of nine verbal information processing items. The third component consisted of six object-imagery items. We were particularly interested in the spatial-imagery dimension as it had larger correlations with performance in mathematics compared to object-imagery and verbal information processing. Items that loaded on the spatial-imagery factor were mostly related to connecting devices or computers or playing construction games and included: "I can connect two electronic devices", "I am good at playing 3D action video games", and "I am very good at construction games". Thus, the content validity of the instrument may be an issue that needs to be addressed to improve the instrument.

Among the 15 questions designed to assess spatial-imagery, only three questions were directly related to the encoding and processing of information from school mathematics. These three items loaded on separate factor: (1) I can easily imagine and rotate three-dimensional figures in my mind; (2) I am good at solving geometry problems with 3-D figures; (3) It is easy for me to solve geometry problems.

DISCUSSION AND CONCLUSION

Research Question 1: To what extent are cognitive styles (verbal information processing, spatial-imagery and object-imagery) related to spatial visualisation and performance on mathematics tasks?

We used the C-OSIVQ self-reporting instrument to obtain measures of students' cognitive styles in terms of verbal information processing, object imagery, and spatial imagery. There were significant correlations between the spatial imagery information processing and the two measures of ability, namely spatial visualisation ability (as measured by the PFT) and mathematics problem solving performance (the MPI).

Given the correlation between spatial imagery and performance in the two measures of ability, we investigated further the ways in which different ability levels varied in this dimension of cognitive style. In the MPI, the high performers had higher mean spatial imagery scores than the average and low performers, a result that tallies with previous research findings (van Garderen, 2006). This may suggest that higher mathematical performance is associated with schematic processing of information as is characteristic of spatial imagery. Conversely, the mathematics ability level of students may give an indication of their preferred cognitive style. Similarly, when we split students by level of spatial visualisation ability, those with high spatial visualisation ability had high mean spatial imagery score. The foregoing observations allow us to conclude that high performers in mathematics have higher spatial visualisation ability and have preference for spatial imagery. These observations align with previous research (Hegarty & Kozhevnikov, 1999), which underlined the relation between spatial imagery and success in mathematics problem solving. By considering a relatively larger sample, this study clarifies further how problem solvers having different ability levels vary in terms of using spatial imagery.

Research Question 2: To what extent does the self-reported C-OSIVQ instrument provide valid measures of cognitive style?

We re-analysed the factor structure and the content validity of the C-OSIVQ instrument in an attempt to understand the extent to which it measures cognitive style in relation to encoding and processing mathematical information. The three dimensions of cognitive style were inter-correlated. This may suggest that if a person has preference for a certain way of processing information (e.g., using pictures rather than making a schematic representation) then that does not mean s/he does not use the verbal-analytical way of interpreting information. Such inter correlations were also observed by Blazhenkova and Kozhevnikov (2009).

Given that this instrument had been validated, we expected a higher number of items to load more consistently on each of the three dimensions. However, only half of the items loaded on the three components, with more items loading on the spatial-imagery component. This prompted us to analyse the content of the items with specific focus on the spatial-imagery component as it is more related to mathematics performance. The 15 items in this dimension consist of three categories of situations: 9 items associated with connecting devices and construction games, 3 items associated to school geometry and 3 items associated to schematic representations. The close resemblance of the 9 items related to connecting devices and computers put into question the content validity of the instrument in relation to the spatial-imagery dimension. Further, only three items in the questionnaire are directly related to school mathematics. As Blazhenkova and Kozhevnikov (2009) indicated, self-reports allow for the assessment of subjective aspects of imagery. Nevertheless, self-reports should assess what they are purported to measure.

Implications of the study and future research

Given the correlation between spatial-imagery and performance in problem solving in mathematics, it is important to support students to develop habits of representing mathematical information more schematically. Not all students may have a propensity for this mode of encoding and processing mathematical information. Instructions should encourage students to represent relations in problems schematically so that such a mathematical behaviour becomes a habit of mind.

There is a need for a more extensive conceptualisation of cognitive style, aligned to learning mathematics. This can be accomplished by more empirical work, especially in terms of qualitative studies. Merely extrapolating ideas from psychology to mathematics may not serve the purpose of mathematics educators. It appears that there is a need to improve the content validity of the self-reported cognitive style instrument. This is important if cognitive style is to be regarded as having practical pedagogical implications among pragmatic educators.

References

Anderson, K.L., Casey, M.B., Thompson, W.L., Burrage, M.S., Pezaris, E., & Kosslyn. S.M. (2008). Performance on middle school geometry problems with geometry clues matched to three different cognitive styles. *Mind, Brain, and Education 2*(4), 188-197.

Blazhenkova, O., Becker, M., & Kozhevnikov, M. (2011). Object-spatial imagery and verbal cognitive styles in children and adolescents: Developmental trajectories in relation to ability. *Learning and Individual Differences, 21*(3), 281-287.

Blazhenkova, O., & Kozhevnikov, M. (2009). The new object-spatial-verbal cognitive style model: Theory and measurement. *Applied Cognitive Psychology, 23*, 638-663.

Chrysostomou, M. Pitta-Pantazi, D., Tsingi, C., Cleanthous, E., & Christou (2013) Examining number sense and algebraic reasoning through cognitive styles. *Educational Studies in Mathematics, 83*, 205-223.

Ekstrom, R. B., French, J. W., & Harman, H. H. (1976). *Manual for kit of factor-referenced cognitive tests*. Princeton, NJ: Educational Testing Service.

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4), 684-689.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children*. Chicago: University of Chicago Press.

Lowrie, T. (2013). Cross Cultural Comparison of Student Sense-making: The development of a mathematics processing framework. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow* (Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia, eBook, pp. 751-754). Melbourne, Australia, MERGA, Inc.

McKelvie, S. J. (1994). Guidelines for judging psychometric properties of imagery questionnaires as research instruments: A quantitative proposal. *Perceptual and Motor Skills, 79*, 1219-1231.

Pitta-Pantazi, D., & Christou, C. (2010). Spatial versus object visualisation: The case of mathematical understanding in three-dimensional arrays of cubes and nets. *International Journal of Educational Research, 49*, 102-114.

Riding, R. J. (1991). *Cognitive Styles Analysis*. Birmingham: Learning and Training Technology.

van Garderen., D. (2006). Spatial visualization, visual imagery, and mathematical problem solving of students with varying abilities. *Journal of Learning Disabilities, 39*(6), 496-506.

STUDENT UNDERSTANDING OF PROOF AND PROVING: IS INTERNATIONAL COMPARISON POSSIBLE?

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Research on proof and proving includes international assessments of argumentation competencies, comparisons of proofs in school contexts, and application of research results from one national and cultural context to others. There are a number of obstacles to such international comparisons, arising from linguistic, epistemological, cultural, and educational differences, which are outlined in this article. Ways in which these obstacles are addressed or ignored, and the degree to which they are avoidable are discussed, and an example is given of an international comparison in which these obstacles informed the research methodology in a useful way.

INTRODUCTION

In recent decades there has been an increasing emphasis on student understanding of mathematical *processes*, among which proving is included. This is reflected in the increasing number of research studies on proving (Reid & Knipping, 2010, p. xiii), and in influential policy documents, such as the NCTM Principles and Standards (2000) as well as in national mathematics curriculum documents. A parallel development is a shift in focus of international mathematics assessments from curriculum content to mathematical processes and competencies. This article explores the degree to which it is possible to extrapolate research results across international and intercultural borders, and especially whether it is possible to compare student understanding of proving across such borders.

There are a number of studies that are often cited as evidence for "students' low level in understanding and building mathematical proofs (Galbraith, 1981; Fischbein, 1982; Senk, 1985; Martin and Harel, 1989; Chazan, 1993; Battista and Clements, 1995; Zaslavsky and Ron, 1998; Healy and Hoyles, 2000; Recio, 2000)" (Recio & Godino, 2001, p. 83). Note that the majority of these studies, and others on proof and proving reported at PME and in international mathematics education journals, were conducted in English speaking countries. However, for reasons discussed below, there is good reason to be cautious about applying these findings to other national and cultural contexts. As Bell (1976) notes, "Viewed internationally, the proof aspect of mathematics is probably the one which shows the widest variation in approaches" (p. 23).

INTERNATIONAL ASSESSMENTS

The concerns outlined in this article apply to any kind of research on proof and proving that is conducted or applied across borders, but international mathematics assessments

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are a special focus, because of the increasing emphasis on comparing proving and other mathematical processes using such assessments.

This emphasis is especially clear in the PISA assessments which focus on "competencies" such as "argumentation" which includes

> knowing what mathematical proofs are and how they differ from other kinds of mathematical reasoning; following and assessing chains of mathematical arguments of different types; ... and creating and expressing mathematical arguments (OECD, 2009, p. 106).

In the most recent assessment framework, the terminology of "competencies" has been dropped, and the process of proving appears implicitly as part of a more general process. "This process of employing mathematical concepts, facts, procedures, and reasoning includes activities such as … reflecting on mathematical arguments and explaining and justifying mathematical results" (OECD, 2013, p. 29).

This interest in international assessment of mathematical processes raises the question of what obstacles might exist to doing so, and how they might be overcome. This paper explores four such obstacles: word usage, epistemological perspectives, cultural differences, and educational differences.

DIFFERING USAGES OF THE WORDS "PROOF" AND "PROVING"

As Godino and Recio (1997, Recio & Godino, 2001) point out, the words "proof" and "proving" are used differently in foundations of mathematics and mainstream mathematics, to refer on one hand to a purely formal derivation from explicit axioms and on the other to refer to a deductive but only partly formal argument that "convinces someone who knows the subject" (Davis and Hersh, 1981, p. 40). In mathematics education there is a further variation of usage, both in reference to the object called a "proof" and the process of "proving".

The object called a proof might be either a written text or a convincing argument. The NCTM uses "proof" in the first sense when they write, "High school students should be able to present mathematical arguments in written forms that would be acceptable to professional mathematicians" (NCTM, 2000, p. 58). "Proof" has been used to mean "convincing argument" by a number of authors, including Mason, Burton, and Stacey (1985) and Davis and Hersh (1981).

"Proving" can refer to three distinct processes: reasoning deductively, reasoning "to remove or create doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241) or "collective processes in which students and teacher develop the proof together" (Knipping, 2004, p. 73).

Clearly, when assessing proof and proving the meaning of these words must be made explicit. This is true in reporting research and especially in international assessments. As noted above, the PISA frameworks have become less explicit over time about what is being assessed related to proving, but it may be that there are more explicit unpublished documents and item analyses that ensure that consistent meanings of proof and proving are used. But readers of the results are left not knowing what exactly is assessed that might be considered "proving" by the assessment designers.

EPISTEMOLOGIES OF PROOF

Balacheff (2008) describes "epistemologies of proof" which relate to the concept of proof more generally, and especially the connection between proof, truth, and validity.

> Our epistemology of proof (the relationship we have with truth and validity) first shapes our research framework, even before the choice of a problématique (i. e., the choice of the relevant questions and research problems), and the choice of a theoretical framework and its related methodology. (p. 502)

Balacheff does not include Fischbein among the researchers whose epistemology he describes, but Fischbein's work offers a very clear example of how a researcher's epistemology guides the choice of research questions, theories and methods. Fischbein (1982) asserts that, "the concept of formal proof is completely outside the main stream of behavior" (p. 17).

> A new "basis of belief", a new intuitive approach, must be elaborated which will enable the pupil not only to understand a formal proof but also to believe ... in the a priori universality of the theorem guaranteed by the respective proof. (p. 17)

On the basis of this epistemology of proof, Fischbein and Kedem (1982) designed a study to test the hypothesis that most students "do not have a clear idea of what a formal mathematical proof means" (p. 128). They tested this by presenting students with proofs, and then asking if additional checks would increase their confidence. They found that the majority felt that additional checks would increase their confidence, and hence did not understand mathematical proof as "further checks are superfluous since a formal proof guarantees *a priori* the absolute validity of the statement" (p. 131).

Starting from a different epistemology, one might propose different research questions such as, "Why do students use examples to increase their confidence when given a proof?". Chazan (1993) interviewed students to explore more deeply their reasons for distrusting proofs and their attitudes towards examples. Some of the reasons students gave for using examples reflect a nuanced view of proof; for example, they noted that the assumptions used in the proof might be wrong.

While it may be possible to be explicit about usages of the words "proof" and "proving" and to agree on specific usages in the context of an international assessment, it is less clear that epistemologies of proof can be agreed.

> Is a consensus possible? By consensus I mean at best a common theoretical framework, at least a glossary guaranteeing a minimal set of shared meanings. The deadlock on the route towards achieving such a programme is our own epistemology of mathematical proof. … Indeed, researchers themselves cannot avoid involving in their work their own epistemology of mathematical proof and, beyond it, their own epistemology of mathematics. (Balacheff, 2008, p. 508)

This suggests that an international assessment must either adopt a number of incompatible epistemologies that come with differing meanings of key concepts, or adopt a single epistemology, excluding those whose epistemologies differ. Adopting a single epistemology makes it possible to consistently assess proving but at the cost of narrowing the perspectives represented. This could create biases in cases where particular epistemologies are dominant in some national or cultural contexts more than in others.

CULTURAL DIFFERENCES

As the PISA frameworks quoted above indicate, the process of proving is often related to argumentation in general. But the nature of argumentation varies across cultures. In the West, it is associated with a 'struggle' to 'defend' a claim, words that suggest conflict. Sekiguchi and Miyazaki (2000) provide an insightful description of the process of "hanashi-ai" which they see as the counterpart to argumentation in the West, but which lacks any connotation of conflict.

> The word [hanashi-ai] means mutual conversation or consultation, and does not signify a war. Because people try to avoid direct confrontation, they try to put their opinions ambiguously so that they can withdraw or change them easily when others indicate opposition (Nakayama, 1989). As a result, people in "hanashi-ai" do not usually bring up such full logical defense devices like "grounds," "warrants," and "backing." Even in those situations where the social exchange model is working, people tend to avoid bringing up logical armaments because they feel that arguing logically is impersonal ("katakurushii"). (Sekiguchi & Miyazaki, 2000, Communication and Argumentation in Japanese and Western Cultures section, para. 10)

Because of such cultural differences, differences in epistemologies of proof exist between cultures, and these differences cannot be overcome by agreeing to particular usages of words. For example, defining a 'proof' as a 'convincing argument' gives it a different meaning in Japan than in English speaking countries.

EDUCATIONAL DIFFERENCES

For almost everyone the primary exposure to mathematical proof is in school. But school systems differ in many ways, some of which are clearly relevant to international comparisons of proof and proving. This can be illustrated by looking at one of the few traditional proof tasks that have been included on an international assessment.

One item on the Third International Mathematics and Science Study (TIMSS-95) for advanced mathematics students in their final year of secondary school asked the students to write a proof (see Figure 1). In most countries the percentage of correct answers is close to the international average (34%) and the percentage of wrong or partial answers is higher (TIMSS 1998a, 1998b). However, in some countries (e.g., France, Switzerland, Russia) about half the students responded correctly, and about a third gave wrong or incomplete answers.

The results from Greece indicate that national differences in schooling are a factor. There 65% of the students were able to construct a valid proof, and only 6% gave wrong or incomplete answers. The remainder did not attempt the item either because they ran out of time or because they skipped it. This suggests that for these students proof is an all or nothing affair; either they know the proof and write it correctly or they recognise that they do not know it and write nothing. There is a continued emphasis on Euclidean geometry in the Greek mathematics curriculum (Kuzniak & Vivier, 2009) which has been the case since the beginning of organised schooling (Toumasis, 1990). It is reasonable to conclude that exposure to similar proofs in school accounts for the results of the Greek students on this assessment item. In other countries with a less strong emphasis on traditional Euclidean geometry (which includes most English speaking countries) students may not be able to predict in advance whether they will be able to construct such a proof and so they are more likely to attempt it and fail.

In the ΔABC the altitudes BN and CM intersect at point S. The measure of $\angle MSB$ is 40° and the measure of $\angle SBC$ is 20°. Write a PROOF of the following statement:

"ΔABC is isosceles."

Give geometric reasons for statements in your proof.

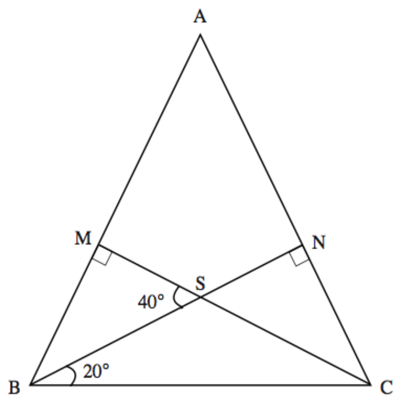

Figure 1: Item K18 from the released Advanced Mathematics items, TIMSS 1995, reproduced from TIMSS, 1998c, p. 89.

CONCLUSIONS

The obstacles described above vary in seriousness. Differences in word usage could be addressed by increased awareness of such differences and explicitness in published research, curriculum and policy documents and international assessment frameworks. It may even be possible to agree on specific usages for purposes on international comparisons. Differences in epistemologies of proof are more serious. It is possible to be more aware of and explicit about our epistemologies, but as Balacheff (2008) notes, consensus is not possible. Epistemologies of proof can be seen as an aspect of culture, and cultural differences in general, including differences in the nature of argument and justification may be the most difficult obstacles to overcome in conducting research and making comparisons internationally. Much of culture is implicit, and it is impossible to be clear to others about assumptions we ourselves are unaware we are making. Finally, there are educational differences which in some ways would seem to be the most easily overcome obstacles, but which experience indicates may not be. One effect of international assessments like TIMSS and PISA is an increased interest in how education is conducted in other national contexts. However, this interest has not translated into a significantly better understanding of different educational systems. Too often a simple answer is sought to the complicated question of what is different about education in Finland, Shanghai, or elsewhere. When no single variable is found that accounts for differences in performance, interest wanes. But it is possible to trace some effects of educational differences. For example, the emphasis on Euclidean geometry proofs in Greece is documented, and seems to account well for the different pattern of answers on the proof item in TIMSS-95. By asking more specific questions about differences in performance, and expecting more complicated answers, it may be possible to convert educational differences from an obstacle to conducting international comparisons, into a useful way to gain insight into the effects of education on learning.

SOME CAUSE FOR HOPE

Given the obstacles outlined above, there seems to be little point in attempting to measure differences in student performance or understanding of proof and proving. However, the very same obstacles create a research opportunity, to better understand how linguistic, epistemological, cultural, and educational differences between countries influence teaching related to proof. There have been a few studies that have attempted to do so, and which suggest possible approaches. Cabassut (2005) compared textbook proofs and Knipping (2004) compared classroom teaching in France and Germany. They describe differences in practice, and in accounting for these differences they take linguistic, epistemological, cultural, and educational differences into consideration. Similarly, Hemmi, Lepik, and Viholainen (2013) compared the role of proof-related competences in Estonian, Finnish, and Swedish curricula and consider linguistic, epistemological, cultural, and educational differences in accounting for differences observed in the development of proof in the curricula compared.

Grundey (2014) offers another case of comparative research on proof, in which the expectation of linguistic, epistemological, cultural, and educational differences guided the methodology. Grundey conducted a design experiment in which instructional materials were used to reveal students' understandings of proof, not in comparison to a normative expectation but rather in comparison to other students. The teaching then attempted to influence the students' understandings. Two classrooms each in Germany and Canada were involved and this allowed further comparisons of the initial understandings of the students and the changes that occurred during the teaching. By attempting to make explicit differences between students, both within and across classrooms, Grundey incorporated those differences into the knowledge she gained of their understandings of proof. As a result, differences that could have been obstacles became sources of insight.

References

Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: An essay on the case of proof. *ZDM The International Journal on Mathematics Education 40*(3), 501-512. doi: 10.1007/s11858-008-0103-2

Bell, A. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics, 7*(1-2)*,* 23-40.

Cabassut, R. (2005). Argumentation and proof in examples taken from French and German textbooks. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education. Sant Feliu de Guixols, Spain* (pp. 391-400). Retrieved from http://ermeweb.free.fr/CERME4/CERME4_WG4.pdf.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*(4), 359-387.

Davis, P. J., & Hersh, R. (1981). *The mathematical experience.* Boston: Birkhäuser.

Fischbein, E. (1982). Intuition and proof. *For the Learning of Mathematics, 3*(2), 9-18.

Fischbein, E., & Kedem, I. (1982). Proof and certitude in the development of mathematical thinking. In A. Vermandel (Ed.), *Proc. 6th Conf. of the Int. Group for the Psychology of Mathematics Education,* (pp. 128-131). Antwerp.

Godino, J., & Recio, A. (1997). Meaning of proofs in mathematics education. In E. Pekhonen (Ed.), *Proc. 21st Conf. of the Int. Group for the Psychology of Mathematics Education.* (Vol. 2, pp. 313-320). Lahti, Finland.

Grundey, S. (2014). *Beweisvorstellungen und eigenständiges Beweisen. Entwicklung und vergleichend empirische Untersuchung eines Unterrichtskonzepts am Ende der Sekundarstufe.* Doctoral dissertation: University of Hamburg

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), *Research in collegiate mathematics education III.* (Issues in Mathematics Education, Volume 7, pp. 234-282). Providence RI: American Mathematical Society.

Hemmi, K., Lepik, M., & Viholainen, A. (2013). Analysing proof-related competences in Estonian, Finnish and Swedish mathematics curricula: Towards a framework of developmental proof. *Journal of Curriculum Studies, 45*(3), 354-378.

Knipping, C. (2004). Argumentations in proving discourses in mathematics classrooms. In G. Törner, R. Bruder, A. Peter-Koop, N. Neill, H.-G. Weigand, & B. Wollring (Eds.), *Developments in Mathematics Education in German-speaking Countries. Selected Papers from the Annual Conference on Didactics of Mathematics, Ludwigsburg, March 5-9, 2001,* (pp. 73-84). Hildesheim: Verlag Franzbecker.

Kuzniak, A., & Vivier, L. (2008). A French look on the Greek geometrical working space at secondary school level. In V. *Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.) Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education. January 28th - February 1st 2009, Lyon (France)* (pp. 686-695). Retrieved from http://ife.ens-lyon.fr/publications/edition-electronique/cerme6/wg5.pdf

Mason, J., Burton, L., & Stacey, K. (1985). *Thinking mathematically. Revised edition.* New York: Addison Wesley.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Organisation for Economic Co-operation and Development [OECD]. (2009). *PISA 2009 assessment framework: Key competencies in reading, mathematics and science*. OECD.

Organisation for Economic Co-operation and Development [OECD]. (2013). *PISA 2012 assessment and analytical framework: Mathematics, reading, science, problem solving and financial literacy*. OECD.

Recio, A., & Godino, J. (2001). Institutional and personal meanings of mathematical proof. *Educational Studies in Mathematics, 48*(1), 83-99.

Reid, D., & Knipping, C. (2010). *Proof in mathematics education: Research, learning and teaching*. Rotterdam: Sense.

Sekiguchi, Y., & Miyazaki, M. (2000). Argumentation and mathematical proof in Japan. *The Proof Newsletter.* Retrieved from http://www.lettredelapreuve.org/OldPreuve/Newsletter/000102Theme/000102ThemeUK.html

Third International Mathematics and Science Study [TIMSS] (1998a). *Data almanacs for achievement items: Percent correct on the advanced mathematics items: Final year of secondary school.* Retrieved from http://timss.bc.edu/timss1995i/TIMSSPDF/pc_alm95/m12alm95.pdf

Third International Mathematics and Science Study [TIMSS] (1998b). *Data almanacs for achievement items: Percent of responses by math item categories: Final year of secondary school.* Retrieved from http://timss.bc.edu/timss1995i/TIMSSPDF/dr_alm95/cism1295.pdf

Third International Mathematics and Science Study [TIMSS] (1998c). *TIMSS mathematics and science items: Released set for population 3 (final year of secondary school), Advanced Mathematics Items.* Retrieved from http://timss.bc.edu/timss1995i/TIMSSPDF/CitemAdM.pdf

Toumasis, C. (1990). The epos of Euclidean geometry in Greek secondary education (1836–1985): Pressure for change and resistance. *Educational Studies in Mathematics, 21*(6), 491-508.

PROSPECTIVE ELEMENTARY TEACHERS' DIAGNOSTIC PROCEEDING IN ONE-ON-ONE DIAGNOSTIC INTERVIEWS: FACETS OF DATA COLLECTION AND ATTENTION

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The focus of the research presented in this paper is on cognitive diagnostic strategies prospective elementary mathematics teachers (PTs) use in the reflection of one-on-one interviews about arithmetic problems with children in Grade 1. Thereby, it responds to the detected lack of knowledge regarding qualitative facets of diagnostic proceeding in interview assessments. Results of a part-study reveal that collecting information to supply interpretation and conclusion within a diagnostic micro-process may vary concerning the choice of information or concerning the type of collecting. The discussion takes up the relevance of these findings for teacher education and touches "hidden" high-leverage practices in terms of diagnostic attention.

INTRODUCTION

The challenges of every-day classroom situations include the design of appropriate learning opportunities which refer to adaptive teaching competence and include diagnostic competence (cf. Wang, 1992). To meet these demands, beginners and experienced teachers benefit from a constructivist view on their students' individual progress in developing mathematical concepts: A powerful method to gain particular information on children's mathematical conceptions is provided with diagnostic one-on-one interviews which stem back to the clinical method of interviewing developed by Jean Piaget (cf. Ginsburg, 2009). Standardised task-based interviews enable one to assess the range and depth of children's thinking as (in-service) teachers actively explore qualitative facets of children's approaches to mathematical tasks. Prepared interview tools and empirically based growth points for the analysis may guide through these one-on-one interviews and thereby foster teachers' professional development (e.g., ENRP task-based assessment interview/CMIT/EMBI; cf. Clarke, 2013; Bobis et al., 2005; Peter-Koop, Wollring, Spindeler, & Grüßing, 2007).

Additionally, there is a need to sensitise *prospective* elementary mathematics teachers (PTs) to the varieties, ranges and depth of young children's mathematical thinking and to qualify them for informal formative assessment. In this sense, preparing, conducting and analysing students' mathematical conceptions in one-on-one interviews offers substantial learning opportunities and supports the development of PTs' diagnostic attitude (cf. Peter-Koop & Wollring, 2001; Prediger, 2010; Sleep & Boerst, 2012). Yet, qualitative facets of the diagnostic proceeding during a one-on-one interview have scarcely been studied so far. This includes facets of data collection, i.e., the question how actions or utterances are taken up before being used for interpretation.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 73-80. Hobart, Australia: PME.

THEORETICAL FRAMEWORK

The concept of diagnostic competence and domains of teacher knowledge

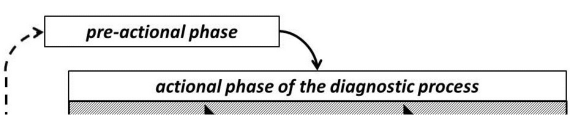
Recent studies on (the development of) diagnostic competence mainly focus on measuring the accuracy of teachers' judgments (cf. Südkamp, Kaiser, & Möller, 2012). With an emphasis on those numerical indicators, diagnostic competence is most often "operationalized as the correlation between a teacher's predicted scores for his or her students and those students' actual scores" (Helmke & Schrader, 1987, p. 94). In this concept, questions of *qualitative* aspects of diagnostic competence and its acquisition remain unanswered, and *processes of diagnosing* by which teachers evaluate an individual student's learning development are unaccounted for.

Ball et al. (2008) suggest that *pedagogical content knowledge* (PCK) includes knowledge about *common* mathematical conceptions or misconceptions that are frequently encountered in the classroom. Options to achieve this kind of knowledge may arise from analysing *individual* cases which refers to *knowledge of content and students* (KCS) defined as subdomain of PCK (Ball et al. 2008, p. 403). Thus, the capability of "eliciting and interpreting individual students' thinking" can be found among the set of "high-leverage practices" with which novices should be familiarised (cf. Ball et al., 2009; Cummings Hlas & Hlas, 2012). Sleep & Boerst (2012, p. 1039) conceptualise this particular "high-level practice" as subcomponent of the domain "assessing student thinking". In this sense, analysing an individual's mathematical concept may contribute to a deeper understanding of widespread (mis)conceptions. It may develop KCS, improve a teacher's practices in terms of diagnostic attention and expertise, and thereby enrich his or her diagnostic competence.

Modeling phases of the diagnostic process

In the field of elementary mathematics education research (which deals intensely with *qualitative* aspects of children's wide-ranging learning developments), expertise in this area reaches beyond teachers' accuracy in measuring children's achievements. It additionally includes rather vague aspects like diagnostic sensitivity, curiosity, an interest in children's emerging understanding and learning or the aptitude to gather and interpret relevant data in non-standardized settings (e.g., Prediger, 2010). Following this process-oriented attitude towards diagnostic competence, activities of formative assessment in a one-on-one interview can be seen as a multidimensional cyclic process (Klug, 2011; Klug, Bruder, Kelava, Spiel, & Schmitz, 2013). According to this model, a *pre-actional phase* (e.g., considerations of preparing diagnostic activities; choice of tasks/methods) prepares for an *actional phase* (including data collection and data interpretation) which is followed by a *post-actional phase*. The latter implies taking the necessary action from data collection/interpretation which leads to the design/the evaluation of a concept for an individual support in a repeated run through phases of this diagnostic *macro-process*.

Focusing on *micro-processes within the actional phase of diagnosing*, **collecting** data, **interpreting** and drawing further **conclusions** have deep impact on the diagnose via



an interview and are based on different kinds of knowledge (e.g., KCS, see Figure. 1).

Fig. 1: Differentiating the micro-process in the actional phase of diagnosing

When conducting a one-on-one interview, there is no direct access to students' conceptions. Instead and in terms of cognitive activity, those conceptions "must be reconstructed by interpreting their utterances" (Prediger, 2010, p. 76). Facets of *interpreting within the actional phase of the diagnostic process* in a one-on-one interview have already been discussed and shown in detail by Reinhold (2014): Those findings support the notion that cognitive elements of PTs' ways of proceeding in diagnostic interviews often resemble basic processes in qualitative data analysis. Here, cognitive elements of diagnostic strategies which were reconstructed from PTs' external (verbal) articulation in re-interviews included different sub-categories of interpreting, namely *contrasting*, *enriching*, *isolating*, *coding,* and *supporting.*

Collecting as a source for interpretation and conclusion

The elements of PTs' *interpretation* interact with facets of *collecting* and *concluding* in distinct types of diagnostic strategies (Reinhold, 2014). Yet, the implications of *collecting* or "gathering information" (Klug et al., 2013, p. 39) are still implicit concerning details of *collecting within the actional phase of the diagnostic process*. Collecting valuable information is obviously of high importance as this information is the source for interpretation and conclusion. Sleep and Boerst (2012, p. 1038) point out that the available information initially relies on the (previous) choice of tasks for the diagnostic situation as tasks "yield sound and useful information about student learning of particular content" (Sleep & Boerst , 2012, p. 1038). For one-on-one interviews, these tasks are usually chosen in the pre-actional phase, but they influence opportunities for data collection in the actional phase as well. Moyer and Milewicz (2002) identified general questioning categories (check-listing/instructing/ probing and follow-up questions) used by PTs while collecting data in one-on-one interviews. Furthermore, interpreting within any diagnostic situation is based on a substantial perception of the diagnostic situation. This "includes the ability to structure the situation cognitively, the ability to change the focus of attention and the willingness and ability to adopt other perspectives" (Barth & Henninger, 2012, p. 51).

Thus, attention and the capability to focus this attention tend to be crucial prerequisites for collecting within the actional phase. Attending as integral element of "professional noticing of children's mathematical thinking" defined by Jacobs, Lamb, and Philipp (2010) refers to the skill of "being able to recall the details of children's strategies" (p.

172).

In the actional phase of diagnosing in a one-on-one interview, noticing and collecting includes the motivation to listen and watch, the ability to observe with keen eyes, and the capability to detect important details or to value particular aspects in children's utterances or actions. Yet, little is known about the *facets of collecting* PTs use in one-on-one interviews they prepare and conduct with children: *How* is all this information "gathered" and *what kind of information* is it that tends to be interesting for those who "act systematically", for those who interpret and conclude?

RESEARCH QUESTIONS

Aiming at an empirically grounded theoretical framework for a qualitative view on PTs' cognitive activities in one-on-one interviews with children, the main purpose of the project ***diagnose: pro*** is to detect traits of diagnostic strategies: We intend to find out what cognitive elements characterise the PTs' diagnostic strategies when they diagnose individual arithmetic approaches in one-on-one mathematics interviews with first-graders and try to reconstruct how these strategic elements interact. Former reports on this research project already exemplified facets of interpreting within the micro-process in the actional phase of diagnosing (cf. Reinhold, 2014). This paper is a further excerpt from this larger study and directs the attention to facets of collecting PTs use in their diagnostic proceeding:

- What *kind of information* is collected to supply any kind of interpretation and conclusion during the actional phase of the diagnostic process?
- What *differences in the way this information is collected* can be detected?
- (How) Do differences concerning the choice of collected information or in the way of collecting *influence the type of diagnostic strategies* which can be reconstructed from interviews?

METHODS

In the sense of theoretical sampling (Corbin & Strauss, 2008), data collection intended to capture the range of PTs' practices and proceedings and focused on re-interviews of one-on-one diagnostic interviews. All PTs attended mathematics methods courses in the last year of their university studies (Master of Education). In cooperation with an elementary school, these courses provided the opportunity to prepare, conduct and analyse individual diagnostic interviews with up to six first-graders per PT. Drafts for these interviews were prepared at the beginning of the course where the PTs could make use of theoretical work on concepts of arithmetic learning trajectories and the method of task-based mathematics interviews (e.g., Peter-Koop et al., 2007). Until fall 2013, 7 PTs from these courses agreed to take part in retrospective interviews which focused on the video-recording of an interview they had conducted shortly before. With deliberately general advice at the beginning of the retrospective interviews, the PTs were asked to "analyse the interview" while watching the video-recording. The interviewee was requested to stop the video at any scene in order to comment on the diagnosis he or she would derive from this specific situation. If comments were rather short or pure in detail, the interviewee was asked to explain what knowledge, information or evidence warranted his or her uttered hypothesis. These retrospective analyses of diagnostic interviews offered the chance to narrow the focus and to pay attention to details. In this sense, data collection obviously differed from real-time practice in an interview which requires being concurrently aware of many more details.

As all interviews' analyses are based on Grounded Theory methodology, codes are derived from data via open, axial and selective coding or contrasting comparison of the data. Use of the software ATLAS.ti enables direct coding of video-data. To approach the aim of capturing identified characteristics of diagnostic proceeding in whole range ("saturated", Corbin & Strauss, 2008, p. 143), we also include data which consists of written comments of 31 PTs (collected in 2011) and video/audiotaped peer-talks among 28 PTs about video-scenes of diagnostic interviews (collected in 2012).

FINDINGS

Results of the study reveal that collecting information within the actional phase of a diagnostic micro-process may vary concerning the type of collecting and concerning the choice of information, as the following excerpts from the re-interviews (n=7) exemplary display. In *our* process-oriented analyses of the *PTs'* process-oriented analyses we took into account that facets of data collection may include observations which are not mentioned by the PTs: Subconsciously grasped information (e.g., on a child's hidden insecurity or motivation while working on a task) could also have an influence on a conclusion which is drawn, later on. In this sense, we are restricted to focus on the mentioned items. Besides, there is no way to tell data collection in the interview from data collection which can *definitely* be assigned to the re-interview.

Collecting: From observing to tracking

PTs' data collection was coded as *observing* when we considered the PTs to watch closely what was happening in the diagnostic situation. All PTs did listen attentively to the child's utterances. They paid attention to significant details, but they most often (also) noticed the (singular) occurrence of micro-incidents which were only loosely connected. In this sense, data collection included various details (see list in table 1) and often ended up in collections which resembled a "colorful bunch of flowers".

On a higher level, facets of collecting coded as *tracking* refer to the skill of following a series of activities or utterances. This includes to follow a child's action over a longer sequence and to maintain attentive during the diagnostic situation. This can be seen in the following protocol of Lisa's re-interview on an interview with 6-year old Sam. He is asked to take five chips (one side blue, the other side red) and comment on possible ways of displaying an addition with this material. Sam starts with spreading the chips and starts to sort them "Three red ones and two blue ones" as Lisa stops the video:

Lisa (01:51): To comment on this, I'd say he separated red and blue from the beginning and named what was lying on the table.

Later on, Lisa tracks this idea and collects further information from subsequent situations which refer to this issue (sorting and considering position of colors).

Lisa (02:16): Here, it is clear that he separated the colours from the beginning.

Lisa (10:20): We wanted them to find that sorting the possible additions helps to find all of them, yes and he is arranging them in any kind of structure, but… not the one we had intended them to find, and no one could find out if he had an idea of how to sort it. But in a way he does sort the possible arrangements because in this corner here, the blue ones are closer together. In the next row, the blue ones stick closely together, too, and there the red ones.

Collecting in the sense of recognising or sorting

PTs' data collection was coded as *recognising* when they repeatedly identified details they had already noticed in previous situations. In contrast to *tracking*, this was restricted to single incidents. *Sorting* in PTs' data collection was identified when PTs found/intentionally searched for groups or patterns in children's utterances or actions.

What kind of data is mentioned in re-interviews on one-on-one interviews?

A further analysis of PTs' comments on children's work also reveals a wide range of mentioned details as exemplified in Table 1 in which all interview excerpts are translated into English by the author.

Collected	Example
verbal utterance	"This boy, he was able to identify the summands and he said "This number and this number equals this number." (Anne)
activity	"He's drawing a circle around this piece of the pattern." (Pam)
(in)correctness of solution	"He was supposed to draw a circle around repeating parts of the pattern, but he failed." (Pam)
(elements of) strategy	"He used counting strategies, saw 4 and continued counting from that first summand." (Sue)
eye movement	"He hesitated and looked the other way." (Anne)
(subtle) movements of lips, head or hands	"I see he is nodding and I guess he's counting up to five here." (Lisa)
emotional state	"I got the impression he'd start crying." (Anne)
interviewer's behaviour	"Okay, I liked what I did in this situation as we decided to accept 'wrong' answers, too." (Sue)

Table 1: Various sources for interpretation: What is collected?

DISCUSSION

The study responds to the detected lack of knowledge regarding qualitative facets of diagnostic proceeding in one-on-one interviews and thereby contributes to strengthen the "power of task-based one-on-one interviews" (Clarke, 2013) in daily practice. Even if the reported findings are restricted to a certain type of tasks (arithmetic issues) and that they refer to a small number of participants (n=28 in peer-talks; n=7 re-interviews), the study takes “a look behind the scenes” of PTs’ diagnosing.

PTs’ attention was most often attracted by children’s obvious or prominent activities or utterances. Items were also collected if the PTs found surprising deviations from what they had expected before. Furthermore, other incidents obviously exactly matched what they had expected. This emphasises the importance of KCS (e.g. knowledge of common (mis)conceptions) as both deviation and alignment can only be stated if there is knowledge which can be used for this comparison. Additionally, this underlines the close relationship between collecting data and reasoning about the collected details (*interpreting* and *concluding*). Yet, this relationship does not necessarily appear as a linear process. Instead, PTs may run through these intertwined micro-processes in circles: a type of diagnostic strategy we call a *branched interpretation.* At the same time, we detect other diagnostic strategies, namely the strategy of being a *descriptive collector*, when the PTs focus on collecting and describing the child’s actions and neglect both interpreting and concluding (see Reinhold, 2014). This reveals “hidden” diagnostic practices which have to be uncovered in order to make them explicit. They are assumed to be of great importance for teacher education and further investigations in the project *diagnose:pro* will explore e.g., how elements of diagnostic strategies and types of strategies can be taken up in discussions of university courses. Awareness in this domain (including awareness of “strategic diagnostic tools”) may contribute to appropriate interpretations of children’s utterances in interviews and help to identify “high-leverage diagnostic practices” to cope with diagnostic challenges in class.

References

Ball, D. L., Thames, L. & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.

Ball, D. L., Sleep, L., Boerst, T. & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *The Elementary School Journal*, 109(5), 458-474.

Barth, Ch., & Henninger, M. (2012). Fostering the diagnostic competence of teachers with multimedia training – A promising approach? In I. Deliyannis (Ed.), *Interactive Multimedia* (pp. 49-66). Rijeka, Croatia: InTech.

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R. B., Young-Loveridge, J., & Gould, P. (2005). Supporting teachers in the development of young children’s mathematical thinking. *Mathematics Education Research Journal*, 16(3), 27-57.

Clarke, D. (2013). Understanding, assessing and developing children’s mathematical thinking: task-based interviews as powerful tools for teacher professional learning. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17-30). Kiel: PME.

Corbin, J., & Strauss, A. L. (2008). *Basics of qualitative research: grounded theory procedures and techniques* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Cummings Hlas, A., & Hlas, Ch. S. (2012). A review of high-leverage teaching practices: Making connections between mathematics and foreign languages. *Foreign Languages Annals*, 45(s1), s76-s97.

Ginsburg, H. P. (2009). The challenge of formative assessment in mathematics education: children' s minds, teachers' minds. *Human Development*, *52*, 109-128.

Helmke, A., & Schrader, F.-W. (1987). Interactional effects of instructional ability and teacher judgment accuracy on achievement. *Teaching and Teacher Education*, *3*(2), 91-98.

Jacobs, V. R, Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169-202.

Klug, J. (2011). *Modelling and training a new concept of teachers' diagnostic competence*. Darmstadt: TU Darmstadt.

Klug, J., Bruder, S., Kelava, A., Spiel, Ch., & Schmitz, B. (2013). Diagnostic competence of teachers: A process model that accounts for diagnosing learning behavior tested by means of a case scenario. *Teaching and Teacher Education*, *30,* 38-46.

Moyer, P. S., & Milewicz, E. (2002). Learning to question: categories of questioning used by preservice teachers during diagnostic mathematics interviews. *Journal of Mathematics Teacher Education*, *5*, 293-315.

Peter-Koop, A., & Wollring, B. (2001). Student teacher participation in interpretative classroom research. *Mathematics Teacher Education and Development, 3,* 4-15.

Peter-Koop, A., Wollring, B., Spindeler, B., & Grüßing, M. (2007). *Elementar Mathematisches BasisInterview*. Offenburg: Mildenberger.

Prediger, S. (2010). How to develop mathematics for teaching and for understanding. *Journal of Mathematics Teacher Education*, *13*(1), 73-92.

Reinhold, S. (2014). Qualitative facets of prospective elementary teachers' diagnostic competence: micro-processes in one-on-one diagnostic interviews. In C. Nicol, S. Oesterle, P. Liljedahl , & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 5, pp. 41-48). Vancouver, Canada: PME, 2014.

Sleep, L., & Boerst, T. A. (2012). Preparing beginning teachers to elicit and interpret students' mathematical thinking. *Teacher and Teacher Education, 28*, 1038-1048.

Südkamp. A., Kaiser, J., & Möller, J. (2012). Accuracy of teachers' judgments of students' academic achievement: a meta-analysis. *Journal of Educational Psychology*, *104*, 743-752.

Wang, M. C. (1992). *Adaptive teaching strategies: building on diversity*. Baltimore: Brookes.

EARLY YEARS PROSPECTIVE TEACHERS' SPECIALISED KNOWLEDGE ON PROBLEM POSING

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This paper focuses on primary prospective teachers' specialised knowledge for teaching (MTSK) while posing a problem for a given expression in the context of division. Using a particular expression, where the result was an infinite repeating decimal, we found some special relationships between different sub-domains of prospective teachers MTSK. From the fact that we gave prospective teachers a particular starting expression to pose a problem we also discuss some findings concerning problem posing as a way to develop teachers' knowledge.

INTRODUCTION

Internationally it is more or less accepted that problem solving is one of the core aspects in and for developing mathematical reasoning. At the same level should be problem posing, as good problems need to be posed to allow solvers to develop their mathematical ability and knowledge. Although problem solving and posing should be considered at the same level, most research focuses on problem solving, and of the few studies that consider problem posing, most focus on students, ignoring the teacher's role and knowledge in students' learning.

Considering that teachers' knowledge is a crucial factor in students' learning (e.g., Grossman, 2010), and the fact that prospective teachers need to acquire a deep understanding of the mathematical concepts (Tichá & Hošpesová, 2013), problem posing is perceived as a way to access (and develop) problem posers' mathematical knowledge (considering the different specificities of such knowledge). It is also perceived as a way to enrich the understanding of the content of teachers' knowledge and the relationships between its different aspects/sub-domains. Amongst different possible ways of perceiving teachers' knowledge, we consider the Mathematics Teachers' Specialised Knowledge (MTSK) (Carrillo et al., 2013) conceptualization. We assume that identifying, understanding and developing such knowledge (in and for problem posing) would allow teachers to teach with a different focus and understanding to that with which they have been taught (e.g., Cooney, 1994), as well as conceptualize tasks for teachers' training that would allow (and focus on) developing such specialised knowledge.

Considering one of the core tasks of teaching is giving sense to students' productions, teachers' knowledge should include a broad range of strategies and representations for problem solving that could help them successfully develop such an endeavour (Ribeiro, Mellone & Jakobsen, 2013). In that sense, considering that one of students' difficulties

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concerns number sense and problem solving in the context of the operations, in particular division (e.g., Fosnot & Dolk, 2001), and that students' difficulties are aligned with teachers' knowledge (e.g., Hill, Rowan & Ball, 2005), we perceive it as crucial to focus teachers' training in those aspects. A starting point can be identifying critical mathematical situations linked with the specificities of teachers' knowledge while posing problems (using a practice-based approach), having as an end point the design of tasks that would allow the development of teachers' specialised knowledge.

In this paper we expand previous work that identified particular features and dimensions involved in prospective teachers' knowledge. Here in particular we focus on the following question: what problems (kind, content, nature) do early years prospective teachers pose from a given division expression and what characteristics of MTSK does such a process bring to front?

MATHEMATICS TEACHERS' SPECIALISED KNOWLEDGE ON PROBLEM POSING IN DIVISION

Sharp, Garofalo and Adams (2002) consider that students' difficulties in understanding concepts involved in operations and algorithms are grounded in teachers' approaches that focus on remembering and on solving exercises. In order to allow for the connection of concepts, procedures, symbols, and their semantic referents when working on the idea of the operational composition of a number (Subramaniam & Banerjee, 2011) with students, a conceptual understanding is essential: this would allow them to link number sense and the algorithm, considering one of the number sense dimensions to be the meaningful use of symbols and mathematical language. Developing such conceptual understanding and multiplicity of connections in students' knowledge is only possible if teachers themselves have the knowledge that would allow them to pose mathematically rich problems, exploring them in a mathematically demanding practice (Ribeiro & Carrillo, 2011).

Teachers' knowledge is perceived with its particularities associated with the specificities of the tasks of teaching, considering that such tasks are developed with the aim of allowing students to understand what they do, why, and what it is for. In the context of this work, we consider the MTSK conceptualization (Carrillo et al., 2013), aiming also at contributing to enrich the knowledge of the content of its sub-domains. Such sub-domains are perceived as a relevant starting point for designing tasks for the mathematical preparation of teachers and for doing research on what inputs to teacher training and teacher knowledge produce effects on practices and students. The MTSK conceptualization considers three sub-domains in both Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). Besides needing a rich and ample knowledge of examples, strategies, and representations for problem solving (Chapman, 2012), teachers need also to possess a specific mathematical knowledge that would allow them to pose mathematically rich problems that do not concern PCK. Here we will focus only on the SMK sub-domains: Knowledge of Topics (KoT), Knowledge of the Mathematical Structure (KSM) and Knowledge of Mathematics Practice (KMP).

Knowledge of Topics (KoT) includes teachers' knowledge linked with, amongst others, phenomenology, meanings, definitions and examples. It relates to dimensions characterizing the specific mathematical content, complementary to the disciplinary mathematical content. Considering division, teachers' knowledge needs to include knowing how to perform the operation (with or without using an algorithm), having interpretative knowledge that would allow them to give sense to students' answers and comments, and to anticipate such possible answers outside their own primary space of solutions (Ribeiro et al., 2013). It corresponds, for example, to knowing that 44.7 is an incorrect answer for 536:12 when using an algorithm, as well as to the (im)possibility of using, in the problem for the previous expression, contexts linked with different variables (continuous or discrete) or different types of problems ((im)possible, with or without a mathematical context, with (in)sufficient data).

The Knowledge of the Mathematical Structure (KSM) concerns teachers' knowledge of an integrated system of connections, allowing them to understand and develop advanced concepts from an elemental standpoint and elemental concepts from approaches considering an advanced mathematical standpoint. Concerning problem posing and division it is related, amongst other things, to number sense and operation (e.g., Slavit, 1999) as well as to the mathematical whys that allow for considering division as measurement, addition, subtraction or multiplication. It includes also, for example, possible connections between division and fractions and the existence of quantities expressed in terms of infinite repeating or non-infinite repeating decimal and the possibilities of its representation in terms of measurement of a segment – linking it with the History of Mathematics.

The Knowledge of Mathematics Practice (KMP) includes knowing and creating mathematics (in the sense of the syntactic knowledge of Schwab, 1978), as well as aspects concerning mathematical communication.

When thinking in problem posing when a given division expression is presented, teachers' knowledge involved – both accessed and potentialities for its development – is aligned with the possible connections (nature, type) and with the advanced and elemental concepts it allows to be explored, leading to the creation of "new" mathematical knowledge and awareness. In the context of division, such knowledge is linked with number (and operation) sense and indirectly with fractions, and as these aspects/topics are pervasive in many other mathematical domains they are crucial and strategic mathematical topics of inquiry (Ribeiro et al., 2013). In that sense it is fundamental to discuss with teachers the choice, importance and role of the contexts chosen at the time of posing problems, allowing considering problem posing as a way of identifying the conceptual understanding of students (Silver & Cai, 1996). It is also fundamental at the moment of allowing teachers (and students) to pose problems and in giving sense to their productions (Ribeiro et al., 2013).

CONTEXT AND METHOD

We present part of a broader study aimed at obtaining a deeper understanding of teachers' knowledge and abilities and of conceptualizing tasks allowing the development of such knowledge.

Here we focus on data gathered during a non-compulsory course for early years prospective teachers (PTs) on their last year of graduation (3rd). (In order to become teachers (from kindergarten or primary) all prospective teachers are required to complete a Master's degree – two semesters for kindergarten and three semesters for primary (from Year 1 to Year 6)).Eighteen prospective teachers from a Portuguese University participated in such a course. The focus of the course was on teachers' specialized knowledge on different mathematical topics and its "evolution" from kindergarten till primary (students aged 3 till 12). The first author was the lecturer of such a course and all the classes were audio and video recorded and all the prospective teachers' written productions were scanned at the end of each class. One of the proposed tasks concerned division and problem posing. The first part of the task asked prospective teachers to find the solution for a set of given divisions (with or without using an algorithm – it was a solver option) and afterwards to pose a problem that could be solved using such expressions, indicating also the grade level they consider to be most adequate to pose such a problem. The second part of the task required them to solve the problems they had posed as they assumed students would solve them and on the third part, a set of students' productions was given and prospective teachers were required to give sense to those non-standard solutions – this last part was designed following previous work on interpretative knowledge (Ribeiro et al., 2013). After the three parts were solved (individually) a mathematical discussion was orchestrated in whole group.

Our focus here will be on the first part of the task (find the result to a given expression and pose a problem that could be solved using such an expression). Solving the first question of the first part of the task (find a solution to the expression) requires knowledge that, supposedly, any Portuguese fifth grader should have. The starting amount to be divided was always the same (536) and here, due to space constraints, we will focus only on the second (of five) given expressions (536:12). The choice of this particular expression (where the result is an infinite repeating decimal) is justified by the possibilities for obtaining a deeper understanding of prospective teachers' knowledge of division and number sense (in particular concerning the links with infinite repeating decimal numbers) as well as the knowledge involved in posing problems. For the kind of problem they considered, we started using Leung and Silver's (1997) categorization, but from the specificity of having a starting situation a new category emerged. Thus, we started looking at problems posed by prospective teachers, considering that they could be: (i) *not a problem* (a loose sentence or just a description – Peter has 536 marbles and 12 cookies); (ii) *not a mathematical problem* (it contains a questions not directly linked with mathematics – Peter is going to travel from Faro to Coimbra, corresponding to 536 km in 12 hours. In what country is Peter?); (iii) *impossible* (no answer can be given, even with complementary information – Peter's school has 536 students and he wants to divide them equally into 12 classes. How many students in each class?); (iv) *insufficient* (when it can be solved with complementary information – Peter has 536 toys to share with 12 friends. How many toys does each friend get? (Does Peter have some left?)) and (v) *sufficient* (it can include some extra information – Peter owns a company with 12 employers, each producing 536 litres of biodiesel each hour. If such production is going to be split amongst 12 recipients for selling, what is the minimum capacity of each recipient?). The newly emergent category concerns the correspondence between the given starting situation (in our case the given division) and the problem posed. We consider it as a *lack of correspondence*.

In the following section we start by presenting and discussing prospective teachers' answers to the expression (in the sense of knowing to find a correct answer). The answers contain rich evidence of their MTSK of division. Then we discuss the problems posed, their nature and links with prospective teachers' MTSK.

RESULTS AND DISCUSSION

When finding the solution to the operation 536:12, all PTs used *the* algorithm (the one traditionally used in Portugal) and presented three different types of solutions, using four types of different numerical representations: (a) 44 (seven PTs); (b) 44.7 (2 PTs); (c) $44.\bar{6}$; 44 (6) or 44… (six PTs); 44.66 (three PTs). The three answers in (c) are considered of the same type as we assume they intend to represent the same quantity (operational representation, in the sense of Subramaniam and Banerjee (2011), considering the answer as an infinite repeating decimal). Such diversity of solutions reveals different aspects of these prospective teachers' mathematical knowledge included at KoT and linked with number sense and operation (e.g., Slavit, 1999). It is surprising that despite using the algorithm, two of the PTs present as a correct answer for the given division 44.7. Such results call our attention to the need for a deeper discussion of the focus of the given training (they had already taken one course focusing on numbers and operations), as being able to give a correct solution for such division is a content of year five, which these prospective teachers will be teaching in some time.

Concerning the second part of the task (to pose a problem that can be solved using the given division), all prospective teachers tried to pose a problem that they considered adequate to the given expression. Most of prospective teachers also considered the posed problems to be adequate for a much higher grade level than they really were, according to the National Curriculum. Analysing and reflecting upon the posed problems gives some insights on these prospective teachers' mathematical knowledge, mainly concerning number sense and operation; the role of the considered variables and the (in)ability to connect the two stages of the task (most of the correct answers to the division did not lead to a correct problem).

Although all the prospective teachers posed problems, 13 of them posed impossible problems. Such impossibility comes from the considered contexts/variables (e.g., *Peter's school has 536 students and he wants to share them equally into 12 classes. How many students in each class?*). Such impossible problems were found to be associated with the three types of solutions, (a), (b) and (c), previously mentioned, revealing some critical features of the mathematical knowledge that would allow them to connect different kinds of variables (outside the space of statistics – also a specific "theoretical" content of one of the courses they had already taken), different ways of representing the same quantity, and the possible problems to pose in order to explore such quantities (KFLM).

Only one prospective teacher (the one who gave 44 as an answer) posed an *insufficient* problem (*Tiago's mother has 12 children, including Tiago. She received a bonus at work (536 euros) and wants to share the money amongst her twelve children. How many euros does each child get? And how many euros are left to the mother to buy herself a t-shirt?*). It is considered an insufficient problem, to which different kinds of connections can be associated (KSM) – the multiplicity of possible answers is linked with the decimal basis of the euro, but also with the fact that, physically, there are only hundredths (cents) – and thus it is also a good candidate for a starting point for elaborating tasks allowing discussing and developing teachers' interpretative knowledge (e.g., Ribeiro et al., 2013), leading also to a deeper discussion and understanding of teachers' KSM (e.g., functions and optimization).

The remaining four prospective teachers posed problems included in the *sufficient* category. In the first part of the task all of these prospective teachers gave as an answer an infinite repeating decimal, but when posing the problem only one of them made the correspondence between the given expression, the solution presented and the problem posed. That led to a new sub-category in the sufficient problems: *absence of correspondence*. A typical example of problems included in such a sub-category is: *A baker made 536 bread loaves and divided them amongst 12 bags. How many loaves are there in each bag? Are some left out?* From a "simple" mathematical point of view (and considering that it was proposed as adequate for year 4) the problem contains sufficient information to be solved, but it requires integers as answers. If aimed at expanding the domain of integers, it would also be a good candidate for developing professional learning tasks (Smith, 2001), focusing particularly on connections with fractions and exploring their pervasiveness in different topics (KSM). The only problem posed that could be considered as sufficient and with correspondence was: *There are 536 litres of water for 12 gardens. How many litres does each garden receive?*

These results, revealing the need for a change in the focus of teachers' training, reinforce Tichá and Hošpesová's (2013) ideas on the need for focusing on problem posing in teachers' training in order to allow the introduction of prospective teachers to the teaching of mathematics, and in particular to the development of their MTSK that would enable them to create an awareness of its role (as teachers) and of the role of their knowledge in and for such teaching.

CONCLUDING REMARKS AND FUTURE POSSIBIBLE FOCUSES

The first part of the task gives insights on PTs knowledge (difficulties) concerning number sense (Slavit, 1999). The problem posing part reveals PTs' difficulties concerning mainly the role and connections of the kinds of variables and contexts considered (and their implications for solving the problem with correspondence to their own solution to the given expression). These kinds and natures of the identified aspects of knowledge "should" also inform on aspects to be the focus of change in the teachers training program. The PTs knowledge revealed, both on division and problem posing, is mainly included in KoT, and albeit it should have been developed during their period as students – as problem posing (and solving) should be one of the core aspects in a mathematical class – it was not knowledge developed during their training, which will limit the learning opportunities these PTs will provide to their students, continuing the vicious circle. In order to allow PTs to allow their future students to develop the ability to solve problems, good problems need to be posed and explored, involving contexts and mathematically demanding practices (Ribeiro & Carrillo, 2011).

The need for further research seems evident, as well as its potential for developing a deeper understanding of the content of the different sub-domains of MTSK – aimed both at enriching the conceptualization theoretically and in conceptualizing tasks for accessing and developing such knowledge. A focus on problems sufficient and with (an absence of) correspondence would also allow for the opening of a window for accessing and developing teachers' knowledge on posing problems and complementarity and on giving sense to problems posed by others, contributing to developing their knowledge on giving sense to the reasoning that leads to such elaboration (Ribeiro et al., 2013). It would be, thus, a good starting point for elaborating professional learning tasks (Smith, 2001), allowing discussing teachers' MTSK in contingency moments (Rowland, Huckstep, & Thwaites, 2005), contributing to focusing training on where it is actually needed.

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References

Carrillo, J., Climent, N., Contreras, L. C., & Muñoz-Catalán, M. C. (2013). Determining Specialized Knowledge for Mathematics Teaching. *Proceedings of CERME 8* (pp. 2985-2994). Antalya: ERME.

Chapman, O. (2012). Practice-based conception of secondary school teachers' mathematical problem-solving knowledge for teaching. In T. B. Tso (Ed.), *Proceedings of the 36th PME* (Vol. 2, pp. 107-114). Taipei, Taiwan: PME.

Cooney, T. J. (1994). Research on teacher education: In search of common ground. *Journal for Research in Mathematics Education, 25*, 608-636.

Fosnot, C. T., & Dolk, M. (2001). *Young Mathematicians at work: Constructing multiplication and division*. Portsmouth, NH: Heinemann.

Grossman, P. L. (2010). Learning to Practice: the design of clinical experience in teacher preparation. *NEA policy brief, May*.

Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematics knowledge for teaching on student achievement. *American Education Research Journal, 42*(2), 371-406.

Leung, S.-k. S., & Silver, E. A. (1997). The role of task format, mathematics knowledge, and creative thinking on the arithmetic problem posing of prospective elementary school teachers. *Mathematics Education Research Journal, 9*(1), 5-24.

Ribeiro, C. M., & Carrillo, J. (2011). Discussing a teacher MKT and its role on teacher practice when exploring data analysis. In B. Ubuz (Ed.), *Proceedings of the 35th PME* (Vol. 4, pp. 41-48). Ankara, Turkey: PME.

Ribeiro, C. M., Mellone, M., & Jakobsen, A. (2013). Prospective teachers' knowledge in/for giving sense to students' productions. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th PME* (Vol. 4, pp. 89-96). Kiel, Germany: PME.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*, 255-281.

Sharp, J., Garofalo, J., & Adams, B. (2002). Children's development of meaningful fraction algorithms: A kid's cookies and a puppy's pills. In Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions: 2002 Yearbook* (pp. 18-28). Reston: NCTM.

Silver, E., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*, 521-539.

Slavit, D. (1999). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics, 37*(3), 251-274.

Smith, M. S. (2001). *Practice-based professional development for teachers of mathematics*. Reston VA: The National Council of Teachers of Mathematics.

Stake, R. E. (2005). Qualitative Case Studies. In N. Denzin & Y. Lincoln (Eds.), *The Sage handbook of qualitative research* (pp. 443-466). Thousand Oaks: Sage Publications.

Subramaniam, K., & Banerjee, R. (2011). The Arithmetic-Algebra Connection: A Historical-Pedagogical Perspective. In J. Cai & D. Knuth (Eds.), *Early Algebraization: A Global Dialogue from Multiple Perspectives* (pp. 87-105): Springer.

Tichá, M., & Hošpesová, A. (2013). Developing teachers' subject didactic competence through problem posing. *Educational Studies in Mathematics, 83*, 133-143.

THE CULTURE OF RATIONALITY IN SECONDARY SCHOOL: AN ETHNOGRAPHIC APPROACH

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The authors introduce the notion of 'Culture of Rationality' (CQ) in order to research –based on the Toulmin Model, with an ethnographic focus and with a case study– whether argumentation standards of disciplinary affirmations exist in a regular mathematics classroom. In the class observed, the researchers discovered a CQ (e.g., regularities in terms of the type of arguments, and in the trajectories of participation) that enables foreseeing future actions of the teacher involved.

ANTECEDENTS AND THE PROBLEM PROPOSAL

One prototypical trait of mathematics is its rationality, if by this one understands the set of justification standards based upon which a certain community of mathematicians habitually sustains mathematical statements.

The research presented here is interested in studying a general phenomenon, namely determining the existence of a rationality in the mathematics classroom –as happens in disciplinary mathematics. The authors specifically raise the idea of examining whether in regular mathematics classrooms there are standards of sustentation of disciplinary affirmations that are made therein in a systematic and consistent manner (rather than just on certain points or barely raised at all). In order to gather empirical evidence, an exploratory case study is carried out (Stake, 1999); for the analysis, the authors have adopted an ethnographic focus (Berteley, 2000), whereby it is a matter of determining the possible rationality of the class studies by way of direct empirical observation and of the recovery of voices, actions and meanings of the actors. Moreover an ethnological focus is also assumed (Berteley, 2000) –resorting to a more longitudinal-type study – pursuant to which the authors seek to identify the habits and patterns related to rationality. Consistent with this cultural perspective, rationality is conceived of here as a sub-culture (contained in the school culture), which in this research is called 'Culture of Rationality'. The latter construct is part of the interpretative framework based upon which the authors identify, describe and interpret (Denzin & Lincoln, 1994) within the last paragraph of this paper, the possible components that determine and give structure to the rationality of mathematics class in the study. In order to undertake the analysis of the backings that take the shape of arguments in the class, the authors have used the Toulmin Model (1974).

There is much in the way of research on the rationality of mathematics education that is based on the Habermas Model, in turn based on epistemic, teleological and communicative rationality. Amongst others, there are those of Boero and Planas (2014) and that of Morselli and Boero (2009). There are also several works that resort of

Toulmin, for instance the work of Krummheuer (1995), that of Yackel (2002) or that of Martinez and Pedemonte (2014).

INTERPRETATIVE FRAMEWORK

Epistemic Schemes: Their definition

The mechanisms to which a person or a community habitually resorts to sustain mathematics facts, are denominated by Rigo (2013) as 'epistemic schemes'. In this paper, she defines epistemic schemes of a mathematics type (e.g., instantiations of rules) as well as epistemic schemes based on extra-mathematical considerations, such as the operational scheme, by way of which validity is granted to a rule by resorting to the authority of mathematics.

Functional interpretation of arguments: The Toulmin Model

According to the Toulmin Model (1974), an argument is comprised of three elements, namely: Claim (C, a conclusion whose merits are attempting to be established), Data (D, facts to which one appeals in order to provide the foundations of the affirmation) and Warrant (W, by way of which one can account for the rules, principles or licenses of inference that authorize going from one evidence to an affirmation). Backing also exists (B, and it supports the guarantee offering its theoretical, practical or experimental foundations).

The Culture of Rationality: A characterisation

The following are, inter alia, the components of the Culture of Rationality:

- CR.i. *Standards of sustentation.* The baggage of arguments that a community habitually activates in order to sustain affirmations or mathematics facts. They is a matter of recurring practices and the most well accepted practices of argumentation or sustentation that arise in a community. The arguments are integrated by the epistemic schemes (both mathematics and extra-mathematical) that appear in C, D, W and B.
- CR.ii. *Trajectories of participation and distribution of responsibilities.* This refers to the person who gives out the C, D, W and B, and sanctions those participations. The trajectories of participation are made up of a succession of interventions of the class actors in the argumentation process.

TECHNIQUES AND INSTRUMENTS OF EMPIRICAL RESEARCH

The research reported here is of the interpretative (Denzin & Lincoln, 1994) and ethnographic type (Berteley, 2000); it is furthermore based on a case study (Stake, 1999). In order to determine the case, observations were made without any intervention of three teachers in a school that has a reputation for academic prestige in the area. Teacher Noemí, who has two years of service experience, was chosen because she presented the greatest tendency toward mathematical justification. When she was observed, she was teaching a mathematics class to a group of first year secondary school students that was made up of 42 students. During the first stage, she was observed throughout 5 sessions, while during a second stage she was observed during 6 sessions (that were analysed in this paper), and during a third stage she was again observed during another 5 sessions. The classes were video-taped and transcribed.

THE CULTURE OF RATIONALITY OF A REGULAR MATHEMATICS CLASS - EMPIRICAL STUDY

For the analysis presented here, the authors examined a didactic sequence that dealt with proportional distribution. The didactic sequence, given in six modules of 50 minutes each, was fragmented into episodes. In each of said episodes, one or several arguments were proposed in order to provide backing to a single affirmation. For this report, the authors analysed 33 episodes and 68 arguments.

First level of reconstruction: One case analysis

In the mathematics classroom, what is meant by arguing or sustaining mathematics affirmations, that is what is meant by taking part in practices of rationality (and of educating oneself in that rationality and of learning on the basis of it) is created and recreated in daily school activities (Cf. Berteley, 2000; Stake, 1999). This is why in the section below the authors present the analysis of a fragment of class given by a teacher who took part in the study. In that analysis, what frequently takes place in the classroom on a daily basis is revealed and pinpointed. It is a matter of an empirical referent made up of three arguments that are part of one and the same episode, which properly illustrates the habitual means of sustentation in the class observed. The transcription of the class arguments appears in the first column of Table 1 and its functional interpretation, based on the Toulmin Model (1974), is included in the second column.

43 T: If we have to make 12 liters of lemonade, how many lemons and how many teaspoons of sugar do we need? 58 S: It's 48 lemons and 24 teaspoons of sugar. 60 S: Because they'll be doubled.	Argument 1 D1: Instantiation of an intuitive rule (IPM) → C1: For 12 liters of lemonade, we need 48 lemons and 24 teaspoons of sugar W1: Use of a rule B1: Mathematical. Properties of proportionality (basic knowledge)
76 T: Doubled? Why? Before, you had 6, now you're being asked for 12, the amount is being doubled, right? We multiply 6 by 2, 12; 24 by 2 … Liters of lemonade / Number of lemons / Teaspoons of sugar 3 / 12 / 6 6 / 24 / 12 12 / 48 / 24 (*Table that the teacher writes on the blackboard*) 83 T: We can see that it's [the quantity of liters of water] doubling, we realize that we have to	Argument 2 D2: i) D1; ii) Confidence in the conversion of register; iii) Reflection and explanation of the why of the processes and relations; iv) Semi-inductive scheme → C2 = C1 W2: Viability or justification of the validity of the a IPM (in the context of a notion of proportionality); property of the isomorphisms

		double the quantity of lemons and the quantity of teaspoons of sugar. If, for example, [the water] were tripled, if we have the data for 3 and we're asked for 9, do you agree that it's tripled? Three times three? Nine, right or not? By how much would we have to multiply the 12?	B2: Mathematical. Properties of proportionality and of isomorphisms (intermediate knowledge)
85	T:	Pay attention. Look at the table. At the beginning, they gave us 3 liters of water for 12 lemons and 6 teaspoons of sugar. If the number of liters of water rises, what happens to the lemons? Do they increase or decrease?	
86	S:	They increased.	
87	T:	Right, they went up too. If the quantity of liters of water goes up, what happens to the teaspoons of sugar?	
89	T:	Aha, they rose. If the quantity of liters of water decreases, what would happen to the lemons?	
90	S:	They go down.	
91	T:	They go down. And the teaspoons of sugar?	
92	S:	They go down.	
93	T:	That relationship is called a proportionality relationship. If one goes up, the other does too. We did it. They go up by the same amount, right or not right? If I multiply one of the items by 2, the other items has to be multiplied by 2 as well. If they decrease, they decrease equally, agreed?	
134	S:	Well I just multiply this times that, and I divide it by this [12 times 24 into 6] Liters of lemonade — Number of lemons 6 → 24 12 → 48 Liters of lemonade — Teaspoons of sugar 6 → 12 12 → 24	Argument 3 D3: Instantiation of a school rule R3 → C3 = C1 W3: Use of a rule B3: Extra-mathematical, operating

Table 1: Record of the class and analysis of arguments using the Toulmin Model.

In the first argument that appears in Table 1, a student provided as evidence (D1) instantiation of an intuitive rule (the Isomorphic Property of multiplication –IPM, for its acronym. Vergnaud, 1989); The warrant that provided backing is the use of a rule (W1), which in turn was backed (at B1) by a mathematics backing and the basic notion of proportionality. In the second argument, conveyed by the teacher, there is evidence (D2) of a different level than that of (D1) which was facilitated by the student in the preceding argument. The teacher took advantage of the student's intervention to carry out a case analysis that enabled her to introduce and justify (by way of a semi-inductive scheme at D2.iv) the concept of proportionality. In order to accomplish this, she based herself on three considerations related to said notion of proportionality, namely: that if quantities of a space of measure are doubled or tripled, the same has to be done with the other spaces of measure (IPM for basic cases, at 83); that the quantities of spaces of measure either increase or they decrease (a property of the isomorphisms that, albeit not defining proportionality, is generally used in basic education as an essential hallmark, at 85-92); and that said variation between spaces occurs in "couples", perhaps referring to the case of IPM application to a scalar (93). The connection between evidence and affirmation was sustained in W2, composed of an epistemic scheme, that of IPM viability, which was supported by mathematical backing (B2), specifically, in proportionality properties which imply a medium knowledge. The third argument was provided by a student, who used a school rule as evidence (D3), rule of three (R3); the argument's warrant (W3) was the use of that school rule and the backing (B3) an extra-mathematical operational scheme, since the application of said rule was supported by the confidence the boy had in mathematical formulae. Table 2 shows a summary of this fragment of class.

C	D (Data)		W (Warrant)	B (Backing)	NA
2 (St)	St	Instantiation of an intuitive rule IPM	Used IPM	Proportionality Rules (basic knowledge)	IRI-M
	T	Instantiation of an intuitive rule IPM; Confidence in the conversion of register; Reflection and explanation about the why of processes and relations; Semi-inductive scheme	Viability or justification for the validity of IPM(in the context of a proportionality idea)	Proportionality Rules (basic knowledge)	V-M
	St	Instantiation of a school rule R3	Used R3	Operational scheme	IRE-EM

Table 2: Summary of the analysed episode. (C: Claim, NA: name of argument, T: teacher, and St: student).

Salient characteristics of the rationality practices in Noemí's class are then: Trajectory of participations (first a student, who suggested the affirmation and first evidence; next the teacher, who enriched and deepened the student's statement with her argument; then another student, who provided different evidence); the quality of participations (rule instantiation, in the case of the children; conceptual explanation of rules or of rule usage, or an explanation of their viability, from the teacher); and the type of arguments (mostly, although not always, backed by mathematical considerations). The episode under examination also provides clarity about how the teacher negotiates her own rationality practices —an objective that, by way of dialogical exchange, involves the students by means of constant questions, not only about what but also about why— and how this enculturates her students in that rationality.

The features of practices of rationality in this section are notable, in the next one they will be substantiated with an additional level of analysis, a numerical one. This examination is intended to show that the features described here are a concrete expression of the sustentation patterns that delineate and make up the Culture of Rationality of the observed class.

Second level of reconstruction: Pattern identification

The recurring arguments formulated in Noemí's classroom are described in table 3. Also included in that table are the type of backing on which the argument (mathematical or extra-mathematical) is supported, the frequency with which it occurred in class, and the actor who formulated. The relatively high incidence of these arguments in the observed class suggests ties with sustentation rules (CR.i) which shape and update the Culture of Rationality in said class.

Argument	Description of the Argument	Backing	Teacher	Students
IRI	Instantiation of an intuitive rule (IPM or unit value)	Mathematical	3	20
IRE	Instantiation of a school rule (R3)	Mathematical Extra-mathematical	1 3	0 3
EGCP	Generic conceptual explanation of the process. Processes that can be generalized to other cases are explained, and mathematical content and meanings are made explicit.	Mathematical	3	0
EGUR	Generic conceptual explanation of the usage of a rule. The functioning of a rule is made explicit.	Mathematical Extra-mathematical	9 5	5 0
V	Viability or justification of a rule or a process.	Mathematical	4	1
RAI	Repetition for institutional endorsement	Mathematical Extra-mathematical	3 1	0 0
R	Reasonable considerations	Mathematical Extra-mathematical	2 2	0 0
CRI	Check of an intuitive rule	Mathematical	3	0

Table 3: Recurring arguments in the observed class.

Another rule that possibly makes up the Culture of Rationality of the observed class (CR.i, and that also emerges from the analysis of quantities in table 3) makes reference to a balanced division of the number of arguments that the students gave and those that the teacher gave: while nearly 45% were provided by them, the remaining (55%) were given by the teacher. It is interesting that 80% of the arguments given by students involve the instantiation of a rule.

Without any doubt, one of the most salient features of the rules of sustentation (CR.i) of the Culture of Rationality that reigns in Noemí's class is the observed tendency towards backing based on mathematical considerations: of the 39 arguments given by the teacher, 28 of them had mathematical backing (nearly 72%); of the 29 arguments provided by the students, 26 had mathematical backing (nearly 90%), and of the 68 arguments in total, 54 (nearly 80%) were backed by mathematics. It stands out that nearly 12% of the arguments are conceptual (V and EGCP).

The predominant or more remarkable trajectories of participation in Noemí's class appear on Table 4 (CR.ii). The first column denotes the first argument that was given in class for backing a mathematical statement and the second, the argument that followed it and the basis for justifying that statement.

Student	Teacher	Totals
IRI-M	IRI-M	1
	EGUR-M	2
	EGPR-M	2
	V-M	2
	RAI-M	3
	CRI-M	1
IRE-EM	EGUR-EM + R-EM	1
	EGUR-EM	1
	RAI-EM	1
EGUR-M	EGUR-M + R-M	1
	EGUR-M	3
EGUR-EM	EGUR-M	1
Student	**Student**	**Totals**
IRI-M	V-M	1
Teacher	**Teacher**	**Totals**
IRI-M	R-EM	1
EGUR-EM	V-M	1
IRE-EM	CRI-M	1
EGUR-M	CRI-M	1

Table 4: Trajectories of participation

In that Table 4 it is possible to detect some regularities that, as in previous cases, very possibly define one of the patterns of the Culture of Rationality in Noemí's class (in CR.ii): in nearly 56% of the arguments (38) that were formulated in the didactic sequence, the students provided the first evidence, with the intention of letting the type of strategy they used to back their affirmation be known; in these cases, the teacher's participation ensued, who, as previously mentioned, provided depth with her complementary comments, with the purpose of endorsing the child's participation, even if not directly, as well as to explain and make relations and concepts involved in the rules used by the student explicit (for which she resorted to the EGUR and the EGCP, among others).

FINAL REMARKS

Using the ethnographic analysis shown here, based on the notion of Culture of Rationality, there is interest in discovering the rationality that –from the necessarily interpretational view of the investigators– dominates the ordinary mathematics classroom, in attempts to distinguish what occurs there from what is desirable.

Without a doubt it would be ideal for students in a classroom to be educated in the Culture of Rationality that school mathematics define in the curriculum, and that as students make scholarly progress, this rationality leans more towards that of disciplinary mathematics. However, to achieve these objectives, it is currently essential, among other things, that teacher and those involved in his formation approach the classroom with an open view so that they may discover and take conscience of the rationality that truly dominates the classroom and based on which students are enculturated (Morselly & Boero, 2009) on a daily basis.

Every school and every mathematics class has its own culture in terms of the knowledge, beliefs and values of its participants. The culture will differ from one class to another. Although with this in mind it is 'rationally' to be expected, it continues to be astounding how well articulated her practice is and the clear and recognizable presence of a rationality structure, with systemic and clear backing rules, and trajectories of participation that are consistent with said rules. This offers the junction between the terrain of pure description and that of prediction, because although the culture is not deterministic, the regularity in the teacher's practices –integrated and organized in the Culture of Rationality as an interpretative construct– they provide the well founded possibility of foreseeing the actions and decisions of the mentor from a reasonable range of options. This way, the Culture of Rationality and the theoretical-methodological instruments suggested and applied here may become useful tools for teachers and the people involved in their formation to achieve the objectives posed in the preceding paragraph.

References

Berteley, M. (2000). Conociendo nuestras escuelas (Knowing our Schools). México D. F.: Paidós.

Boero, P., & Planas, N. (2014). Habermas' construct of rational behaviour in mathematics education: New advances and research questions. In P. Liljedahl, C. Nicol, S. Oesterle & D. Allan (Eds.), Proc. of the Joint Meeting of PME 38 and PME-NA, 36 (Vol. 1, pp. 205-235). Vancouver, Canada: PME.

Denzin, N. K., & Lincoln, Y. S. (1994). Introduction. In N. K. Denzin & Y. S. Lincoln (Eds.), Handbook of qualitative research (pp. 1-18). California: Sage Publications.

Krummheuer G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), The emergence of mathematical meaning: Interaction in classroom cultures (pp. 229-269). Hillsdale, NJ: Lawrence Erlbaum Associates.

Martinez, M. V., & Pedemonte, B. (2014). Relationship between inductive arithmetic argumentation and deductive algebraic proof. Educational Studies in Mathematics, 86(1), 125–149.

Morselli, F., & Boero, P. (2009). Proving as a rational behaviour: Habermas' construct of rationality as a comprehensive frame for research on the teaching and learning of proof. In V. Durand, S. Soury y F. Arzaello (Eds.), Proceedings of the 6th Congress of the European Society for Research in Mathematics Education (CERME 6), pp. 211-220, Lyon (France).

Rigo, M. (2013). Epistemic schemes and epistemic states. A study of mathematics convincement in elementary school classes. Educational Studies in Mathematics 84(1), 71-91.

Stake, R. E. (1999). Investigación con estudio de casos (Investigation with case study). Madrid: Ediciones Morata.

Toulmin, S. E. (1974). The Uses of Argument. New York: Cambridge University Press.

Vergnaud, G. (1989). Multiplicative structures. In J. Hiebert & Behr (Eds.), Number concepts and operations in the middle grades (Vol. 2, pp. 141-161). Reston: NCTM.

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation? Journal of Mathematical Behavior, 21(4), 423–440.

"BORROWING FROM THE NEIGHBOUR" - PRESERVICE TEACHERS' INTERPRETATIONS OF STUDENT ERRORS

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Using Variation Theory, preservice teachers' specialized content knowledge (SCK) of place value was qualitatively analysed. The notions of discernment, critical features, and critical aspects helped to characterize PSTs' SCK by revealing the variety of ways they interpreted a student's erroneous solution to a subtraction task.

INTRODUCTION

Building upon Shulman's notion of pedagogical content knowledge (1986), Ball, Thames, and Phelps (2008) developed the Mathematical Knowledge for Teaching framework. The framework identifies three discrete but related subdomains of both subject matter and pedagogical content knowledge. One of the subdomains of subject matter knowledge is referred to as Specialized Content Knowledge, or SCK. It describes the knowledge needed by teachers to analyse students' solutions, recognize the existence of multiple solution strategies, understand unusual student solutions, and ultimately analyse student errors for misconceptions (Ball et al., 2008). Teachers use their SCK to analyse and make sense of students' approaches and errors in order to maximize the learning potential of tasks (Sullivan, Clarke, & Clarke, 2013). Because selecting tasks and analysing students' thinking on those tasks is a significant aspect of teaching practice, determining approaches to support preservice teachers (PST) in developing SCK is important. While there have been attempts to characterize and improve PSTs' SCK (e.g., Morris, Hiebert, & Spitzer, 2009; Rayner, Osana, & Pesco, 2013), the literature lacks a coherent framework for examining PSTs' SCK to analyse student errors. We propose Variation Theory (Marton & Booth, 1997) as one such framework; it is a versatile theory of learning and has been used to inform task development for PSTs (Nicol & Bragg, 2009), analyse the mathematical content implemented in lessons (Fermsjö et al., 2014), and assess student learning following a lesson (Olteanu, 2014).

This paper highlights the potential of Variation Theory as an analytical tool to examine PSTs' SCK of place value. It draws upon data collected from a larger mixed methods study focused on understanding the role of conceptual and procedural knowledge in how PSTs made sense of a hypothetical student's erroneous solution to a subtraction task. This paper extends that research to examine PSTs' knowledge from a more expansive qualitative perspective. It seeks to better understand the kinds of interpretations PSTs offered and in doing so provides insight into how PSTs might be supported in their learning of SCK in their teacher education programs.

REVIEWING THE LITERATURE

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 97-104. Hobart, Australia: PME.

SCK requires conceptual knowledge, procedural knowledge, and an understanding of how conceptual and procedural knowledge are interrelated (Ball et al., 2008). It seems reasonable, therefore, to examine the development of SCK in terms of conceptual and procedural knowledge. To better understand PSTs' SCK, Royea and Osana (2012b) designed a study to determine the effects of various lesson sequences on elementary PSTs' knowledge of place value and multi-digit arithmetic. Building on the mathematical knowledge acquisition research on K-12 students (Rittle-Johnson & Koedinger, 2009), the study compared the effects of three different lesson sequences (teaching concepts before procedures, teaching procedures before concepts, and iterating teaching between concepts and procedures) for developing PSTs' conceptual and procedural knowledge, and SCK. Using a three condition pretest-posttest experimental design, the results of the study were analysed both quantitatively (Royea & Osana, 2012b) and qualitatively (Royea & Osana, 2012a). Where statistically significant changes in the PSTs' knowledge were revealed by the quantitative analysis, qualitative analyses were performed to better understand the nature of these improvements. More specifically, Grounded Theory techniques (Corbin & Strauss, 2007) were used to openly analyse the data to look for common themes across the PSTs. The themes that emerged were then used to code the data. For SCK, a statistically significant main effect of time was revealed and four qualitative themes emerged from the PSTs' interpretations of the hypothetical student's work on a subtraction task. That is, the PSTs' explanation of the student's work was either: (a) strictly procedural in nature, as indicated by PSTs' use of terminology associated with a procedural understanding of place value such as "borrowing" or "taking from" and references to procedural steps; (b) strictly conceptual, as indicated by terminology associated with conceptual understanding, such as "regrouping" and explanations about conceptual rationales; (c) a combination of procedural and conceptual explanatory elements, but with no connections made between the concepts and procedures; or (d) a combination of procedural and conceptual elements with connections between the concepts and procedures used (Royea & Osana, 2012a).

Although examining PSTs' SCK strictly in terms of conceptual and procedural knowledge provided some understanding of the nature of their knowledge, such a focus is also limited. First, while being distinct at certain levels, conceptual and procedural knowledge are closely related and have been shown to be difficult to separate in practice (Hiebert & Wearne, 1996). Therefore, it is simplistic to conclude that PSTs lack conceptual knowledge or coordinated conceptual and procedural knowledge when they use exclusively procedural language in articulating their thinking. Among other explanations, it is possible that their procedural language was learned from previous instruction, and is thus the most direct way for them to express their thinking. This account this does not preclude their understanding of more than the procedures they tend to favour in their explanations. Furthermore, aside from potential flaws in the criteria used by Royea and Osana (2012a) to investigate the nature of the knowledge used by PSTs in their interpretations of the student's response, the Grounded Theory analysis did not provide a complete picture of the PSTs' SCK either. Even though Suzuka et al. (2009) explicitly delineate the important role such knowledge plays in teachers' interpretations of students' mathematical productions, the Grounded Theory analysis did not reveal the variety of ways the PSTs interpreted the student's misconceptions. This led us to the question: What other analytical tools are available to describe PSTs' SCK as measured by their analysis of a hypothetical student's erroneous solution to a multi-digit subtraction task?

THEORETICAL FRAMEWORK

As a type of knowledge uniquely required for the work of teachers, SCK plays a vital role in teaching for understanding. The SCK tasks relevant to the present study include being able to understand the conceptual underpinnings of mathematical procedures, acknowledge multiple solution strategies, interpret students' mathematical productions, and use appropriate mathematical language. Variation Theory has promise as an analytical tool to describe these aspects of PSTs' SCK. Variation Theory views learning as a function of the learner's awareness. It places particular emphasis on how the learner's attention is drawn to aspects of the *object of learning* (Marton & Booth, 1997). Learning is characterized as seeing an *object of learning* with a more thorough understanding, within a wider perspective, or even in a completely new way (Lo, 2012). Discernment, simultaneity, and variation are three integral components of Variation Theory (Marton & Booth, 1997). As an analytical tool, the notion of discernment is of particular interest. Discernment refers to an ability to go from a holistic experience of an object to experiencing the different parts or features of an object. All objects have a multitude of features. The way any given object is understood is determined by which critical features are placed in focus. As a result, teachers and students may see the same object in dramatically different ways. To create the necessary conditions for students to understand an object in the way the teacher desires, identification of the object's critical features is needed. While a critical feature is the value of a dimension of variation, a critical aspect refers to a dimension of variation that needs to be developed to see an object in the desired way (Marton & Booth, 1997). For example, to understand ¼ as a part-whole relationship, at least two aspects of the fraction, or *object of learning*, need to be discerned: (a) the numerator represents the number of parts being referred to; and (b) the denominator represents the number of parts into which the whole is equally divided. Some students, however, see "¼ of a pizza" as *one pizza divided into four parts* instead of *one part of a pizza that is divided into four parts* (Mack, 1995). For these students, their understanding of the numerator represents a *critical aspect*.

A teacher's ability to identify and draw out critical features of something to be learned affects student learning. At the same time, discerning critical features can be particularly difficult for a teacher because these features are often taken for granted (Lo, 2012). Therefore, in addition to having a deep understanding of mathematics, teachers must be able to discern the critical features of an object of learning and furthermore identify the critical aspects responsible for student learning difficulties. In terms of SCK, this means that teachers would draw upon this form of knowledge to adequately discern critical features and aspects from students' mathematical work. Analysing students' mathematical productions to identify their understandings and misconceptions requires identifying the critical features and aspects of a specific mathematical task. Although Marton and Booth (1997) indicate that teachers should empirically determine critical features through carefully designed assessments, Lo (2012) points out that most critical features can be discerned through careful reflection and analysis of students' work.

For our study, we described the PSTs' SCK in terms of the critical features and aspects discerned while performing an error analysis of a subtraction problem with regrouping. The participating PSTs analysed the hypothetical student's solution before and after completing the intervention in the larger study (Royea & Osana, 2012b). Only the PSTs' interpretations of the student's work after the intervention were analysed here using Variation Theory to explore how the theory may be used to provide a more complete portrait of their SCK.

METHODOLOGY

Participants were 31 PSTs enrolled in a four-year elementary teacher education program at an urban, English language university in Canada. All the PSTs were registered in the first of three required teaching mathematics methods courses and had completed the intervention as described in the Royea and Osana (2012b) study. The majority of the PSTs were female ($n = 30$) with the ages ranging from 19 to 43 years. All of the PSTs but one reported having some teaching experience in the form of internships, tutoring, or classroom teaching. The item that was used for this extended analysis was a task designed to assess a specific aspect of the participants' SCK: their ability to interpret an elementary students' erroneous solution to a multi-digit, vertically presented, subtraction with regrouping problem (see Figure 1).

The standard procedure for the operation in the subtraction task is to decompose the hundred into 90 tens and 10 ones, add the 10 ones to the existing 4 ones, and then subtract the 9 ones to arrive at a difference of 95. In the case of the hypothetical student depicted in the written work (Figure 1), it appeared that the student either decomposed the hundred into a group of 10 or may have decomposed the hundred into tens and ones but neglected to record the number of groups of ten that remained. Being able to recognize if the student's answer was correct or incorrect requires common content knowledge (Ball et al., 2008), but analysing the student's mathematical production to determine the nature of the error requires flexible thinking and meaning making that is unique to SCK (Ball et al., 2008; Suzuka et al., 2009). We examined the PSTs' written analyses of the student's solution to identify the critical aspects and features of the subtraction problem that were discerned by the PSTs. To demonstrate adequate SCK, PSTs were required to provide a reasonable explanation of the student's work as evidenced by the student's mathematical production while also identifying the critical aspect that needs to be developed for the student. In this case, the critical aspect for the student is the need to recognise the positional values of the numbers at regrouping. We took detailed notes about the features and aspects discerned by the PSTs as indicated by their written interpretation of the student's work.

Instructions: Look at the solutions below produced by an elementary school student. Indicate if the student got the right answer. If the student solved the problem correctly, explain the steps used. If the student solved the problem incorrectly, describe the mistake(s) made by the student.

The student's work: $\begin{array}{r} 104 \\ -\ \ 9 \\ \hline 5 \end{array}$

Figure 1. Multi-digit Subtraction SCK Item

RESULTS

SCK includes the knowledge and skills required to be able to unpack and repack mathematical knowledge in order to appropriately evaluate and address students' mathematical understandings, misunderstandings, and "why" questions. When we used the notions of discernment, critical features, and critical aspects to analyse the PSTs' interpretations of the student's solution, we uncovered elements of their SCK that were not made explicit by our previous Grounded Theory analysis. Specifically, examining the PSTs' responses in term of critical features revealed the variety of ways that PSTs actually interpreted the student's erroneous solution.

PSTs either analysed what the student had done using the evidence provided in the student's work, described what the PST thought the student should have done, or provided an unacceptable response. Fourteen of the PSTs' responses were judged to have adequately demonstrated appropriate SCK in their analysis of the student's work. Responses that we considered adequate demonstrations of SCK were those that reasonably and clearly explained what the student may have done based on the limited evidence that was provided and used appropriate mathematical language. For example, the PST quoted below discerned why regrouping was required and provided an explanation that was reasonable based on the evidence provided in the student's work. There were no drastic leaps in the interpretation or errors in the mathematical language used.

> The student recognized that 9 could not be subtracted from 4 and therefore borrowed from the 1 in the hundreds column and the 0 in tens column at the same time in order to add ten ones to the 4 in the ones column but the student forgot or didn't know to write down that there are nine tens left in the tens columns. Because of that, the answer is off by 90.

Most of the PSTs discerned at least some of the critical features, even if their responses did not always reflect well-developed SCK. Table 1 presents the frequencies of the types of interpretations of the student's work.

Type of PST Response	Description	Frequency
Analysed Student Response based on Evidence Provided	Regrouped tens and hundreds simultaneously	14
	Thought was borrowing tens rather than hundreds	2
Described What Student Should have Done	Two phases of regrouping required	13
Unacceptable	Inaccurate/Unclear	2
	Total	31

Table 1: Distribution of Features of Student Solution Discerned

PSTs' responses that analysed the student's work based on the evidence provided were further categorized as either interpreting that: (a) the student regrouped the tens and hundred simultaneously but did not record the number of tens that remained; or (b) the student thought he/she was borrowing from the tens but borrowed from the hundreds. Fourteen of the PSTs in this study interpreted the student's work as containing the regrouping error described in (a) and 2 interpreted the student's work as borrowing from the wrong column as described in (b). Below is an example of a PST who indicated that the student had regrouped the tens and hundreds simultaneously. We did not consider the response adequate in terms of demonstrating well-developed SCK because the mathematical language used was not sufficient.

> [the student]…borrowed from the neighbour to subtract 9 from 14 but cancelled out the 100 when it should have become a 9 in the tens place.

Rather than analysing the student's work for what the student actually did, 13 PSTs described instead what the student should have done to get the correct answer. All of these PSTs indicated that to get the correct answer the student should have regrouped in two phases. That is, for the first phase the student should have regrouped the hundreds to tens, and for the second phase, from tens to the ones. For example:

> ...[the student] understood that 9 couldn't be subtracted from 4… and borrowed from the 1 hundred directly to the ones column when he should have borrowed from the 1 hundred to the tens to make the zero a ten and then from the tens to the ones to make 9 and 14. Here, they didn't have the 9 to bring down to the answer.

We suggest that describing what the student should have done indicates incomplete SCK because it does not take two important components into account: acknowledging the possibility of multiple solutions to the same task and understanding unusual student solutions (Ball et al., 2008).

Two of the PSTs provided responses to the task that were considered unacceptable because the responses were unclear or inaccurate. For example, the following PST response does not explain what the student may have done and reverses the digits in the one's place. Furthermore, it is unclear and does not use appropriate language.

> [The student] took the 1 and the 0 and perceived it as a ten when he could have just subtracted the 4 from the 9. The student does not understand that the 10 comes from the value of the number to its left and is transferred to the number on the right.

DISCUSSION

Using different theoretical perspectives to examine the same data can shed new light on research findings (Cobb & Yackel, 1996). In our study, a Variation Theory-based analysis complemented the previous Grounded Theory analysis of PSTs' SCK (Royea & Osana, 2012a). While the Grounded Theory analysis provided insight on the conceptual and procedural nature of PSTs' SCK, using the notions of discernment and critical features helped reveal the particularities of the PSTs' interpretations of the student's work. That is, the Variation Theory perspective helped characterize PST developing conceptions of SCK as their understandings of the critical features of the task emerged. Using Variation Theory in this way also provides insight on the features that remain critical to the PSTs' SCK development such as acknowledging multiple solution strategies and understanding unusual solutions. Insights on the features that remain critical to PSTs' SCK can inform teacher educator's pedagogical decisions when selecting and designing tasks that use patterns of variations to help develop PSTs' mathematical knowledge for teaching. At the same time, a single, brief, written analyses of a student's work provides only limited information about PSTs' SCK. Future research investigating a variety of tasks requiring this knowledge and extended contact and discussion with PSTs would yield more information on the potential value of using Variation Theory to analyse and develop SCK. Consistent with the view that Variation Theory can be used to better understand the relationship between mathematics, teaching, and student learning (Runesson, 2013), our work demonstrates that Variation Theory can be productively used to characterize SCK.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What mak es it special? *Journal of Teacher Education*, *59*, 389-407.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3/4), 175-190.

Corbin, J., & Strauss, A. (2007). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Los Angeles, California: Sage.

Fermsjö, R., Maunula, T., Magnusson, J., Eriksson, H., Svanteson, J., & Ryberg, U. (2014). A mathematics lesson on measurement - analysed through different lenses; learning activity and variation theory. In S. Oesterle, C. Nicol, P. Liljedahl, & D. Allan (Eds.) *Proceedings of the Joint PME 38 and PME-NA 36*, Vol. 6, pp. 69. Vancouver, Canada: PME.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in mutlidigit addition and subtraction. *Cognition and Instruction*, *14*¸ 251-284.

Lo, M. L. (2012). *Variation theory and the improvement of teaching and learning*. Gothernburg, Sweden: Act Universitatis.

Mack, M. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, *26*, 422-441.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Lawrence Erlbaum

Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education*, *40*(5), 491-529.

Nicol, C., & Bragg, L. (2009). Designing problems: What kinds of open-ended problems do preservice teachers pose? In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis. (Eds.), *Proceedings of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp, 225-232. Thessaloniki, Greece: PME.

Olteanu, C. (2014). Improving teacher education by improving the experience of the object of learning. In S. Oesterle, C. Nicol, P. Liljedahl, & D. Allan (Eds.) *Proceedings of the Joint PME 38 and PME-NA 36*, Vol. 6, pp. 188. Vancouver, Canada: PME.

Rayner, V., Osana, H. P., & Pesco, D. (2013) A comparison of instruction that develops preservice teachers' SCK of division with fractions. In M. Martinez, A. Castro Superfine (Eds.), *Proceedings of 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Chicago, IL: University of Illinois at Chicago.

Rittle-Johnson, B., & Koedinger, K. R. (2009). Iterating between lessons on concept and procedures can improve mathematics knowledge. *British Journal of Educational Psychology*, *79*, 483-500.

Royea, D. A., & Osana, H. P. (2012a). *A qualitative analysis of the effects of lesson sequencing on preservice teachers' place-value knowledge*. Paper presented the Canadian Society for the Study of Education, Victoria, British Columbia.

Royea, D. A., & Osana, H. P. (2012b). *The effects of lesson sequencing on preservice teachers' place value knowledge.* Paper presented at the Canadian Society for Brain, Behaviour and Cognitive Science (CSBBCS/SCSCC) Annual Meeting in Kington, ON.

Runesson, U. (2013). Focusing on the object of learning: A case study of teachers' inquiry into teaching and learning mathematics. *Perspectives in Education*, *31*(3), 170-183.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Sullivan, P., Clarke, D., & Clarke, B. (2013). *Teaching with tasks for effective mathematics learning*. New York, NY: Springer.

Suzuka, K., Sleep, L., Ball, D. L., Bass, H., Lewis, J. M., & Thames, M. H. (2009). Designing and using tasks to teach mathematical knowledge for teaching. In D. S. Mewborn & H. S. Lee (Eds.), *Scholarly practices and inquiry in the preparation of mathematics teachers* (7-24). San Diego, CA: Association of Mathematics Teacher Educators.

HOW CHALLENGING TASKS OPTIMISE COGNITIVE LOAD

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This theoretical paper argues that the reform in mathematics towards more problem-based learning can be made consistent with cognitive load theory through the use of carefully designed challenging tasks. It is argued that such tasks can provide the benefits of problem-based approaches whilst being cognisant of the issue of cognitive overload. Possible directions for future research are suggested.

CONTEXT: A REFORM IN MATHEMATICS TOWARDS PROBLEM-BASED APPROACHES

Over the past several years, there have been calls to reform mathematics education in Australia to increase the amount of time students spend engaged in deep problem solving (e.g., Hollingsworth, Holden, & McCrae, 2003). This reform has paralleled similar developments in other countries, particularly the United States. For example, teachers have been encouraged to utilise more cognitively demanding tasks to better engage students in rich mathematical discussions (e.g., Stein, Engle, Smith, & Hughes, 2008). As part of this reform process, it has been argued that traditional lesson structures (i.e., teacher explanation, followed by student practice and correction) are inherently inadequate for meeting contemporary mathematical learning objectives (Sullivan et al., 2014). Instead, reform-oriented teaching approaches have frequently employed a triadic lesson structure: Launch, Explore, Discuss (Stein et al., 2008). The lesson begins with the launch phase during which the teacher introduces students to the task, which generally represents a challenging problem to be solved. During the explore phase, students work on the problem, sometimes collaboratively, while the teacher provides support and guidance. Finally, after students have spent sufficient time engaged with the problem, the lesson enters the discuss phase, during which time various student-generated approaches to the problem and possible solutions are discussed. The teacher generally finishes by offering some form of summary comment (Stein et al., 2008).

There is some support for the notion that this Launch-Explore-Discuss lesson structure, which can be characterised as a form of problem-based learning, more effectively meets the contemporary aims of mathematics education. For example, there is empirical evidence to suggest that higher-order mathematical goals, such as the ability to reason and think critically, are more likely to be realised when students are given an opportunity to explore concepts prior to direct instruction (Marshall & Horton, 2011). Furthermore, building a lesson around students first tackling a cognitively demanding task may improve student persistence, as students work through the “zone of confusion” (Sullivan et al., 2014, p. 11).

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 105-112. Hobart, Australia: PME.

However, this recent emphasis on problem-based learning in mathematics is not without its critics, particularly within certain branches of educational psychology. In particular, some cognitive load theorists have argued that launching a lesson with a cognitively demanding activity, which is not explicitly linked to teacher instruction and prior learning, is problematic (Sweller, Kirschner, & Clark, 2007). This argument, which is briefly elaborated below, is based on the idea that our working memory has limited capacity to process novel information, and is therefore easily overloaded when required to solve an unfamiliar problem (Sweller, 2010).

CRITIQUE OF PROBLEM-BASED APPROACHES

It has been asserted that an understanding of human cognitive architecture should lead to the unequivocal rejection of problem-based, and other "minimally-guided", approaches to learning (Kirschner, Sweller and Clark, 2006, p. 75). Specifically, Sweller, and colleagues argue that such pedagogical approaches are less effective than traditional learning approaches that rely more on carefully scaffolded direct instruction. The relative ineffectiveness of minimally-guided approaches is thought to be due to unnecessary and irrelevant cognitive load (extraneous cognitive load) brought about by poor instructional design, and/or the overly ambitious nature of the learning objectives resulting in the cognitive load inherent in the learning task (intrinsic cognitive load) being too high (van Merrienboer & Sweller, 2005). In either case, it is contended that adopting minimally-guided approaches tends to result in cognitive overload. This unsustainably high load in turn impedes the formation of new schemas, thus undermining learning (Sweller, 2010).

This assertion outlining how cognitive load theory establishes the superiority of direct instruction over minimally-guided approaches is not uncontroversial and has attracted a number of critical commentaries (e.g., Schmidt, Loyens, van Gog, & Paas, 2007). Sweller and colleagues, however, maintain that proponents of minimally-guided approaches are choosing to ignore contemporary knowledge of human cognition when designing instruction (Sweller et al., 2007):

> The process of discovery is in conflict with our current knowledge of human cognitive architecture which assumes that working memory is severely limited in capacity when dealing with novel information sourced from the external environment but largely unlimited when dealing with familiar, organized information sourced from long-term memory. If this view of human cognitive architecture is valid, then by definition novices should not be presented with material in a manner that unnecessarily requires them to search for a solution with its attendant heavy working memory load rather than being presented with a solution (Sweller et al. 2007, p. 116).

However, the current paper will contend that Sweller and his colleagues' critique of minimally-guided approaches is an overreach, as it *does not apply* to some of the more nuanced approaches to problem-based learning that have evolved in mathematics education. Specifically, the current paper will advance several arguments in support of the notion that launching a lesson with a challenging problem is in fact consistent with our knowledge of human cognitive architecture, provided that the tasks themselves meet particular criteria. Moreover, this analysis will be couched in language and ideas central to cognitive load theory.

WHAT ARE CHALLENGING TASKS?

Sullivan and Mornane (2013) describe challenging tasks as complex and absorbing problems with multiple solution pathways. Such problems are presented to the entire class, with the teacher encouraging all students to make an attempt at the problem. After a student has spent some time in the 'zone of confusion' and remains unsure how to proceed, he or she is given access to 'just in time' support through 'enabling prompts' (Sullivan, Mousley, & Zevenbergen, 2006). Enabling prompts reduce the intrinsic cognitive load of the task through changing how the problem is represented, helping the student connect the problem to prior learning and/ or removing a step in the problem (Sullivan et al., 2006). Students who complete the problem early are given access to an 'extending prompt'. This is designed to expose the student to an additional task that is more challenging, however requires them to use similar mathematical reasoning, conceptualisations and representations as the main task.

Consequently, challenging tasks can be viewed as a subset of problem-solving tasks that meets specific criteria. Adapted from the work of Sullivan and his colleagues (e.g., Sullivan & Mornane, 2013), criteria relevant to the issue of optimising cognitive load are presented below.

The task must:

- be solvable through multiple means (i.e., have multiple solution pathways) and may have multiple solutions;
- involve multiple mathematical steps (i.e., as opposed to a single insight facilitating completion of the problem);
- have at least one enabling prompt and one extending prompt developed prior to delivery of the lesson;
- involve students having primary control over how they are able to approach the task and when they are able to access enabling and extending prompts, within some constraints established by the teacher.

HOW CAN CHALLENGING TASKS REDUCE EXTRANEOUS COGNITIVE LOAD?

This section introduces two effects discussed in the cognitive load literature which have been linked empirically with extraneous cognitive load. It is argued that challenging tasks possess particular structural characteristics that allow them to leverage off these effects, reducing extraneous cognitive load relative to more teacher-directed learning approaches.

Goal-free (and means-free) effect

One of the earliest ideas within cognitive load theory to gain empirical support was the notion that goal-free tasks can reduce extraneous cognitive load through reducing reliance on a cognitively taxing means-end analysis (Sweller, 1988). Sweller argued that the absence of established schema require individuals to problem solve through adopting a means-end analysis. Although he acknowledges this may be an efficient means of solving a problem, he argues that it places a substantial strain on working memory. Specifically, he suggests that using a means-end analysis requires the problem solver to continually hold in mind several elements simultaneously, including the original problem state, the end goal state, how the two states relate to one another, strategies and operators that could bridge the two states and any sub-goals that the problem solver needs to reach as he or she works through a problem. He suggested that this substantial extraneous load inhibits learning, because building an appropriate schema to understand how the relevant concepts interrelate and solving the problem are not compatible goals. Sweller suggested that to circumvent this issue, instructors should provide students with goal-free problems, which allow them to more comprehensively explore and comprehend a concept. Empirical support for the goal-free effect is well established within the literature (e.g., Bobis, Sweller, & Cooper, 1994).

Challenging tasks are open-ended in the sense that they may have multiple solutions. This may result in lower extraneous cognitive load, as described by the goal-free effect. Perhaps more importantly, the fact that challenging tasks have multiple solution-pathways means that they may have a lower (extraneous) cognitive load compared with traditional learning approaches, which emphasise algorithms and 'one-best method'. This may be termed a 'means-free effect'. The rationale is similar to the goal-free effect. Essentially, through ensuring that there are multiple viable pathways to a particular solution, instructors are increasing the probability that learners have some prior knowledge of strategies that can bridge the problem and solution states. Moreover, it is likely that the search time for locating an appropriate strategy is reduced, as learners only have to recall one of the multiple means of solving the problem to proceed. Similarly, there is likely to be less emphasis on reaching a specific sub-goal, and even when a particular sub-goal is still vital to solving the problem, there are almost certainly multiple pathways for reaching that sub-goal. To summarise, this enhanced connectivity between the problem and solution states reduces the cognitive load required to productively engage in the problem, and, therefore, enhances the likelihood of learning occurring.

Reducing/ removing the expertise reversal effect

Many of the mechanisms and techniques that have been associated with lower extraneous cognitive load when learners are novices have a paradoxical effect when learners are more expert (Kalyuga, 2007). For example, Renkl and Atkinson (2003) argued that as learner experience with a particular problem type increased, they should progress from worked examples, to completion problems, and finally to fully intact problems. They demonstrated empirically that attempting to provide experts with more scaffolding than they required actually inhibited their learning. This perhaps counterintuitive finding within the cognitive load literature has been termed “the expertise reversal effect” (Kalyuga, 2007, p. 509).

The expertise-reversal effect has been attributed to another phenomenon within cognitive load theory, termed the redundancy effect (Kalyuga, 2007). Specifically, requiring experts to process additional information intended to support learning but irrelevant to their learning needs unnecessarily burdens their working memory, resulting in an extraneous cognitive load. It has been suggested that, in order to reduce the expertise-reversal effect and optimise how much support is provided to learners, learning environments need to be tailored so they can adapt to learner expertise (Kalyuga, 2007). Challenging tasks include enabling prompts to provide scaffolding for a problem for those students who require it. Students are primarily responsible for determining if and when they should access these prompts. Structuring support in this manner can reduce the likelihood of the expertise reversal effect inhibiting learning.

While it can be argued that all problem-based approaches by definition reduce the expertise-reversal effect because their low-support approach fundamentally caters to the needs of experts, challenging tasks appear to do so without compromising the level of support offered to non-expert learners. Through the withholding of information, which would otherwise simplify or breakdown the problem (i.e., not automatically providing all students with the enabling prompts), experts are not provided with potentially redundant information.

A further strength of the challenging task approach is that no initial judgements need be made by the teacher in relation to the expertise of the student, and therefore the level of scaffolding and support they will require. Instead, students self-select based on their perceptions of the difficulty of the task. Although teachers clearly have a role in encouraging students who are struggling unproductively with a task to access an enabling prompt, this self-determination increases the accuracy with which expertise is identified. This in turn should serve to further reduce the expertise-reversal effect, in comparison to less precise ways of determining expertise with a given task (e.g., relying on past test scores). In a more general sense, the use of prompts potentially optimises the level of challenge inherent in the task (i.e., the intrinsic cognitive load).

HOW CAN CHALLENGING TASKS OPTIMISE INTRINSIC COGNITIVE LOAD?

Intrinsic cognitive load is determined by the extent to which the various elements inherent in a particular learning task interact (Sweller, 2010); in other words, task complexity (Schnotz & Kurschner, 2007). A large number of interacting elements requiring simultaneous information processing suggests a high intrinsic cognitive load. In addition to task complexity, intrinsic cognitive load is also determined by the extent of the learner’s expertise with similar tasks (which will impact on subjective task complexity) and the level of outside support provided to tackle the task (Schnotz & Kurschner, 2007). In contrast to extraneous cognitive load, the level of intrinsic cognitive load is considered fixed for an individual with a given level of expertise.

Changing the level of intrinsic cognitive load can only be achieved through altering the task, which in turn would imply different learning objectives (Sweller, 2010).

To maximise learning, intrinsic cognitive load needs to be at an appropriate level as determined by the interaction between the complexity of the problem and the expertise of the learner (Sweller, 2010). If intrinsic cognitive load is too high, students will become overloaded and learning will not occur. However, if intrinsic cognitive load is too low, learning is also undermined. Not only is cognitive capacity underutilised, but as Schnotz and Kurschner (2007) argue, more expert learners may choose to disengage and 'tune out' if the challenge inherent in the task is inadequate. Essentially this last point is an alternative interpretation of the expertise-reversal effect discussed earlier.

It is proposed that challenging tasks can optimise the level of intrinsic cognitive load through learners utilisng enabling prompts and extending prompts on a 'just in time' basis. In the first instance, accessing sequenced enabling prompts can reduce the amount of interactivity amongst the elements of the task until the task is at an appropriate level of challenge for a given learner's expertise. For example, consider a challenging task for a Grade 2 student: "Can you add all of the digits from one to nine together, and explain your approach to a partner?" The first enabling prompt may represent the task for the student as a number sentence (1+2+3+4+5+6+7+8+9=), making the problem to be solved far less opaque and unfamiliar. The second enabling prompt may remind students that they do not need to add numbers in the order they are first presented in. The third enabling prompt may ask students to consider if they can see any number bonds equalling ten, and the fourth enabling prompt may provide students with some examples of number bonds equally 10 taken from the problem (i.e., 1+9; 2+8). In contrast, an extending prompt essentially attempts to increase the number of interacting elements to make the problem more challenging. For example, modifying the above challenging task so that multi-digit numbers need to be added (e.g., "Can you add all of the numbers from eleven to twenty together?") introduces additional place-value elements to the task (i.e., adding multi-digit numbers; understanding place-value to 3-digits).

It needs to be noted that in using prompts to enable the activation of requisite knowledge and facilitate the creation of new 'intermediate' knowledge, the nature of the problem has been changed and therefore the learning objectives of the task have been somewhat altered. However, although students are in reality working on slightly different problems, critically they have a similar experience in having worked on the same challenging task. This enables them to actively participate in the discussion component of the lesson, and reflect on the key mathematical concepts explored. Indeed, systematically modifying intrinsic cognitive load by reducing or increasing the number of elements and/or the interactions between elements *without undermining the primary learning objective* has some precedent within cognitive load theory (see the part-whole approach; van Merrienboer, Kester & Paas, 2006).

Consequently, if a particular learning objective is a central focus of a lesson, it should not be compromised by any of the enabling prompts. For example, in the task outlined above, if the primary learning objective was for students to be able to translate worded problems into number sentences, then the first enabling prompt, which effectively does this for the student, is clearly not appropriate.

SUMMARY AND FUTURE RESEARCH DIRECTIONS

Whilst there is some evidence that the reform in mathematics education towards problem-based learning has been efficacious (e.g., Marshall & Horton, 2011), other authors cite evidence that problem-based approaches impose too high a cognitive load, and therefore undermine learning (e.g., Sweller et al., 2007). This paper has argued that teaching with challenging tasks can provide the benefits of problem-based approaches (e.g., higher order thinking, persistence) whilst being cognisant of the issue of cognitive overload. There are at least two lines of future research suggested by the arguments put forward in this paper.

Firstly, the contention that enabling and extending prompts effectively modify the intrinsic cognitive load of a task so that it is optimised for a given learner could be examined in a classroom context. This would require multiple measurements of cognitive load to be taken during a particular lesson, as well as data around whether students perceive the level of challenge on offer as optimal. Changes with respect to learners' perceptions of cognitive load and challenge optimality could then be examined in relation to time.

Secondly, student learning outcomes achieved in classrooms adopting the Launch-Explore-Discuss lesson structure could be contrasted with student learning outcomes achieved by classrooms adopting more traditional lesson structures (i.e., lessons beginning with a period of teacher-facilitated instruction). This would get to the heart of the debate by addressing concerns about whether problem-based learning contexts generate extraneous cognitive load, therefore undermining student learning. Any such study would need to ensure that both classroom types essentially contained the same content and pedagogy, with lesson structure being the only factor allowed to vary.

References

Bobis, J., Sweller, J., & Cooper, M. (1994). Demands imposed on primary-school students by geometric models. *Contemporary Educational Psychology, 19*(1), 108-117.

Hollingsworth, H., McCrae, B., & Lokan, J. (2003). *Teaching mathematics in Australia: Results from the TIMSS 1999 video study.* Australian Council for Educational Research.

Kalyuga, S. (2007). Expertise Reversal Effect and Its Implications for Learner-Tailored Instruction. *Educational Psychology Review, 19*(4), 509-539.

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational psychologist, 41*(2), 75-86.

Marshall, J. C., & Horton, R. M. (2011). The relationship of teacher-facilitated, inquiry-based Instruction to student higher-order thinking. *School Science and Mathematics, 111*(3), 93-101.

Renkl, A., & Atkinson, R. K. (2003). Structuring the transition from example study to problem solving in cognitive skill acquisition: A cognitive load perspective. *Educational psychologist, 38*(1), 15-22.

Schmidt, H. G., Loyens, S. M., Van Gog, T., & Paas, F. (2007). Problem-based learning is compatible with human cognitive architecture: Commentary on Kirschner, Sweller, and Clark (2006). *Educational psychologist, 42*(2), 91-97.

Schnotz, W., & Kurschner, C. (2007). A Reconsideration of Cognitive Load Theory. *Educational Psychology Review, 19*(4), 469-508.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340.

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A., Roche, A., & Walker, N. (2014). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education*, *17*(1), 1-18.

Sullivan, P., & Mornane, A. (2013). Exploring teachers' use of, and students' reactions to, challenging mathematics tasks. *Mathematics Education Research Journal, 25*, 1-21.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. *International Journal of Science and Mathematics Education, 4*(1), 117-143.

Sweller, J. (2010). Element interactivity and intrinsic, extraneous, and germane cognitive load. *Educational Psychology Review, 22*(2), 123-138.

Sweller, J., Kirschner, P. A., & Clark, R. E. (2007). Why minimally guided teaching techniques do not work: A reply to commentaries. *Educational psychologist, 42*(2), 115-121.

Van Merriënboer, J. J., Kester, L., & Paas, F. (2006). Teaching complex rather than simple tasks: Balancing intrinsic and germane load to enhance transfer of learning. *Applied Cognitive Psychology, 20*(3), 343-352.

van Merriënboer, J. J., & Sweller, J. (2005). Cognitive Load Theory and Complex Learning: Recent Developments and Future Directions. *Educational Psychology Review, 17*(2), 147-177.

MEASUREMENT ESTIMATION IN PRIMARY SCHOOL: WHICH ANSWER IS ADEQUATE?

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Measurement estimation is seen as an important part of mathematics learning, although still very little is known about children's abilities in this respect. To make matters worse, criteria for the adequacy of estimates are arbitrarily chosen and differ in studies on this topic. If teachers have to evaluate students' estimation performances, they need criteria, too. In this paper, we first present some of those studies and their criteria for adequacy. These criteria are evaluated and prepared for discussion by applying them to data from an interview study with 4th grade students estimating length and capacity. This empirical basis for discussion is complemented by data of expert opinions about the items and children's results.

INTRODUCTION

Mathematics in primary school is usually seen as a discipline of precision. Children have to learn how to calculate correctly. Even lessons in measurement encourage students to measure as accurately as they can. For this reason, students gain a one-sided picture of mathematics as a discipline. But as Freudenthal stated already in 1978, there are two different 'worlds of mathematics' that have to be known by students: one in which precision is virtuous and one in which it is vicious.

Depending on the context and the questions one tries to answer, numbers and measures have to be more or less accurate. Whereas calculation is a procedure in the exact world of mathematics, estimation is an integral part of the second view of mathematics. Therefore, estimation recently gained more attention in the curricula of different countries such as Germany or Taiwan (see Huang, 2014).

In our study, we are mainly interested in strategies fourth-graders use to estimate length and capacity (see Ruwisch & Heid, 2015). Interpreting the answers quantitatively as well, we realised that there is no clear criterion that allows us to decide, whether an estimation is a good one. Although some studies about measurement estimation had also included a quantitative analysis (e.g. Swan & Jones, 1980; Hildreth, 1980; Siegel, Goldsmith, & Madson 1982; Clayton 1992; Jones, Forrester, Gardner, Andre, & Taylor, 2012; Huang, 2014), the authors used different criteria for this decision. This fact motivated us to take a closer look at those criteria and evaluate them here by applying them to our data.

THEORETICAL BACKGROUND

Estimation processes in mathematics lessons can be divided into three different contents: computational estimation, numerical estimation, and measurement estimation (O'Daffer, 1979; Sowder, 1992). We will restrict ourselves to the last one

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in this paper. Measurement estimation is a mental process that is thought to be analogous to real measurement processes but without handling a measurement tool (Bright, 1976; Sowder, 1992). Most research in measurement estimation is focused on lengths (see Sowder, 1992; Jones et al., 2012). Since our own study also deals with length and capacity, we will mainly focus on items in these measurement areas.

Adequacy of estimated measures: the terminology

In the literature dealing with the adequacy of estimated measures, there is also no agreement concerning the terminology. Most researchers use 'accuracy' (e.g., Swan & Jones, 1980; Siegel et al., 1982; Jones et al., 2012; Huang, 2014). In our opinion, this term overemphasises the aspect of precision and correctness. Huang (2014) also uses 'acceptability', a term that already includes the scope for decision making by the researcher. Other researchers use 'reasonable estimates' (Clayton, 1992) or 'reasonableness' (Siegel et al., 1982) as well, but even these terms differ in their meaning. Whereas Siegel et al. (1992) call comprehensible estimations 'reasonable', Clayton (1992) emphasises the complex situation that has to be taken into account when deciding the adequacy of estimations. The term 'adequacy' which is used in this paper, focuses on the equivalence between the estimation and the real measure, and may also evoke the association of precision. In German the word '*angemessen*' is used as a synonym for 'adequate'. '*Angemessen*' literally means 'to be measured with reference to something else'. In this sense, the adequacy of estimations is dependent on a reference point. So one of our questions is: Which reference point(s) can be useful to decide, whether an estimated measure is adequate?

Criteria for adequacy of estimated measures in the literature

In 1980, Swan and Jones reported about their measurement estimation studies from the seventies. 780 elementary school children (Grades 4 to 6) participated in 1971, and 304 did so in 1977. Every child had to provide written answers to eight estimation problems. Four of these problems dealt with length: "two distance intervals one of which was between 50 and 75 meters in length, the other 5 to 10 meters in length. […] two heights, one of which was about 20 meters tall, and the other shorter (such as a flagpole)." (Swan & Jones, 1980: 299). As the authors admitted, they arbitrarily judged an estimate within a maximum deviation of 25 % from the real value as 'accurate'. Although the students performed better in 1977, only 13 to 39 % gave an 'accurate' estimate of the lengths under these conditions. Junior high school students (Grades 7 to 8) performed significantly better but still poor: 21 to 50 %. Since the authors did not present their raw data, no conclusions about the deviations from the real values can be drawn.

In 1980, Hildreth published his PhD dissertation about the use of estimation strategies for length and area. Since we were not able to access the entire dissertation, raw data and detailed results of this study with 24 fifth-graders, 24 seventh-graders, and 24 college students cannot be reported here. Nevertheless, it can be stated that Hildreth measured the estimation ability by "the number of items on which the relative error was less than 1/3" (phdtree.org/pdf/24304583). Thus, a good estimation deviates within a 33 % range from the real value.

In 1982, Siegel et al. reported about skills in estimating length and numerosity. Six different types of estimation problems in four contexts were presented to 20 children of each grade (Grades 2 to 8). Two problem types dealt with numbers only, two others with length only. The remaining two problem types asked for a combination of estimating numbers as well as lengths and to calculate them. Siegel et al. differentiated between 'accuracy' and 'reasonableness'. Whereas an 'accurate' estimation was defined as a maximum deviation of 50 % from the actual value, the authors scored an estimation 'reasonable', if it was "plus or minus an order of magnitude of the actual value" (217). Since the authors were interested in the different problem types no overall data were given in the paper. Unreasonable answers only were given if the estimation process got difficult (e.g., in the combined estimation problem type). Nevertheless, benchmark problems dealing only with length were performed much better than the other problem types – no unreasonable answers were observed here – and older students performed better in all problem types than did younger students. Again, no raw data are given, so no conclusions about the adequacy of the criteria are possible.

In a recent study Huang (2014) used a two-step process to score the estimated measures of 72 fourth-, fifth-, and sixth-graders. In her study she presented 12 problems that required the estimation of length and area. In scoring the children's answers, she differentiated between 'accurate' and 'acceptable'. 'Accuracy' was defined as a maximum deviation of 10 % from the real value and scored by 2 points, whereas 'acceptability' was defined as a maximum deviation of 25 % from the real value and scored by 1 point. In length-estimation the children could achieve a maximum of 12 points. The results show that on average fourth-graders achieve of 5.91 points, whereas fifth- and sixth-grader did slightly but not significantly better. Again, no conclusion about the adequacy of the evaluating process is possible due to the fact of missing raw data. Nearly the same process is used by Hogan and Brezinski (2003). They decided to use a three-step scoring: 3 points for an answer within a range of 10 %, 2 points within 10 to 20 % and 1 point within 20 to 30 %. Since measurement estimation was a very small part of the whole study with college students as participants, no further information will be presented here.

Although there are some other suggestions how to evaluate the adequacy of estimations – Lörcher (2000) defined accuracy by an interval from the half to the double of the real value; Clayton (1992) proposed a logarithmic model, but applied it to numerosity only – we will focus on the criteria mentioned above.

METHOD

Measurement estimation tasks

Our tasks for estimating length and capacity were constructed with reference to Bright's (1976) typology of requests in estimating length. First of all, it can be differentiated if a suitable measure has to be given to a representative or if a suitable representative has to be found to a given measure. In each case the (possible) representatives can be physically present or absent as well as the unit itself may be visible or not (for more details see Ruwisch & Heid, 2015).

If the representatives are given and physically present, it can clearly be said how long, wide, tall or high they actually are, when a subject is asked to estimate their lengths. This applies equally for the estimation of the capacity of objects: If the representatives are given and physically present, it can clearly be said how much capacity they actually take.

Therefore, the answers to these tasks will be chosen for discussing our question concerning the adequacy of estimations given by the children. The following objects were presented to estimate their lengths: the diameter of the head of a wooden bug (5 mm), the length of a piece of chalk (8 cm), the length of a book with an unusual format (46 cm), the height of the table (70 cm), and the height of the room (3 m). The following objects were presented to estimate their capacities: a test tube (10 ml), a small glass (100 ml), a vase (300 ml), a carafe (500 ml or 1 l), and a big pot (3.5 l).

Sample

One hundred and thirty fourth-graders from 13 primary schools in the north of Germany were involved in this part of the study, but not every child estimated all tasks given above. As the data in Table 1 show, the total numbers of answers differ from 77 (test tube) to 128 (table).

Tasks for estimating length			Tasks for estimating capacity		
Item	Length	Number of answers	Item	Length	Number of answers
bug	5 mm	117	test tube	10 ml	77
chalk	8 cm	112	glass	100 ml	116
book	46 cm	95	vase	300 ml	115
table	70 cm	128	carafe a)	500 ml	80
room	3 m	88	carafe b)	1 l	44
			pot	3.5 l	117

Table 1: Total numbers of answers to each item

Although estimation should be part of the curriculum since 2004, none of the teachers participating in this study fostered it in their classes. All students were familiar with the measurement of length, and had already gone through one unit about capacity during this school year. All children were interviewed individually during the second half of the school-year; the whole interviews lasted about 20 to 25 minutes (see Ruwisch & Heid, 2015 for more details).

On the purpose of comparison, 17 mathematics educators who participated in a conference workshop estimated themselves the ten items given above. Afterwards, they were asked to evaluate given ranges of deviations. They should choose that range they think to be adequate for the evaluation of estimates given by 4th grade students.

Data

Table 2 shows the minimum and maximum estimations that were given by any child.

Tasks for estimating length				Tasks for estimating capacity			
Item	Actual length	Min. estimate	Max. estimate	Item	Actual length	Min. estimate	Max. estimate
bug	5 mm	0.35 mm	15 cm	test tube	10 ml	1 ml	200 ml
chalk	8 cm	1 cm	15 cm	glass	100 ml	1 ml	1 l
book	46 cm	3 cm	90 cm	vase	300 ml	3 ml	2 l
table	70 cm	8 cm	1.30 m	carafe a)	500 ml	2 ml	2 l
room	3 m	2 m	6 m	carafe b)	1 l	500 ml	3.5 l
				pot	3.5 l	200 ml	10 l

Table 2: Maximum deviations from the real values

For almost all objects an underestimation of nearly 100 % can be found. Only one item of the lengths (room: 40 % deviation) and one item of the capacities (big carafe: 50 % deviation) show better values. Looking at the overestimations, a greater variety can be stated: Whereas the overestimations of the lengths differ by 80 to 100 % from the real values, the maximum deviations of the capacities range between 100 and nearly 2,000 %.

The same tendencies can be seen in the extremes of the experts' estimations, although the deviations are much smaller.

If we do not focus on the extremes, but on the means of deviations in the children's estimations, it can be stated, that on average the lengths were mostly underestimated, whereas the capacities were underestimated as well as overestimated.

Length overestimated: room (+2%).
Length underestimated: table (-6%), book (-17%), chalk (-20%), and bug (-30%).
Capacities overestimated: glass (+7%), small carafe (+20%), test tube (+68%).
Capacities underestimated: vase (-11%), big carafe (-15%), pot (-26%).

Looking at the means of the positive values of deviations, the estimations of lengths show a very uniform picture with the positive exception of the room: bug (M 37.8%; SD 27.3), chalk (M 31.8%; SD 22.0), book (M 32.2%; SD 23.1), table (M 31.0%; SD 21.0), room (M 18.2%; SD 21.4). Perhaps the height of the room is a known value for a greater number of children. The values of the estimated capacities show much greater mean deviations as well as very high standard deviations: test tube (M 113.4%; SD 248.1), glass (M 93.6%; SD 95.0), vase (M 71.2%; SD 44.4), small carafe (M 42.2%; SD 38.2), big carafe (M 27.2%; SD 36.8), pot (M 41.1%; SD 25.1). Again, it may be that the carafes are better known than a test tube.

All in all, the results of the experts show less extreme deviations and were in total closer to the real values. But they are more likely to overestimate than to underestimate. Since only 17 experts participated, no means and standard deviations are given here.

THE DATA FROM THE PERSPECTIVE OF DIFFERENT CRITERIA

Overall application of different criteria

Tables 3 and 4 show the overall results of the children, and the experts. All estimations were accumulated and evaluated by the criteria mentioned above.

Criteria of 'accuracy'	10 %	25 %	33 %	50 %	> 50%
Length (total # answers: 540)					
absolute	115	255	325	442	98
relative	21.3 %	47.2 %	60.2 %	81.9 %	18.1 %
Capacity (total # answers: 549)					
absolute	92	161	166	277	272
relative	16.8 %	29.3 %	30.2 %	50.5 %	49.5 %

Table 3: cumulated 'accurate' answers of the children using different criteria

The results of the children as well as of the experts show, that the estimation of lengths is easier than the estimation of capacities.

Criteria of 'accuracy'	10 %	25 %	33 %	50 %	> 50%
length (total # answers: 85)					
absolute	48	76	78	81	4
relative	56.5 %	89.4 %	91.8 %	95.3 %	4.7 %
Capacity (total # answers: 81)					
absolute	24	45	50	64	17
relative	29.6 %	52.9 %	58.8 %	79.0 %	21.0 %

Table 4: Cumulated 'accurate' answers of the experts using different criteria

Concerning the different criteria, there is nearly no difference for the children's results between 25 % or 33 % in the capacity-condition, whereas it gives a good differentiation in the application to the estimated lengths and also to the experts' results, if this differentiation is necessary.

Ranges of deviations from the perspective of different groups of students

Table 5 shows the deviation-ranges of the estimation that were given by the best quarter and the best half of the children.

Deviations in estimating length			Deviations in estimating capacity		
Item	Best quarter	Best half	Item	Best quarter	Best half
bug	10 %	35 %	test tube	20 %	77 %
chalk	13 %	25 %	glass	48 %	85 %
book	11 %	30 %	vase	33 %	65 %
table	14 %	28 %	carafe a)	0 %	40 %
room	0 %	13 %	carafe b)	0 %	10 %
			pot	21 %	43 %

Table 5: Ranges of deviations of the best 25 % (50 %) estimations

Again, the results for the room and the carafes show that these estimations have been easy for at least the best quarter of students. It also becomes clear that the items differ in their difficulty especially in the capacity-condition.

DISCUSSION

Looking at the data and the application of the criteria, the following suggestions have to be discussed:

- It seems necessary to use different criteria for the evaluation of estimates in different measurement areas. The children and the experts gave better estimations for lengths than for capacities. The 17 experts also chose smaller ranges for lengths as adequate for evaluating children's estimations. But: Which ranges are adequate for which measurement area?
- A multi-step evaluation seems to be more adequate than a single-step one. But: How many steps should be differentiated? Is the number of steps different in different measurement areas?
- Since even our items differed in their difficulty, we seem to need different evaluations for them. But we are not sure yet, if there is a medium bandwidth in every measurement area in which it is easier to estimate. Do we have to define such bandwidths and use different criteria for evaluation if an item is in it or not?
- Last but not least: How many items have to be estimated to get a realistic picture of a child's performance? How do we take the age of the child into account?

Nevertheless, the overall question remains, if the decision about an adequate estimate is a normative one or if it may be solved experimentally. But: Should a poor result get a good evaluation because it's the average of performance?

References

Bright, G. W. (1976). Estimation as part of learning to measure. In D. Nelson & R. E. Reys (Eds.), *Measurement in school mathematics* (pp. 87-104). Reston: NCTM.

Clayton, J. G. (1992). Estimation in schools. MPhil-thesis in Education. London: University of London. Retrieved from http://www.johngclayton.co.uk/website%20files/Output complete%20text.pdf [15-02-14].

Freudenthal, H. (1978). *Vorrede zu einer Wissenschaft vom Mathematikunterricht*. München: Oldenbourg.

Hildreth, D. J. (1980). *Estimation Strategy Use in Length and Area Measurement Tasks by Fifth and Seventh Grade Students.* Abstract retrieved from PhDTree http://phdtree.org/24304583-estimation-strategy-use-in-length-and-area-measurement-tasks-by-fifth-and-seventh-grade-students/ [22-09-14].

Hogan, T. P. & Brezinski, K. L. (2003). Quantitative estimation: One, two, or three abilities? *Mathematical Thinking and Learning, 5*(4), 259-280.

Huang, H.-M. E. (2014). Investigating children's ability to solve measurement estimation problems. In S. Oesterle; P. Liljedahl,; C. Nicol & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36*, Vol. 3, pp. 353-360. Vancouver, Canada: PME.

Jones, M. G.; Forrester, J. H.; Gardner, G. E.; Andre, T. & Taylor, A. R. (2012). Students' accuracy of measurement estimation: context, units, and logical thinking. *School Science and Mathematics, 112*(3), 171-178.

Lörcher, G. A. (2000). Zur Entwicklung der Zahlvorstellung. In M. Neubrand (Ed.), *Beiträge zum Mathematikunterricht: Vorträge auf der 34. Tagung für Didaktik der Mathematik 28.2.-3.3.2000*, pp. 402-405.

O'Daffer, P. (1979). A case and techniques for estimation: Estimation experiences in elementary school mathematics – essential not extra! *Arithmetic Teacher, 26*(6), 46-51.

Ruwisch, S. & Heid, M. (2015). Children's use of strategies in estimating length and volume. Poster presentation. Hobart, Australia: PME. [for submisson 15-03-01].

Siegel, A. W.; Goldsmith, L. T. & Madson, C. R. (1982). skill in estimation problems of extent and numerosity. *Journal for Research in Mathematics Education, 13*(3), 211-232.

Sowder, J. (1992). Estimation and Number Sense. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 371-389). Reston: NCTM.

Swan, M. & Jones, O. E. (1980). Comparison of students' perceptions of distance, weight, height, area, and temperature. *Science Education, 64*(3), 297-307.

SELF-EXPLANATIONS AND GESTURES

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To investigate the role of gestures for the identification of self-explanations, thinking aloud sessions of 33 undergraduate students who learned with worked-out examples dealing with complex numbers were analysed. Two coding procedures were applied to the recorded video data. During the first one, solely the verbal (audio track of the recordings) and written data (spontaneous notes of the participants) of the recordings was available to the raters. Nonverbal video data were included only in the second procedure. A comparison of the coding results revealed distinctly different numbers of identified self-explanations. In addition, an example of a self-explanation that was not identified until the second coding procedure is presented.

A plethora of studies show that learners who engage in self-explaining – a specific cognitive process that occurs during learning (Chi 2000) – have statistically significant higher learning outcomes than learners who do not self-explain (cf. Wylie & Chi 2014). Research on self-explanations usually analyses written texts or verbal protocols from learners who think aloud. Gestures and other body movements are not regarded in those analyses (see literature review below). In this paper, consequences of an inclusion of gestures into self-explanation analyses based on thinking aloud protocols are shown.

LITERATURE REVIEW

Self-Explanations

Self-explaining is "a constructive activity that engages students in active learning and insures that learners attend to the material in a meaningful way" (Roy & Chi. 2005, p. 272). *Self-explanations* are defined as productions by self-explaining that consists of "units of utterances" which are meant to be verbal (Chi, 2000, p.165). Instances of self-explanations are, for example, integrating different representations, inferencing from depicted data, explaining the goal of an operation, or activating prior knowledge (Chi, Bassok, Lewis, Reimann, & Glasser, 1989; Renkl 1997; Aleven & Koedinger , 2002).

Studies that investigate self-explanations with thinking aloud protocols follow a more or less similar procedure. The participants are asked to think aloud while they are learning with materials such as worked-out examples, tasks or texts. These learning sessions are recorded and transformed into protocols based on verbal expressions. Actions or gestures performed during learning are not considered in the protocols (the reviewed studies are: Chi et al. (1989), Chi and VanLehn (1991), Chi, de Leeuw, Chui, and LaVancher (1994), Pirolli and Recker (1994), Bielaczyc, Pirolli, and Brown (1995), Recker and Pirolli (1995), Renkl (1997), Renkl, Stark, Gruber and Mandl (1998), Neuman and Schwarz (1998), Stark (1999), Neuman, Leibowitz, and Schwarz (2000), Wong, Lawson and Keeves (2002), Renkl (2002), Ainsworth and Loizou

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 121-128. Hobart, Australia: PME.

(2003), McNamara (2004), Renkl, Schworm, and Hilbert (2004), Butcher (2006), Ainsworth and Burcham (2007) and de Koning, Tabbers, Rikers, and Paas (2011)).

Gestures

A growing body of research verifies the important role that gestures play in mathematical thinking and communicating (e.g., Alibali & DiRusso, 1999; Goldin-Meadow & Singer, 2003; Radford, 2009; Edwards, Ferrara, & Moore-Russo, 2014). In this paper gestures are seen as “body movement[s] fulfilling communicational function” (Sfard, 2008, p. 194). Different studies describe the learners’ subtle use of gestures in combination with verbal utterances when talking about functions, fractions, numbers and other topics (Arzarell, Paola, Robutti, & Sabena, 2008; Edwards, 2008; Yoon, Thomas, & Dreyfus, 2011).

A special case of mathematical communication is thinking aloud in individual sessions. Research studies that investigate the use of gestures in thinking aloud settings found a frequent use of gestures accompanying verb phrases with “individual differences in the use of gesture in both communication and inference” (Hegarty. Mayer, Kriz & Keehner, 2005). Having analysed students’ behavior while they were solving gear problems, Schwartz and Black (1996) reported on numerous pointing and tracing gestures that could be observed during the solving process. Emmorey and Casey (2001) found out that more gestures occurred if an experimenter to whom the participants have to explain their solutions was in sight.

RESEARCH QUESTION

From these findings it could be hypothesised that gestures – shown in nonverbal video data – could play a role for the identification of self-explanations. The following analysis follows the question:

How does the consideration of gestures influence the identification of self-explanations?

To investigate the role of gestures for the identification of self-explanations, the definition of a self-explanation is slightly modified: Instead of defining self-explanations as units of exclusively verbal utterances, also nonverbal utterances like gestures and representations of utterances like drawings and writings are taken into account (Nemirovsky & Ferrara, 2008, p. 162).

METHODS AND CATEGORIES

Participants, materials and procedure

Due to the research focus, the data collection follows a common procedure that was derived from the literature review of self-explanations (see above and Ericsson & Simon, 1993). The participants were 33 undergraduate students (22 female, 11 male) from a German university. Worked out examples were chosen as learning material because they allow a structured analysis of self-explanations and are used in many of the reviewed research studies. The chosen worked-out examples addressed the multiplication of complex numbers in Cartesian and polar coordinates as well as the transformation from one form into another. The students were unfamiliar with the transformation and multiplication of complex numbers, and obtained a brief introduction in essentials of complex numbers.

During the intervention phase, the participants worked individually with three worked-out examples printed on paper without a time limit. They were asked to learn from the worked-out examples, to signal when they were finished and to think aloud during their work. The use of a ‘cheat sheet’ with definitions and formulas, a triangle ruler and a calculator application were permitted. The intervention phase was audio- and videotaped; the notes of each participant were collected afterwards. No guidelines on taking notes or gesturing were given.

Data analysis

A two-phase qualitative content analysis was applied to investigate the collected data (Mayring, 2010, p. 59; Lamnek, 2010, p. 460). During the *pilot phase*, the recorded learning sessions of two students who were no participants of the main study, were analysed. The units of analysis were combinations of utterances, written notes, gestures and actions. The preliminary category scheme derived from the literature review was adapted to the empirical findings of this phase (Mayring, 2010, p. 62).

The unit of coding of the *main phase* consisted of the video sequences from 33 participants described above. Two coding procedures were applied to all of the 33 recorded learning sequences by two coders familiar with the topic. (1) The first coding procedure was based only on the audio track and the written notes of each participant (verbal and written data, no video data was available for the raters). (2) The second coding procedure was based on all available data collected during the intervention phase (verbal, written and nonverbal video data).

The inter-rater reliability of both coding procedures was ascertained based on classifications of 10 % of the data that was coded by both raters. Due to possible random matches, Cohen’s Kappa was chosen for the calculation of the inter-rater reliabilities. The first procedure features a Kappa of 88.93%, the second one 89.2%.

Category Scheme

Self-explanation: A segment was coded as self-explanation if it fitted into one or more of the identified self-explanation categories. No distinctions between the different instances were made. For each participant, an overall self-explanation score was calculated. Typical self-explanation categories are (see also literature review): activating prior knowledge for explanations of steps, calculations, representations; integrating different representations, e.g., symbolic and geometrical; drawing inferences from information depicted in the worked-out examples and the ’cheat sheet’.

No self-explanation: No self-explanation could be identified.

RESULTS

First, the coding results are depicted. Second, a short protocol gives an impression of a self-explanation that can only be identified based on all available data.

Coding Results

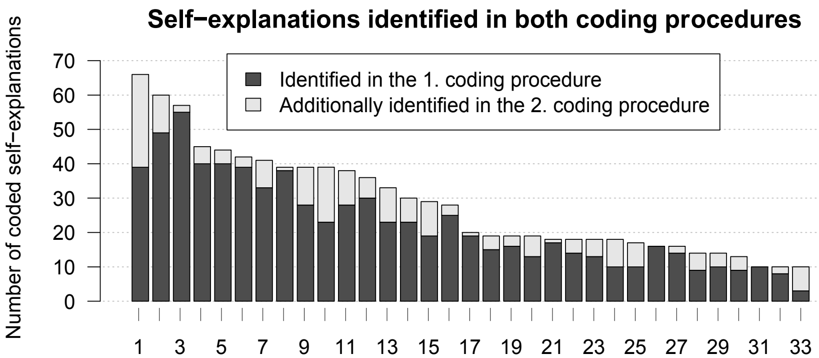


Figure 1: Results of the two coding procedures, sorted by number of coded self-explanations after the second procedure. The coding results of the second coding procedure are the individual sums of dark and light grey bars.

In summary, 738 self-explanations could be identified *during the first coding procedure* based on verbal and written data (Fig. 1, dark grey bars). The mean for the self-explanations is 22.36 and the standard deviation is 13.1. The individual results range between 3 (participant 33) and 55 self-explanations (participant 3). *During the second coding procedure,* which was based on all available data, 935 self-explanations could be identified with a mean of 28.33 and a standard deviation of 15.29 self-explanations. The individual results range between 10 self-explanations (participants 31, 32 and 33) and 66 (participant 1).

Therefore, 197 self-explanations were not identified in the first procedure. During the second procedure, 5.97 additional self-explanations per participant could be identified with a standard deviation of 5.29 self-explanations. These self-explanations were distributed unequally among the participants with a minimum of 0 (participants 26 and 31) and maximum of 27 (participant 1).

Moreover, two phenomena could be identified in the video data. First, the participants performed actions. They used the triangle ruler or the calculator. Such actions reveal connections between information in the material or integrations of different representations. In combination with verbal and written data, self-explaining processes could be reconstructed. From 197 self-explanations, 31 self-explanations could be identified because of the above described actions.

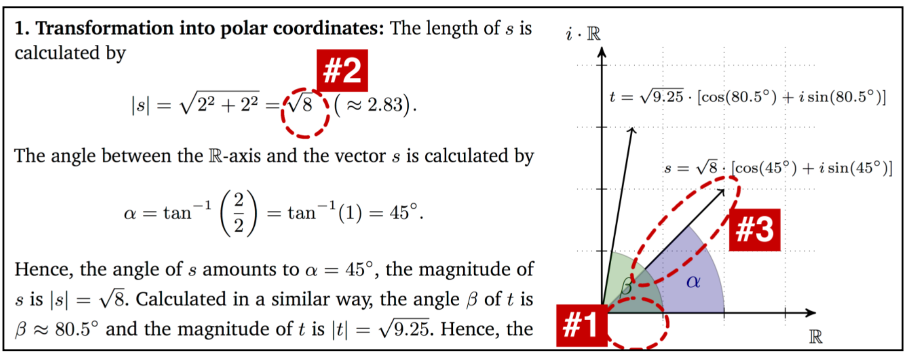
The remaining 166 self-explanations were coded because of the visible gestures in the recordings. All but three of these gestures had mainly deictic functions and therefore connected verbal utterances with words, numbers or symbols on the example sheets or the 'cheat sheet'. The question whether the participants were pointing or tracing to themselves or the video camera has to remain open.

An example of a specific self-explanation only coded based on all available data

The following transcript gives insight into the way gestures allow to identify self-explanations in the second coding procedure that were not categorised during the first coding.

Brian tries to verify a part of the worked-out example which features three solution steps: i) transformation of two complex numbers given in Cartesian coordinates into trigonometric form, ii) multiplication of these numbers in symbolic representation, iii) geometrical representation of the result. Brian then deals with the transformation of the complex number $s = 2 + 2i$ (Figure 2).

> **Brian**: In the example with (points with right middle finger at #1) …(points with pencil in his left hand to #2) square root of 8, (traces along #3 with right index finger) that is the magnitude of the vector. Exactly (*lifts the pencil*), you get the magnitude, if you …



1. Transformation into polar coordinates: The length of s is calculated by

$$|s| = \sqrt{2^2 + 2^2} = \sqrt{8} \ (\approx 2.83).$$

The angle between the $\mathbb{R}$-axis and the vector s is calculated by

$$\alpha = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = 45°.$$

Hence, the angle of s amounts to $\alpha = 45°$, the magnitude of s is $|s| = \sqrt{8}$. Calculated in a similar way, the angle β of t is $\beta \approx 80.5°$ and the magnitude of t is $|t| = \sqrt{9.25}$. Hence, the

Figure 2: Snippet of Brian's example sheet. Dashed lines with identification numbers indicate locations to which pointing or tracing gestures refer.

First, Brian points at #1 – at this point it is unclear, if he is referencing to the R-axis, the angle β, the angle α, or the whole coordinate system. With the pencil in his left hand, he points at the symbol 'square root of 8' (#2) and says "square root of 8, that is the magnitude of the vector". Both speech and gestures combined, it seems as if he names the transformation of vector s as "the example" here. It could be concluded that his first gesture (#1) was a pointing at the angle α and the vector s.

His words can be found reordered in the written text of the example in which it says "the magnitude of s is |s| = √8". The word "vector" is in the text, too. While he is speaking, he traces along vector s in the coordinate system (#3) and links the magnitude to the length of the arrow. Thus, he integrates different representations: the symbol from the text and the length of the geometrically represented vector.

Without recognising this gesture, this brief scene could not be identified as self-explanation in the first coding procedure. However, based on all available data and especially the video track, it could be identified as one.

DISCUSSION AND PERSPECTIVES

The difference between the two coding procedures reveals to what extent the inclusion of video data influences the coding results in the reported study. The results also reveal individual differences of gesture use in thinking aloud sessions and hence, are in line with the findings of Hegarty et al. (2005). Nonetheless, it remains unclear on which personal factors the individually different use of gestures depends.

The observed gestures were almost exclusively pointing and tracing gestures that locate areas on the example sheets. To what extent this depends on the content to be learned or on the presence of worksheets, remains unclear. For example, Edwards (2008) reports a smaller proportion of pointing gestures but more iconic-physical (referring to concrete actions), iconic-symbolic (referring to symbolic inscriptions) and metaphoric (referring to abstract ideas, see also McNeill 1992) gestures when teachers talk about fractions without learning material.

The depicted transcript highlights how subtle pointing and tracing gestures are coordinated with speech and how they allow the identification of self-explanations. For future studies it would be interesting to find out in how far these phenomena appear in other domains and with participants of different age.

References

Ainsworth, S., & Burcham, S. (2007). The impact of text coherence on learning by self-explanation. *Learning and Instruction*, *17*, 286–303.

Ainsworth, S., & Loizou, A. T. (2003). The effects of self-explaining when learning with text or diagrams. *Cognitive Science*, *27*, 669–681.

Aleven, V., & Koedinger, K. (2002). An effective metacognitive strategy: learning by doing and explaining with a computer-based cognitive tutor. *Cognitive Science*, *26*(2), 147–179.

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive Development, 14*, 37–56.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2008). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Bielaczyc, K., Pirolli, P., & Brown, A. L. (1995). Training in self-explanation and self-regulation strategies: Investigating the effects of knowledge acquisition activities on problem solving. *Cognition and Instruction, 13*, 221–252.

Butcher, K. (2006). Learning from text with diagrams - promoting mental model development and inference generation. *Journal of Educational Psychology, 98*(1), 182–197.

Chi, M. T. H. (2000). Self-explaining expository texts: The dual processes of generating inferences and repairing mental models. In R. Glaser (Ed.), *Advances in instructional psychology*, (pp. 161–238). Mahwah, NJ: Lawrence Erlbaum Associates.

Chi, M. T. H., Bassok, M., Lewis, M. W., Reimann, P., & Glaser, R. (1989). Self-explanations: How students study and use examples in learning to solve problems. *Cognitive Science, 13*, 145–182.

Chi, M. T. H., de Leeuw, N., Chiu, M.-H., & LaVancher, C. (1994). Eliciting self-explanations improves understanding. *Cognitive Science, 18*(3), 439–477.

Chi, M. T. H., & VanLehn, K. (1991). The content of physics self-explanations. *The Journal of the Learning Sciences, 1*(1), 69–105.

de Koning, B. B., Tabbers, H. K., Rikers, R. M. J. P., & Paas, F. (2011). Improved effectiveness of cueing by self-explanations when learning from a complex animation. *Applied Cognitive Psychology, 25*(2), 183–194.

Edwards, L. D. (2008). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics, 70*(2), 127–141.

Edwards, L. D., Ferrara, F., & Moore-Russo, D. (2014): *Emerging Perspectives on Gesture and Embodiment in Mathematics*. Charlotte, NC: Information Age Publishing.

Emmorey, K. & Casey, S. (2001). Gesture, thought and spatial language. *Gesture, 1*, 35–50.

Ericsson, K. A. & Simon, H. A. (1993). *Protocol Analysis*. Cambridge, MA: MIT Press.

Goldin-Meadow, S., & Singer, M. (2003). From children's hands to adults' ears: Gesture's role in the learning process. *Developmental Psychology, 39*(3), 509–520.

Hegarty, M., Mayer, S., Kriz, S., & Keehner, M. (2005). The role of gestures in mental animation. *Spatial Cognition and Computation, 5*(4), 333–356.

Lamnek, S. (2010). *Qualitative Sozialforschung*. Basel: Beltz, Weinheim.

Mayring, P. (2010). *Qualitative Inhaltsanalyse*. Basel: Beltz, Weinheim.

McNamara, D. S. (2004). SERT: self-explanation reading training. *Discourse Processes, 38*(1), 1–30.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought.* Chicago: University of Chicago Press.

Nemirovsky, R., & Ferrara, F. (2008). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70*(2), 159–174.

Neuman, Y., Leibowitz, L., & Schwarz, B. (2000). Patterns of verbal mediation during problem solving: A sequential analysis of self-explanation. *The Journal of Experimental Education, 68*(3), 197–213.

Neuman, Y., & Schwarz, B. (1998). Is self-explanation while solving problems helpful? The case of analogical problem-solving. *British Journal of Educational Psychology, 68*, 15–24.

Pirolli, P., & Recker, M. M. (1994). Learning strategies and transfer in the domain of programming. *Cognition and Instruction, 12*(3), 235–275.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*, 111–126.

Recker, M. M., & Pirolli, P. (1995). Modeling individual differences in students' learning strategies. *The Journal of the Learning Sciences, 4*(1), 1–38.

Renkl, A. (1997). Learning from worked-out examples: A study on individual differences. *Cognitive Science, 21*(1), 1–29.

Renkl, A. (2002). Worked-out examples: instructional explanations support learning by self-explanations. *Learning and Instruction, 12*(5), 529–556.

Renkl, A., Schworm, S., & Hilbert, T. S. (2004). Lernen aus Lösungsbeispielen: Eine effektive, aber kaum genutzte Möglichkeit, Unterricht zu gestalten. In M. Prenzel, & J. Doll (Eds.), *Bildungsqualität von Schule* (pp. 77–92). Münster: Waxmann.

Renkl, A., Stark, R., Gruber, H., & Mandl, H. (1998). Learning from worked-out ex- amples: The effects of example variability and elicited self-explanations. *Contemporary Educational Psychology, 23*, 90–108.

Roy, M., & Chi, M. T. H. (2005). The self-explanation principle in multimedia-learning. In R. E. Mayer (Ed.), *The Cambridge handbook of multimedia learning* (pp. 272–286). Cambridge, U.K: Cambridge University Press.

Schwartz, D. L., & Black, J. B. (1996). Shuttling between depictive models and abstract rules: Induction and fallback. *Cognitive Science, 20*, 457–497.

Sfard, A. (2008). What's all the fuss about gestures? A commentary. *Educational Studies in Mathematics, 70*(2), 191–200.

Stark, R. (1999). *Lernen mit Lösungsbeispielen: Einfluss unvollständiger Lösungsbeispiele auf Beispielelaboration, Lernerfolg und Motivation.* Göttingen: Hogrefe.

Wong, R. M. F., Lawson, M. J., & Keeves, J. (2002). The effects of self-explanation training on students' problem solving in high-school mathematics. *Learning and Instruction, 12*, 233–262.

Wylie, R., & Chi, M. T. H. (2014). The Self-Explanation Principle in Multimedia Learning. In R. E. Mayer, (Ed.), *The Cambridge Handbook of Multimedia Learning* (pp. 413–432). Cambridge, U.K: Cambridge University Press.

Yoon, C., Thomas, M. O. J., & Dreyfus, T. (2011). Grounded blends and mathematical gesture spaces: developing mathematical understandings via gestures. *Educational Studies in Mathematics*, 78(3), 371–393.

SHIFTING THE EMPHASIS TOWARD A STRUCTURAL DESCRIPTION OF (MATHEMATICS) TEACHERS' KNOWLEDGE

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Despite the wide range of various conceptualisations of (mathematics) teachers' knowledge, the literature is restricted in two interrelated respects: (1) the focus is (almost always) limited to the subject matter content, and (2) the form and nature of teachers' knowledge seem not to have been noticed by researchers working in the field. The paper seeks to address these gaps by (a) broadening the current perspective to include an epistemological, cognitive, and didactical lens on the knowledge base for teaching mathematics, and (b) going beyond what the teachers' knowledge is about to take account of how the knowledge is structured and organised. The theoretical work presented here intends to stimulate discussion about the structural description of this kind of knowledge.

CONCEPTUALISATIONS OF TEACHERS' KNOWLEDGE: MAPPING THE TERRAIN

Over the past decades, several interesting approaches, partly distinct and partly overlapping, in conceptualising the *knowledge base for teaching* have been developed; the majority of them follow Shulman's (1986, 1987) distinction between subject matter knowledge, pedagogical knowledge, pedagogical content knowledge, and knowledge of various aspects of the educational setting (including knowledge of the educational context). The frameworks and models that shape the landscape in research on teachers' knowledge are at various levels of *specificity* – ranging from general to discipline-, domain-, and concept-specific frameworks (see, Scheiner, 2015).

Quite a few *general* frameworks contributed to the field, particularly in (a) shifting the attention to subject matter knowledge *for* teaching (in addition to subject matter knowledge per se) (Shulman, 1987), in (b) providing insights into critically important determinants of what teachers do and why they do it, namely teachers' *resources* (including knowledge), *orientations* (including beliefs), and *goals* (Schoenfeld, 2010), and in (c) highlighting the multiple dimensions of *teachers' proficiency*, including, but not limited to, knowing students as thinkers and learners (Schoenfeld & Kilpatrick, 2008). The latter contribution builds the bridge to discipline-specific frameworks since Schoenfeld and Kilpatrick initially developed the framework of teachers' proficiency in the context of *mathematics*.

A substantial body of research work is located in mathematics education, providing both *discipline*- and *domain*-specific frameworks and models (e.g., Ball, Thames & Phelps, 2008; Baumert et al., 2010; Blömeke, Hsieh, Kaiser, & Schmidt, 2014;

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 129-136. Hobart, Australia: PME.

Fennema & Franke, 1992; Kilpatrick, Blume, & Even, 2006; Rowland, Huckstep, & Thwaites, 2005; Tatto, Schwille, Senk, Ingvarson, Peck, & Rowley, 2008). These frameworks and models of *knowledge for teaching mathematics* can be understood as elaborating rather than replacing Shulman's (1986; 1987) contribution to the field. The approaches taken, and the conceptualisations of mathematics teachers' knowledge proposed, are not inclusive, nor are the identified dimensions of mathematics teachers' knowledge mutually exclusive. In contrast, the identified dimensions are *complementary*, and provide, taken together, a more *refined* picture of the knowledge base for teaching mathematics (see, Scheiner, 2015).

Notice that, with few exceptions (e.g., Even, 1990), researchers have almost overlooked *concept*-specific frameworks. However, from the author's perspective, investigating teachers' knowledge at the level of specific concepts is an important issue that needs particular attention in future research efforts.

MOVING BEYOND PAST AND CURRENT TRENDS IN RESEARCH ON MATHEMATICS TEACHERS' KNOWLEDGE

As described in detail elsewhere (Scheiner, 2015), several trends can be identified in past and current practices in research on mathematics teachers' knowledge. For the purposes of this paper, the attention is drawn to two particular trends:

(1) Although the discipline-specific frameworks mentioned above differ in detail, many of them converge in efforts to further *extend* and *refine* the construct of subject matter knowledge (SMK) and pedagogical content knowledge (PCK).

(2) With few exceptions, the literature tends to a particular orientation, namely the idea of a teachers' capacity to *unpack subject matter knowledge* in ways that are accessible to their students.

In more detail, the literature suggests that subject matter knowledge (SMK), for instance, can be further extended and refined in *qualitatively different* sub-dimensions such as Bromme's (1994) distinction between school mathematical knowledge and academic content knowledge. However, of particular importance and interest are contributions that reflect the idea that there is *unique* content knowledge for teaching mathematics. For instance, the notion of 'specialised content knowledge' introduced by Ball and her colleagues is described as pure content knowledge "that is tailored in particular for the specialised uses that come up in the work of teaching" (Hill et al., 2008, p. 436). In this sense, and in contrast to Shulman (1986) treating 'SMK *for* teaching' as equivalent to PCK, these considerations lead to the claim that there is pure mathematical knowledge specialised for teaching mathematics. Thus, it seems reasonable to distinguish between *mathematical content knowledge per se* (MCK per se) and *mathematical content knowledge for teaching* (MCK for teaching) (see, Scheiner, 2015).

However, recent approaches in the literature on the knowledge base for teaching mathematics center their focus on the *subject matter content* and articulate the importance of the central teaching task that is making the mathematics content accessible to students. In the literature on mathematical knowledge for teaching, these recent practices are reflected in the metaphor of 'teachers' *unpacking* of mathematics content in ways accessible to their students'. The author argues that this dominating *content-oriented* focus can be traced back to Shulman's (1987) conceptualisation of PCK as the capacity of '*transforming*' subject matter of the discipline to subject matter of the school subject. To put it in other words, most of the contributions in the 'mathematical knowledge for teaching' literature tend to be associated with a particular 'school of thought', namely Shulman's (1987) idea of a teacher's capacity for transformation of the subject matter – the capacity to deconstruct one's own knowledge into a less polished final form where critical components are accessible and visible.

Drawing on recent theoretical reflections on conceptualising (mathematics) teachers' knowledge (e.g., Scheiner, 2015), the work calls to broaden the perspective to include an epistemological, a cognitive, and a didactical dimension (see, Figure 1), in addition to a content dimension.

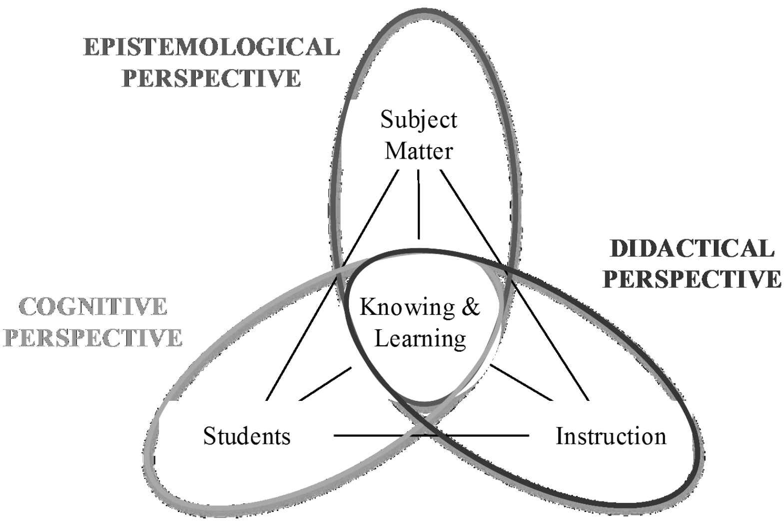

Figure 1: The epistemological, cognitive, and didactical perspective

The *epistemological* dimension refers to knowledge about the epistemological foundations of mathematics and mathematics learning (see, Bromme, 1994). For instance, Harel (e.g., 2008) calls for teachers' knowledge of epistemological issues involved in the learning of specific mathematical concepts including knowledge of epistemological obstacles. The *cognitive dimension* refers to knowledge of students' cognitions (Fennema & Franke, 1992), in particular, knowledge of students' common conceptions, knowledge of students' cognitive difficulties involved in concept construction (Harel, 2008), and the interpretation of students' emerging thinking (Ball et al., 2008). In other words, it includes knowledge of how students think, learn, and acquire specific mathematical knowledge (Fennema & Franke, 1992). The *didactical* *dimension* refers to what Shulman (1986, p. 9) described as knowledge of "the most useful ways of representing and formulating the subject that make it comprehensible to others", including teachers' illustrations and alternative ways of representing concepts (and the awareness of the relative cognitive demands of different topics) (Rowland et al., 2005) and knowledge of the design of instruction (Ball et al., 2008).

These various dimensions (epistemological, cognitive, and didactical) are considered as useful *lenses* in investigating (mathematics) teachers' professional knowledge, in particular, in describing the interconnectedness of knowledge of subject matter, knowledge of students' understanding, and knowledge of instructional strategies. These three resources (subject matter, students' understanding, and instruction) should be directed towards the same goals (i.e., learning goals) and reinforce each other rather than working past each other. However, this is often challenging to achieve. Often what is missing is a central theoretical framework or model about *knowing* and *learning* which guides the process and around which the three resources can be coordinated. From this perspective, a model of cognition and learning may serve as a cornerstone that brings cohesion to subject matter, students' understanding, and instruction (see, Fig. 1).

Bringing these perspectives into focus, several extensions and refinements of Shulman's initial categories of subject matter knowledge and pedagogical content knowledge can be identified, namely (a) knowledge of students' mathematical thinking and understanding (KSU), (b) knowledge of learning mathematics (KLM), (c) knowledge of teaching mathematics (KTM), (d) mathematical content knowledge per se (MCK per se), and (e) mathematical content knowledge for teaching (MCK for teaching).

In summary, the teachers' knowledge base can, and should, be examined from a range of angles using different lenses, including an epistemological lens (knowledge of learning mathematics), a cognitive lens (knowledge of students' mathematical thinking and understanding), a didactical lens (knowledge of teaching mathematics), and a content-oriented lens (MCK per se and MCK for teaching).

A STRUCTURAL DESCRIPTION OF TEACHERS' KNOWLEDGE: THE NATURE AND FORM

In the past, the literature concentrated its focus on what the teachers' knowledge is about. In doing so, the literature limited its attention to the *content* teachers do or should possess. What is missing in the current landscape of the conceptualisation of mathematics teachers' knowledge are efforts in going beyond what the teachers' knowledge is about to include a *structural description* of teachers' professional knowledge. Of course, several perspectives for theoretical reflection on the nature and form of teachers' knowledge can be presented (Scheiner, accepted), including those concerning the *nature* of the knowledge such as

(a) *source* What are the constituent knowledge bases?

(b) *development* Does the transformation of subject matter knowledge per so to subject matter knowledge for teaching takes place by the individual teacher situated in the act of teaching or is it supported by educators and curriculum?

(c) *specificity* Is the knowledge general, subject-, domain-, or topic-specific?

as well as those concerning the *form* of the knowledge such as

(i) *degree of integration* Does the amount of knowledge in each knowledge domain matter most or the degree of integration?

(ii) *size* Does the knowledge comes in pieces, units, or schemes? Is the knowledge stable and coherent or contextually-sensitive and fluid?

From the author's perspective, the major issues that need better resolution if we are to understand teachers' acquisition of an integrated knowledge base are questions concerning the *nature* and *form* of teachers' professional knowledge. In the following, new avenues for theoretical reflection on these issues are outlined. The objective of such theoretical reflection is evolving – aiming to make new theoretical extensions and innovations.

Teachers' knowledge as a complex system of 'knowledge atoms'

Although the various frameworks and models on the construct of mathematics teachers' knowledge have provided crucial insights on what mathematics teachers' knowledge is about, several of the discipline-specific frameworks represent conceptualisations of mathematics teachers' knowledge by a very general approach that seem ad hoc. The author, by contrast, does not believe in the existence of a general framework on teachers' knowledge but rather thinks that in investigating the form and nature of teachers' knowledge various frameworks may be discovered, which will be quite specific to particular mathematical concepts and individuals.

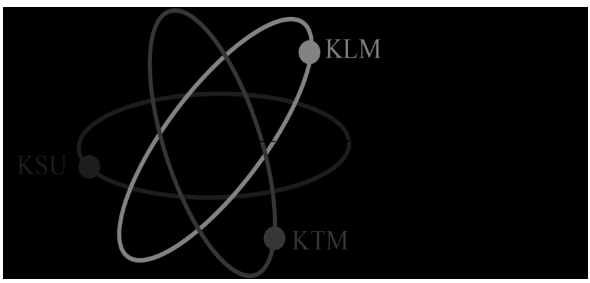

Figure 2: The 'knowledge atom'

The author calls for paying attention to investigating what in this paper is called 'knowledge for teaching mathematics' considered as a pool of personal and private constructed pieces of knowledge that have been transformed along a variety of knowledge bases identified in previous research investigating the multidimensionality of teachers' knowledge. In more detail, this work emphasises to view the professional knowledge for teaching mathematics as the repertoire of '*knowledge atoms*' that have been transformed along (1) knowledge of students' mathematical thinking and understanding (KSU), (2) knowledge of learning mathematics (KLM), and (3) knowledge of teaching mathematics (KTM), taking (4) mathematical content knowledge per se (MCK per se) and (5) mathematical content knowledge for teaching (MCK for teaching) as the cornerstones (see, Fig. 2). Notice that (i) the notion of 'transformation' implies that the constituent knowledge bases are inextricably combined into a new form of knowledge that is more powerful than the sum of its parts (*degree of integration*). (ii) In contrast to Shulman and his proponents' work, it is KSU, KLM, and KTM, together with MCK per se and MCK for teaching that build the knowledge dimensions that serve as the constituent knowledge bases for teaching mathematics (*source*). (iii) The notion of 'knowledge atom' indicates that knowledge is of a microstructure, highly context-sensitive, and concept-specific and has to be considered as of a fine-grained size (*specificity* and *size*). (iv) The notion of 'repertoire' indicates that knowledge is personal and private and that teacher education programs can only provide (as good as possible) rich resources for building up a fruitful repertoire of knowledge atoms (*development*).

The above mentioned considerations draw on the '*knowledge in pieces*' framework developed by diSessa (e.g., 1993), in particular taking the view of knowledge as microstructures coming in a loose structure of quasi-independent, atomistic knowledge pieces. Form the author's perspective, the 'knowledge in pieces' framework provides a rich resource on which to explore these, and related, issues.

NEW PRACTICES IN RESEARCH ON TEACHERS' KNOWLEDGE: MODELING TEACHERS' KNOWLEDGE AT THE 'KNOWLEDGE LEVEL'

As stated in the previous section, with few exceptions, past and current research seems to have skipped describing and characterising the structure and organisation of teachers' knowledge. One of the aims of this work was to progress toward a structural description of teachers' knowledge, and the previous section may have moved in that direction. Since the lack of a theoretical foundation of an adequate description concerning the *form* and *nature* of teachers' knowledge is recognised, research is needed that looks at knowledge (and processes of knowledge development) in fine-grained detail, through which a theoretical framework evolves. A structural description of teachers' knowledge is, at least from the author's perspective, an ongoing process that is always subject to new information and insights. With this, the objective of such research is evolving – by simultaneously developing theory and empirical research. Though a comprehensive theory is targeted, seeking not 'grand theory' but "humble theory" (diSessa & Cobb, 2004) with multiple cycles of revision and extension seems to be appropriate.

Research efforts on the way to a suitable description concerning the *form* and *nature* of teachers' knowledge should take place at the background of well-established practices in research on teachers' professional knowledge describing and identifying what the knowledge is about (concerning *content*). From the author's perspective, it is time to move toward new practices in research on teachers' knowledge that examine in a dialectic way both (1) the nature of certain kinds of teachers' knowledge (theory development, concerning *form*) and (2) what people know of that kind (empirical work, concerning *content*).

Research is needed that aims to *model* (mathematics) teachers' knowledge at the '*knowledge level*', for instance, by drawing on the methodological approach employed by researchers working with the 'knowledge in pieces' framework (diSessa, Sherin, & Levin, in process), namely *knowledge analysis*. Within the wide range of types of methodologies in 'knowledge analysis', in terms of time-scale, empirical and theoretical focus, in particular, microanalytic and microgenetic methods provide a good target for a complex, integrated, and dialectical research design. From the author's perspective, *knowledge analysis* may challenge the boundaries of what is known, and may provide a rich resource for a more complete and nuanced understanding of teachers' knowledge.

REFERENCES

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Baumert, J., Kunter, M., Blum, W., Brunner, M, Voss, T., Jordan, A., Klusmann, U., & Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, *47*(1), 133-180.

Blömeke, S., Hsieh, F.-J., Kaiser, G., & Schmidt, W. H. (Eds.). (2014). *International perspectives on teacher knowledge, beliefs and opportunities to learn. TEDS-M results*. Heidelberg: Springer.

Bromme, R. (1994). Beyond subject matter: A psychological topology of teachers' professional knowledge. In R. Biehler, R. W. Scholz, R. Strässer, & B. Winkelmann (Eds.). *Mathematics didactics as a scientific discipline: The state of the art* (pp. 73-88). Dordrecht: Kluwer.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction*, *10*(2-3), 105-225.

diSessa, A. A., & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *Journal of the Learning Sciences*, *13*(1), 77-103.

diSessa, A. A., Sherin, B., & Levin, M. (in process). Knowledge analysis: An introduction. (in preparation).

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, *21*(6), 521-544.

Fennema, E., & Franke, K. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-163). New York: Macmillan.

Harel, G. (2008). A DNR perspective on mathematics curriculum and instruction. Part II: With reference to teacher's knowledge base. *ZDM – The International Journal on Mathematics Education*, *40*, 893-907.

Kilpatrick, J., Blume, G., & Allen, B. (2006). *Theoretical framework for secondary mathematical knowledge for teaching*. University of Georgia & Pennsylvania State University.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255-281.

Scheiner, T. (2015). Lessons we have (not) learned from past and current conceptualizations of mathematics teachers' knowledge. Research paper presented at the *9th Congress of European Research in Mathematics Education* (CERME). Prague, Czech Republic: CERME.

Scheiner, T. (accepted). Conceptualization of teachers' knowledge: Shifting the attention to the nature and form. Paper to be presented to the *16th Biennial Conference of the European Association for Research on Learning and Instruction*. Limassol, Cyprus: EARLI.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Schoenfeld, A. H., & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching mathematics. In D. Tirosh & T. Wood (Eds.), *International handbook of mathematics teacher education* (Vol. 2, pp. 321-354). Rotterdam, The Netherlands: Sense Publishers.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*, 1-22.

Tatto, M. T., Schwille, J., Senk, S., Ingvarson, L., Peck, R., & Rowley, G. (2008). *Teacher education and development study in mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics. Conceptual framework.* East Lansing, MI: Teacher Education and Development International Study Center, College of Education, Michigan State University.

EFFECTS OF ENJOYMENT AND BOREDOM ON STUDENTS' INTEREST IN MATHEMATICS AND VICE VERSA

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Enjoyment, boredom, and interest are important for students' learning. To clarify the interplay between these affective variables, data from an interventional study of 119 ninth graders were analysed. Interest was assessed before and after, and emotions (enjoyment and boredom) were assessed during the five-lesson teaching unit. The results showed that (1) students who enjoyed their lessons were less bored than students who did not enjoy their lessons (2) enjoyment, but not boredom, during a teaching unit depended on students' initial interest in mathematics, and (3) students' initial interest and enjoyment during the teaching unit predicted their interest at posttest, but boredom did not influence students' interest at posttest.

INTRODUCTION

Emotions and motivational orientations such as interest are important for students' learning (Zan, Brown, Evans, & Hannula, 2006) and are related to students' performance in mathematics (Schukajlow, accepted; Schukajlow & Krug, 2014a). However, we do not know much about the development of interest or the role of emotions in this process. In the current study, I address this research gap by examining the interplay of interest, enjoyment, and boredom in the framework of a short-term intervention with regard to the teaching of modelling competency. The research questions were about the relation between students' enjoyment and boredom in mathematics classes, the importance of students' initial interest for enjoyment and boredom in mathematics classes, and the influence of initial interest, enjoyment, and boredom on interest at posttest.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Interest

Interest characterises a relation between a person and an object such as mathematics. Interested learners engage with the object of their interest over time (Hidi & Renninger, 2006). Models of interest development assume that this motivational variable develops from the situational interest that can be captured, for example, in the so-called interest-dense situations of a given moment (Bikner-Ahsbahs, 2004) by individual (or personal) interest. Individual interest is comprised of cognitive and affective aspects. Cognitive aspects include the attribution of personal significance to object-related activities and feelings of competence in the target domain. Emotions refer to the affective aspects of interest. Positive emotions occur when a person engages with an object of interest, whereas negative emotions do not accompany such an engagement. According to this conception, interested students enjoy doing mathematics and are not bored when solving mathematical problems.

Students' interests change during the school years. In most studies, students tend to report a decline in their interest in mathematics and science from primary to secondary school. In Frenzel et al.'s (2012) study, which assessed the cognitive and affective aspects of interest in student interviews, students in grade 5 frequently verbalised the affective aspects and rarely verbalised the cognitive aspects of interest in comparison with students in grade 9. This finding is in line with the four-phase theory of interest development (Hidi & Renninger, 2006). In the initial phase, which often occurs in the early grades, positive feelings are crucial for triggering interest. In the second and third phases, other variables such as knowledge and reengagement in the domain accompany interest development. Finally, students achieve a self-generated phase of interest and can regulate their interest-related activities on their own. We know from research in other domains that only a few students attain a well-developed level of interest in school. Thus, initiating situations that stimulate positive, and prevent negative, emotions during mathematics lessons is important for improving students' interest in both primary and secondary school.

Enjoyment

Enjoyment is one the most frequently reported positive emotions in the classroom. Students' enjoyment was found to be related to effort and performance (Schukajlow & Krug, 2014a) and was found to predict self-regulation skills and academic achievements (Ahmed, van der Werf, Kuyper, & Minnaert, 2013). According to the control-value theory of achievement emotions (Pekrun, 2006), enjoyment is a positive activating emotion and can affect whether students will engage and reengage with the enjoyable content. In this way, enjoyment might not only accompany interest development but may also have a positive influence on it. Self-concept has been identified as an important predictor of students' enjoyment (Goetz, Frenzel, Hall, & Pekrun, 2008). Another valuable factor for the development of students' academic enjoyment may be the solving of demanding, authentic problems or cooperation during the learning process (Pekrun, 2006).

Boredom

Similar to enjoyment, achievement boredom is an activity-related emotion that accompanies learning. Feelings of boredom are not simply a lack of interest or enjoyment. If students are not interested in mathematics or do not enjoy mathematics classes, they may feel very different negative emotions such as anger or frustration, but they are not always bored with it. Self-perceived levels of boredom depend to a large extent on students' general experiences in school and in particular on their experiences in specific school subjects (Jablonka, 2013). Boredom is one of the negative deactivating emotions and is reported more frequently during learning than anxiety, anger, frustration, hopelessness, and shame (Ahmed, van der Werf, Minnaert, & Kuyper, 2010). Boredom results from a lack of controllability over actions (Pekrun, 2006) and in most studies has been found to be negatively related to performance in mathematics (see summary by Schukajlow, accepted).

Affect measurement

One important characteristic of measures of affect are their trait-like vs. state-like nature. Trait-like scales assess the construct in general, that is, over time. A sample item representing enjoyment as a trait is: "I enjoy mathematics." State-like scales collect data with regard to a specific point in time: "I enjoyed mathematics class *today*." The two potential ways to assess affect differ in their stability and sensitivity. The trait-like scales are more stable and show low sensitivity with regard to interventional programs, whereas the state-like scales show minor changes in the affective measures and are sensitive to treatment.

Items that measure affect assess different dimensions of the constructs such as cognitive or emotional ones for interest and describe typical situations or activities. For mathematics, one of the key activities is problem solving. Thus, self-reported items for the measurement of affect often refer to the solving of problems, to mathematical reasoning, or recently – in task-specific questionnaires – even demonstrate sample problems for students (Krug & Schukajlow, 2013; Schukajlow et al., 2012).

The relationships between interest, enjoyment, and boredom

Most studies have found a positive relationship between interest and enjoyment. The value of correlations between interest and enjoyment for young secondary school students depends on the measures used to assess the affective constructs and ranges from low for task-unspecific questionnaires (Ahmed, Minnaert, Van der Werf, & Kuyper, 2008), to high, for task-specific questionnaires (Schukajlow et al., 2012).

The subjective psychological state of disinterest in response to low levels of arousal accompanies the state of boredom (Vogel-Walcutt, Fiorella, Carper, & Schatz, 2012). Because of its aversive and avoidance-oriented nature, boredom is incompatible with interest or enjoyment. Thus, a negative relationship between boredom and interest or enjoyment can be expected. A low, but statistically significant negative correlation between enjoyment and boredom, was reported for 7th graders (Ahmed et al., 2010). For university students studying the social sciences, boredom during lessons was negatively related to intrinsic motivation, which is closely related to interest (Pekrun, Goetz, Daniels, Stupnisky, & Perry, 2010). No results were found that addressed this issue in school students in the mathematics domain.

Research questions

The research questions derived from the theoretical framework I addressed were:

1) Is students' boredom during mathematics classes related to their enjoyment?
2) How important is students' initial interest for enjoyment and boredom during mathematics classes?
3) To what extent do students' initial interest, as well as enjoyment and boredom during mathematics classes, influence interest at posttest?

METHOD

One hundred and nineteen German ninth graders from 6 middle-track school classes (62% female; mean age=15.2 years) were asked about their initial interest before a 5-lesson-long teaching unit, about their enjoyment and boredom during the teaching unit, and about their interest after the teaching unit (see Fig. 1). During the teaching unit, students solved modelling problems with, vs. without, missing information in group work and were asked to find one vs. two solutions for each problem. At least one person from the research group was present to administer the tests and to observe the implementation of the treatment. All students' solutions were collected. Analyses of the reports and solutions showed that students worked on the modelling problems as intended (for more information, see Schukajlow & Krug, 2014b).

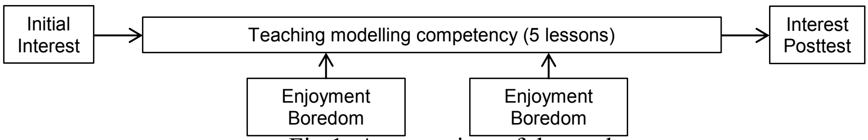

Fig.1: An overview of the study

Measures

Interest, enjoyment, and boredom were assessed with scales used in other studies and each consisted of 3 statements that were answered on 5-point Likert scales ranging from 1 (strongly disagree) to 5 (strongly agree). Sample items are "I am interested in mathematics," "I enjoyed task processing," or "Task processing was boring." The Cronbach's alpha reliabilities were .80 and .74 for interest (on the pretest and posttest), .84 and .82 for enjoyment, and .85 and .86 for boredom (at the first and second measurement points during the teaching unit) (cf. Fig. 1). Enjoyment and boredom were measured twice during treatment and were aggregated into a mean value.

RESULTS

The means and standard deviations for interest, enjoyment, and boredom are presented in Table 1. Students' interest in mathematics at pretest and posttest was slightly under and students' enjoyment during the teaching unit was slightly above the theoretical mean of 3. Most of students were not bored during instruction.

	initial interest	interest at posttest	enjoyment	boredom
Mean (SD)	2.56(.940)	2.71(.94)	3.40(.87)	1.95(.81)

Table 1: Means and standard deviations for interest, enjoyment, and boredom.

The first research question was about the relation between boredom and enjoyment during the teaching unit. The analysis supported my expectation of a negative relation between the two emotions: The correlation between enjoyment and boredom was moderate and negative (-.51, see Table 2). Thus, students who enjoyed the task processing did not feel bored when solving the problems.

	initial interest	interest at posttest	enjoyment	boredom
initial interest	1	.60*	.24*	.07
interest at posttest		1	.36*	.00
enjoyment			1	-.51*
boredom				1

Note: $*p<.05$

Table 2: Pearson correlations between interest, enjoyment, and boredom.

Second, the effects of initial interest on students' enjoyment and boredom during the teaching unit were analysed. As initial interest was measured before the teaching unit, the correlations between initial interest and emotions might be interpreted as regressions. This analysis partially confirmed expectations about the impact of initial interest on emotions - there were positive effects of initial interest on enjoyment but not on boredom. Thus, students with higher initial interest in mathematics enjoyed the task processing more than students with low interest did. However, interested and uninterested students showed equal amounts of boredom during task processing.

The third research question concerned the effects of initial interest, enjoyment, and boredom on students' interest at posttest. To answer this question, a linear regression analysis with interest at posttest as the dependent measure and initial interest, enjoyment, and boredom as the independent measures was applied. Forty one per cent of the variance in students' interest at posttest was explained by the hypothesised regression model ($R^2=.41$). Students' initial interest was revealed to be the most powerful predictor of interest at posttest ($\beta=.52$, $p<.05$). Furthermore, enjoyment but not boredom during the teaching unit affected interest at posttest (enjoyment: $\beta=.30$, $p<.05$; boredom: $\beta=.11$, $p>.10$). This result was partly in line with the theoretically derived assumptions. The analysis indicated the importance of students' initial interest and their enjoyment during task processing for students' interest at posttest. Students' boredom while solving mathematical problems did not negatively influence their interest at posttest.

SUMMARY AND DISCUSSION

The current paper investigated the interplay between interest, enjoyment, and boredom using student questionnaires administered before, during, and after the teaching unit with regard to the enhancement of students' interest in solving real-world problems.

Descriptive findings revealed a high level of enjoyment and a low level of boredom during the teaching unit. Enjoyment measured in other studies was clearly under 2.5 (between 1.98 and 2.36 by Ahmed et al., 2013) and boredom was over 2.0 (between 2.19 and 2.64 by Ahmed et al., 2013). One possible explanation for this finding may be the processing of cognitively stimulating tasks with a connection to the real world and cooperative group work during the teaching unit (Schukajlow & Krug, 2014b).

In line with the results of other studies (Ahmed et al., 2010; Pekrun et al., 2010), a negative relationship between enjoyment and boredom was found. Indeed, as expected according to the control-value theory of achievements emotions, the avoidant, aversive, and low-arousal psychological state of boredom is incompatible with students' enjoyment (Pekrun et al., 2010; Vogel-Walcutt et al., 2012). Thus, stimulating students' enjoyment decreases the level of boredom they feel during task processing and vice versa.

Students' initial interest in mathematics was expected to be an important factor that would positively influence enjoyment and negatively influence boredom because interested students enjoy engaging with their object of interest and are not bored with it. In line with theoretical considerations and previous empirical results on the correlation between the two variables (Ahmed et al., 2008; Schukajlow et al., 2012), initial interest in mathematics positively affected enjoyment during the teaching unit. One implication of this finding is that it is necessary to improve students' interest in mathematics so that they can achieve greater enjoyment while solving mathematical problems. Theories of interest suggest that interest-dense situations while students learn can capture their situated interest in the classroom, which can be developed into a stable individual interest in mathematics over time (Bikner-Ahsbahs, 2004). Stimulating learning materials and opportunities for students to engage in social interactions while solving mathematical tasks are important features of learning environments that offer opportunities for interest development. Experiences of competence while solving mathematical problems, which can be improved, for example, by teaching students to provide multiple solutions to real-world problems, has been revealed to be a crucial factor that positively affects students' individual interest (Schukajlow & Krug, 2014b).

An unexpected result of the present study was a zero correlation between prior interest and boredom. However, previous findings on the negative connection between the two affective variables have been based on samples of university students from the social science domain. Thus, the connection may be different for school students and in the domain of mathematics. Another explanation for the zero correlation between interest and boredom may be the specific kind of task (real-world problems) that was used in the current study. It is possible that students' interest in mathematics emerges from a positive relationship with intra-mathematical tasks, which are often solved in the regular mathematics classroom. Conversely, real-world problems are rarely solved in school. Students' interest in intra-mathematical tasks is connected with their interest in real-world problems, but the two are not identical (correlation of .68 by Schukajlow et al., 2012). Thus, the influence of initial interest on boredom may be different if intra-mathematical tasks are used in the classroom.

Finally, I found positive effects of initial interest and enjoyment but not boredom on students' interest at posttest. The positive effect of prior interest on interest at posttest found in other studies was also confirmed in the present study and showed that interest in mathematics remains stable over time (Schukajlow & Krug, 2014b). Students' enjoyment while solving problems during the teaching unit was also found to be a valuable predictor of their interest. Students who enjoyed solving the mathematical problems reported higher interest than students who did not enjoy the task processing. As students' self-concept was previously shown to be an important factor for students' enjoyment (Goetz et al., 2008), fostering their self-concept can affect their enjoyment and by affecting their enjoyment, it may also positively affect students' interest. The use of authentic mathematical tasks, cooperation during the learning process, as well as teacher enjoyment and enthusiasm have also been found to be valuable factors that influence students' enjoyment (Pekrun, 2006), and according to the results of the present study, these factors could also affect their interest in mathematics. How to improve enjoyment and interest is an important open question for future studies.

The main limitations of the current study are that we applied an intervention with a short duration and that we used problems that differed from the specific kinds of problems that are usually solved in the classroom. Different results may occur in long-term studies and if students are asked to solve other kinds of problems that are more typically found in mathematics classrooms.

Summarising the results of the present study, I would like to emphasise that a close reciprocal connection between the positive emotion of "enjoyment" and interest was found. Initial interest influenced enjoyment during task processing, and enjoyment while learning mathematics affected students' interest after the teaching unit. The analysis of the students' negative emotion "boredom" revealed a different pattern. Boredom was related to enjoyment but was not related to interest. This result enhances the importance of overcoming a simplistic view of emotions with regard to their value as positive or negative (Hannula, Pantziara, Wæge, & Schlöglmann, 2009). More research on specific emotions using quantitative and qualitative methodology is essential as each emotion may have its own dynamic and might show different relations to other motivational and achievement factors.

REFERENCES

Ahmed, W., Minnaert, A., Van der Werf, G., & Kuyper, H. (2008). Perceived social support and early adolescents' achievement: The mediating roles of motivational beliefs and emotions. Journal of Youth & Adolescence, 39, 36-46.

Ahmed, W., van der Werf, G., Kuyper, H., & Minnaert, A. (2013). Emotions, self-regulated learning, and achievement in mathematics: A growth curve analysis. Journal of Educational Psychology, 105(1), 150-161.

Ahmed, W., van der Werf, G., Minnaert, A., & Kuyper, H. (2010). Students' daily emotions in the classroom: Intra-individual variability and appraisal correlates. British Journal of Educational Psychology, 80(4), 583-597. doi: 10.1348/000709910x498544

Bikner-Ahsbahs, A. (2004). Interest-dense situations and their mathematical valences. Paper presented at the Tenth International Congress in Mathematics Education, Copenhagen.

Frenzel, A. C., Pekrun, R., Dicke, A.-L., & Goetz, T. (2012). Beyond quantitative decline: Conceptual shifts in adolescents' development of interest in mathematics. Developmental Psychology, 48(4), 1069-1082.

Goetz, T., Frenzel, A. C., Hall, N. C., & Pekrun, R. (2008). Antecedents of academic emotions: Testing the internal/external frame of reference model for academic enjoyment. Contemporary Educational Psychology, 33(1), 9-33.

Hannula, M. S., Pantziara, M., Wæge, K., & Schlöglmann, W. (2009). Introduction. multimethod approaches to the multidimensional affect in mathematics education. Paper presented at the CERME 6, Lyon.

Hidi, S., & Renninger, K. A. (2006). The four phase model of interest development. Educational Psychologist, 41(2), 111–127.

Jablonka, E. (2013). Boredom in mathematics classrooms from Germany, Hong Kong and the United States. In B. Ubuz, Ç. Haser & M. A. Mariotti (Eds.), Proceedings of the Eight Congress of the European Society for Research in Mathematics Education (pp. 1885-1894). Ankara, Turkey: Middle East Technical University.

Krug, A., & Schukajlow, S. (2013). Problems with and without connection to reality and students' task-specific interest. In A. M. Lindmeier & A. Heinze (Eds.), Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 209-216). Kiel, Germany: PME.

Pekrun, R. (2006). The control-value theory of achievement emotions: Assumptions, corollaries, and implications for educational research and practice. Educational Psychology Review, 18, 315-341.

Pekrun, R., Goetz, T., Daniels, L. M., Stupnisky, R. H., & Perry, R. P. (2010). Boredom in achievement settings: Exploring control-value antecedents and performance outcomes of a neglected emotion. Journal of Educational Psychology, 102(3), 531 - 549.

Schukajlow, S. (accepted). Is boredom important for students' performance? In N. Vondrova & J. Novotna (Eds.), Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education. Prag, Szech Republic: Charles University of Prag.

Schukajlow, S., & Krug, A. (2014a). Are interest and enjoyment important for students' performance? In C. Nicol, S. Oesterle, P. Liljedahl & D. Allan (Eds.), Proceedings of the Joint Meeting of PME 38 and PME-NA 36 (Vol. 5, pp. 129-136). Vancouver, Canada: PME.

Schukajlow, S., & Krug, A. (2014b). Do multiple solutions matter? Prompting multiple solutions, interest, competence, and autonomy. Journal for Research in Mathematics Education, 45(4), 497-533.

Schukajlow, S., Leiss, D., Pekrun, R., Blum, W., Müller, M., & Messner, R. (2012). Teaching methods for modelling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations. Educational Studies in Mathematics, 79(2), 215-237.

Vogel-Walcutt, J. J., Fiorella, L., Carper, T., & Schatz, S. (2012). The definition, assessment, and mitigation of state boredom within educational settings: a comprehensive review. Educational Psychology Review, 24, 89-111.

Zan, R., Brown, L., Evans, J., & Hannula, M. S. (2006). Affect in Mathematics Education: An Introduction. Educational Studies in Mathematics, 63(2), 113-122.

VALIDATION OF PROOFS AS A TYPE OF READING AND SENSE-MAKING

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We present results on the proof validation behaviours of 16 U.S. undergraduates after taking an inquiry-based transition-to-proof course. Participants were interviewed individually towards the end of the course using the same protocol used by Selden and Selden (2003). We describe participants' observed validation behaviours and provide descriptions of their evaluative comments and their sense-making attempts. We make the, perhaps counterintuitive, suggestion that taking an inquiry-based transition-to-proof course emphasising proof construction, in which validation was modelled, may not enhance students' abilities to judge the correctness of other students' proof attempts.

INTRODUCTION

We consider proof validation as a type of reading and sense-making within the genre of proof. In line with reading comprehension researchers, we view reading as an active process of meaning-making in which readers use their knowledge of language and the world, including the mathematical world, to construct situation models of texts in light of their backgrounds and experience (e.g., Kintsch, 2004).

Past validation studies include: first-year Irish undergraduates' validations and evaluations (Pfeiffer, 2011); U.S. undergraduates' validations at the beginning of a transition-to-proof course (Selden & Selden, 2003); U.S. mathematics majors' validation practices across several content domains (Ko & Knuth, 2013); U.S. mathematicians' validations (Weber, 2008); and comparison of U.K. novices' and experts' validation behaviours, using eye-tracking (Inglis & Alcock, 2012). In contrast, we considered students' validation behaviours after having taken a course in proof construction that emphasised validation, something not done by the other studies.

THEORETIAL PERSPECTIVE

We view the proof construction process as a sequence of mental or physical actions in response to situations in a partly completed proof. This process, even when accomplished with few errors or redundancies, contains many more actions, or steps, than appear in the final written proof and cannot be fully reconstructed from a final written proof. For example, actions, such as "unpacking" (Selden & Selden, 1995) the conclusion to see what one is being asked to prove, or drawing a diagram, may not appear in the final written proof, and hence, are often unavailable to students for later consideration and reflection.

Many proving actions appear to be the result of the enactment of small, automated situation-action pairs that have been termed *behavioural schemas* (Selden, McKee, &

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 145-152. Hobart, Australia: PME.

Selden, 2010). A common beneficial proving behavioural schema consists of a situation where one has to prove a universally quantified statement like, "For all real numbers x, P(x)" and the action is writing into the proof something like, "Let x be a real number," meaning x is arbitrary but fixed. Focussing on such behavioural schemas, that is, on small habits of mind for proving, has two advantages. First, the uses and interactions of behavioural schemas are relatively easy to examine. Second, this perspective is not only explanatory but also suggests concrete teaching actions, such as the use of practice to encourage the formation of beneficial schemas and the elimination of detrimental ones. (See Selden, McKee, & Selden, 2010, pp. 211-212).

While a number of proof construction actions have been investigated, thinking about proof validation actions is still in its infancy. However, it seems reasonable to conjecture, based on the extant proof validation literature (e.g., Inglis & Alcock, 2012; Selden & Selden, 2003; Weber, 2008) that examination of the overall structure of a proof is crucial for determining whether a given proof attempt, if correct, actually proves the statement (theorem) that it sets out to prove. In addition, it also seems that a careful line-by-line reading of a proof attempt is useful for determining whether individual assertions are warranted, either explicitly or implicitly (e.g., Weber & Alcock, 2005). However, one often also wants to "get a sense of" a proof—What makes it work? What are the key ideas? These questions refer to the explanatory function of proof. Our specific research question was: Would having taken a transition-to-proof course emphasising proof construction, in which validation was modelled, enhance university students' abilities to judge the correctness of other students' proof attempts?

ROLE OF VALIDATION

Although we are focussing here on the validation practices of U.S. undergraduates who are at least in their second-year of mathematics study, validation has a role to play throughout mathematics students' education and in mathematicians' practice.

Holders of U.S. bachelor's degrees in mathematics are normally expected, not only to know considerable mathematics content, but also to be able to construct moderately complex proofs and to solve moderately non-routine problems. Indeed, one major way that an individual's mathematical knowledge of a theorem is sometimes taken to be warranted is by the ability to "produce" a proof, not in a rote way, but in the way a mathematician would produce it, namely, with understanding (Rodd, 2000). However, constructing or producing proofs appears to be inextricably linked to the ability to validate them reliably, and a "proof" that could not be validated would not provide much of a warrant. Pre-service secondary mathematics education majors and in-service secondary mathematics teachers also need to be able to validate proofs reliably because school mathematics curricula are likely to place increasing emphasis on justification and proof (e.g., Common Core State Standards for Mathematics, 2014).

In addition, validation appears to play a fundamental role in mathematicians' practice. While some mathematicians can sometimes obtain conviction in other ways (Weber, Inglis, Mejia-Ramos, 2014), mathematicians' belief in the general reliability and unproblematic nature of validation supports the assurance needed to use a theorem in later work. That is, once a theorem is proved one can expect it to "stay proved."

RESEARCH SETTING

The course, from which the interviewees came, has been taught by the authors for several years at a U.S. Ph.D.-granting university. It is meant as a second-year university transition-to-proof course for mathematics and secondary education mathematics majors, but is often taken by a variety of other majors and by more advanced undergraduate students. The course is taught in a very modified Moore Method way (Mahavier, 1999). The students are given course notes with definitions, questions, requests for examples, and statements of theorems to prove.

The students in this course prove the theorems in the course notes outside of class and present their proofs in class on the blackboard and receive extensive critiques. These critiques consist of careful line-by-line readings and validations of the students' proof attempts, often with corrections and insertions of missing warrants. In a sense, the second author models proof validation for the students. The students are aware that being asked to present their proof attempts does not necessarily mean that these are correct, but rather that their proof attempts probably provide interesting points for the second author to discuss. This validation is followed by a second reading of the students' proof attempts, indicating how the proofs could be written in "better style" to conform to the genre of proofs. Once these corrections and suggestions have been made, the student, who made the proof attempt, is asked to write it up carefully, including the indicated corrections and suggestions, for duplication for the entire class. Given these careful critiques of student work—consisting of the line-by-line checking of students' proof attempts (i.e., modelling proof validation), followed by a second reading to help with the "style" in which proofs are written, and finally, a carefully rewritten final proof—we expected that the students might "adopt" some of the second author's techniques of validation and be able to implement them in their own proving attempts.

METHODOLOGY

Sixteen of the 17 students enrolled in the course opted to participate in the study for extra credit. Of these, 81% (13 of 16) were either mathematics majors, secondary education mathematics majors, or were in mathematics-related fields (e.g., electrical engineering, civil engineering, or computer science). Interviews were conducted outside of class during the final two weeks of the course. The students signed up for convenient one-hour time slots. They were told that they need not study for this extra credit session. The protocol was the same as that of Selden and Selden (2003).

Upon arrival, participants were first informed that they were going to validate four student-constructed "proofs" of a single number theory theorem (see Figure 1), indeed, that the proof attempts that they were about to read were submitted for credit by students, who like themselves, had been in a transition-to-proof course. The participants were asked to think aloud and to decide whether the student-constructed proof attempts were indeed proofs. Participants were encouraged to ask clarification questions and were informed that the interviewer would decide whether a clarification question could be answered.

For any positive integer n, if n^2 is a multiple of 3, then n is a multiple of 3.

"Proof (a)": Assume that n^2 is an odd positive integer that is divisible by 3. That is $n^2 = (3n + 1)^2 = 9n^2 + 6n + 1 = 3n(n + 2) + 1$. Therefore, n^2 is divisible by 3. Assume that n^2 is even and a multiple of 3. That is $n^2 = (3n)^2 = 9n^2 = 3n(3n)$. Therefore, n^2 is a multiple of 3. If we factor $n^2 = 9n^2$, we get 3n(3n); which means that n is a multiple of 3. ■

"Proof (b)": Suppose to the contrary that n is not a multiple of 3. We will let 3k be a positive integer that is a multiple of 3, so that 3k + 1 and 3k + 2 are integers that are not multiples of 3. Now $n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Since $3(3k^2 + 2k)$ is a multiple of 3, $3(3k^2 + 2k) + 1$ is not. Now we will do the other possibility, 3k + 2. So, $n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ is not a multiple of 3. Because n^2 is not a multiple of 3, we have a contradiction. ■

"Proof (c)": Let n be an integer such that $n^2 = 3x$ where x is any integer. Then $3|n^2$. Since $n^2 = 3x$, nn = 3x. Thus 3|n. Therefore if n^2 is a multiple of 3, then n is a multiple of 3. ■

"Proof (d)": Let n be a positive integer such that n^2 is a multiple of 3. Then n = 3m where $m \in \mathbf{Z}^+$. So $n^2 = (3m)^2 = 9m^2 = 3(3m^2)$. This breaks down into 3m times 3m which shows that m is a multiple of 3. ■

Figure 1: The student-constructed proof attempts that the participants saw

The interviews were audio recorded. The participants wrote as much or as little as they wanted on the sheets containing the student-constructed proof attempts. Participants took as much time as they wanted to validate each proof, with one participant initially taking 25 minutes to validate "Proof (a)".

The interviewer, who is the first author, answered an occasional clarification question, such as the meaning of the vertical bar in $3|n^2$, but otherwise only took notes, and handed the participants the next printed page when they were ready for it. The data collected included: the sheets on which the participants wrote, the interviewer's notes, and the recordings of the interviews.

These data were analysed and tallies were made of such things as: the number of correct judgments made by each participant individually; the percentage of correct judgments made by the participants (as a group); the validation behaviours that the interviewer observed; the validation comments that the participants made; the amount of time taken by each participant to validate each proof attempt; the number of times each participant reread each proof attempt; the number of participants who underlined or circled parts of the proof attempts; and the number of times the participants substituted numbers for n.

RESULTS

Given that validation can be difficult to observe, it is remarkable how verbal and forthcoming the participants in this study were. All participants appeared to take the task very seriously and some participants spent a great deal of time validating at least one of the student-constructed proof attempts. For example, LH initially took 25 minutes to validate "Proof (a)" before going on, and VL initially took 20 minutes to validate "Proof (b)". Here LH and VL are pseudonyms.

Participants' Evaluative Comments

The participants sometimes voiced what they didn't like about the student-constructed proof attempts. For example, CY objected to "Proof (b)" being referred to as a proof by contradiction. He insisted it was a contrapositive proof and twice crossed out the final words "we have a proof by contradiction". Fourteen (87.5%) mentioned the lack of a proof framework, or an equivalent, even though they had been informed at the outset that the students who wrote the proof attempts, unlike them, had not been taught to construct proof frameworks.

Participants seemed to be bothered by: (1) lack of clarity in the way the student-constructed proof attempts were written. Some referred to parts of the proof attempts as "confusing", "convoluted", "a mess", or not "making sense" (68.75%); (2) the notation, which one participant called "wacky"; (3) the fact that "Proof (d)" started with n, then introduced m, and did not go back to n; (4) not knowing what the students who had constructed the proof attempts knew or were allowed to assume; (5) having too much, or too little, information in a purported proof. For example, one participant said there was "not enough evidence for a contradiction" in "Proof (b)"; (6) the "gap" in "Proof (c)" which was remarked on by six participants.

Some Participants' Local and Overall Comments

Local comments on "Proof (a)":"[I] don't like the string of = s." (MO). "$3n+1$, if $n=1$, is not odd, [rather it] would be even." (KW). "This [pointing to $n^2 = 9n^2$] isn't equal." (AF).

Overall comments on "Proof (a)": "[It] needs more explanation -- I can't see where they are going." (CL). "[The] first case doesn't seem right." (CY). "Not going where they need to go." (KW). "Not a proper proof". (FR). "Partial proof". (MO).

Local comments on "Proof (b)": "Not seeing the closing statement." (FR). "Not a proof because we don't introduce n, but we use n." (KK).

Overall comments on "Proof (b)": "[This makes] a lot more sense to me [than "Proof (a)"]" (CL). "[It's] not written well." (SS). "[I] feel like it's a proof because [they're] showing that the two integers in between are not multiples of 3" (AF).

Local comments on "Proof (c)": Commenting on the use of the universal quantifier with *x*, "[The bit about] where *x* is *any* integer worries me" (CJ).

Overall comments on Proof (c)": "Just can't get my head around [it]." (CY). "Need more information. Don't buy it." (CJ). "[This one is] closer [to a proof] than the others." (KK). "Sound proof". (MO).

Local comments on "Proof (d)": "Why would you use *m*? ... [It's] kind of confusing with that *m*." (LH).

Overall comments on "Proof (d)": "[He is] putting [in] more information than needs to be [there]. [This does] not help his proof." (MO). "Not a strong proof." (LH).

What Participants Said They Do When Reading Proofs

In answer to some final debrief questions, all participants said that they check every step in a proof or read a proof line-by-line. All said they reread a proof several times or as many times as needed. All, but one, said that they expand proofs by making calculations or making subproofs. In addition, some volunteered that they work through proofs with an example, write on scratch paper, read aloud, or look for the proof framework. All of these actions can be beneficial. Indeed, it is quite reasonable to suspect something might be wrong with a proof, if in an initial line-by-line reading, one or more logical implications cannot be warranted by the reader (Weber & Alcock, 2005). Such a situation calls for a rereading, or a rethinking, of the proffered argument.

In addition, ten (62.5%) said they tell if a proof is correct by whether it "makes sense" or they "understand it". These are cognitive feelings that, with experience, can be useful. Four (25%) said a proof is incorrect if it has a [single] mistake, and four (25%) said a proof is correct "if they prove what they set out to prove." These last two views of proof call for some caution during implementation.

INTERPRETATION OF RESULTS

Participants' comments did not seem to focus primarily on whether the theorem had been proved. Rather, these included evaluative comments about whether they liked the student-constructed proof attempts, found them confusing or unclear in some way, or were lacking in some details or information.

According to the reading comprehension literature (e.g., Kintsch, 2004; Zwaan & Radvansky, 1998), unless reading is done totally superficially, the reader makes a situation model of the text being read. We conjecture that the participants in our study may have been attempting to make a "situation model" of each of the proofs, that is, they were trying to understand, and make sense of, where the authors of the proof attempts were "coming from". Perhaps that is why they made comments about the student-constructed proof attempts not "making sense", having "wacky" notation, or being "confusing", "convoluted", or "a mess". Students are not unique in their interest in understanding proofs. As Rav (1999) has stated, one important reason that mathematicians read proofs is to expand their understanding.

In addition to interpreting their task as first making sense of what they were reading, probably due to their prior experiences with reading and making situation models more generally, we conjecture that the participants in this study might have felt it important, perhaps even necessary, to gain a top-level view of each proffered argument, that is, to be able to comprehend it holistically (Mejia-Ramos, Fuller, Weber, Rhoads & Samkoff, 2012, pp. 10-11) before making a judgment on its validity. Indeed, Selden & Selden (2003, p. 5) said of an ideal validation that, “Towards the end of a validation, in an effort to capture the essence of the argument in a single train-of-thought, contractions of the argument might be undertaken.” Thus, perhaps the participants implicitly felt that making sense of the other students’ proof attempts, that is, of where the student authors’ were “coming from”, was a prerequisite to being able to judge whether they were indeed proofs.

DISCUSSION AND TEACHING IMPLICATIONS

In answer to our research question, the participants in this study took their task very seriously, but made fewer final correct judgments (73% vs. 81%) than the undergraduates studied by Selden and Selden (2003) despite, as a group, being somewhat further along academically. In this study, 56% (9 of 16) of the participants were in their fourth-year of university, whereas just 37.5% (3 of 8) of the undergraduates in the Selden and Selden (2003) study were in their fourth year.

Because the participants in this study were completing an inquiry-based transition-to-proof course emphasising proof construction, in which validation had been modelled extensively by the second author, we conjectured they would be better at proof validation than those at the beginning of a transition-to-proof course (i.e., those studied by Selden and Selden, 2003), but they weren’t. We have tentatively concluded that if one wants undergraduates to learn to validate “messy” student-constructed proof attempts, in a reliable way, one needs to teach validation explicitly, perhaps through validation exercises or activities.

We stress this because it may seem counterintuitive. We note that, as students, most mathematicians have received considerable implicit proof construction instruction through feedback on assessments and on their dissertations. However, most have received no explicit validation instruction, but are apparently very skilled at it; for otherwise, they would submit some invalid proofs for publication.

Finally, we note that, at least in the U.S., many future teachers of secondary or tertiary mathematics take a transition-to-proof course. Thus, for future pedagogical purposes, it would be useful for today’s mathematics and mathematics education majors to be taught to distinguish between a proof being valid (i.e., guaranteeing the truth of the claimed theorem) and having additional positive or negative features.

References

Common Core Standards Initiative (2014). *Common Core State Standards for Mathematics* [CCSS-M]. Downloaded March 11, 2014 from http://www.corestandards.org/Math/.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, *43*(4), 358-390.

Kintsch, W. (2004). The construction-integration model of text comprehension and its implications for instruction. In R. Ruddell & N. Unrau (Eds.) *Theoretical models and processes of reading*. 5th Edition, International Reading Association.

Ko, Y.-Y., & Knuth, E. J. (2013). Validating proofs and counterexamples across content domains: Practice of importance for mathematics majors. *Journal of Mathematical Behavior*, *32*, 20-35.

Mahavier, W. S. (1999). What is the Moore Method? *PRIMUS*, *9*, 339-354.

Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, *79*(1), 3-18.

Pfeiffer, K. (2011). *Features and purposes of mathematical proofs in the view of novice students: Observations from proof validation and evaluation performances*. Doctoral dissertation, National University of Ireland, Galway.

Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica*, *7*, 5-41.

Rodd, M. (2000). On mathematical warrants: Proof does not always warrant, and a warrant may be other than a proof. *Mathematical Thinking and Learning*, *2*, 221-244.

Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioural schemas and the proving process. *International Journal of Mathematical Education in Science and Technology*, 41(2), 199-215.

Selden, A., & Selden, J. (2003). Validations of proofs written as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*(1), 4-36.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*, 123-151.

Weber, K. (2008). How do mathematicians determine if an argument is correct? *Journal for Research in Mathematics Education*, *39*(4), 431-439.

Weber, K., & Alcock, L. (2005). Warranted implications to understand and validate proofs. *For the Learning of Mathematics*, *25*(1), 34-38 & 51.

Zwaan, R. A., & Radvansky, G. A. (1998). Situation models in language comprehension and memory. *Psychological Bulletin*, *126*(2), 162-185.

PROMOTING MATHEMATICS TEACHER NOTICING DURING MENTORING CONVERSATIONS

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Mathematics teacher noticing is important for improving teaching. The ability to notice instructional events and to respond appropriately to these events can be challenging. Mentoring conversations can be structured to enhance teacher's ability to notice issues of mathematical content and student learning. This paper presents a case study of mentoring conversations between a mentor and a teacher on a Primary 4 lesson on comparing decimals in a Singapore classroom. Based on data which include observations of mentoring sessions and classroom lessons, field notes and semi-structured interview, findings from this study suggest that mentoring conversations direct teacher to pay attention on relevant issues during lesson planning. This enhances teacher's ability to notice during teaching.

INTRODUCTION

Learning to notice classroom events and students' mathematical thinking is important for improving teaching. This skill empowers teachers to work towards building connections between learners and mathematical content. Mason (2011) advocates noticing as an intentional act and one needs to sensitise oneself so as to notice opportunities in the future and to be able to select a preferred action based on a collection of alternative actions instead of habitual reactions. Professional development efforts to build teachers' capacity to notice often involves showing video clips of classroom teaching to teachers, and asking them to notice certain features of instruction (Miller, 2011; Star & Strickland, 2008; van Es, 2011). In this paper, we provide another perspective of professional development on noticing by studying the discussion points during mentoring conversations to raise the teacher's awareness to attend to students, to the mathematics content and to make decisions based on the instructional situations. One of the key questions addressed in this study is the impact of mentoring conversations on the processes of teacher noticing (attending, interpreting and deciding).

THEORETICAL CONSIDERATIONS

Teacher Noticing

This study is underpinned by the theoretical construct of noticing. Although there are different notions of mathematical noticing, scholars on teacher noticing, such as Mason (2002) and Erickson (2011), suggest that what teachers notice has direct bearings on

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 153-160. Hobart, Australia: PME.

their pedagogical responses. Based on the concept of situational awareness, noticing is essential for effective teaching (Miller, 2011). Extending the work of Goodwin (1994) on professional vision, which was described as ways of making sense of events that are of interest to specific groups, the construct of noticing has evolved to encompass the ability to notice significant events and decide how to respond based on what is noticed (Jacobs, Lamb, & Philipp, 2010). In this process, teachers maintain an "awareness of awareness" (Mason, 2011, p. 43), meaning that teachers are cognizant about the extent to which they are conscious about classroom happenings. When noticing, teachers draw attention to students' thinking in instances that are most pertinent for improving instruction.

While teaching a lesson, teachers who notice are cognizant about student understanding and misconceptions occurring in the classroom (Miller, 2011). In a classroom where multiple events are happening simultaneously, it requires expertise to identify noteworthy aspects of a classroom situation. Choy (2013) makes a distinction between more productive and less productive noticing and highlights the potential of collaborative teacher learning in enhancing the productivity of mathematical noticing. He advocates directing teacher's attention to key mathematical ideas and students' learning difficulties related to these concepts during lesson planning as an approach to support teachers' ability to notice mathematical features (Choy, 2014).

Mason (2002) describes noticing as a set of practices that work together to improve teachers' sensitivity to new responses during teaching situations. These practices include reflecting systematically; recognising choices and alternatives; preparing and noticing possibilities; and validating with others (Mason, 2002). To develop professional practice, teachers must first develop their own sensitivities and awareness in order to stay attuned to fresh possibilities when they are needed and be alert to such a need through awareness of what is happening at any given time. Mason therefore highlights the need for advanced preparation to notice and the use of prior experience to enhance noticing in order to have a different act in mind. In this paper, we refer to teacher noticing as a teacher's noticed moment, her understanding of that moment, and her response to that moment.

Teacher Noticing and Mentoring

Scholars on noticing have emphasised the need to explicitly teach preservice teachers to notice because they are initially quite weak at observing classroom events and interpreting student understanding (Star & Strickland, 2008). At the same time, research has also called for the support of inservice teachers to deepen their noticing capabilities (Jacobs et al., 2010). Building on the noticing framework by van Es and Sherin (2008), Jacobs and colleagues propose a structure for teachers to better understand and act on their students' mathematical conceptions and practices. They characterise noticing into three interrelated phases: attending, interpreting and deciding. Attending is about noting aspects of a mathematical moment as a way to gather meaningful evidence. Interpreting involves coordinating the observed actions (attending) with what is known about mathematical development in a particular area. Deciding refers to conceiving (and executing) an effective strategy drawn from the interpretation of a student's mathematical thinking.

Noticing is not just an individual cognitive process. According to Goodwin (1994), "The ability to see a meaningful event is not a transparent, psychological process, but is instead a socially situated activity" (p. 607). He illustrated how the discourses and tools of a discipline shape the ways professionals make sense of, or notice, complex events. For example, in the 1992 trial of Rodney King, Goodwin illustrated how a police expert was critical in shaping what features the jurors noticed that led to their verdict of an acquittal.

Murray (2001) broadly defined mentoring as a one-on-one relationship between an experienced and less experienced person for the purpose of learning or developing specific competencies. Mentoring is based on the idea that individuals make meaning of knowledge within a social context and as a result of interactions with others (St George & Robinson, 2011). Situating the emergence of what teachers notice within the lesson and structures of the classroom, the mentor plays a key role in pointing the way, offering support, and challenging ideas (Daloz, 1983). In grounding noticing within a collaborative mentorship, we posit that in the presence of a mentor, a less experienced teacher will be empowered to notice specific mathematical details and students' possible misconceptions during lesson planning. This advanced preparation to notice will serve as the bedrock upon which the mentee construct ways to respond with a different act in mind during teaching situations.

METHODOLOGY

This paper presents two vignettes drawn from a case study which formed part of a larger exploratory study on mentoring of teachers to teach low progress learners in primary mathematics in Singapore. It uses data from a full-cycle of mentoring sessions which comprised pre-lesson discussion, lesson observation and post-lesson discussion as shown in Figure 1. The mathematics mentor had 15 years of teaching experience and good mathematics pedagogical content knowledge. She had participated in a workshop on mentoring skills based on Figure 1 and four sessions of networked learning which were facilitated by the researchers. The mentor had weekly mentoring sessions with the teacher for about four months. The teacher had about two years of teaching experience and was new to teaching mathematics at Primary 4.

We adopted an experimental model to teaching as a systematic approach to learning from teaching (Hiebert, Morris, & Glass, 2003). In this theoretical model, teachers view lessons as experiments to inquire and make sense of their teaching in order to improve their knowledge and practice. Mentoring as a collaborative inquiry, therefore, supports the underlying assumption that the key to learning to teach is the ability to plan lessons that are aligned to specific learning goals, and to monitor the effectiveness of the lesson based on evidence collected during implementation. To facilitate conversations between the mentor and teacher, we introduced a mentoring framework

(Figure 1) to focus discussions on identifying readiness of students and the specifics of mathematical concepts the teacher wanted to teach. This is premised on the belief that for mentoring to be beneficial and effective, it calls for specific discussion points pertaining to the subject area (Curran & Goldrick, 2002). A study by Hudson (2009) on mentoring pre-service primary mathematics teachers also highlighted the need for a set of specific mentoring practices for the mentors to focus on.

Figure 1 shows the various discussion points which the mentor used to direct the teacher to notice key mathematical ideas and students' learning difficulties related to these concepts. These discussion points guided the mentor to focus the mentoring conversations on what the teacher needed to attend to in terms of the mathematical content and students' learning during lesson preparation and actual lesson. The conversations with the mentor aimed to empower the teacher to interpret instructional events more meaningfully and to provide the teacher with an increased repertoire of potential instructional decisions.

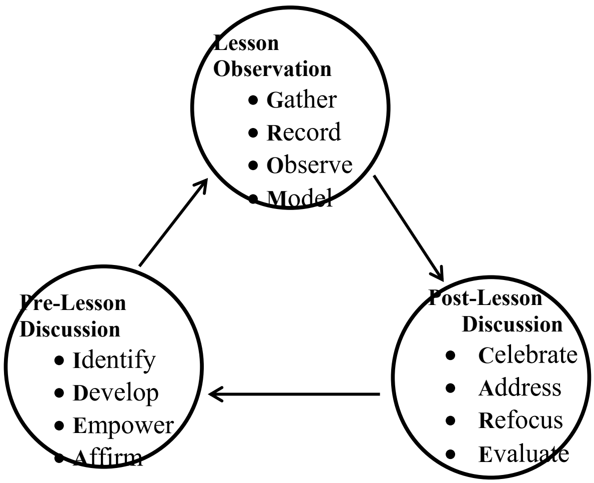

Figure 1: Mentoring Framework for Math Mentors

The authors took on the role of non-participant observers, made field notes and audio recordings of the mentoring sessions and video recording of the lesson conducted by the teacher. Semi-structured interviews with the mentor and teacher were also audio-recorded. The transcripts of the audio-recordings and video-recording were parsed into episodes according to the processes of noticing, namely attending, interpreting and deciding. This paper focuses on the findings drawn from mentor-teacher discourses from one pre-lesson discussion, one lesson observation which was followed by a post-lesson discussion, as well as semi-structured interview with mentor and teacher.

RESULTS AND DISCUSSION

To investigate the impact of mentoring conversations on teacher noticing, we discuss two note-worthy episodes; one that occurred during pre-lesson conversation and one during the lesson. The first episode illustrates discussion points directed by the mentor to empower the teacher to see the salient points in comparing decimals during lesson planning. The second episode focuses on teacher noticing during the actual lesson and her sensitivity to respond as a result of her advanced preparation to notice during the pre-lesson discussions on issues of mathematical concept of comparing decimals and student learning.

Vignette 1: Lesson Planning on Comparing Decimals

During the pre-lesson discussion on a lesson on comparing decimals up to three decimal places, the teacher highlighted the importance of building students' factual fluency in expressing a fraction with a denominator of 10 or 100 as a decimal before proceeding to the concept of comparing decimals. Students' difficulties in recognising tens with tenths and hundreds with hundredths and the concept that tenth is greater than hundredth were also discussed in great detail. The teacher then suggested using a menu of food with prices listed to engage students to relate comparing decimals to their everyday experiences of buying food.

> Teacher: I will ask them, with $2.50 what can you buy from this list of items?
>
> Mentor: There is only one item that is cheaper than $2.50 or less than $2.50.
>
> Teacher: Ya, why is that so? So over here I will be able to elicit the word 'more than' and 'less than' when they are trying to decide which item is more expensive. So, this will be able to lead us to the lesson objectives to compare decimals.
>
> Mentor: I see (pause) there are some key words…more than, less than, more expensive (pause) but I couldn't link it to the tenths, hundredths and thousandths for comparing decimals….So, using the concept of money, would it be effective in achieving your lesson objective?

The concept between comparing money and comparing decimals is a subtle one. Through the mentor's questions, the teacher noticed that, in money, the dot separates the dollars and the cents. So, given two amounts of money ($2.50 and $2.05), it is about comparing the values of dollars and the value of cents. In comparing decimals (2.50 and 2.05), the key concept is on the place value and that decimals are part of the base-ten system of numeration. The teacher subsequently noticed the importance to focus students' attention on the mathematical concepts of place value and decimals; the digit before the decimal point indicates the number of wholes and the digit written after the decimal point represents the fractional/decimal parts.

The teacher's noticing was less productive in the beginning as she seemed more concerned about eliciting words such as 'more than' rather than focussing on students' conception of place value. However, her noticing became productive when the mentor directed her attention to applying the concepts of place value when comparing decimals. This led her to redesign her instructions to emphasis the meaning of tenths, hundredths and thousandths. Although she still believed in the importance of the menu as a strategy to help her students appreciate the relevance of comparing decimals, it was used only for the purpose of illustrating the idea of comparing which students experienced in their everyday situations, such as buying food. Parallel to the study by Choy (2014), the findings in Vignette 1 also highlight the importance of productive noticing during lesson preparation because it sensitises teachers to think about the key mathematical idea, students' possible misconceptions, and the various instructional strategies to deal with these problems.

Vignette 2: Responding to Students' Thinking during Classroom Observation

During the lesson, although her students were able to give the correct answers to her questions, the teacher noticed that some of them may not be sure of the underlying mathematical concepts in comparing decimals. The transcript below shows her probing her students' thinking when they said that 0.1 is greater than 0.01.

Teacher: 0.1, yes. Do you want to explain to me why do you say that 0.1 is greater than 0.01?

Student: It has less digits.

Teacher: If you say a decimal has less digits, does it tell us that it must be a greater decimal?

Noticing students' misconception that a decimal with fewer digits is greater, the teacher needed to interpret the content from the perspective of the student and made a decision to relate students' understanding of 0.1 and 0.01 to their prior knowledge of fractional parts. This was modelled with a whole chocolate bar being shared by 10 people and a whole chocolate bar being shared by 100 people. Following this, the teacher used the example of comparing 1.2 and 1.20 to further address the misconception that 'a bigger decimal will have fewer digits' and to deepen students' conceptual understanding of decimals.

During the post-lesson discussion with her mentor, the teacher reflected that she needed to make the concepts of place value very explicit to her students and the process of comparing the whole number parts first before the decimal parts, starting from the left.

Teacher: Because their mindset is such that less digits means to say more tenths, no hundredths or no thousandths. That's why, so I think the common misconceptions would be, you know, the number of digits in a decimal will affect their understanding. The most important thing is to have them compare the digits in the similar place value. That's the key takeaway.

Analysis of the transcript of the semi-structured interview showed that the teacher attributed this mathematical noticing to the questions by her mentor.

Teacher: …sometimes when I craft my lesson, there are some parts whereby I didn't realise the misconceptions certain students have….having a mentor actually will be able to point out for you … like this topic, requires more probing of students' understanding of tenths and hundredths and concept of comparing.

Hence, it can be argued that advanced preparation to notice has sensitised the teacher to listen to students' mathematical reasoning and make sense of what she heard in order to respond appropriately to her students' thinking (Mason, 2011). Attending to the mathematical aspects of students' reasoning, therefore, provided the teacher with insights into students' thinking (Jacobs et al., 2010) which led her to make a meaningful interpretation based on the evidence of students' verbal explanation. Having attended and interpreted this relevant information, the teacher made a thoughtful instructional decision which resulted in enhancing students' learning.

By describing what and how she intended to teach, the negotiation with her mentor provided a foundation for the teacher to recognise in-the-moment when a similar incident (place value of decimals) began to emerge. Through the lens of her mentor, she developed the awareness to stay attuned to fresh possibilities in the future and hence, to be more adaptive to instructional events. As she became more attuned to how her students perceive the learning, she was also better prepared to engage in more productive mathematical noticing.

CONCLUSION AND IMPLICATIONS

Teaching noticing is specialised and it is not a natural extension of being observant in daily life. As it does not naturally develop with teaching experience, the processes of attending, interpreting and deciding must be deliberately refined through practice (Jacobs et al., 2010). The two vignettes show that mentoring conversations encouraged the teacher to notice specific mathematics in students' reasoning and facilitated her understanding of that moment as well as her response to that moment. As teachers and mentors reflect systematically, explore various alternatives and validate their practices with one another, mentoring conversations are avenues to develop teacher's mathematical noticing.

Despite the limitations of a case study, this paper highlights the value and potential of mentoring conversations to develop teacher noticing to improve teaching, and provides a start towards understanding the role mentors can play in building teachers' capacity to connect instructional decisions to interpretations based on attending to evidence for effective mathematics teaching.

References

Choy, B. H. (2013). Productive Mathematical Noticing: What it is and why it matters. In V. Steinle, L. Ball & C. Bardini (Eds.), *Proceedings of 36th annual conference of Mathematics Education Research Group of Australasia.* (pp. 186-193). Melbourne, Victoria: MERGA.

Choy, B. H. (2014). Teachers' productive mathematical noticing during lesson preparation. In C. Nicol, P. Liljedahl, S. Oesterle & D. Allan (Eds.)*, Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 2, pp. 297-304). Vancouver, Canada: PME.

Curran, B. & Goldrick, L. (2002). *Mentoring and supporting new teachers.* Issue Brief. National Governors' Association, Washington, D.C. Center for Best Practices.

Daloz, L. (1993). Mentors: Teachers who make a difference. *Change, 15*, 24–27.

Erickson, F. (2011). On noticing teacher noticing. In M .G. Sherin, V. R. Jacobs & R. A. Philipp (Eds.), *Mathematics teacher noticing: seeing through teachers' eyes* (pp. 17 – 34). New York: Routledge.

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96*, 60 –633.

Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An "experimental" model for teaching and teacher preparation in mathematics. *Journal of Mathematics Teacher Education, 6,* 201 – 222.

Hudson, P. (2009). Mentoring pre-service teachers in primary mathematics. *The International Journal of Learning. 16*, 119–132.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*, 169–202.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: RoutlegeFalmer.

Mason, J. (2011). Noticing: Roots and branches. In M .G. Sherin, V. R. Jacobs & R. A. Philipp (Eds.), *Mathematics teacher noticing: seeing through teachers' eyes* (pp. 35 – 50). New York: Routledge.

Miller, K. (2011). Situation awareness in teaching: What educators can learn from video-based research in other fields? In M .G. Sherin, V. R. Jacobs & R. A. Philipp (Eds.), *Mathematics teacher noticing: seeing through teachers' eyes* (pp. 51 – 65). New York: Routledge.

Murray, M. (2001). *Beyond the myths and magic of mentoring*. San Franciso: Jossey Bass.

Star, J. & Strickland, S. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education, 11,* 107–125.

St George, C. & Robinson, S. (2011). Making mentoring matter: Perspectives from veteran mentor teachers. *The Delta Kappa Gamma Bulletin*, 24–28.

van Es, E. A. (2011). A framework for learning to notice students' thinking. In M. G. Sherin, V. R. Jacobs & R. A. Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers' eyes (pp. 134–151). New York: Routledge.

Van Es, E. A. & Sherin, M. G. (2008). Mathematics teachers' learning to notice in the context of a video club. *Teaching and Teacher Education, 24*, 244–276.

TRANSFORMATION OF STUDENTS' VALUES IN THE PROCESS OF SOLVING SOCIALLY OPEN-ENDED PROBLEMS

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Bishop (1991) pointed out the importance of research on values in mathematics education. Based on this idea, Shimada and Baba (2012) developed three "socially open-ended" problems, in which students' values play an important role (Baba 2010). We gave each of the problems to fourth graders once per month, and identified characteristics of classroom interaction. However, an issue remains regarding how students appreciated others' values and transformed their own values in the classroom interaction (Shimada & Baba 2012). The objective of this research is to study this issue. As a result of the analysis, we identified four characteristics of the transformation of students' values such as diverse mathematical models, existence of implicit values, transformation of values, and change of mathematical models.

RESEARCH BACKGROUND

Baba (2012a) points out that there are two types of research on values: one involving a cultural and/or historical analysis of the values which students and teachers hold; and the other involving the taking up values appearing through problem-solving and even positively developing them through interaction. This paper is based on the latter kind of research. In the previous study (Shimada & Baba, 2012) we identified three values that we should foster in mathematics education: mathematical values, social (human) values and personal values. In this paper, we focus on social (human) values and explore the transformation of students' values through classroom interaction. There are three interrelated reasons for this exploration.

The first reason relates to whether or not students transform their values. Seah (2012, p. 1) described values as being extremely internalised and stable (see Krathwohl, Bloom & Masia, 1964) within an individual, and they are usually not acquired overnight. We understand that adults' values become more stable after encountering a variety of values through various experiences in their lives. Thus it is of interest whether or not students' values are transformed in problem-solving during a mathematics lesson.

The second relates to whether or not we are able to evaluate a transformation in students' values. If we observe students' values during a mathematics lesson, we must be able to evaluate those values. This evaluation does not mean determining whether students' values are good or bad, especially when we deal with social values. Thus, from the educational point of view, it is necessary for the teacher to grasp how students transform their values during a mathematics lesson.

The third relates to how the research into the transformation of students' values during

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problem-solving takes place. The difficulty of studying the transformation of students' values is that values usually stay implicit without being spoken out loud. Of the study of values, it may be said that we have been studying them with the aim of clarifying values that are implicit. Seah, one of the leaders of the Third Wave international research project on values, stated the following in an overview of research on values:

> The researching of values in the mathematics classroom has traditionally been approached using the research methods of questionnaires, observation, and/or interviews. … By the late 2000s, values were also identified through content analyses of artefacts such as photographs and drawing, often followed by participant interviews which served to clarify initial findings or questions. (Seah, 2012, pp. 2-3)

In this paper, we document and research the transformation of students' values as they appear in the problem-solving process.

RESEARCH OBJECTIVE

The objective of this paper is to study the transformation of students' social values as they occur through the teaching of problem-solving. This is associated with one purpose of the international comparative survey, the Third Wave, that is, how these values are negotiated by the students and the teacher. In the classroom, the students introduce their values and interact with each other, and they may or may not transform their values.

Thus, the following research questions are set up: 1. What models are created for what values? 2. How do implicit values become apparent? 3. Are values transformed? 4. Are mathematical models changed?

RESEARCH METHODOLOGY

Overview of the class: The first author carried out a problem-solving lesson using the socially open-ended problem[*1] "Hitting the target" with fourth graders in a private elementary school in Tokyo on March 12, 2013. The problem is shown in Figure 1.

"Hitting the target:" At a school cultural festival, your class offers a game of hitting a target with three balls. If the total score is more than 13 points, you can choose three favorite gifts. If you score 10 to 12 points, you get two prizes, and if you score 3 to 9 points, you get only one prize. A first grader threw a ball three times and hit the target in the 5-point area, the 3-point area, and on the border between the 3-point and 1-point areas. How do you give the score to the student?

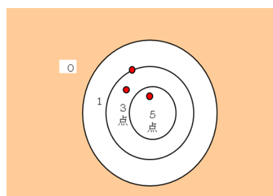


Figure 1: Problem-solving task

There were 38 students, comprising 19 boys and 19 girls. The first author is a teacher who has specialised in mathematics education, with 40 years of teaching experience. In this study, he taught the same fourth grade class in the same school in which Shimada and Baba (2012) had practiced, employing almost the same class design. In other words, the lesson follows the sequence of provision of a problem, individual solutions, presentation and discussion of the mathematical models and reasons, and finally selection of one model with its reason at the end by each other.

The research method on the transformation of the students' values

Shimada and Baba (2012) reported that some of the students' values exist both implicitly and explicitly in a problem-solving lesson. Furthermore, they proposed a method for the students to become aware of the implicitness of values through comparisons. This paper will also employ this method of comparing, because "Values do not often appear in daily life, but when we are challenged, and we meet opposing values, we can be conscious of the implicitness of values for the first time. This is the nature of values." (Baba, 2012b, p.1)

We let the students write the mathematical models and the reasons on a worksheet at the beginning of problem solving. The students' values appear in the reasons written by the students (Shimada & Baba, 2012). Of course, in some cases, students' values remain implicit at the beginning stage. However, through the process of sharing ideas in the classroom, the implicit values become explicit, in contrast to different and sometimes opposing values.

After the students present their models, we set the scene where students were to select models and write the reasons for their selection on a worksheet. As Seah (2007) has noted, "Selection and decision-making are the important keys." We even ask the students during the class: "Why did you choose this idea?" In this research we consider the "transformation of students' values" through the comparison of mathematical models and the reasons at the beginning and at the end of the lesson.

Analysis of classroom interaction and students' data

The analysis of classroom interaction and the worksheets revealed four characteristics of students' cognition and transformation of values such as diverse mathematical models, existence of implicit values, transformation of values, and transformation of mathematical models.

Diverse mathematical models for the same values

The first characteristic is that diverse models are being created with the same values. Table 1 summarizes students' values and mathematical models after students engaged in initial self-resolution. For example, the value "kindness to the first grader" is given with diverse mathematical models such as "a. 5+3+3," "b. 5+3+(3+1)," "c. 5+3+3+1+1," and "d. 5+3+2." The same thing can be said for the value of "fairness and equality." All numbers in Table 1 are percentages except those in parenthesis. In the "Percentage of explicit values" column, the fractions in parenthesis show (the number of students who expressed the values explicitly)/(the total number of students who wrote the mathematical model). The models are categorised into a, b, c, d, e, f and g with typical examples.

Mathematical model (Typical example)	Values	Percentage of explicit values	Type-wise Percentage of explicit values
a. 5+3+3 b. 5+3+(3+1) c. 5+3+3+1+1 d. 5+3+2	Kindness to the first grader (Specific person)	92.9 (13/14) 100.0 (1/1) 100.0 (1/1) 100.0 (2/2)	94.4 (17/18)
e. 5+3+2 f. 5+3+1 g. 5+3+3	Fairness and equality to the whole class (all students)	0.0 (0/9) 0.0 (0/10) 0.0 (0/1)	0.0 (0/20)

Table 1: Students' Values and Mathematical Models at the Initial Self-Resolution Time (n = 38)

Implicit values became apparent through comparison with other values

The second characteristic is that implicit values become apparent through comparison with other values as stated in the method. Looking at Table 1, we can see that the value of "kindness to the first grader" (94.4%) is elicited more than the values "fairness and equality" (0.0%). We found that the latter implicit values are manifested in the following protocols.

1	T1:	So, please make a presentation on how you think about this problem. S.J., please.
2	S.J.1:	I gave 3 points to the first grader, so I wrote 5+(3×2)=11, 11 points, because the first grader should be welcomed. (Fig. 1)
3	2:	Does anyone have any questions for S.J.?
4	S1:	I think that we do not have to write parentheses in the expression because we calculate the multiplication before the addition.
5	S.J.2:	Thank you, I understand. I will rewrite it as 5+3×2=11.
6	T3:	And who did you think of, S.J.?
7	S.J.3:	I thought of the first grader.
8	T4:	I will write the words "kindness to the first grader" next to S.J.'s idea. Next, please present your idea, K.K.
9	K.K 1:	The ball is on the boundary of 3 points and 1 point. I give 1 point because the 1-point area of the ball is larger than the 3-point area of the ball. So, 5+3+1=9, 9 points. (Fig. 2)

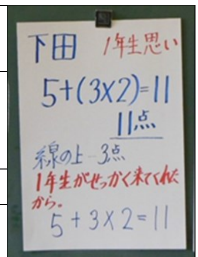
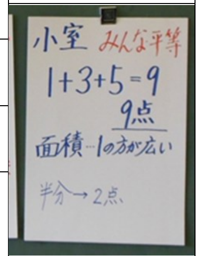


Fig. 2: K.K.'s

10	T5:	Does anyone have any questions for K.K.?	
11	S2:	What points will you give to the first grader when the ball reaches the middle just above the line?	
12	K.K. 2:	I will give 2 points.	
13	S3:	What points will you give to the first grader when the ball reaches the middle of just above the line of 1 point and 0 points?	
14	K.K. 3:	I will give 0.5 points.	
15	T6:	S.J. gave 3 points to the first grader. And who did you think of, K.K.?	
16	K.K. 4:	I thought about all the people who play the game. I want to be impartial to all people.	
17	T7:	So I will write the words “fairness to all people” next to K.K.’s idea.	



Analysis of the above transcripts reveal that the value “kindness to the first grader” is explicit in S.J.’s presentation, so in T3, the teacher confirmed S.J.’s values, and wrote it in T4. On the other hand, we cannot explicitly see values in K.K.1. In his opinion of considering the area coverage, we can see the mathematical interpretations and models, but we cannot see who K.K. is thinking about, so the teacher uses S.J.’s explicit values to contrast with K.K.’s. In this way, the teacher made the implicit values apparent in K.K.4: “I thought about all the people who play the game. I want to be impartial to all people.” The teacher employs the same method in dealing with other opinions including implicitness.

The mathematical models of the students in the class are shown in Table 2 below, in order of presentation.

Student	Mathematical model	Explanation
K.R.	5＋3＝8, 8+3＝11	I’ll give 3 points to the first grader, but 2 points to a third or fourth grader, 1 point to a fifth or sixth grader.
M.H.	3÷2＝1.5, 1÷2＝0.5, 1.5+0.5＝2, 5+3+2＝10	Dividing 3 by 2 gives 1.5. Dividing 1 by 2 gives 0.5. Adding 1.5 and 0.5 gives 2. It becomes 10 when I add 5 and 3 and 2.
T.R.	5+3＝8, (1+3)÷2＝2, 8+2＝10	The ball is on the boundary of 3 and 1. It becomes 4 by adding 1 and 3, then it becomes 2 by dividing 4 by 2. It becomes 10 when I add 8 and 2.
Y.S.	3-1＝2, 2+3+5＝10	Subtracting 1 from 3 gives 2 because the ball is on the boundary of 3 points and 1 point.

A.K.	5+3+3+1＝12	I'll give 4 points by combining 1 and 3 for the first grader.
K.U.	1+3＝4, 5+3＝8, 8+4＝12, 12+1=13	I'll give 4 points combining 1 point and 3 points for the first grader. I give 1 point with a further bonus.

Table 2: Students' mathematical models presented in a classroom after S.J. and K.K.

Some students transform their values from the initial **self-resolution time to the final selection time**

The third characteristic regards the existence of both students who transform their values and those who do not. Table 3 is a cross-tabulation table showing the relationship between the values at the initial self-resolution time and the final selection time. All numbers are percentages except those in parenthesis. The fractions in parenthesis show (the number of students who expressed the values both at the initial self-resolution time and the final selection time) / (the number of all students in the class).

		Values at the final selection time		
	Values	Fairness and equality	Kindness to the first grader	Total
Values at the initial self-resolution time	Fairness and equality	36.8 (14/38)	15.8 (6/38)	52.6 (20/38)
	Kindness to the first grader	15.8 (6/38)	31.6 (12/38)	47.4 (18/38)
	Total	52.6 (20/38)	47.4 (18/38)	100.0 (38/38)

Table 3: Values at the initial self-resolution time and the final selection time (n = 38)

Table 3 shows the values by types, at the initial self-resolution time and at the final selection time. The percentage of students who selected the values "fairness and equality" at the self-resolution time and selected the value of "kindness to the first grader" at the final selection time is 15.8%. About one-third of students who selected different values at both times is (6/38+6/38=12/38, 12÷38≒0.32).

Table 4 below shows 1/2 (3÷6=1/2) of students, U.K., T.M., N.M., who selected the value of "fairness and equality" at the self-resolution time and transformed it at the final selection time, supporting K.U.'s idea (Table 2). They thought, such as "The first grader will be happy and come here again." On the other hand, K.U. himself transformed his idea to the idea of T.R. (Table 2) at the final selection time. He transformed his value after knowing the value of "fairness and equality," and stated "I think that it is nice to give two points because of equality."

Name	Transformation	Reason of the transformation
U.K.	To K.U.'s idea	The first grader will be happy and come here again.
T.M.	To K.U.'s idea	It is good for us to give bonus to the first grader.
N.M.	To K.U.'s idea	It is good for us to be kind to the first grader.
K.U.	To T.R.'s idea	I think that it is nice to give two points because of equality.

Table 4: Four students who transformed their values and their reasons

Some students remain with the same values but change the mathematics models with the same values

The fourth characteristic is the existence of students who did not transform their values but changed the mathematical models. Table 5 is a cross-tabulation showing the relationship between mathematical models at the initial self-resolution time and the final selection time against the same values. Numbers are percentages except those in parenthesis. The fractions in parenthesis show, for example, in the case of 5/14, (the number of students who expressed the same mathematical models at both times with respect to the values "fairness and equality") / (the number of all students who expressed the same values "fairness and equality" at both times).

		Mathematical models at the final selection time		
		Same	Different	Total
Mathematical models at the self-resolution time	Fairness and equality	35.7 (5/14)	64.3 (9/14)	100.0 (14/14)
	Kindness to the first grader	25.0 (3/12)	75.0 (9/12)	100.0 (12/12)
	Total	30.8 (8/26)	69.2 (18/26)	100.0 (26/26)

Table 5: Mathematical Models at the Initial Self-Resolution Time and the Final Selection Time with the Same Values

The percentage of students who selected the same value "fairness and equality" at both times but changed their mathematical models is 64.3%. However, the percentage of students who selected the same value "kindness to the first grader" at both times but selected different mathematical models is 75.0%. Overall, the percentage of students who changed mathematical models with the same values is 69.2%. From this fact alone, it can be said that social interaction has had an impact on the students.

Table 6 shows M.H. changed the mathematical model because of the beauty of formula.

Name	Transformation	Reason of the transformation
M.H.	To T.R.'s model	Because the friend's formula is beautiful.
T.Y.	To K.U.'s model	More and more the first grader will come to do the game.
K.R.	To K.U.'s model	I was more strongly the value of "kindness to the first grader"

Table 6: Three students who transformed the mathematical models and their reasons

M.H. described about mathematical value. On the other hand, 6 students (e.g., T.Y., K.R.) supported K.U.'s model. K.U.'s idea is accepted by several students.

CONCLUSION AND FUTURE ISSUES

We have analysed transformations of values within a lesson and concluded that more than half of students did not transform their values. However, many of them modified mathematical models according to their values. This illustrates the stability of values within an individual (Seah, 2012, p. 1), and at the same time, the possibility of gradual or small scale transformation of values, which necessitates with change of mathematical models. This variability that a teacher experiences in the classroom when introducing problems that have implications for exploring students' values related to mathematics will indicate ttransformation may occur gradually over a longer period. Next we would like to discuss a more sensitive and long-term transformation.

*1 A socially open-ended problem is one (Baba, 2010) which has been developed to elicit students' values by extending the traditional open-ended approach (Shimada, 1977).

References

Baba, T. (2010). *Socially open-end approach and critical mathematics education.* Paper presented at EARCOME 5, Tokyo.

Baba, T. (2012a). Issues in Mathematics Education from the Perspective of Values. *Proceedings of JSME 45th Annual Conference*, 1, pp. 1219-1220. [In Japanese.]

Baba, T. (2012b). *Research on values in mathematics education (1): using the framework of the international research project on values, the "Third Wave."* Paper presented at the 36th JASME. [In Japanese.]

Becker, J.P. & Shimada, S. (Eds.) (1997). *The open-ended approach in mathematics education: a new proposal for teaching mathematics.* Reston: National Council of Teachers of Mathematics. [Translation of Shimada, Ed. (1977).]

Bishop, A. (1991). *Mathematical enculturation: a cultural perspective on mathematics education.* Dordrecht: Kluwer Academic Publishers.

Seah, W.T. (2007). Qualities co-valued in effective mathematics lessons in australia: preliminary findings. *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, 4,* 161-168.

Seah, W.T. (2012). *Identifying values in mathematics learning and teaching.* Document in the study group of values in Hiroshima University.

Shimada, S. (Ed.) (1977). *Open ended approach in mathematics education: A proposal for lesson improvement.* Tokyo: Mizuumi Press. [In Japanese.]

Shimada, I. & Baba, T. (2012). Emergence of students' values in the process of solving the socially open-ended problem. *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics*, Vol. 4, pp. 75-82.

A THEORETICAL FRAMEWORK FOR CURRICULUM DEVELOPMENT IN THE TEACHING OF MATHEMATICAL PROOF AT THE SECONDARY SCHOOL LEVEL

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The aim of this paper is to construct a theoretical framework for curriculum development in the teaching of mathematical proof at the secondary school level. To accomplish this aim, we first search for, through the review of related literature, the principal aspects of mathematical proof that should be taken into consideration for the framework. In particular, we consider the idea of "local organization" introduced by Freudenthal (1971) and the idea of "mathematical theorem" proposed by an Italian research group (Mariotti et al., 1997). In terms of these ideas, we then develop a framework for teaching mathematical proof and examine elements of the framework in line with mathematics curricular content in Japan. Examples and implications for curriculum development are also discussed.

MATHEMATICAL PROOF IN CURRICULUM

Traditionally, the teaching of mathematical proof was relegated to geometry at the secondary school mathematics level. It might be a well-known fact that the majority of students were unable to construct valid proofs. Currently, however, there seems to be a general trend towards including proof and proving at all levels of school mathematics (e.g., NCTM, 2000). Therefore a number of research studies carried out at all levels of mathematics have been reported the teaching and learning of proof and proving in light of explanation, reasoning, argumentation, and so on (e.g., Mariotti, 2006; Stylianou et al., 2009; Reid & Knipping, 2010; Hanna & de Villiers, 2012). In retrospect, what does such an endeavour mean for improving the teaching of mathematical proof at the secondary school level? We think that it is necessary to consider—from the perspective of important results of earlier research on proof and proving—a more synthesised approach to the mathematical or formal proof in curriculum. We cannot ignore the influences of curricular content and sequencing when we analyse students' constructions of mathematical proofs (Hoyles, 1997). However, because of "the huge variation in when proof in introduced and how it is treated in different countries" (Hoyles, 1997, p. 7), only a few attempts have so far been made at a broader discussion of curricular content and sequencing of mathematical proof that could be explicitly introduced at the secondary level in some countries, including Japan. There is room for argument on this point.

This paper reports on part of an ongoing research project regarding the developmental study of the teaching of mathematical proof throughout six years (Grades 7-12) of secondary schooling in Japan. It focuses on proposing a theoretical framework for curriculum development in the teaching of mathematical proof. For this reason, we must draw attention to the theoretical perspectives with a few examples, but the discussion of empirical aspects of the framework would take us beyond the scope of this paper. Although the present study is targeting Japanese secondary school mathematics, in developing a framework we attempt to synthesise multiple theoretical perspectives well known within the international mathematics education community in order to enable the framework to be comparable with those in other countries. Thus, the research questions in this paper are as follows: *What kinds of teaching contents should be included in the secondary curriculum for the teaching of mathematical proof?* and *What kinds of evolution should be envisioned in the course of the curriculum?*

THEORETICAL PERSPECTIVES

"Proof" and "Demonstration"

What is meant by "mathematical proof"? There is the distinction often made in some countries between "proof" and "demonstration". For example, Balacheff (1987) describes the French distinction between "prevue" and "démonstration" as follows:

> We call proof an explanation accepted by a given community at a given moment… Within the mathematical community only explanations adopting a particular form can be accepted as proofs. They are an organised succession of statements following specified rules: a statement is known to be true or is deduced from those which precede it using a deductive rule taken from a well defined set of rules. We call such proofs "démonstrations". (Balacheff, 1987, p. 148: English translation cited from Reid & Knipping (2010, pp. 32-33))

In Balacheff's sense, "démonstration" in French can be translated as "mathematical proof" in English, and it is distinguished from "proof". Although "most English writers do not use 'proof' and 'mathematical proof' in the same way as Balacheff does" (Reid & Knipping, 2010, p. 33), within the Japanese mathematics education community, we sometimes make a similar distinction between "proof (*shoumei*)" and "mathematical proof *(ronshou*)" (e.g., Hirabayashi, 1991; Japan Society of Mathematical Education, 1966). Thus, in this paper we would like to use the word "mathematical proof" in the special sense of "démonstration" as Balacheff says.

The distinction between proof and mathematical proof implies that these words are often discussed in relation to the statements or theorems to be proven and the system of mathematics in which the proof is carried out. We, therefore, attempt to consider organisation or systematisation of statements as the principal aspects of mathematical proof. In order to do so, the idea of "local organization" (Freudenthal, 1971) and the idea of "mathematical theorem" (Mariotti *et al*., 1997) are taken into account.

Local Organization

Freudenthal (1971; 1973) proposed the idea of local organization and emphasised the significance of mathematical activities based on the local organization in geometry. Local organization is an important didactic idea proposed as distinguished from the idea of global organization based on the axiomatic system:

> Indeed, a student who never exercised organising a subject matter on local levels will not succeed on the global one. (Frendenthal, 1971, p. 426)
>
> In general, what we do if we create and if we apply mathematics, is an activity of local organization. Beginners in mathematics cannot do even more than that. Every teacher knows that most students can produce and understand only short deduction chains. They cannot grasp long proofs as a whole, and still can they view substantial part of mathematics as a deductive system. (ibid., p. 431)

What Freudenthal means by local organization is shown by this example of the proof of the perpendicular bisectors of a triangle. Consider a question by the teacher: "draw the bisectors of AB and BC, which intersect at M; look where the bisector of AC passes". Freudenthal provides the analysis of the following proof:

> The proof rests on the property of the bisector of XY being the set of all points equidistant from X and Y, which may have been recognised by symmetry arguments. M is on the bisector of AB whence
>
> $MA = MB$;
>
> M is on the bisector of BC where
>
> $MB = MC$
>
> From both follows
>
> $MA = MC$,
>
> whence M is on the bisector of AC. (Freudenthal, 1971, p. 429)

In his view, students need not be able to prove the equidistance property of the perpendicular bisector, because this property may be, for students who do not have the idea of a relational system, taken for granted, and it "cannot contribute anything to the understanding of the circumcircle theorem" (ibid., p. 430). In line with Freudenthal's idea, Hanna and Jahnke (2002) proposed a distinction between "small theory" and "large theory", and they remarked that "instead of building a large theory (namely, Euclidean geometry) in the course of the curriculum, it seems to be more appropriate to work in several small theories" (p. 3). Here it is important to note that the property taken for granted in the local organization or small theory is consistent with the theorem proven in the global organization or large theory. We think that such a distinction can be one of the principal aspects of teaching mathematical proof that should be taken into consideration when developing a curriculum.

Mathematical Theorem

In order to elaborate on the relationship between mathematical proof and local organization, we consider another important theoretical perspective—the idea of "mathematical theorem" proposed by the Italian research group (Mariotti et al., 1997;

Mariotti, 2006: Antonini & Mariotti, 2008). According to the characterisation by Mariotti et al. (1997), a mathematical theorem consists of a system of relations between a *statement*, its *proof*, and the *theory* within which the proof make sense. Indeed, in mathematicians' mathematical practice, a mathematical assertion such as a proposition and its validation is always considered in a certain theoretical context such as geometrical, arithmetic, algebraic, and other contexts; "the existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense" (Mariotti et al., 1997, p. 182). We consider that these three elements—*statement*, *proof*, and *theory*—that characterise a mathematical theorem can be principal aspects of teaching mathematical proof that evolve throughout secondary school mathematics. We think that, in particular, the nature of *theory* can be well characterised by the idea of local organization.

ELEMENTS OF A THEORETICAL FRAMEWORK OF TEACHING MATHEMATICAL PROOF

The methodology we adopt in the present study is that of synthesising the theoretical perspectives mentioned in the previous section and of examining the contents and levels of mathematical proof in terms of "statement", "proof", and "theory" in line with mathematics curricular content in Japan. In this way, we develop a framework for teaching mathematical proof that allows us to design a curriculum.

Contents of "Statement", "Proof", and "Theory"

We first attempt to identify the *contents* of "statement", "proof", and "theory" respectively. Here we lean on logical points of view to identify the different kinds of "statement" that could be included in secondary mathematics. We think that there are four kinds of propositions: a) singular proposition, b) universal proposition, c) existential proposition, and d) other proposition such as negative proposition. Although these four kinds of propositions are included in both primary and secondary school curriculum in Japan, the distinctions between them—such as distinct universal from existential proposition—are not explicitly taught even at the secondary level.

We next consider the contents of "proof" to be types of proof such as: a) direct proof, b) indirect proof, and c) mathematical induction, which are included in the secondary school curriculum. As far as indirect proof is concerned, it is formally introduced in Grade 10 in Japan, but informally students spontaneously produce indirect argumentation (Anotinini & Mariotti, 2008). Therefore it is necessary to examine how we could deal with indirect proof progressively in the course of the curriculum.

In general, the contents of "theory" are both mathematical theory—Euclidean geometry, number theory, and so on—and the logical inference rules, such as modus ponens, conjunctive inference, and so on. In particular, the latter is referred to "meta-theory" (Antonini & Mariotti, 2008). This distinction also becomes important in discussing secondary school mathematics. Although "mathematical theory" can be explicit teaching content, "meta-theory" remains implicit at the secondary level in

Japan. In order to understand what "meta-theory" is like, let us show a proof by contradiction as an example (see Antonini & Mariotti (2008) for detailed analysis).

Statements: Let a and b be two real numbers. If $ab = 0$, then $a = 0$ or $b = 0$.

Proof: Assume that $ab = 0$, $a \neq 0$, and $b \neq 0$. One can divide both sides of the equality $ab = 0$ by a and by b, obtaining $1 = 0$. It is a contradiction ($1 \neq 0$). Therefore $a = 0$ or $b = 0$.

Theory: Properties of equality, real numbers.

Meta-theory: Law of excluded middle, law of double negation, modus ponens, etc.

Levels of "Statement", "Proof", and "Theory"

We then attempt to identify the *levels* of "statement", "proof", and "theory" respectively. As far as levels of "statements" are concerned, there are two different kinds of educational evolution in terms of the setting of a proof. One level is about the *object* that the statement refers to. It seems reasonable to suppose that there are two levels: i) an object of the real world, and ii) an object of the mathematical world. For example, in the beginning stage of learning geometry, if the statement (probably a singular proposition) refers to "a written triangle", the object of investigation is in the real or material world. At a higher stage, if the statement (probably a universal proposition) refers to "any triangle", the object of investigation is in mathematical world. Another evolution is about the *formulation* of the statement, because the same statement is able to have different representations. In the course of curriculum, it seems that there are three levels of formulation of the statement: i) figure, manipulation, and gesture; ii) ordinary language and word; and iii) mathematical word and symbol. In the case of the universal proposition, for example, the statement can be formulated as "the sum of the interior angles of *any* triangle is 180°". This formulation is the second level, although the universal quantifier is not represented as the symbol "$\forall$", which is the third level. In Japanese language, we rarely say "*any* triangle" or "*all* triangles" in a textbook or geometry class. Although the third-level formulation is not dealt with in the current curriculum, we think that the progressive formulation of the statement can be a crucial point of the curriculum development in this research project.

Concerning the levels of "proof", we consider two different kinds of evolution. Since these have been discussed in Balacheff's (1987) categories of proof so far, similar categories can be applied to our framework as levels of "proof"; that is, the *validation* and *formulation* of "proof". Since the same may be said about the formulation levels of the statement, here we just mention validation levels. It is fair to say that there are three levels of validation: i) explanation, ii) mathematical proof, and iii) formal proof. "Explanation" includes a discourse by informal reasoning, such as inductive and abductive reasoning. Although both "mathematical proof" and "formal proof" are considered as *intellectual proof* in Balacheff's sense, "formal proof" is based on *naïve formalist* language such as symbolic logic. And "mathematical proof" that can be an accepted discourse in the mathematicians' community which means a simplified version of "formal proof". For the consideration of a transition from one level to a higher level, well-known Balacheff's subcategories—*naïve empiricism*, *crucial experiment*, *generic example*, and *thought experiment*—may be useful.

We rely on Frendenthal's idea of local organization, or on "small theory" and "large theory" by Hanna & Jahnke (2002), in order to characterise different levels of "theory". By focusing on the *nature* of each system within which the proof is carried out, we propose three levels of "theory" as follows: i) logic of the real world, ii) local theory, and iii) (quasi-) axiomatic theory. The first level is not the main focus of the study in secondary mathematics. If one accepts that a geometric property is to be true by means of physical experiment or measurement based on the real world, it can be interpreted that the nature of "theory" is based on "logic of the real world". The distinction between "local theory" and "(quasi-) axiomatic theory" is rather important in secondary schools. The former can be the main focus of study in lower secondary school. We put the label "quasi-" onto "axiomatic theory", because it is not relevant to deal with a globally organised axiomatic system explicitly in secondary school mathematics. As a result, Table 1 provides a summary of the framework that resulted from considering contents and levels of three elements. Additionally, in the next section, since the transition from "local theory" to "quasi-axiomatic theory" can be a key to the curriculum development in upper secondary school, we attempt to draw a brief sketch of such a crucial transitional aspect by means of a mathematics textbook.

	Statement	***Proof***	***Theory***
	a. Singular proposition b. Universal proposition c. Existential proposition d. Others	a. Direct proof b. Indirect proof c. Mathematical induction	a. Normal theory (e.g., algebra, geometry, calculus, etc.) b. Meta-theory (e.g., modus ponens, etc.)
	Object i. An object in the real world ii. An objects in the mathematical world **Formulation** i. Figure, manipulation, gesture ii. Ordinary language, word iii. Mathematical word, symbol	**Validation** i. Explanation ii. Mathematical proof iii. Formal proof **Formulation** i. Figure, manipulation, gesture ii. Ordinary language, word iii. Mathematical word, symbol	**Nature of system** i. Logic of the real world ii. Local theory iii. (Quasi-) axiomatic theory

Table 1: A framework for curriculum development in the teaching of mathematical proof—contents and levels

EXAMPLES AND IMPLICATIONS FOR CURRICULM DEVELOPMENTS

Let us consider the introduction of mathematical induction [MI] as an example to illustrate the nature of "local theory" and "quasi-axiomatic theory". MI is a teaching content that is included in the teaching unit of sequence in upper secondary school in Japan. MI as a teaching material is a kind of capstone in this teaching unit, which consists of the following items in a textbook that has been mostly used in an 11th Grade class. In Table 2, there is space only for the items (left side) and some excerpts of concrete statements (right side), though proofs are also described in the textbook.

§1. Arithmetic sequence and geometric sequence 1.1 Sequence and the general term 1.2 Arithmetic sequence 1.3 Arithmetic series 1.4 Geometric sequence 1.4 Geometric series	The sum of S_n of the first n terms of an arithmetic sequence with the first term a and common difference d is given by the following formula. $S_n = \frac{n}{2}\{2a+(n-1)d\}$
§2. Other kinds of sequence 2.1 The sigma notation Σ 2.2 Difference of sequence 2.3 The sum of various series	By using above formula and given identical equation, the following equations are proven. $1+2+\cdots+n=n(n+1)/2$ $1^2+2^2+\cdots+n^2=n(n+1)(2n+1)/6$
§3. Mathematical induction 3.1 Recurrence relation 3.2 Mathematical induction	The following statements are proven by mathematical induction. - The sum of the first n positive integers is $n(n+1)/2$ - The sum of the first n^2 positive integers is $n(n+1)(2n+1)/6$

Table 2: Outline of the teaching unit "sequence" and some excerpts from a textbook

On the one hand, the contents of §2 can be seen as proof and proving at the level of "local theory", because the accepted formula (e.g., the sum of S_n of the first n terms) and/or given identical equation are deductively used for proving the statements (e.g., $1+2+\cdots+n=n(n+1)/2$). But part of the formula used in the proof has been acquired by a generic pictorial explanation that cannot be accepted as mathematical proof (it may be at the level of "logic of the real world"), and ready-made identical equations (e.g., $k^3-(k-1)^3=3k^2-3k+1$) without proof are used for proving the statement (e.g., $1^2+2^2+\cdots+n^2=n(n+1)(2n+1)/6$). On the other hand, contents of §3 can be seen as proof and proving at the level of "quasi-axiomatic theory" because the statements (some of them are the same statements proved in §2) are proven by appeal to the Principle of Mathematical Induction (Peano's fifth axiom for the foundation of natural number) that permits the application of "a meta-theory" (i. e., modus ponens, etc.) to establish the truth of the statement about the elements of sets that can be placed in one-to-one correspondence with the set **N** (cf. Tall et al., 2012, p. 39). What does it imply for further developmental research? Although the appeal to Peano's axiom is usually implicit in the proof method of MI, it may be worthwhile at this point to relate to the other aspects such as the formulation of "statement" or the validation of "proof", and to investigate how more-precise mathematical words might affect students' proof and proving at the level of "quasi-axiomatic theory" for the sake of curriculum development.

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References

Antonini, S. & Mariotti, M. A. (2008). Indirect proof: what is specific to this way of proving? *ZDM-The International Journal of Mathematics Education*, *40*(3), 401-412.

Balacheff, N. (1987). Processus de preuves et situations de validation. *Educational Studies in Mathematics*, *18*(2), 147-176.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*(3/4), 413-435.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

Hanna, G. & Jahnke, H. K. (2002). Another approach to proof: arguments from physics. *ZDM-The International Journal of Mathematics Education*, *34*(1), 1-8.

Hanna, G., & de Villiers, M. (Eds.), *Proof and proving in mathematics education: The 19th ICMI study*. New York: Springer.

Hirabayashi, I. (1991). Reviews and issues on the teaching contents of geometry. In N. Noda, et al. (Eds.), *A course of new practice and examples in junior high school mathematics teaching: volume 3, geometry* (pp. 3-34). Tokyo, Japan: Kaneko-shobou. (in Japanese)

Hoyles, C. (1997). The curricular shaping of students' approaches to proof. *For the Learning of Mathematics*, *17*(1), 7-16.

Mariotti, M. A. (2006). Proof and proving in mathematics education, In A. Gutiérrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: past, present and future* (pp. 173-204). Rotterdam, The Netherlands: Sense Publishers.

Mariotti, M. A. *et al*. (1997). Approaching geometry theorems in contexts: from history and epistemology to cognition. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the Psychology of Mathematics Education*, *Vol. 1*, 180-195, Lahti, Finland: PME.

NCTM (2000). *Principles and standards for school mathematics*. Reston, VA, USA: NCTM.

Japan Society of Mathematical Education (Ed.) (1966). *Modernization of mathematics education*. Tokyo, Japan: Baihukan. (in Japanese)

Reid, D. A. & Knipping, C. (2010). *Proof in mathematics education. Rotterdam*, The Netherlands: Sense Publishers.

Stylianou, D. A. *et al*. (Eds.) (2009). *Teaching and learning proof across the grades: A K-16 perspective*. New York, Routledge: NCTM.

Tall, D. *et al*., (2012). Cognitive development of proof, In Hanna, G., & de Villiers, M. (Eds.), *Proof and proving in mathematics education: The 19th ICMI study* (pp. 13-49). New York: Springer.

USE OF I-POEMS TO UNCOVER ADOLESCENTS' DYNAMIC MATHEMATICS IDENTITY WITHIN SINGLE-SEX AND COEDUCATIONAL CLASSES

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The purpose of this study is to inform our understanding of the middle grade female and male students' mathematics identity construction within single-sex and coeducational mathematics classrooms. Students' mathematics identities were understood as being composed of and shaped by an interplay of "voices," which were gleaned from participants' I-statements within their personal narratives. Results support the notion that one's mathematics identity is complex, and suggest that being in a single-sex or coeducational classroom does not shape participants' multiple voices differently. On the other hand, participants' gender and teacher influenced identity construction differently.

SITUATING THE PROBLEM

The manner in which one views and portrays him or herself in relation to mathematics, or one's mathematics identity, is an abstract and complex construct (Cobb, Gresalfi, & Hodge, 2009) that is multiple, dynamic, and constantly in flux (Bishop, 2012). In this study, I seek to gain an understanding of how female and male middle grade students narrate their mathematics identity within single-sex and coeducational mathematics classroom settings within the same public coeducational school in the United States. Therefore, the perceptually salient feature of gender is utilised in segregating students into these classes, which is a visible marker that has historically situated female students as an "outsider" within a subject area deemed more suitable for males (Bartholomew, Darragh, Ell, & Saunders, 2011). Relevant to the conference theme, in the United States, researchers are exploring this institutionalised practice and *Climbing Mountains* and *Building Bridges* in regards to ideologies and beliefs concerning the purpose of single-sex education held by parents, teachers, administrators, and policymakers, as well as the possible impacts of single-sex education on students such as academic achievement (e.g., Pahlke, Hyde, & Mertz, 2013) and academic self-concept (e.g., Sullivan, 2009). This study focuses on the following research questions, (1) What voices shape the mathematics identities of female and male students? (2) How are these voices embodied in different types of class composition (i.e., all-female, all-male, and coeducational)? This study will contribute to the scholarly work on single-sex education because there is no known study that explores students' mathematics identity within such settings, as well as to the scholarly work on mathematics identity as it has been historically overlooked in mathematics education (Bishop, 2012).

Unlike other countries such as Australia (e.g., Carpenter & Hayden, 1987), single-sex education, particularly, single-sex mathematics classrooms within coeducational public schools in the United States were non-existent until 2006 when the federal government made revisions to Title IX legislation (USDOE, 2006). Additionally, a majority of research has been conducted within single-sex private and public school settings rather than single-sex classes in coeducational public schools (e.g., Pahlke, Hyde, & Mertz, 2013). Research involving single-sex education is diverse, inconclusive, and still evolving in that validated and replicated results have yet to transpire (Pahlke, Hyde, & Allison, 2014). Therefore, this study relied on the scholarly research grounded in gender when considering the possible voices shaping participant's mathematics identity. For instance, one voice considered was student's self-confidence in mathematics, defined as "one's perceptions of their ability do well in mathematics and to learn mathematics quickly" (Else-Quest, Hyde, & Linn, 2010, p. 117). Research in general suggests that boys tend to report higher levels of self-confidence in their mathematical abilities than girls do (e.g., Else-Quest et al, 2010), as well as attribute their success in mathematics to ability, while girls tend to attribute their success in mathematics to effort (e.g., Gilbert, 1996). Self-confidence in this study is also viewed as a performative act in which students take an active or passive role (Hardy, 2007); acts possibly being shaped by the extent students feel accepted and included (or not) within the mathematics classroom.

THEORETICAL FRAMEWORK

This study utilises the notion of "voice" as the basic unit for understanding female and male's dynamic mathematics identity. The work of Gilligan (1982) emphasised that of the missing *voice* from human development theories, the voice of women. Omitting the voice and experience of women positioned them as the subordinate gender and established a dichotomy of "appropriate" gender roles, resulting in women feeling pressured to conform and reject their "true" sense of self. Additionally, through the cumulative work of Bakhtin (1981), it can be argued that one's *voice(s)* is composed of and shaped by the words, utterances, and social languages not of our own, but of others' that have lived before us and that presently live with us; termed hybridisation. Yet in this process, students negotiate the authoritative voice of others, such as teachers, parents, and mathematics curriculum to be internally persuasive; making it one's own *voice* through language populated within their own meanings, intentions, and accents. Evans (2008) extends Bakhtin's notion of hybridisation to contend that individuals have a lead voice that is a hybrid of *voices,* rather than that of language, composed of all societal influences and voices of others that affect who we are. These multiple voices are vying for audibility and are constantly in interplay with one another, rejecting voices, accepting voices, and creating new voices. In summary, these theoretical perspectives guided my understanding and analysis of female and males' dynamic mathematics identity as constructed by their narrated *voices.*

METHODOLOGY

This research employed narrative inquiry, a qualitative research method that privileges the experiences and voices of individuals and serves as a means for researchers to gain an understanding of individual's voices as "truth" (Connelly & Clandinin, 1990). More specifically, this study relied on descriptive narratives because the intent is not to explain why something has happened, but to render the multiple voices of one's mathematics identity as narrated by participants.

Participants

Participants were twelve 7^{th} grade students (6 males, 6 females) enrolled in one of the mathematics class types, all-female, all-male, or coeducational, and instructed by one of two teachers in a coeducational middle school located in the southeast region of the United States. Twelve participants were purposively selected based on several criteria: returned consent forms, class composition (i.e., single-sex or coeducational), gender of participant, and results from the *Mathematics as a Gendered Domain* instrument (Forgasz, Leder, & Kloosterman, 2004). Utilising this instrument reduced selection bias, as participants were not selected based on researchers or teachers' subjectivities.

Data sources

Mathematics identity is a construct that cannot be easily observed, but best represented by how participants talk about themselves as a mathematics student within a single-sex or coeducational mathematics classroom (Polkinghorne, 1988). Therefore, the primary data source for this study was semi-structured interviews. As an example used to gain an understanding of participants' mathematics identity, they provided at least three adjectives that described themselves as math students and explained how each adjective described them. The interviews lasted between 20-30 minutes, conducted during their enrichment period (11:00-11:50), and took place in the school's library. The interviews were transcribed verbatim.

Data analysis

The interviews were analysed using I-poems, which are part of a four-step interview analysis technique known as the Listening Guide (Brown, Debold, Tappan, & Gilligan, 1991). The Listening Guide was developed to provide a safe space for females to speak freely about their experiences such as silence and depression (Beauboeuf-Lafontant, 2008). I-poems were utilised as an evocative and potent way to capture participants' narrative of self ("I") as they spoke of themselves as mathematics students within a single-sex or coeducational classroom. They provided participants with a space to be partially heard and not silenced through the research process (Bhattacharya, 2008). This is a two-step process. One, the researcher underlined every participant's use of "I" along with the verb and any accompanying important words or phrases. Two, each "I" phrase was taken out of context and positioned on a separate line of a poem in the same sequential order of the text. Stanzas were subsequently created to represent the varying voices narrated by participants.

RESULTS

The first research question asked what voices shape the mathematics identities of female and male students. Table 1 provides a few voices, along with an accompanying example, expressed by participants.

Voice	Defined	Example
Voice of Oscillation	Voice expresses contradictory statements in regards to mathematics abilities	I am good I am super I get A's I am a little not good I am bad I don't get it
Voice of Assist	Voice expresses giving mathematical or non-mathematical help to others	I am helpful I feel good I can help I like to help
Voice of Invisibility	Voice expresses taking a passive role in the classroom	I have stage fright I don't like I would rather not I don't like talking
Voice of Outsider	Voice expresses being "picked on" by one's peers; feelings of being "the other"	I dislike the people I felt like nobody liked me I don't know
Voice of Effort	Voice expresses working hard and efficiently in order to be successful in mathematics	I am willing I will put forth I do the homework I spend a lot of time
Voice of "Good" Student	Voice expresses actions that are associated with being a "good" student	I am always focused I am always turning I am always on time I am always there

Table 1. Sample voices that emerged from participants' I-poems.

The second research question explored how voices were embodied in different types of class composition (i.e., all-female, all-male, and coeducational). To illustrate the interplay of voices shaping one's mathematics identity (Evans, 2008), I will briefly discuss Matthew, a male enrolled in a single-sex class. His I-poems expressed an interplay of voices composed of Voice of Pride, Voice of Subordinate, Voice of Manipulation, and Voice of Outsider. Below is an abbreviated version of Matthew's I-poems. Readers are encouraged to *listen* to Matthew's voices and form a relationship with Matthew, rather than simply read the I-poems as a form of data.

All I got to do is go over it once
I like to prove how smart
I go the harder way
I am top dog
I always compete
I feel smart

I dislike the teacher
I feel bad
I have a negative attitude
I can't be myself

I was trying to act out
I would get moved
she thought I was quiet
I will pretend to raise
she thinks I know the answer

I think they are jealous
I wouldn't volunteer
I don't want to see
I don't want people
I make mistakes

Expounding upon Matthew's I-poems, Matthew's use of first person indicated a dislike for the teacher because of her inability to teach him effectively. This view of his teacher was being shaped by his Voice of Pride or his abundant confidence in himself as a mathematics learner. His Voice of Pride also narrated his feeling as a victim of his teachers' instructional methods and procedures, and interactions with him as a mathematics student (i.e., Voice of Subordinate). For instance, he expressed how his teacher had a tendency to make him feel bad when she disregarded his questions such as, is infinity a rational number. Therefore, Matthew articulated a Voice of Manipulation in which he purposefully experienced behaviours to control or influence his teacher. Additionally, Matthew situated himself as an outsider in relation to his peers (i.e., Voice of Outsider), which similar to the Voice of Subordinate, this voice was being shaped by his Voice of Pride.

When examining the voices composing participant's mathematics identity, researchers deduced that participants' voices did not differ based on class composition. In other words, being enrolled in a single-sex or coeducational mathematics classroom had little to no influence in shaping the interplay of voices of participants' mathematics identity *differently.* The voices shaping participants' mathematics identities are more a function of students' personal experiences as a mathematics student. However, when considering the voices as distinct entities, differences were noted in respect to the gender of the participants. For example, four of the six female participants expressed a

Voice of Assist (see Table 1), while none of the six male participants expressed this voice. This voice expressed accepting and participating in the societal gender norms expected of females, voices expressing an ethic of care and relationships (Gilligan, 1982). As an additional example, male participants were the only to self-narrate the Voice of Effort rather than a voice expresses a natural mathematical ability (Gilbert, 1996). Results also suggests a difference in voices based on participants' teachers; thus, reinforcing the impact of teachers' instructional practices, interactions and dispositions on how students' perceive themselves as learners (e.g., Horn, 2008). One teacher instructed participants (n = 6) that expressed negative voices, such as Voice of Invisibility, while all the participants (n = 6) instructed by the other teacher expressed the Voice of "Good" Student.

CONCLUSION

Insights from analysis of the I-poems reinforced the idea that mathematics identity is a complex (Cobb et al., 2009) and individualistic construct as no two participants expressed the same interplay of voices (Evans, 2008). In considering the mathematics identity of one participant, Matthew, we gain some sense of how the interplay of his multiple voices is vying to be heard. For instance, his Voice of Pride and his Voice of a Victim seem to be in competition, influencing one another in a play for this lead voice in narrating his mathematics identity (Evans, 2008). In addition, the findings suggest that across the three class types, participants in this study are more similar than they are different. Any noted distinction is apparent in regards to participants' gender and teacher. This suggests that student's voices, and hence their mathematics identity, is an interplay between the local context (i.e., teacher) and global context (i.e., gender).

The notion of students' mathematics identity being composed of multiple voices present and challenge researchers to study identity differently and to continue building theoretical ideas that explain how adolescents negotiate and narrate subject-specific identities, particularly through the use of I-poems; therefore, building upon our current understanding of adolescents' mathematics identity. This study also contributes to the current scholarship on single-sex education within the United States in particular, which as noted above is inconclusive (Pahlke et al., 2014). Based on the findings from this study, one continuing line of research is to explore how teachers' mathematics identity and teacher identity is in accordance or conflict with the mathematics identity of their students and what this means in regards to shaping students' voices.

References

Bartholomew, H., Darragh, L., Ell, F., & Saunders, J. (2011). 'I'm a natural and I do it for love!: exploring students' accounts of studying mathematics. *International Journal of Mathematical Education in Science and Technology, 42*(7), 915-924.

Bakhtin, M. M. (1981). *The dialogic imagination* (C. Emerson & M. Holquist, Trans.). Austin, TX: University of Texas Press.

Bishop, J. P. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom. *Journal for Research in Mathematics Education, 43*(1), 34-74.

Beauboeuf-Lafontant, T. (2008). Listening past the lies that make us sick: a voice-centered analysis of strength and depression among black women. *Qualitative Sociology*, *31*(4), 391-406. doi: 10.1007/s11133-008-9113-1

Bhattacharya, K. (2008). Voices lost and found: Using found poetry in qualitative research. In M. Cahnmann-Taylor & R. Siegesmund (Eds.), *Arts-based research in education: Foundations of practice* (pp. 83-88). New York, NY: Routledge.

Brown, L. M., Debold, E., Tappan, M., & Gilligan, C. (1991). Reading narratives of conflict and choice for self and moral voices: A relational method. In W. M. Kurtines & J. L. Gewirtz (Eds.), *Handbook of Moral Behavior and Development*, (Vol. 2, pp. 25-61). Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.

Carpenter, P., & Hayden, M. (1987). Girls' academic achievements: Single-sex versus coeducational schools in Australia. *Sociology of Education, 60*(3), 156-167.

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education, 40*(1), 40-68.

Connelly, F. M., & Clandinin, D. J. (1990). Stories of experience and narrative inquiry. *Educational Researcher, 19*(5), 2-14.

Else-Quest, N. M., Hyde, J. S., & Linn, M. C. (2010). Cross-national patterns of gender differences in mathematics: A meta-analysis. *Psychological Bulletin, 136*(1), 103-127.

Evans, F. (2008). *The multivoiced body: society and communication in the age of diversity.* New York: Columbia University Press.

Forgasz, H. J., Leder, G. C., & Kloosterman, P. (2004). New perspectives on the gender stereotyping of mathematics. *Mathematical Thinking and Learning, 6*(4), 389-420.

Gilbert, M. C. (1996). Attributional patterns and perceptions of math and science among fifth-grade through seventh-grade girls and boys. *Sex Roles, 35*, 489-506.

Gilligan, C. (1982). *In a different voice: Psychological theory and women's development.* Cambridge, MA: Harvard University Press.

Hardy,T. (2007). Participation and performance: Keys to confident learning in mathematics?. *Research in Mathematics Education, 9*(1), 21-32.

Horn, I. S. (2008). Turnaround students in high school mathematics: Constructing identities of competence through mathematical worlds. *Mathematical Thinking and Learning, 10*, 201-239.

Pahlke, E., Hyde, J. S., & Allison, C. M. (2014). The effects of single-sex compared with coeducational schooling on students' performance and attitudes: A meta-analysis. *Psychological Bulletin*. Advance online publication. doi:10.1037/a0035740

Pahlke, E., Hyde, J. S., & Mertz, J. E. (2013). The effects of single-sex compared with coeducational schooling on mathematics and science achievement: Data from Korea. *Journal of Educational Psychology, 105*(2), 444-452.

Polkinghorne, D. E. (1988). *Narrative knowing and the human sciences.* Albany, NY: State University of New York Press.

Sullivan, A. (2009). Academic self-concept, gender and single-sex schooling. *British Educational Research Journal, 35*(2), 259-288. doi:10.1080/01411920802042960

U.S. Department of Education. (2006). *Nondiscrimination on the basis of sex in education programs or activities receiving federal financial assistance; final rule* (34 CFR Part 106). Retrieved from http://www2.ed.gov/legislation/FedRegister /finrule/2006-4/102506a.pdf.

THE ENGAGEMENT OF STUDENTS WITH HIGH AND LOW ACHIEVEMENT LEVELS IN MATHEMATICS

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Student engagement is a significant issue in mathematics classrooms. It plays a major role in students' enjoyment, interest and future participation in mathematics study. Interviews with 37 high and low achieving Year 7 students from ten high schools explored perceptions of their engagement and achievement levels in mathematics. Results indicate that highly engaged students were more 'alike' in terms of their attitudes towards mathematics, strategies for learning the subject, interest and behaviour regardless of differences in their achievement levels. Conversely, the profiles of high and low achieving students disengaging from mathematics were less alike. These findings shed light on student engagement in mathematics because they disentangle characteristics of engagement from those attributed to achievement.

BACKGROUND

Student disengagement in mathematics is a significant issue for education (McPhan, Morony, Pegg, Cooksey, & Lynch, 2008; Tytler, Osborne, Williams, Tytler, & Cripps Clark, 2008). Recent reports (McPhan et al., 2008) note concerns about the proportion of students achieving acceptable levels of proficiency in mathematics and the poor level of student engagement in the subject. In particular, there is concern about decreasing interest, participation and achievement in mathematics as students move from primary and through lower secondary years of schooling. There is an emphasis on the need for a thorough understanding of the relationship between engagement and achievement in mathematics as this is expected to guide teachers' practices that are motivationally supportive and effective for promoting student engagement in mathematics.

This study focused on understanding individual and classroom level factors that influence engagement and achievement through in-depth interviews with Year 7 students (approximately 11 to 13 years of age). One of the key aims of the study was to understand the influence of salient factors on levels of engagement distinct from students' achievement levels in mathematics, specifically asking:

What individual and classroom level factors do students perceive as influencing their engagement, motivation and achievement in mathematics?

Addressing this question required eliciting students' feelings towards mathematics, their views about their mathematical abilities, their thoughts about what takes place and reports of their behaviour in mathematics classrooms—and the effect this has on their engagement and achievement in mathematics

Student Engagement and Motivation

Although researchers adopt different conceptual approaches when investigating engagement and motivation, the relationship between the two is broadly agreed upon. Motivation is concerned with the psychological processes that are underlying sources of energy displayed by visible engagement characteristics (Skinner & Pitzer, 2012). Motivation is considered to underpin engagement, creating a cycle in which motivation and engagement are inherently linked to learning outcomes (Martin, 2012). However, although they are linked, motivation and engagement are viewed as distinct: motivation is viewed as encompassing the internal, private and unobservable factors of the outer, public and observable engagement.

Certain features are shared between engagement and motivation because of the underlying sources of energy that are reflected in engagement characteristics. For example, persistence (being an adaptive motivation) may be observed by time spent on tasks and asking questions, which are also characteristics of behavioural engagement. Often, engagement is more obviously connected to the learning environment because it reflects an individual's interaction within contexts (Fredricks, Blumenfeld, Friedel & Paris, 2005) as the underlying motivational processes may be harder to determine.

The relationship between motivation and engagement is an important one although it is at times difficult to distinguish between the two and ultimately how they are differentiated depends on the conceptualisations and definitions applied by researchers. As noted by Reeve (2012), the focus taken by researchers influences their perspectives because "those who study motivation are interested in engagement mostly as an outcome of motivational processes, whereas those who study engagement are mostly interested in motivation as a source of engagement" (p.151).

THEORETICAL FRAMEWORKS

The present study sought to consider engagement and motivation together, using two complementary frameworks. One provides detailed definitions of types of engagement and the other identifies a range of motivational factors underlying engagement. Importantly, by investigating these together adds clarity about specific motivational sources of student engagement in mathematics.

The first framework describes and clarifies types of student engagement (Fredricks, et al., 2005; Fredricks, Blumenfeld, & Paris, 2004), framing it as a meta-construct incorporating behavioural, cognitive and emotional engagement elements. The second framework organises dimensions of academic motivation and engagement within a higher and lower order construct framework (Martin, 2003; 2007). The multidimensional approach incorporates 11 motivational factors relevant to adaptive and maladaptive student behaviours, emotions and cognitive strategies. In some cases researchers consider student engagement from a specific motivational perspective such as self-efficacy (Schunk & Mullen, 2012), in other cases, from a selection of motivation constructs, for example, autonomy, competence relatedness and meaningfulness (Turner, Christensen, Kackar-Cam, Trucano & Fulner, 2014). The approach reflected by Martin's Motivation and Engagement Wheel (2007) aims to comprehensively consider the influence of motivational constructs on engagement more broadly than research with particular perspectives.

The overarching theoretical orientations of Martin's Motivation and Engagement Wheel (2003; 2007) and Fredricks et al., (2004) 'Types of Engagement' align and draw together motivational theories and engagement factors related to academic achievement outcomes. Martin (2007) notes that motivation and engagement play a large part in students' energy and drive to participate and learn at school, affecting their interest and enjoyment in what they do at school including their academic outcomes. Therefore, identifying the factors that influence engagement levels in students is important, and the frameworks used in this study aim to provide a basis for a deeper inquiry into the nature of engagement in mathematics learning.

Student Perceptions and Beliefs

Both student perceptions and beliefs are relevant to this study. Perceptions are understood to be personal in nature because individuals interpret factors and respond to their environments through personal 'filters' as they process information (Broadbent, 1958). Beliefs are defined as "psychologically held understanding, premises, or propositions about the world that are felt to be true" (Richardson, 1996, p.103) and also vary according to the bearer of the beliefs. Therefore, there is an intricate process between perceptions and beliefs. The beliefs an individual holds are filtered by what they perceive or notice, similarly, how an individual perceives objects and events are filtered by how they construct their beliefs of (knowledge) and in (values) phenomena (Philip, 2007). Consequently, for this study, amongst other things, students were asked to report their beliefs about their mathematical achievement and their perceptions of mathematics teaching. Student self-reports were considered an appropriate method for sourcing student perceptions and beliefs because students are able to "accurately report on their own engagement and environments" (Reschly & Christenson, 2012, p. 9) and critically, they can provide rich data about aspects of engagement that are not easily observable such as emotional and cognitive factors.

METHODOLOGY

This study was initially nested within a larger project that utilised a validated Motivation and Engagement Survey [MES] (Martin, 2007) with over 1600 middle year students (Years 5 to 8, approximately 10 to 15 years of age). Mathematics achievement was assessed using an adaption of the Wide Range Achievement Test 3 (Wilkinson, 1993) so as to be administered across a range of age groups. Reliability, descriptive and distributional properties, and confirmatory factor analytic results of the quantitative component have been published elsewhere (Martin, Bobis, Anderson & Way, 2011).

The MES and the achievement test were undertaken in Time 1 (May 2008) and Time 2 (May 2009) by students in 47 comprehensive primary and secondary schools from one district in a major capital city on the east coast of Australia. Targeted Year 7 students were invited for interview based on the size of the shifts in engagement and disengagement over a period of 1 school year (between Time 1 and Time 2). From Time 1 and Time 2 quantitative data, Year 7 students were grouped into one of four categories: low achieving + disengaged (LAD), low achieving + engaged (LAE), high achieving + disengaged (HAD) and high achieving + engaged (HAE). Thirty-seven Year 7 students drawn from each of 10 secondary schools involved in the quantitative component of the project were interviewed. The numbers of students interviewed from each category are presented in Table 1.

	Disengaged Students	Engaged Students
Low Achieving	10	10
High Achieving	8	9
Total	18	19

Table 1: Number of disengaged and engaged students interviewed

The students were purposefully grouped into one of the four categories because the researchers were interested in understanding the perceptions of students with varying levels of engagement and achievement in mathematics. Therefore the interviews focused on understanding individual and classroom level factors that influenced engagement and achievement through in-depth semi-structured interviews (approximately 30 to 40 minutes in duration) with students. The interview questions related to four key themes: (1) student beliefs about their mathematics achievement; (2) student emotions toward mathematics; (3) student perceptions of mathematics teaching; and (4) student behaviours while learning mathematics. Student responses to questions were transcribed and then coded by the researcher, using a second researcher to establish an average inter-rater reliability accuracy rate of 95.8%.

The four themes captured the main characteristics of student perceptions towards mathematics that they brought to the classroom including their attitudes, feelings, behaviours, and beliefs about their mathematical ability. By using the student groupings of LAD, LAE, HAD and HAE the themes could be viewed from two different perspectives—levels of engagement and levels of achievement. The benefit of this was to understand the characteristics of students with varying levels of engagement in mathematics untangled from their achievement levels.

RESULTS

Analysis of interview data indicated that a complex mix of factors influenced student engagement levels in mathematics. However when examined through the lens of the different student groupings—LAD, HAD, LAE and HAE—some generalities pertaining to each group were revealed. The results of the two *engaging* groups both low and high achieving are presented first, followed by findings from the *disengaging* groups – both low and high achieving students. Summaries of the findings are represented in Figure 1.

Engaging Students (LAE and HAE)

Engaged students portrayed positive feelings towards mathematics even when they found concepts challenging or difficult. Low achieving students reported enjoying learning new concepts and liked the challenges the work presented once they were able to ‘get it’. High achieving students enjoyed learning more complex concepts and solving problems with multiple parts, reported a history of liking mathematics and talked about the importance of mathematics for their future. High achieving students also expected that mathematics required effort and frequently equated this as contributing to their achievement.

Importantly, the overall results indicated that low levels of student engagement in mathematics are not always aligned with low levels of achievement and vice versa, as is often commonly assumed in educational contexts. Further, engaged students were more ‘alike’ in terms of their attitudes towards mathematics, strategies for learning the subject, interest and behavioural engagement in mathematics regardless of differences in their achievement levels. Conversely, high and low achieving students disengaged from maths, were less alike. They reported a greater variety of strategies for learning mathematics with high achieving student strategies very different to their lower achieving disengaged counterparts. However, common amongst disengaged students was their general lack of interest in mathematics and their lack of confidence to achieve well in the subject.

Disengaging Students (LAD and HAD)

The disengaged group included students of both low (LAD) and high mathematical achievement levels (HAD). The interviews specifically sought to explore if the reasons for disengagement of high achieving students were different to those of low achieving students. The negative feelings about learning mathematics for low achieving disengaged students’ were strongly linked to their lack of general understanding of mathematics concepts with some reporting experiences of frustration and anger. Although many low achieving disengaged students maintained their efforts in mathematics they often ‘forgot’ what they were taught, knew they did not really ‘get’ what was going on and ‘felt lost’. On the contrary, high achieving disengaged students found mathematics learning more challenging than they had previously done so in elementary school and the increased effort (persistence) required to master mathematical concepts in secondary school made mathematics less enjoyable. Some students reported an indifferent attitude towards the subject and others found repetitive instruction boring which also reduced their interest in learning mathematics. A summary of the key results for each student group reflecting their beliefs about their mathematical ability, feelings about and behaviours towards mathematics, as well as their perceptions of mathematics teaching, are presented in Figure 1.

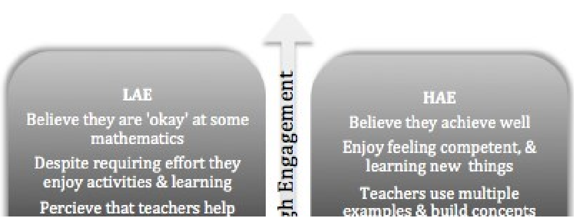

Figure 1: Profiles of engaging and disengaging students in mathematics

DISCUSSION AND CONCLUSION

The findings of this research significantly enhance and extend our understanding of the engagement construct by revealing that nuances surrounding student engagement in mathematics are differentiated for low and high achieving students. Student engagement in learning is seen as an important predictor of general academic achievement and positive academic outcomes (Fredricks et al., 2004; Lutz, Guthrie, & Davis, 2010), and even necessary for constructive educational beliefs and behaviours (Middleton & Midgley, 2002). Yet as is evident from this research, not all students who were highly engaged experienced high achievement. This is apparent by the existence of the LAE group — students who were identified as engaged in mathematics but experienced low levels of achievement. Understanding the reasons why students remain engaged in learning mathematics despite low achievement is important information for mathematics educators.

The study drew attention to the contrary case, where high levels of student achievement were not necessarily indicative of high levels of engagement—evident by the high achieving disengaged (HAD) group of students. This group of students were, despite their relative success in mathematics, identified as 'disengaging' from mathematics. Increases in student disengagement (indicated by disinterest and lack of participation in mathematics beyond compulsory requirements) during secondary school are of concern in Australia and whilst achievement in mathematics is related to these declines it is only one of the influencing factors (Forgasz, Barkatsas, Bishop, Clarke, Keast, Tiong Seah et al., 2008; McPhan et al., 2008). Uncovering reasons for students' low levels of engagement when their achievement is high in mathematics is central for understanding the underlying needs of these students. Understanding students' needs and the degree to which they are being met is necessary to halt the shift towards disengagement and to re-engage students in mathematics. In particular, knowing that a variety of teaching practices will be required to address student disengagement, with the more effective ones being dependent upon the achievement level of the students in question, is a crucial practical implication of the findings presented.

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References

Broadbent, D.E. (1958). *Perception and communication*. London: Pergamon Press.

Forgasz, H., Barkatsas, A., Bishop, A., Clarke, B., Keast, S., Tiong Seah, W., et al. (Eds.). (2008). *Research in Mathematics Education in Australia 2004-2007* (Vol. 12). Rotterdam: Sense Publishers.

Fredricks, J., Blumenfeld, P., Friedel, J., & Paris, A. (2005). School engagement. In K. A. Moore & L. H. Lippman (Eds.), *What Do Children Need to Flourish: Conceptualizing and Measuring Indicators of Positive Development* (pp. 305-321). NY: Springer Science+Business Media.

Fredricks, J., Blumenfeld, P., & Paris, A. (2004). School engagement: potential of the concept, state of the evidence. *Review of Educational Research*, *74*(1), 59-109.

Lutz, S., Guthrie, J., & Davis, M. (2010). Scaffolding for engagement in elementary school reading instruction. *The Journal of Educational Research*, *100*(1), 3-20.

Martin, A. (2003). The Student Motivation Scale: Further testing of an instrument that measures students' motivation. *Australian Journal of Education, 47*(1), 88-106.

Martin, A., Bobis, J., Anderson, J., & Way, J. (2011). Multilevel variance in psycho-educational phenomena: Exploring differences across constructs, *German Journal of Educational Psychology (Zeitschrift fur Padagogische Psychology)*, *25*(1), 49-61.

Martin, A. (2007). Examining a multidimensional model of student motivation and engagement using a construct validation approach, *British Journal of Educational Psychology*, *77*(2), 413-440.

McPhan, G., Morony, W., Pegg, J., Cooksey, R., & Lynch, T. (2008). *Maths? Why not?* Canberra: Department of Education, Employment and Workplace Relations.

Middleton, M., & Midgley, C. (2002). Beyond motivation: Middle school students' perceptions of press for understanding in math. *Contemporary Educational Psychology*, *27*, 373-391.

Philipp, R.A. (2007). Mathematics teachers' beliefs and affect. In F.K. Lester Jr, (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol.1, pp.257-315). Charlotte, N.C: Information Age Publishing.

Reeve, J. (2012). A self-determination theory perspective on student engagement. In A. Christenson, A. Reschly & C. Wylie (Eds.), *Handbook of research on student engagement.* New York: Springer.

Schunk, D. H., & Mullen, C. A. (2012). Self-efficacy as an engaged learner. In A. Christenson, A. Reschly & C. Wylie (Eds.), *Handbook of research on student engagement.* New York: Springer.

Skinner, E.A., & Belmont, M.J. (1993). Motivation in the classroom: Reciprocal effects of teacher behaviour and student engagement across the school year. *Journal of Educational Psychology*, *84* (4), 571-581.

Skinner, E.A., & Pitzer, J.R. (2012). Developmental dynamics of student engagement, coping and everyday resilience. In A Christenson, A. Reschly & C. Wylie (Eds.), *Handbook of research on student engagement*. New York: Springer.

Turner, J.C., Christensen, A., Kackar-Cam, H.Z., Trucano, M., & Fulmer, S.M, (2014). Enhancing students' engagement: Report of a 3-year intervention with middle school teachers. *American educational research journal*, *51* (6), 1195-1226.

Tytler, R., Osborne, J., Williams, G., Tytler, K., & Cripps Clark, J. (2008). *Opening up pathways: Engagement in STEM across the Primary-Secondary school transition.* Canberra : Department of Education, Employment and Workplace Relations.

Wilkinson, G.A. (1993). Wide Range Achievement Test. Wilmington, DE: Wide Range Inc.

RESEARCH ON MATHEMATICAL ARGUMENTATION: A DESCRIPTIVE REVIEW OF PME PROCEEDINGS

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Mathematical argumentation and proof (MA&P) traditionally are major topics of mathematics education in secondary and tertiary education. Although many studies focus on MA&P it remains unclear how they contribute to a coherent understanding of MA&P processes. We have analysed PME research reports focusing on MA&P published 2010 to 2014 to determine the different prerequisites as well as goals of argumentation and proving processes investigated within these reports. Results indicate that research on MA&P covers a broad range of processes, sub-skills and knowledge facets, but that individual reports predominantly address only singular aspects. A holistic approach to MA&P, taking into account the whole process or multiple sub-skills, is rare. We discuss implications for future research of MA&P.

INTRODUCTION

Mathematics is a proving science, and mathematical argumentation and proof (MA&P) therefore are central to mathematical activity (Ubuz, Dincer, & Bulbul, 2012). Many standard documents worldwide put forward MA&P as one central goal of mathematics learning (CCSSI, 2010), especially in secondary and tertiary education. Accordingly, mathematics education research has traditionally approached this field from various perspectives. It is widely agreed that MA&P comprise complex skills that integrate diverse individual cognitive prerequisites and different kinds of mathematical activities. From our understanding, an aim of MA&P research must be an increasingly coherent understanding of these diverse facets, since otherwise effective support of MA&P processes is not possible.

The purpose of this review is to analyse current research on mathematical argumentation and proof in secondary and tertiary education. To cover the diversity of MA&P extensively, we have based our analysis on existing theoretical frameworks of scientific reasoning which highlight prerequisites, processes and goals of MA&P: Predictors for mathematical argumentation skills (Ufer, Heinze, & Reiss, 2008), epistemic activities in scientific argumentation (Fischer et al., 2014), and argumentative and proving activities (Giaquinto, 2005).

THEORETICAL FRAMEWORK

Argumentation and proof

According to Balacheff (1999), there are two meanings of argumentation within the field of mathematics. Thus, mathematical argumentation can be considered a discursive activity aimed at convincing a listener. On the other hand, based rather on

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 193-200. Hobart, Australia: PME.

Toulmin's view, argumentation is an activity which is aimed at the generation, exploration and validation of conjectures and hypotheses in terms of their objective and individual rationality (Pedemonte, 2007). For our review we adopt this second view. Accordingly, mathematical proof is seen as a more formal form of mathematical argumentation, which is subject to (mostly implicit, social, and possibly changing) norms of the mathematical community. This difference between argumentation and proof is nicely put by Pedemonte (2008, p. 385):

> "There is a 'structural gap' between argumentation and proof because in argumentation inferences are based on content while in proof they follow a deductive scheme (data, claim, and inference rules)."

Sub-skills and knowledge facets

The success of mathematical argumentation and proving depends on individual prerequisites like domain-general and domain-specific knowledge facets, beliefs and more overarching skills. Over the last decades researchers have proven a variety of such sub-skills and knowledge facets to be predictive for mathematical argumentation skill (e.g., Ufer, Heinze, & Reiss, 2008), which therefore often are called predictors. For this review we adopt a framework of predictors worked out in Ufer, Heinze and Reiss (2008) that a) is well based on research from the last decades, b) is not limited to a specific mathematical area, and c) allows separation of domain-specific and more domain-general predictors. The framework contains six main predictors. *Methodological knowledge* is knowledge of the nature and the functions of proof as well as the acceptance criteria for a valid proof (Healy & Hoyles, 2000). *Mathematical knowledge base* consists of basic conceptual and procedural knowledge in the field of mathematics (Ufer et al., 2008). Mathematical strategic knowledge is knowledge about cues within mathematical tasks and problems that indicate which concepts and representation systems can be used productively (Weber, 2001). *Problem-solving skills* consist of domain-general and domain-specific problem solving skills and strategies (Schoenfeld, 1985). Finally there are *beliefs* about the mathematical content and nature of mathematics (Leder, Pehkonen, & Törner, 2002; Schoenfeld, 2010) as well as *affective aspects* like emotions and motivation towards mathematics (Hannula, 2006).

Similar approaches to consider complex skills together with various predictors can also be found for self-regulated learning (De Corte, Verschaffel, & Eynde, 2000), mathematical problem solving in general (Schoenfeld, 1985) or mathematical proof in geometry (Chinnappan, Ekanayake, & Brown, 2011), with very similar predictors.

Epistemic Activities

Besides their predictors, we describe MA&P processes by analysing their sub-activities with a framework that has been proposed by Fischer et al. (2014). It describes eight such "epistemic activities" (Table 1) from an interdisciplinary viewpoint that allow comparisons among different domains and topics. The idea is that cognitive aspects of individual MA&P processes can be described in terms of these basic activities. Albeit the linear presentation of these activities, they do not need to occur in this specific order, can be iterated in cycles and not necessarily are all present in an argumentative process.

Epistemic Activity	Description
Problem identification	Perceiving a mismatch concerning the explanation of a problem and building a problem representation
Questioning	One or more initial questions are identified
Hypothesis generation	Possible answers to the questions are derived from models, theoretic frameworks, …
Construction and redesign of artifacts	Development of a prototypical object, axiomatic system or another object used in order to work on the problem
Evidence generation	Evidence for the hypothesis is generated
Evidence evaluation	Evaluating evidence according to some norms
Drawing conclusions	Integrating different pieces of evidence, reevaluating the initial claim considering the new evidence
Communicating and scrutinizing	Sharing and discussing individual reasoning and argumentation within a community

Table 1: Overview of epistemic activities (Fischer et al., 2014).

Argumentative and proving activities

Not only the individual cognitive sub-activities within an argumentative process can be distinguished, but also the overall goal of the reasoning process with reference to task contexts. Mejia-Ramos and Inglis (2009) introduced a framework of argumentative and proving activities based on work by Giaquinto (2005). They divide argumentative activities associated with mathematical proof into the three categories *construction of novel arguments*, *reading arguments* and *presenting arguments*, each with a few sub-categories. Even though this distinction sounds very similar to some of the epistemic activities, it refers to the *overall goal* of MA&P processes, not the sequence of activities within this process.

AIM AND RESEARCH QUESTIONS

The goal of this review is to analyse which aspects of MA&P have been investigated in the last 5 years within the PME community, and to identify patterns that might yield directions for future research in understanding and supporting MA&P as a complex individual skill. The review was therefore guided by the following questions:

- To which extent does research on MA&P consider the different predictors, sub-activities, and goals of MA&P processes?
- Which combinations of predictors and epistemic activities are being considered in MA&P research? Can research gaps be identified with regard to a comprehensive understanding of MA&P processes?

THE CURRENT STUDY

Literature selection, coding and analysis

We decided to restrict our review to PME research reports (RRs) published from 2010 to 2014, because we considered them to be a fair representation of latest good-quality, international mathematics education research. All 782 RRs of the PME proceedings from 2010 to 2014 were selected as data basis for the review. This selection bears the danger of overlooking research that is not published within the PME proceedings, but includes research that is of good quality and is not limited to journal publications, thus giving a more extensive picture of the activities in the community. A similar approach was taken by Matos (2013) for his literature review.

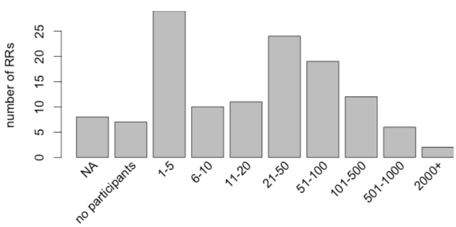
Based on an initial coding of the research topic and grade level, we selected those 129 RRs for detailed analysis which studied MA&P and which were situated in secondary or tertiary education. The focus on reasoning and argumentation in secondary and tertiary education is due to the fact that it differs considerably from that in pre-primary and primary education, particularly proof is rather non-existent.

A coding scheme was created to categorize these RRs according to the predictors investigated, the epistemic activities studied, and the type of reasoning activity (according to its goal) in the study. Reading the RRs completely, we coded the main research foci of each report with respect to the three theoretical frameworks. For each predictor we coded if it was a variable central to the RRs (e.g., it was the sole focus of the report), if it was considered substantially (e.g., it was analysed together with other predictors), if it was only mentioned (e.g., as a variable to be controlled), or if it did not occur at all. The goals of MA&P processes were coded in the categories *argument construction*, *argument reading*, and *argument presentation* where possible, but also codes *not explicit* and *multiple goals* were introduced. Moreover, we coded for each epistemic activity if it was focused in the report. The notion of "focused" is very important to understand the whole coding process. For example, if participants of a study were talking or discussing a problem only for purposes of the study (e.g., to foster collaboration or as a "thinking aloud" technique) this would neither be coded as the activities *proof presentation* or *communication and scrutinizing*, nor as the goal *proof presentation*. After several steps of refinement, the coding reliability reached an acceptable level with a mean inter-rater reliability of $\kappa_{Mean} = 0.77$ (SD = 0.15). Except for the interrater reliabilities of the epistemic activities *drawing conclusions* ($\kappa = 0.56$) and *communicating and scrutinizing* ($\kappa = 0.46$) all IRRs were acceptable (above $\kappa = 0.64$).

In an additional step the results of the descriptive analysis were backed up by considering examples of reports from the respective categories in order to ensure coding validity and to gain qualitative insight.

RESULTS

A total of 532 (68%) articles were situated in secondary (44%) or tertiary (24%) education and 160 (20%) RRs focused on MA&P. The intersection of both groups contained 129 (16%) RRs, which met the inclusion criteria, and were coded in detail. Comparing the research methods of these RRs (57% qualitative, 26% quantitative, 11% mixed methods, and 6% theoretical) to the ones found by Matos (2013), the selected reports are quite representative for PME RRs in terms of research methods. The same holds for the RRs' distribution of participants (Figure 1, left side).



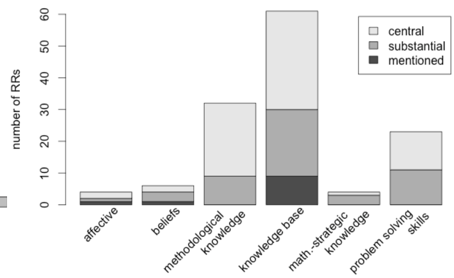
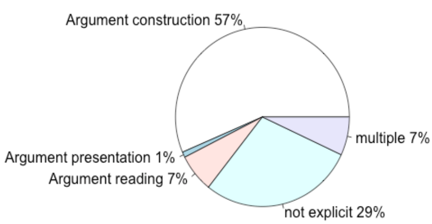


Figure 1: Distribution of number of participants (left) and use of predictors (right).

Starting our analysis with the predictors, *mathematical knowledge base* was studied by far most often (47% of the RRs; Figure 1, right side). Only 17% considered *methodological knowledge* and 18% *problem solving*. *Strategic knowledge*, *beliefs* and *affective aspects* were studied even less frequently (3%, 5%, and 3%, resp.). All in all only 22% of the RRs considered at least two of these predictors simultaneously, over two thirds of these cases focused on the predictor *mathematical knowledge base* in combination with any one predictor.



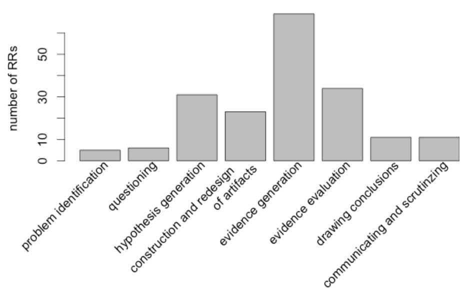


Figure 2: Frequencies of argumentative (left) resp. epistemic activities (right)

Regarding the goals of the argumentative processes (Figure 2, left side), almost 60% of the RRs focused on *argument construction*, 1% on *argument presentation* and 7% on *argument reading*. The number of reports including two or more of these goals is also low with 7%. Almost a third of the RRs (29%) could not be associated with one of these three activities, for example because they were theoretical. These results resemble those of Mejia-Ramos and Inglis (2009), who found *argument construction* in 64% of their sample, but no contribution on *argument presentation*.

In line with this focus on the dominating goal of *argument construction*, *evidence generation* was the most frequently studied epistemic activity (Figure 2, right side), followed by *hypothesis generation* and *evidence generation*. Nevertheless, all epistemic activities were studied at least in some form in some RR. A qualitative analysis of the RRs focusing at least one epistemic activity (96 of 129 RRs) revealed four main clusters (named A, B, C, D) of RRs, a finding also supported by a cluster analysis. Two of these clusters (A, D) focus on one epistemic activity only, the others (B, C) on several. Cluster A focuses solely on *evidence generation* and constitutes the largest cluster with 32% of the 96 RRs. A representative of this cluster is a RR on unjustified assumptions in geometry proofs, where students' written geometry proofs were analysed for these assumptions. The second largest cluster with 30% of the RRs is cluster B, the conjecturing cluster that focuses on the activities of *hypothesis generation*, *construction and redesign of artefacts* and *evidence generation*. A representative of this cluster is a videotaped interview study of the ways successful provers use examples when exploring and proving conjectures given to them. The third biggest cluster with 24% of the RRs is cluster C, the "complete" process cluster, which incorporates RRs looking at multiple epistemic activities at once. A representative of this cluster is a RR on the role of dynamic geometry on the process of exploration, conjecturing and proving geometrical problems. Finally, the smallest cluster with only 14% of the RRs is cluster D, the evaluation cluster, which focuses on the epistemic activity of *evidence evaluation*. A representative of this cluster is an eye-tracking study of the role of pictures while reading proofs.

These clusters also differ in the sample sizes and the applied research methods. The mean sample size in cluster A is 85, whereas the other clusters have mean sample sizes of 50 and below, with cluster D having the smallest mean sample size of 32. Although all clusters predominantly contain qualitative RRs, the percentages are especially high in the complete (C) and conjecturing (B) clusters with 77% resp. 79%. Apparently a qualitative approach is used more often when having a wider perspective on MA&P and/or focusing on several epistemic activities.

Data also reveal a strong connection between the processes and goals of MA&P investigated. Thus, RRs with a focus on *argument construction* predominantly studied the activities of *hypothesis* and *evidence generation*, whereas the RRs on *argument reading* or *presentation* focused exclusively on *evidence evaluation*. Especially in the case of *argument presentation* this seems surprising as a focus on *communicating and scrutinizing* would be obvious.

DISCUSSION

The aim of our review was to analyse the inclusion and combination of different predictors, sub-activities, and goals of MA&P in research on MA&P in PME and how it contributes to a comprehensive understanding of MA&P. The results reveal that perspectives on MA&P are often restricted to very specific aspects such as single epistemic activities or one or few predictors. Initially, such more focused analyses are necessary as a first approach to better understand complex skills. Nevertheless, MA&P require the coordination of multiple processes and knowledge facets. Even though taking a broader perspective of MA&P poses major methodological problems, e.g., in terms of sample size or time for testing or analysis, it is important to find ways to study the complex interactions of the often disconnected aspects described in existing research. This may include studies comparing the influence of different predictors or research on the coordination of different epistemic activities during MA&P processes.

We also find that MA&P are mostly researched in situations where *argument construction* is the main goal of the activity. This may be one reason why certain epistemic activities resp. their combinations are studied in more detail than others. However, Meija-Ramos and Inglis (2009) suggested that *argument presentation* and *argument comprehension* may be more important in learning settings than *argument construction*. Certainly, the relative importance of different goals of MA&P and different epistemic activities has to be seen in conjunction with the overall aims of mathematics instruction that may be more focused on *argument construction*. Still, we cannot expect to gather a comprehensive understanding of MA&P while having blind spots in our research.

Despite these imbalances and potential research gaps it must be underlined that, with over 20% of the RRs focusing on argumentation and proof, we have a sound basis of research on the separate aspects of MA&P. Thus, it may be time to build on that basis and to start studying the relations and interactions between the different facets of MA&P in order to obtain a coherent picture as well as more detailed knowledge how to foster MA&P effectively.

References

Balacheff, N. (1999). Is argumentation an obstacle? Invitation to a debate... *International Newsletter on the Teaching and Learning of Mathematical Proof*. Retrieved from http://www.lettredelapreuve.it/OldPreuve/Newsletter/990506Theme/990506ThemeUK.html

CCSSI. (2010). Common Core State Standards for Mathematics. Retrieved from http://www.corestandards.org/assets/CCSSI_Math Standards.pdf

Chinnappan, M., Ekanayake, M., & Brown, C. (2011). Specific and General Knowledge in Geometric Proof Development. *SAARC Journal of Educational Research*, *8*, 1–28.

De Corte, E., Verschaffel, L., & Eynde, P. O. (2000). Chapter 21 - Self-Regulation: A Characteristic and a Goal of Mathematics Education. In M. Boekaerts, P. R. Pintrich, & M. Zeidner (Eds.), *Handbook of Self-Regulation* (pp. 687–726). San Diego: Academic Press.

Fischer, F., Kollar, I., Ufer, S., Sodian, B., Hussmann, H., Pekrun, R., … Eberle, J. (2014). Scientific reasoning and argumentation: Advancing an interdisciplinary research agenda. *Frontline Learning Research*, *4*, 28–45.

Giaquinto, M. (2005). Mathematical Activity. In P. Mancosu, K. Jørgensen, & S. Pedersen (Eds.), *Visualization, Explanation and Reasoning Styles in Mathematics* (Vol. 327, pp. 75–87). Springer Netherlands.

Hannula, M. S. (2006). Affect in mathematical thinking and learning. *New Mathematics Education Research and Practice*, 209–232.

Healy, L., & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education*, *21*(4), 396–428.

Leder, G. C., Pehkonen, E., & Törner, G. (2002). *Beliefs: A hidden variable in mathematics education?* (Vol. 31). Dordrecht: Kluwer Academic Publishers.

Matos, J. F. (2013). Trends and opportunities for research methodologies within the PME community. In A. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 51–66). Kiel, Germany: PME.

Mejia-Ramos, J. P., & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol. 2, pp. 88–93). Taipei, Taiwan

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*(1), 23–41.

Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM. The International Journal on Mathematics Education*, *40*(3), 385–400.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. New York: Academic press.

Schoenfeld, A. H. (2010). *How we think : a theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Ubuz, B., Dincer, S., & Bulbul, A. (2012). Argumentation in undergraduate math courses : A study on proof generation. In T.-Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 163–170). Taipeh, Taiwan: PME.

Ufer, S., Heinze, A., & Reiss, K. (2008). Individual predictors of geometrical proof competence. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of the 32nd Conference of the International Group for the Psychology of Mathematics Education and the XX North American Chapter* (Vol. 1, pp. 361–368). Morelia, Mexico: PME.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119.

PISA'S REPORTING OF MATHEMATICAL PROCESSES

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The OECD's 2012 PISA survey reported for the first time on student proficiency in processes of doing mathematics, in addition to the scores for overall mathematical literacy and four content categories. The three process categories (Formulate, Employ and Interpret) are derived from the mathematical modelling cycle, emphasising PISA's focus on what students can do with the mathematics they know. The paper describes this change and reviews initial country-level results. It then presents an empirical investigation of the categorisation by examining responses to multiple choice items to find the sources of major cognitive demand. This demonstrates the general success of the new categorisation and further underlines the difficulty that students have with formulating problems mathematically. Further research is recommended.

AIMS

The aims of this paper are to describe and investigate some conceptual and practical aspects of the scores that are reported by the OECD's PISA mathematics survey. Before 2012, PISA mathematics outcomes were reported as an overall score of mathematical literacy (for every participating country/economy and for nominated subgroups within them), and also scores for four content categories (see below). From PISA 2012, mathematics also reported scores in computer-based mathematics and in three process categories, which are the subject of this paper. This paper reviews the reasons for reporting the process categories, describes how it is done with reference to the mathematical modelling cycle, reviews some initial findings and then discusses some criteria by which the success of the new venture could be judged. The paper then demonstrates one method for empirically evaluating the success, working at the item level, and provides suggestions for ways in which this investigation could be improved and extended. Since PISA is very important for educational policy, PISA-related research by mathematics educators is encouraged.

THE NEED TO REPORT MATHEMATICAL PROCESSES

As described in the first reports of the results of OECD's PISA 2012 survey (OECD 2014a), the aim of the PISA project is to assess the "key knowledge and skills that are essential for full participation in modern societies" noting in particular that "modern economies reward individuals not for what they know, but for what they can do with what they know" (p. 24). For mathematics, PISA therefore has an emphasis on *mathematical literacy*, defined (in part) as "an individual's capacity to formulate, employ and interpret mathematics in a variety of contexts" (OECD 2014a p. 37). Real world challenges are therefore inherent to PISA.

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For all PISA surveys to date, mathematical breadth is ensured by having approximately equal numbers of PISA items in each of four content categories (Quantity, Space and Shape, Change and Relationships, Uncertainty and Data). In addition to reporting overall scores for mathematical literacy, PISA surveys in which mathematics has been the major survey domain have also reported scores for these four content categories (OECD 2014a). Variation in the scores across the content categories have provided useful information for educational jurisdictions, helping them interpret overall scores and sometimes pointing to desired or accidental differences in curriculum emphasis as a whole or for students of PISA age. For example, Shanghai-China, Chinese Taipei, Korea and Macao-China, all of which have extremely high overall scores, do particularly well on the Space and Shape scale (19 or more score points higher than their 2012 overall scores) and relatively poorly on Uncertainty and Data (11 or more points below) (OECD 2014a). In PISA 2012, five of the ten lowest performing countries/economies (hereafter abbreviated just to 'countries') had scores in Change and Relationships 11 or more points below their overall scores (OECD 2014a).

Since PISA 2006, student performance in science has been reported as an overall score as well as three knowledge of science scores (earth and space, living systems, physical systems), a score on knowledge about how science is conducted, and three scientific competencies (Identifying scientific issues, Explaining phenomena scientifically, and Using scientific evidence). When, in 2012, mathematics became the survey's major domain for the second time and received a major review, it was a priority that mathematics follow the science lead and provide detailed reporting on students' proficiencies in the processes of doing mathematics, not just content categories. More detailed reporting gives educational jurisdictions better information about the strengths of their students. But how should reporting on processes be done?

By the late 1980's, components of the process side of mathematics made their way into curriculum documents. Among many such initiatives, the first NCTM Standards (NCTM 1989) listed valuing mathematics, being confident, mathematical problem solving, communicating mathematically and reasoning mathematically. The Australian Curriculum Profile (Curriculum Corporation 1994) divided 'working mathematically' into investigating, conjecturing, using problem solving strategies, applying and verifying, using mathematical language and working in context. In Denmark, the KOM project led by Mogens Niss described a set of eight competencies for use in curriculum design and assessment (reasoning, problem handling, modelling, mathematical thinking, representation, symbols and formalism, communication, using aids and tools). This scheme, in various guises, has been adopted by PISA mathematics and has been present in the frameworks for all the PISA surveys (Niss 2015). A modified version of the scheme is now used to predict the difficulty of items during test construction (Turner, Blum & Niss 2015). However, despite its influence on PISA's conceptualisation, this scheme could not be used for formally reporting on the processes of doing mathematics. Eight components are too many for strong statistics and multiple competencies are involved in solving nearly all problems, whereas the psychometric model requires that items be allocated to only one category. What to do?

PISA's emphasis on mathematical literacy means that items are set in real-world contexts. Hence, mathematising the real world and using mathematical modelling to solve problems have always been foundations of PISA, although variously named in the various surveys (Stacey & Turner 2015). Within the mathematics education world, mathematical modelling has for many years been widely described by means of the modelling cycle (Blum and Niss 1991). For PISA 2012 the modelling cycle provided the basis of the new reporting of student proficiency in three mathematical processes. A simple version of this cycle is used, as shown in Figure 1. The label "mathematical processes" was arrived at after a great deal of discussion in the PISA 2012 Mathematics Expert Group, taking into consideration translation into many languages and also existing terminologies within PISA and OECD documents. The three processes, which are defined and illustrated in OECD (2014a) are:

- Formulating situations mathematically
- Employing mathematical concepts, facts, procedures, and reasoning
- Interpreting, applying and evaluating mathematical outcomes.

Figure 1 shows that the processes correspond to the arrows in the diagram. Formulate and Interpret move between the real world and the mathematical world, whereas the process Employ operates within the mathematical world. In the real world, there are concepts such as value for money; in the mathematical world this becomes a measure such as dollars per kilogram. Three, not four, processes were defined, with interpreting mathematical results in real world terms and evaluating the real world solution against the problem requirements combined, because items solved under PISA conditions are unable to incorporate any deep evaluation of solutions against real world criteria.

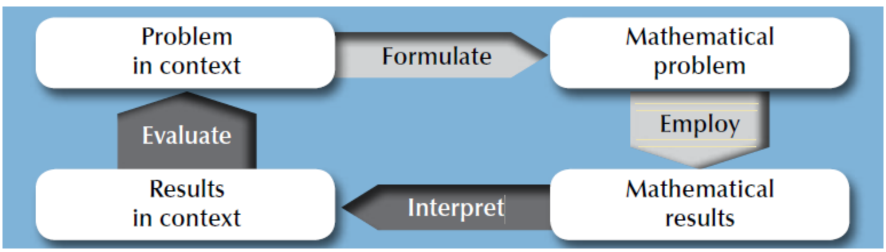

Figure 1. Mathematical modelling cycle showing PISA processes. (OECD 2014a)

In order to report scores for each process for PISA 2012, items were allocated to the process that experts judged to be the source of greatest cognitive demand of the item. This requires mathematical analysis of the item and judgements about solution paths that 15-year-olds are likely to take and difficulties they are likely to encounter. It is often the case that a solution involves more than one process and mathematics from more than one content category. For example, PM942Q02 Climbing Mt Fuji Q2 (OECD 2013) requires students to assemble information about walking speeds, distance, breaks, etc., and formulate a mathematical model in order to find an appropriate starting time for a walk up Mt Fuji. However, finding the starting time also requires employment of significant intra-mathematical skills. Hence, decisions – in this case between Formulate and Employ – have to be made ‘on balance’. In professional life, working even once around the mathematical modelling cycle can be a substantial undertaking, but many PISA items are just a small fragment of this. Identifying exactly where they best fit requires thought. This issue is explored by Stacey (2015). In the PISA survey, approximately 50% of the items belonged to Employ, and 25% to each of the Formulate and Interpret processes (OECD 2014a). The items in each process category are balanced across content categories, the nature of the real contexts and cover the full range of difficulty.

The purpose of this paper is to investigate some aspects of the reporting by processes. In the next section, we review some of the first published results to demonstrate that interesting insights into students’ mathematical proficiency have already emerged. Then we report an empirical investigation into how the categorisation of items has operated in practice. This is a rich field for further investigation.

PATTERNS IN MATHEMATICAL PROCESS SCORES

The first report of results of PISA 2012 (OECD, 2014a) shows there are interesting patterns in the process scores. The correlations between overall mathematical literacy scores and the seven other reported scores (four for content, three processes) are all high. The average OECD score for overall mathematical literacy is 494, for formulating 492, for Employ 493 and for Interpret 497. This means that Formulate items were found to be more difficult than the average item, and Interpret items were easier. However, nine of the ten top performing countries scored more than three points higher in Formulate. (This and the following comparisons are relative to the country’s own scores, not the OECD average). Top performers are generally Asian countries which stereotypes might have predicted had their greatest strength instead in calculation and hence in the Employ process, not Formulate. Interestingly the four highest performing countries scored relatively low on Interpret. Netherlands, Denmark and Sweden scored relatively high on both Formulate and Interpret (where real world contexts matter), and low on the intra-mathematical Employ. Non-Asian English speaking countries (Canada, Australia, New Zealand, UK, USA) were relatively stronger in Interpret only. Nine European countries were lower in Formulate but higher in both Employ and Interpret. The biggest difference between boys and girls was in Formulate. These results warrant further investigation.

AN EMPIRICAL INVESTIGATION INTO PROCESS CATEGORISATION

This section reports on an initial investigation into the categorisation of items to processes, using empirical results from PISA 2012. Consistent and meaningful categorisation is essential to ensure the usefulness of results. PISA’s Mathematics Expert Group was responsible for categorisation with initial suggestions from item writing teams. The task is to identify where the item is best placed within the modelling cycle and if more than one process is involved, to make a decision about the source of greatest cognitive demand for most students in most countries.

What empirical evidence from PISA results might assist in testing these expert categorisations? Looking at the most common errors could give information about the source of greatest cognitive demand. PISA items use various types of multiple choice and constructed response formats. Simple multiple choice, where students answer one question by selecting from four or five options, seemed to provide the best possibility to understand students' thinking from the data published. Therefore, these items were examined to identify what process caused the most common errors. There are many untested assumptions behind this approach (some highlighted below) but it does seem important to see what information can come from the empirical data.

Method

There were 32 simple multiple choice items in the main 2012 survey, with 7 allocated to Formulate, 13 to Employ, and 12 to Interpret. On average 3.2% of responses were missing, with no item having more than 10% missing. To locate items where it was likely that there had been one main error, two criteria were chosen. A main error was defined as an incorrect option chosen by over 25% of students for four options or 20% for five options, *or* an incorrect option selected by more than half of the incorrect students (including missing responses). So, for example, the OECD success rate for PM942Q01 Climbing Mt Fuji was 46.93%. Option E was selected by 21.01% (over 20%) so it was regarded as the main error. The success rate for PM918Q05 Charts was 76.67%, so that 23.33% of students were not correct. Option C was selected by more than half of these students (14.84%) so was deemed the main error.

Once main errors were identified, the two authors independently examined them to determine the process behind them. Both authors were familiar with the definitions of processes through their work on PISA 2012. For example, in PM942Q01 Climbing Mt Fuji (OECD, 2013) students must work out an average number of climbers per day given there are 200 000 climbers between July 1 and August 27. The main error (Option E) resulted from calculating with 27 days, rather than (31 + 27) days. Both authors decided this is a Formulate error, as it arises during the translation from real world situation to mathematical form. Of course amongst almost 500 000 PISA students, there will be many reasons for selecting Option E, but we expect this failure of Formulate is probably the most common. This empirical evidence supports the categorisation of PM942Q01 as Formulate. The authors rated items independently, discussed disagreements and uncertainties, and then made final decisions.

RESULTS AND DISCUSSION

Table 1 shows the 15 items from the main 2012 survey that met the criteria for having a main error. There was never more than one main error. Six items met both criteria. The table gives the names and identifiers of items (non-shaded items have been released and can be found in OECD (2013)); the process allocated for PISA 2012; the average percent correct (Table A2, OECD 2014b); the main error (with average percent responses); the number of countries given the item (with number of OECD countries in brackets); and the authors' decision on the process of the main error. The percents are averages for the OECD countries doing the items. Two OECD countries and 15 not from the OECD chose to include booklets of easier items in their rotation to get a better measure of low performing students. Because two countries seems small (even involving thousands of students), the average percents were calculated for this set of 17 countries. With this new data, five of the eight items in Table 1 met the criteria and the other three failed by 0.4%, 1.04% and 2.12% respectively. It was decided, therefore, to retain all items. Note that because the populations for the OECD average differ across these items, percent correct is not a reliable guide to item difficulty here.

Item ID	Abbreviated Name	Allocated Process	% correct	Main error	Countries (OECD)	Error Process
PM564Q01	Chair Lift	Formulate	46.11	C (42.93)	71 (34)	Formulate
PM942Q01	Climbing Fuji	Formulate	46.93	E (21.01)	17 (2)	Formulate
PM982Q04	Employment	Formulate	51.45	B (37.64)	71 (34)	Formulate
PM800Q01	Comp. Game	Employ	88.39	B (8.34)	71(34)	Formulate
PM915Q01	CO2 tax	Employ	40.18	A (30.25)	71(34)	Interpret
PM918Q05	Charts	Employ	76.67	C (14.84)	54 (32)	Formulate
PM957Q01	Helen Cyclist	Employ	52.91	A (28.91)	17 (2)	Formulate
PM957Q02	Helen Cyclist	Employ	36.86	B (25.61)	17 (2)	Employ
PM961Q03	Chocolate	Employ	44.68	A (25.83)	17 (2)	Employ
PM985Q02	Which car?	Employ	37.48	A (49.29)	17 (2)	Employ
PM423Q01	Tossing Coins	Interpret	79.05	C (10.58)	71 (34)	Employ
PM918Q01	Charts	Interpret	87.27	D (7.87)	54 (32)	Interpret
PM939Q02	Racing	Interpret	38.14	B (52.42)	17 (2)	Interpret
PM948Q01	Part time work	Interpret	85.91	D (8.69)	17 (2)	Interpret
PM991Q01	Garage	Interpret	65.14	D (20.86)	17 (2)	Interpret

Table 1. Items, allocated processes, average results and error process categories.

Table 1 shows that the allocated process and main error process coincide for 10 of the 15 items examined. This is supportive evidence that the major source of cognitive demand in the item arose from the allocated process. There was complete agreement for the Formulate and Interpret categories except for item PM423Q01. Rethinking this item led us to believe that both the allocated and main error process should have been Employ and that the item was wrongly categorised initially.

The agreement between allocated and the current post-hoc analysis of categories was not strong for Employ items with four of seven items mismatched. Three of the four main errors were Formulate errors. PM918Q05 Charts (OECD, 2013) illustrates what happened. Students had to extrapolate a trend in five data points and the Mathematics Expert Group expected that the proportional reasoning and getting all the calculations correct would be the major challenge, hence Employ. However students making the main error probably misinterpreted the word 'trend' (a Formulate error) and therefore thought that they only had to read one data point and do no calculations. Hence they had no opportunity to make the expected Employ errors. This item is discussed in detail by Stacey (2015). It was the same situation in the other two Employ items above with Formulate main errors: students making the common Formulate error had almost no intra-mathematical work to do.

Further analysis of the mismatched PM915Q01 CO2 tax, allocated to Employ with main error Interpret, showed the usefulness of examining the reasons behind all the options, not just the main error. In that case, whilst the identified main error was Interpret (30.25%), the authors judged that the main difficulty of a correct solution and the errors leading to the two other options were all within Employ, and hence applied to 65.22% of students. The allocated Employ process therefore is justified. Hence these four mismatched Employ items demonstrate two reasons why the main error is not necessarily a guide to the source of greatest cognitive demand for the item. The Employ category may be most affected by this, since different solution paths can vary so much in the intra-mathematical work they require.

CONCLUSION AND FUTURE DIRECTIONS

In summary, the results in Table 1 support most of the expert allocations and the analysis has led to deeper understanding of the items. Furthermore, the finding that nearly half of the Employ items had Formulate main errors demonstrates again that 15-year-olds find formulation of a mathematical model from the real world situation difficult. The activity of analysing the main error processes has demonstrated some reasons why this empirical evidence does not prove that the correct or incorrect process allocation has been made. At best it provides support. In particular, it has shown that the most significant driver of cognitive demand may depend on the solution path, which is often linked to the ability of the student. The need to look more widely across all options was also highlighted. Data will reflect the actual options offered and their process errors, which has implications for test item writers.

Because PISA is very influential in educational policy, more detailed research by mathematics educators into items and their functioning, categorisation, results and the conceptual underpinnings is to be encouraged. The new reports on mathematical processes provide an additional dimension for understanding country and subgroup differences and open new areas for research. The country patterns are interesting, but there are more patterns to find and links to curriculum, pedagogy or other factors can be explored. The wide range of publicly available PISA information supports this.

The empirical work presented above is a beginning that could certainly be improved. Items, including all their options, should be examined more closely (see CO2 tax example). Investigation beyond multiple choice items is important, such as with the 'double digit coded' constructed response items, where answers are coded for both correctness and method used or errors made. Actual student responses from PISA scripts or written or verbal smaller scale studies could be used. There are also fundamental questions such as whether there are other links between the items in each of the process categories that better explain the results. With the process category reporting a new dimension of analysis of PISA outcomes becomes possible, and new opportunities open to investigate mathematics teaching and learning.

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References

Blum, W. & Niss, M. (1991). Applied Mathematical Problem Solving, Modelling, Applications, and Links to Other Subjects. *Educational Studies in Mathematics*, *22*, 37-68.

Curriculum Corporation (1994). *Mathematics: A curriculum profile for Australian schools*. Carlton: Curriculum Corporation.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, Va.: Author.

Niss, M. (2015). Mathematical Competencies and PISA. In K. Stacey & R. Turner (Eds.), *Assessing Mathematical Literacy* (pp. 35-56). Dordrecht: Springer.

Organisation for Economic Co-operation and Development (OECD). (2014a). *PISA 2012 Results: What Students Know and Can Do – Student Performance in Mathematics, Reading and Science (Volume I, Revised edition).* PISA, OECD Publishing.

Organisation for Economic Co-operation and development (OECD). (2014b). *PISA 2012 Technical Report.* PISA, OECD Publishing.

Organisation for Economic Co-operation and Development (OECD). (2013). *PISA 2012 Released Mathematics Items.* Paris: OECD Publishing.

Stacey, K. & Turner, R. (Eds.) (2015). *Assessing mathematical literacy*. Dordrecht: Springer.

Stacey, K. (2015). The real world and the mathematical world. In K. Stacey & R. Turner (Eds.) *Assessing Mathematical Literacy* (pp. 57 - 84). Dordrecht: Springer.

Turner, R., Blum, W., & Niss, M. (2015). Using competencies to explain mathematics item demand: A work in progress. In K. Stacey & R. Turner (Eds.), *Assessing Mathematical Literacy* (pp. 85-116). Dordrecht: Springer.

SELF-GENERATED REPRESENTATIONS ARE THE KEY: THE IMPORTANCE OF EXTERNAL REPRESENTATIONS IN PREDICTING PROBLEM-SOLVING SUCCESS

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Many primary school children face barriers when solving word problems, in particular with regard to the process of reaching the target state from the initial state. A training program that encourages children to construct external representations and specifically use them to find solutions has been proved to positively influence their problem-solving success. One goal of our study was to find out which external representations (e.g., sketches, tables) are good predictors of problem-solving success. The results concerning this research question are presented in this paper. Additionally, findings regarding the effects of the representation training on the predictors will be discussed on the conference.

THEORETICAL BACKGROUND

Mathematical word problems

Mathematical word problems are characterized by a demanding and complex mathematical structure. Several mutually interdependent conditions make the process of arriving at the target state from the initial state difficult. The problem solver is faced with a barrier; the solution is not readily accessible (Hussy, 1993). Indeed, finding the solution requires thinking processes that specifically enable problem solvers to establish new links between the properties of the initial and target states (Hussy, 1993). If they can perform the transformation in their memory, then it is not a problem for them (Franke & Ruwisch, 2010; Hussy, 1993; Winter, 2000). Problem barriers can present challenges. Challenging, yet solvable tasks require a greater willingness to work hard, which enables problem solvers to reach the zone of the next development (Vygotskij, 2002). Problem solvers have to "pay painstaking attention to detail" in order to find new links within the information (Kurt Reusser cited from Rasch, 2001, 42). To solve the word problem, the student has to find a different way of approaching the situation in the task by reflecting on the content, considering other structures, and finding different relationships between the elements (Rasch, 2001; Verschaffel, Greer, & De Corte, 2000).

External representations as a way of overcoming barriers

The process of constructing external representations produces mental pictures. Students producing representations express their individual thoughts, show which contents they have in mind, and give them an external appearance (Bruner, 1996). This enables the recipients to reconstruct the mental pictures of the persons who produced them (Schnotz, Baadte, Müller, & Rasch, 2010). Externalizations are considered to be "benefits" both in psychology and mathematics education (Bruner, 1996; Franke & Ruwisch, 2010; Schnotz et al., 2010). From a psychological perspective, external representations relieve the working memory, because the task conditions no longer have to be kept in mind. The free capacities can be used for finding the solution (Schnotz et al., 2010; Sweller, 2005). From a constructivist point of view, these help the problem-solving process if they are individually constructed by the problem solvers. By means of a dynamic analysis, the students uncover new links within the available information, restructure it, and overcome the existing barriers. The main didactic advantage is considered to be the creative phase of the problem-solving process. By externalizing the content, it is possible to identify relationships between the initial and target states. The problem is structured, sub-goals are recorded, and the search area is reduced (Franke & Ruwisch, 2010). Empirical studies (cf. Rasch, 2001) have confirmed that external representations, as an approach for supporting problem solving, help to overcome problem barriers.

Findings from current learning research

In spite of their manifold advantages, external representations are rarely used by novices for solving word problems. They struggle to externalize their mental models (Hohn, 2012; Rasch, 2001). To recognize and accept the fact that producing and using representations is helpful, students have to acquire experience working with them, independently discover their benefits, and learn how to identify suitable representations for the respective problems (Rasch, 2001). In this way, they develop a repertoire of external representations which they can draw on to work on the problem they are trying to solve and they can decide which representation is appropriate for them and their task (Cox, 1999; Kindfield, 1994). Novices have a lack of ability and skills as well as confidence for constructing appropriate representations, which is why they require support to put their mental models onto paper. Thus, there is a need for training in order to exploit the benefits of external representations (Fehse, 2001). Empirical studies (cf. collection in Fehse, 2001) have shown that depictive representation skills can be trained and that they enhance problem-solving success.

RESEARCH QUESTIONS

There is a gap in learning research regarding the role of individual external representations in problem solving. For teaching practice it is of interest to know which representations help students to successfully solve problems. As our study aim is to generate findings that are useful for primary school mathematics instruction, we developed the following research questions:

- Which representations are good predictors of success in solving mathematical word problems?
- Do trained classes perform differently to untrained classes regarding the good predictors?

METHOD AND DESIGN

Design of the intervention study

The study focused on a training program that fosters the construction of representations to enhance success in solving difficult word problems. In total, 366 third-graders from 20 classes participated in the empirical study. Ten of these classes completed the representation training program. Every student solved one word problem in one lesson per week over a period of twelve weeks. Students taking part in the training were encouraged to construct external representations and integrate them into their problem-solving process. At the beginning of each lesson, the class discussed and analyzed the problem-solving protocols of four students: a sketch, a calculation, a table, and a reasoning statement. Based on this range of problem-solving approaches, they learned that a task could be solved in several different ways. The important properties of the respective representations were analyzed and emphasized. In the next step, the third-graders independently solved a new word problem, applying their newly acquired knowledge. Students working on the same tasks in a regular mathematics lesson were not explicitly encouraged to generate external representations. Their mathematics teachers were not aware of the content of the training program.

The analysis is based on a total of 1071 problem-solving logs of 357 third-graders from Rhineland-Palatinate, Germany. The student protocols, which were collected after the 12-week intervention in a post-test, provide the basis for evaluating the problem-solving success and the self-generated representations of the third-graders. The students worked on a self-constructed problem-solving test made up of three word problems from different task areas (Sturm & Rasch, in press). During the test, neither the trained nor the untrained classes were explicitly asked to use external representations.

Method of the intervention study

Coding of problem-solving success: Problem-solving success, i.e., whether the word problem was successfully solved, was measured dichotomously: (1) correct solution and (0) incorrect solution. If the solution was only partly correct or if only some of the steps had been correctly carried out, the solution was still marked as incorrect. This dependent variable focuses on the end product and not on the process.

As the problem-solving test involved solving three tasks, problem-solving success was aggregated across all three tasks. As a result, the following number of points could be scored: (0) no task, (0.33) one task, (0.66) two tasks, and (1) all three tasks were correctly solved.

Coding of students' external representations: Based on qualitative and quantitative content analyses, external representations were divided into four main categories: sketches, tables, calculations, and reasoning statements. In addition, specific properties were identified that described the individual main categories in more detail and put them into concrete terms. These properties made up the sub-categories for the different types of representations.

Properties of sketches:

- *Represented state*: What is actually represented? Initial state; target state; approach; initial state and approach; target state and approach; initial and target state; initial state, target state and approach.
- *Structure*: Surface characteristics are shown and the incorrect task structure (decorative sketch); correct sketch-text relationship but the incorrect task structure (more than surface characteristics); correct sketch-text relationship but the incomplete task structure; all conditions were correct.
- *Depiction of the situation*: The sketch depicted the situation of the task; the sketch did not depict the situation of the task.
- *Systematic approach*: The sketch reflected an unsystematic approach; the sketch reflected a systematic approach.
- *Monitoring the conditions*: The conditions of the task were not controlled in the sketch; the conditions of the task were controlled in the sketch.

Properties of tables:

- *Dimensions*: One column; two columns; more than two columns.
- *Depiction of the situation*: The columns depicted the situation of the task; the columns did not depict the situation of the task.
- *Use*: tables as trial-and-error record; tables as "working-out" record.
- *Systematic approach*: The table reflected an unsystematic approach; the table reflected a systematic approach.
- *Constancy of a condition*: None of the conditions were kept constant in the table; one condition was kept constant in the table.
- *Monitoring the conditions*: The conditions of the task were not controlled in the table; the conditions of the task were controlled in the table.

Properties of calculations:

- *Number of calculations*: One calculation was used; more than one calculation was used.
- *Written, mathematical language*: The equals sign was misused; the equals sign was used correctly.
- *Depiction of the situation*: The calculation was not labeled with the situation of the task; the calculation was labeled with the situation of the task.
- *Use*: Calculation(s) fulfilled: working-out function; entry function; control function; entry and control function.
- *Systematic approach*: An unsystematic approach was applied to the calculation; a systematic approach was applied to the calculation.
- *Constancy of a condition*: None of the conditions were kept constant in the calculation; one condition was kept constant in the calculation.

- *Monitoring the conditions*: The conditions of the task were not controlled in the calculation; the conditions of the task were controlled in the calculation.
- *Diversity of arithmetic operations*: Use of one arithmetic operation; use of more than one arithmetic operation.

Properties of reasoning statements (cf. Neumann, Beier, & Ruwisch, 2014):

- *Mathematical structured reasoning:* Description of mathematical findings without reasoning; description of mathematical findings with partial reasoning; description of all mathematical findings with complete reasoning; reasoning contains generalizing aspects.
- *Written language used for structured reasoning*: Descriptive verbalization without structured reasoning; reason-and-consequence relationship with structured reasoning but with no link to the task; reason-and-consequence relationship with structured reasoning but with a link to the task; complete and consistent reasoning with a link to the subject matter of the task.
- *Correctness and completeness*: None of the conditions were verbalized in the text; one condition was verbalized in the text; all conditions were verbalized in the text.

The categorical, non-dichotomized predictors were dummy coded, producing 32 predictors to be measured from the 22 properties.

Additional influencing factors: Problem-solving success is not only determined by the representation selected. We also expected problem-solving success to vary depending on whether the children had participated in the training program or not (influence of the group). Children’s intelligence was measured using the Coloured Progressive Matrices test, their text comprehension using the ELFE 1-6 test, and their abilities and skills in mathematics using the HRT 1-4 test. The data collection took place at the end of the second grade. Moreover, we also recorded the children's native language by asking whether they mainly spoke German or another language at home.

RESULTS

In our analyses, we only considered properties that were found in more than 5 % of the students' solutions. Three items that violated this assumption were eliminated: ‘initial state and approach’ and ‘target state and approach’ of the sketch property *represented state* and the 'entry and control function' of the calculation property *use.* A principal axis factor analysis was conducted on the 29 items to identify clusters of variables and reduce the data set. The Kaiser-Meyer-Olkin measure verified the sampling adequacy of the analysis, KMO = .73 (‘middling’ according the Hutcheson & Sofroniou, 2009), but the KMO values for the individual item ‘control function’ of the calculation property *use* and the item ‘working-out record’ of the table property *use* were smaller than .5, which is below the acceptable limit of .5 (Field, 2013). These items were excluded from the analysis. In a next step, a second principal axis factor analysis was conducted on the 27 items with orthogonal rotation (varimax). KMO = .77 and all KMO values for the individual items were greater than .57, which is above the acceptable limit of .5. An initial analysis was run to obtain eigenvalues for each factor in the data. Seven factors had eigenvalues above Kaiser's criterion of 1; together they explained 71.53% of the variance. The scree plot showed inflexions that justified retaining four factors. Due to the discrepancy, we conducted a parallel analysis according to Horn. We retained the four factors because the parallel analysis confirmed the result of the scree plot.

	b	*SE B*	*ß*	*p*
Step 1				
Constant	-.496	.108		$p < .001$
Group	.076	.014	.239	$p < .001$
Language	.048	.033	.067	$p = .149$
Intelligence	.019	.004	.232	$p < .001$
Text comprehension	.004	.004	.052	$p = .284$
Mathematical abilities	.004	.001	.360	$p < .001$
Step 2				
Constant	-.353	.103		$p = .001$
Group	.034	.014	.108	$p = .018$
Language	.054	.031	.076	$p = .077$
Intelligence	.014	.004	.173	$p < .001$
Text comprehension	.001	.003	.011	$p = .810$
Mathematical abilities	.004	.001	.289	$p < .001$
Sketches	.112	.020	.239	$p < .001$
Tables	.382	.086	.196	$p < .001$
Calculations	.387	.083	.207	$p < .001$
Reasoning statements	.039	.018	.093	$p = .030$

Note. $R^2 = .35$ for step 1; $\Delta R^2 = .10$ for step 2 (p < .001).

Table 1: Linear model of predictors of problem-solving success

The items that cluster on the same factor suggest that Factor 1 represents the sketches, in particular only the properties of sketches load on this factor. The same applies to the other three factors tables, calculations and reasoning statements. Except for the calculations properties *depiction of the situation* and *use* ('entry function'), all the other properties loaded > .4 on the respective factors. All Characteristics of a factor, whose loadings were > .4 were aggregated. First, the influencing factors (group, language, intelligence, text comprehension, and mathematical abilities) were included in the regression model. In a second step, the aggregated factors followed respectively. Table

1 shows the linear model of predictors of problem-solving success. The first model explained 35% of the variance. In addition, 10% more of the variance was explained by the use of sketches, tables, calculations, and reasoning statements. In summary, our model explains almost 50% of the variance.

RELEVANCE FOR SCHOOL TEACHING AND DISCUSSION

For the purpose of teaching practice, it is not surprising that sketches are the strongest predictors of problem-solving success. Their importance in mathematic classrooms, especially for solving word problems, is emphasized in mathematics education. The construction of sketches rank among the most important heuristic strategies in problem-solving (Franke & Ruwisch, 2010; Winter, 2000). This strategy is taken up and implemented by many textbooks. For instance "create a picture of the situation" is a typical instruction for students while solving word problems. However, constructions of tables and suitable calculations are rarely considered, while reasoning statements are not encouraged at all. Results confirmed this given the fact that reasoning skills revealed to be the weakest predictor of problem-solving success. This might be due to the fact that third-graders are novices in reasoning and still have to develop reasoning skills. It is of interest for instructors to know which representations should be integrated into lessons and which properties they should have to help children solve word problems. Research in mathematics education investigated questions such as: What impact does a helpful sketch have? What has to be included when producing tables? What types of calculations lead to problem-solving success? How does a reasoning statement have to be structured for the problem solver to reach his or her goal? Our project includes analyses on the properties of representations used to solve word problems and, therefore, helps provide answers to these questions.

Solving difficult word problems is like climbing a mountain. At the beginning there is a barrier, comparable with a big mountain. Students cannot imagine that they are able to manage to reach the top. But if they succeed, they are satisfied and proud of their performance. External representations can help to build bridges for students to overcome barriers and be successful in problem-solving.

References

Bruner, J. (1996). *The culture of education*. Cambridge, England: Harvard University Press.

Cox, R. (1999). Representation construction, externalised cognition and individual differences. *Learning and Instruction, 9,* 343-363.

Fehse, E. (2001). *Unterstützung von Kohärenzbildung beim kooperativen und individuellen Lernen mit externen Repräsentationen.* (Doctoral dissertation). Retrieved from http://www.freidok.uni-freiburg.de/

Field, A. (2013). *Discovering statistics using IBM SPSS statistics*. London: Sage.

Franke, M., & Ruwisch, S. (2010). *Didaktik des sachrechnens in der grundschule* [Research on solving word problems in primary school] (2nd ed.). Heidelberg, Germany: Spektrum.

Haffner, J., Baro, K., Parzer, P., & Resch, F. (2005). *HRT 1-4. Heidelberger rechentest.* [Heidelberger numeracy test]. Göttingen, Germany: Hogrefe.

Hussy, W. (1983). *Denken und problemlösen* [Thinking and problem solving]. Stuttgart, Germany: Kohlhammer.

Hutcheson, G., & Sofroniou, N. (2009). *The multivariate social scientist: Introductory statistics using generalized linear models*. London, England: Sage.

Hohn, K. (2012). Gegeben, gesucht, lösung? Selbstgenerierte repräsentationen bei der bearbeitung problemhaltiger textaufgaben [Given, searched, solution? Self-generated representations while working on word problems] (Doctoral dissertation). Retrieved from *http://d-nb.info/1028021070/34*

Kindfield, A. C. H. (1994). Biology diagrams: Tools to think with. *Journal of the Learning Sciences, 3,* 1-36.

Lenhard, W., & Schneider, W. (2006). *ELFE 1-6. Ein Leseverständnistest für Erst- bis Sechstklässler* [A Reading Comprehension Test for 1st-6th-Graders]. Göttingen, Germany: Hogrefe.

Neumann, A., Beier, F., & Ruwisch, S. (2014). Schriftliches begründen im mathematikunterricht [|Written reasoning in mathematic classrooms]. *Zeitschrift für Grundschulforschung, Bildung im Elementar- und Primarbereich, 7*(1), 20-32.

Rasch, R. (2001). *Zur arbeit mit problemhaltigen textaufgaben im mathematikunterricht der grundschule* [Working with word problems in primary school mathematics instruction]. Hildesheim, Germany: Franzbecker.

Raven, J. C., Bullheller, S., & H. O. Häcker (Eds). *CPM. Raven's progressive matrices and vocabulary scales. Coloured progressive matrices*. Göttingen, Germany: Beltz.

Sturm, N. & Rasch, R. (in press). Forms of representation for solving mathematical word problems – development of an intervention study. In W. Schnotz, A. Kauertz, H. Ludwig, A. Müller, J. Pretsch (Eds), *Multidisciplinary Research on Teaching and Learning* (pp. 201-223). Basingstoke, England: Palgrave Macmillan.

Schnotz, W., Baadte, C., Müller, A., & Rasch, R. (2010). Creative Thinking and Problem Solving with Depictive and Descriptive Representations. In L. Verschaffel, E. De Corte, J. Elen, & T. DeJong (Eds), *Use of External Representations in Reasoning and Problem Solving* (pp. 11–35). Amsterdam, Netherlands: Elsevier.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making Sense of Word Problems.* Lisse, Netherlands: Swets & Zeitlinger.

Vygotskij, L. S. (2002). *Denken und Sprechen* [Thinking and Communicating]. Weinheim, Germany: Beltz.

Winter (2000). *Sachrechnen in der Grundschule* [Word Problems in Primary School]. Bielefeld, Germany: Cornelsen.

DYNAMIC AND STATIC NATURE OF UNIVERSITY MATHEMATICS LECTURING

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This paper extends the notion of a dynamic model of teacher knowledge from a school to a university setting. I consider both mathematics content knowledge and mathematics pedagogical knowledge in the context of university mathematics lecturing. The dynamics of mathematical knowledge of six research mathematicians are analysed. Findings from open-ended, semi-structured interviews suggest dynamic changes are pertinent more with regard to research mathematicians' mathematics pedagogical knowledge relative to mathematics content knowledge. Further, research mathematicians describe that some mathematical knowledge is static.

INTRODUCTION

Understanding the nature of mathematical knowledge in teaching is instrumental to understanding how teachers hold and use knowledge (Ball, 1988; Elbaz, 1981). Ball and Bass (2000) contend that the use of mathematical knowledge in teaching is often 'taken for granted' and thus the nature of mathematical knowledge in teaching remains unexamined. Researchers (Elbaz, 1981; Fennema & Franke, 1992) agree on the dynamic nature of mathematical knowledge, which continually changes and develops. However, it is important to look at how it changes and what experiences contribute to the change and growth of lecturers' mathematical knowledge (Fennema & Franke, 1992) particularly in university teaching, which *"remains a largely unexamined topic in mathematics education"* (Speer, Smith & Horvath, 2010, p. 100). In this paper, I extend Fennema & Franke's (1992) dynamic model of teacher knowledge from a school to a university setting focussing on the two components of mathematics content knowledge and mathematics pedagogical knowledge. The main research question is: what are the dynamics of university research mathematicians' mathematical knowledge?

THEORETICAL BACKGROUND

Dynamic nature of teacher knowledge

Many researchers (Davis & Renert, 2013; Elbaz, 1981; Fennema & Franke, 1992; Hashweh, 2005; Meredith, 1995) have discussed the notion of the dynamic nature of mathematical knowledge in teaching. Elbaz (1981) proposed teacher knowledge as dynamic and interactive 'held in active relationship to practice and used to give shape to that practice' (p. 48). However, the notion of the dynamic nature of mathematical knowledge was widely discussed following Shulman's (1986) theoretical framework on teacher knowledge. The need is to conceive mathematical knowledge for teaching as dynamic rather than seeing it as one-dimensional and static (Meredith, 1995).

Hashweh (2005) proposes interaction between different knowledge categories within the knowledge used in teaching. From a participatory framework, Davis and Renert (2013) discussed teachers' disciplinary knowledge of mathematics having a dynamic, evolving form. Fennema and Franke's (1992) model of dynamic teacher knowledge was based on conceptualising school mathematics teaching. It has four components: mathematics content knowledge, knowledge of mathematics pedagogy, knowledge of students' cognition and teachers' beliefs. They propose that teachers' knowledge of mathematical content is connected to the knowledge of mathematics pedagogy and to students' cognition. They argue that these combine with teachers' beliefs to decide teacher's classroom practice. Teachers' knowledge is seen as process of development where these four components interact and develop in the classroom. Also, teachers change their existing knowledge and create new knowledge in the developing context of teacher knowledge. However, the changes in dynamics of the different components are not clear in this model and additionally, whether some components are more dynamic than others is not discussed. This paper extends the dynamic teacher model from school to university setting and investigates the dynamics of mathematical knowledge of university research mathematicians focussing on two components from the model proposed by Fennema and Franke (1992).

Focussing on two components - extending dynamic model of teacher knowledge to university mathematics lecturing

The choice to focus on two components from the model is based on a hypothesis comparing school and university settings. The two components are mathematics content knowledge and mathematics pedagogical knowledge. The hypothesis is that compared to school teachers', university mathematicians are strong in mathematics content knowledge and lack significant formal mathematics pedagogical knowledge. Further, to extend Fennema and Franke's (1992) model from school setting to university lecturing; it needs to take into account the main differences between school and university setting, namely teacher qualifications and mathematics subject knowledge. Because most school teachers will have formal teacher training qualifications, it is reasonable to assume that they will have better understanding of mathematics pedagogical knowledge. Most university lecturers will not have formal teaching qualifications or training in teaching (Wood et al., 2011). This leads us to assume that university lecturers may lack significant formal mathematics pedagogical knowledge. The other factor is subject knowledge. Studies have revealed the limited mathematics content knowledge of school teachers (Ma, 1999). So, there are good reasons to think that university research mathematicians are stronger in their mathematics content knowledge (Speer & Wagner, 2009) because of their advanced mathematical qualifications and knowledge they gain from research studies. The dynamics of university research mathematicians' knowledge are studied, considering these variations in mathematics content knowledge and mathematics pedagogical knowledge between school and university settings. I propose that the changes in dynamics of research mathematician's mathematics pedagogical knowledge are more pertinent relative to mathematics content knowledge.

METHODOLOGY

This paper uses data collected from six research mathematicians at a university mathematics department in New Zealand. The study used semi-structured interviews. Six mathematicians (RM1, RM2, RM3, RM4, RM5 and RM6) are all researcher mathematicians with more than 20 years experience. None of them have a formal teaching qualification.

Interviews are a useful method to infer research mathematician's experiences of the changing nature of mathematical knowledge (Ball & Bass, 2000). These interviews were designed to be open-ended and semi-structured to give freedom to the participant in sharing experiences and perceptions (Bryman, 2012). The interview questions were framed to understand the dynamics of research mathematicians' mathematical knowledge comparing their experiences as novice lecturer to those as an experienced research mathematician. This orientation was chosen because the dynamic conceptualisation of mathematical knowledge develops over time and not at a specific point of time (Fennema & Franke, 1992). Also, we already know that there exists a difference in mathematical knowledge of novice and experienced teachers (Leinhardt & Smith, 1985). The semi-structured interview questions were; 'Has the mathematical knowledge you use in your lecturing changed comparing as a novice lecturer and experienced research mathematician?', 'How will you describe that change?', 'What has changed?'

The interviews were audio-recorded and transcribed for the purpose of this study. The episodes from the transcripts were separated into categories and were coded to analyse the interview data (Bryman, 2012) based on constructs from the literature relating to the goal of the study.

RESULTS AND DISCUSSION

Using the two components of mathematics content knowledge and mathematics pedagogical knowledge, data from the research mathematicians seem to suggest some changes to our dynamic teacher knowledge model.

The dynamic nature of mathematical knowledge

When comparing change in mathematical knowledge as novice lecturer and experienced research mathematician, RM1 said;

> "But mostly some things haven't changed from that time [novice lecturer]. Umm...we have been having ways of thinking about it now, but certainly I have no more mathematics that I use in my teaching now than I did then." [RM1]

This quote is interpreted as RM1 suggesting that something (some knowledge) has not changed from that time (as novice lecturer). I suggest that this is probably the mathematics content knowledge based on the hypothesis of this study. However, mathematics pedagogical knowledge has changed, as confirmed by the discussion with RM1 on lecturing practice.

> "It [lecturing practice] did not change my ability to do mathematics, but it did change my ability to teach". [RM1]

RM2 shares the same line of thought on the changing nature of mathematical knowledge. RM2 perceives change in mathematical knowledge from two perspectives and thinks that actual relevant knowledge (or mathematical content knowledge) is not significantly different, but what he uses in lecturing (which is mathematics pedagogical knowledge) has changed.

> "I would say, slightly more, I have a much better idea of how things fit in, but in terms of actual knowledge, yeah, I would say it is probably very similar when I first started out [as novice lecturer]... but I have more of other views, sort of point of views of what's being done, which is of different strengths I guess, but in terms of actual relevant knowledge, is not significantly different." [RM2]

RM3 does perceive changes in his mathematical knowledge from novice lecturer to present and asserts that the changes are a positive increase in mathematical knowledge. *'Having experience in lecturing a variety of courses'* and *'actually doing mathematics'* and *'research in mathematics'* all increased his mathematical knowledge.

> " I've had more experience teaching quite a variety of courses, so umm, it has been said that the best way to learn something is to actually teach it, so various times, especially earlier on, I have taught some things that actually made quite a lot. Yeah, and then just actually doing mathematics, doing research in mathematics over the years has and been exposed to mathematics in the ways I mentioned, earlier on, has increased my mathematical knowledge." [RM3]

It is suggested here that '*I've had experience in teaching a variety of courses'* and '*I have taught some things that actually made quite a lot'* have brought more changes in RM3's mathematics pedagogical knowledge. Mainly because RM3 being an expert mathematician, it is assumed that the changes to RM3's mathematical knowledge are more to his pedagogical knowledge rather than content knowledge (based on the hypothesis). Another reason could be that RM3 might not be taking content knowledge different from pedagogical knowledge as this study assumes.

He further goes on and says,

> " ...But pretty much I would say it [change] has been a development of what I originally had." [RM3]

This is not 'new knowledge substituting old knowledge' as Fennema and Franke's (1992) dynamic model on teacher knowledge argues.

Similarly, RM4 perceives change his mathematical knowledge, the change being increase.

> "Sure, I mean, well, first because I have a broader knowledge of mathematics, I see more the interaction between different fields of mathematics that I was not seeing before and so

especially when I teach mathematics students, it is better, because then I can more easily point out to them, look what you are doing here seems to be something like discrete mathematics and you can also use that in some other fields like this one and this one and this one." [RM4]

RM4 says that his expert mathematics content knowledge is useful for making more changes to his mathematics pedagogical knowledge. This is because as RM4 uses his broader mathematics content knowledge in lecturing; he transforms his content knowledge to make his pedagogical knowledge better. The changes are more brought about to his pedagogical knowledge through lecturing practice.

These analyses show that research mathematicians' do see dynamic changes both in their mathematics content knowledge and mathematics pedagogical knowledge. Also, the research mathematicians specifically attribute more dynamic changes to their mathematics pedagogical knowledge relative to mathematics content knowledge. Lecturing practice is identified to be an influencing factor in the dynamics of research mathematicians' knowledge. The development of knowledge is seen as adding up to and developing the existing expert knowledge rather than substituting existing knowledge with new knowledge. This is different from Fennema and Franke's (1992) dynamic model on teacher knowledge where they propose that school teachers change their existing knowledge and create new knowledge. The underlying reason for their suggestion is probably because of the assumption that school teachers lack sufficient mathematics content knowledge which is not necessarily true in the case of university research mathematicians. So, extending Fennema and Franke's (1992) dynamic model of teacher knowledge from school to university suggest some differences in the dynamics of university research mathematicians' pedagogical knowledge. Furthermore, extending the model to university lecturing indicates a static component of mathematical knowledge.

The static nature of mathematical knowledge

A static component associated with the mathematical knowledge of research mathematician is identified.

Research mathematician RM2 says he had lost some expertise in his mathematical knowledge now; for no longer using or practising that knowledge in lecturing.

"I guess I had particular expertise in calculus [as novice lecturer] and certain things which I no longer have because I am not practising that so much." [RM2]

Thus, the lack of practise of lecturing is indicated as a factor influencing the static nature of mathematical knowledge.

Further RM2 asserts the static nature as follows,

"...but most of that knowledge [mathematical knowledge I had as novice lecturer] is not relevant to anything that we teach because mathematics is such a huge subject and what we teach in university is just a tiny bit." [RM2]

As learners of mathematics during their formal mathematics learning, mathematicians learn a lot of mathematics. But when they start practising university mathematics by lecturing one or two courses, they limit their opportunity to make advances in the other areas of mathematics they have expertise in. This causes expert mathematicians to lose some mathematics.

RM3 identified some mathematical knowledge not developing; the reason being not using and practising some mathematical knowledge in lecturing thus making that mathematical knowledge '*stagnant*'.

> "I probably have forgotten some as well over the time. I guess my mathematical knowledge has changed, some is forgotten and some is learned, I have forgotten some old stuff, learned some new stuff probably." [RM3]

The static nature of mathematical knowledge in this study is characterised as the non-developing mathematical knowledge for various reasons such as being lost expertise in for not using or practising in lecturing. However, it is reasonable to ask if the forgotten knowledge is able to return and be put into practice. The answer is sometimes it may be possible, as in the case of RM2 and RM3; if they can practise that mathematical knowledge in lecturing again. But there are other factors that make it difficult. RM5 shares the same line of thought about the forgotten mathematical knowledge, but also shares the thought that some knowledge is retrievable, but some difficult.

RM5 describes the static nature as follows,

> "If I had to go and learn about something which I had studied as an undergraduate and forgotten, then I could probably pick up some kind of reasonable text book and find out again more quickly. But, there are some other things I did when I was a PhD student and I think I would find it harder to do that because they are more; technically more complicated. It is because I have forgotten, although some of these things I never really understood very well in the first place." [RM5]

Probably reading mathematics books and practising that knowledge in lecturing may develop static mathematical knowledge the mathematicians had expertise in, but there are some that are forgotten for they are more complicated mathematics and the mathematicians do not have extra time to invest in redeveloping that mathematical knowledge.

Research mathematician RM6 talks about how habituation leads to stagnant knowledge.

> "I mean, the first time is obviously, you know what it is [to lecture] and the second time, probably there will be some fairly big changes and then each subsequent time the changes get smaller and smaller and smaller. It would be sort of the first to the second and second to the third, are most of the changes happened, and in subsequent times, you just mostly make some little tweaks here and there to adjust that they [students] did not quite understand. So I can do it a little bit better or motivate, but in after the first two times, I would say it didn't change that much." [RM6]

RM6 is talking about mathematics pedagogical knowledge rather than mathematics content knowledge. After lecturing, we can imagine that RM6 uses student feedback and other information to adjust his mathematics pedagogical practice. He is aware that changes occur more when starting lecturing a new course, and as the lecturing continues, he feels that the process of change narrows and becomes stagnant. As the pedagogical change approaches a certain level, there is still the possibility to make changes, but overall it is getting closer to a static practice, and static knowledge underlying it.

CONCLUSION AND IMPLICATIONS

In summary, this study extends the notion of the dynamic nature of teacher knowledge to university lecturing focussing on the two components of mathematics content knowledge and mathematics pedagogical knowledge. This study highlights the dynamic nature of mathematical knowledge of university research mathematicians based on the hypothesis that they have better mathematics subject knowledge and lack significant formal mathematics pedagogical knowledge. Findings from interviews with research mathematicians suggest that they identified their mathematical knowledge as dynamic and developing, thus useful in developing existing mathematical knowledge rather than substituting existing knowledge. The dynamics are more pertinent in research mathematicians' mathematics pedagogical knowledge relative to their mathematics content knowledge which is consistent with the hypothesis. Further, this study identified the static nature of some mathematical knowledge that is stagnant, and may lead to losing some mathematical knowledge. The static nature of mathematical knowledge happened in some areas which they no longer practised or used in lecturing. Additionally, habituation in lecturing is found to lead some mathematics pedagogical knowledge to approach stagnant nature. Lecturing practice is found to be one of the main factors influencing university mathematicians' dynamic and static nature of mathematical knowledge. More study is needed to understand how other components from the dynamic teacher model works in university settings and what other factors contribute to dynamics of mathematical knowledge. Further, understanding nature of mathematical knowledge has implications to professional development of university lecturers; by providing necessary opportunities for growth and development of new mathematical knowledge and hindering factors leading to static mathematical knowledge.

References

Ball, D. L. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics, 8*, 40-48.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83-104). Westport CT: Ablex Publishing.

Bryman, A. (2012). *Social Research Methods* (4th ed.). New York: Oxford University Press.

Davis, B., & Renert, M. (2013). Profound understanding of emergent mathematics: broadening the construct of teachers' disciplinary knowledge. *Educational Studies in Mathematics, 82*, 245-265. doi: 10.1007/s10649-012-9424-8

Elbaz, F. (1981). The teacher's "practical knowledge: Report of a case study. *Curriculum Inquiry, 11*(1), 43-71.

Fennema, E., & Franke, L. M. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook on research on mathematics teaching and learning* (pp. 147-164). New York: NY: Macmillan.

Hashweh, M. Z. (2005). Teacher pedagogical constructions: A reconfiguration of pedagogical content knowledge. *Teachers and teaching: Theory and practice, 11*(3), 273-292.

Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*(3), 247-271.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah NJ: Lawrence Erlbaum Associates.

Meredith, A. (1995). Terry's Learning: some limitations of Shulman's pedagogical content knowledge. *Cambridge Journal of Education, 25*(2), 175-187. doi: 10.1080/0305764950250205

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Speer, N. M., Smith, D. A., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior, 29*, 99-114.

Speer, N. M., & Wagner, J. F. (2009). Knolwledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education, 40*(5), 530-562.

Wood, L. N., Vu, T., Bower, M., Brown, N., Skalicky, J., Donovan, D., . . . Bloom, W. (2011). Professional development for teaching in higher education. *International Journal of Mathematical Education in Science and Technology, 42*(7), 997-1009.

PERFORMANCE AND STRATEGY USE IN COMBINATORIAL REASONING AMONG PRE-SERVICE ELEMENTARY TEACHERS

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In our study we analyse strategies observed in combinatorial problem-solving. The study was carried out among the students of the Elementary Teacher Training College of Eötvös Loránd University (N = 128). Our research shows the variety of strategies appearing during the solution of tasks with the same mathematical structure, but with different wording. In line with our most important hypothesis, building mental models for the tasks corresponding to both the text of the word problem and the combinatorial structure are essential. However, as revealed by this investigation, students often build superficial models, i.e. they choose one of the algebraic expressions (e.g., factorial, binomial coefficients) from their high-school repertory, and use it without sense-making of the problem.

INTRODUCTION

Combinatorics is a branch of mathematics which has an obvious practical use; on the other hand combinatorics is especially suitable for students to perform mathematical activities via simple experiments and observations already in their early development. In Hungary today along the development of combinatorics, the issue of developing combinatorial thinking has again have been put into the limelight. In the entrance examination tasks secondary-school students have to solve combinatorics a lot more dominant than it is present in the guidelines of the Frame Curriculum. The relatively small number of hours allocated to combinatorics and the low proportion of such tasks in the textbooks justify the importance of investigating the topic.

In Hungary, at the end of the 1960s researchers started to think about teaching combinatorics (Varga, 1968), and the topic was included in general education at the start of the school-year in 1978 (Halmos & Varga, 1978). Tamás Varga and his colleagues defined the levels of teaching combinatorics, which have been built in the methodology of Hungarian mathematics. In accordance with the level of development of students the levels of combinatorial tasks built on each other are the following (Pintér, 2013):

- Differentiating the cases
- Listing all possible cases as brainstorming
- Regular listing
    - Two types of representations: objects – images (drawings, letters, tables, graphs)

    - Strategies: (change, fixing, cyclicality)
- Applying formal methods
    - Two types of representations: objects – images
    - Strategies (multiplication, addition, one-to-one mapping, recursion)
- Recognition of structures

During the elaboration of the topic and the solution of the tasks, finding the best fit model is essential (see Godino, Batanero, & Roa, 2005). Therefore it is not worthwhile requiring task solution always on the highest possible level available for the age group. Pupils will understand and find abstract models and formal methods when simpler lists with a lower number of elements and systemisations are deducted on a manipulative (and later on an image level) in many types of specific forms. Often the same task occurs on every level: by increasing the number of elements from the specific objective activity through image depiction we reach the application of symbolic methods (Varga & Dumont, 1973).

Two main questions have to be taken into consideration when solving combinatorial calculation problems: Have we added different cases? Have we added all the cases? Combinatorial ideas from a very young age appear and can be developed (English, 1991; 1993) however we can find typical errors at all levels of learning combinatorics (Batanero, Navarro-Pelayo, & Godino, 1997).

The role of combinatorics is special in mathematics teaching. Solving combinatorial tasks helps the flexibility of thinking in approaching problems, and selecting representations. The majority of the tasks cannot be solved mechanically, but need critical thinking, thus activating metacognitive skills, strategic planning, thus improving mathematical performance. (Csíkos, Szitányi, & Kelemen, 2012) There is hardly any topic which could improve the skills of pupils so extensively as combinatorics (Sriraman & English, 2004, p. 182).

Combinatorial problems are generally given as word problems. Research on maths word problems, according to Verschaffel, Greer and Torbeyns (2006), has moved in recent years into the direction of research within the framework of general problem-solving. Theories of problem solving have long ago acknowledged the importance of metacognitive processes (planning, monitoring and evaluation strategies). One important aspect of metacognitive knowledge is students' conscious use of different types of drawings while solving the problem (Van Meter & Garner, 2005).

The hypothesis that analogical thinking has an important role in knowledge acquisition has been verified in previous investigations. Analogies are usually considered to be development tools of inductive thinking (Nagy, 2000) Learning via analogy means that in a new situation we try to recall a known, successfully solved similar situation, so that we can reconstruct the successful solving strategy and apply it to the present situation. (Gick & Holyoak, 1990) In our research, the possibility to build analogies in the area of combinatorics has been investigated.

Research questions

- What is the difficulty and teachability of the topic assessed among elementary school teacher training college students?
- What level of performance do pre-service elementary teachers have on a set of deep-structurally similar combinatorics tasks?
- What kinds of combinatorial reasoning strategies they have? Do they recognise the analogous structures? What kinds of visual aids do they give themselves?
- What is the connection between their performance and strategy use?

METHODS

Sample

Our research was conducted at the Eötvös Loránd University's Faculty of Primary and Pre-School Education. There were 128 students involved who represented not only the student pre-service elementary teacher population of that Faculty but a much wider student population. The members of that much wider population is characterised by having a diploma and they do not consider themselves as mathematics teachers but rather school subject generalists.

Elementary teachers are required to possess combinatorial reasoning structures and at the same time to be able to solve problems by means of simple strategies which are expected to be used by their pupils. At the beginning of their tertiary studies, they already have learnt combinatorics in the high schools: permutations, variations and combinations are listed among the target topics of the maturation exam. They have to know the n! or $\binom{n}{k}$ $\binom{n}{k}$ expressions. At the time of the data collection they had not yet encountered further combinatorics studies at the university.

Measures

Questionnaire

The questionnaire addressed students' attitudes towards combinatorics and to other mathematical sub-domains. Besides attitude measures students were asked how difficult they consider teaching combinatorics, and how they self-assessed their level of knowledge in combinatorics.

A test of combinatorial reasoning

The test consisted of seven tasks that were deep-structurally similar to each other. Five out of the seven tasks was built on same combination structure chose of 7 elements 2, while the other two tasks were built on the $\binom{8}{2}$ or $\binom{7}{2}$ +7 scheme. Two examples are given here:

> Anna, Béla, Cili, Dóra, Erik, Fanni and Gizi are drinking champagne. What is the sum of the toasts if everyone is clinking with everyone else?

In a domino set the maximum number of spots on each side is six. There can be identical number of spots on two sides of a domino, and any side may remain empty. What is the number of dominos in the set?

Video footages and audio-files

13 students were asked to fill in the questionnaire and the test in front of a video camera. The analysis of these video footages enabled us to more reliable categorise the students' answers on the paper-and-pencil tests.

Analysis

Students' solutions were assessed according to two aspects. Besides the dichotomous assessments of the rightness of the solution, their strategies were coded as follows:

1 - Staircase-like visualisation: This method was often applied in the first problem. Those, for whom the selection of element pairs is not primary, addition is performed according to the following: A toasts 6 people, B only five, C four, thus the number of cases is 1+2+3+4+5+6. In our problems, most often this correspondence was found. An example is shown in Figure 1 from an answer given to this strategy.

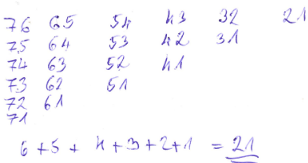

Figure 1: An illustration of the "Staircase-like visualisation" strategy

2 - Polygon as visual aid: The Figure 2 shows that pictures are a bit more difficult to use and follow when the elements are depicted as the vertices of a polygon.

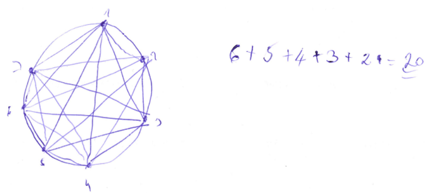

Figure 2: An illustration of the "Polygon as visual aid" strategy

In this case, in case of accurate addition the 6+5+4+3+2+1 structure, or $\frac{n(n-1)}{2}$. appears. The following is a quotation from an interview:

"I recalled the drawing. Indeed, the drawing example, how many are there? 7, yes. I absolutely recalled the linking drawing, and theoretically everybody toasts 6 people, we divide it by two, because we add everything twice, therefore 6×7/2 is 21, theoretically."

However, the well-chosen representation does not always lead to the accurate result, the following two examples show this:

"From one point I can draw 6 lines, so this is 7×6, and the points I have already used should be deducted, so it is 6x6."; "It can be solved in different ways, when we draw the lines of the polygon defined by these 7 points, we get a polygon with 7 sides. Then we draw the diagonals, and so on… And we know the number of diagonals, n-3×n, this means 4×7=28 + 7 sides"

3- Lines drawn: As seen in Figure 3, the person solving the problem places the 7 elements on one line, of which she selects two by connecting them. This solution is a little problematic, as it is not easy to follow and add the drawn lines.

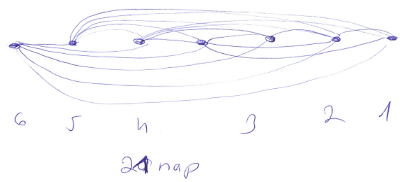

Figure 3: An illustration of the "Lines drawn" strategy

4 - Graphs: A technical process often applied was to build a graph, which is not connected. An example is shown in Figure 4 from an answer given to the 2nd problem.

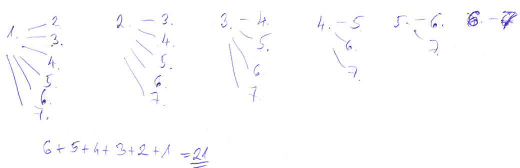

Figure 4: An illustration of the "Graphs" strategy

5 - Listing cases: It occurred that the person solving the problem listed all the cases, but we could not find any structure.

6 - Calculation solely with numbers: The students tried to recall a formula based on earlier studies, or operated with the four basic operations from the numerical data. For example:

> "I am still calculating in my head and I recall an image, automatically in my head 7 choose 5 and 7 choose 2, and I am trying to arrange in my head how accurate my thinking is. As we have to think aloud, and we tend to be shy, therefore I will start. 7 choose 5, this is 7! multiplied by 2!"

7 - Recursion: A learnt method often applied in the solution of combinatorial tasks: When you cannot understand the problem, solve it with a smaller number of elements. In our task system there was only one problem suggesting this method.

RESULTS

Concerning pre-service elementary teachers' beliefs about the difficulty of combinatorial knowledge and about the difficulty of teaching the topic, the correlation is strong (.68), i.e., the stronger the person considers their combinatorial problem-solving skills, the more they like teaching the topic ($p < .005$). Correlation between belief on own combinatorial knowledge and the results achieved in the test proved to be also significant (.58). As compared to other mathematical domains, combinatorial knowledge and teaching of the topic is apparently at the worst place, as seen in Table 1.

Domain	Own knowledge	Difficulty of teaching
Algebra	3.59	3.41
Combinatorics	*2.90*	*2.58*
Functions	3.35	3.20
Geometry	3.84	3.83
Logic	3.57	3.14
Number theory	3.45	3.32
Sets	4.14	4.23

Table 1: Students' beliefs on a five-point Likert-scale about their knowledge and about the difficulty of teaching mathematical domains

The system of the seven tasks has a marginally acceptable reliability (Cronbach's α = .64). Taken the small number of items into account, this reliability coefficient suggests that the task system measure a psychological construct, i.e. a subsystem of combinatorial reasoning.

The mean solution rates for the seven tasks are shown in Table 2.

Task	Mean (%)	SD
1.	79	41
2.	24	43
3.	93	26
4.	84	37
5.	82	39
6.	55	50
7.	51	50

Table 2: Mean solution rate and standard deviation in each task

To explore the relation between the success of the solution and the applied strategy first we examined whether the average of those using unidentified strategies was different from the others. In tasks 5 and 6, significant differences were found.

The frequency of the strategies used in each task is shown in Table 3.

Strategy/Task	1	2	3	4	5	6	7
1 - staircase	93	80	96	100	100	100	100
2 - polygon	79			100	89	78	0
3 - line	91				100	100	50
4 - graphs	88	80	100	92	100		83
5 - cases	100	65		93	91		50
6 - calculation	60	2	56	44	77	79	42
7 - recursion			100				
No strategy identifiable	79	16	88	70	66	25	48

Table 3: Solution rates for each task according to observed strategies (%)

As for the consistency of strategy use, correlation between the number of tasks where the student used an identifiable strategy and the number of right solutions is .24, which is significant at $p = .007$ level.

Six people solved the problems by only looking at them, without using any visual aid or written calculations. Besides them, the connection between identifiable strategy use and the number of right solution is obvious: the more somebody used some visible strategy, the higher score they achieved. Except for these six persons, the correlation above is .42, which is significant on a level of $p < .001$.

DISCUSSION

The first research question concerns the assessment of the topic of combinatorics. Results show that combinatorics has the worst place compared to other fields of mathematics considering the belief related both to own knowledge and the teaching of the topic. Our task sheet contained problems with the same structure, but with different contents. Therefore it is not true that in case someone acquired one type of process, they could solve the tasks better. The majority of our students used image depiction when solving the problems.

The variety of methods used in the solution (an average of 3.84 different strategies used) suggests the analogy among the tasks was difficult to recognise for students. The lack of analogies used points to the difficulty of recognising the combinatorial structures the tasks were based on. Finally, our research has shown that the use of recognisable strategies are associated with better performance. It is worth being noted that it indicates the use of metacognitive strategies.

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References

Batanero, C., Navarro-Pelayo, V. & Godino, J. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics, 32,* 181-199.

Csíkos, C., Szitányi, J. & Kelemen, R. (2012). The effects of using drawings in developing young children's mathematical word problem solving: A design experiment with third-grade Hungarian students. *Educational Studies in Mathematics, 81,* 47-65.

English, L. D. (1991). Young children's combinatorics strategies. *Educational Studies in Mathematics, 22*, 451-474.

English, L. D. (1993). Children's strategies for solving two-and three-dimensional Combinatorial Problems. *Journal for Research in Mathematics Education, 24,* 255-273.

Godino, J., Batanero, C., & Roa, R. (2005). An onto-semiotic analysis of combinatorial problems and the solving processes by university students. *Educational Studies in Mathematics, 60*, 3-36.

Halmos, M., & Varga, T. (1978). Change in Mathematics Education since the late 1950's – Ideas and realisation, Hungary. *Educational Studies in Mathematics, 9,* 225-244.

Gick M. & Holyoak K. J. (1990). Analogical problem solving, *Cognitive Psychology, 12,* 306-355.

Pintér, K. (2013). A kombinatorika tanítása [Teaching of combinatorics] In E. Herendiné Kónya (Ed.), *A matematika tanítása az alsó tagozaton [Teaching of mathematics in elementary level]* (pp.219-237). Szeged: Mozaik.

Nagy, L. (2000). Analógiák és az analogikus gondolkodás a kognitív tudományok eredményeinek tükrében. [A survey of research on analogies and analogical thinking in cognitive science] *Magyar Pedagógia, 100,* 275-302.

Sriraman, B., & English, L. D. (2004). Combinatorial mathematics: Research into practice. *Mathematics Teacher*, *98*(3), 182-191.

Van Meter, P. & Garner, J. (2005). The promise and practice of learner-generated drawing: Literature review and synthesis. *Educational Psychology Review, 17*, 285–325.

Varga, T. (1968). Matematika kicsi kortól kedvvel [Mathematics for small age with enthusiasm] *Élet és Tudomány, 25*(10), 637-647

Verschaffel, L., Greer, B. & Torbeyns, J. (2006). Numerical thinking. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 51-82). Rotterdam: Sense Publishers.

USING METAPHORS TO ASSESS STUDENT MOTIVATION AND ENGAGEMENT IN MATHEMATICS

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This study examined the use of metaphors as a viable tool for assessing primary students' motivation and engagement in mathematics. Data were gathered from 20 Year 6 students from one classroom via a metaphor task and focus group. Metaphors were inductively and deductively analysed. Information from the focus group was used to validate qualitative interpretations. Of interest is the degree to which elements in established theoretical frameworks of motivation and engagement were depicted in student metaphors. Aspects of emotional engagement were most notably expressed, followed by cognitive and behavioural aspects such as self-belief, persistence, valuing of mathematics, and uncertain control. The implications for assessing student motivation and engagement are addressed.

INTRODUCTION

Teachers routinely use a wide range of information about their students when planning for instruction; including how motivated and engaged they are to learn (Bobis, Anderson, Martin & Way, 2011). Such information not only influences the teaching strategies teachers adopt, but their responses to students and the efforts they make in their teaching (Hadrè, Davis & Sullivan, 2008).

Numerous research endeavours have investigated student motivation and engagement, mainly utilising surveys (Fredricks, Blumenfeld & Paris et al., 2004; Martin, 2007). While validated quantitative instruments provide results comparable to established standards, they are often not practical for regular classroom use – usually requiring a specific skill set to analyse and interpret, and considerable lag-time for collated results to be returned, thus making them of little use for immediate instructional needs. In the absence of suitable alternative data gathering instruments, teachers rely on intuitive assessments of student motivation and engagement. Such assessments have been proven unreliable, particularly regarding student emotional engagement — students are easily able to mask their true motivations and levels of engagement (Hadrè et al., 2008). Viable alternative tools are needed to assist teachers make accurate judgements about the nature of their students' motivation and engagement in mathematics. This paper reports on research that explored the viability of using metaphors as a tool for teachers to easily and accurately assess upper primary students' motivation and engagement in mathematics.

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MOTIVATION AND ENGAGEMENT: CONCEPTUAL FRAMEWORKS

The terms motivation and engagement are often used interchangeably, possibly due to their close relationship. While acknowledging their connectedness, we differentiate between them, defining *motivation* as an individual's intention or willingness to act, and *engagement* as the actual involvement (Gettinger & Walter 2012). Furthermore, engagement is conceived as a meta-construct, incorporating three interrelated types: behavioural, emotional and cognitive engagement (Fredricks et al., 2004). Behavioural engagement refers to active involvement and participation; emotional engagement encompasses positive and negative reactions; and, cognitive engagement refers to the intellectual investment one is willing to employ in efforts to learn (Fredricks et al., 2004). While adopting these three categorises of engagement, we recognise that they often overlap, such as when intense negative emotions (e.g., anxiety) incorporate cognitive aspects (a 'mental block' during a test). With this in mind, we also drew upon Martin's (2007) multidimensional motivation and engagement framework to guide our investigation (see Figure 1).

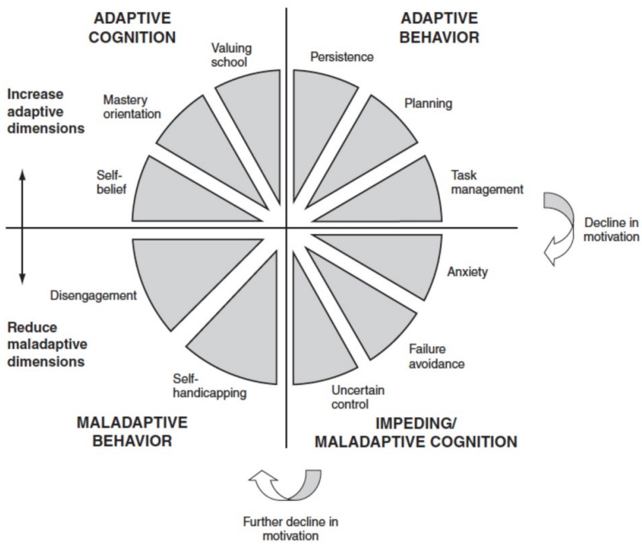

Figure 1: Motivation and Engagement Wheel (Martin, 2010, p. 9).

Drawing upon theory and research, Martin (2007, 2010) developed the Motivation and Engagement Wheel to integrate a number of theoretical perspectives and articulate a framework that is readily accessible to researchers and practitioners (see also Bobis et al., 2011). The upper quadrants contain adaptive cognitive and behavioural factors such as self-belief, mastery orientation, valuing of school, persistence, planning and study management. The lower quadrants include maladaptive behavioural and impeding cognitive factors, such as disengagement, self-handicapping, uncertain control, failure avoidance and anxiety. Importantly, affective factors are not prominent in the Wheel, and neither Martin's (2007) nor Fredricks et al.'s (2004) frameworks specifically relate to mathematics. However, when considered in tandem, the two cover a breadth of behavioural, cognitive and affective motivational and engagement factors that are clearly relevant to mathematics.

It is imperative teachers understand student motivation and engagement in mathematics, as they are key agents for improving educational success (Bobis et al., 2011). There is a need to provide teachers with accessible, low-tech tools that will provide viable assessments of their students' motivation and engagement in mathematics.

ASSESSING STUDENT MOTIVATION AND ENGAGEMENT

Extensive research has been undertaken to investigate and assess student motivation and engagement through the use of both quantitative and qualitative methods. Most quantitative approaches utilise different forms of surveys and scales, originally designed to measure motivation and engagement in only generic terms (Hardré et al., 2008). Drawbacks to this approach include the non-return of tests, self-reporting of perceptions, laborious analysis of data and generalised questions that usually yield a numeric score or set of scores as the sole information about students' motivation and engagement. They can, however, provide data that describes the nature of existing conditions at a specific point in time from a very large number of students.

Conversely, qualitative methods usually encompass observations, interviews and open-ended response questionnaires (Hardré et al., 2008; Turner, Warzon, & Christensen, 2011). Although these methods capture rich detail, they present trade-offs, including that they are time-consuming and human-resource intensive to both implement and analyse (Fredricks et al., 2004). Hence, they typically consist of small-scale sample sizes, yielding results that may be incorrect to generalise to the larger population. However, qualitative methods usually provide enriched understandings of motivation and engagement levels that can be student, subject, and even topic-specific.

Recently, Cai and Merlino (2011) expressed a need for more practical, classroom-friendly tools to help teachers analyse students' dispositions towards mathematics due to a "near-universal lament of low student motivation" (p. 147). They introduced a strategy involving short metaphor tasks as a way to effectively assess student motivation and engagement. Such tasks are considered particularly appropriate for use with middle and high school students (rather than younger students) due to their ability to think metaphorically. There is wide agreement regarding the benefits of metaphors and what they can expose about students (e.g., Solomon & Grimley, 2011). For instance, Lakoff and Johnson (2003) considered metaphors capable of revealing expressions of feelings and thoughts which expose deep underlying meanings that might otherwise remain undetected.

This study examined the use of metaphors as a viable tool for teachers to easily and accurately assess upper primary students' motivation and engagement in mathematics. In particular, it addressed: (1) What do metaphors reveal about students' engagement in mathematics?; and (2) How do these results align with established theories of motivation and engagement?

METHODOLOGY

The current study was nested within a larger project that aimed to explore the mathematical engagement of Year 5 to 7 students from schools located in the northern suburbs of a large city on the east coast of Australia. Several teachers and their classes were invited to take part in a case study component of the larger project. Case studies were intended to provide in-depth information of a small number of classes using qualitative methods (Yin, 2009). The mixed-ability Year 6 class (n=20; 11 to 12 year old students) involved in the current investigation was one of the case study classes. It was ultimately chosen due to the teacher's willingness and availability to participate in the case study component.

Data collection methods, process and analysis

Facilitated by the classroom teacher, the Year 6 students undertook a mathematics metaphor task. The metaphor task required students to complete a written response to the statement: "If mathematics was a food, it would be …". They were also required to provide a written explanation for their choice of food. Cai and Merlino (2011) recommended using prescribed topics (including food) because some students invent metaphors that do not lend themselves to rich descriptions. It was considered that 'food' would be a familiar topic to Year 6 students and would instigate a richness and range of responses that could be analysed in terms of their mathematical engagement.

In the same week, five students were purposefully selected to participate in an hour-long focus group interview conducted at the school and facilitated by one of the researchers. Based on their metaphors and discussions with the class teacher, students were selected so as to obtain a representative sample of varying mathematical dispositions and achievement levels in mathematics. The main purpose of the focus group was to provide student explanations of their metaphors that would help validate our interpretations. Hence, students were provided with a copy of their metaphor and asked to further explain why and how it reflected their view of mathematics. They were encouraged to give examples from the mathematics classroom to elaborate their meanings. These examples provided a springboard for other students to contribute to the discussion. The focus group was video recorded to enable follow-up clarifications to occur. A second researcher observed the focus group and took field notes.

Metaphors were analysed using both inductive and deductive approaches. They were first transcribed to a spreadsheet to assist analysis. For the inductive analysis, we categorised each metaphor to determine the kinds of foods students used and why. Metaphors were then individually read several times and examined for commonalities and differences by two researchers. The researchers discussed and agreed upon several prevalent themes. Throughout the inductive process both researchers viewed relevant segments of the focus group video and consulted the field notes to assist their interpretations and help settle any disagreements in interpretations.

The deductive analysis was guided by Fredricks et al.'s (2004) and Martin's (2007) theoretical frameworks. Each metaphor was interpreted and categorised according to the presence of any words, implied feelings or ideas related to the three types of engagement and/or pre-determined definitions for each aspect on the Wheel by the first-named researcher. Hence, each metaphor was considered to exhibit, or not exhibit, an aspect of the Wheel – scoring a point for each aspect present to a total of 7 points for adaptive and 5 points for maladaptive aspects. A second researcher independently followed the same process. The two analyses were compared, revealing an overall interrater reliability of 86% agreement and a 95% agreement on emotional and some cognitive engagement aspects. A metaphor was only considered to exhibit a particular aspect if both researchers were in agreement. Interpretations were drawn upon and validated with the assistance of the field notes and video recording of the focus group.

RESULTS AND DISCUSSION

Findings derived from the inductive analysis of the metaphors are presented first. Due to space restrictions, only selected metaphors are drawn upon to support our interpretations. Most of the metaphors (18 out of 20) included affective responses towards mathematics (see Table 1 for examples of affective responses).

Category/ Sub-category	Metaphor example	Frequency
Level of enjoyment		
Like/love	Chocolate. I love chocolate and think it is nice and smooth and creamy, like maths.	4
Variable enjoyment	Ice-cream. Sometimes you love maths and look forward to it. Where as sometimes it can be hard or melted and very yuck and boring. Plus the good thing about ice cream is trying new flavours. With maths you can try new topics. *Yum*!	14

Table 1: Affective responses included in metaphors.

While no metaphors displayed an absolute and definitive dislike for mathematics, the majority revealed variable levels of enjoyment (a mix of both positive and negative affective responses). Twelve students conveyed their variable affective responses by juxtaposing words such as, "rotten and ripe", "bruised or ripe", "hard" or "soft", and "sour or sweet", indicating that the emotional engagement of these students with mathematics was capable of changing quiet dramatically.

Students' conceptions about the nature and structure of mathematics were also clearly depicted in their metaphors, ranging from a narrow view of mathematics (e.g., "Meat", "Fish" and "Banana") by the majority (15) to five students who depicted mathematics as dynamic and multi-layered (e.g., "Lasagne. Maths is complicated just like lasagne.

You need to eat your way through all the layers and you need to work through a maths problem to find the answer…"). Cognitive demands associated with mathematics were also depicted in over half the metaphors (e.g., "Fish is hard to eat but is very flavoursome, but there can be bones… which are hard to eat"). Thirteen metaphors depicted strategies for dealing with difficult mathematics content, including 'persistence/effort' ("Apple is hard to get at, at first, but once you bite into it, it is so sweet and tasty you want more…"), 'practice' ("…it is hard to eat at first but the more you practice the easier it gets"), and 'chunking' ("Watermelon. It's nice to eat and you can break it up into chunks…").

Using the deductive analysis process described earlier, metaphors were determined to depict a total of 61 occurrences of motivation and engagement aspects contained in the selected theoretical frameworks (Fredricks et al., 2004; Martin, 2007) (see Table 2).

Type and Aspect	Frequency
Adaptive Cognitions and Behaviours	
Self-belief	8
Learning Focus	5
Valuing of mathematics	5
Planning	2
Task Management	4
Persistence	8
Emotional Engagement	
Liked/Positive	4
Variable	14
Maladaptive and Impeding Cognitions and Behaviours	
Anxiety	1
Failure Avoidance	3
Uncertain control	7
Self-handicapping	0
Disengagement	0

Table 2: Frequency of each aspect's occurrence in student metaphors.

As evident from Table 2, emotional elements (e.g., "love", "like", "enjoy", "dislike") were the most prevalent to be depicted in the metaphors, but as determined during the inductive analysis, the majority of student metaphors expressed variable emotional engagement rather than an absolute emotional response towards mathematics. This is consistent with Solomon and Grimley's (2011) finding that Year 5 and 6 "children expressed a wide range of strong feelings about mathematics" (p. 700) in their drawings. It is also consistent with our conceptions of motivation and engagement that view them as variable and highly malleable constructs (Bobis et al., 2011).

There were 18 occurrences of metaphors exhibiting adaptive (positive) behaviours and cognitions. The two most frequently occurring adaptive aspects were 'self-belief' and 'persistence' (8 occurrences each), followed by 'valuing of mathematics' and 'learning focus'. Conversely, there were fewer negative aspects depicted – only 11 occurrences of maladaptive/impeding cognitions and behaviours, with 'uncertain control' proving to be the most common issue (7 occurrences) expressed. There was one occurrence of 'anxiety' noted and no instances of 'self-handicapping' or 'disengagement'. It is possible that the selected topic of the metaphor (food) made it more likely that some theoretical aspects would be expressed more often than others. For instance, students could easily relate to enjoy eating or not enjoy eating certain foods (emotional engagement), or to consider the presence of "bones in fish" or "seeds in watermelon" as being out of their control (impeding cognitive engagement), but few students might associate 'anxiety' with eating a food. Such limitations may be overcome if students respond to two or three different metaphors (Cai & Merlino, 2011). Another suggestion by Solomon and Grimley (2011) is to supplement metaphor information with that gained from other sources, like focus groups (such as the current study) or student interviews.

SUMMARY AND CONCLUSIONS

The analysis of metaphors revealed rich, mathematics-specific information about the class as a whole and about individual student's motivation and engagement. The deductive analysis of the metaphors especially contributed to our understanding of student engagement by directing our attention to specific behavioural, cognitive and emotional aspects contained in each of the theoretical frameworks. Albeit a result of the selected topic for the metaphor, they were chiefly beneficial in providing insights into students' varying emotional responses towards mathematics – an area of motivation and engagement research that has lagged behind other areas partly due to difficulties associated with conventional quantitative methods being able to viably assess it (Martin, 2007). The metaphors also revealed insights into students' beliefs and conceptions about the nature of mathematics, and their cognitive and behavioural strategies for learning it and dealing with challenging problems. The analysis shows that metaphors can be used to detect very specific aspects of individual students' motivation and engagement in mathematics, which make it a viable, low-tech tool suitable for use by classroom teachers. However, to truly realise the full power of metaphors as an assessment tool, teachers also need to have in-depth knowledge of motivation and engagement. Future research could investigate the skills and knowledge that teachers require to successfully implement and analyse metaphors within a classroom context.

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References

Bobis, J., Anderson, J., Martin, A., & Way, J. (2011). A model for mathematics instruction to enhance motivation and engagement. In D. Brahier (Ed.), *Motivation and Disposition: Pathways to Learning Mathematics* (pp. 31-42). Reston, VA: National Council of Teachers of Mathematics.

Cai, J., & Merlino, F. J. (2011). Metaphors: A powerful means for assessing students' mathematical disposition. In D. J. Brahier & W. R. Speer (Eds.), *Motivation and Disposition: Pathways to Learning Mathematics*. Reston: National Council of Teachers of Mathematics.

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School Engagement: Potential of the Concept, State of the Evidence. *Review of Educational Research, 74*(1), 59-109.

Gettinger, M., & Walter, M. (2012). Classroom strategies to enhance academic engaged time. In S.L. Christenson, A.L. Reschly., & C. Wylie (Eds). *Handbook of research on student engagement* (pp. 653-673). New York: Springer.

Hardré, P. L., Davis, K. A., & Sullivan, D. W. (2008). Measuring teacher perceptions of the "how" and "why" of student motivation. *Educational Research and Evaluation, 14*(2), 155-179.

Lakoff, G., & Johnson, M. (2003). *Metaphors we live by.* Chicago: University of Chicago Press.

Martin, A., J. (2007). Examining a multidimensional model of student motivation and engagement using a construct validation approach. *British Journal of Educational Psychology*, *77*, 413-440.

Martin, A. J. (2010). *Building classroom success: Eliminating academic fear and failure*. London: Continuum.

Solomon, C., & Grimley, M. (2011). Metaphors used by Year 5 and 6 children to depict their beliefs about maths. In J. Clarke, B. Kissane, J. Mousley, T. Spence, & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices (Proc. of the 23rd biennial conf. of AAMT Inc. and the 34th annual conf. of MERGA Inc., pp. 693-701)*. Adelaide: MERGA.

Turner, J. C., Warzon, K. B., & Christensen, A. (2011). Motivating mathematics learning: Changes in teachers' practices and beliefs during a nine-month collaboration. *American Educational Research Journal, 48*(3), 718-762.

Yin, R. (2009). *Case study research: Design and methods* (4th ed.). California: SAGE Publications.

MATHEMATICS TEACHER EDUCATORS' PURPOSES FOR K-8 CONTENT COURSES

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This report provides empirical findings from a study that examined the purposes of eight experienced mathematics teacher educators, who taught mathematics content courses for prospective K-8 teachers. The data revealed 15 common purposes, aligned to providing the opportunity to develop prospective teachers' pedagogical content knowledge and subject matter knowledge. Two of the purposes aligned with the pedagogical content knowledge (knowledge of curriculum and instructional strategies) are elaborated in this paper. Implications from this study contribute to the literature on mathematics teacher educators' purposes and provide insights into the teacher educators' classroom practices from the K-8 content courses.

INTRODUCTION

Research suggests many prospective teachers do not receive adequate experiences from their teacher education programs in order to develop deep, conceptual knowledge of the mathematics they will teach (e.g., Greenberg and Walsh 2008). As a result, the Conference Board of Mathematical Sciences (CBMS) (2012) proposed that all institutions preparing elementary teachers offer and require at least nine credits of mathematics content courses designed specifically for this population and focused on mathematical relevance, depth, and breadth, concluding that "teaching elementary mathematics requires both a wide range of pedagogical skills and considerable mathematical knowledge'' (p. 55). Research also suggests developing *pedagogical content knowledge* (PCK) and *subject matter knowledge* (SMK) is critical for prospective teacher education (An, Kulm, & Wu, 2004, Blömeke, Suhl, & Kaiser, 2014). In support of these efforts, various mathematics teacher education curriculum materials have been developed to help address these issues, however, very little research exists on what content (and how) are being taught in these courses, what PCK and SMK aspects are being emphasised and addressed, and what goals and purposes do the mathematics teacher educators (henceforth referred to as teacher educators) have in mind when teaching these courses.

Some research efforts have examined the development of mathematics teacher educators, their practice, self-studies, professional development, teacher educator collaborations, and more recently, the differences between the knowledge of K-12 teachers and teacher educators (e.g., Even, 2008; Goodell, 2006; Superfine & Li, 2014, Taylor, 2013; Tzur, 2001). Although these efforts represent a useful start, additional research and development work are needed in order to accumulate an empirical and conceptual knowledge base for mathematics teacher education. Superfine and Li (2014) recommend researching teacher educators' reflection on their practice and this

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could provide “insights into the potential mathematical and pedagogical purposes of those interactions” (p. 313). Building off of this recommendation, we present a conceptual depiction of teacher educator purposes for teaching K-8 content courses for prospective teachers based on empirical data collected during a case study of what eight teacher educators said during an initial interview. More specifically, we sought to understand, document, and investigate the research question: What purposes do experienced teacher educators have for prospective teachers to develop knowledge about teaching mathematics in a K-8 content course for teachers? We define *purposes* as what teacher educators want prospective teachers to learn from K-8 content courses. More specifically, the teacher educators’ professional and personal intentions, that may or may not be included in the course syllabus and/or curriculum, for prospective teachers’ specific knowledge development and learning outcomes.

THEORETICAL FRAMING FOR THE STUDY

Ball and Bass (2000) suggest that teachers’ mathematical knowledge is important, and that ‘‘simply looking at the math problem or considering the content on which students are working does not lead to a sufficient appreciation of the specific mathematical knowledge or sensibility that it takes to teach that problem or that content’’ (p. 91). An, Kulm, and Wu (2004) also argue that prospective teachers’ knowledge of pedagogy is especially important in mathematics teacher preparation programs. We frame this work in the research perspectives focused on teachers’ knowledge development. In particular, this study is grounded in the perspective that aims at capturing teacher educators’ purposes that support the development of prospective teachers’ PCK and SMK. For the purpose of this paper, the focus of analysis is on teacher educators’ purposes specifically supporting prospective teachers’ development of PCK and SMK domains.

Pedagogical content knowledge, originally coined by Shulman (1986), is defined as teachers’ knowledge about “the most useful ways of representing and formulating the subject that make it comprehensible to others” (p. 9). Grossman (1990) built on Shulman’s work and identified four central domains of PCK: knowledge of curriculum, knowledge of instructional strategies, knowledge of students’ understanding, and knowledge of assessment. Magnusson, Krajcik, and Borko (1999) further modified Grossman’s perspective by adding a fifth element of PCK: orientation towards teaching.

In contrast, *subject matter knowledge* contains common content knowledge (i.e., math knowledge and skills used in professions other than teaching), knowledge at the mathematical horizon (i.e., awareness of mathematical connections between topics), and specialised content knowledge (Ball, Thames, & Phelps, 2008). The authors argue that specialised content knowledge is a critical domain of SMK, which entails the type of mathematical knowledge that is specifically unique to teaching and is “not typically needed for purposes other than teaching” nor used in professions other than teaching (p. 400).

Accordingly, this work is framed in the perspective that teacher educators have various purposes aimed to provide the opportunities for prospective teachers to develop necessary knowledge bases, specifically attending to PCK, as well as SMK. It is critical that the teaching practices utilised by teacher educators, during teacher preparation courses, provide the opportunity for prospective teachers to develop the necessary knowledge that will enable prospective teachers to become effective mathematics teachers and successful educators.

METHODOLOGY

This study is a case study (Stake, 2005), where the "case" is a group of eight *experienced* teacher educator volunteers (5 males; 3 females) from five different universities in the Eastern portion of the U.S. who regularly teach content courses for prospective K-8 teachers. We define *experienced* as: a) having at least a Master's degree; b) having at least 20 years of K-12 teaching experience and teaching mathematics content to K-12 teachers; and c) being professionally active by attending/presenting at local, state, and national professional meetings. We treated the group of eight teacher educators as a single prototypical case, which allowed us to make claims about the nature of their purposes for teaching K-8 content courses as a whole. Data for the project were gathered through 1-hour semi-structured initial interviews, during which participants were asked about their educational background, their purposes for the K-8 content course they teach (i.e., intentions for small group and whole group instruction), whether (and how) explicit they were with prospective teachers about their purposes, and the approaches they used to engage prospective teachers to address the identified purposes. Interviews were audio-recorded, transcribed, and coded using constant comparison analysis (Corbin & Strauss, 2008).

A total of 326 codes emerged from the data analysis. They were arranged under 15 different purposes mirroring the knowledge domains of PCK and SMK. Instances of teacher educators' purposes were identified through interview responses as reflection on various tasks the teacher educators used to engage prospective teachers with course content. Two researchers independently coded each interview. Researchers met throughout the coding process to compare, verify, and finalise the codes. Through several iterations of sorting the purpose codes, a coding dictionary was created from the data to define and illustrate each purpose. The researchers collaborated to refine descriptions of specific purposes the teacher educators articulated.

RESULTS

Across the U.S., most content courses for prospective teachers are treated as a regular college mathematics course, hosted and taught by mathematics faculty in the mathematics department (Greenberg & Walsh, 2008). As we analysed the data from the classrooms of experienced teacher educators, we noticed numerous K-8 connections with regard to students' learning, curriculum, and classroom connections. In fact, we identified 15 different purposes that experienced teacher educators utilised via these K-8 connections in effort to develop prospective teachers' PCK and SMK.

For the purpose of this paper, we present the results from two (out of 15) purposes related to PCK: 1) knowledge of instructional strategies; and 2) knowledge of curriculum.

Know about instructional tools used in K-8 teaching

Teacher educators mentioned they desire for prospective teachers to know about instructional tools used in K-8 teaching—a purpose that addresses the *PCK component of providing the opportunity for prospective teachers to develop their knowledge of instructional strategies*. That is, they incorporate models, physical manipulatives, and representations in their content courses to articulate mathematical concepts studied and taught at the K-8 level. A common theme among the teacher educators was that in their content course, they wanted to provide the opportunity for prospective teachers to develop multiple approaches, representations, and tools for learning and teaching mathematics. They indicated they used instructional tools (i.e., physical models and/or manipulatives) to help prospective teachers make better conceptual connections of mathematical concepts and become familiar and comfortable with using the tools in their future classroom. All study participants perceived the use of manipulatives as an integral part of the mathematics content courses. For example, one teacher educator shared, "I want the [prospective teachers] to come away [from the course] understanding the power of physical models…that they feel comfortable in seeing how to use those physical models when they're working with kids" [Ian]. Similarly, another teacher educator commented,

> I think [prospective teachers] get the message that the answer isn't always good enough. They realise they're going to be teaching children. They're going to have to be explaining things. They're going to need a deeper understanding. They kind of get that, and so they seem to get the message by the end [of the course] that the process of being able to explain "what and why" is what's important… and that manipulatives can provide a visual for helping them explain the "what and why" to their students. [Trina]

Every teacher educator in our study mentioned his/her personal and professional intentions and purposes for their students to be well equipped mathematically and pedagogically for K-8 teaching, in which (the teacher educators believed) that K-8 models and manipulatives play a critical role. Furthermore, they indicated that they primarily used K-8 instructional tools to extend prospective teachers' mathematical thinking to go beyond "the answer," to model and make better sense of mathematical concepts, and to be able to construct more accurate and thorough mathematical explanations and justifications of their work.

Expose to policy documents on curriculum, content, and teaching

Teacher educators indicated that they want to expose prospective teachers to K-8 standards and policy documents on curriculum, content, and teaching—this purpose aligns to the *PCK component of providing the opportunity to develop prospective teachers' knowledge of K-8 school curriculum*. Two different types of K-8 curriculum connections were articulated by the teacher educators. One, teacher educators discussed the scope and sequence (i.e., specific grade bands) of where prospective teachers might encounter the mathematical topics they were learning during the content course in K-8 school curriculum. For example, one teacher educator talked about division of fractions and the grade levels at which this topic is typically introduced to children. He verbalised,

> We mainly talk about how the content we're covering relates to what the students they are going to have in class have to do. For example, we talk about how modelling the division of fractions is actually something that appears in 5th grade, so [school children] are going to be asked to do these things that we are doing in class. [Oliver]

The second K-8 connection the teacher educators articulated was that they specifically addressed mathematics teaching practices described in K-8 standards documents. They wanted prospective teachers to know key processes and proficiencies for the type of mathematical thinking and reasoning K-8 students should engage in, which teacher educators modelled for them during content courses. The teacher educators mentioned several documents that helped them to make these connections: *Standards for mathematics practices* (CCSS, 2010); *Principles and standards for school mathematics* (NCTM, 1989); and *Adding it up: Helping children learn mathematics* (NRC, 2001). Teacher educators shared that they either directly referenced these documents or selected a few focal points from the documents to discuss with prospective teachers. For example, one teacher educator stated,

> I [want] my students to be familiar with the NCTM Process Standards and now the Standards for Mathematical Practice of the Common Core…I love the new buzzword of "sense making." I am somewhat explicit with them about that. [I say to them that] math makes sense, math had better make sense, and it had better make sense to you if you're going to teach it to kids. [Ethan]

Teacher educators also indicated that these documents play a dual purpose in their content courses: a) they help to unveil and put forth a few "practical" suggestions to the prospective teachers about teaching and learning K-8 mathematics, and b) they help teacher educators to model the methods and practices (described in these documents) directly with prospective teachers. Teacher educators shared that they structure the learning opportunities in their content courses to specifically address these standards through course activities so that their prospective teachers are able to experience the mathematical learning echoed in these documents firsthand.

In the study, a total of 15 different purposes (i.e., teacher educators' personal and professional intentions) were identified from interviews, which indicated classroom opportunities for prospective teachers to develop PCK or SMK during mathematics content courses (see Table 1).

Teacher Educator's Purposes for Developing PCK for Prospective Teachers is to:	Teacher Educator's Purposes for Developing SMK for Prospective Teachers is to:
Know about instructional tools used in K-8 teaching	Understand mathematical concepts at a deeper level and articulate the why behind the concepts and formulas
Expose them to policy documents on curriculum, content, and teaching	Develop multiple ways and/or approaches to solve mathematical tasks
Know about K-8 experiences/experiences K-8 students have	Have concrete experiences (e.g., manipulatives) to develop conceptual understanding of mathematical concepts
Experience mathematical success and confidence	See mathematics conceptually
Change their attitude to a positive one towards the subject of mathematics	Experience mathematical learning in different ways
Change their attitude towards teaching math	Know K-8 mathematical concepts they will teach
Have fun with math and see that math can be fun	Develop and improve their mathematical explanations and language
Engage in collaboration	

Table 1: Summary of teacher educators' mathematical and pedagogical purposes for teaching K-8 content courses

CONCLUDING REMARKS

Results indicate that teacher educators not only focus their content courses on developing the mathematical knowledge of prospective teachers, but on providing the opportunity for prospective teachers to develop four components of PCK: knowledge of curriculum, knowledge of instructional strategies, knowledge of students' understanding, and orientation towards teaching. The experienced teacher educators used the processes of reconceptualising, revisiting, revising, and re-learning mathematics in the course while making connections to K-8 students' learning, teaching, and curriculum as channels to develop prospective teachers' PCK. We did not find any data indicating that teacher educators provided the opportunity to develop prospective teachers' knowledge of assessment—a fifth component of PCK.

The highlighted purposes are representative of eight experienced teacher educators' reasons for engaging prospective teachers in various mathematical learning experiences throughout K-8 content courses. These 15 purposes may help teacher educators of all experience levels to design, plan, and teach courses for prospective teachers. The list of purposes may not be exhaustive and may vary across different settings based on the experiences of teacher educators; however, these empirical data provide a foundation on which other teacher educators may build their practice.

With this study, we join others (e.g., Superfine & Li, 2014; Taylor, 2013) in providing new insights into the knowledge and purposes that teacher educators draw on to enrich the learning experiences of prospective teachers. Ultimately, this study serves as a window for engaging teacher educators in professional conversations about specific purposes embedded in the teaching of content courses to explore further questions: (a) What PCK connections are critical to make in the content courses? (b) What SMK connections are essential to address in content courses for prospective teachers? and (c) How do we help university faculty (especially the non-educators) in making these PCK and SMK connections in the content courses?

References

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education, 7*(2), 145-172. doi: 10.1023/B:JMTE.0000021943.35739.1c

Ball, D. L.& Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport, CT: Ablex.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-407.

Blömeke, S., Suhl, U., & Kaiser, G. (2014). Teacher Education Effectiveness: Quality and Equity of Future Primary Teachers' Mathematics and Mathematics Pedagogical Content Knowledge. In *International Perspectives on Teacher Knowledge, Beliefs and Opportunities to Learn* (pp. 91-114). Springer Netherlands.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research*. Los Angeles, CA: Sage Publications.

Even, R. (2008). Facing the challenge of educating educators to work with practising mathematics teachers. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 4, pp. 57-73). Rotterdam, The Netherlands: Sense Publishers.

Goodell, J. (2006). Using critical incident reflections: a self-study as a mathematics teacher educator. *Journal of Mathematics Teacher Education, 9*(3), 221-248. doi: 10.1007/s10857-006-9001-0

Greenberg, J., & Walsh, K. (2008). *No common denominator: The preparation of elementary teachers in mathematics by America's education schools*. Washington, DC: National Council on Teacher Quality.

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.

Magnusson, S., Krajcik, J., & Borko, H. (1999). Nature, sources, and development of pedagogical content knowledge for science teaching. In J. Gess-Newsome & N. G. Lederman (Eds.), *Examining Pedagogical Content Knowledge* (pp. 95-132). Dordrecht, The Netherlands: Kluwer Academic Publishers.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Stake, R. E. (2005). Qualitative case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *The sage handbook of qualitative research* (3rd ed., pp. 443-466). London: Sage Publications.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.doi: 10.3102/0013189X015002004

Superfine, A. C. & Li, W. (2014). Exploring the mathematical knowledge needed for teaching teachers. *Journal of Teacher Education, 65* (4), 202-214. doi: 10.1177/0022487114534265

Taylor, C. E. (2013). Facilitating prospective teachers' knowledge of student understanding: The case of one mathematics teacher educator. In Lindmeier, A. M. & Heinze, A. (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 273-280. Kiel, Germany: PME.

The National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common Core State Standards [Mathematics]*. National Governors Association Center for Best Practices, Council of Chief State School Officers, Washington D.C.

Tzur, R. (2001). Becoming a mathematics teacher-educator: Conceptualising the terrain through self-reflective analysis. *Journal of Mathematics Teacher Education*, *4*(4), 259-283. doi: 10.1023/A:1013314009952

WHICH CONTINUATION IS APPROPRIATE? KINDERGARTEN CHILDREN'S KNOWLEDGE OF REPEATING PATTERNS

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This study explores kindergarten children's ability to identify possible continuations of repeating patterns. Children were presented with two ABB patterns, one which ended with a complete unit of repeat and one which ended with a partial unit. Children were then shown, for each pattern, four possible continuations of the pattern, two appropriate continuations and two inappropriate continuations. Results indicated that more children were able to continue the pattern which ended with a complete unit than the pattern ending with a partial unit. The role of the unit of repeat in children's performances is discussed.

INTRODUCTION AND BACKGROUND

Mathematics has been described as the "science of patterns with theory built on relations among patterns and on applications derived from the fit between pattern and observation" (Steen, 1988, p. 611). The importance of engaging young children with pattern activities is supported by mathematicians, mathematics education researchers, and curriculum developers (Sarama & Clements, 2009). To begin with, pattern exploration and recognition may support children as they learn a variety of mathematical skills developed at this age. For example, recognising repeating patterns may help children develop skip counting, such as 5, 10, 15, 20, 25, 30 ... where the ones digit forms the pattern 5, 0, 5, 0, … Recognition and analysis of patterns are important components of young children's' intellectual development as they provide a foundation for the development of algebraic thinking and provide children with the opportunity to observe and verbalise generalisations as well as to record them symbolically (Threlfall, 1999). While there are several types of patterns, this study focuses on repeating patterns and preschool children's ability to identify possible appropriate continuations of a repeating pattern.

Repeating patterns are patterns with a cyclical repetition of an identifiable 'unit of repeat'. For example, a pattern of the form ABBABBABB… has a (minimal) unit of repeat of length three. Lüken, Peter-Koop, & Kollhoff (2014) found that preschool children's repeating pattern abilities have an influence on their mathematical competencies at the end of first grade. Warren and Cooper (2007) suggested that repeating patterns may be used as a stepping stone for learning the concept of ratio. Describing how two patterns, such as "red, red, blue, red, red, blue" and "step, step, clap, step, step, clap" are the same and how they are different, could help children focus on underlying structures and introduce children to the powers of algebra (Rivera, 2013). The cyclical nature of repeating patterns also sets the stage for investigating

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 249-256. Hobart, Australia: PME.

oscillating patterns in other mathematical contexts, such as repeating decimals and rational numbers, and trigonometric models (Rivera, 2013).

Seo and Ginsburg (2004) found that during natural play, such as block activities, young children engage in pattern activities such as building block towers with an ABAB pattern. However, while most children by the end of kindergarten will be able to copy a repeating colour pattern, few will be able to extend or explain it (Clarke & Clarke, 2004). If they can copy a pattern, does it mean that they recognise the structure of the pattern? Papic, Mulligan, & Mitchelmore (2011) found that some preschool children may be able to draw an ABABAB pattern from memory by recalling the pattern as single alternating colours of red, blue, red, blue, basically recalling that after red came blue and after blue came red. However, when shown a more complicated pattern such as ABBC, they could not replicate the pattern. Rittle-Johnson, Fyfe, McLean, & McEldoon (2013) found that when young children were asked to duplicate or extend an ABB pattern, some children could not produce more than one unit of repeat correctly while others reverted to producing an ABAB pattern.

In the above mentioned studies, children were only presented with repeating linear patterns which ended in a complete unit. However, repeating patterns, such as repeating decimals, do not always present themselves by ending in a complete unit. When dividing one by seven on a calculator, students might receive a solution of 0.142857142857142. Students need to recognise the pattern and surmise that after the two comes an eight, etc. Another occasion of repeating patterns which do not always end with a complete unit may be seen in modulo n problems. For example, if the first car in a row is yellow, the second, blue, and the third red, and this repeats consistently, then what colour is the 26th car? Zazkis and Liljedahl (2006) suggested that experience with repeating patterns and recognising the unit of repeat will assist children in solving such problems. In the above problem, the length of the unit of repeat is three, there are eight complete units and two extra cars in a row of 26 cars and thus, the 26th car is the second element in the unit of repeat, which is blue.

This study explores kindergarten children's ability to identify appropriate continuations for repeating patterns that do not end in a complete unit. Our first question is: Is there a difference between children's ability to extend a pattern which ends with a complete unit of repeat and their ability to extend a pattern which ends with a partial unit? Previous studies requested children to extend a given repeating pattern by adding one element at a time. However, requesting children to extend a pattern, one element at a time, may reinforce a recursive approach, which in turn may hinder children's successful generalisation (Orton & Orton, 1999). This study investigates whether children are able to look ahead and consider three or four elements at a time. Our second question is: Given different possibilities of extending a pattern by adding three or four elements at a time, are children able to choose an appropriate extension? Related to the second question we ask a third question: When extending a pattern, do children tend to extend the pattern so that it ends with a complete unit or do they accept that the pattern may end with a partial unit?

METHODOLOGY

According to the mandatory Israel National Preschool Mathematics Curriculum (INPMC, 2008) by the end of kindergarten, children should be able to identify create, extend, describe patterns. That being said, at the time of the study, the curriculum was still fairly new and preschool teachers were only just becoming familiar with the standards. Informal interviews with some preschool teachers revealed that most of the patterning activities taking place in the kindergartens consisted of children drawing borders or frames for pictures, albeit borders which were made up of repeating patterns. Few activities aimed to develop children's appreciation for pattern structure or for the unit of repeat in a repeating pattern.

Participants in this study were 156 kindergarten children between the ages of 5-6 years. All of the kindergartens were located in middle to low socioeconomic neighbourhoods in urban locations. A researcher sat with each child individually in a quiet corner of the kindergarten and recorded all verbal utterances and gestures.

The tools of this study were two pictorial linear repeating patterns, each presented on a strip of paper, each with the same unit of repeat – ABB. This unit was chosen so as not to be too simple, and so that children would not merely attach themselves to alternating pictures, as was found in Papic, et al.'s study (2011). Pattern One (see Figure 1) included three instances of the repeating unit (ABBABBABB), ending with a complete unit. Pattern Two (see Figure 2) included three instances of the repeating unit and in addition the first two elements of the unit of repeat (ABBABBABBAB). In other words, Pattern Two ended with a partial unit.

The activity began by placing Pattern One on the table and then placing the four continuations also on the table and asking the child: Are there any continuations which are appropriate to place here? The researcher demonstrated the meaning of her question by placing each continuation, one at a time, at the end of the pattern on the blank space, and saying each time: Is this appropriate? Is this appropriate? Is this appropriate? Is this appropriate? She then placed the four continuations to the side of the pattern and let the child choose. After the child chose a continuation, the researcher took it off the table and asked if there was another appropriate continuation. This was repeated until the child replied that no more continuations were appropriate. The same procedure was followed for Pattern Two.

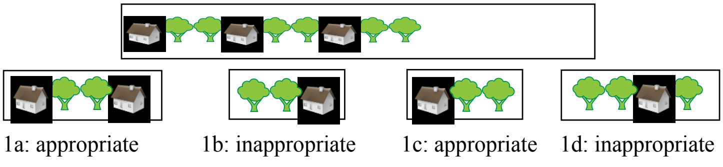
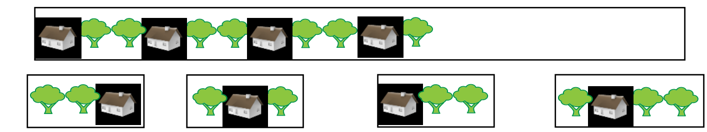


Fig. 1: Pattern One and four continuations shown to the children

Regarding the continuations for Pattern One, both 1a and 1c are appropriate ways of continuing the pattern. Both continuations begin in the same way as the pattern begins, which might encourage children to choose these continuations. However, continuation 1c is basically the unit of repeat and so choosing 1c will end the pattern in a complete unit. On the other hand, choosing continuation 1a will end the pattern in a partial unit. Regarding inappropriate continuations, both 1b and 1d are inappropriate. Continuation 1b is a reflection of the unit of repeat, while continuation 1d may be considered as the beginning of the reflection of the presented pattern. Guez-Sandler (2010) found that when asked to extend patterns, some children reflect the given pattern instead. Regarding the continuations for Pattern Two, both 2b and 2d are appropriate ways of continuing the pattern. Choosing continuation 2d will continue the pattern so that it ends in a complete unit while choosing continuation 2b will continue the pattern but will end it with a partial unit. Continuations 2a and 2c are both inappropriate choices. Continuation 2c is exactly the unit of repeat while continuation 2a is the reflection of the unit of repeat. We were interested in exploring if children would choose continuation 2c, perhaps recognizing it as the unit of repeat, perhaps looking for a continuation which begins in the same manner as the pattern, but disregarding that it was inappropriate for the continuation of a pattern which ended with a partial unit.



2a: inappropriate 2b: appropriate 2c: inappropriate 2d: appropriate

Fig. 2: Pattern Two and four continuations shown to the children

RESULTS

Of the 156 children who participated in this study, 20 children, for each of the patterns presented, chose all four continuations as appropriate for the pattern. These children were excluded from the final analysis because it was thought that they might have merely chosen each continuation until all the possibilities ran out. One of those children actually took each possible continuation in hand and placed it under the presented pattern in such a way so that it always matched up to some part of the presented pattern. He did not seem to understand, despite the demonstration by the researcher, that the continuation was supposed to be placed in the blank space at the end of the strip. The following sections describe the results for the rest of the participants, first for each pattern separately, including some statements made by children as they worked on the activity, and then for the two patterns together.

Pattern One

From Table 1 we see that nearly two-thirds or more of the children succeeded in recognising which continuations were appropriate and which continuations were not appropriate. We also note that there was no difference between children's ability to correctly choose appropriate continuations and their ability to correctly not choose inappropriate continuations. Thus, the task of choosing appropriate continuations was neither simpler nor more difficult than not choosing inappropriate continuations.

Continuation	1a appropriate	1b inappropriate	1c appropriate	1d inappropriate
Frequency (%) of correct responses	65	65	79	76

Table 1: Frequency of correct responses to Pattern One (N=136)

A paired-samples t-test was used to investigate any difference between the children's reactions to the two appropriate continuations and if there was a difference between the children's reactions to the two inappropriate continuations. Results indicated that significantly more children correctly chose continuation 1c, the unit of repeat which would end the pattern in a complete unit (M=.79, SD=.41) than those who chose correctly continuation 1a, which would end the pattern with a partial unit (M=.65, SD=.48), t(135)=2.78, p<.01). A significant difference was also found between correctly not choosing continuation 1d (M=.76, SD=.43) and correctly not choosing continuation 1b (M=.65, SD=.48), t(135)=2.59, p=.01), i.e., more children incorrectly chose the reflection of the unit of repeat than the reflection of the presented pattern.

Pattern Two

Results for each of the continuations in Pattern Two are summed up in Table 2. Approximately 60% of the children responded correctly to each continuation. As with Pattern One, the task of choosing appropriate continuations was neither simpler nor more difficult than the task of not choosing inappropriate continuations.

Continuation	2a inappropriate	2b appropriate	2c inappropriate	2d appropriate
Frequency (%) of correct responses	69	59	57	61

Table 2: Frequency of correct responses to Pattern Two (N=136)

Paired-samples t-tests were carried out to compare children's choices of continuations. Results indicated that no significant difference was found between choosing the appropriate continuations of 2b and 2d. However, significantly more children correctly did not choose continuation 2a (M=.69, SD=.46) than continuation 2c (the unit of repeat) (M=.57, SD=.50), t(135)=2.17, p<.05).

Children's gestures and utterances during the activity

When deciding whether or not to choose some continuation, some children merely seemed to guess, while others had some strategy. One strategy was to physically move each continuation to the end, trying it out before deciding whether or not it was appropriate. Another strategy aligned each continuation with the pattern's beginning to see if it matched. One child chose continuations based on the last element of the pattern. Since both patterns ended with a tree, he claimed that all of the continuations which began with a tree were appropriate, disregarding the aspect of the pattern.

Some children's verbal utterances hinted at their recognition of a unit of repeat while others merely reflected what they were looking at. For example, C8 would place a continuation at the end of the pattern and then read out loud each element of the pattern from the beginning. On the other hand, when C88 correctly chose 1a for Pattern One, he said, "Because it has two trees and here is a house (pointing to the house.) Similarly, C28 correctly did not choose 2a for Pattern Two and said, "Because there are two trees here." Both of these children hinted at the unit of repeat by noting two trees and a house and did not merely say "tree, tree, house." Likewise, C5 correctly did not choose continuation 1b for Pattern One and said, "Here, there are four trees." She noticed that if 1b would be placed as the continuation, the pattern would not continue because instead of there being two trees, there would now be four trees. A few children specifically alluded to the completion or the incompletion of a unit. For Pattern One, C106 correctly chose 1a and said, "… and you need to continue (with) tree, tree." In other words, while she did choose this appropriate continuation, it seemed to bother her that the unit was not complete and so she stated that it should be continued more by adding another two trees. Likewise, for Pattern Two, C108 did correctly choose continuation 2b but added, "If there wasn't a tree here at the end, it would be best." On the other hand, C102 incorrectly did not choose continuation 2b and said, "It is not appropriate because you are left with an extra tree at the end." It seems to bother him that the pattern will not end in a complete unit. When he correctly chose 2d, he said, "This is appropriate. It is house, tree, tree."

Comparing the two patterns

The difference between the results of the two patterns may be seen in Table 3, which presents the frequencies of those who chose all appropriate continuations as well as correctly did not choose inappropriate continuations. As can be seen from Table 3, few children responded correctly to all of the possible continuations for both of the patterns. For each child, we configured a grade based on the number of correct choices made (0, 1, 2, 3, or 4) and a mean score was configured for the group per pattern. Results of a paired-samples t-test found that children scored significantly higher ($M= 2.96$, $SD=1.20$) on Pattern One than they did for Pattern Two ($M=2.46$, 1.17), $t(135)=2.82$, $p<.01$). In other words, children had greater success extending the pattern which ended with a complete unit, than for the pattern which ended in a partial unit.

	Pattern One	Pattern Two	Patterns One and Two
Frequency (%) of perfect scores	41	22	15

Table 3: Frequency (%) of perfect scores per pattern (N=136)

SUMMARY AND DISCUSSION

This study focused on children's ability to identify appropriate extensions for two ABB patterns, one that ended with a complete unit and one that did not. Results indicated that more children made correct choices regarding all of the possible continuations for Pattern One than for Pattern Two and the average score was greater for Pattern One than for Pattern Two. In other words, in answer to our first research question, being presented a pattern which ends with a complete unit, as opposed to ending with a partial unit, impacts on children's choices of appropriate continuations.

In regards to the second research question, children's success at choosing appropriate continuations of more than one element was dependent on the type of continuation. Regarding appropriate continuations, for Pattern One, more children accepted the continuation which ended the pattern with a complete unit than the continuation which ended the pattern with a partial unit. This seems to support the first finding in that it shows children's preference for patterns that end with a complete unit. However, we also note, that this way of extending the pattern, basically consisted of choosing the unit of repeat. It is possible that some children chose this continuation because they recognised it as the unit of repeat, which is commendable, but does not always allow for correctly extending the pattern. In fact, choosing the unit of repeat as a continuation for the second pattern was incorrect. Regarding the second pattern, no difference was found between children's choices of appropriate continuations. When the pattern was not presented ending with a complete unit, children did not necessarily choose the continuation that would end it with a complete unit, despite children's statements which show a preference for ending a pattern with a complete unit. Thus, in answer to the third question, perhaps if children are presented with a pattern which ends with a partial unit, they more readily accept extending it so that it will still end with a partial unit.

Regarding inappropriate continuations, on the first pattern, many children incorrectly chose the reflection of the unit of repeat. On the second pattern, many children incorrectly chose the unit of repeat. These results hint at the centricity of the unit of repeat. It could be that children recognised the unit of repeat, in its presented form or in its reflected form, and automatically chose it, without discerning if it was an appropriate way to extend the pattern in the specific case. While we acknowledge, as do other researchers (e.g., Zazkis & Liljedahl, 2006), the significance of the unit of repeat in supporting children's knowledge of structure, these results suggest that this recognition might not be enough. In addition to the unit of repeat, children need to recognise the sequencing aspect of the pattern and how to continue a pattern from any point. Educators agree that patterning is an essential activity for young children, one that can help children seek out structure and generalisations. Part of recognising the structure may be recognising not only the basic elements of the unit of repeat, but also where in the unit of repeat a pattern left off and how to continue the pattern, not only one element at a time as in recursive reasoning (Orton & Orton, 1999) but several elements at time. This study adds to existing knowledge by showing how the dominance of the unit of repeat, in conjunction with a tendency to end patterns with a complete unit, may actually hinder children's ability to see the sequencing and general structure of a pattern. Activities such as those presented here may help promote children's recognition of the unit of repeat, the overall structure of the pattern, as well as an appreciation for recognising location within the structure.

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References

Clarke, D. M., & Clarke, B. A. (2004). Mathematics teaching in Grades K-2: Painting a picture of challenging, supportive, and effective classrooms. In R. N. Rubenstein & G. W. Bright (Eds.), *Perspectives on the teaching of mathematics* (pp. 67-81). Reston, VA: NCTM.

Guez-Sandler, L. (2010). *Identifying repeating patterns at preschool*. Unpublished master's thesis, Tel Aviv University, Tel Aviv, Israel.

Israel national mathematics preschool curriculum (INMPC) (2008). Retrieved from http://meyda.education.gov.il/files/Tochniyot_Limudim/KdamYesodi/Math1.pdf

Lüken, M., Peter-Koop, A., & Kollhoff, S. (2014). Influence of early repeating patterning ability on school mathematics learning. In Liljedahl, P., Nicol, C., Oesterle, S., & Allan, D. (Eds.). (2014). *Proc. PME 38/PME-NA 36* (Vol. 4, pp. 137-144). Vancouver: PME.

Orton, A. and Orton, J. (1999). Pattern and the approach to algebra. In A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp. 104–120). London.

Papic, M., Mulligan, J., & Mitchelmore, M. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education*, *42*(3), 237-269.

Rittle-Johnson, B., Fyfe, E. R., McLean, L. E., & McEldoon, K. L. (2013). Emerging understanding of patterning in 4-year-olds. *Journal of Cognition and Development*, *14*(3), 376-396.

Rivera, F. (2013). *Teaching and learning patterns in school mathematics: Psychological and pedagogical considerations*. Dordrecht: Springer.

Sarama, J. & Clements, D. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. London, England: Routledge.

Seo, K.H., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. In D. H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 91–104). Hillsdale, NJ: Erlbaum.

Steen, L. A. (1988). The science of patterns. *Science*, 24, 611-616.

Threlfall, J. (1999). Repeating patterns in the primary years. In A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp. 18–30). London, England: Cassell.

Zazkis, R., & Liljedahl, P. (2006). On the path to number theory: Repeating patterns as a gateway. In R. Zazkis and S. R. Campbell (Eds.). *Number Theory in mathematics education: Perspectives and prospects* (pp. 99-114). New Jersey: Erlbaum.

THIRD-ORDER VIABILITY IN RADICAL CONSTRUCTIVISM

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In this paper, we will address the methodological problem of extending second-order models in radical constructivism. As a solution, we propose to convert second-order models to third-order viable first-order models. This conversion consists of identifying what information the students could not precisely access, in the case that their behaviors were the most rational in the situation. Because of this conversion, any converted model is expected to be viable, not only for the observer (first-order viable) and for the observed subject (second-order viable), but also for other persons (third-order viable). We will discuss the educational implications.

INTRODUCTION

Radical constructivism (RC) is a philosophy of knowing which assumes:

> [1-a] Knowledge is not passively received, either through the senses, or by way of communication; [1-b] knowledge is actively built up by the cognizing subject. [2-a] The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability; [2-b] cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality. (von Glasersfeld, 1995a, p. 51; Numbering added for citation)

One of the recent contributions of RC to mathematics education is the study of how *second-order models* are developed, and what potential impact RC may have on practice (Ulrich, Tillema, Hackenberg, & Norton, 2014). A second-order model is a model of a particular student's thinking processes, used to explain the observer's experience (Steffe & Thompson, 2000, p. 205). The reflective use of second-order models can provide strong guidance for teachers and researchers (Thompson, 2000, pp. 303–304). However, according to Sánchez Gómez's (2014) comment on Ulrich et al. (2014), the validity of the extension of second-order models for a particular student to new students is not methodologically warranted.

Although Tillema, Hackenberg, Ulrich, and Norton (2014) claimed that Sánchez Gómez's interpretation was "different from [Tillema et al.'s] understanding of the purpose of creating second-order models and the nature of these models" (p. 355), this does not seem to be a valid counterargument against Sánchez Gómez (2014) from the RC perspective itself. Following the RC principle [2-a] cited above, any interpretation should be viable for the interpreter. RC should not be able to claim that Sánchez Gómez misinterprets. In this paper, we will address the methodological problem of extending second-order models in RC (the extension problem). For this, we will start with a review of the nature of knowing in RC.

NATURE OF KNOWING IN RC

The concept of *viability* is the most important concept in this paper. For students, the condition that *pieces of knowledge reflect the absolute truth* is neither necessary nor sufficient for their use. Rather, students seem to use them if they are *viable*; that is, "if they fit the purposive or descriptive contexts in which [the students] use them" (von Glasersfeld, 1995a, p. 14). RC is "uninhibitedly instrumentalist" (p. 22). The term *viable* is not a synonym for the terms *true* or *valid*. The observation that a particular piece of knowledge is viable for the subject does not mean that the person has a particular justified belief. Rather, it means that in a particular situation, the subject has the *disposition* to *make a decision* to use a particular cognitive tool.

This instrumentalist view of knowledge becomes more clear with the concept of *knowledge-how* in the sense of Ryle (1949). For example, the reason that one can speak logically is not that one can recall the rules of inference and apply them, it is because one implicitly knows how to speak in such a way. Such implicit knowledge is described as *knowledge-how*, while a propositional knowledge is described as *knowledge-that*. With this terminology, we can say that RC does not acknowledge any piece of *knowledge-that* because we cannot have access to the absolute truth. Rather, RC only acknowledges *knowledge-how*, and regards any type of knowledge (e.g., ideas, strategies, cognitive structures, or models) as *knowledge-how*.

It is noteworthy that even cognition like "seeing ... as ..." or "recognizing ... as ..." is treated as *knowledge-how*. For example, suppose that a subject uses a stone to drive a nail into a wall because s/he cannot immediately access a hammer (cf. von Glasersfeld, 1995b, p. 374). Let *S* be the subject. Seeing a stone as a hammer is *S*'s *knowledge-how*. The reason that *S* saw the stone as a hammer is not that *S* volitionally decided to see a stone as a hammer, and so actually saw the stone as a hammer. It is because *S* implicitly knew how to see the stone as a hammer, for example, how to decide which parts of the stone would correspond to the face, or the grip of a hammer. If *S* knew only how to see a small and hard substance as a stone, the stone would be only a stone for *S*.

S cannot arbitrarily construct any *knowledge-how* which *S* wants, because the environment constrains the viability of *S*'s *knowledge-how* (von Glasersfeld, 1990, p. 24). However, note that *S* can arbitrarily construct any *knowledge-how* as long as the constraints are not violated. Whatever *S* learns from the fact, is what *S* selectively and hypothetically constructs. In the above example, the expectation that the stone can be used as a hammer is an ill-grounded hypothetical construct. Generally speaking, *S* actively uses, not only justified knowledge, but also hypothetical knowledge when trying to achieve a particular goal. In this paper, we will call this characteristic of knowledge use as *the hypothetical nature*.

In summary, any *knowledge-how* construction and use are valid for *S*. In this sense, even a young, uneducated child is regarded as a mini scientist (or a mini mathematician). This view has shed light on the nature of children's construction of knowledge. However, it diminishes the distinction between naïve, and sophisticated, knowledge construction. Especially within the context of second-order models, any methodological critique of the use of second-order models becomes invalid, because any temporal knowledge construction is scientifically valid. In the next section, we will address this problem.

REAL PROBLEMS IN EXTENDING SECOND-ORDER MODELS

The purpose of building a second-order model is "to organize his or her experience in a way that helps him or her effectively interact with multiple students at different stages of reasoning, often at the same time" (Tillema et al., 2014, p. 356). That is, building and using second-order models is the observer's *knowledge-how*.

Let us take an example of extending second-order models from Ulrich et al. (2014). They used the models of *two composite units* (two units of units) and *only single composite unit* (one unit of units) to explain responses from a sixth grade student (Charice). Two problems were given to her for promoting a meaning of powers.

> *The Two-Suit Card Problem*: You have the Ace through King of hearts (13 cards). Your friend has the ace through King of spades (13 cards). You and your friend make two-card hands by drawing a card from your hand, then drawing a card from your friend's hand, and putting them together. Use an array to show how many different two-card hands you could make.
>
> *The Password Problem*: students are creating two-character passwords for their computer account at school (e.g., "FD" is an example password). They can choose from the characters A through N to create the password. How many two-character passwords are possible (Assume "FD" and "DF" count as different passwords)? (p. 333)

The teacher/researcher expected Charice to solve each problem with two composite units. The two sets of 13 hearts and 13 spades are regarded as two units of units, because we must choose one from each of them in the Two-Suit Card Problem. The two sets of 14 characters are regarded as two units of units because we must choose one from each of them in the Password Problem. For the first problem, the teacher gave Charice all of the hearts in a deck of cards, and for the second problem, the teacher presented Charice with 14 cards on which one of the letters A through N was printed. Although Charice easily solved the first problem, she could not solve the second problem, and expressed that there is no number that is multiplied by 14. Because she seems to make passwords by choosing from a single set of 14 characters, her thinking is constrained by the model of only single composite unit (pp. 333–334).

This extension of the second-order model is valid due to the hypothetical nature of knowledge use. It is, in fact, hypothetical, but reasonable and promising. Although Tillema et al. (2014) claimed that Sánchez Gómez's (2014) interpretation was different from theirs, we can now properly understand both Sánchez Gómez's and Tillema et al.'s interpretations. The former viewed the hypothetical nature of the extension as a methodological problem, while the latter accepted the risk of the potentially invalid extension for possible future benefit.

"A drowning man will clutch at a straw." That is, when a person must make a decision without enough justified knowledge, s/he tends to use any knowledge, even ill-grounded knowledge, in order to make the decision. Using ill-grounded knowledge and risking biased decisions is not always irrational, because making no decision and taking no action may make the situation worse than making the wrong decision. In the case of extending the second-order model, as cited above, the purpose is to promote effective interactions with students. No matter how likely the extension of a model is to be invalid, it is more rational for teachers to extend it, and to interact with their students, than to make no decision and take no action.

Even if so, we cannot say that any methodological critique of extending second-order models is meaningless. Any extension of second-order models is idiosyncratically rational and valid for the extender himself or herself, while it is not always viable for others. Thus, as a methodological critique, we can ask the following question: *How likely is the second-order model to be second-order viable?* In RC, *first-order viability* is the viability of a piece of knowledge for the knowledge holder, while *second-order viability* is the viability of the piece of knowledge "not only in [the knowledge holder's] own sphere of actions but also in that of the other" (von Glasersfeld, 1995a, p. 120). Simply speaking, we can say that Ulrich et al.'s (2014) extension was not viable for Sánchez Gómez. This does not necessarily mean that he misunderstood Ulrich et al.'s intention to promote effective interactions with students. Rather, it means that he did not think that he would extend and use the second-order model in the same way if he were the teacher of Charice. For example, Ulrich et al. (2014) wrote that the teacher/researcher

> … asked [Charice] to elaborate on her observation, which opened the way for her to continue thinking about a solution to the problem. As she moved forward in her solution, she determined that she could pair A with each of the 13 other letters, then concluded that A could also be paired with itself so that A could be paired with 14 letters, and eventually that each of the 14 letters could be paired with 14 other letters. (p. 335)

The above quotation expresses only what decision the teacher actually made. It does not include the information on why she determined to teach in such a way. It is implicit from the reader's point of view how the extended second-order model works when the teacher made the decision. The proverb "a drowning man will clutch at a straw" is second-order viable because we share the implicit assumption that there is nothing but the straw around the man. We naturally think that we would clutch at a straw if we were drowning. On the other hand, the second-order model of only one composite unit does not necessarily have high second-order viability because we cannot assume that there are no different second-order models. Thus, in the next section, we will discuss how we can make the model to be second-order viable.

FROM SECOND-ORDER VIABILITY TO THIRD-ORDER VIABILITY

A possible reason that the second-order model of only one composite unit does not have high second-order viability is that it does not explain why some students think in such a way. Any second-order model is problematic for the same reason.

This problem is similar to Confrey's (1991) critique of using the label *misconception*:

> Labeling a student's model as a *misconception* fails to take in consideration the perspective

> of the student, for whom the belief may explain all instances under consideration, and fail only in cases to which s/he is not privy. [...] Finally, others have chosen more simply *conception*, which omits any indication that the perspective may deviate considerably from the expert's position. (p. 121)

Although the term *second-order model* does not have a modifier like *mis*, labeling a student's thinking as a second-order model is often equal to *misconception*. The second-order model has provided the distinction between correct and incorrect thinking. It has not provided the explanation of idiosyncratic rationality for students. Unless we identify the students' idiosyncratic reason that they think with only single composite unit, we still implicitly keep the label *mis*.

For RC, it is important to explain idiosyncratic rationality. Based on the RC principle [2-b], all decisions are idiosyncratically rational. The fact that a person made a particular decision means that there was at least a moment when s/he thought that it was the most rational decision, even if it is later understood to be irrational, based on new information. Since human beings have only a limited capacity to deal with incoming information, we cannot deal with too much information at one time. We become, however, able to deal with a great deal of data at once if we acquire the ability to abstract and mathematise information. Thus, in mathematics education, we should assume that novices might not know what information is important to them, while focusing on that which is trivial; but the novices will always behave in the most rational way *from their own point of view*. Lacking the knowledge of what information is important is not necessarily careless; it is a result of overconcentration on other pieces of information. This characteristic of novices is referred to as *local rationality*. In contrast, experts' rationality, developed by dealing regularly with relatively large amounts of information, is referred to as *global rationality*.

Although the use of second-order models fails to explain the local rationality of students, there is one possible solution to this problem. It is to convert the already existing second-order models to *the observer's first-order models*. This would be achieved by identifying the information the students were not able to access, provided their behavior was otherwise rational, given the information they *did* have.

For example, in case of Charice, the teacher (i) presented the Password Problem to Charice, (ii) demonstrated a way of creating two-letter passwords with a set of 14 cards, and (iii) asked Charice if she could make a chart to solve for the total number of passwords. Then, (iv) Charice wrote down the list of 14 characters, and stopped solving (p. 333). In this case, Charice's response would be considered rational, even from our perspective, if step (i) did not exist. The teacher's question at step (iii) seems to shift Charice's interest from the Password Problem to the question itself. At this moment, she lost the need to solve the Password Problem, and suddenly needed to make a chart. According to the assumption of local rationality, Charice probably over concentrated on creating a chart. This situation is one in which the information presented at steps (i) and (ii) became inaccessible.

Suppose that we were she, and that we could not access the precise information presented at steps (i) and (ii). Then, for making a chart, we would have to recall how we had made similar charts. Although we had made them by choosing two units (e.g., hearts and spades in the Two-Suit Card Problem) until now, we could find only one unit (a set of 14 cards). We would not notice that we used only one set twice previously, because the information that we used a set twice was inaccessible because of our current assumption. As a result, we would be confused, because we could not make a chart in the same way as before. In this way, we find that we ourselves would also use only one composite unit, if important information suddenly became inaccessible.

There are two advantages to the above conversion. First, the converted model enables the teacher to empathise with the students. The model of only one composite unit is converted from a second-order model for explaining the students' behavior, to a first-order model for explaining the observer's virtual experience. While second-order models are only first-order viable, the converted models are not only first-order viable, but also second-order viable for the observer, because it is viable not only for the observer, but also for the students. Because of this second-order viability, it is easier for the observer to understand the students' thinking with the converted models, than with the corresponding second-order models.

If we understand the local rationality of the students, the question of why there are such students is easily answered. The reason that there are students modeled by the model, is that some teachers' behavior unintentionally causes them to lose focus on the important information. For example, in case of Charice, the reason that she used only one composite unit is that the teacher's question at the step (iii) unintentionally caused her to lose the focus on the information in steps (i) and (ii). Although, of course, there is the possibility that the student is careless, attributing the cause of the student's behavior to the teacher's behavior makes it easy for the teacher to empathise with the student, and to consider what to do next.

The second advantage is that the converted model is expected to be not only second-order viable for the observer, but also second-order viable for the third person, like the readers of research papers. For example, although the second-order model of only one composite model does not seem to be viable for Sánchez Gómez, the converted model is viable, even for him, because it provides him with a method to empathise with the student. If it is still not viable for him, the reason is not that the converted model itself lacks viability, but that he cannot accept the assumption of local rationality. The conversion includes the process of explaining novices' local rationality so that even experts can understand it. Thus, as long as the nature of local rationality is assumed, any converted model is expected to be viable not only for the observer (i. e., the first person "I") and for the observed subject (i. e., the second person "you"), but also for other persons (i.e., the third persons "they"; e.g., the readers of the research papers). Second-order viability is stronger than first-order, and this new viability is stronger than second-order. Therefore, we will call it *third-order viability*.

Third-order viability is the key concept for solving the extension problem. The first-order viable second-order models are retrospectively built after some observation. Since it strictly depends upon observation, the models are fragile. Since they are not related to any other information, we are constrained to use them without any supplemental information. On the other hand, the third-order viable first-order models are assimilated into the observer's existing knowledge when they are converted from the corresponding second-order models. That is, much of the observer's past experience will support using the third-order viable first-order models. Although it is never safe, in the sense that they are only approximate models of absolute reality, it is useful in that the observer can use them in accordance with his or her own empirically, well-tested, viable, existing knowledge. In the next section, we will discuss how to use third-order viable first-order models as educational tools.

EDUCATIONAL IMPLICATIONS AND CONCLUSION

Before discussing the implications of using third-order viable first-order models for education, note that we do not intend to criticise Charice's teacher in the discussion below. According to the assumption of local rationality, we believe that the teacher's real-time practice was done to the best of her ability and understanding, based on her experience. We do not believe that the teacher should have done anything differently. Here, we will discuss what we could do in similar circumstances as Charice's teacher.

Even if we discover the cause of the students' behavior, we must keep in mind that eliminating that cause is not always the best way to improve the lesson. For example, the cause of Charice's behavior seemed to be the teacher's question as to whether Charice could make a chart. However, if the teacher presented only the Password Problem itself, and provided no support to solve it, then Charice could not know what to do. Since ancient times, it has been well known that introducing sub-questions in assignments is one of the most effective ways of supporting students. To cease introducing sub-questions would be ineffective.

Let us elucidate the model of Charice's thinking: From the hypothesis that she lost the need to solve the Password Problem because of the requirement to make a chart, it is deduced that she was not ready to make a chart. In fact, no one can *a priori* determine what a given problem will require one to do. It is determined after solving the problem. Thus, generally speaking, a student needs to notice, by himself or herself, that making a chart is a useful solution for this problem.

Keeping in mind the above, we can provide a useful approach to teaching the Password Problem in the future. A possible situation in which a student notices the usefulness of creating a chart is one in which s/he must make new passwords one after another. For example, suppose that (i) students engage in a game; (ii) it requires them to make new passwords by turns; and (iii) one wins the game by making more passwords than the other students make. In the game, the students may randomly create passwords, but gradually they will realise that it becomes more difficult to create new passwords according to the rules the longer the game lasts. Then, they will realise that a system for generating these passwords is required. The need to make a chart will arise. If they make many passwords by themselves, we can also expect them to notice on their own that the pile should be used twice. In this case, the role of the teacher will not include prompting them to make a chart. Rather, the teacher would (i) find the first student who makes a chart, (ii) share the information that that particular student is creating and using a chart, and (iii) encourage the students to consider what kind of chart would be the best for winning the game. This approach would be expected to help the students to understand the usefulness of tables as a preliminary step towards understanding powers.

In this paper, as a solution of the methodological problem of extending second-order models, we proposed to convert second-order models to third-order viable first-order models. However, the paper does not provide a general strategy for converting second-order models. The method of conversion still depends on each second-order model. Developing a practical strategy is an issue to be addressed in the future.

References

Confrey, J. (1991). Learning to listen: A student's understanding of powers of ten. In E. von Glasersfeld (Ed.), Radical constructivism in mathematics education (pp. 111–138). The Netherlands: Springer.

Ryle, G. (1949). The concept of mind. University of Chicago Press.

Sánchez Gómez, P. J. (2014). Methodological issues of second-order model building. *Constructivist Foundations, 9*(3), 344–346.

Steffe, L. P., & Thompson, P. W. (2000). Interaction or intersubjectivity? A reply to Lerman. *Journal for Research in Mathematics Education, 31*(2), 191–209.

Thompson, P. W. (2000). Radical Constructivism: Reflections and Directions. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism in action: Building on the pioneering work of Ernst von Glasersfeld*, (pp. 291–315). London: Routledge.

Tillema, E. S., Hackenberg, A. J., Ulrich, C., & Norton, A. (2014). Authors' response: Interaction: A core hypothesis of radical constructivist epistemology. *Constructivist Foundations, 9*(3), 354–359.

Ulrich, C., Tillema, E. S., Hackenberg, A. J., & Norton, A. (2014). Constructivist model building: Empirical examples from mathematics education. *Constructivist Foundations, 9*(3), 328–339.

Von Glasersfeld, E. (1990). An exposition of constructivism: Why some like it radical. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Monographs of the Journal for Research in Mathematics Education, #4*, (pp. 19-29). Reston, VA: NCTM.

Von Glasersfeld, E. (1995a). *Radical constructivism: A way of knowing and learning*. London: The Flamer Press.

Von Glasersfeld, E. (1995b). Sensory experience, abstraction, and teaching. In L. P. Steffe (Ed.), *Constructivisim in education* (pp. 369–383). Hillsdale, NJ: Lawrence Erlbaum.

THE ROLE OF STUDY MOTIVES AND LEARNING ACTIVITIES FOR SUCCESS IN FIRST SEMESTER MATHEMATICS STUDIES

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Research on the transition from school mathematics to university mathematics study has identified major challenges for students. Students' study prerequisites as well as their study behaviour are repeatedly mentioned when trying to explain the difficulties in this transition. We present a study with N=333 university mathematics students that analysed the relations between students cognitive and motivational study prerequisites, their learning behaviour, and study success in the first semester. We propose a specific conceptualization of students' motives to choose a mathematics programme. We find that specific motives play an important role in the activation of learning activities that go along with study success. Implications for research as well as for support of beginning university mathematics students are discussed.

PROBLEMS AT THE SECONDARY-TERTIARY TRANSITION

Problems of students entering university mathematics education have been reported for many years and in many countries (de Guzman, Hodgson, Robert, & Villani, 1998; Heublein, 2014). A substantial amount of students (up to 50%) drop out of university mathematics programmes, mostly at a very early stage (Heublein, 2014). There are numerous theoretical and explorative studies into reasons for these difficulties (e.g., de Guzman et al., 1998; Rach, 2014), which usually identify two central challenges explaining students' problems. Firstly, the nature subject *mathematics* is considered to change dramatically in the transition from school to university. This includes a shift in the main goal of mathematical activity from a focus of applying mathematics at school (Dörfler & McLone, 1986) towards building up a scientific theory at university. Moreover, the nature of the mathematical theory itself changes from a mostly "locally ordered" system of perception-based statements towards an axiomatic theory with definitions, axioms and a central role of deductive proofs (e.g., Tall, 2008) as well as formal-symbolic representations (e.g., Clark & Lovric, 2008). Secondly, the nature of mathematical learning opportunities changes. For mathematics teachers it is central to explicate also the genesis of mathematical rules and concepts and to support students in building up a *concept image* of new concepts. In contrast to that, university mathematics teaching often focuses on the product aspect of mathematical activity like definitions, theorems, and proofs (e.g., Clark & Lovric, 2008; Siebert, Rach, & Heinze, 2013).

Accordingly, it is repeatedly argued that beginning university students have to adapt their cognitive and meta-cognitive learning behaviour to the new learning goals and learning opportunities (e.g., de Guzman et al., 1998; Thomas & Klymchuk, 2012). In order to support students in this transition, it is thus vital to understand how their

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learning behaviour relates to their prerequisites at the beginning of university study and their success in the transition process.

THEORETICAL FRAMEWORK AND PRIOR RESEARCH

Learning prerequisites for university mathematics study

Beginning mathematics students vary considerably in several prerequisites that are considered to be necessary for a successful transition to university study (Eilerts, 2009; Rach, 2014). Final school qualification grades, a very general measure of (not exclusively cognitive) learning prerequisites, have been shown to be predictive for study success (e.g., Eilerts, 2009; Rach, 2014; Trapmann, Hell, Weigand, & Schuler, 2007). Apart from these very general indicators, sound understanding of mathematical concepts from the school context has been put forward as an important prerequisite (de Guzman et al., 1998) and was empirically shown to have a positive effect (Eilerts, 2009; Rach, 2014). From the affective-motivational side, interest and self-concept in mathematics have been proposed theoretically and studied empirically as learning prerequisites of university mathematics study, yet not yielding a conclusive pattern of positive effects. For example, interest in mathematics goes along with study success in Eilert's (2009) study, but not in Rach's (2014). Instead, Rach (2014) found that a positive self-concept in mathematics reduced the danger of drop-out during the first semester. Summarizing, there are some consistent results showing a positive effect of cognitive learning prerequisites on study success, while there is no clear pattern for student prerequisites like interest and self-concept. Moreover, motivational dispositions of students towards their studies are often mentioned as possible reasons for problems in the transition to university mathematics (e.g., de Guzman et al., 1998), but empirical evidence of such their effects is scarce at the best.

The role of study motives in university study

In self-determination theory (Deci & Ryan, 1985), motives are conceptualized as types of reward, which individuals value in an activity. Following this concept, *study choice motives* are defined here as the potential reward which students anticipate from choosing a specific study programme. These motives can be considered an important prerequisite of study success: Models of self-regulated learning (cf. Zimmerman, 2011) usually describe the role of such overarching motives as an important factor in individuals' choices of learning strategies and activities. It is very plausible that the motive for a student to enrol in a university mathematics program directs his/her learning activities towards specific learning opportunities and contents offered in the programme. In current research, such motives are usually conceptualized as intrinsic vs. extrinsic motives (Blüthman, Lepa, & Thiel, 2008) and it was shown that students' actual choice of study (e.g. Retelsdorf & Möller, 2012 for teacher programmes) reflects their self-reported study choice motives, that extrinsic motives increase the probability of programme drop-out, and that intrinsic motives go along with study satisfaction (Blüthmann et al., 2008).

What actually should count as an intrinsic vs. extrinsic motive to choose a mathematics programme is not easy to decide. If a student chooses a mathematics programme in order to learn how to solve complex everyday problems with mathematical tools, this might quite well be considered intrinsic to mathematics when viewed from the perspective of school mathematics (Dörfler & McLone, 1986). When taking the nature and content of university mathematics programmes into account, this can be doubted since this motive is clearly not well aligned with the content and learning opportunities of the programme. Departing from this reasoning, we focus four motives for studying mathematics: A motive of good *financial and professional perspectives*, a motive to *apply mathematics* to complex real world problems, a motive to *engage in mathematics problems*, and a motive to become acquainted with the methods and ideas of *scientific mathematics*. These motives can be perceived as increasingly more aligned with the content and learning opportunities in the first semester of a university mathematics programme. Following models of self-regulated learning, it can be expected that individual motives which are aligned with the nature of such a programme go along with more effective individual learning behaviour and higher study success.

Describing students' learning behaviour

A typical approach to describe students' learning behaviour in research on self-regulated learning is to survey students' learning strategies. Rach's (2014) summary of current research as well as her own study indicate that the resulting empirical evidence is not conclusive. In particular, general learning strategies surveyed by questionnaires were sometimes found to be unrelated to study success. Effects are usually only detected if questionnaires asked for very specific and concrete study behaviour. For example, Rach and Heinze (2012) found that students solving weekly exercise tasks or self-explaining others' solutions to these tasks were more successful in the first semester of a university mathematics programme than other students. Moreover, students who solved these tasks on their own had high self-concept and high mathematics knowledge at the start of the first semester university (Rach, 2014).

Chi (2009) proposes a different approach to describe learning behaviour: She introduced three kinds of so-called *learning activities*, which are described by observable student behaviour, but are explicitly related to specific cognitive processes: *Active learning activities* are described by primarily physical actions, which include mainly the activation or storing of existing knowledge. Constructive learning activities produce new information beyond the presented information. Interactive learning activities correspond to a substantive dialogue with a partner, building on and extending the partners' ideas. Chi's main assumption is that learning becomes increasingly effective from active over constructive to interactive learning activities. Solving the tasks on weekly exercise sheets, which are part of most German university mathematics programmes (Rach, 2014), is an example of a constructive learning activity.

GOALS OF THE STUDY AND RESEARCH QUESTIONS

The main goal of this study was to understand the relationship between students' cognitive and motivational learning prerequisites, their learning behaviour and their success in the first semester of mathematics study. More precisely, we addressed the following questions:

- Do students' self-reported study choice motives relate to their choice of a more or less application-oriented mathematics programme?
- How do students' affective and cognitive learning prerequisites relate to their learning activities? We expected, in particular, that more mathematics-related study motives should go along with more sustainable (constructive and interactive) learning activities.
- How do students' learning prerequisites and learning activities relate to their study success? Apart from a strong relation between cognitive learning prerequisites and study success, we expected also that more mathematics-related study motives and more sustainable learning activities should go along with higher study success. Moreover, we expected that less mathematics-related study motives would relate to lower study success.

Construct and example item	Cronbachs α
Study motive – professional perspectives (3 items) I chose math because I will earn much money.	.80
Study motive – application of mathematics (3 items) I chose math because I will learn to solve complex application problems.	.78
Study motive –engage in mathematics problems (3 items) I chose maths because I like to think about maths questions.	.77
Study motive –mathematics as a science (3 items) I chose maths because I want to learn about research in mathematics.	.66
Learning activities – active (4 items) When I do not know how to solve a task, I ask a teacher for advice.	.70
Learning activities – constructive (4 items) In lectures, I try to check if it makes sense to me what the lecturer says.	.52
Learning activities – interactive (4 items) I discuss my solutions to exercise tasks with other students.	.78

Table 1: Questionnaire instruments and reliability coefficients

DESIGN AND METHODS

We conducted a longitudinal survey study with N=333 (169 female) first semester mathematics students enrolled in a regular mathematics programme (N=94) or a financial mathematics programme (N=238) at the University of Munich, Germany. We do not focus on programme or gender differences in this contribution for reasons of space, but these variables were used as control variables.

During the first lecture of the semester, we conducted a pre-test on calculus knowledge (e.g., concepts of limit and derivatives, example item: *Prove that the function* $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ *has no derivative at* $x=0$, WLE-Reliability .54, Rach, 2014) and surveyed students' high school qualification grade (HSQG [values were recoded so that higher grades correspond to higher school achievement]) as well as their study motives. After four weeks, we surveyed students' self-reported learning activities. See table 1 for constructs, example items, and reliability coefficients. At the end of the term, we collected students' scores on the final exam of the analysis lecture. The same lecture was part of both study programmes. Moreover, as an additional measure of constructive learning activities, we scanned and counted all exercise sheets that were handed in by each student to receive written feedback. Datasets were linked using anonymous personal codes for each student. For regression analyses, we estimated missing data with the FIML algorithm.

RESULTS

To validate our measures we compared the study choice motives reported by students from the more application-oriented financial mathematics programme to those of the regular mathematics programme (table 2).

Motive	Perspectives	Application	Problems	Science
Regular math.	*1.44 (.84)*	*1.46 (.80)*	2.23 (.66)	2.43 (.45)
Financial math.	*1.92 (.73)*	*1.99 (.77)*	*1.67 (.68)*	2.07 (.52)
ANOVA	$p<.001$ $\eta^2=.07$	$p<.001$ $\eta^2=.09$	$p<.001$ $\eta^2=.12$	$p<.001$ $\eta^2=.09$

Table 2: Means, standard deviations, and ANOVAs for study choice motives of students in the regular mathematics and in the financial mathematics programme (scaling 0: strongly disagree, 1: disagree, 2: agree, 3: strongly agree)

As expected, students from the more application-oriented financial mathematics programme reported stronger motives related to *professional perspectives* and *application*, while students in the regular mathematics programme agreed more to motives of *engaging in mathematics problems* and *mathematics as a science*.

We used separate regression analyses to study the relation between students' learning prerequisites and their self-reported active, constructive and interactive learning activities. Cognitive learning prerequisites showed no significant relations to students' reports of learning activities and all learning prerequisites together explained a substantial amount of variance only in constructive learning activities (R^2=.33). Reports of constructive learning activities were significantly negatively related to a perspective motive (β=-.29, p<.001) and significantly positively related to both, content- and science-related motives (β=.27, p<.01 resp. β=.21, p<.01). For active and interactive learning prerequisites, no significant relations to learning motives could be uncovered and the overall variance explanation by learning prerequisites was low (R^2=.05).

To predict study success, as described by students' score on the end-of-semester analysis exam, we used a stepwise regression approach. For space reasons, we focus on the result of the final, full regression model here. Students' learning prerequisites explained 51% of variance in study success, which increased to 54% when including students' learning activities to the model. In the final model, students' cognitive learning prerequisites (calculus knowledge, HSQG) showed a significant positive relation to students' success (calculus knowledge: β=.34, p<.001; HSQG: β=.28, p<.001). Of students' study choice motives, only an application-related motive showed a significant relation to study success (β=-.13, p<.01). Only students' constructive learning activities related significantly to study success (β=.17, p<.05).

The number of exercise sheets handed in by each student was correlated significantly to self-reported constructive, but not active or interactive learning activities (r=.25, -.02, and .01 for constructive, active, and interactive learning activities). To support our results on students' constructive learning activities, we repeated both regression analyses, replacing self-reported learning activities by the number of exercise sheets: The number of handed-in exercise sheets was predicted significantly by students' high school qualification grades (β=.34, p<.001), but not by their prior calculus knowledge (β=.06, p>.05). Only a science-related study motives related significantly to the number of handed-in exercise sheets (β=.20, p<.01). Study success was predicted positively by the number of handed-in sheets (β=.25, p<.001), with almost unchanged results for the learning prerequisites compared to the previous analysis.

DISCUSSION

The goal of this study was to analyse relations between students' cognitive and motivational learning prerequisites, their learning activities, and their study success during the first semester. To start with, the results indicate that self-reported study motives differentiate clearly between two mathematics programmes that vary in terms of application-orientation, which provides validation our motive measures.

We found that students' self-reported study choice motives provided specific predictive information about their study behaviour and were partially predictive for study success. In particular, clearly extrinsic motives (professional perspectives) went along with less constructive learning activities. Those motives that are well aligned with the nature of university mathematics programmes (e.g. engaging in mathematics problems) were connected to more constructive learning activities. Moreover, we found a direct connection between a motive to apply mathematics and reduced study success. This is particularly interesting, since this motive can be considered intrinsic to mathematics as a school subject, but not well aligned with the nature of university mathematics programmes. These results underpin the view that only those motives can support in coping with the transition from school to university mathematics, which are in line with the programme under study.

As hypothesized by Chi (2009), self-reported constructive learning activities were connected to higher study success. This result was largely confirmed when using a different, correlated measure of constructive learning, the number of voluntarily handed-in exercise sheets. Thus, the idea of surveying constructive activities by a self-report questionnaire can be considered sufficiently valid for our purposes. Given these stable results, it might be worth to investigate means to direct students towards constructive learning, in particular if they start their study with less adequate motives.

Yet, the results are less assuring for active and interactive learning activities. Self-reports of both kinds of activities were neither significantly connected to cognitive and motivational learning prerequisites nor to study success. As for active learning activities, additional analyses show at least that these go along with reduced drop-out from the analysis course. Following Chi (2009), it is also plausible that active learning is not strongly connected to study success. Nevertheless, also interactive learning activities were connected to almost no other measure in our current analyses (apart from gender, with females reporting more interactive learning activities). We would like to discuss two related explanations: First, it is well documented that even if students engage in collaborative forms of learning, this does not imply that they really engage in substantial discussions, taking up and developing the contributions of their partner (Kollar, Fischer, & Slotta, 2006). What students report in the questionnaire might partially account for less effective or superficial collaboration strategies. Second, and connected to that, we must admit that reconsidering our questionnaire items for interactive learning activities made us doubt that these really addressed effective, deep collaborative activities in a clear way (cf. table 1). Thus, it seems warranted not to take strong conclusions before improved survey instruments or ways to study students in real interaction instead of using self-reports are available.

Summarizing, our results indicate that students' motives for choosing a university mathematics programme are connected to their learning behaviour and study success. In particular, goals that relate to the nature of mathematics programmes as described in the literature on the secondary-tertiary transition (Clark & Lovric, 2008; de Guzman et al., 1998; Rach et al., 2012) go along with sustainable learning activities. Theories of self-regulated learning (Zimmerman, 2011) form a plausible framework for the effects of these motives. In view of our results, it might seem vital to provide future mathematics students with sound information about the nature and goals of mathematics university programmes and about the differences to school mathematics during their decision for a programme. Moreover, it seems promising to provide support for students during the transition phase to adapt their own goals and learning activities to the specific demands and nature of the programme.

References

Blüthmann, I., Lepa, S., & Thiel., F. (2008). Studienabbruch und -wechsel in den neuen Bachelorstudiengängen. *Zeitschrift für Erziehungswissenschaft, 3*, 406-429.

Clark, M., & Lovric, M. (2008). Suggestion for a theoretical model for secondary-tertiary transition in mathematics. *Mathematics Education Research Journal, 20*(2), 25–37.

Chi, M. T. (2009). Active, constructive, interactive: A conceptual framework for differentiating learning activities. *Topics in Cognitive Science, 1*(1), 73-105.

Deci, E. L., & Ryan, R. M. (1985). *Intrinsic motivation and self-determination in human behavior*. New York: Plenum.

Dörfler, W., & McLone, R. (1986). Mathematics as a school subject. In B. Christiansen, A. Howson & M. Otte (Eds.), *Perspectives on Mathematics Education*, 49-97. Reidel: Dordrecht.

Eilerts, K. (2009). *Kompetenzorientierung in der Mathematik-Lehrerausbildung: Empirische Untersuchung zu ihrer Implementierung*. Zürich, Münster: LIT.

Guzman, M. de, Hodgson B., Robert, A., & Villani, V. (1998). Difficulties in passage from secondary to tertiary education. In G. Fischer & U. Rehmann (Eds.), *Proceedings of the International Congress of Mathematicians,* Vol III, 747-762.

Heublein, U. (2014). Student Drop-out from German Higher Education Institutions. *European Journal of Education 4*, 497-513.

Kollar, I., Fischer, F., & Slotta, J. D. (2007). Internal and external scripts in computer-supported collaborative learning. *Learning and Instruction, 17*(6), 708-721.

Rach, S. (2014). *Charakteristika von Lehr-Lern-Prozessen im Mathematikstudium: Bedingungsfaktoren für den Studienerfolg im ersten Semester*. Münster: Waxmann.

Rach, S., & Heinze, A. (2012). Studying mathematics at the university: The influence of learning strategies. In B. Ubuz (Ed.). *Proc. 35th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 9-16. Ankara, Turkey: PME.

Retelsdorf, J., & Möller, J. (2012). Grundschule oder Gymnasium? Zur Motivation ein Lehramt zu studieren. *Zeitschrift für pädagogische Psychologie, 26*(1), 5-17.

Siebert, U., Rach, S., & Heinze, A. (2013). Teaching quality of mathematics university courses. In A. Lindmeier & A. Heinze (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, p. 269. Kiel, Germany: PME.

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), 5–24.

Thomas, M. & Klymchuk, S. (2012). The school-tertiary interface in mathematics: teaching style and assessment practice. *Mathematics Education Research Journal, 24*(3), 283–300.

Trapmann, S., Hell, B., Weigand, S., & Schuler, H. (2007). Die validität von schulnoten zur vorhersage des studienerfolgs - eine metaanalyse. *Zeitschrift für pädagogische Psychologie, 21*(1), 11-27.

Zimmerman, B. (2011). Motivational sources and outcomes of self-regulated learning and performance. In B. Zimmerman & D. Schunk (Eds.). *Handbook of Self-Regulation of Learning and Performance*, 49-64. New York & London: Routledge.

THE INAPPROPRIATE APPLICATION OF NATURAL NUMBER PROPERTIES IN RATIONAL NUMBER TASKS: CHARACTERIZING THE DEVELOPMENT THROUGH PRIMARY AND SECONDARY EDUCATION

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The natural number bias is known to explain many difficulties learners have with understanding rational numbers. The research field distinguishes three aspects where natural number properties are inappropriately applied in rational number tasks: density, size, and operations. A comprehensive test was constructed to characterize the development of 4th to 12th graders' natural number bias. This test was administered to 1343 elementary and secondary school students. Results showed an overall natural number bias that was weakest in size tasks, somewhat stronger in operations tasks, and by far the strongest in density tasks. An overall decrease of the strength of the natural number bias – but no disappearance except for size tasks – could be found with grade.

INTRODUCTION

A good understanding of rational numbers is an essential part of mathematical literacy, which is not only important in learners' school career, but also in their everyday experiences. Although a good rational number understanding is found to be very important, many people have great trouble understanding the different aspects of rational numbers (e.g. Vamvakoussi, Van Dooren, & Verschaffel, 2012). Recent research literature ascribes many of the difficulties learners have with the understanding of rational numbers to a phenomenon called the natural number bias.

The natural number bias is described as the tendency to (inappropriately) apply natural number features in rational number tasks (e.g. Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). While the origin of the natural number bias is still a matter of debate, there is large consensus in the literature that before children are introduced to rational numbers, they already have formed an intuitive idea of what a number is, which is primarily based on natural numbers (Vamvakoussi & Vosniadou, 2010). Indeed, in their daily experiences, children encounter natural numbers much more often than rational numbers (one example is finger counting). This intuitive idea of numbers as natural numbers is confirmed and systematized in the first years of mathematics education that a learner goes through (Greer, 2004). When rational numbers are then introduced in the classroom (mostly in the middle years of primary education), the principles and features of natural numbers are no longer always applicable, but learners continue to apply them. So, learners are found to make systematic mistakes specifically in rational numbers tasks where reasoning purely in terms of natural numbers results in an incorrect solution (these will later be denoted as incongruent items). At the same

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 273-280. Hobart, Australia: PME.

time, much higher accuracy levels are found in rational number tasks where reasoning merely in terms of natural numbers also results in a correct solution (denoted as congruent items). In the research literature, three main aspects of the natural number bias are distinguished: density, size, and operations.

The first aspect concerns the dense structure of rational numbers. Natural numbers are characterized by discreteness: You can always point out the successor number of any given number (for example: after 4 comes 5). Rational numbers, on the contrary, are characterized by a dense structure: There is no such thing as a successor number of a given rational number, as there are always infinitely many numbers between any two rational numbers. Still, learners are reported to think that for instance between 2/5 and 3/5, there are no other numbers (Vamvakoussi & Vosniadou, 2010).

The second aspect is related to the numerical size of rational numbers. Learners have the wrong assumption that, as is the case with natural numbers, "longer decimals are larger", "shorter decimals are smaller" and "a fraction's numerical value increases when its denominator, numerator, or both increase" (Meert, Grégoire, & Noël, 2010; Resnick et al., 1989).

The third aspect of the natural number bias concerns the effect of arithmetic operations. While addition and multiplication with natural numbers will always result in a larger number and division and subtraction will always result in a smaller number, these rules do not longer necessarily apply in the case of rational numbers (for example 0.4×9 will result in an outcome smaller than 9), but learners still wrongly assume them to be true (Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2014).

In line with the above, a theoretical framework that has been frequently used to explain the natural number bias is the conceptual change theory, and more specifically Vosniadou's framework theory approach towards conceptual change (Vosniadou, 1994; 2006; Vosniadou, Vamvakoussi, & Skopeliti, 2008). The main assumption of this theory is that learners gradually tend to organize their daily experiences in quite coherent framework theories (Vamvakoussi & Vosniadou, 2010). When learners are confronted with new information which is not in line with their framework theory, they will have more difficulties to understand this information than when the new information affirms or extends their initial framework theory. In the former cases, conceptual change is needed: Learners need to accommodate their initial framework to the new incompatible information. This accommodation is typically not an all at once process. Learners often attempt to assimilate the new information without completely revising the assumptions of their initial framework theories, which often results in inconsistencies or misconceptions (Vosniadou et al., 2008).

THE PRESENT STUDY

The overall goal of this study was to characterize the development of the natural number bias in all three aspects (density, size, and operations) across the wide span between 4th and 12th grade. By doing so, we addressed various issues that – despite the extensive attention that the topic of rational number understanding recently has received – were not covered by research so far.

Since no comprehensive test instrument was available to measure the natural number bias, a secondary goal of our study was to create such a comprehensive paper-and-pencil test. We administered this paper-and-pencil test to learners from 4th until 12th grade, with the aim to investigate:

- The overall occurrence of a natural number bias.
- The relative strength of this bias in the decimal vs. fraction format.
- The relative strength of the natural number bias across the density, size, and operations aspects of rational number understanding.
- The evolution with age of the natural number bias as a whole and specifically within each of the three aspects.

METHOD

Participants

Data were collected in 21 schools (9 primary schools and 12 secondary schools) from different parts of Flanders, Belgium. This resulted in a representative sample of 1343 learners distributed over 4th grade (n = 213), 6th grade (n = 230), 8th grade (n = 293), 10th grade (n = 302), and 12th grade (n = 305).

Design

Starting from a broad literature review and an exploration of the Flemish mathematics curriculum, a comprehensive paper-and-pencil test (the Rational Number Sense Test, further abbreviated as 'RNST') was constructed with the aim to measure learners' natural number bias in rational number tasks. The test contained items addressing the three aforementioned aspects of the natural number bias (density, size, and operations), with items presented in fraction or decimal form or using a combination of both. The test further contained open and multiple choice questions and items of a varying difficulty degree for each of the aspects, in order to tap the full range of learners' natural number bias. The reliability of the test instrument was high (Cronbach's alpha = .87).

In this paper we report data from 63 items that were solved by learners from every grade (the higher grades received additional items that were more sophisticated, but these are not considered in the current paper). In total there were 15 density items (2 congruent and 13 incongruent), 33 size items (15 congruent and 18 incongruent), and 15 operations items (2 congruent and 13 incongruent). Further, 37 items involved fractions, 23 decimal numbers, and 3 items allowed a fraction or a decimal number as an answer. Examples of congruent and incongruent items are given in Table 1.

	Congruent item	Incongruent item

Density	Write a number between 1/4 and 3/4	Write a number between 3.49 and 3.50
Size	Choose the largest number: 14/18 or 29/31	Choose the largest number: 3/9 or 2/5
Operations	Is 50 × 3/2 bigger or smaller than 50?	Is 72 × 0.99 bigger or smaller than 72?

Table 1: Examples of test items

Analysis

Because we had a repeated measures design, data were analysed using the Generalised Estimation of Equations (GEE) in order to correct for repeated (and thus likely correlated) measures within participants. The degree of difference in accuracy on the congruent versus incongruent items was seen as an indicator of the strength of the natural number bias.

RESULTS

Congruency Main Effect

A significant main effect of congruency was found $X^2(1, N = 1343) = 1456.13$, $p < .001$. The accuracy level for congruent items (87.9%) was significantly higher than for incongruent items (66.8%) for the whole group of participants. Given this difference, our results clearly confirm an overall occurrence of a natural number bias. In what follows, we will investigate the impact of the representational format, the aspect of the natural number bias and the grade level on the strength of the natural number bias, which implies that we will look specifically at the interaction between congruency and the aforementioned variables.

Congruency × Representation Interaction Effect

No significant interaction effect between representation and congruency could be found, $X^2(1, N = 1343) = 0.03$, $p = .87$. This indicates that the natural number bias was equally strong in rational number tasks with decimal numbers as with fractions.

Congruency × Aspect Interaction Effect

A significant interaction effect between congruency and aspect was found $X^2(3, N = 1343) = 439.86$, $p < .001$. The odds ratios and their 95% confidence intervals showed that the natural number bias was weakest in size tasks (OR = 1.48, 95% CI [1.40, 1.57]), somewhat larger in operations tasks (OR =1.66, 95% CI [1.58, 1.74]) and clearly largest in density tasks (OR = 11.48, 95% CI [10.01, 13.17]).

Congruency × Grade Interaction Effect

A significant interaction effect between congruency and grade was found $X^2(8, N = 1343) = 998.95$, $p < .001$. The odds ratios and their 95% confidence intervals of each grade level are shown in the upper right panel in Figure 1. The strength of the natural number bias was not significantly different for 4th graders (OR = 5.16, 95% CI [4.72, 5.63]) and 6th graders (OR = 5.66, 95% CI [5.12, 6.25]). However, the natural number bias was significantly weaker in 8th graders (OR = 3.85, 95% CI [3.45, 4.31]) and even significantly weaker in 10th graders (OR = 2.52, 95% CI [2.24, 2.83]) and 12th graders (OR = 2.26, 95% CI [1.98, 2.59]). The strength of the natural number bias did not significantly differ between these latter two grades.

Congruency × Grade Interaction Effect for Every Aspect

Besides the overall evolution, Figure 1 provides an overview of the evolution of the strength of the natural number bias for each aspect separately.

Density

Only a limited number of density tasks were provided in the test of the 4th graders. Consequently, the data from the density tasks of the 4th graders were not used in the current analysis. There was a significant interaction effect between congruency and grade $X^2(3, N = 1130) = 15.14$, $p < .01$. The odds ratios showed that the natural number bias was very strong in all grades: 6th graders (OR = 25.71, 95% CI [18.95, 34.89]), 8th graders (OR = 24.02, 95% CI [15.77, 36.58]), 10th graders (OR = 11.69, 95% CI [7.33, 18.65]), and 12th graders (OR = 8.05, 95% CI [4.92, 13.18]).

Size

There was a significant interaction effect between congruency and grade, $X^2(4, N = 1343) = 80.81$, $p < .001$. The odds ratios showed that the strength of the natural number bias was largest in 4th graders (OR = 3.56, 95% CI [3.20, 3.97]), somewhat smaller in 6th graders (OR = 1.85, 95% CI [1.60, 2.14], and nearly absent in 8th graders (OR =1.23, 95% CI [1.05, 1.44]). No natural number bias for size items was found in the odds ratio of the 10th (OR = 0.97, 95% CI [0.82, 1.15]) and 12th graders (OR =0.87, 95% CI [0.72, 1.07]).

Operations

There was a significant interaction effect between congruency and grade, $X^2(4, N = 1343) = 80.90$, $p < .001$. The odds ratio showed that the strength of the natural number bias was largest in 4th graders (OR = 4.49, 95% CI [3.74, 5.41]), somewhat smaller in 6th graders (OR = 2.45, 95% CI [2.04, 2.96], 8th graders (OR =2.11, 95% CI [1.68, 2.65]), and 10th graders (OR = 1.71, 95% CI [1.35, 2.16]), and nearly absent in 12th graders (OR =1.43, 95% CI [1.08, 1.87]).

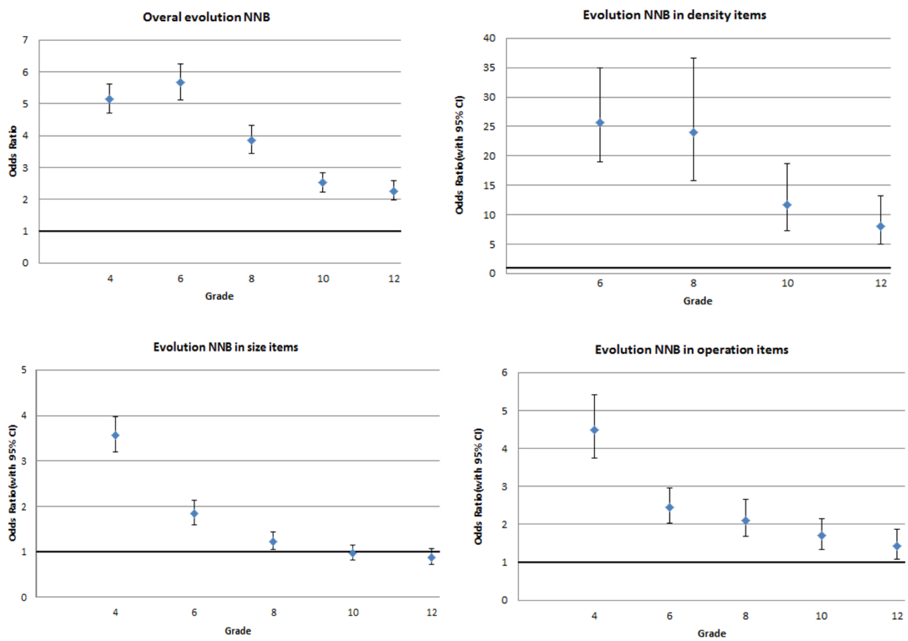

Figure 1: Overall evolution and evolution per aspect of the strength of the natural number bias as represented by the odds ratio (and 95% confidence interval) of accuracy for congruent and incongruent items (an odds ratio of 1 indicates an absent bias)

CONCLUSION AND DISCUSSION

Based on the existing literature and an analysis of the Flemish curriculum, we created and validated a comprehensive test instrument that enabled us to systematically and directly compare the strength of the natural number bias in the different aspects (density, size, and operations) and representations (fractions, decimal numbers) between 4th and 12th grade. By administering the new test instrument to a large group of 4th to 12th graders, we first of all found that there was a clear natural number bias, as shown in the significantly higher accuracy to congruent than to incongruent items. Second, it was found that this bias was equally strong in tasks with decimal numbers as with fractions. This is an interesting finding, particularly because the available theoretical and empirical literature contains evidence that different kinds of natural number-based errors may occur in items involving these two representations. Third, our results indicated that learners' natural number bias decreases with grade and that it develops over a period of at least six years, without completely disappearing at the end of secondary education. Fourth, results showed that the natural number bias was weakest in size tasks, somewhat stronger in operations tasks, but by far the strongest in density tasks. This is in line with previous research (see for example Vamvakoussi et al., 2012). Fifth, while an evolution can be found in learners' understanding of all three aspects of the natural number bias, they continue to struggle particularly with the operations and density aspect. The dense structure of rational numbers remains especially difficult to grasp, even for students at the very end of secondary education.

Besides the above-mentioned theoretically relevant findings, the present study also resulted in another valuable outcome, namely a valid and reliable test instrument that measures learners' natural number bias in rational number tasks. This test could be useful in future research in this domain. This RNST could, for example, be used to conduct (internationally) comparative research. It could further be used as an effectiveness measure in intervention studies aimed at improving learners' rational number understanding. Lastly, the RNST could also be used to investigate the relation between learners' rational number understanding and other aspects of learners' number sense and/or other learner characteristics.

One of the limitations of our study was that we only used performance data collected in a collective paper-and-pencil test. With a view to investigating the interference of learners' natural number knowledge in rational number tasks, it would be interesting to complement these performance data with data about learners' reaction times. As shown by numerous studies (see for example Vamvakoussi et al., 2012), reaction time data are effective in investigating the natural number bias. The core finding of these reaction time studies is that it takes more time to respond correctly to incongruent than to congruent tasks. The main advantage of reaction times, therefore, is that they still can shed light on the natural number bias even in learners who no longer make errors.

We finally turn to an important educational implication that emerges from this study. Given that the understanding of rational numbers has been shown to relate to later general mathematics achievement, it is quite worrying that the majority of learners have troubles understanding the several aspects of rational numbers, some of which even last until the end of secondary school. Consequently, the acquisition of rational number understanding – and particularly of the understanding of the differences with natural numbers – deserves more attention in the mathematics class. In this respect, we note that errors committed by learners may be partly caused by formal instruction, or at least by the fact that they are not sufficiently addressed by instruction. Debou and Verschetze (2012) systematically investigated the three most often used textbooks for elementary school mathematics in Flanders. Their analysis showed that textbooks pay almost no explicit attention to the (conceptual) differences between natural and rational numbers, but rather tend to only point to similarities between both. We believe that if textbook designers and teachers have a thorough understanding of the natural number bias, they will be better able to address the natural number bias, for instance by pointing the learners systematically to differences between natural numbers and rational numbers.

References

Debou, E., & Verschetze, L. (2012). *De overgang van natuurlijke naar rationale getallen. Een curriculumanalyse van drie Vlaamse wiskundemethoden [From natural to rational numbers. A curriculum analysis of 3 Flemish mathematics textbooks] (Unpublished master's thesis)*. Katholieke Universiteit Leuven, Faculteit Psychologie en Pedagogische Wetenschappen, Belgium.

Greer, B. (2004). The growth of mathematics through conceptual restructuring. *Learning and Instruction, 14*, 541-548. doi:10.1016/j.learninstruc.2004.06.018

Meert, G., J., Grégoire, J., & Noël, M-P. (2010). Comparing the magnitude of two fractions with common components: Which representations are used by 10- and 12-year-olds? *Journal of Experimental Child Psychology, 107*, 244–259.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20*, 8-27.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior, 31*, 344-355. doi:10.1016/j.jmathb.2012.02.001

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*, 181–209. doi:10.1080/07370001003676603

Van Hoof, J., Lijnen, T., Verschaffel, L., & Van Dooren, W. (2013). Are secondary school students still hampered by the natural number bias? A reaction time study on fraction comparison tasks. *Research in Mathematics Education, 15*, 154-164. doi:10.1080/14794802.2013.797747

Van Hoof, J., Vandewalle, J., Verschaffel, L., & Van Dooren, W. (2014). In search for the natural number bias in secondary school students' interpretation of the effect of arithmetical operations. *Learning and Instruction*. Advance online publication. doi:10.1016/j.learninstruc.2014.03.004

Vosniadou, S. (1994). Capturing and modelling the process of conceptual change. *Learning and Instruction, 4*, 45-69. doi:10.1016/0959-4752(94)90018-3

Vosniadou, S. (2006). Research Forum: The conceptual change approach in the learning and teaching of mathematics: An introduction. In J. Novotna, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (pp. 155-184). Prague: PME.

Vosniadou, S., Vamvakoussi, X., & Skopeliti, I. (2008). The framework theory approach to conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 3-34). Mahwah, NJ: Lawrence Erlbaum.

TEACHER REPRESENTATIONS OF FRACTIONS AS A KEY TO DEVELOPING THEIR CONCEPTUAL UNDERSTANDING

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This paper presents a case study of one Year 6 teacher who, over a period of one year, developed a deeper understanding of equivalent fractions. Evidence from assessment of teacher pedagogical content knowledge, interviews and classroom observation reveals that the exploration of representations of fractions was central to the teacher's growth in knowledge for teaching. Also apparent was a change in pedagogy, from a focus on procedural understanding to an emphasis on developing conceptual understanding. We propose that the deliberate exploration of representations may be a key to increasing both the mathematical and pedagogical knowledge of teachers, supporting their ability to foster conceptual understanding in their students.

INTRODUCTION

As well as possessing sufficient mathematics knowledge, teachers must be able to skilfully design learning experiences, including the use of representations, explanations, examples, and identify and address student misconceptions (Ball, Lubienski & Mewborn, 2001; Baumert & Kunter, 2010). The blending is often referred to as pedagogical content knowledge or PCK (Schulman, 1987). One indicator of a deficit in teacher knowledge and PCK appears to be the under-utilisation of a variety of representations of mathematics concepts and an over-reliance on symbolic notation (Ball, Thames, & Phelps, 2008; Isak, 2008). This suggests that developing a teacher's understanding of representations in a mathematics topic may be a productive focus for professional learning. However, as pointed out by Lee, Brown and Orill (2011), although much attention has been given to researching representations associated with students' conceptual understanding, limited attention has been given to teachers' facility with representations. This study explores the proposition that attending to a teacher's understanding of representations of fractions can have a direct influence on pedagogy.

DEVELOPING UNDERSTANDING: THEORETICAL BACKGROUND

Central to the thinking guiding this study is the premise that the goal of mathematics education is deep conceptual understanding of mathematics concepts and relationships. Insufficient conceptual understanding of mathematics content increases a teacher's reliance on procedural understanding. Deeper understanding of fraction concepts allows teachers to use "representations to attach meaning to mathematical procedures" as part of their pedagogy (Charalambous, 2010, p. 273) and so attend to conceptual understanding.

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We use the expression 'development of understanding' to reflect constructivist learning theory, in which the learner actively responds to experiences by continuously reorganising and building knowledge (von Glaserfeld, 1987). Compatible with the constructivist view of learning is the *Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding* (Pirie & Kieren, 1989, 1994). The Pirie-Kieren Theory consists of eight layers usually depicted as a set of nested rings and are named, from the centre outwards: Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring, and Inventising. The layers of knowing build from a learner's existing knowledge towards more sophisticated generalisation and abstraction, yet do not indicate a hierarchical linear progression, because the learner will constantly cycle back through preceding layers as the tentative new understanding is challenged by unfamiliar contexts. In this 'folding back' to an inner layer the learner brings new insights and seeks to refine or expand the earlier thinking (Pirie & Kieren, 1989, 1994; Pirie & Martin, 2000). The model is not restricted to the learner's internal thought processes but encompasses social interactions, concrete experiences and actions (Pirie & Martin, 2000).

Of particular interest in this study are the inner layers of Image Making, Image Having and Property Noticing where the formation of mental imagery plays a critical transitional role towards abstraction. Like Wright (2014), we interpret the 'Image making' layer to include the use of concrete materials and the manipulation of representations. However, we acknowledge that the image-making process involves 'reflecting' on existing knowledge and memories, not only the mental images of actual objects (Wright, 2014).

While the Pirie-Kieren theory describes the growth of understanding in young learners (school students), we see parallels in the development of understanding in teacher-learners. Importantly, the Pirie and Kieren model helps to interpret the significance of shifts in the ways a teacher represents fractions in terms of conceptual understanding.

UNDERSTANDING FRACTIONS

Siegler, Fazio, Baily and Zhou (2013) describe a comprehensive understanding of fractions to be composed of conceptual knowledge, procedural knowledge, as well as non-symbolic and symbolic knowledge. Conceptual knowledge of fractions includes "…understanding of the properties of fractions: their magnitudes, principles, and notations", and procedural knowledge requires "…fluency with the four fraction arithmetic operations (Siegler et al., 2013, p14). Non-symbolic knowledge refers to concrete models and various representations, and symbolic knowledge is competence with conventional written expressions of fractions.

Unfortunately, teachers tend to rely on symbolic notation or use one only one type of representation (typically area diagrams) in their teaching, rather than selecting the most appropriate representation from a range of options (Ball, et al., 2008; Isak, 2008). Consequently, the lack of purposeful exploration of the relationships between multiple representations can inhibit understanding the properties of fractions.

Previous research has demonstrated that the teaching of fractions is typically limited to the part-whole construct as modelled by area diagrams such as shaded parts of circles, which provides limited scope for comparing magnitudes of fractions, understanding equivalence, and operations with fractions (Clarke & Roche, 2009; Lamon, 2007). Recent research points to the importance of developing an understanding of the multiplicative structure of equivalent fractions as a basis for deducing a mental strategy rather than simply being taught a procedure (Wong & Evans, 2007, 2010).

This study examined one teacher's use of representations for equivalent fractions and the growth of her understanding – using the research question: What is the relationship between changes in the teacher's use of fractions representations and the development of her understanding?

METHODOLOGY

The study is situated within a larger project that supported teachers' professional learning as they strove to improve student engagement with, and understanding of, mathematics concepts and processes. A collaborative inquiry approach to teacher learning was taken, in which emphasis was placed on supporting teachers to be self-directed learners (Timperley, Wilson, Barrar & Fung, 2007).

The motivation for this particular case study arose from analysis of the data from a set of written tasks completed twice by all the project teachers. Along with changes in PCK, the researchers observed changes in the drawn representations of fractions used by the teachers in their task responses (Way, Bobis, Anderson & Cameron, 2013). This prompted a deeper investigation of the reasons for the representational changes, drawing on the additional data collected for the project's case study teachers. Due to space restrictions, only data pertaining to the first task (See Table 1) is reported here.

The case teacher, referred to as Lyn, was an early career teacher (4 years experience) in a non-government primary school (Catholic School system) in the metropolitan area of a large city in Australia. At the time of the study Lyn was teaching a mixed-ability Year 6 class of 23 children (approx. 12 years of age).

Scenario: Student Task	Teacher Task
Which fraction is larger 2/3 or 5/6? Draw and write something to explain your reasons.	Give three examples of student responses (correct or incorrect) to this task and comment on the likely thinking behind each response.

Table 1: Task to assess teacher pedagogical content knowledge on equivalent fractions

Design and methods

An instrumental case study approach (Stake, 2003) was selected as being compatible with the primary goal of gaining insight into a particular phenomenon – that is, the change in a teacher's use of representations of fractions and the growth of understanding. In this type of case study the researcher is focused on interpreting the available qualitative data using deductive 'pattern matching' techniques (Yin, 2009) under the guidance of a pre-determined proposition or conceptual framework, rather than attending to the full range of details of the case in an open-ended approach. This meant that the data in this study was deliberately searched for references to the two major 'variables' of representations and understanding, and other pieces of information excluded. As with all case studies the process of triangulation played an important role – comparing the information from different data sources – as the more convergent the data, the stronger the confidence in interpretation of meaning.

Three data sources were searched to identify instances explicitly relating to representations and understanding, then pooled and compared to create a description: a) PCK assessment task (See Table 1), completed in February and again in June; teacher interviews from post-lesson observation (June), small group (November), and the final reflection (December); c) classroom lesson observation and video (early June).

RESULTS AND DISCUSSION

Initial procedural understanding

Lyn began the project with a procedural understanding of equivalent fractions, which she came to realise had been influencing the way she taught fractions to her class.

> I didn't have a great understanding of fractions and I looked at the text book sort of things to help me and that's how I taught my lessons…. So to start with when I was first teaching fractions I was just teaching …- when you're changing it to an equivalent function whatever you do at the bottom you do at the top - procedural stuff. (Group interview Nov 2013)

The responses to the PCK task (Time 1-Feb) also indicated limited deep understanding. For example, one of Lyn's responses was a drawn pair of circles, one divided into three parts (2/3 shaded) and the other six parts (5/6 shaded). The comments on student thinking did not explain how the diagram might be used, or deal with the concept of equivalence. Another response for the task gave the procedure of multiplying the numerator and denominator, expressed in both words and symbolic notation. This aligns with Isak's (2008) findings that teachers mostly use symbolic procedures, or use drawn diagrams in a basic illustrative manner, rather than as a conceptual tool.

Exploration of representations and emergence of conceptual understanding.

The use of the grid overlays to represent fractions (See Figure 1), introduced by the researchers in the May workshop, was new to Lyn, and her efforts to make sense of the model proved to be important for increasing her conceptual understanding.

> I can learn something myself and I think the biggest thing for me was having those overlays and actually seeing how equivalent fractions are made from another fraction. It was just amazing - it actually took me along time to get my head around it myself but once I got my head [around it]…. it was like oh, that's what it is. (Final interview Dec)

The Time 2 PCK assessment task (June) was administered five weeks after Lyn's first encounter with the grid overlays, and one week after the observed lesson Lyn drew a rectangle with vertical divisions to show thirds, and described the repartitioning of the whole into sixths by drawing a horizontal dividing-line across the rectangle. Her comment on student thinking referred to noticing that "5/6 is greater than the equivalent fraction [for 2/3] of 4/6".

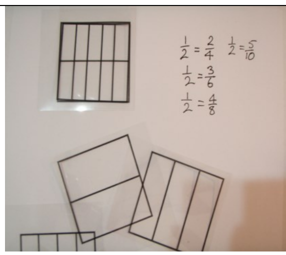	A set of grid overlays consists of squares divided into equal parts to represent fractions, printed on transparencies. A grid is formed by overlaying one square with another – one with horizontal bars, the other with vertical bars. For example, overlaying the square showing halves with the square showing fifths creates a grid showing tenths. The halves and fifths are still observable, allowing comparison of fraction size or determination of equivalent fractions.

Figure 1: Grid overlays.

Working with the grid overlays provided Lyn with the opportunity to return to the Image-making and Image-having layers as described in the Pirie-Kieren theory. This then allowed the properties of equivalent fractions to be 'noticed'. Lyn's use of a single diagram to show both fractions in the comparison problem clearly demonstrates reference to the same whole (Lee, Brown & Orill, 2011) and the multiplicative relationship between the fractions (3 being a factor of 6), signifying a conceptual approach to the problem rather than procedural (Wong & Evans, 2011).

Translating new understanding into teaching strategies

Lyn was motivated to share her new comprehension of equivalent fractions with her students and used the grid overlays as the basis for an “exploration and discovery lesson” observed by one of the researchers. She commented that the students “…could actually see the equivalent fractions forming in front of them” (Post lesson interview June). Reflecting on the student responses, Lyn decided she would like to “… give them some more time on their actual overlays and drawing and finding of fractions” and planned further lessons with the grid representation working with “specific denominators such as half and twelfths.” (Post lesson interview June)

After several lessons using the grid overlays (not observed by the researchers) Lyn moved back to the written tasks that many of the students had found difficult at the start of the year. She attributes gains in the students’ conceptual understanding to the use of representations.

> But then introducing the overlays it was allowing the kids to see for themselves how an equivalent fraction is made and from there they were then able to do written tasks and all the other stuff. I've got the conceptual ideas forming…. So with that I'm using more hands-on, more concrete materials. From there, they're understanding different concepts….. (Final interview Dec)

Through her teaching interactions, Lyn continues her own development of understanding (Pirie & Martin, 2000), while recognising the critical role the representation plays in her students’ learning.

SUMMARY AND CONCLUSIONS

The findings suggest that when Lyn began the project she did not possess a robust representation or imagery for the comparison of fraction magnitude and the relationship between equivalent fractions, and that this manifested in a reliance on symbolic procedures and inadequate models in her teaching. The exploration of the grid-overlays representation allowed Lyn to develop a deeper conceptual understanding of fractions, and inspired changes in pedagogy.

We propose that the deliberate exploration of representations of mathematics concepts may be a key to increasing both the mathematical and pedagogical knowledge of teachers, supporting their ability to foster conceptual understanding in their students. Lyn’s case provides an example that can inform further investigations.

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References

Ball, D., Lubienski, S. & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers’; mathematical knowledge. In V. Richardson (Ed).

Handbook of Research on Teaching: 4th Edition (pp.433-456), Washington: American Educational Research Association.

Ball, D., Thames, M. & Phelps, D. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Baumert, J. & Kunter, M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Journal, 47*(1), 133-180.

Charalambous, C. Y. (2010). Mathematical knowledge for teaching and task unfolding: An exploratory study. *The Elementary School Journal, 110*(3), 247-278.

Clarke, D., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics, 72*(1), 127-138.

Lamon, S. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Charlotte, NC: Information Age Publishing.

Lee, S., Brown, R., & Orrill. (2011). Mathematics teachers' reasoning about fractions and decimals using drawn representations, *Mathematical Thinking and Learning*, *13*(3), 198-220.

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For The Learning of Mathematics, 9*(3), 7–11.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*, 61–86.

Pirie, S., & Martin, L. (2000). The role of collecting in the growth of mathematical understanding. *Mathematics Education Research Journal, 12*(2), 127–146.

Siegler, R., Fazio, L., Bailey, D. & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Science, 17* (1), 13-19.

Shulman, L.S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1)*,* 1–22.

Stake, R. (2003). Case studies. In N. Denzin & Y. Lincoln (Eds.), *Strategies of qualitative inquiry* (2nd ed.). (pp. 134-164). Thousand Oaks, CA: SAGE.

Timperley, H., Wilson, A., Barrar, H., & Fung, I. (2007) *Teacher professional learning and development: Best evidence synthesis iteration.* Wellington, New Zealand: Ministry of Education.

Von Glasersfeld, E. (1987). Learning as a constructive activity. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics,* (pp. 3–18). Hillsdale: Lawrence Erlbaum Associates.

Way, J., Bobis, J., Anderson, J. & Cameron, L. (2013). Representations of concepts as a catalyst for change in teacher pedagogical content knowledge. *Procedia Social and Behavioral Science, 106*, 2248-2258.

Wong, M. & Evans, D. (2007). Students' conceptual understanding of equivalent fractions. In J. Watson & K. Beswick (Eds), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Group of Australasia, pp. 824-883). Adelaide: MERGA.

Wong, M. & Evans, D. (2011). Assessing students' understanding of fraction equivalence. In J. Way & J. Bobis (Eds.), *Fractions: Teaching for Understanding*, (pp. 91-98). Australian Association of Mathematics Teachers: Adelaide.

Wright, V. (2014). Frequencies as proportions: Using a teaching model based on Pirie and Kieren's model of mathematical understanding. *Mathematics Education Research Journal, 26* (1), 101-128.

Yin, R. (2009). *Case study research: Methods and design* (4th ed.). Thousand Oaks, CA: SAGE.

PRESERVICE TEACHERS' TEMPERATURE STORIES FOR INTEGER ADDITION AND SUBTRACTION

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Ninety-eight elementary and middle school preservice teachers posed eight stories for integer addition and subtraction number sentences. Stories that were posed about temperature were analysed using a modified Marthe's (1979) framework for integer problem types. This framework was modified based on the stories provided by the preservice teachers. This paper reports on the problem types utilized by the preservice teachers. Results highlight that preservice teachers do not frequently use some problem types. Also, results may indicate that some number sentence types (e.g., -23 – -5=☐) support different problem types (e.g., State-State-Translation).

INTRODUCTION

Negative numbers and operations about them are notoriously challenging. Piaget (1948) reflected on this noting that, "Everyone knows the difficulty that secondary students [and even university students!] have in understanding the algebraic rules of signs – 'minus times minus equals plus'" (p. 104). This is further complicated when our students are asked to reason conceptually about integers or apply integers to various contexts.

BACKGROUND LITERATURE

Research focusing on student thinking about integers and the ways young children reason has gained recent momentum in the field (e.g., Bofferding, 2014; Bishop et al., 2014). Within this increased research about integers, a subset has focused on understanding the role of context within the realm of student thinking (Stephan & Akuyz, 2012; Whitacre et al., 2014). Yet, research situated in making sense of how preservice teachers (PSTs) reason about and use integers has mainly focused on their solution strategies to integers arithmetic problems (Bofferding & Richardson, 2013; Chrysostomou & Mousoulides, 2010). However, making sense of integers within contexts is important for PSTs to develop, as they will be teaching this topic to their students in the future.

Both children and PSTs have difficulties with creating contexts for integer operations (Kilhman, 2009; Mukhopadhyay, 1997; Wessman-Enzinger & Mooney, 2014). Mukhopadhyay (1997) asked 32 students in grades 5, 6, and 7 to solve problems involving negative integers and tell a story that matched the equations. Four case studies were provided that demonstrated that students struggled to generate stories. She hypothesized that this was attributed to the various mental models the students were possibly employing. Similarly, Kilhman (2009) asked 99 PSTs to solve and describe their thinking for number sentences (e.g., -8 – -3= ☐). Of the 99 PSTs, Kilhman found

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that only 23 utilized a model or context to explain the mathematics and did so with either number lines and/or temperature to explain their reasoning.

Research with children has found that stories they pose for integer addition and subtraction can be classified in to the Conceptual Models for Integers of Addition and Subtraction (CMIAS), or ways of mathematically reasoning about and using the integers (Wessman-Enzinger & Mooney, 2014). Temperature was found to be a useful context for developing two CMIAS, *Translation* and *Relativity*. For example, *translation* concepts surface when a context suggests increasing or decreasing a temperature. *Relativity* conceptions develop because the temperature scale itself is a relative scale with an arbitrarily, although intentionally, selected zero.

THEORETICAL PERSPECTIVE

For informing the translation conceptual model, it is important to understand contextual problem types that may support those ways of thinking. Marthe (1979) , in the first paper published about negative integers in PME proceedings, classified different problem types for additive structures for integers. The first category was S_iTS_f, where the initial state (S_i) is translated (T) to the final state (S_f). Marthe then described that S_i, T, or S_f could represent the unknowns in any given problem. A second category was $T_1T_2T_3$. He described T_1, T_2, and T_3 as "transformations," although they can also be described as linear translations. From this problem type, Marthe described that there are three subsequent problems that can be posed, where T_1, T_2, or T_3 are unknowns having differing magnitudes and signs. For example, $T_1T_2T_3$ with T_2 unknown, T_1 and T_3 with opposite signs, and $|T_1| < |T_3|$, could be contextualized as, "A car makes an initial journey of 20 km upstream. Then it makes a second journey. If it had made only one journey from its starting-point to its destination, it would have made a journey of 25 km downstream. Describe the second journey" (Marthe, 1979, p. 156). Marthe also included a category, SSS, which is composed of all states and no translations.

Although Marthe did not provide examples for temperature, his problem types are applicable to the context of temperature. The dropping of temperature can be compared to travelling downstream; and, a relative position on a stream can be compared to the relative position on a temperature scale. Identifying the types of contextualized that PSTs pose can shed light into the difficulties that they may have and ways to support their learning as future mathematics educators.

Research Question

What kinds of stories about temperature do PST elementary and middle school teachers pose for integer addition and subtraction number sentences?

METHODOLOGY

Ninety-eight elementary and middle school PSTs participated in a study focusing on integer addition and subtraction while enrolled in an introductory mathematics content course. The authors, who are also professors for this course, analysed the written tasks from their students. The mathematics content course is designed to promote conceptually-oriented discourse around number and operations. To prepare PSTs to become mathematics educators, they are encouraged solve problems in multiple ways, present their own solution strategies, and understand the reasoning of others (Cobb & Yackel, 1996).

Data Collection

Data was collected across two academic semesters, Fall 2013 and Spring 2014. The PSTs were given 8 integer addition and subtraction number sentences (i.e., 16 – 4 = ☐, -17 + 12 = ☐, 18 + -13 = ☐, 8 – 20 = ☐, -2 – 3 = ☐, -14 – -20 = ☐, -6 + -9 = ☐, -23 – -5 = ☐) and asked to pose stories that they thought best matched these number sentences. This task was given to the PSTs prior to instruction on integer operations and reasoning about integers contexts in the course. Although other integer tasks were given to the PSTs, this preliminary task is the focus of this paper.

Data Analysis

The 98 PSTs posed 8 stories each for a total of 784 stories. Of these, 108 (13.8 %) were posed utilizing the context of temperature. The 108 stories about temperature constitute the unit of analysis for this study. These 108 temperature stories were organized by integer addition and subtraction problem types and examined for themes. The authors identified themes such as: the realism of context, mathematical correctness, and the consistency of the problem type to the story. Although these themes became codes, this paper reports on the themes about problem types that emerged and were guided by Marthe's (1979) macro-problem types (i.e., STS, TTT, SSS).

The authors used constant comparative methods (Glaser & Strauss, 1967) to analyse the temperature stories. After an initial pass through of the temperature stories, the authors discussed and agreed on modifying Marthe's STS problem type by extending it to *State-State-Translation* (SST) and *State-State-Distance* (SSD). Our SST is similar to Marthe's problem type STS with T unknown, but we felt that making the permutations of the letters explicit captured an imperative difference in the problem types SST and STS. For SST, the S's represent two relative numbers and T represents a translation from one relative number to another. We also wanted to make directed distance explicit, which is why we added the problem type SST and differentiated it from SSD, where the D represented distance without established direction. Although one may mathematically argue that all distance is directed, the PSTs posed stories in a way that direction was relative. We maintained Marthe's problem types TTT and SSS. After the authors agreed upon these modified problem types (i.e., STS, SST, SSD, TTT, SSS), the authors coded each of the 108 temperature stories posed by the PSTs with these codes. The authors agreed on 92 of the 108 codes or 85.2% of the time. All of the disagreements were negotiated and resolved.

RESULTS & DISCUSSION

Results are reported by problem type (i.e., STS, SST, SSD, TTT, and SSS). Though no stories were posed by the PSTs about temperature for 16 – 4 = ☐, the other seven number sentences each had ten or more stories about temperature posed for them.

STS Problems

When a story was posed with a relative temperature and a translation with the second relative temperature as the unknown, it was considered to be an STS problem. The PSTs posed STS problems the most for the number sentences -17 + 12 = ☐ and 8 – 20 = ☐ (See Table 1).

Number Sentence	STS Problem Type
-17 + 12 = ☐	21/23 (91.3%)
18 + -13 = ☐	8/14 (57.1%)
8 – 20 = ☐	11/13 (84.6%)
-2 – 3 = ☐	13/16 (81.3%)
-14 – -20 = ☐	11/16 (68.8%)
-6 + -9 = ☐	12/16 (75%)
-23 – -5 = ☐	6/10 (60%)

Table 1: Number Sentence and STS Problem Type

A few common examples of STS problems that the PSTs posed for these number sentences are shown below.

PST 31: The temperature is 8 degrees and then it goes down 20 degrees. What is the temperature now?

PST 91: In New York the temperature was -17°F in the morning. If the temperature went up 12°F, what is the temperature?

PSTs posed STS problems the least for the problem types 18 + -13 = ☐ and -23 – -5 = ☐ . Some common examples of the STS problem posed for 18 + -13 = ☐ are shown below.

PST 26: During the day, the temperature was 18 degrees. By the end of the day, the temperature decreased by 13 degrees. What temperature was it by the end of the day?

PST 29: The temperature is 18 degrees and it goes up by -13 degrees. What is the temperature?

PST 18: The temperature is 18°. By tonight it will drop -13. What will the temperature be?

Although each of these examples is considered to be a STS problem type, there are notable distinctions between these stories. PST 26’s story is for 18 – 13 = ☐, which is equivalent to 18 + -13 = ☐. Although PST 29 posed a story that is mathematically equivalent to 18 + -13, it is not realistic to talk about temperature increasing by a negative number. However, PST 18 posed a story that is not mathematically equivalent to 18 + -13 = ☐ and instead posed a story for 18 – -13 = ☐, which is also not realistic.

Number sentences like -17 + 12 = ☐ and 8 – 20 = ☐ seem to support STS problems more than number sentences like 18 + -13 = ☐ and -23 – -5 = ☐.

SST and SSD Problems

When a story was posed with two give relative temperatures and the translation is unknown, it was considered to be an SST problem. Whereas, when the story was provided with two relative temperatures and a distance, without direction, it was considered to be an SSD problem. Although not mathematically correct, PST 25 posed a SST problem for the number sentence -17 + 12 = ☐.

> PST 25: It was 12° outside Wednesday. It was 17 below zero degrees Thursday. How much had the temperature dropped since Wednesday?

In this story both the temperatures are provided, and the question provides a distinct direction by indicating Wednesday to Thursday. Interestingly, the SST problem type was utilized more for problems like -17 + 12 = ☐, rather than other more reasonable number sentences, like -14 – -20 = ☐ (see Table 2). For example, PST 74 provided an example of an SST problem that works well.

> PST 74: It is -23°F in Antarctica and it is -5 degrees in Illinois. What is the difference between Antarctica's temperature and Illinois?

The distinguishing feature of the stories posed that were consider to the SSD problem type is that no direction is provided in the stories. The most common number sentence used for the SSD problem type was -14 – -20 = ☐ (see Table 2). A common story for this number sentence and problem type is shown below.

> PST 4: One day in New York it is -14 degrees out. In Maine the same day it was -20 degrees. What is the difference between the two states' temperatures?

Again, the direction is not established. Thus, one could use both number sentences -14 – -20 = ☐ and -20 – -14 = ☐ to describe this story. -14 – -20 = 6 could be described as representing 6 degrees warmer in New York. Whereas, for -20 – -14 = -6, the -6 could be described as representing 6 degrees colder in Maine. The SSD problem type was used more for problem types like -14 – -20 = ☐ and -23 – -5 = ☐. This could point to evidence that these number sentences could promote use of these problem types more. Overall, many PSTs did not use either the SST or SSD problem types frequently. Instead of SST or SSD problem types for number sentences like -14 – -20 = ☐ , PSTs would often pose stories like:

> PST 14: The temperature last night was -14°F. When I woke up, it had gone up 20°. What is the temperature right now?

Although -14 – -20 = ☐ is mathematically equivalent to -14 + 20 = ☐, the stories posed for both are not equivalently appropriate. PSTs need to be able to pose stories for -14 – -20 = ☐ that are appropriate and realistic, rather than just posing mathematically equivalent stories for -14 + 20 = ☐.

Number Sentence	SST Problem Type	SSD Problem Type
-17 + 12 = ☐	2/23 (8.7%)	0/23 (0%)
18 + -13 = ☐	1/14 (7.1%)	1/14 (7.1%)
8 – 20 = ☐	1/13 (7.7%)	1/13 (7.7%)
-2 – 3 = ☐	1/16 (6.3%)	2/16 (12.5%)
-14 – -20 = ☐	0/16 (0%)	5/16 (31.3%)
-6 + -9 = ☐	0/16 (0%)	1/16 (6.3%)
-23 – -5 = ☐	1/10 (10%)	3/10 (30%)

Table 2: Number Sentence and SST & SSD Problem Types

TTT and SSS Problems

None of the PSTs in this study posed stories for the TTT problem type presented by Marthe (1979). However, some PSTs posed stories where two relative temperatures were provided and a third relative temperature was unknown, a SSS problem type.

The SSS problem type was only used for the number sentences 18 + -13 = ☐ and -6 + -9 = ☐. The following examples are stories considered to be the SSS problem type for the number sentence -6 + -9 = ☐.

PST 85: It is -6 degrees outside at 12 pm. At 12 am another -9 degrees is added. How many degrees is it at 12 am?

PST 91: It is -6°F in Bloomington and -9°F in Chicago. What is the sum of the two temperatures?

Number Sentence	SSS Problem Type	Other
-17 + 12 = ☐	0/23 (0%)	0/23 (0%)
18 + -13 = ☐	2/14 (14.3%)	2/14 (14.3%)
8 – 20 = ☐	0/13 (0%)	0/13 (0%)
-2 – 3 = ☐	0/16 (0%)	0/16 (0%)
-14 – -20 = ☐	0/16 (0%)	0/16 (0%)
-6 + -9 = ☐	2/16 (12.5%)	1/16 (6.3%)
-23 – -5 = ☐	0/10 (0%)	0/10 (0%)

Table 3: Number Sentence and SSS & Other Problem Types

Piaget (1952) referred to temperature as an example of how he defined intensive quantities, pointing out that temperatures cannot be summed. Here, the context of temperature, the SSS problem type is not appropriate. Perhaps some of the PSTs posed these types of stories for number sentences like 18 + -13 = ☐ and -6 + -9 = ☐ because there are not many temperature contexts that are appropriate for these number sentences. It seems that the SSS problem type is not even appropriate for the Translation CMIAS. Three stories that used wind chill incorrectly were classified as other because there wasn't enough information to determine if the PSTs intended the problem to be TTT or STS. For example, a PST wrote, "It's 18 degrees outside. There is a wind chill of -13 degrees. How warm does it feel outside?"

CONCLUSION

Most of the PSTs posed temperature stories that were the STS problem type. Although it is certainly important for PSTs to be able to make sense of integer addition and subtraction with the STS problem type, it is also important for PSTs to utilize other problem types like SST, SSD, and TTT. This study also provided evidence that certain integer addition and subtraction number sentences facilitate reasoning about integers in different ways and with different problem types. Number sentences like 18 + -13= ☐ and -23 – -5= ☐ are potentially rich areas for robust discussions with PSTs given their struggles for appropriate problem types for these number sentences. For example, if it is -23 degrees in Chicago and -5 degrees in Boston, then discussions about how both -5 – -23= ☐ and -23 – -5= ☐ could fit that story would be productive. Because certain number sentences may support some problem types more than others, a variety of number sentences should be given when teaching integers to facilitate reasoning in different ways. Identifying PSTs' conceptions, like we did in this paper, is instrumental to develop rich instructional tasks and understand how to prepare them to become future mathematics educators.

References

Bishop, J. P., Lamb, L L. C., Philipp, R. A., Whitacre, I., & Schappelle, B. P. (2014). Using order to reason about negative integers: The case of Violet. *Educational Studies in Mathematics*, *86*, 39–59.

Bofferding, L. (2014). Negative integer understanding: Characterizing first graders' mental models. *Journal for Research in Mathematics Education, 45*(2), 194–245.

Bofferding, L. & Richardson, S. E. (2013). Investigating integer addition and subtraction: A task analysis. In M. Martinez & A. Superfine (Eds.), *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 111–118). Chicago, IL: University of Illinois at Chicago.

Cobb, P. & Yackel, E. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27,* 458-477.

Chrysostomou, M., & Mousoulides, N. (2010). Pre-service teachers' knowledge of negative numbers. In M. Pinto & T. Kawakaki (Eds.), Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 265–272). Belo Horizonte, Brazil: PME.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. New York, NY: Aldine.

Kilhamn, C. (2009). Making sense of negative numbers through metaphorical reasoning. In C. Bergsten, B. Grevholm, & T. Lingefjärd (Eds.), *Perspectives on mathematical knowledge. Proceedings of madif6* (pp. 30–35). Linköping, Sweden: SMDF.

Marthe, P. (1979). Additive problems and directed numbers. In D. Tall (Ed.), *Proceedings of the 3rd Conference of the International Group for the Psychology of Mathematics Education* (pp. 317–323). Coventry, England: PME

Mukhopadhyay, S. (1997). Story telling as sense-making: Children's ideas about negative numbers. *Hiroshima Journal of Mathematics Education, 5*, 35–50.

Piaget, J. (1948). *To understand is to invent: The future of education*. New York, NY: The Viking Press, Inc.

Piaget, J. (1952). *The child's conception of number*. London, UK: Routledge & Kegan Paul, Ltd.

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education, 43*(4), 428–464.

Wessman-Enzinger, N. M., & Mooney, E. S. (2014). Making sense of integers through story-telling. *Mathematics Teaching in the Middle School, 20*(4), 203–205.

Whitacre, I., Bishop, J. P., Lamb, L. L. C., Philipp, R. A., Bagley, S. & Schappelle, B. P. (2015). 'Negative of my money, positive of her money': Secondary students' ways of relating equations to a debt context. *International Journal of Mathematical Education in Science and Technology*, *46*(2), 234-249.

EXPLORING EARLY SECONDARY STUDENTS' ALGEBRAIC GENERALISATION IN GEOMETRIC CONTEXTS

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Exploring multiple representations of a relationship between two variables has been shown to help students learn conceptually about functions in algebra. This paper discusses an investigation of 102 Year 7 students, prior to their study of algebra, which considered their making sense of a linear functional relationship in two geometric contexts - through generalisation of a 2-dimensional figural growing pattern and through representation of its variables on a Cartesian plane. This provided the opportunity to consider how algebraic and geometric understandings might reinforce each other in different representations of the same function. Students used a variety of ways to describe the structure of the figural pattern and 40% were able to use this to create an explicit rule, with over 20% already using a symbolic expression or full equation. Two thirds of the students created an incorrect graphical representation of the relationship and the other third drew either a column or line graph. Implications for middle-school algebra learning are discussed.

Understanding functions is foundational to Calculus, an important area of algebra used in many economic, science, and engineering domains. A key aspect of learning about functions as relationships between two variables is generalisation: the ability to notice and express mathematically this relationship. One route to developing students' ability to generalise has been through the use of geometric growing patterns. Even young students have been able to learn about variables and functions in the context of varying quantities of things they can see (Blanton & Kaput, 2004), rather than learning rote procedures for creating equations from numerical sequences. There is still much to understand, however, about "how students can be assisted in becoming aware of structure in patterns and in using symbols to express these patterns" (Kieran, 2007, p. 729). The generalisation of patterns and their representation symbolically and graphically are mentioned in curriculum for the middle years of schooling, such as the *Principles and Standards for Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000). Students are also expected to learn coordinate geometry and systems such as the Cartesian plane. This paper discusses an investigation of Year 7 students' use of algebraic generalisation in two geometric contexts: their visualisation of the geometric structure of a pattern to then represent its linear functional relationship symbolically, and creating a representation of the variables on a Cartesian plane to describe spatially their ideas about the functional relationship.

BACKGROUND

There are a variety of strategies for teaching pattern generalisation and developing students' functional thinking but one key focus is "the interaction of context, multiple representational forms, and technological tools" (Confrey & Smith, 1994, p. 32). These

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 297-304. Hobart, Australia: PME.

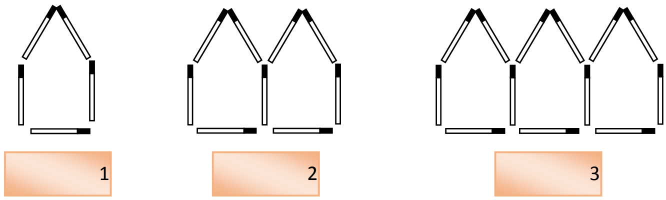
are seen as crucial in supporting students' development of functional thinking since they help students to explore the nature of functions conceptually, spatially, and symbolically. "The idea of a function embodies multiple instances, all collected within a single entity (e.g., a list, table, graph), a process that also involves generalising – answering the question, 'What is it that all these instances have in common?'" (Kaput, 1999, p. 146). This study focuses on the figural representation of a linear function with a 2-dimensional geometric growing pattern, and students' ability to visualise, generalise, and symbolise it algebraically. It also investigates their subsequent ability to use spatial reasoning to represent and notice features of the same function to see how they might describe algebraic ideas.

Students' recognition of functional relationships seems to be enhanced by asking them first to describe the features of a geometric pattern before expressing these algebraically (MacGregor & Stacey, 1995). Warren and Cooper (2008) used concrete materials to create the structure of growing patterns and found that specific questioning highlighted for students the relationship between the two variables: a pattern item quantity and its position number in the sequence. The first part of the task discussed in this paper asked students to extend the pattern (shown in Figure 1), describe their visualisation of its geometric structure in words, and then generalise a relationship between the number of houses and the number of matchsticks needed.

There are two approaches to generalising quantifiable aspects of a growing pattern. Finding the next item using step-by-step drawing or counting is referred to "near generalisation"; creating a general rule for any item in the pattern is known as "far generalisation" (Stacey, 1989, p. 150). These are also termed co-variation and correspondence respectively (Confrey & Smith, 1994). A co-variation approach which describes the relationship between successive items in a pattern is also known as recursive generalisation or a local rule (Mason, 1996). A correspondence approach describes the relationship between any item position number in the pattern/sequence and a quantifiable aspect of that item – also known as explicit generalisation or a direct or closed or relational rule. Figure 1 provides an example of these two approaches using the geometric growing pattern used in this study.

The second part of the task asked students to create a graph of the growing pattern (with a provided Cartesian plane labelled with the two variables). The students' responses to these two aspects of the task were analysed using a learning progression framework (Figure 2) adapted from an empirically substantiated instruction theory about geometric growing pattern generalisation (Markworth, 2010). A higher level represents more sophisticated functional thinking, showing the progression from co-variation (recursive) to correspondence (explicit) approaches and the subsequent application to multiple representations.

Matchstick houses



Co-variation: One house has 5 matchsticks and with each extra house 4 new matchsticks are added – the total numbers of sticks for each item in the pattern are 5, 9, 13, 17…
Correspondence: Each group of houses has 4 matchsticks per house plus 1 extra for the left wall of the first house ($t = 4n + 1$)

Figure 1: Two approaches to describing a geometric growing pattern

1. Extend a growing pattern by identifying its physical structure, features that change, and features that remain the same (*figural reasoning*).
2. Identify quantifiable aspects of items that vary in a geometric growing pattern.
3. Articulate the linear functional relationship between quantifiable aspects of a growing pattern by identifying the change between successive items in the sequence (*co-variation* or *recursive generalisation*).
4. Generalise the linear functional relationship between aspects of a growing pattern by:
    - 4.1 describing the relationship between a quantifiable aspect of an item and its position in the sequence (*correspondence* or *explicit generalisation*); or
    - 4.2 using symbols or letters to represent variables; or
    - 4.3 representing the generalisation of a linear function in a full, symbolic equation.
5. Apply an understanding of linear functional relationships between variables to further pattern analysis and multiple representations.

Figure 2: Learning progression framework for generalisation of geometric growing patterns (Wilkie, 2014, adapted from Markworth, 2010, p. 253)

RESEARCH DESIGN

This paper describes part of a design-based research project with six Year 7 mathematics teachers and their classes which aimed to investigate how students can be assisted in developing their functional thinking through noticing geometric structure in patterns and exploring how variables and relationships can be represented in multiple ways. An initial survey of students (n = 102), prior to any algebra teaching that year, sought insight into their prior algebraic thinking to focus the teachers' understanding and to inform the subsequent preparation of learning tasks for the teaching experiment. The students attended a large independent school in Melbourne and their mathematics teachers all agreed to participate in the project that year. Student learning and teacher learning were joint goals of the research (Gravemeijer & van Eerde, 2009). Three key aspects of the overall methodology were instructional planning and design, ongoing analysis of classroom events, and retrospective analysis (Cobb, 2000).

The survey was developed by the author from adaptations of examples of problem types (e.g., Markworth, 2010; Mason, 1996; Rivera, 2010; Stacey, 1989). The task discussed in this paper asked students to extend, describe, and generalise the matchstick houses pattern (Figure 1). Their responses were analysed and assigned a score based on the levels from the learning progression framework (Table 1). A process of cyclic check-scoring with pairs of researchers had been used in previous research to increase the reliability of results and the effective use of this framework as a rubric (Wilkie, 2014). These established principles were applied to this study.

The types of visualisations students demonstrated were categorised and analysed in terms of the students' subsequent ability to generalise explicitly the functional relationship. They were also asked to represent the pattern graphically, and to explain what they noticed about their graph's spatial features.

RESULTS AND IMPLICATIONS

The following section focusses on four aspects of the matchstick houses task.

Levels of functional thinking evidenced in pattern generalisation

Table 1 presents the results of students' attempts to visualise spatially, and then generalise algebraically, the matchsticks pattern with scores from the previously presented learning progression framework.

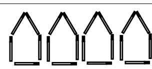
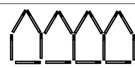
Score on learning progression	Description	Percentage of students	Illustrative example
Pre-1	Did not demonstrate ability to extend pattern	2.9%	
1	Extend pattern correctly by drawing	17.6%	
2	Use quantifiable aspects of pattern to find num. matchsticks in 7 houses	15.7%	"5+4+4+4+4+4+4=29 matchsticks" "29; add 4 each time"
3	Use co-variation (recursive) approach to find num. matchsticks in 17 houses and to describe recursive rule for any num. houses	19.6%	"Your first house will have 5 sticks so then for every new house you plus 4" "Every time you add a new house, all you need to do is add 4 matchsticks each time"
4.1	Use correspondence (explicit) approach to describe in words the rule to find num. matchsticks for any num. houses	18.6%	"You simply times the amount by 4 and then plus one for the originating matchstick" "Draw or do the amount of houses minus 1 and times by 4 and plus 5 at the end"
4.2	Represent rule as an expression using symbols/letters	14.7%	"(4 × ?) + 1" "n × 5 - (n - 1)" "N × 4 + 1"
4.3	Represent rule with full, symbolic equation	6.9%	"a × 4 +1 = b" "4 × x + 1 = ?"
Unscored response	Unclear student response	2.9%	"Count the middle house first then the sides of the other houses"; "5 + (4 × number of other houses)" "Find out how much is in a number that goes into it or is close to it"
No response		1.0%	

Table 1: Students' highest level of generalisation on the matchsticks task (n = 102)

It can be seen that 40% of the students demonstrated a correspondence (explicit) approach to the task, with just over 20% using symbols or letters in an expression or equation. Another 20% used a recursive (co-variation) approach to find the number of matchsticks for 17 houses. Nearly 16% could find the number of matchsticks for 7 houses but not 17 houses. Nearly 20% of the students demonstrated their ability simply to extend the pattern correctly, but not to use either a recursive or explicit strategy for generalisation. Nearly 3% of the students were unable to extend the pattern correctly and a further 3% made an unclear response to the task. Examples of symbols that the students used included question marks, underscores, *N*, *n*, *a*, *b*, *x*, *Y*, and *h*. Some students used a mixture of symbols and words such as '4 × ? + 1 = how many matchsticks' or '(number × 4) + 1'. One student defined their question mark as '? = number of houses' and then wrote '(4 × ?) + 1 ='. According to the Australian curriculum, students of this age are expected to be able to generalise explicitly, but not necessarily to use symbolic representation yet (Australian Curriculum Assessment and Reporting Authority [ACARA], 2009); this study found that less than half the students could generalise explicitly, but just over 20% of students could already use symbols or pronumerals in their functional rules.

Noticing the geometric structure and subsequent generalisation

By asking students to explain in writing how they visualised the matchstick structure, it was possible to gain insight into their spatial reasoning and to relate their visualisation descriptions to their subsequent generalisation. There were four noticeable types of visualisations that led to different but equivalent expressions of the rule. Table 2 presents the percentage of students who described their visualisation a certain way and then used it to generate a correct explicit rule (words or expression or full equation).

It can be seen that the most common type of visualisation was Type 3 – one matchstick for the first wall and then 'broken' houses made up of four matchsticks each. A few students described the structure according to one type but then represented the rule as Type 3. One student created a table of values to explain how they saw the structure and then wrote an explicit rule. A few unusual uses of pronumerals emerged from the students' geometric reasoning but space limits their inclusion in this paper.

Representing the pattern on a Cartesian plane

The ability to apply an understanding of functions to multiple representations is considered to demonstrate flexible and conceptual functional thinking. In addition to exploring the function through a geometric growing pattern and by creating a symbolic rule, the students were asked to draw a graph of the number of matchsticks for 1 to 10 houses (with Cartesian axes provided and labelled). Interestingly none of the students correctly plotted discrete points for each ordered pair of values. Nearly 18% drew an accurate column graph and nearly 14% drew an accurate line graph. Two thirds of the students made an incorrect graphical representation and 3% made no response. Although use of the Cartesian plane (all four quadrants) is explicitly included in the

Australian curriculum for students at this level (ACARA, 2009), it seems that students may not have experienced its use in a conceptual context involving pairs of quantities of things; instead it appears that for this task they relied on their experiences with statistical representation (in the form of column graphs).

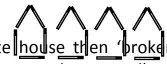
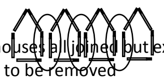
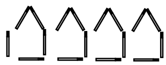
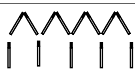
Type of visualisation	**Illustrative example**	**Students who wrote a correct explicit rule**
TYPE (1)  One complete house then 'broken' houses (or vice versa: one complete at end)	"5 matches is needed to make the first house therefore there will be one less house to make, the rest of the houses need 4 matches."	8.8%
TYPE (2)  Complete houses all joined but extra matchsticks in between to be removed	"Original pattern has 5 matchsticks. Houses join sides (one less than the number of houses."	2.0%
TYPE (3)  One wall then all 'broken' houses	"times 4 because that's how many matchsticks per house, + 1 because that was the starting matchstick."	19.6%
TYPE (4)  Numbers of roofs, walls, and floors	"roof -> double number of houses sides of house -> number of house plus 1 base -> the number of houses *plus them together*"	4.9%
TRANSITION BETWEEN TYPES / OTHER STRATEGY		4.0%

Table 2: Students' different types of visualisation and subsequent ability to write correct explicit rule (n = 102)

Noticing geometric aspects related to the functional relationship

Although none of the students correctly drew a set of points for the two variables on the axes provided, nearly one third did produce an accurate column or line graph. In terms of noticing aspects of the functional relationship from the spatial features of the graph, the most frequent response from these students was that the graph was increasing by 4 (14% of the total number of students). Nearly 9% referred to the graph as increasing or going up and 4% referred to the rate of change of the graph, e.g., "the number of matchsticks is increasing 4 per one house". Relating the slope of a graph (gradient) to the rate of change of a function is an important conceptual idea that connects the spatial features of a function's graphical shape to its symbolic equation (for a linear function, m in $y = mx + c$). It appears that some students were able to use their graph to notice spatially the change in one variable as compared to the other. Samples of other students' explorations of spatial features of functional relationships via representing growing patterns using a Cartesian plane will be presented at the conference.

CONCLUSION

A survey of 102 Year 7 students provided an opportunity to examine how students' ability to make sense of the structure of a geometric growing pattern related to their subsequent ability to generalise it explicitly. It was found that students visualised and described the structure in four different ways and 40% were subsequently able to use their spatial understanding of the geometric pattern to create an explicit rule for the functional relationship. This is an expected ability in algebra for students at this age according to numerous middle-school curricula. Just over 20% also demonstrated the successful use of symbols and pronumerals for representing variables in their rules. From this evidence of many students' ability both to define and use pronumerals in expressions and equations, it appears that it would be worthwhile to incorporate this aspect of algebra learning for primary pre-service and practising teachers to be able to develop students' conceptual knowledge. Although the use of pronumerals to represent variables is not introduced in the Australian curriculum until Year 7 (ACARA, 2009), it appears that students may be ready for this important aspect of algebra before reaching secondary school. Further research on effective contexts for helping students learn about variables conceptually at this age would be valuable, to address the disengagement many students experience when learning algebra in more abstract ways at secondary school (Greenes, Cavanagh, Dacey, Findell, & Small, 2001).

The study also provided opportunity to see if and how students could connect their understandings of the spatial features of another representation (on a Cartesian plane) to the same functional relationship and notice algebraic ideas from it. None of the students created a discrete set of points for each of the matchstick houses items (1 – 10) even though the use of such a coordinate system is prescribed for upper primary students in the Australian curriculum (ACARA, 2009). Many students' experience with graphing appeared to be limited to column graphs in the domain of statistical representation. It is unclear whether students have explored the use of Cartesian plane but did not link this to the type of context in the survey, or if this aspect of geometry in the curriculum is in fact being covered at primary levels. Nonetheless a small proportion of students were able to describe the gradient, rate of change or slope of their graphs. It would be worthwhile to consider ways to connect students' algebraic and geometrical understandings to help them explore functional relationships spatially and provide another helpful conceptual context for their learning of algebra.

References

Australian Curriculum Assessment and Reporting Authority [ACARA]. (2009, January, 2011). *The Australian curriculum: Mathematics.* Retrieved from http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10

Blanton, M., & Kaput, J. (2004). Elementary grades students' capacity for functional thinking. In M. Høines & A. Fuglestad (Eds.), *Proceedings of the 28th annual meeting of International Group for the Psychology of Mathematics Education* (pp. 135-142): IGPME.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, NJ: Lawrence Erlbaum Associates.

Gravemeijer, K., & van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *The Elementary School Journal, 109*(5), 510-524.

Greenes, C., Cavanagh, M., Dacey, L., Findell, C., & Small, M. (2001). *Navigating through algebra in prekindergarten - grade 2*. Reston, VA: The National Council of Teachers of Mathematics.

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Erlbaum.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 707-762). Charlotte, NC: National Council of Teachers of Mathematics, Information Age Publishing.

Markworth, K. A. (2010). *Growing and growing: Promoting functional thinking with geometric growing patterns*. Unpublished doctoral dissertation, University of North Carolina at Chapel Hill. Available from ERIC (ED519354).

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Dordrecht: Kluwer Academic Publishers.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Rivera, F. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics, 73*(3), 297-328.

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. L. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 133-160). New York: Taylor & Francis Group.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics, 20*(2), 147-164.

Warren, E., & Cooper, T. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics, 67*(2), 171-185.

Wilkie, K. J. (2014). Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra. *Journal of Mathematics Teacher Education, 17*(5), 397-428.

PERFORMANCE OR PROGRESS? INFLUENCES ON SENIOR SECONDARY STUDENTS' MATHEMATICS SUBJECT SELECTION

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In the context of a reported decline in the proportion of Australian students opting to study higher-level mathematics, this qualitative study explored school leaders' perspectives on their school-based approaches for influencing student subject choice. Dweck's (2007) mindsets framework was used to analyse and discuss findings from the interviews of seven leaders. Tensions both within and between individuals were related to the pressure to consider student mathematics performance and the desire to promote learning for its own sake.

Globally there seems to be a decline in the proportion of tertiary education students in Science, Technology, Mathematics and Engineering (STEM) fields. In 2006, a working group comprising of representatives from 16 countries in the Organisation for Economic Development published some findings and recommendations for the decline in student interest in STEM areas (OECD, 2006). The report indicated that there was a decline over the decade (1994-2003) in various countries in the absolute numbers of students in tertiary mathematics courses. Australia was one of the countries in the report. In these last two decades there has been a national decline in the proportion of Australian students electing to study intermediate and advanced mathematics in senior secondary years of schooling (McPhan, Morony, Pegg, Cooksey, & Lynch, 2008). This has implications for students' eligibility for and ability to study in many of the tertiary STEM courses. While there has been an increase in the absolute number of students taking lower level mathematics subjects, the number of students selecting intermediate level mathematics subject has remained relatively stable, but the number opting to take advanced level mathematics subject has declined (Kennedy, Lyons, & Quinn, 2014). The concern is that Australia, as with other OECD economies, needs STEM expertise in upcoming generations to support its economic well-being and international competitiveness (OECD, 2006).

Dweck's (2007) theoretical framework on mindsets was used in this study to investigate leaders' school-based approaches for influencing student subject choices and how these might relate to their beliefs about mathematics teaching and learning. The following section provides details on the context for the study by reviewing research on Australia's declining enrolments and by providing an overview of the literature on factors that are implicated.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 4, pp. 305-312. Hobart, Australia: PME.

BACKGROUND

In Victoria, Australia, senior secondary students in Years 11 and 12 study a number of subjects to obtain a Victorian Certificate of Education (VCE) for tertiary education (Victorian Curriculum and Assessment Authority [VCAA], 2015). In the various research studies found on factors affecting student VCE mathematics subject selection, there were common interacting themes related to student-based factors, school-based factors, and other external factors, discussed in the following paragraphs.

Students' beliefs about, attitudes and motivation towards school mathematics were found to be important influences in their decisions to continue with mathematics into their senior secondary and tertiary years. Beliefs about mathematics included students' self-perception of ability and perceptions of subject difficulty and its usefulness (Kennedy et al., 2014). Attitudes such as students' interest and liking in the subject, affected by their prior experiences and previous achievement in mathematics, were also factors (McPhan, et al., 2008). Mathematics teachers and how they teach mathematics affect how students perceive school mathematics – their beliefs and self-concept – and consequently their subject choices (McPhan et al., 2008). Career advisers also play a crucial role in informing students about their subject selection in accordance to their career options and to maximise university entrance scores for their choice of university courses (McPhan et al., 2008). In a national study on factors affecting the educational and occupational aspirations of young Australians, parental influences and academic performance were the two strongest predictors of occupational aspirations (Gemici, Bednarz, Karmel, & Lim, 2014). University entrance requirements and future career requirements are also critical considerations in students' decisions about studying mathematics (Gemici et al., 2014). As many Australian universities no longer require students to study advanced level mathematics as a pre-requisite, even for STEM courses (McPhan et al., 2008), this external incentive to study advanced level mathematics has been removed. School programs such as the availability of subjects impact students' subject choices. In a nationwide survey in 2005, only 63.8% of Australian secondary schools offer advanced level mathematics subject (Harris & Jensz, 2006).

Overall, studies on students' reasons for studying or not studying mathematics implicated student, school and external factors. In a number of these factors, the teachers' influence and the school context play an important part in the formation and the development of students' beliefs, motivation and attitudes, and consequently their decisions about subject choices for mathematics.

The study described here was designed with the consideration of the literature reviewed and sought to incorporate elements that were highlighted as important for investigating student enrolments. The following section describes its design in detail.

RESEARCH DESIGN

As an initial phase of a larger study, the perspectives of Victorian mathematics leaders (coordinators, head of mathematics), who have a direct influence on their schools' policies and programs relating to student VCE subject enrolments, were explored using a qualitative case study approach. The research questions were: (a) What are secondary mathematics school leaders' perceptions about the reasons for and the ways their schools address the issue of declining percentages of VCE students choosing intermediate and advanced mathematics subjects? (b) How do these perceptions inform our understanding of the influences on students' mathematics subject choices, using Dweck's (2007) theoretical framework?

In-depth interviews of the mathematics leaders were conducted and audio-recorded. After obtaining ethics approval from the university, schools with different profiles (government, independent, or catholic sectors, co-educational and single-sex, from different social economic status [SES] backgrounds; details to be provided in conference presentation) were approached by the researchers via email or phone. A one-hour face-to-face interview with the mathematics leader was conducted for those who agreed to participate, with the exception of one leader who preferred to provide written responses to the interview questions.

The interview transcripts were coded using Nvivo qualitative analysis software. An initial coding framework was developed jointly by the researchers. Each researcher then coded two interviews individually and then discussed results to ensure consistency. The coding framework was then modified for the next round of coding. This cyclical process was repeated multiple times for the full set of interviews. Themes emerged through the coding process, which were then analysed using Dweck's (2007) theoretical framework of mindsets.

Theoretical framework used for the analysis

Dweck (2007) defined the *fixed mindset* as incorporating the belief that one's qualities are carved in stone – that one has a fixed amount of intelligence, a certain personality, and certain moral character. Teachers with a fixed mindset tend to believe that "intelligence is a static trait" (Dweck, 2010, p. 26) and that they have no influence on their students' basic intellectual capabilities. She defined the *growth mindset* as incorporating the belief that qualities are able to be cultivated through one's effort, and although everyone differs "in their initial talents, attitudes, interests, or temperaments", each person can grow through application and experience (Dweck, 2007, p. 7). It includes the belief that a person's true potential is both unknown and unknowable, a focus on learning through effort, and the value of resilience in the face of setbacks. Studies on teachers' mindsets found that low-achieving students were more likely to improve their progress with teachers who demonstrated a growth mindset (Dweck, 2007). Although the two mindsets are conceptualised as *either/or*, Dweck (2007) cautioned that it is possible for an individual to hold different mindsets in different areas; her research found that the type of mindset held in a particular area is likely to guide an individual's decisions in *that* area.

Dweck's (2007) framework on mindsets was considered a helpful analytic tool for the study because it provides a useful way to focus on the leaders' perspectives on their school's management of students' subject selection and how these might relate to their beliefs about mathematics teaching and learning. Additionally, it accommodates the multidirectional influences of beliefs among students, leaders, teachers, and parents, and also allows for the possibility of an individual holding different types of mindsets for different aspects of their beliefs. Nonetheless, this study acknowledges that the interplay between so many perspectives and how these correlate with or even mismatch what actually happens with a school's enrolment process are obviously more complex in reality.

DISCUSSION

The following discussion focuses on the findings of preliminary analysis - two types of school-based approaches for influencing student subject selections in Years 11 and 12 that the leaders raised in their interviews – and relates these to Dweck's fixed and growth mindsets. Pseudonyms are used.

Providing information and recommendations

Six of the seven leaders described some type of input from the school for Years 9 and 10 students when choosing Years 11 and 12 subjects. This input might come from the students' mathematics teachers, careers staff, and a school coordinator or adviser. Several schools provided meetings with staff and students and parents to discuss options together; sometimes students had multiple meetings with different staff. Students' Year 10 results, either midway through the year or towards the end of the year, were used as a key indicator of what subject/s might be most appropriate. The rationale behind these school-based strategies seemed to vary, depending in part on the leader's own beliefs and mindset about mathematics learning. Several leaders indicated that their stance or that of their school advisers was one of the encouragement: "trying to do the highest level of maths which you're capable" (Diana) or pushing "brighter students" to do intermediate and advanced level mathematics subjects (Angela). One leader encouraged Year 10 students to give higher-level mathematics a go as "you can always drop down" and "you don't want people underrating themselves" (Eddie). Quite a few leaders referred to this ability to "drop down" from a higher level to an easier level of mathematics – "it's possible to go that direction, it really is impossible to go the other" (Eddie).

One leader indicated that some independent schools in Victoria encourage their brighter students to do the *easier* mathematics subject to try and maximise their university entrance score. Since final scores are allocated by rank according to a Gaussian (Normal) distribution for each unit (VCAA, 2015), it was expressed that this strategy was not only unfair to other students who needed to do the easy mathematics, it also discouraged students from learning mathematics at an appropriate level for them – a type of "dumbing down" (Angela).

It can be seen that there are competing influences on a student's subject choices. Encouraging students to choose a higher level of mathematics, "to give it a go" (Barb), seems to relate to a growth mindset, to challenge themselves and focus on learning and mastery of mathematics. Yet although arguments about maximising one's university score might relate to a fixed mindset and its focus on performance, the pressure on young people to think strategically about their future options is a significant one. One leader also explained that students, who enjoy mathematics and would do well in a higher level subject, nonetheless opt out due to pragmatic reasons that limited their choices to a certain number of subjects in Year 12 to maximise their scores. She described their teachers as "lamenting" over their understandable that disappointing choice to give up mathematics (Barb).

Some leaders expressed that conflicting advice might be given to students by teachers and career advisers. One leader said "I have actually learned a few things this year about what Careers tells them that's different from what we tell them as maths teachers" (Fiona). She described that

> Careers recommend everyone to do a VCE maths, particularly if they don't know what they want to do, because it leaves options open. But, we know well and truly there are some students who are just not going to be successful at what we offer.

While some leaders described encouraging some students – particularly capable ones – to challenge themselves (growth mindset), there was also an expressed concern about other students choosing a subject that their teacher believed would not result in successful outcome – "it's just not viable, they're just not going to (be successful) and they stress and then teachers stress and parents stress, and it's not worth it" (Fiona). This response resonates with previous research on mathematics teachers which found that in the face of a student's poor results, teachers demonstrating a fixed mindset tended to respond by saying that not everyone could be good at mathematics (Dweck, 2007).

Most leaders indicated that although students are given recommendations, their school policy is that students are free to go against them. Two leaders expressed particular concern about "migrant" or "EAL" (English as an Additional Language) students. One leader came from a high SES school and the other from a low SES school yet both indicated that many of these students "often want to do this subject (intermediate level mathematics) even if they are not capable enough" (Geena). One leader suggested that this was because "students have high expectations and their future studies require Maths Methods (intermediate level subject)" (Geena). The other leader related the issue to the design of examinations that "have increased in the level of English literacy" particularly the "extended response questions" which EAL students struggle to understand and therefore "just leave them blank" (Angela). She felt that these students were nevertheless opting for the more challenging subjects because "they're prestigious". In this situation, the students' desire for progress in a harder subject (growth mindset) is at odds with the leaders' belief that they may not have the capability to perform well (fixed mindset). Only one of the leaders indicated that their school imposed restrictions on students and did not allow them to go against the school's decision for their subject choice. This is discussed in the next subsection.

Establishing school structures and pathways

Several leaders described school-based structures that incorporated some form of streaming or tracking to provide pathways for students' mathematics study, with the proviso that once the students reached Year 11, they could nonetheless still choose their subjects. One school leader did not advocate or employ streaming at all until midway through Year 10 where two streams (based on Levels 10 and 10A in the Australian Curriculum) are offered to students. This was viewed as an encouragement strategy to "build their confidence" and enable students who "would've bailed at the end of Year 10" to achieve some sort of success: "actually maths is not too bad; I'll stick with it." It also allowed "brighter kids to really start to extend themselves" (Barb). She saw this approach in terms of promoting a growth mindset – an opportunity to learn. Another leader (Angela) also kept mixed-ability classes but until the end of Year 10, with separate programs for withdrawing students for support or extension. Yet she described pressure on her from her mathematics teaching staff to stream Year 10 into two levels and to prevent students in the lower stream from choosing the higher levels of mathematics in Year 11. Angela, unlike Barb, described her reluctance to stream Year 10 because it promoted a fixed mindset – she felt that streaming focussed students too much on their performance rather than on progress. She described her staff insisting that they did not have time to extend the "high fliers" if they had mixed-ability classes in Year 10 but she speculated that they were unwilling to take responsibility for their weaker students. She was also critical of many schools preventing students from completing Year 12 subjects earlier in Year 11 and she explicitly provided this option at her school to enable ongoing progress for those students who were ready and willing.

Although Angela sent out letters of concern to students at the end of Year 11 about low results, she resisted her staff's interest in introducing a "cut-off" (an E grade in Year 11 would mean not being allowed to continue in Year 12). She felt that it was important to maintain students' ability to choose, rather than sending a message that it was too late to make progress. She wished she could have more students "not caring about results but whether or not they're improving". This focus on progress not performance epitomises the growth mindset. Yet how to manage and maintain this mindset alongside pressures for university entrance and the current education system structure that focuses on performance, remains a challenge for those who want students to master mathematics for the inherent sake of their learning and development.

Another leader, Diana, described her school's structure (being trialled for the first time that year) that did in fact impose restrictions on a student's subject choices, based on Year 9 examination results and some "shuffling" in Year 10. Students were placed into three streams based on their performance and only the students in the top stream were permitted to study the intermediate and higher-level mathematics in Years 11 and 12. The Year 9 teachers were asked to remind students about this "first little fork in the road for them" as a way to address "mucking [fooling] around and being lazy". A flowchart showing these pathways was shown to lower secondary students to show them how their performance in Year 9 might "cut off the whole top row" (Diana). The leader described this approach as an encouragement strategy since she believed that students appeared to work harder earlier on, yet the criterion for students was clearly about performance. Strategies for targeting younger students were also raised by two other leaders.

CONCLUSION

From the interviews with seven secondary school mathematics leaders, it can be seen that schools employ a variety of strategies for influencing and guiding students' study choices in the senior years and include input from multiple sources – the leader, the mathematics teachers, the careers staff, coordinators, and parents. These strategies were often described by the leaders in terms of encouraging students to aim high and challenge themselves (growth mindset), or discouraging students from attempting a subject in which they were not deemed capable of succeeding (fixed mindset). Nonetheless, a majority of leaders indicated that students could override the recommendations and choose for themselves. One school was trialling a structure that disallowed students with low results from accessing the higher-level mathematics and another leader described pressure from the teaching staff to introduce a similar system in Year 11.

There was the sense that the growth mindset was described by leaders when discussing the importance of enjoying mathematics learning through seeing its usefulness or relevance, or valuing opportunities for students to try and challenge themselves by attempting a harder mathematics subject, at least to begin with. But this was countered by strong concerns about some students not heeding the warning of teachers that they may be aiming too high and that realistically they were not capable of achieving success, which can be related to a fixed mindset. This tension between the need to consider performance and the desire to promote progress – learning for its inherent sake – was noticeable. There was also the sense that the leaders experienced tension through differences between them and the careers staff, or between them and their mathematics teaching staff, or between them and parents.

Many believe that Australia needs increasing STEM expertise. Understanding their own mindsets and the mindsets of others may provide school leaders with insights into multiple perspectives on mathematics teaching and learning. This might help them develop effective strategies for encouraging more students to take higher level mathematics, not only at secondary levels but also in future learning. School's curricular structure and content, and assessment modes and opportunities for mathematical experiences at lower year levels might "channel" students to certain academic pathways. Mobility between pathways, particularly upward mobility, might be limited. Such streams or pathways seemed more likely to determine students' mathematics subjects in the middle years rather than later on. What is needed is a growth mindset and also ways to resolve the tension between the need for performance and the desire for progress. Further research would be worthwhile to explore these tensions from the perspectives of students and with reference to their own mindsets in mathematics.

References

Dweck, C. S. (2007). Mindset the new psychology of success: How we can learn to fulfill our potential. New York: Ballantine Books.

Dweck, C. S. (2010). Mind-Sets and Equitable Education. *Principal Leadership, 10*(5), 26-29

Gemici, S., Bednarz, A., Karmel, T., & Lim, P. (2014). *The factors affecting the educational and occupational aspirations of young Australians*, NCVER, Adelaide. Retrieved from http://www.lsay.edu.au/publications/2711.html

Harris, K-L., & Jensz, F. (2006). The preparation of mathematics teachers in Australia: Meeting the demand for suitably qualified mathematics teachers in secondary schools. Report prepared for the Australian Council of Deans of Science. Retrieved from http://www.cshe.unimelb.edu.au/research/disciplines/docs/Prep_Math_Teach_Aust.pdf

Kennedy, J., Lyons, T., & Quinn, F. (2014). The continuing decline of science and mathematics enrolments in Australian high schools. *Teaching Science, 60*(2), 34-46.

McPhan, G., Morony, W., Pegg, J., Cooksey, R., & Lynch, T. (2008). *Maths? Why not?* Report prepared for the Department of Education, Employment and Workplace Relations. Retrieved from http://www.aamt.edu.au/index.php/Activities-and-projects/ Previous-projects/Maths-Why-Not

Organisation for Economic Co-operation and Development [OECD] (2006, May 4). *Evolution of student interest in science and technology studies policy report.* Retrieved from http://www.oecd.org/science/sci-tech/36645825.pdf

Victorian Curriculum and Assessment Authority [VCAA] (2015). *Victorian Certificate of Education (VCE)*. Retrieved from http://www.vcaa.vic.edu.au/Pages/vce/index.aspx

USING CRITICAL INCIDENT TECHNIQUE TO INVESTIGATE PRE-SERVICE TEACHER MATHEMATICS ANXIETY

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High level of mathematics anxiety in pre-service primary teachers affects both their current study, and their future teaching of mathematics. This paper proposes Critical Incident Technique as an appropriate research method, and reports how it was used in a teacher education course to increase pre-service primary teachers' understanding of the impact of previous mathematics classroom experiences on their identities as learners and teachers of mathematics. The results also provided insights for teacher educators and teachers into strategies that could prevent or diminish their students' mathematics anxiety. The discussion highlighted the need for teacher educators to be aware of the perspectives of PSTs, the importance of verbalisation and the sharing of emotions, and outlined recommendations for further research.

INTRODUCTION

This paper demonstrates the use of the Critical Incident Technique (CIT), a robust research technique that is widely used for problem solving. This study applied CIT in order to investigate primary pre-service teachers' (PSTs') images of themselves as learners and future teachers of mathematics. The PSTs were asked to recall critical incidents in their mathematics learning, and examine their associated emotions. These written reflections were used to identify factors contributing to primary PSTs' mathematics anxiety (mathematics anxiety).

Critical incident reflections are descriptions of events that people remember as being meaningful in their experience. As this research sought to investigate factors that had an influence on PSTs, this paper defines "critical incident" in the sense of an incident that the participant selects and defines as having had an impact (Hughes et al., 2007); whereas some researchers, for example, Vandercleyen, Boudreau, Carlier, and Delens (2012) define a 'critical incident' as any incident on which the PST critically reflects.

Research methods are needed which will identify strategies to elucidate the impact of mathematics anxiety. The paper suggests CIT as a method in teacher education to investigate the issue of mathematics anxiety. It contributes to an ongoing project investigating the use of CIT and bibliotherapy to address PST mathematics anxiety (Wilson, 2014), and aims to assist PSTs with mathematics anxiety to perceive their past experiences differently and re-evaluate their potential to become effective teachers of mathematics.

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

This study is based in the interpretive paradigm, which holds that people socially and symbolically construct their realities. Blumer (1969) coined the term "symbolic

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interactionism" for the idea that people's actions are based on the meaning things have for them, derived from social interaction and modified through interpretation.

Emotional responses are not determined by objective reality but by interpretation of events - by subjective reality. The 'transactional model of emotion' (Lazarus, 1991) links motivational, social and cognitive dimensions. According to Lazarus, a lived experience consists of contextual and personal factors, which determine whether the event will be appraised: primarily as harmful or threatening (negative emotion), or challenging or beneficial (positive emotion); and secondly, for likely future outcomes, and their coping strategies.

Emotion disrupts cognitive processes, but PSTs learn when their beliefs, knowledge and skills are challenged. The self-analysis of an emotionally-charged experience is an opportunity to analyse their past actions and emotions, and the process of writing can be used to reflect on their actions and decisions.

Causes of mathematics anxiety in PSTs

Anxiety towards mathematics in PSTs has been widely publicised as an international issue. Mathematics anxiety refers to feelings of tension and fear in mathematical situations in school and in everyday life. High mathematics anxiety impacts on performance and achievement in mathematics (Sheffield & Hunt, 2006; Stubblefield 2006). Primary PST's mathematics anxiety has important impacts. It affects not only their current study but also their future teaching. Hence, identifying and addressing primary PSTs' perceptions of these experiences, is a critical aspect of their education.

Previous researchers have investigated the sources of mathematics anxiety, using a range of methods. Hoyles (1982) using semi-structured interviews, identified three areas for explanations of anxiety particularly related to mathematics - those derived from the nature of the subject mathematics, based on the influence of past experiences in mathematics and the self-concept of ability in the subject, and, concerned with how mathematics is taught and learned (including teacher pace and pressure). A number of researchers have used PSTs' mathematics autobiographies (Ellsworth & Buss, 2000; Sliva & Roddick, 2001) to identify themes such as the powerful effect of teachers, the ways mathematics was presented (relevance, comprehension, and emphasis on skills and memorisation); and fear failure, and avoidance, in mathematics experiences. More recently, Lutovac and Kaasila, (2009), using an autobiographical interview with a student, identified that the teacher was the main character in her memories of negative experiences. Teachers who are hostile, hold gender biases, or embarrass students in front of peers play a powerful role in mathematics anxiety (Vukovic, Keiffer, Bailey, & Harari, 2013). The perceptual changes that occur as a result of mathematics classroom experiences are persistent and enduring.

> "People who claim that they were born without mathematical ability will often admit that they were good at the subject until a certain grade, as though the gene for mathematics carried a definite expiry date. Most people will also recall an unusual coincidence: that the year their ability disappeared, they had a particularly bad teacher." (Mighton, 2004, p. 20)

Reflective thinking and Critical Incident Technique in PST education

Reflective thinking is important to identify the assumptions that underlie thoughts and actions. Researchers have suggested scaffolds to elicit detailed reflections; for example, Gibbs' (1988) reflective model can be summarised under six headings: Description – what happened? Feelings – what were you feeling? Evaluation – what was good or bad about the experience? Analysis – what sense can be made of the situation? Conclusion – what else could you have done? and, Action plan – if the situation arises again what would you do?

Although Flanagan (1954) developed CIT to establish facts in situations where the critical incidents relied on accurate and truthful reporting, he later adapted his technique, and CIT has been widely used to solve problems in education, health, and industry, by focusing on real-life incidents. In particular CIT was modified to include individual perspectives and affective responses (Chell, 1998).

> Critical incident technique (CIT) is a well proven qualitative research approach that offers a practical step-by-step approach to collecting and analysing information about human activities and their significance to the people involved. It is capable of yielding rich, contextualized data that reflect real-life experiences. (Hughes et al., 2007 p. 49)

The exploration of critical incidents can to challenge participants' concepts of self. When analysing a critical incident, individuals ask: Why did I view the original situation in that way? What assumptions about it did I make? How else could I have interpreted it? What other action(s) might I have taken that could have been more helpful? What will I do if I am faced again with a similar situation? (Serratt, 2010) These questions can extend and elaborate the Analysis, Conclusion and Action plan sections of Gibbs' (1988) reflective model.

Critical incidents focus on participants' lived experience. The method allows researchers to examine common situations, shared by a group, from the individuals' unique perspectives, and in their own words. CIT permits a degree of replication, in that the context and outcomes may be apparent in other PSTs' experiences. At the same time CIT provides the opportunity to identify and analyse even quite rare events, which may have devastating effects on vulnerable people (Pedersen, 1995).

Flanagan (1954) was concerned to make sure that descriptions were factually correct because he was trying to identify good procedures, and accurate reporting was essential. However, where critical incidents are descriptions of vivid events that people remember as being meaningful, it is not important if the interpretation is correct, as the way a person perceives an event is real in its consequences.

> … like all data, critical incidents are created. Incidents happen, but critical incidents are produced by the way we look at a situation: a critical incident is an interpretation of the significance of an event. To take something as a critical incident is a value judgment we make, and the basis of that judgment is the significance we attach to the meaning of the incident (Tripp, 2012, p 8.)

The process of writing helps PSTs reflect on their perception of the event and its impact on their construction of themselves as a learner of mathematics. The study used CIT to access the narrative or storied nature of experiences, as narratives are important for meaning construction (Ricoeur, 1985). The aim was to understand the meaning PSTs attach to lived experiences. Instead of researchers selecting which parts of mathematics autobiographies to analyse for themes, in CIT the participant chooses the experience and identifies the impact. Participants were not guided to the selection of a negative experience, so their choice provided comparative data on PSTs' positive and negative responses. By asking the research question: How do PSTs describe their mathematics experiences? the researcher used critical incidents identified by PSTs to illuminate key factors in the development of mathematics anxiety.

METHODOLOGY

The research aimed to examine the range of ways mathematics anxiety is experienced within a given context, with a range of participants. Given the complex nature of the phenomenon, and the aim of the study to access the narrative or storied nature of experience, a qualitative approach was appropriate. The interpretive tradition is characterised by prioritising lived experiences, with a focus on meaning of interactions and events. Erickson (1986) argued that it be used for answering questions like "What is happening, specifically, in social action that takes place in this particular setting? What do these actions mean to the actors involved in them, at the moment the actions took place?" (p. 121). These are clearly the type of questions asked in this study. The researcher attempted to understand mathematics anxiety by accessing the meaning that the participants gave to it, and to develop insights into lived experience from point of view of the participant. The quotations from PST narratives and vignettes reflect real experiences, chosen to illustrate themes identified by the literature.

Procedure

Ethics approval was based on accepted informed consent procedures. PSTs who agreed to participate in the study wrote a description during a tutorial of a critical incident (positive or negative) from their own school mathematics education that impacted on the way they thought about themselves as learners and future teachers of mathematics. Two important aspects of the research method were that PSTs chose the incident, and that they could choose a positive or negative experience. The participants were 268 primary PSTs studying mathematics education units from Bachelor of Education (Primary) degree courses, at an Australian university. Data were collected from the perspectives of the participants, using their own words. Pseudonyms were used for privacy.

Critical incident data analysis

The traditional, binary analysis using Lazarus' model of emotion, was used to identify ratios of positive and negative responses. A preliminary analysis based on the themes identified by other researchers was commenced, and further thematic analysis is in progress.

RESULTS AND DISCUSSION

The critical incidents described occurrences that were pivotal not only personally, but also potentially had an impact professionally. They were related to situations that impacted on and potentially interfered with the PSTs' beliefs, and identities. The binary analysis showed that the majority of the critical incidents were perceived as threats. Of the 268 PSTs, 236 (88%) wrote incident reflections. Of these, 102 (39%) were negative, 157 (61%) were negative and 2 (1%) described a neutral incident.

The thematic analysis of the critical incidents identified similarities with themes from the existing literature. Of the 236 PSTs, 135 (57%) wrote about the teacher. Of the 140 comments about the teacher, 46 (33%) were positive and 94 (67%) were negative. To be coded as teacher, comments had to specifically mention the teacher. If a comment mentioned two teachers, in separate years, both were counted separately. The emphasis on the role of the teacher reflected findings from other researchers (Ellsworth & Buss, 2000; Sliva & Roddick, 2001; Lutovac & Kaasila, 2009).

In addition, themes of the cycle of fear failure and avoidance, the students' perceptions of the nature of mathematics, their self-image as a learner of mathematics, and the influence of parents, were consistent with the themes identified from mathematics autobiographies (Sliva & Roddick, 2001; Ellsworth & Buss, 2000). The themes show specific links to Hoyles' (1982) second and third categories. Themes are illustrated using quotations from the PSTs' transcripts.

The role of the teacher

Many PSTs recognised the lasting impact on individual teachers. For example Amanda wrote: "I never had a teacher that taught. They used the textbook and board and said, 'I've taught you'". Another theme that emerged from interactions with the teacher was shame and humiliation. PSTs recalled experiences where the teacher made them feel embarrassed in front of their peers, for example, "I felt all the students at the tables were watching me and thinking I was stupid" (Patsy). Josh, another PST, recalled an incident from Year 8:

> On one occasion the teacher made me complete problem in front of the entire class on the whiteboard. I had absolutely no idea what I was doing and yet the teacher still made me complete the task. I tried to attempt the problem and it made me a joke in front of all the other students. It was a humiliating and degrading experience.

These feelings of humiliation have strong links to avoidance. Previous researchers identified that PSTs retain intense memories of their experiences with disabling teachers. (Ellsworth & Buss, 2000; Sliva & Roddick, 2001; Wilson & Thornton, 2008).

Cycle of fear, failure, and avoidance

Feelings of embarrassment gave way to resignation and a sense of inadequacy –for example, Joyce wrote: "Can anyone blame a girl for wanting to stick to what they feel they can cope with – rather than risking the humiliation of tackling the unknown connections between big ideas". This demonstrates the cycle of fear, failure and avoidance (Sliva & Roddick, 2001) and is similar to reflections reported in previous research (Wilson, 2014). When an incident is perceived as a threat, the outcome can be lack of action, emotion focused coping, or the strategy of minimisation.

The PSTs accepted blame, and felt inadequate, struggling with a lack of understanding. “We never understood what the formulas were or why they worked” (Joyce). “If I did finally work out how, as soon as the question changed slightly, I wouldn’t be able to do them” (Christine). Some responses showed the coping mechanisms that some PSTs used in situations that they found extremely stressful. “I didn’t understand and everything began to move away too quickly. I questioned and questioned but still couldn’t come to an understanding, so I quit.” (Hilary)

Nature of mathematics and ways mathematics was presented

The accounts highlighted the prevalence of a right/wrong dichotomy in school mathematics, and the discomfort that comes from mistakes. Mandy explained “this is how I viewed mathematics, as long as I knew the set of rules and applied them appropriately then I didn't really need to know why. To me mathematics was all about getting the right answer.” Kay wrote:

> Every morning we had an A4 sheet of multiplications. That just wrecked me. We were timed to do it. I couldn't do it and everyone else could. I still get anxious when papers are handed out in class and with multiplication.

Debbie said she “was able to retain the formula, and put the correct variable in it but I did not really understand the concept”.

Parents

Although comments about the influence of parents and families tended to be less common, some PSTs felt pressured by parental expectations:

> I was okay up until Year 9 when I was taken into the 5.3 pathway and I could deal with it for a bit but it got VERY overwhelming. Anyway, my parents wanted me to stay in 5.3- it was a big thing for them that I was 'excelling' but in truth I was drowning (Danielle).

> Dad was good at mathematics. Mum was not. I got blessed with mum's background. Mum tried to help. Dad could do it straight away. He said: “why are you crying, this is the answer.” Dad yelled the roof off. He couldn't see why I couldn't understand. I said: “I get it” to stop him. Then I wouldn't ask him. (Marilyn)

CONCLUSIONS

The purpose of this paper was to explore CIT as a mechanism to encourage and analyse prospective primary teachers’ reflections on key aspects of their mathematics learning experiences, and to better understand the impacts of these incidents on their anxiety about mathematics. The technique helps PSTs to reflect on their construction of themselves as a learner of mathematics, as a result of their perception of that event. CIT can provide the catalyst for introducing a contradictory consciousness (Gramsci, 1971) to question previous assumptions that they made as a student and stimulate different ways of reflecting on past experiences. Understanding their appraisal processes and coping strategies helps them to reassess their anxiety towards mathematics and their previous evaluations of themselves as potential teachers.

The findings contribute to teacher educators' knowledge and understanding of the experiences of PSTs and their context, and how CIT could be incorporated into teacher education courses. Teacher educators need to know about the experiences of PSTs, and the importance of verbalisation and sharing of their emotions. The research also provides insights for school teachers on how their actions may be interpreted by students, and strategies to help avoid stimulating students' mathematics anxiety.

Future research will investigate the application of the critical incident techniques used in the study, in combination with bibliotherapy, to investigate their potential to combat mathematics anxiety in PSTs. Additional analysis will explore the themes of shame and humiliation that have arisen so strongly in the accounts.

References

Chell, E. (1998). Critical incident technique. In G. Symon, & C. Cassell (Eds.). *Qualitative methods and analysis in organizational research* (pp. 51-72). London: Sage.

Ellsworth, J., & Buss, A. (2000). Autobiographical stories from preservice elementary mathematics and science students: Implications for K-16 teaching. *School Science and Mathematics, 100*(7), 355-364.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *The third handbook of research on teaching* (pp. 119-161). New York: Macmillan.

Flanagan, J. (1954). The critical incident technique. *Psychological bulletin*, 51(4), 327-358.

Gibbs, G. (1988). *Learning by doing: A guide to teaching and learning methods*. Oxford: Further Education Unit, Oxford Brookes University.

Harrison, J., & Lee, R. (2011). Exploring the use of critical incident analysis and the professional learning conversation in an initial teacher education programme. *Journal of Education for Teaching: International research and pedagogy, 37(2),* 199-217.

Hoyles, C. (1982). The pupil's view of mathematics learning. *Educational Studies in Mathematics 13*, 349-372.

Lazarus, R. (1991). *Emotion and adaptation*. New York, NY: Oxford University Press.

Lutovac, S. & Kaasila, R. (2009). Using narratives as innovative tools in Finnish teacher education. Proceedings of the Conference: 'Development of Competencies in the World of Work and Education". University of Ljubljana, Slovenia. http://www.decowe.com/static/uploaded/htmlarea/files/Using_Narratives_as_Innovative_Tools_in_Mathematics_Education_Course_.pdf

Mighton, J. (2004). *The myth of ability: Nurturing mathematical talent in every child.* Toronto: The Text Publishing Company.

Pedersen, P. (1995). *The five stages of culture shock.* Westport, CN: Greenwood Press.

Ricoeur, P. (1985). *Time and narrative.* Chicago: Chicago University Press.

Sheffield, D., & Hunt, T. (2006). How does anxiety influence math performance and what can we do about it? *MSOR Connections*, *6(4),* 19–23.

Sliva, J. & Roddick, C. (2001). Mathematics autobiographies: A window into beliefs, values, and past mathematics experiences of preservice teachers. *Academic Exchange Quarterly, 5*(2), 101.

Stubblefield, L. (2006). Mathematics anxiety among GED recipients in four-year institutions. *Journal of Mathematical Sciences and Mathematics Education, 1(2),* 19–23.

Tripp, D. (2012). *Critical incidents in teaching: developing professional judgement*. Classic ed. Milton Park, England: Routledge.

Vandercleyen, F., Boudreau, P., Carlier, G., & Delens, C. (2012). Pre-service teachers in PE involved in an organizational critical incident: emotions, appraisal and coping strategies, *Physical Education and Sport Pedagogy*, DOI:10.1080/17408989.2012.732564.

Vukovic. R., Keiffer, M., Bailey, S., & Harari, R. (2013). Mathematics anxiety in young children: Concurrent and longitudinal associations with mathematical performance. *Contemporary Educational Psychology, 38* (1) 1-10. Retrieved from http://www.sciencedirect.com/science/journal/0361476X/38/1.

Walsham, G. (1995). The emergence of interpretivism in IS research. *Information Systems Research, 6*(4), 376-394.

Wilson, S. (2014). "Fail at mathematics and you fail at life": Learned barriers to equal opportunities - mathematics anxiety and Quality of Life. In R. Brown & R. Faragher (Eds.), *Quality of Life and Intellectual Disability: Knowledge application to other social and educational challenges***.** Hauppauge, NY: Nova Science Publishers, Inc.

Wilson, S. & Thornton, S. (2008). “The factor that makes us more effective teachers”: Two pre-service primary teachers’ experience of bibliotherapy. *Mathematics Teacher Education and Development, 9,* 22–35.

Wilson, S. & Raven, M. (2014). “Change my thinking patterns towards mathematics”: A bibliotherapy workshop for pre-service teachers’ Mathematics Anxiety. In *Curriculum in focus: Research guided practice.* (Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia. Sydney: MERGA.) pp. 645-652.

USING NETWORK ANALYSIS TO CONNECT STRUCTURAL RELATIONSHIPS IN EARLY MATHEMATICS ASSESSMENT

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Awareness of Mathematical Pattern and Structure (AMPS) has been described as a general construct that underpins early mathematical development. Five structural groupings of concepts that contribute to AMPS were assessed through a Pattern and Structure Assessment (PASA) interview conducted with 818 Kindergarten and Grade 1 students. Network analysis was applied to map relationships between the levels of structural development and structural groupings coded from student responses. The network analysis revealed a complex web of interrelationships between students obtaining high and low AMPS levels and within and between particular structural groupings. The analysis showed also that responses in counting-based structural groupings may have masked the difficulties encountered in other structural groupings.

INTRODUCTION

Structural thinking can emerge from, or underlie mathematical concepts, procedures and relationships. Mason, Stephens and Watson (2009) view structural thinking as more than simply recognising elements or properties of a relationship but also as having deeper awareness of how those are used, explicated or connected. Over the past decade the Australian *Pattern and Structure* project has investigated, in a series of related studies, the development of patterning and structural relationships among 4 to 8 year olds across a range of mathematical concepts (Mulligan, Mitchelmore, English, & Crevensten, 2013). These studies have shown that *Awareness of Mathematical Pattern and Structure* (AMPS) underpins the development of relationships in mathematics and can enable simple forms of generalisation from an early age. An interview-based *Pattern and Structure Assessment* (PASA) and a *Pattern and Structure Mathematics Awareness Program* (PASMAP) have been developed and evaluated (Mulligan et al., 2013; Mulligan, Mitchelmore, & Stephanou, 2015).

In initial PME28 and PME29 reports a descriptive study of 103 first graders found that levels of structural development were consistent across a wide range of mathematical domains (see Mulligan et al., 2013). There was a high positive correlation (0.944) between student performance on 39 PASA tasks and four levels of structural development: pre-structural, emergent, partial, and structural (later extended to advanced structural). Multiplicative structure (including unitising and partitioning) and spatial structuring were found critical to development of pattern and structure. Individuals tended to show a single level across their responses, but the influence of task difficulty was evident.

At PME33, Mulligan and colleagues reported a 2-year longitudinal evaluation study of 316 Kindergartners (see Mulligan et al., 2013). They found highly significant differences on the PASA between PASMAP students and the ‘regular’ group at the retention point ($p < 0.002$) and increased levels of structural development for PASMAP students. The study produced a valid instrument (PASA) and a Rasch scale of AMPS. While there was wide variance in student AMPS levels across items, there was a positive correlation between high AMPS levels and the successful solution of a broad range of novel mathematical items (tasks), including items involving growing patterns and multiplicative reasoning, not usually expected of 5 and 6 year olds.

Research questions

Further research questions arose concerning: (i) the coherence of proposed Structural Groupings (SGs) of PASA items (ii) the extent to which variance in student AMPS levels was reflected consistently across the SGs, and (iii) the influence of task difficulty on consistency of structural levels of student responses. In order to address these questions, a new validation study was conducted to examine complex connectivity within AMPS using revised forms of PASA with Kindergarten and Grade 1 students. This paper aims to provide insights into such connectivity using an innovative approach based in network theory. Network analysis was used to map connections between the SGs for students with high or low AMPS levels in order to test a theoretical prediction that some students who performed at a high level in particular SGs may not have performed well in other SGs.

NETWORK ANALYSIS

Network theory, a modern development of graph theory, has proven useful in examination of complex connectivity of concepts apparent in student learning in mathematics (e.g., Woolcott, Chamberlain, Scott, & Sadeghi, 2014). Mowat and Davis (2010), working from within the assumptions of Lakoff’s embodiment theory (e.g., Lakoff & Núñez, 2000), developed a theoretical approach to complex learning in mathematics using network theoretical approaches. Network connectivity may underpin the development of mathematics expertise, with student failures related to inadequate development of concept connectivity (see e.g., Woolcott et al., 2014).

The exploration of networks and the connectivity of nodes within them, using empirical data, has developed rapidly in recent years (e.g., Newman, Barabási, & Watts, 2006). Network analysis offers significant potential, largely because the rules governing the relationships within such networks remain independent of the nature of the subjects being linked (Hanneman & Riddle, 2005) and because of rapid advances in software for analysing big data (e.g., Borgatti, 2012).

Although network analysis has been applied widely in a number of differing disciplines, from economics to neural pathways (Newman et al., 2006), there has been little such application in education (e.g., Kop & Hill, 2008). Network analysis, however, has been used, without software, to examine the development of learning in school mathematics (Strom, Kemeny, Lehrer, & Forman, 2001). At PME38, Woolcott et al. (2014) applied network analysis to the examination of mathematics multiple choice items and student responses, both within and across Grades 3 to 6. Such analysis illustrated complex connectivity between concepts derived from curriculum outcomes and concepts inherent in items, for example between words, symbols and graphics. This complex connectivity suggests opportunities for educators to recognise and better utilise conceptual relationships across mathematics.

METHOD

The study employed a purposive sample of 396 Kindergarten and 422 Grade 1 students from two metropolitan schools and represented students from a diverse range of cultural and socio-economic contexts. The PASA interviews were conducted consistently following protocols from previous studies (see Mulligan et al., 2013). Six trained interviewers piloted protocols for conducting the interviews and coding responses, with inter-rater reliability 0.82. The interviewers coded student responses to each item as one of five structural levels according to PASA descriptors: pre-structural (L1); emergent (L2); partial (L3); structural (L4); and, advanced structural (L5).

PASA - The pattern and structure interview-based assessment instrument

There were two forms of PASA used in interviews, PASA1 (14 items) and PASA2 (16 items), with 7 common items (see Table 2). The items in each of PASA1 and 2 were based on five SGs (see Table 1 and description following the table).

Sequences (SG1) involves recognising a (linear) series of objects or symbols arranged in a definite order or using repetitions, such as repeating and growing patterns and number sequences.

Structured Counting (SG2) involves counting in groups, such as counting by 2s or 5s or on a numeral track with the equal grouping structure recognised as multiplicative.

Shape and Alignment (SG3) involves recognising structural features of two- and three-dimensional (2D & 3D) shapes and graphical representations, and constructing units of measure, such as colinearity (horizontal and vertical coordination), similarity and congruence and such properties as equal sides, opposite and adjacent sides, right angles, horizontal and vertical, parallel and perpendicular lines.

Equal Spacing (SG4) involves partitioning of lengths, other 2D or 3D spaces and objects into equal parts, such as constructing units of measure. It is fundamental to representing fractions, scales and intervals.

Partitioning (SG5) involves the division of lengths, other 2D or 3D spaces, objects and quantities, into unequal or equal parts, including fractions and units of measure.

Structural grouping	PASA1	PASA2	Structural grouping	PASA1	PASA2
SEQUENCES (SG1)	Item number		SHAPE & ALIGNMENT (SG3)	Item number	
Repeating pattern	1	—	Grid completion	11	11
Border pattern	2	2	EQUAL SPACING (SG4)		
Spatial pattern continuation	7	7	Distance	14	—
Visual memory 1 - Triangular array	3	3	The ruler	—	14
Growing pattern continuation	—	13	The clock	10	10
STRUCTURED COUNTING (SG2)			Bar chart	—	15
Visual memory 2 - Rectangular array	8	8	PARTITIONING (SG5)		
Skip counting by 3s	6	6	Partitioning length thirds	5	1
Groups of four	9	—	Comparing triangles	—	12
Ten frames	13	5	Partitioning money	4	4
Hundred chart	—	9	Comparing capacities	12	16

Table 1: Structural Groupings (SGs) of PASA1 and 2 items.

The development of the AMPS construct is based on the interrelationships between these SGs where some features are salient across SGs and some are more integral to a particular SG. For example, Sequences (SG1) and Structured Counting (SG2) both involve the idea of equal groups or units represented in a linear way. These SGs may be linked to Shape and Alignment (SG3) where students may count using a 2D grid. Equal Spacing (SG4) and Partitioning (SG5) both involve division into equal parts. A student's AMPS level influences how these interrelationships may occur for that individual.

Statistical analysis

Primary analysis, based on coded student responses for one of the five structural levels, resulted in an AMPS scale (see e.g., Mulligan et al., 2015). Data for each of PASA1 and 2 for each individual were organised in a matrix (item by structural level) with matrix construction following Woolcott et al. (2014). A statistical analysis was conducted on responses in this matrix to determine item difficulty in each of PASA1 and 2 as a complement to network analysis. The percentage of responses at each level for each item was reviewed and, if this percentage was greater than 33%, the response across the cohort at this level was considered as 'high frequency' (see Table 2 in Results and Discussion).

Network analysis: Structural Groupings (SGs)

Each matrix was analysed using network software (Borgatti, 2012) in order to visualise (map) pairwise connections between items for students who obtained level 5 and/or 4 (L5/L4) in some items and level 2 and/or 1 (L2/L1) in others. If students, for example, obtained L4 in a PASA1 item, then a network map (graphical representation) was generated that showed connections to the other items in which these students also obtained L4 and, as well, connections also to items in which these students also obtained L1.

There were four maps generated to show all such connections, with a supplementary set of 24 maps providing more significant connections on a 10% to 60% basis—for example, where 10% of the students who obtained L4 on item 8 also obtained L4 or L2 on other items (e.g., see Figure 1). If at least one student obtained the same level on each item in an entire SG, then this was considered a ‘coherent grouping’. Network maps were used in conjunction with the statistical analysis to examine variance across levels, and coherence within SGs.

RESULTS AND DISCUSSION

Statistical analysis

There was variance across levels for most students: no student obtained a single level in all items in either PASA1 or 2; and, no student obtained only L4 or only L5 in either assessment. This variance appeared biased toward high levels in SG1 (Sequences) and SG2 (Structured Counting). There was a high frequency, for example, for numbers of students who performed at L5/L4 in PASA1 in the SG1 and SG2 items 1 & 7 and 6, 8, 9 & 13, respectively (Table 2). In contrast, there were higher frequencies in L2/L1 in the SG4 (Equal Spacing) and SG5 (Partitioning) items 4, 5 & 12 and 10, respectively (Table 2). In PASA2, there were also high frequencies in L5/L4 in SG1 and SG2 and high frequencies in L1/L2 in SG4 and SG5, but there were also high frequencies in the single item SG3 (Shape and Alignment) at L5/L4.

	PASA1		PASA2
Level	High frequency (>33%) of responses in items	Common items	High frequency (>33%) of responses in items
1	4, 5	4, 5 (=1)	1, 4, 12
2	8, 9, 10, 11, 12	12 (=16)	14, 16
4	1, 6, 7, 9, 12	6, 7, 12 (=16)	6, 7, 11, 16
5	1, 7, 8, 13	8	6, 8, 9, 11

Table 2: Summary of high frequency levels of responses in PASA1 and 2 items.

The difference between the frequency of responses at L5/L4 and L2/L1 for SG1 and SG2 may be attributed to the relative difficulty of the items or to the way that students used these structures. For example, in item 1 in SG1 the model of the repeating pattern provided the structure of the unit of repeat and hence this item was easier than those in which a student had to construct the unit (Mulligan et al., 2015). Similarly, SG2 items involved a student using structured representations (e.g., number track), where the structure is partially developed. In some cases, a student identified the structural features of the representation, such as identifying equal spaces, but others managed to elicit a correct response without utilising the structural feature. Hence, it was possible to use superficial strategies to obtain a L5/L4 response without attending to deep structural features (e.g., see Mulligan et al., 2013).

Network analysis

This paper focuses on three selected features of the network maps: connections from high (L5/L4) to low (L2/L1) AMPS levels; connections within SGs; and, connections between SGs, these being observed at L5/L4 with students connected to L2/L1.

First, using all connections, the network maps showed that in PASA1 and 2 there were some students who obtained L5/L4 in one item who also obtained L1/L2 in at least one other item. Although the items were similar to those of high frequency in the statistical analysis, an averaged value, the network maps provide evidence that some individuals obtained several of the high frequency items connected at either L4/L5 or L2/L1. This variance, however, was not consistent across the cohort when more detailed connectivity maps (10%-60%) were examined (e.g., see Figure 1).

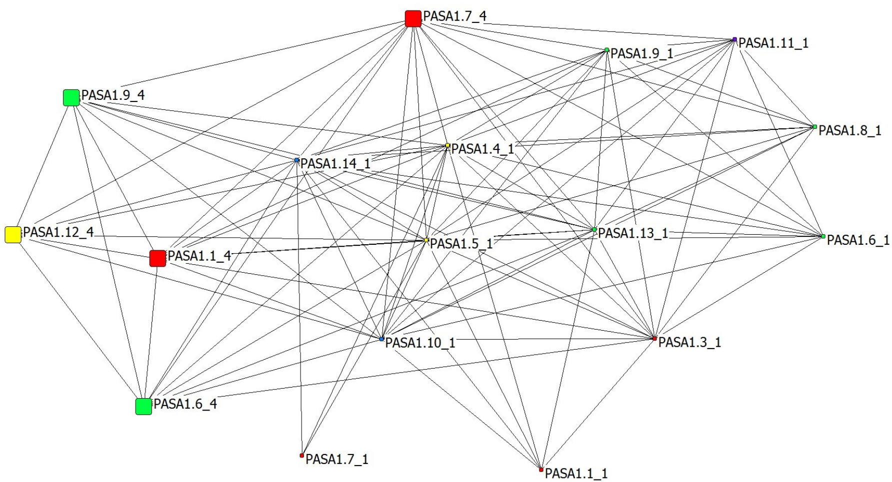

Figure 1: Network map showing detail of connections (10%) from L4 to L1 in PASA1. The annotation PASA1.6_4 indicates PASA1, item 6 at L4.

In PASA1, there were no connections of L5 with L2/L1 in any of the detailed maps and in PASA2, connections of L5 items with L2/L1 were seen only in one detailed map

(10%), for the counting-based items in SG2 and item 11 in SG3 (at L5). This was expected since very few students obtained the most advanced structural level (L5). What distinguishes students at L5 is that they can generalise.

In PASA1 there were, however, L4 to L2/L1 connections for SG1 and SG2, as well as for item 12 in SG5, but only in the 10% map (see Figure 1). This connectivity was similar in PASA2, except that additional connections were found for item 10 in SG4 (clock) and item 16 in SG5 (capacity). This may reflect item difficulty, i.e., item 12 (=16 in PASA2) involved comparing the capacity of cups where the cup size was given. Students may, therefore, have used estimation and superficial counting strategies. These findings again reflect the statistical analysis, but pinpoint individuals with connections across these SGs.

Second, in both PASA1 and 2 there were both coherent groupings and connections within SGs at L5, for maps of all connections of L5 with L2/L1, but not in the detailed maps. This was similar for all connections of L4 with L2/L1, but in detailed maps there were connections within the counting-based SG1 and SG2.

Third, in PASA1 and 2, maps for all connections of L5 with L2/L1, there were connections between some items across all SGs, but predominantly between items across SG1, SG2 and SG3. There were, however, no connections between SGs in the more detailed maps. This may reflect the small number of students at L5. In PASA1 and 2, maps for all connections of L4 with L2/L1 showed connections between the coherent groupings SG1 and SG2, and these also connected to SG3. Connections between items across all SGs were seen in all maps of L4 with L2/L1. This finding is consistent with other forms of connectivity shown for the SGs linking simple repetitions, growing patterns and counting sequences.

IMPLICATIONS AND FUTURE RESEARCH DIRECTIONS

This paper provides only a limited description of particular forms of connectivity seen in the network analysis, but it does provide exemplars of important interrelationships between mathematical concepts. The application of network theory outlined here draws on extensive research on complex connectivity in mathematics (e.g., Lakoff & Núñez, 2000; Mowat & Davis, 2010) and the exemplars provide an initial examination of whether network analysis is functional in the context of early mathematical development. The innovative combination of statistical and network analysis illustrated in this paper provided insight into the complex connectivity in the construct AMPS (Mulligan & Mitchelmore, 2009). This complementary approach indicates that students who obtain high AMPS in the counting-based SG1 and SG2 may also obtain high AMPS in items where they can use superficial counting strategies. Importantly, an arguably significant number (10% or more) of these students obtained also L2/L1 in items of other SGs, such as SG4 and SG5.

A more comprehensive study is being undertaken to examine whether the theoretical SGs are as meaningful as the initial analysis here suggests. Such a study will be multidimensional and make use of network analysis as a complement to other statistical methods, such as Rasch analysis (e.g., Mulligan et al., 2015). This may shed more light on the interconnectivity of theoretical SGs since these sometimes overlap. This may also facilitate the disentanglement of the influence of item difficulty and student awareness of pattern and structure in their responses—task difficulties may depend partly on prior knowledge, skills and experiences as well as on AMPS. This may be complicated by differential individual student development across the structural elements being examined. A more comprehensive study may also enable examination of the limitations of using a small numbers of items in each SG. Additionally, longitudinal network analysis of the growth of student AMPS over time could provide new direction for establishing pedagogies promoting more coherent structural development in early mathematics.

References

Borgatti, S. (2012). *UCINET: Graph Visualization Software, v6.509*. Harvard: Analytic Technologies.

Kop, R., & Hill, A. (2008). Connectivism: Learning theory of the future or vestige of the past? *International Review of Research in Open and Distance Learning*, *9*(3), 1-13.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.

Mason, J., Stephens, M. & Watson, A. (2009). Appreciating structure for all. *Mathematics Education Research Journal*, *2*(2), 10-32.

Mowat, E., & Davis, B. (2010). Interpreting embodied mathematics using network theory: Implications for mathematics education. *Complicity: An International Journal of Complexity and Education*, *7*(1), 1-31.

Mulligan, J. T., & Mitchelmore, M. C. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21*(2), 33-49.

Mulligan, J. T., & Mitchelmore, M. C., & Stephanou, A. (2015). *Pattern and Structure Assessment (PASA): An assessment program for early mathematics (Years F-2) teacher guide*. Australian Council for Educational Research. Melbourne: ACER Press.

Mulligan, J. T., English, L. D., Mitchelmore, M. C., & Crevensten, N. (2013). Reconceptualising early mathematics learning: The fundamental role of pattern and structure. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 47-66). New York: Springer.

Newman, M. E. J., Barabási, A. -L., Watts, D. J., (2006). *The Structure and Dynamics of Networks*. Princeton, NJ: Princeton University Press.

Strom, D., Kemeny, V., Lehrer, R., & Forman, E. (2001). Visualizing the emergent structure of children's mathematical argument. *Cognitive Science*, *25*(5), 733-773.

Woolcott, G., Chamberlain, D., Scott, A., & Sadeghi, R. (2014). Mapping concept interconnectivity in mathematics using network analysis. In P. Liljedahl, C. Nicol, S. Oesterle & D. Allan (Eds.), *Proceedings of PME38/PME-NA36* (Vol. 5, pp. 385-392). Vancouver, Canada: PME.

SOME FEATURES OF MATHEMATICS ANXIETY FROM COGNITIVE NEUROSCIENCE FOR THE FUNCTIONAL TASKS

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This study used EEG in the functional tasks to investigate mathematics anxiety. We divided the participants into two groups of undergraduate students, twenty of high mathematics anxiety (HMA) and twenty of low mathematics anxiety (LMA). All participants solved the functional tasks composed of the quadratic function with graphical and algebraic representations. To this end, HMA group had longer reaction time and larger amplitude than LMA group. In addition, both LMA and HMA groups had larger amplitude in graph-to-algebra tasks than algebra-to--graph tasks. We suggest that mathematics anxiety must be studied through EEG in more specific ways to help students to lessen their MA.

INTRODUCTION

In the modern research, the topic of mathematics anxiety (MA) in mathematics education, which is experienced during learning mathematics has been interested since the mid-19th century. Especially in Korea, the problem of the affective domain is getting more attention than the cognitive domain in the mathematics achievement in relation to building up the character of education. By the advance of the tools in the brain science technology, it becomes possible to measure EEG that had been hard to check before.

This study was to find some features of MA by brain-based measurement in order to understand clearly what happens in our brains during doing mathematical tasks, and to provide a way to study MA with implications for the future of mathematics education. For the purpose of the study, in order to investigate the relationship MA and EEG measurement in the tasks of function, the research questions were set as follows: Firstly, what percent showed in the correct answers and the reaction time between the MA groups? Secondly, what differences of functional thinking in EEG showed between the graph-to-algebra and the algebra-to-graph translations of the HMA and LMA groups?

THEORETICAL FRAMEWORK

1. Definition of MA

In this study, we defined that MA is individual's feeling of anxiety in situations where it is necessary to solve a mathematical problem.

2. Brain-based Research

One of the research of MA through the EEG during execution of the arithmetic is Colome, Nucez-Pena, and Suarez-Pellicioni (2013). In their study, the researchers analysed the differences of people who were evaluated as high mathematics anxiety (HMA) when solving arithmetic problems by EEG. Since it was P600, and P3b waves that were known related to logic, decision, and cognition in brain, the researchers analysed them on the EEG by event-related brain potentials (ERPs). This experiment illuminated the research of Faust (1996) by EEG. Faust (1996) had argued that children who had HMA needed more time to solve simple or comlpex addition than children who had LMA. Regardless of MA, there are some research using brain-based measurement in the functional tasks, which are Waisman et al. (2013), and Thomas et al. (2010). These studies suggested how the functional tasks were constituted and practiced. In the study, Thomas et al. (2010) classified functional concepts into four formats because this functional concept is too complex and limited portions that could be measured by EEG. They used linear function and quadratic function made into eight tasks which were constituted in four formats, i.e., graph to graph, algebra to algebra, graph to algebra, and algebra to graph.

Waisman et al. (2013) developed the tasks of algebra and graph for children divided into five groups who were genius children, normal children, children having good mathematics practice, children not, and super genius children. They tried to find out factors about the genius by ERPs. In results, the children of more genius or more having more good mathematics practice had smaller EEG width. This meant that when solving problems, HMA group had more loads in the brain than LMA group.

METHODS

1. Participants

To achieve this research purpose, the participants were 40 of undergraduate students attending one university which located in Kyunggi State, Korea. They consisted of 20 of students who enrolled in the department of natural science and 20 in the department of humanities and society. Each participant completed the MASS before taking the mathematical tasks. After the data were collected, we analysed statistically using SPSS after using the brain wave analysing program, ERP. An average of MASS of the total of 40 students was shown 2.99 points. Therefore students were divided into the low mathematics anxiety (LMA) group and the high mathematics anxiety (HMA) group based on 3.0. On the contrary, the MA of students of social sciences was scored at average 3.7 which was much higher than the students of natural science. In particular, MA in the factor of “test and performance” had the highest score on average 4.0. They showed the lowest core, 3.0 in the factor of 'motivation'.

2. Research Instruments

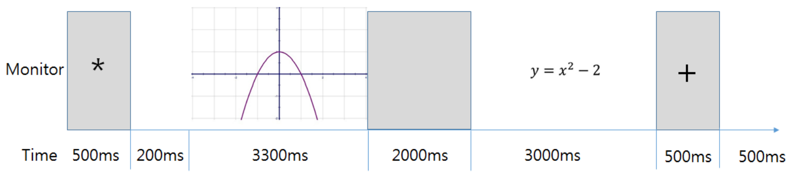
1) Mathematics Anxiety Scale for Student

Ko, & Yi (2011) revised a Mathematics Anxiety Scale for Students (MASS), referring to MARS, MAS, MAQ, MARS for Adolescent students developed in the previous research in order to identity mathematics anxiety per factor and to help Korean students figure out what factors of MA affected more so that the reasonable intervention can be provided. It was constructed in a total of 65 items of the questionnaire. The MASS was divided into four domains with twelve factors. Its alpha value of Cronbach per factor was proved from 0.7 to 0.9. We revised some words and phrases to make them proper to the undergraduate students.

2) Functional tasks

Functional tasks were composed of $a=\{1, -1\}$, $p=0$, $q=\{-2, -1, 0, 1, 2\}$ in standard type $y=a(x-p)^2+q$. Each participant solved a total of 20 of graph-to-algebra and 20 of algebra-to-graph problems. Thus, each participant solved a total of 40 numbers of problems.

The progressive order of functional tasks was as follows. After informing the beginning with a mark, "*" presented for 500ms, there is a blank time of 200ms. After the graph or algebra was presented for 3300ms, there is a blank time of 2000ms. Then, After the algebra or graph was presented for 3000ms, then it was supposed to inform the end with a mark, "+" for 500ms. Finally, there was a blank time of 500ms before the next problem was presented. Therefore, the total time to solve one set of problems was



10000ms.

Fig. 1: Time table of one graph-to-algebra format

3. Methods of Measuring EEG

The machine was the Brain Products, V-AMP developed by Brain Vision standard Inc. The electrode is composed 16 channels. Their Professional Recorder software and analysis program were used as an Analyzer. For the method of the measurement EEG, we used the event-related brain potentials (ERPs) measurement method in the case of using the visual stimuli. Because ERP experiments are performed by opening the eyes, eye movement can affect the measurement of EEG so we used the filtering function of Analyzer to remove the unrelated signals.

EEG measurements were carried out in a state of controlling the movement of the subject's body in a quiet environment. Interrupts in EEG generated by the influence of electromagnetic waves of the PC and the monitor used to measure the EEG, were reduced by allowing some distance about 1.5m between the monitor and a participant. Also, the experiments were carried out by using a shielding fiber. After the participant entered the experiment room and had the time of 10 minutes to adjust the electrode to wear the EEG senser, and another 10 minutes of time are needed to stabilise EEG. When brain waves were stable on the screen, they were asked to raise their hand to run the E-Run program and executed the task. After the task performance was saved in the EEG, we removed the EEG senser and conducted simple interviews with participants.

RESULTS

1. Research question 1

	MA	RT
	N=40	
MA	-	
RT	.681*	-

* p<.05 (Mathematics Anxiety=MA, Reaction Time=RT)

Table 1: Correlation analysis of MA and RT in algebra to graph format

	MA	PCA
	N=40	
MA	-	
PCA	.131*	-

* p<.05 (Mathematics Anxiety=MA, Percent of Correct Answers=PCA)

Table 2: Correlation analysis of MA and PCA in algebra to graph format

As we see in the table above, the relationship of MA and RT was statistically significant in algebra to graph format (r=.681, p<.05). That is, MA and RT were found to have a correlation of about 46% (r^2 =.46), having a positive correlation. This is the same as the result of Colome (2013) that the process of the group of HMA in determining the working memory took longer. Contrariwise, MA and PCA were not correlated significantly (r=.131, p<.05).

In the graph to algebra format, the average of RT and the average of the HMA group were 2.105 second and 89.793% respectively. The answers of 47 problems out of a total of 460 problems were incorrect. Among 47, in 20 of correct problems they chose incorrect answers and in 27 of incorrect problems they did in an opposite way.

In graph to algebra format, the average of RT and the average PCA of LMA group were 1.576 second and 90.699% respectively. The answer of 32 problems out of a total of 340 problems is wrong. Among 32, in 14 of correct problems they chose incorrect answers and vice versa in 18 problems.

	MA	RT
	N=40	
MA	-	
RT	.552*	-

* $p<.05$ (Mathematics Anxiety=MA, Reaction Time=RT)

Table 3: Correlation analysis of MA and RT in graph to algebra format

	MA	PCA
	N=40	
MA	-	
PCA	.269*	-

* $p<.05$ (Mathematics Anxiety=MA, Percentage of Correct Answers=PCA)

Table 4: Correlation analysis of MA and PCA in graph to algebra format

As we see in the table above, the relationship of MA and RT was statistically significant in graph to algebra format (r=552, $p<.05$). That is, MA and RT were found to have a correlation of about 30% (r^2 =.30), which had a positive correlation. Also, MA and PCA had a weak correlation (r=.269, $p<.05$).

	HMA group	LMA group
algebra to graph	93.044%	92.069%
graph to algebra	89.793%	90.699%

Table 5: PCA by the degree of MA in formats

	HMA group	LMA group
algebra to graph	1.567 sec	1.201 sec
graph to algebra	2.105 sec	1.576 sec

Table 6: RT by the degree of MA in formats

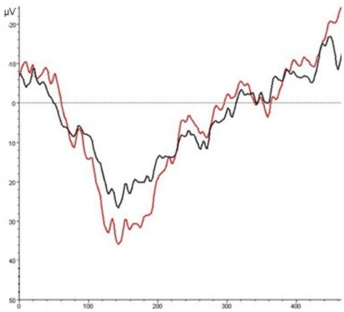
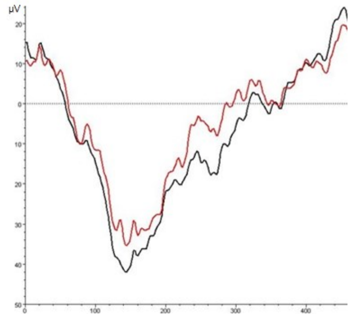
As a result, HMA group responded more slowly than the LMA group when performing the graph to algebra format tasks

2. Research question 2.

In this study, we analysed P300 which was a brain wave that corresponds to 200ms ~ 400ms from stimuli. "The P300 is considered to be an endogenous potential, as its occurrence links not to the physical attributes of a stimulus, but to a person's reaction to it. More specifically, the P300 is thought to reflect processes involved in stimulus evaluation or categorization” (referred to Wikipedia). We tried to observe EEG of two processes. One process is that students watched the graph of a quadratic function first and memorised it, then verified whether an algebraic expression related to the graph through visual stimuli was true or not. We named this process the graph to algebra format. The other process is the reverse of these two representations which we named the algebra to graph format.

 Fig. 2: EEG in algebra to graph tasks	 Fig. 3: EEG in graph to algebra tasks
black: HMA, red: LMA	black: HMA, red: LMA

As shown above, the result of EEG in both algebra to graph format and graph to algebra format was that the amplitude of HMA group was shown larger than LMA group. In the algebra to graph format, the maximum amplitude of LMA group was -28µV and the maximum amplitude of HMA group was –36µV (See Fig. 1). In the graph to algebra format, the maximum amplitude of LMA group was –35 µV and the maximum amplitude of HMA group was –42µV (See Fig. 2). It was analysed that students with HMA showed greater loads on working memory when solving two kinds of tasks. This result is matched to the previous study.

 Fig. 4: The difference of S1 and S2 in LMA group	 Fig. 5: The difference of S1 and S2 in HMA group
black: S1, red: S2	black: S1, red: S2

The EEG outcome when performing algebra to graph format is referred to S1. The EEG outcome when performing graph to algebra format is referred to S2. Figure 4 shows that the difference of minimum of S1 and S2 is 11 in LMA group. Figure 5 shows that the difference of minimum of S1 and S2 is 14 in HMA group. And the amplitude of S2 was larger than S1 both LMA and HMA group. This means that there are more loads on working memory when performing the graph to algebra format than the algebra to graph format.

CONCLUSION

The correlation between the percent of correct answers and the math anxiety did not show any significant difference regardless of the groups of mathematics anxiety, since the participants had the skills to understand the quadratic function as college students. In addition, the participants who had mathematics anxiety or not had the lower percent~~age~~ of correction and the longer reaction time at the tasks from graph to algebraic equation than the tasks from algebraic equation to graph. Also, as a result of the analysis of the EEG brain waves of a HMA group and LMA group, HMA group's brain waves were recorded higher amplitude. So we founded HMA group was using more working memory of brain to solve the same task than LMA group. Through this analysis, we found math anxiety with psychological pressure affected by brain nerves. In light of this result, we suggest that teacher must construct teaching materials carefully and provide it by realising the tasks from graph to algebraic equation are much harder tasks. In addition, when performing functional tasks based on the fact that there were significant differences in reaction time, we should be careful that sufficient time be provided. In particular, it needs more careful concern to the HMA group.

Of course, this research has many limitations and restrictions. However, before the present study, there has been no study about MA through functional tasks using EEG. We suggest that research should be studied about MA using the EEG in a more specific way. It needs more understanding of MA with sufficient samples, and more technically advanced neuroscience devices.

References

Colome, A., Nunez-Pena, M. I., & Suarez-Pellicioni, M. (2013). Mathematical anxiety effects on simple arithmetic processing efficiency: An event-related potential study. *Biological Psychology, 94*, 517-526.

Faust, M. W. (1996). Mathematics anxiety effects in simple and complex addition. *Mathematical Cognition*, *2*(1), 25-62.

Kaput, J. J. (1998*)*. Representations, inscriptions, descriptions, and learning: A kaleidoscope of windows. *Journal of Mathematical Behavior*, *17*(2), 265–281.

Ko, H. K., & Yi, H. S. (2011). Development and validation of a mathematics anxiety scale for students. *Asia Pacific Education Review, 12*(4), 509-521.

Thomas, M. O., Wilson, A. J., Corballis, M. C., Lim, V. K., & Yoon, C. (2010). Evidence from cognitive neuroscience for the role of graphical and algebraic representations in understanding function. *ZDM*, *42*(6), 607-619.

Waisman, I., Leikin, M., Shaul, S., & Leikin, R. (2014). Brain activity associated with translation between graphical and symbolic representations of functions in generally gifted and excelling in mathematics adolescents. *International Journal of Science and Mathematics Education, 12*(3), 669-696.

CHINESE SECONDARY TEACHERS' AND STUDENTS' PERSPECTIVES OF EFFECTIVE MATHEMATICS TEACHING: THE UNDERLYING VALUES

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This paper explores the underlying values of effective mathematics teaching perceived by both secondary teachers and students in Chinese Mainland. Four secondary mathematics teachers and twenty-four students accepted an invitation to express their perceptions of effective mathematics teaching and of effective mathematics teachers. Some common characteristics of effective teaching like fun, multiple methods, involvement, and examples are valued by both students and teachers. Some discrepancies in the perceptions of effectiveness also existed.

INTRODUCTION

Effective mathematics teaching has long been a focus of attention in mathematics education around the world. NCTM (2000) proposed that "effective mathematics teaching requires understanding what students know and need to learn, and then challenging and supporting them to learn it well" (NCTM, 2000, p. 16). However, the construct of effective teaching is often ill-defined and subject to differing interpretations (Wilson, Cooney, & Stinson, 2006). Most research implied either that effective teaching is a well-agreed-upon construct or that it is so relative to any classroom context that it is meaningless to establish a single definition (Anthony & Walshaw, 2008; Atweh & Seah, 2008). In various international comparative studies, such as TIMSS and PISA, they generally arrive at similar conclusions that effective teaching is more about responding to and valuing the socio-cultural aspect of the learning environment than it is about adopting particular teaching methods (Hollingsworth, Lokan, & McCrae, 2003; OECD, 2004).These different socio-cultural contexts might be a major factor to explain why the definition of effective teaching is ill-defined in research. Each person could have his/her own perceptions of effectiveness, which – we will argue here – is regulated by what s/he values in effective teaching.

Values are general guide for the behaviour emerging from one's experience and relations in his/her life (Raths, Harmin, & Simon, 1987). Values in mathematics education are inculcated thought the nature of mathematics and individual experience, and thus become the personal convictions that an individual regards as being important in the process of learning and teaching mathematics (Seah & Kalogeropoulos, 2006).Values related to mathematics education express the extent to which we value aspects of classroom norms and practices that relate to the teaching/learning of school mathematics (Bishop, 2008).Teachers' and students' values on effective teaching play a subtle yet influential role. Although research has investigated teachers' or students'

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views or beliefs on effective teaching in different cultural contexts (e.g. Cai, Kaiser, Perry, & Wong, 2007; Wilson, Conney, & Stinson, 2005), much of existing research about effective teaching had been investigated in Western cultures. Recently, Chinese students have achieved outstanding performances in TIMMS and PISA. With an increasing interest in unfolding the 'Chinese learner's phenomenon', we are interested in looking into how the Chinese value effective mathematics teaching.

The study reported here is part of the Third Wave Project, an international collaborative research consortium of 21 research teams located across 18 different countries/regions. The Third Wave Project is aimed at identifying specific understandings of what constitutes effective mathematics teaching through harnessing the socio-cultural factors of values from teachers' and students' perspectives (Seah & Wong, 2012). In this study, the socio-cultural perspective of considering values as personal convictions that an individual regards as being important enough to be emphasized will be adopted to help us interpret what constitute effectiveness in the lessons observed, and in turn stimulate the conceptualisation of effective teaching. Specifically, this study seeks to shed light on the following research questions:

> What are Chinese teachers' and students' perceptions of an effective mathematics lesson and an effective teacher?
>
> What might be the underlying values of effective teaching from Chinese teachers' and students' perspectives?

RESEARCH METHODOLOGY

Participants

Four secondary mathematics teachers from Shenzhen city located in the south of the Chinese mainland were invited – and subsequently agreed – to share their perceptions of effective mathematics lessons and effective mathematics teachers through teacher journals prior to lesson visits and semi-structured interviews after lesson visits. Two teachers (A1 and A2) are from a junior secondary school (School A). Both of them have ten years teaching experience each. The other two are from a senior secondary school (school B). One (B1) has taught 5 years and the other (B2) has taught 20 years. A total of 24 secondary students participated in the study. Their ages ranged from 13 to 16 years. Six students are chosen by each teacher according to their performance. Two students are high-achieving, middle-achieving, and low-achieving respectively. The goal of this study involves understanding, describing, discovering, and hypothesis/theory generating, all characteristic of qualitative research (Neuman, 2003). Validity of research findings will be enhanced through triangulation of data sources in two ways, namely through the use of multiple data sources (written documents, visual documents, lesson observations, interviews, and research field notes), and through the inclusion of cross-checking mechanism within each data source.

Data collection

The data collection in this study includes the following sections: open-ended questions through teacher journal and questionnaire, lesson observation, and interviews. Prior to the first lesson visit, each teacher participant was invited to maintain a teacher journal for three weeks. In the journal, each teacher was asked to respond to the following open questions:

- In your opinion, what should an effective mathematics lesson look like?
- What is your vision of being an effective teacher of mathematics?
- Why do you think you have been nominated as a particularly effective teacher of mathematics? And,
- What would make you an even more effective teacher of mathematics?

Each student participant was asked to respond to an open-ended questionnaire. Items in the questionnaire included these two questions:

- A good mathematics lesson should be: _________.
- A good mathematics teacher should be:_________.

All open-ended questions aim to investigate their views of effective mathematics teaching. “Good lesson” or “effective lesson/teaching” is both used in the study.

In order to have a deep understanding of what teachers and students perceived of effective teaching, three mathematics lessons facilitated by each teacher participant were observed. During observation, the six student participants of each teacher would raise a bottle of water when they felt they were learning mathematics particularly well in the lesson. These moments were recorded by observers and a video camera, which provided the basis for the teachers’ and students’ interview sessions. During the students’ group interviews, these students were encouraged to talk about what they found important at the respective moments of effective teaching, the extent to which identified values were unique and personal to them, and the extent to which they emphasised those values in their mathematics learning experiences. In the teacher’s interview, each teacher was expected to talk about the effective teaching moments what they thought. All the data in this study, also in the Third Wave project, were analysed through a multiple-pass approach, utilizing the three-staged open, axial, and selective coding that typifies the grounded theory research approach (Seah & Wong, 2012; Strauss & Corbin, 1990).

RESULTS AND DATA ANALYSIS

The study aimed to identify specific understanding of what constitutes effective mathematics teaching in China, from both students’ and their teachers’ perspectives. In particular, we were interested to achieve this by identifying and considering the underlying values. From the students’ open-ended questionnaire, eight values in effective mathematics teaching were found. They are *fun, involvement, creativity, board writing, multiple methods, explanation, focus and examples*. Indeed, half of the students valued an enjoyable classroom environment (coded as *fun*). While many students valued more on involvement in the classroom, teachers’ clear and detailed explanations were also important to them, which included giving summaries and establishing relationships between mathematical concepts. In particular, the students thought an effective lesson should include both important points and difficult points of content, which was coded as the value of *focus*. Junior secondary students emphasized more on *involvement* than senior secondary students. *Board writing* and *focus* were only valued by senior secondary students.

Further, we investigated these students' perceptions of what an effective teacher should look like. Generally, across the four grades, the students expressed the view that a good mathematics teacher needs to have humour, to be able to give clear expressions, teach in a lively and skilful manner, and to be able to encourage and motivate their students. Particularly, there were 10 students who highlighted the importance of teachers' encouragement. Only junior secondary students proposed a good mathematics teacher should be warm and patient. Here, *patience* means that the teacher could take the time and effort to explain problems to students. It was also found that junior students valued more on teachers' personalities (e.g., having passion, genial, humorous, patient), senior secondary students valued more on teachers' teaching manner (e.g., lively, having focus, motivate students) (see Zhang, 2014).

What are teachers' perceptions of an effective lesson? Our analysis of the teacher journal entries revealed that all teachers emphasized students' active *participation* and students' *understanding* as important factors of an effective mathematics lesson. Particularly, junior secondary mathematics teachers put more focus on students than senior secondary teachers when they talked about an effective lesson. For instance:

> In an effective lesson, each student could benefit. In an effective mathematics classroom, the emphasis is not how much knowledge is taught, but whether each student has understood what the teacher taught. (A1)

> In an effective lesson, students can understand what the teacher says, can solve problems, and can speak out their own thinking. (A2)

Senior secondary mathematics teachers also emphasized the role of students in an effective lesson, but at the same time they also emphasized the teacher's influence.

> An effective lesson should be student-centred. At the same time, teacher is a leader. (B1)

> The teacher should know his students, emphasize their feedback and let them actively involve the activities. The teaching content should be clear, specific, and meet the requirement of examination syllabus. The teaching strategies should be flexible. (B2)

To become an effective teacher, junior secondary teachers held the perception that the teacher should love his/her students and be appreciated by them. Senior secondary teachers valued teachers' professional knowledge and the ability of proactive reflection. Among the four teacher participants, only one senior secondary teacher (B2) thought of himself as a successful teacher. Twenty years of teaching experience might have made him more confident. He also valued his students' performance and peers' evaluations on him with regard to an effective teacher. The other three teachers, however, felt that the passion of teaching and teaching experience would make them become an effective mathematics teacher.

After classroom observations, the teacher and student participants were interviewed about the "effective moments" during the observed lessons. An example of such an effective moment could be found in one of A2's lesson. At that time, A2 had thought there was an effective moment when a student answered a question by using his own method. Particularly, his method was out of her expectation. She was willing to let students speak more.

> This example [about quadratic equation] has two emphases. One is to decide the equal relationship and the other is to decide the proper solution from two roots. His [a student] method is out of my expectation. He is well done and his explanation is also good. So I let him go on even though we spend much more time than my schedule. (A2)

This effective moment was also valued by her students during the group interview. One student remarked that "I think that moment is effective. His method is great! Nobody think of it except him" said by a student of A2. Senior secondary teachers' students also mentioned if their solutions are different from teacher's solution, they would think that moment is very wonderful and effective.

From the interviews, it was also found that students would value some teaching moments which are not valued by the teacher. For instance, in the last five minutes of A2's lesson, when she gave a summary on solving an example of plan geometry, she connected two cases with category discussions. While students had thought that it was effective because she had provided them with another analytic method, A2 did not think so. The teacher commented, "I think at the time their understandings are not deep. So I show the results to them. But I expect they could find solutions by themselves." This mismatch also happened in other lessons. For instance, the senior secondary teacher B1 valued the use of PPT to explain 3D-shapes while her students emphasized more on detailed blackboard writing. "Blackboard writing is important. We can have time to write the notes and review what we have learned", said one student of B1. The senior secondary teacher B2 emphasized the explanation of concepts and mathematical thinking while his students valued more on doing challenging or difficult problems.

DISCUSSION AND CONCLUSION

The present study explored the desires and expectations of a sample of secondary students and teachers in Chinese mainland in relation to the teaching of mathematics. In the light of the results above, we are now better able to develop a picture about what students and their teachers value with regards to effective teaching.

Some common characteristics of effective teaching like *fun, multiple methods, involvement, and examples* are valued by both students and teachers. An effective secondary teacher needs to facilitate students' self-learning. An effective lesson should have clear objects and systematic structure. Teachers' clear expression is thought as a factor of being effective teacher. These expressions not only could be the verbal instructions but also the blackboard writing.

Both students and teachers thought systematic and structured mathematics knowledge are important for an effective mathematics lesson. The value of *focus* was emphasised by both of them. Valuing *focus* could be thought as a main characteristic of Chinese classroom teaching (Li & Huang, 2012) and was not mentioned in other regional studies of Third Wave project (Seah & Wong, 2012). In an effective lesson, the teacher played an important role. The students depended much on their teacher in the class. Their teachers also emphasized teachers' guidance is important for an effective lesson. A teacher-centred mathematics teaching was preferred by both of students and teachers even though they valued students' participations.

At the same time, the analysed data suggest that students' values on effective teaching do not exactly match what the teacher values. A teacher who showed his/her patience, humour to the students, would always be appreciated by their students. Their teacher may value a high peer evaluation and excellent students' performance which are the crucial factors to become an effective/successful teacher. A teacher emphasized the explanation of concepts and mathematical thinking while his students may value more on doing challenging or difficult problems.

Given the small number of participants, we do not propose to generalise these findings here. Although some values of effective teaching have been found in other education systems (see, for example, Seah & Wong, 2012), such as *fun*, *board writing* and *patience*, the nature of the valuing can differ from education system to education system. For example, in Malaysia, students valued *boardwork* as a platform to learn from other students (Lim, 2012), while in Chinese mainland, students wanted to learn more from teachers' detailed and clear writing on the blackboard. This subtle difference in the context of what is being valued similarly (i.e. *boardwork*) is also embodied in the classroom environment. From the questionnaire, although half of the students valued an enjoyment environment (*fun*), we were not able to fully unpack what the real meaning of this enjoyment environment was to be. In Chin and Lin (2000)' study, the value of *fun* was related to interesting mathematics problems which could raise students' curiosity in mathematics. In Seah and Peng's (2012) study, the value of *fun* reported by students implied a fun atmosphere in mathematics or the jokes and/or games during lessons. Hong Kong students valued playing games or doing quiz as means of maintaining a sense of liveliness and an enjoyable environment (Law, Wong, & Lee, 2012). Thus, further investigation is needed to clarify the meaning of each value, especially, how they (students and teachers) interpret their values. At the same time, given the mismatches between what a teacher and what his/her students value, we also suggest that it would be insightful for us to examine how the different values are negotiated in mathematics classroom discourse.

The findings derived from this study have identified students' and teachers' most appreciated and important values. These values are subject to their preferences or choices. Many features of effectiveness indeed reflect the valuing of meaningful and constructive classroom interactions between teacher and students. Bishop (2012) also proposed that understanding what students value in relation to mathematics pedagogy would allow us to facilitate students' growth in terms of levels of mathematical well-being. With a deep understanding of what their students value, teachers are better placed to design more effective pedagogies to promote classroom learning of mathematics.

References

Anthony, G., & Walshaw, M. (2008).Characteristics of effective pedagogy for mathematics education. In H. Forgasz, T. Barkatsas, A. Bishop, B. Clarke, P. Sullivan, S. Keast, W. T. Seah, & S. Willis (Eds.), *Research in mathematics education in Australasia 2004-2007* (pp. 195-222). Rotterdam, the Netherlands: Sense Publishers.

Atweh, B., & Seah, W. T. (2008). *Theorising values and their study in mathematics education*. Paper presented at the Australian Association for Research in Education Conference. November 28, 2007. Fremantle, Australia.

Bishop, A. J. (1999). Educating student teachers about values in mathematics education. In F.-L. Lin (Ed.), *Proceedings of the 1999 International Conference on Mathematics Teacher Education* (Vol. 2, pp. 1-6). Taipei: National Taiwan Normal University.

Bishop, A. J. (2008). Teachers' mathematical values for developing mathematics thinking in classrooms: Theories, research and policy. *The Mathematics Educator, 11*(1/2), 79-88.

Bishop, A. J. (2012). From culture to well-being: A partial story of values in mathematics education. *ZDM—the International Journal on Mathematics Education, 44*(1) 3-8.

Cai, J., Kaiser, G., Perry, B., & Wong, N. Y. (Eds.) (2009). *Effective mathematics teaching from teachers' perspectives: National and cross-national studies*. Rotterdam, The Netherlands: Sense Publishers.

Chin, C., & Lin, F. L. (2000). *Values and values statement emerged in students' preferences on test items: A case study from mathematical induction*. Paper presented at the History and Pedagogy of Mathematics 2000 Conference, Taipei, Taiwan.

Hollingsworth, H., Lokan, J., & McCrae, B. (2003). *Teaching mathematics in Australia: Results from the TIMSS 1999 Video Study*. Camberwell, Vic.: Australian Council for Educational Research.

Law, H. Y., Wong, N. Y., & Lee, L. N. Y. (2012). A study into espoused values in Hong Kong mathematics classrooms. *ZDM - The International Journal on Mathematics Education, 44*(1), 45-57.

Li, Y., & Huang, R. (2012). *How Chinese teach mathematics and improve teaching*. New York: Taylor & Francis.

Lim, C. S. (2012). *Riding the third wave: Negotiating teacher and students' value preferences relating to effective mathematics lesson*. Paper presented at the 12th International Congress on Mathematical Education. 8 July – 15 July, 2012, COEX, Seoul, Korea.

National Council of Teachers of Mathematics.(2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Neuman, W. L. (2003). *Social research methods: Qualitative and quantitative approaches* (5th edition). Boston, MA: Pearson Education.

Organization for Economic Cooperation and Development. (2004). *Learning for tomorrow's world: First results from PISA 2003*. Paris: OECD Publishing.

Raths, L. E., Harmin, M., & Simon, S.B. (1987). Selections from values and teaching. In P.F. Carbone (Ed.), *Value theory and education* (pp. 198-214). Malabar, FL: Krieger.

Seah, W. T., & Kalogeropoulos, P. (2006). Teachers of effective mathematics lessons: What gets valued in their practice? In J. Ocean, C. Walta, M. Breed, J. Virgona & J. Horwood (Eds.), *Mathematics: The way forward* (pp. 279-287). Melbourne, Vic.: The Mathematical Association of Victoria.

Seah, W. T., & Peng, A. (2012).What students outside Asia value in effective mathematics lessons: a scoping study. *ZDM—The International Journal on Mathematics Education, 44*(1), 71-82.

Seah, W. T., & Wong, N. Y. (2012). What students value in effective mathematics learning: a 'Third Wave Project' research study. *ZDM—The International Journal on Mathematics Education, 44*(1), 33-43

Wilson, P., Cooney, T., & Stinson, D. (2005). What constitutes good mathematics teaching and how it develops: Nine high school teachers' perspectives. *Journal of Mathematics Teacher Education, 8*(2), 83–111.

Zhang, Q. P. (2014). The third wave: Chinese students' values in effective mathematics teaching in two secondary schools. *Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education. 18*(3), 209-221.