Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education

Hobart, Australia

July 13-18, 2015

Volume 1

Plenaries, Research Forums, Discussion Groups, Working Sessions, Seminars, Short Oral Communications, Posters

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Editors: Kim Beswick, Tracey Muir, & Jill Fielding-Wells

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Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education

Volume 1

Editors

Kim Beswick, Tracey Muir, & Jill Fielding-Wells

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WELCOME TO PME39:
CLIMBING MOUNTAINS, BUILDING BRIDGES

We are delighted to welcome you to the 39th Annual Conference of the International Group for the Psychology of Mathematics Education, being held in Hobart, Australia. PME39 is being hosted by the University of Tasmania (UTAS), and the theme of the conference is *Climbing Mountains, Building Bridges*. This reflects PME's interest in addressing some of the challenges associated with attaining success in learning and teaching mathematics and developing connections in classrooms, across content and disciplines, and throughout cultures and practices. During the conference, the talks, presentations, and discussions will give insight into these important issues. We invite all participants to contribute actively to the discourse and analysis of ideas, to build bridges, and to help each other in our ascents of mountains, so that our understanding is deepened. We also encourage all of you to foster a welcoming and stimulating atmosphere at the conference, that all participants may feel included as members of the PME community. We extend a special welcome to those attending their first PME conference. Our hope is that the conference will provide a chance to attain some pinnacles and to establish some fruitful connections.

Tasmania's history is simultaneously old and young, with its Aboriginal history dating back tens of thousands of years, and its European history, after some early exploration, dating from the early 1800s. The University of Tasmania recognises the deep history and culture of this island, and wishes to acknowledge the Mouheneenner People, the traditional owners and custodians of the land upon which the university's Sandy Bay campus and the Hobart College were built. We acknowledge the contemporary Tasmanian Aboriginal community, who have survived invasion and dispossession, and continue to maintain their identity, culture and Indigenous rights. We also recognise the value of continuing Aboriginal knowledge and cultural practice, which informs our understandings of history, culture, science and environment; the University's role in research and education, and in supporting the development of the Tasmanian community.

The city of Hobart is Australia's second oldest state capital, and was founded in 1804. It is the smallest of the capital cities with a current population of over 200 000, running along both sides of the Derwent River, a flooded estuary. The city has the impressive kunanyi/Mount Wellington as a backdrop, rising 1271m from the sea, and the river is spanned by the long and gentle arch of the Tasman Bridge. Both the mountain and the bridge are captured in the conference logo.

Australia's mathematics educators have had a long, wide-ranging, and influential impact on international mathematics education and always form a large contingent at international mathematics education conferences, including PME. Australian mathematics educators hosted PME in Sydney in 1984 and Melbourne in 2005, and we are pleased to have another opportunity to welcome those of you have travelled from near and far to be with us. We particularly acknowledge those of you who have grappled with multiple time-zones and a significant change of season!

The Program Committee and the Local Organising Committee want to express our thanks for the support we have received from members of the PME community, including previous conference organisers and Bettina Rösken-Winter, PME's administrative manager. Barbara Jaworski and Stefan Ufer have been constantly available to answer questions. Their advice, suggestions, encouragement, reminders, and understanding have been most helpful.

Finally, on a personal note, I would like to thank the many people who have contributed to what I hope will be a very successful conference. The Program Committee, listed in full later, laboured mightily and with care over many important decisions, including the consideration of all the proposals. The University of Tasmania's maths education group—Rosemary Callingham, Helen Chick, Bruce Duncan, Noleine Fitzallen, Tracey Muir, Robyn Reaburn, and Jill Wells—have all contributed to the planning and preparation. Dawn Cripps provided wonderful liaison with Hobart College, and that institution's generous support of the conference is acknowledged with thanks. Other important assistance was gratefully received from Helen Forgasz, Marj Horne and Linda Page as well as from finance and ICT staff at UTAS. Finally, my thanks go to Lesley Bennell without whom the conference would never have happened: her attention to detail, willingness to go beyond what could reasonably be expected, and capacity to keep track of the important details have been incredible.

Kim Beswick, Conference Chair

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INTRODUCTION

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

History and Aims of PME

The International Group for the Psychology of Mathematics Education (PME) is an autonomous body, governed as provided for in the constitution, whose members are interested in research in mathematics education. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI) and it came into existence at the Third International Congress on Mathematics Education (ICME 3) held in Karlsruhe, Germany in 1976.

Its former presidents have been:

Efraim Fischbein (Israel)	Carolyn Kieran (Canada)
Richard R. Skemp (UK)	Stephen Lerman (UK)
Gerard Vergnaud (France)	Gilah Leder (Australia)
Kevin F. Collis (Australia)	Rina Hershkowitz (Israel)
Pearla Nesher (Israel)	Chris Breen (South Africa)
Nicolas Balacheff (France)	Fou-Lai Lin (Taiwan)
Kathleen Hart (UK)	João Filipe Matos (Portugal).

The current president is Barbara Jaworski (United Kingdom).

The major goals of PME are:

To promote international contacts and the exchange of scientific information in the field of mathematical education;
To promote and stimulate interdisciplinary research in the aforesaid area; and
To further a deeper and more correct understanding of the psychology and other aspects of teaching and learning mathematics and the implications thereof.

Honorary Members of PME

Hans Freudenthal (The Netherlands, deceased)
Efraim Fischbein (Israel, deceased)
Joop van Dormolen (retired)

Website of PME

For more information about International Group for the Psychology of Mathematics Education (PME) as an association, history, rules and regulations and future conferences see its home page at http://www.igpme.org.

PME Membership and Other Information

Membership is open to people involved in active research consistent with the Group's aims, or professionally interested in the results of such research. Membership is on an annual basis and requires payment of the membership fees. For participants of PME39 Conference the membership fee is included in the Conference Deposit; those interested in joining PME can contact the PME administrative manager. PME has between 700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other about their working groups, poster sessions and many other activities. Every year the conference is held in a different country. There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter is issued three times a year, and can be found on the IGPME website. Occasionally PME issues a scientific publication, for example the result of research done in group activities.

PME Administrative Manager

The administration of PME is coordinated by the Administrative Manager

Bettina Roesken-Winter

Email: info@igpme.org

Present Officers of PME

President:	Barbara Jaworski (UK)
Vice-president:	Stefan Ufer (Germany)
Secretary:	Michal Tabach (Israel)
Treasurer:	Olive Chapman (Canada)

Other members of the International Committee

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Marta Civil (USA)
Csaba Csikos (Hungary)
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PME PROCEEDINGS OF PREVIOUS CONFERENCES

PME International

The tables indicate the ERIC numbers of PME conference proceedings.

No.	Year	Place	ERIC number
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943
7	1983	Shoresh, Israel	ED241295
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, Netherlands	ED411130 (vol.l), ED411131 (vol.2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128 (vol.l), ED411129 (vol.2)
13	1989	Paris, France	ED411140 (vol.1), ED411141 (vol.2), ED411142 (vol.3)
14	1990	Oaxtepex, Mexico	ED411137 (vol.1), ED411138 (vol.2), ED411139 (vol.3)
15	1991	Assisi, Italy	ED413162 (vol.1), ED413l63 (vol.2), ED41364 (vol.3)
16	1992	Durham,USA	ED383538
17	1993	Tsukuba, Japan	ED383 536
18	1994	Lisbon,Portugal	ED383537
19	1995	Recife, Brazil	ED411134 (vo1.l), ED411135 (vol.2), ED411136 (vo1.3)
20	1996	Valencia, Spain	ED453070 (vol. 1), ED45307 1 (vol.2), ED453072 (vol.3), ED453073 (vol.4), ED453074 (addendum)
21	1997	Lahti, Finland	ED416082 (vol.1), ED416083 (vol.2), ED4l6084 (vol.3), ED416085 (vol.4)
22	1998	Stellenbosch, South Africa	ED427969 (vol.1), ED427970 (vol.2), ED427971 (vol.3), ED427972 (vol.4)
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	ED452301 (vol. 1), ED452302 (vol.2), ED452303 (vol.3), ED452304 (vol.4)
25	2001	Utrecht, The Netherlands	ED466950
26	2002	Norwich, United Kingdom	ED476065
27	2003	Hawai‘i, USA	ED500857 (vol.1), ED500859 (vol.2), ED500858 (vol.3), ED500860 (vol.4), ISSN: 0771-100X http://www.hawaii.edu/pme27

No.	Year	Place	ERIC number
28	2004	Bergen, Norway	ED489178 (vol.1), ED489632 (vol.2), ED489538 (vol.3), ED489597 (vol.4), ISSN: 0771-100X www.emis.de/proceedings/PME28
29	2005	Melbourne, Australia	ED496845 (vol. 1), ED496859 (vol. 2), ED496848 (vol. 3), ED496851 (vol. 4) ISSN: 0771-100X
30	2006	Prague, Czech Republic	ED496931 (vol. 1), ED496932 (vol. 2), ED496933 (vol. 3), ED496934 (vol. 4), ED496939 (vol. 5) ISSN: 0771-100X http://class.pedf.cuni.cz/pme30
31	2007	Seoul, South Korea	ED499419 (vol. 1), ED499417 (vol. 2), ED499416 (vol. 3), ED499418 (vol. 4) ISSN: 0771-100X
32	2008	Morelia, Mexico	ISBN: 978-968-9020-06-6 ISSN: 0771-100X http://www.pme32-na30.org.mx
33	2009	Thessaloniki, Greece	ISBN: 978-960-243-652-3 ISSN: 0771-100X
34	2010	Belo Horizonte, Brazil	ISSN: 0771-100X http://pme34.lcc.ufmg.br
35	2011	Ankara, Turkey	ISBN 978-975-429-262-6 ISSN: 0771-100X http://www.arber.com.tr/pme35.org
36	2012	Taipei, Taiwan	http://tame.tw/pme36 ISSN: 0771-100X
37	2013	Kiel, Germany	ISBN: 978-3-89088-287-1 ISSN: 0771-100X http://www.pme2013.de/
38	2014	Vancouver, Canada	ISBN: 978-0-86491-360-9 ISSN: 0771-100X http://www.pme38.com

THE REVIEW PROCESS FOR PME39

Research Forum (RF)

The goal of a Research Forum is to create dialogue and discussion by offering attendees more elaborate presentations, reactions, and discussions on topics on which substantial research has been undertaken in the last 5-10 years and which continue to hold the active interest of a large subgroup of PME and PME-NA. A Research Forum is not supposed to be a collection of presentations but instead is meant to convey an overview of an area of research and its main current questions, thus highlighting contemporary debates and perspectives in the field. The Program Committee and the International Committee accepted the topic and co-ordinators of the Research Forum of PME39 on the basis of the submitted proposal. The proposed structure, the contents, the contributors, and the roles of the contributors were reviewed and agreed by the Program Committee.

Discussion Groups (DG)

The objective of a Discussion Group is to provide attendees with the opportunity to discuss a specific research topic of shared interest. The idea of a Discussion Group may be the result of an Ad hoc Meeting or an intensive discussion of a Research Report during the previous conference. Discussion Groups may begin with short synopses of research work, or a set of pressing questions. A Discussion Group is exploratory in character and is especially suitable for topics which are not appropriate for collaborative work in a Working Session because they are not yet elaborate enough or because a coherent research strategy has not been identified. A successful Discussion Group may result in an application for a Working Session one year later. Two proposals were submitted for PME39 and both Discussion Groups were accepted by the IPC subject to minor changes in some cases.

Working Sessions (WS)

The aim of Working Sessions is that participants collaborate in joint activities on a research topic. For this research topic, there must be a clear research framework or research strategy and precise goals so that a coherent collaborative activity is ensured. Ideas for a Working Session can result from Discussion Group sessions of previous conferences where a topic was elaborated upon and a research framework or strategy was developed. Each Working Session should be complementary to the aims of PME and ensure maximum involvement of each participant. Four proposals were submitted for PME39 and all were accepted by the IPC subject to minor changes in some cases.

Research Reports (RR)

Research Reports are intended to deal with topics related to the major goals of PME. Reports should state what is new in the research, how it builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all papers.

The Program Committee received 213 RR papers for consideration. Each full paper was blind-reviewed by three peer reviewers, and then these reviews were considered by the Program Committee, a committee composed of members of the International Committee of PME and the Local Organising Committee. This group read carefully the reviews and also, in some cases, the paper itself. The advice from the reviewers was taken into serious consideration and the reviews served as a basis for the decisions made by the Program Committee. In general if there were three or two recommendations for accept the paper was accepted. Proposals that had just one recommendation for acceptance were looked into more closely before a final decision was made. Of the 213 proposals we received, 132 were accepted, 31 were recommended as Short Oral Communications (SO), and 18 as Poster Presentations (PP). The Research Reports appear in Volumes 2, 3, and 4.

Short Oral Communications (SO)

Short Oral Communications are intended for research that is best communicated by means of a short oral communication instead of a full research report. One hundred and three proposals were submitted. Of these, the IPC accepted 65, recommended 20 as PPs, and the remaining 18 submissions were rejected. In the end, considering resubmissions of Research Reports as Short Orals, 68 Short Orals were accepted. They appear in this volume of the proceedings.

Poster Presentations (PP)

Poster Presentations are intended for information/research that is best communicated in a visual form rather than as a formal paper presentation. Thirty seven proposals were submitted. The IPC accepted 23 proposals and rejected the remainder. In the end, considering resubmissions of Research Reports and Short Oral proposals as Poster Presentations, 40 posters were accepted for presentation. They appear in this volume of the proceedings.

The reviewing process was completed during the second meeting of the International Program Committee at the end of March 2015. Notifications of decisions of the International Program Committee to accept or reject the proposals were available by mid April 2015.

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The PME39 Program Committee thanks the following people for their help in the review process:

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STEM: CHALLENGES AND OPPORTUNITIES FOR MATHEMATICS EDUCATION

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Climbing Mountains, Building Bridges is a rich theme for exploring some of the "challenges, obstacles, links, and connections" facing mathematics education within the current STEM climate (Science, Technology, Engineering and Mathematics). This paper first considers some of the issues and debates surrounding the nature of STEM education, including perspectives on its interdisciplinary nature. It is next argued that mathematics is in danger of being overshadowed, in particular by science, in the global urgency to advance STEM competencies in schools and the workforce. Some suggestions are offered for lifting the profile of mathematics education, with examples drawn from two activities on modelling with data in the sixth grade.

INTRODUCTION

> In metaphorical terms, we need to lift the level of the peaks of the STEM mountain range, and broaden and elevate the whole of the range at the same time. (Marginson, Tytler, Freeman, & Roberts, 2013, p.72).

A focus on advancing STEM (science, technology, engineering, mathematics) in schools and the workforce is escalating across many nations, with its powerful role across multiple sectors being formally recognised (Honey, Pearson, & Schweingruber, 2014; Harrison, 2012; Marginson et al., 2013; The Royal Society Science Policy Centre, 2014). For example, Australia's Chief Scientist emphasised in a recent lecture that STEM is "at the core of almost every agenda," and "the almost universal preoccupation now shaping the world's plans" (Chubb, 2014). In the United States, the 2013 report from the Committee on STEM Education maintained that "The jobs of the future are STEM jobs", with STEM competencies increasingly required not only within, but also outside of, specific STEM occupations (National Science and Technology Council, 2013, p. vi). Developing competencies in the STEM disciplines is thus regarded as an urgent goal of many education systems, fuelled in part by perceived or actual shortages in the current and future STEM workforce and also by outcomes from international comparative assessments (e.g., OECD, 2013).

Further evidence of the vested interest in STEM by researchers, educators, industry leaders, and policy makers can be found in the burgeoning of publications devoted to the field (e.g., Honey et al., 2014; National Research Council, 2014; Purzer, Stroble, & Cardella, 2014; the International Journal of STEM Education; http://www.stemeducationjournal.com/). The biennial international STEM conference (http://stem2014.ubc.ca/) is another example.

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Although global interest in STEM from educational and workforce perspectives has proliferated in recent years, the acronym was actually coined in the United States during the 1990s by the National Science Foundation (USA). The combining of the disciplines was seen as "a strategic decision made by scientists, technologists, engineers, and mathematicians to combine forces and create a stronger political voice" (STEM Taskforce Report, 2014, p. 9). Since this time, the debates and dilemmas surrounding STEM employment shortages and STEM education in general have compounded.

One of the current debates is whether there is, indeed, a global shortage of STEM professionals (e.g., Charette, 2013; Hopkins, Forgasz, Corrigan, & Panizzon, 2014; Smith & Gorard, 2011). In arguing for more evidence for these global claims, Hopkins et al. stressed the need to consider tertiary level enrolment trends in the STEM disciplines taking into consideration, among others, ways in which data are collected, courses are classified, and particular subject areas are targeted. Given the complexities of the data used to make claims about STEM shortages, it would seem difficult to draw definitive conclusions. For example, on the one hand, there are Charette's (2013) extensive analyses of numerous global reports suggesting that the claimed shortages are a myth. On the other hand, there are reports such as that of The Royal Society Science Policy Centre (2014), which conveys employers' concerns regarding the lack of suitable STEM employees and the estimated one million or more STEM professionals and technicians needed in the United Kingdom by 2020.

The debates on deficiencies in the STEM workforce appear entwined with the urgency for improving STEM education in schools. Irrespective of whether there exist or will be employment shortages, the calls for improved STEM education in schools are not unfounded and cannot be ignored. The STEM disciplines permeate so much of our lives that we cannot afford to neglect the current arguments for their advancement, beginning with the earliest years of school. Charette's (2013) claim is especially apt in this regard, namely, we do indeed have a STEM crisis but not necessarily with respect to skills shortages. The crisis lies in STEM literacy, that is, students today are not receiving a solid foundation in science, mathematics, and engineering.

This claim for a literacy crisis is underpinned by industry groups and other organisations emphasising the critical role of STEM education in reforming the economy and fuelling innovation (e.g., the Australian Industry Group, Willox, *The Australian*, 16 Dec., 2014, p. 14). Other reports, such as those from the Australian Office of the Chief Scientist (2013, 2014) and the Australian Council of Learned Academies (Marginson et al., 2013) likewise stress the importance of all students having strong STEM knowledge, skills, and innovative dispositions.

In the remainder of this paper, I first address some of the issues and debates surrounding the nature of STEM education including perspectives on its interdisciplinary nature. I then argue that mathematics is in danger of being overshadowed, in particular by science, in the current international STEM climate. I offer suggestions for lifting the profile of mathematics education and illustrate these ideas by describing two activities that address modelling with data in the sixth grade.

DEFINING STEM EDUCATION

One of the factors contributing to the existing debates is the lack of a globally accepted definition of STEM education. Given different national agendas, such education has been interpreted variously, with some discipline areas being given greater attention than others. In acknowledging the lack of an agreed-upon definition, the Californian Department of Education provides a broad perspective on STEM education, namely,

> [STEM]... is used to identify individual subjects, a stand-alone course, a sequence of courses, activities involving any of the four areas, a STEM-related course, or an interconnected or integrated program of study. (http://www.cde.ca.gov/PD/ca/sc/stemintrod.asp)

Debates on what constitutes STEM education range from David Clarke's (2014) perspective that the four disciplines do not have much in common, to those who advocate commonalities in problem-solving and thinking processes, and more broadly to those advocating a focus on sustained engagement. In his 2014 keynote address at the STEM Conference in Vancouver, Clarke argued that "...it is difficult to recognise that STEM could be the name for a fairly monumental category error. What is it that Science, Technology, Engineering, and Mathematics have in common? One reasonable answer is not much" (http://stem2014.ubc.ca/conference-details/keynote-speakers/). On the other hand, those who embrace definitions of STEM from an interdisciplinary perspective frequently emphasise generic attributes that transcend the disciplines, together with their respective core concepts and skills. The former include critical thinking, problem solving and inquiry processes, teamwork, and design processes, the last of which represents a core engineering link. Other definitions consider STEM education as fostering "sustained engagement with the STEM disciplines where students can become competent contributors and critical participants in a range of STEM-related activities (Burke, Francis, & Shanahan, 2014). Interestingly, Burke et al. consider their approach representative of "the Canadian dialect of STEM education."

An interdisciplinary approach, however, appears to feature most prominently in STEM definitions, with the Californian Department of Education citing the axiom, "the whole is more than the sum of the parts," (http://www.cde.ca.gov/PD/ca/sc/stemintrod.asp) as reflecting this perspective. For example, the STEM Taskforce Report (2014) in the US adopts the strong view that STEM education is far more than a "convenient integration" of its four disciplines, rather, it encompasses "real-world, problem-based learning" that integrates the disciplines "through cohesive and active teaching and learning approaches" (p. 9). The Report argues that the disciplines "cannot and should not be taught in isolation, just as they do not exist in isolation in the real world or the workforce" (p.9). In supporting their stance, the Report defines STEM literacy with respect to each of the disciplines, demonstrating their interconnections (italics added to mathematics), as follows:

> Scientifically literate students use scientific knowledge not only in physics, chemistry, biological sciences, and earth/space sciences to understand the natural world, but they also understand the scientific need for existing and new technologies, how new advances in scientific understanding can be engineered, and how *mathematics* is used to articulate and solve problems.
>
> Technologically literate students understand that technology is the innovation with or manipulation of our natural resources to help create and satisfy human needs and also to learn how to obtain, utilize, and manage technological tools to solve science, *mathematics*, and engineering problems.
>
> Students who are literate in engineering understand how past, present, and future technologies are developed through the engineering design process to solve problems. They also see how science and *mathematics* are used in the creation of these technologies.
>
> *Mathematically literate* students not only know how to analyze, reason, and communicate ideas effectively; they can also mathematically pose, model, formulate, solve, and interpret questions and solutions in science, technology, and engineering (p.9).

STATUS OF MATHEMATICS EDUCATION WITHIN STEM

With the rapid rise of STEM education as an interdisciplinary construct, some researchers have expressed concerns over emerging inequitable discipline representations (e.g., English & Kirshner, 2015; Honey et al., 2014; Moore et al., 2014). As one example, of the 141 regular papers presented at the 2014 STEM conference in Vancouver, 45% were devoted to science, 12% to technology, 9% to engineering, and 16% to mathematics, with the remaining 18% classified as "general" with several papers in this category addressing two or more of the STEM disciplines.

Concerns for the underrepresentation of mathematics cannot be overlooked, especially since influential curriculum documents such as the US *Common Core State Standards for Mathematics* (http://www.corestandards.org/Math/) and the *Next Generation Science Standards* (http://www.nextgenscience.org/) are calling for more in-depth connections among the STEM disciplines. This challenge of maintaining equitable discipline representation is especially germane to our discipline, which I maintain needs to have a stronger presence and role alongside the others.

Although reference to science could be interpreted as encompassing mathematics, I nevertheless argue that there is a real danger that science will overshadow the importance of mathematics in today's world. Indeed, the STEM acronym itself is frequently referred to as simply "science" (e.g., Office of the Chief Scientist, 2014). Even back in 1962, Australia's former Prime Minister, Sir Robert Menzies, identified "the flowering of science" as "the great distinguishing feature of this [then] century apart from wars and political confusions" (cited by Chubb, 2014). Further, the discipline of science seems to dominate many current STEM reports, as Marginson et al. (2013) indicated. Many nations also refer to the role of STEM education as one that fosters "broad-based scientific literacy" with a key objective in their school programs being "science for all" with increased efforts on lifting science education in the primary, junior, and middle secondary school curricula (Marginson et al., 2013, p. 70). Interestingly, Marginson et al. pointed out that STEM discussions rarely adopt the form of "mathematics for all" even though mathematics underpins the other disciplines (as evident in the discipline definitions cited previously). Marginson et al. thus argued that "the stage of mathematics for all should be shifted further up the educational scale" (p.70). Even the rise in engineering education, commencing in the early school years (e.g., Lachapelle & Cunningham, 2014), would appear to be oriented primarily towards the science strand at the expense of mathematics. Nevertheless, alongside the challenges facing mathematics education are opportunities for its advancement.

Mathematical literacy, in particular, has gained increased attention in recent years, albeit with different interpretations and content emphases. The global importance accorded to this literacy is evident in its inclusion as a major domain in the 2012 PISA (Programme for International Student Assessment, OECD, 2013). It is not surprising then, that as nations reflect on their students' mathematical achievements, they are questioning the quality of their curricula and the strategic actions needed to enhance the STEM disciplines. It follows that many nations with high international testing outcomes as well as strong STEM agendas have a well-developed curriculum that concentrates on inquiry processes, problem solving, critical thinking, creativity, and innovation as well as "a strong commitment to disciplinary knowledge" (Marginson et al., 2013, p.110). The need to nurture both the generic skills and in-depth conceptual understanding is paramount.

ELEVATING MATHEMATICS EDUCATION ACROSS STEM

The superior international achievements of STEM-focused nations reflect the mathematical literacy assessed in PISA 2012, with the focus on "meeting life needs ... through using and engaging with mathematics, making informed judgements, and understanding the usefulness of mathematics in relation to the demands of life" (Thompson, Hillman, & De Bortoli, 2013). Mathematical literacy is foundational to STEM education, where a facility in dealing with uncertainty and data is central to making evidence-based decisions involving ethical, economic, and environmental dimensions (Office of the Chief Scientist, 2013). Further, with the exponential rise in digital information within STEM, the ability to handle contradictory and potentially unreliable online data is critical (Lumley & Mendelovits, 2012). More recognition needs to be given to the core role of mathematics in analysing and reasoning with data to make informed decisions and engage in constructive debate about local and global issues (The Royal Society Science Policy Centre, 2014).

With the increasing need to reason effectively with data including entertaining uncertainty and risk, it was timely that the major domain of mathematical literacy within PISA 2012 featured uncertainty and data as one of the four context categories.

Given that many nations are striving to achieve social, cultural and economic prosperity in dealing with a rapidly changing and insecure world, greater recognition needs to be given to the foundational role of mathematics, in particular working with data, in building the required knowledge base. Traditional methods in statistics education, which focus on procedural skills rather than conceptual understanding, are inadequate. As several researchers have indicated, the need to develop new approaches to dealing with uncertainty and data, beginning with the earliest years, is paramount (Bargagliotti, 2014; Batanero, Burrill & Reading, 2011; English & Watson, 2015). One approach to elevating mathematics within STEM is modelling with data, which targets the components of a mathematically literate student defined previously.

MODELLING WITH DATA ACROSS STEM

The terms, modelling, and modelling with data, have been variously interpreted and applied in the mathematics education literature (e.g., Borromeo Ferri, 2013; Doerr & English, 2003; English, 2014; Kaiser & Sriraman, 2006; Lehrer, & Schauble, 2005). It is not the intention of this paper to explore these various interpretations; rather, as used here, modelling with data encompasses a focus on both process and product: (a) It follows a process of inquiry involving comprehensive statistical reasoning that draws upon STEM-based concepts, contexts, and questions; and (b) It generates products, (models) that are supported by evidence and are open to informal inferential thinking, which includes recognising uncertainty, detecting variation, and making predictions. Such models may take different forms depending on the nature of the inquiry (e.g., explanatory documentation, persuasive argument, a representation). Because variation is inherent in data (without variation there would be no need for statistics), models are generated in light of the uncertainty that arises from such variation.

In the remainder of this paper, I report on two quite different activities implemented in sixth-grade classes, the first in Cyprus (English & Mousoulides, in press) and the second in Australia (English & Watson, 2014). Together, the activities target the following interdisciplinary knowledge and processes, which I believe need greater representation across the STEM range.

Exploring, posing, and refining investigative questions within STEM contexts;

Applying discipline-based concepts and engineering design in formulating and solving problems;

Planning and undertaking investigations;

Analysing and representing data in multiple ways;

Developing, applying, and assessing evidence-based models;

Understanding informal inference involving variation and uncertainty;

Critically evaluating data-based arguments and conclusions;

Sourcing, evaluating, and communicating information;

Thinking in creative, flexible, and innovative ways.

Engineering-based Modelling with Data

Given that the first activity, *Rebuilding the 35W Minneapolis Bridge*, is an engineering-based modelling problem, it is worth highlighting the increased focus on engineering design in the *Next Generation Science Standards: For States, by States* (The National Academies, 2014). Broadening the role of engineering design and elevating it to the same level as scientific inquiry, the *Standards* define engineering design practices as those that all citizens should develop. The core features of engineering design encompass three main iterative processes, which have the potential to enhance learning across both science and mathematics: (a) defining problems by specifying criteria and constraints for acceptable solutions, (b) generating a number of possible solutions and evaluating these to determine which ones best meet the given problem criteria and constraints, and (c) optimising the solution by systematically testing and refining, including overriding less significant features for the more important.

Rebuilding the 35W Minneapolis Bridge

Participants. This problem activity was implemented in two 6th-grade classes (12-year-olds, n=48) in a K-6 public school in an urban area of Cyprus. The students had not been exposed to modelling problems of this nature in their regular curriculum.

Method. The activity focused on the 2007 structural failure of the 35W Bridge in Minneapolis, Minnesota (adapted from Guzey, Moore, & Roehrig, 2010). In the first session (35-45 minutes), students studied a newspaper article about the bridge collapse as well a video clip, and answered questions to ensure their understanding of the context and its data. In the second session (1 hr 20 mins - 1 hr 30 mins), students were presented with two tables of data, together with the problem scenario. The first table comprised the key characteristics of the four main bridge types (truss, arch, suspension, cable-stayed), namely, the advantages and disadvantages of each bridge, the span range, the main materials used in construction, and the design effort (low, medium, high). The second table contained two samples of each of the major bridge types and some of their key features including the total length, the number of car lanes, the construction difficulty, and the building costs (in current values).

The problem scenario explained that the Minnesota Public Works Department urgently needed to construct a new bridge in the same location. The bridge was to comprise a highway with a length of approximately 1000 feet, with a deck of four lanes with additional side lanes. The Department required assistance in creating a way (model) for comparing the different bridge types so as to choose the appropriate one to build across each span. Working in small groups of 3-4 (mixed-achievement in school mathematics), the students drew on the given data to generate, refine, and document their models. The groups were to develop a model that (a) included calculating the cost for each one of the four bridge types (using the given characteristics of the four main bridge types) and (b) would enable selection of the best possible bridge type for the reconstruction of the collapsed bridge. All possible factors related to bridge type, materials used, bridge design, safety, and cost were to be taken into consideration. In the final session (40-50 minutes), each student group explained to their peers their model creations and key findings, which they documented in poster format.

Data analysis. Each student group (13 groups in total) was audio taped, while all whole-class discussions were videotaped. The data sources also included students' worksheets and the researchers' field notes. Data were analysed using interpretive techniques (Miles & Huberman, 1994), with detailed analysis of all data sources enabling identification of the mathematization and statistical reasoning processes students applied during solution. Students' cycles of model development, reflecting use of engineering design, were also identified in the analysis.

Sample of results. The models students created varied in the number of problem factors considered (cost per surface unit of bridge deck, aesthetics of the various bridge types, bridge design effort, construction difficulty, length), as well as in students' reasoning with these data, and in the sophistication of the final models generated. I report here on just one student group's model development, which displayed their reasoning with multidisciplinary components.

The group began the problem by excluding a truss-type bridge explaining that, "The collapsed bridge was a truss one" (Student A) and "Selecting the truss type bridge would make people feel insecure and bring back all those bad memories" (Student B)." The group then decided that a cost model for ranking the different bridge types was needed, but after developing an initial model that involved calculating the average cost (money per square feet of deck) for each bridge type, they decided that it was not the most appropriate solution. The group concluded that the substantial variation in their results for bridges of the same type could be addressed by integrating more factors within their initial model. Their reasoning was as follows:

Student C: Our calculations are correct. There is nothing wrong. The cost is very different.

Student D: There are other things (factors) that are important and influence the cost ... for those (bridges) that are close to sea it is more difficult.

Student C: Yes, like in the Golden Gate Bridge. It is so expensive and not that long.

Student B: Cost is not proportionally related to the surface of the bridge (deck), but also the level of difficulty in constructability, just like in the Golden Gate, is an important factor.

On returning to the key characteristics of the four major bridge types (advantages, bridge span etc.), the group came to the conclusion that all types had their advantages as well as disadvantages. The group thus concluded that a suitable bridge type could not be determined from this set of data alone. The students then moved into the next cycle of their model development as they took further data into consideration. Reflecting on their prior discussion on determining an initial cost model also contributed to their progression to a more comprehensive model.

The students' next cycle of model development featured a consideration of engineering, scientific, and societal factors. It was decided that these should be incorporated within their earlier model. These additional data included the necessary extra lanes for bridges, bikes, and pedestrians, as well as the difficulty level of each bridge construction. The last factor was determined by dividing the estimated final cost per ft^2 by 1.5 for the given examples of the four major bridge types. The group referred to this as the "difficult constructability" factor and specifically created this to provide the same basis of comparison for all bridge types.

The group's refined model ranked the bridge types from cable-stayed as most favoured, followed by the arch, truss, and suspension bridge types. In deciding on their final model, however, the students were cognizant of scientific and engineering issues, and thus selected the arch type as the best possible solution. They were still concerned about the stability of a cable-stayed bridge for long span bridges.

Modelling with Data in Developing Statistical Literacy

Participants. The second activity was conducted at the end of a three-year longitudinal study (2012-1014, grades 4-6) on statistical literacy in interdisciplinary contexts, with a focus on informal inference (English & Watson, 2014). For the present activity, four classes of sixth-grade students participated (average age 11 years 10 months, n=89). The students attended a state school situated in an Australian capital city.

Method. A foundational feature of the activity was the investigative process, "Four steps to making decisions with data," which the students had followed in their previous investigations, namely: 1. Posing a question, 2. Collecting data, 3. Analysing (and representing) data, and 4. Making a decision (on the original question), acknowledging uncertainty. Use of the *TinkerPlots* (Konold & Miller, 2011) software program was a key learning feature of the three-year study. The next two activity components involved both whole class discussions as well as small group work.

Are Athletes Getting Better Over Time?

The first component of the activity (2hrs 30mins - 3hrs 25mins) began with a video clip of *Usain Bolt in the London 2012 Olympics 100m Final* (http://www.youtube.com/watch?v=lacjJVxC5d0). The students then considered the general question, "Are athletes getting better over time?" Students quickly realised that the question needed to be refined in order to answer it statistically and meaningfully (corresponding to Step 1 of "Four steps to making decisions with data"). Over the course of the three-year study, students had come to appreciate that statistical questions require carefully planned investigations and any conclusions drawn from the analysis of the data have a certain degree of uncertainty.

On refining the question in their own way, each group recorded the data they would need to answer their question (corresponding to Step 2). Specifically, students were to record: (a) how/where they would find the required data, (b) whether that data would enable them to answer their question, (c) how confident they would feel in answering their question, and (d) whether they considered their question needed further refinement.

Following a class discussion on how required data cannot always be obtained (due to unavailability or in the present case, time constraints), students were supplied with data rather than sourcing these, as would have been preferred. Each group was presented with 12 data sets of various Olympic Gold Medal results for men's and women's freestyle, sprint, running, high, and long jump events. Selecting the appropriate data to answer their question (or if necessary, refining their question first), students were to analyse the data and represent their findings (corresponding to Step 3). Initially they were to sketch a plot of their results, labelling their axes, recording their end points, and indicating the scale they would apply. The students then used the *TinkerPlots* software to generate more detailed representations. On completion of their representation, the students were to respond to the questions, "What does your representation tell you? How does it help to answer your question? How could you improve your representation?"

> Moving to the fourth step, students recorded their responses to the following: "From your analysis, what decision/conclusion have you reached? Explain how you reached this conclusion. What evidence do you have to support your conclusion? How certain of your conclusion are you? Explain your answer." Groups of students shared their conclusions with the class, indicating the data they used, their strategies for analysing their data, and how certain they felt about their conclusion.

Next, students were introduced to a new tool for data analysis, namely, the trend line. The software enabled students to observe improvement over time by drawing a trend line across the data. Using the Text Box feature of the software, students described the "trend" or "relationship" in their chosen data set. The trend line was added to students' existing repertoire of statistical tools, namely, mean, mode, median, and Hat Plot, together with their established understanding of representational features in describing and comparing data sets (e.g., overall shape, outliers, clusters, gaps, etc).

This first component of the activity concluded with a Power Point presentation based on an article from the Technology, Entertainment and Design Conference (namely, http://tedsummaries.com/2014/05/03/david-epstein-are-athletes-really-getting-faster-better-stronger/). The article described how advances in technology have contributed to athletes' improved performances. Students were to subsequently reflect on their prior conclusions and the certainty of their recorded decisions, indicating whether they regarded these as still justifiable.

Sample of results. As not all data from this activity have been analysed at the time of writing, examples are drawn from just two classes.

Group responses to Steps 3 and 4 suggested an awareness and appreciation of trends in the data including any outliers, as well as an appreciation of the uncertainty of conclusions drawn. For example, one group reported in response to the questions of Step 3:

> Our representation shows us that 100m sprinters are generally becoming faster since 1972. It is a gentle decline from 10:14 sec to 9:63 sec at the London 2012 Olympic games. However during 1980 Moscow Olympic games someone won with a time of 10:25 which is a distinct outlier. The representation helped us with our answer as it shows us a clear trend of quicker times. We could improve [our representation] by making the *y* axis, the times, more specific to show the exact times.

Responding to the questions of Step 4, this group explained:

> We have come to the conclusion that 100m sprinters have become quicker from 1972 - 2012. Our graph has clearly shown the trend of quicker times for gold-medalist. In 1972 the time was 10:14. In 2012 the time was 9:63. The graphs shows [sic] a gentle decline (Except for the outlier). [We are] Not extremely certain [of our conclusion]. This is because our time frame does not include all the Olympic games which officially started in 1896.

Let the Selections Begin!

<u>Method</u>. The second component of the activity, *Let the Selections Begin!* (1 hr 30 mins - 2hrs) involved developing models for selecting swimming teams for the 2016 Olympics. Commencing with the question of whether Australian athletes are also improving over time, and if so, whether Australia would be likely to win Gold in the pool at the 2016 Olympics, students again quickly identified difficulties in answering such a broad question. Given that the nature of the activity involved selecting Olympic teams from given data sets, the question needed to be refined substantially; this was achieved through class discussion.

Students were given tables of data (in both printed form and in *TinkerPlots*), for selected swimming competitions during the 2012-2014 time period (personal best times [PB] were included as well as individual race times). Each group was to make its own team selections for the women's or men's 100m freestyle relay event for the Rio 2016 Olympics, choosing the 6 swimmers with the best chance of winning Gold for Australia, and providing justification for their choices in a report. Specifically, students were to report on: (a) The data used and how they were analysed (including any representations) to help their team selection; (b) The athletes selected and reasons for selection; (c) How certain they felt that their selected team would be the "best" and why, and (d) The certainty with which they considered their methods of team selection would apply to other swimming and sports events, and why.

<u>Sample of results</u>. Interesting insights into the students' learning emerged in the final part of the activity where student groups reported on the models they had created. The first set of examples is from one class where 11 groups shared their models for team selection. Several students who questioned their peers' models displayed a critical

analysis of how the models were generated and made requests for clarification of terms used, together with justification for conclusions drawn including the supporting data.

Group 11 explained that, in using the data for the women's 100m freestyle events, they analysed the data "by looking at the athletes' personal best times and how many times they've been to a swimming race." Explaining further that the latter factor referred to the athletes' experience, a class member asked, "When you're talking about experience, what do you mean by that?" followed by, "...which is more important to you, the PBs, the speed, or the experience?" The group indicated the speed.

In describing their model, group 5 stated that they were "75% confident that our team will win the 2016 Rio Olympics." This claim prompted the peer question, "...how do you know that cause you don't really have the teams and their times .... How can you be *any* percent sure?" Further questioning about Group 5's model, which focused primarily on the athletes' personal best times, included "... do you think your team selection would be more accurate if you take more things into account than just PB?" In replying "yes," the group was asked what else they might take into consideration, to which they replied, "the events, like the competition events, so like the Olympic Games, the Australian Swimming Championships..." This response elicited further questions and comments including, "How does the type ... like where the race is, so if it's like the Olympic Games or something else, how does that affect the racer?" On giving this point some thought, one Group 5 member responded, "...they've got different times so they could have. I'm not too sure." The other group member elaborated, "Um, maybe it's like we said. Swimmers, like athletes, can improve over time so maybe we will look at the events as because we think these swimmers could improve." The peer who posed the question concluded, "So you could have, um, considered the time they did the other races [previous competitions] and made a better team."

The second set of examples is from another class (13 groups) where several groups explained why they considered a range of factors in producing their model, not just Personal Best times. Group 8 provides one example:

> And to answer this question we used all times of the [female] swimmers and found the averages. So then we ordered the averages from fastest to slowest and we found according to that, the fastest were ... [selected swimmers]. We also took into consideration their age ... it wouldn't really affect it but just to make sure they were experienced but they were also still like at a good fitness level and ... We focussed on the average, the averages of the times because they basically sum all the swimmers' results and using their Personal Best would not be very accurate as that had, could basically just be chance that they managed to get such a good time.

In expressing uncertainty in their team selection, a group member explained, "So, we're fairly certain that the six women that we chose were the fastest in Australia at 100m Freestyle however, we cannot be completely certain as we do not have all of their results for the races they competed in. And also it's still, there is still an element of chance."

Group 13 considered Personal Best times but indicated more factors needed to be taken into account. In colour coding their data table, the group explained:

> We organised our data set in a manner so we could organise the times in terms of colours and we saw that Cameron and James had exceptional times. Our third selection was Tommaso D'Orsogna because two of his times got into the green colour which signified a time of 48 seconds. His PB was also green but we didn't use this mainly as evidence as there was a lot more components to factor in. We averaged all the competitors, all the swimmers' times in terms of events excluding their PB because it could depend on chance rather than, for example it could be just a good day that they achieved their PB and that's how we selected the other swimmers which were ... [selected swimmers]. The mean was one of our main strategies to choose our swimmers and we are confident with our team, except the decision which included Matthew and Kenneth which had close average times for 100m.

The group expressed confidence in applying their model to other team selections due to its "comprehensive" nature, indicating:

> Our strategy could be used to pick other teams in different countries because it is quite comprehensive and it will depend on the events and you'd need, the more events that they could compete in the more accurate the average could be so, it's one, it could be used, in other teams for choosing. We are quite certain except because this is the best method we could come up with and we also looked at age as well to make sure that um the fitness levels and also that they are around the same age just, and we also noticed that the young swimmers such as Ned McKendry and Samuel Young [aged 22 and 16 years respectively] had not, were not accepted into the team because of slower times.

CONCLUSIONS

The examples presented in this paper for raising the profile of mathematics education are merely touching the surface of opportunities. Mathematics education provides foundational content and processes that bridge the STEM disciplines. Our challenge is to raise awareness of these contributions and increase the mathematical experiences appearing in STEM documents. Modelling with data, just one example, cuts across the disciplines but is not receiving the recognition it warrants, nor is the potential of applying engineering design in enhancing mathematical problem solving and inquiry. With the rapid rise in engineering education drawing heavily on the science curriculum, it is imperative that mathematics does not become the distant relative. My aim for our international community is to lobby in support of our field as a core player in the advancement of STEM. Mathematics needs to be elevated to the peaks of the STEM mountain range, and we must ensure it remains there with its contributions recognised and lauded.

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HOW TO TEACH WITHOUT TEACHING: AN INQUIRY-ORIENTED APPROACH IN TERTIARY EDUCATION

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Tertiary education is at the inflection point. There have been ongoing calls for innovation in tertiary education, where a knowledge transmission model for teaching through traditional lectures is prevalent. This paper serves to share some critical thought and to provide two innovative approaches: the inquiry oriented differential equations (IO-DE) project and calculus flipped learning (FL) project.

"...first and foremost, it should teach those young people to think." (Polya, 1962)

PROLOGUE: KNOWLEDGE VS THINKING

In March 2015, the research team on the Improvement of SNU Education invited all faculty and undergraduate students to participate in a survey designed to understand the educational situation and needs, both current and anticipated, and to gain a grasp on our faculty's interactions with students. The survey closed with 2,251 students and 304 faculty. The results of the survey indicated that both students and faculty were satisfied with the quality of knowledge acquisition through lectures. However, students were not satisfied with enhancing thinking skills through classroom learning.

Thinking as well as learning is an important path to acquiring knowledge. Learning is a partner of teaching in the process of knowing with knowledgeable others. With learning, the process is typically receptive; accepting the object offered. On the other hand, thinking led by curiosity is typically an autonomous, independent, reflective, and generative process. Learning allows students to quickly and easily gain a large amount of knowledge compared to thinking. Though thinking is relatively slower than learning in the acquisition of knowledge, it is necessary and crucial in order to integrate pre-existing knowledge, to utilize knowledge according to a given circumstances and conditions, and to gain new knowledge.

However, there has been much more emphasis on learning rather than thinking even in higher education. The problem, therefore, is that learning more does not guarantee being a better thinker. A number of studies indicate that students do not learn critical, creative thinking or complex skills through traditional lectures (e.g., Arum & Roska, 2011; Bligh, 1972; Bok, 2005; Garrison & Kanuka, 2004). An empirical study conducted by Bao et al. (2009) shows the limitation of content-rich teaching through the collection of data from 5,760 students in four U.S. and three Chinese universities. The study found that Chinese students learn introductory physics topics for five years due to the nationwide college admission exam. In contrast, only one of three high school students in the United States enrolls in the one-year physics course. The results suggest that numerous and rigorous physics courses affect student learning of content knowledge. However, there is no significant difference between the Chinese and U.S. students' scientific reasoning. These results imply that to know more does not necessarily mean better thinking. Also, previous researchers have pointed out that there is a big problem with current teaching methods that deliver a large amount of knowledge without a deep understanding of the process of knowledge generation. How then can university teachers help students to develop mathematical thinking? The aim of this paper is to provide two examples of inquiry-oriented mathematics instruction at the university level and to discuss their implications for the higher-education communities. The first example is called the Inquiry Oriented Differential Equations (IO-DE) project that seeks to find commonality between undergraduate and K-12 mathematics education. The other one is called the Flipped Learning (FL) project designed for multivariable calculus, where students individually watch online lectures prior to class and then engage in classroom learning activities interacting with peers and instructors.

INQUIRY ORIENTED DIFFERENTIAL EQUATIONS (IO-DE) PROJECT

The IO-DE project is an example of a collaborative effort between mathematics educators and mathematicians that seeks to explore the prospects and possibilities for improving undergraduate mathematics education, using differential equations as a case example. In this section, I highlight the theoretical background for the IO-DE project and a summary of quantitative and qualitative studies of the IO-DE project on student learning and how teachers create and sustain an inquiry-oriented learning environment.

Student Inquiry and Teacher Inquiry

While there are clear calls for inquiry in both science and mathematics classrooms, what exactly characterizes an inquiry-oriented classroom is less clear. To clarify the nature of inquiry-oriented classrooms and to provide a more comprehensive perspective on the complexity of teaching and learning, Rasmussen and Kwon (2007) characterize inquiry in terms of both student activity and teacher activity. In particular, students learn new mathematics by inquiry, which involves solving novel problems, debating mathematical solutions, posing and following up on conjectures, and explaining and justifying one's thinking. The first function that student inquiry serves is to learn new mathematics by engaging in genuine argumentation. The second function that student inquiry serves is to empower learners to see themselves as capable of re-inventing mathematics and to see mathematics itself as a human activity. On the other hand, teachers also engage in inquiry. Teacher inquiry centers on inquiring into their students' mathematical thinking and reasoning. Teacher inquiry into student thinking serves three functions. First, it enables teachers to interpret how their students build mathematical ideas. Second, it provides an opportunity for teachers to learn something new about particular mathematical ideas in light of student thinking. Third, it better positions teachers to follow up on students' thinking by posing new questions and tasks.

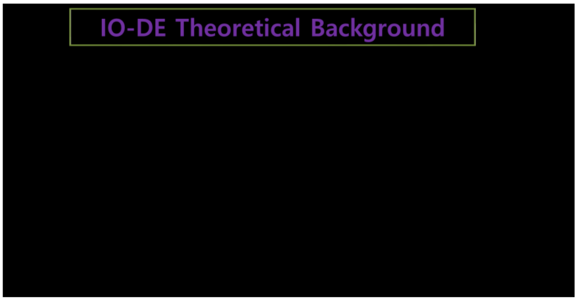

Figure 1: IO-DE Background Theory

Background Theory of IO-DE PROJECT

The theoretical background of IO-DE has originated from the disciplines of mathematics and mathematics education (see Figure 1). Drawing on a dynamical systems point of view, the IO-DE project treats differential equations as mechanisms that describe how functions evolve and change over time. Interpreting and characterizing the behavior and structure of these solution functions are important goals, with central ideas including the long-term behavior of solutions, the number and nature of equilibrium solutions, and the effect of varying parameters on the solution space (Blanchard et al., 1998; Hubbard & West, 1991). Addressing these central ideas draws on graphical, numerical, and analytical techniques, made viable with the use of technology. These techniques utilize a variety of different graphical representations, such as slope fields for first order differential equations and vector fields for systems of differential equations, as well as numerical algorithms such as Euler's method for producing approximate solutions.

A cornerstone of the IO-DE project is adaptation of the instructional design theory of Realistic Mathematics Education (RME) to the undergraduate level. Central to RME is the design of instructional sequences that challenge learners to organize key subject matter at one level to produce new understanding at a higher level (Freudenthal, 1991). In this process, referred to as mathematizing, graphs, algorithms, and definitions become useful tools when students build them from the bottom up through a process of suitably guided re-invention (for illustrative examples and further theoretical development, see Kwon, 2003; Kwon, 2005; Rasmussen, Zandieh, King, & Teppo, 2005).

The mathematization process is embodied in the core heuristics of guided re-invention and emergent models. Guided re-invention speaks to the need to locate instructional starting points that are experientially real to students and that take into account students' current mathematical ways of knowing. The search for such starting points is facilitated by examination of the history of mathematics, as well as students' informal solution strategies and interpretations. The heuristic of emergent models highlights the need for an instructional sequence to be a connected, long-term series of problems in which students create and elaborate symbolic models of their informal mathematical activity (Gravemeijer, 1999). The model is an overarching idea that refers to student-generated ways of interpreting and organizing their mathematical activity, both mental activity and activity with graphs, equations, etc. From the perspective of RME, there is not just one model, but a series of models where students first develop models-of their mathematical activity leading to models-for reasoning about mathematical relationships.

As our research team systematically investigated the learning and teaching in such approaches, we developed three goals that extended contemporary dynamical systems approaches relating to our definition of inquiry.

Accomplishing these three goals was facilitated by conducting research in three related strands: (1) adaptation of an innovative instructional design approach at the undergraduate level, (2) systematic study of student thinking as they build ideas and teacher knowledge to support students' re-invention, and (3) careful attention to the social production of meaning and student identity. These three strands do not represent a linear progression in our research. We conducted research in these three strands concurrently and view the strands as complementary.

Quantitative studies of IO-DE Student Learning

Rasmussen, Kwon, Allen, Marrongelle, & Burtch (2006) conducted an evaluation study to compare the routine skills and conceptual understandings of central ideas and analytic methods for solving differential equations between students in inquiry-oriented classes and traditionally taught classes at four undergraduate institutions in Korea and US. Whereas IO-DE project classes at all sites typically followed an inquiry-oriented format, comparison classes at all sites typically followed a lecture-style format. The assessment consisted of routine skill problems and conceptual understanding problems. Routine skill problems focused on students' instrumental understanding such as the analytic and numerical nature of differential equations. On the other hand, conceptual understanding problems were aimed at evaluating students' conceptual understandings of important ideas and concepts. As shown in Figure 1, there was no significant difference between the two groups on routine problems. However, the IO-DE group did score significantly higher than the comparison group on conceptual problems.

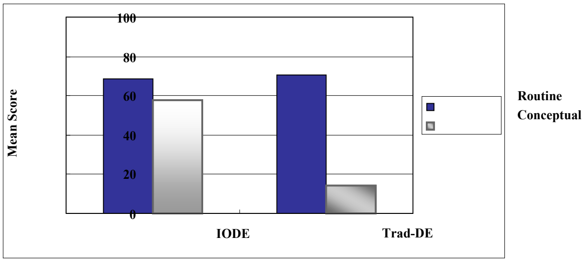

Figure 2: Mean Scores of IO-DE and Comparison Groups on Routine and Conceptual Tests

Further, Kwon, Rasmussen, & Allen (2005) conducted a follow-up study on the retention effect of conceptual and procedural knowledge one year after instruction for a subset of the students from the comparison study. Students' retention of knowledge was compared for the traditional and the IO-DE instructional approach. For the purpose of this analysis, procedurally oriented (PO) items were defined as those questions that were readily solved via analytic/symbolic techniques. Conceptually oriented items were defined into two categories: modelling (M) tasks and qualitative/graphical (QG) tasks, each of which represent important and conceptually demanding thinking in mathematics, in general, and in differential equations in particular. The two modelling tasks involved determining an appropriate differential equation to fit a given real-world situation. The qualitative/graphical tasks involved predicting and structuring the space of solutions. Figure 2 shows that post-test and delayed post-test scores of IO-DE and comparison groups on QG, M, and QG items respectively.

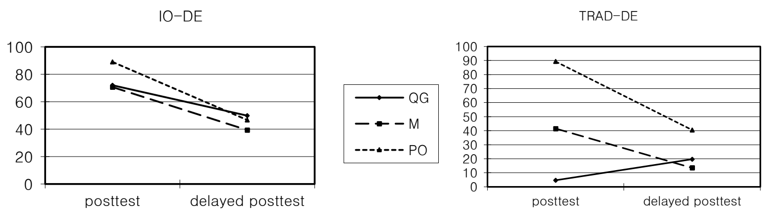

Figure 3: Students' retention of mathematical knowledge and skills in differential equations.

With the limitation of small sample size, our analysis showed that there was no significant difference in retention between the two groups on the procedural oriented items. However, the long-term retention of conceptual knowledge was better for those students who participated in the IO-DE project. We posit that one reason for this difference may be the students'

> participation in the RME guided instructional sequences; the reinvention of the mathematical procedures as they emerge from conceptual knowledge plays a central role in their conceptual understanding and leads to a longer retention effect.

Qualitative studies of IO-DE project

Rasmussen and Blumenfeld (2007) elaborated the emergent model heuristic for student re-invention of solutions to systems of linear differential equations. Their analysis extends the construct of emergent models to situations in which symbolic expressions play a prominent role throughout what is referred to as the model-of / model-for phase. In addition, the analysis connects to the strand on student thinking by highlighting qualitatively different ways that students reason proportionally in relation to the model-of / model-for transition. As such, their article offers a theoretical and empirical grounding for instructional design and for interpreting student thinking.

Marrongelle (2007) illuminates the roles of graphs and gestures in students' re-invention of the Euler method for first order differential equations and how these roles change in students' subsequent use of the Euler method to approximate systems of differential equations. The significance of this article for instructional design and corresponding teacher support material is that it offers a lexicon of student gestures and the relationship of these gestures to student re-invention and use of the Euler method algorithm. Such a lexicon can increase a teacher's knowledge about student thinking, and even suggest ways in which a teacher might intentionally leverage gesturing to support their students' learning.

A second cornerstone of the IO-DE project is research on student thinking and teacher knowledge. For example, research on student cognition in differential equations highlights students' concept images of Euler's method and students' informal or intuitive notions underlying equilibrium solutions, asymptotical behavior, and stability (Artigue, 1992; Rasmussen, 2001; Zandieh & McDonald, 1999; Cho, 2003). Knowledge of such informal or intuitive images has been useful for the IO-DE project because it suggests task situations and instructional interventions that could engage and help reorganize students' informal and intuitive conceptions.

Keene (2007) details student thinking as it relates the use of time as a dynamic quantity. In particular, Keene identifies five distinct ways in which students integrate time as a changing quantity as they progress in their understanding of systems of differential equations. In contrast to earlier research that indicated time may be an obstacle to student understanding of function (Janvier, 1998), Keene details how time based reasoning can promote and further student understanding of solution functions to systems of differential equations.

As our understanding of student thinking evolves, so does our understanding of the kinds of teacher knowledge that are important for promoting student learning. Beyond content knowledge, such knowledge includes awareness of students' informal and intuitive ways of reasoning about central ideas in differential equations, knowledge of pedagogical strategies that can connect to student thinking while moving forward the mathematical agenda (Rasmussen & Marrongelle, 2006)), knowledge of theoretical ideas related to social aspects of the classroom, as well as mathematical knowledge specific to teaching mathematics in general, and differential equations in particular.

Wagner, Speer and Rossa (2007) make a significant contribution to various types of knowledge that IO-DE teachers find useful for inquiry-oriented teaching. The authors argue that the knowledge required for experienced mathematicians to implement effective, reform practices of instruction in their classrooms includes knowledge that differs from the mathematical content knowledge, pedagogical content knowledge, and pedagogical knowledge that support traditional instruction. Their case study of a mathematician implementing the IO-DE curriculum for the first time offers a revealing portrait of the kinds of essential knowledge needed for inquiry-oriented teaching.

In addition to theoretically informed design with extensive classroom based research, the IO-DE project works from the premise that the way in which instructional tasks are constituted is as important as the material itself, and it is toward this aspect that we now turn. An explicit intention of IO-DE project classrooms is to create a learning environment where students routinely offer explanations of and justifications for their reasoning. Because of the strong emphasis on argumentation in inquiry-oriented classrooms, we conjectured that the theoretical constructs arising from research in inquiry-oriented elementary school classrooms would be useful for learning advanced mathematics, such as differential equations. After all, mathematicians engage in similar forms of argumentation when creating new mathematics.

Ju and Kwon (2007) investigated and documented the change in students' beliefs about mathematics, about their relation to mathematics, and about their roles in the classroom practice of mathematics. They conducted an intriguing and innovative discourse analysis that traced shifts from third person perspective to first person perspective as a way to infer changes in students' beliefs. Finally, they pointed to important aspects of the classroom learning environment in transforming students' beliefs, including the instructional materials, students' own cognitive resources, and the role of the teacher. As a whole, this article portrays a powerful analysis of how inquiry-oriented learning environments can meet the challenges facing today's mathematics classrooms.

Further, Kwon et al. (2008) focused on the teacher's revoicing in an inquiry-oriented classroom, because it is one of the discursive strategies that often occurs in the teaching of mathematics, but which has received limited attention in mathematics education research at the undergraduate level. Our analysis shows that a teacher's revoicing can constitute a major repertoire of his or her discursive moves and carries out critical functions in the context of mathematics practice in class. From that perspective, revoicing serves at least three functions in the classroom. First, revoicing functions to highlight specific mathematical ideas and/or provide mathematical content to move forward the mathematical agenda. Second, revoicing functions to honor and empower student thinking. That is, revoicing facilitates the development of students' mathematical identities. Third, revoicing functions to help students understand what constitutes a sufficient explanation or justification. That is, revoicing can serve to promote certain social and socio-mathematical norms (Park et al., 2007).

The implications of the IO-DE project are threefold. First, based on the results of the post-test and the delayed post-test (Kwon, 2005; Rasmussen et al. 2006), the IO-DE students from each of the four institutions outperformed traditionally taught comparison students on the post-test. This result was true for both males and females and for high and low achieving students. This result demonstrates that this instructional approach can be applicable to university mathematics. Secondly and more importantly, the instructional methods and curriculum design approach guided by RME are applicable to promoting student learning in all mathematics classrooms. Thirdly, the IO-DE project can provide a model for how it is that teachers create and sustain inquiry-oriented learning environments in which students gain mathematical power and sophistication.

FLIPPED LEARNING IN MULTIVARIABLE CALCULUS COURSES

Since the early 2000's, blended learning - a convergence of face-to-face classroom learning with technology-mediated online learning - has emerged in response to pedagogical challenges in higher education. The next example of inquiry-oriented mathematics instruction at the university level illustrates how blended learning can offer inquiry-oriented learning environments for students that foster core values of higher education such as critical and creative thinking skills or complex reasoning skills in a multivariable calculus course. A critical issue in the design of blended learning experiences is designing the right blend of learning online and face-to-face (Basham et al., 2010; Gedik, Kiraz & Ozden, 2013). As a newly emerging type of blended learning, flipped learning (FL) represents a means to such a better blend.

Flipped learning is an alternative teaching-learning model in which traditional lectures given by instructors and homework are reversed. Some universities have reported actual cases which actively applied this model (Bishop et al., 2013; Talbert, 2012). Students watch short video clips developed by their instructors or provided from other resources like Massive Open Online Courses (MOOCs) at home before the offline class, while face-to-face time is devoted to classes for exercises, activities, or discussions.

To flip or not to flip: that is not the question

FL aims to elevate students' high-order thinking through the aforementioned learning activities, and features constant, systematic support through online technology in order to achieve its objectives. Thus, FL has two axes: (i) online activities that are supported by technology and feature teacher-centered delivery of knowledge and (ii) face-to-face activities that are focused on learners (Dede et al., 2004; Gannod et al., 2008; Strayer, 2012).

An FL multivariable calculus course enables instructors' explanatory lectures to be replaced by online digital video clips, and the face-to-face class could provide meaningful learning opportunities based on a Realistic Mathematic Education (RME) perspective. From this approach, students learn mathematics by mathematizing the subject matter through examining realistic situations, i.e., experientially real contexts for students that draw on their current mathematical understandings. In this RME approach, the problems precede the abstract mathematics of multivariable calculus, which emerges from students' collaborative work toward solutions. Curricula, as well as the local instruction theory and its justification, are simultanously developed and refined in a gradual, iterative process. Kwon et al. (2013) investigated instructional design principles improving students' argumentations within the proof construction activities in an inquiry-based class of the FL multivariable calculus course. The conclusions of this study support the effectiveness of FL.

In order to investigate what structures are formed by the students' arguments in proof construction activities, we collected data from students' discussions in the face-to-face class. We analyzed their arguments according to Toulmin's argumentation structure (Toulmin, 1958; 2003) and three levels of the argumentation (simple, multiple, and compound) used in the previous research of Kwon et al. (2013). Simon (1995) suggested the term Hypothetical Learning Trajectory (HLT) to refer to *a* prediction of how the students' thinking and understanding will evolve in the context of the learning activities. The HLT is made up of three components: the learning goal, learning activities, and hypothetical learning process. HLT was applied in order to predict the students' arguments in the face-to-face class. The learning goals and activities of the class are focused on proof construction for given mathematical theorems in multivariable calculus. We anticipated the student's argumentation process in proof construction based on Toulmin's framework. Through this, we concluded the Hypothetical Argumentation Structure (HAS) which the student arguments might progress in the actual classes. After each offline class, we compared HAS with the actually implemented argumentation structure in order to reflect on the previous interventions and revise the principles according to the design-based research.

The class was composed of 18 freshmen students, 11 males and 7 females, from the mathematics education major at SNU. I was the lecturer and the researcher. The students took two to three video lectures of 20-30 minutes a week and participated in face-to-face classes which were implemented once a week for 75 minutes. In the face-to-face class, the 18 students were divided into 5 discussion groups of 3 or 4 students

Student argumentation on proof construction

The analysis of student discussion found that the argumentation structure presented by individual students, the groups, and the whole class in the classes gradually developed into a more complicated form as the classes progressed. That is, the argumentation structures transformed from single argumentation to multiple argumentation and compound argumentation as the interventions changed, revealing the improvement of student mathematical reasoning in proof construction. Figure 4 illustrates the compound argumentation structure of the offline class in the seventh week.

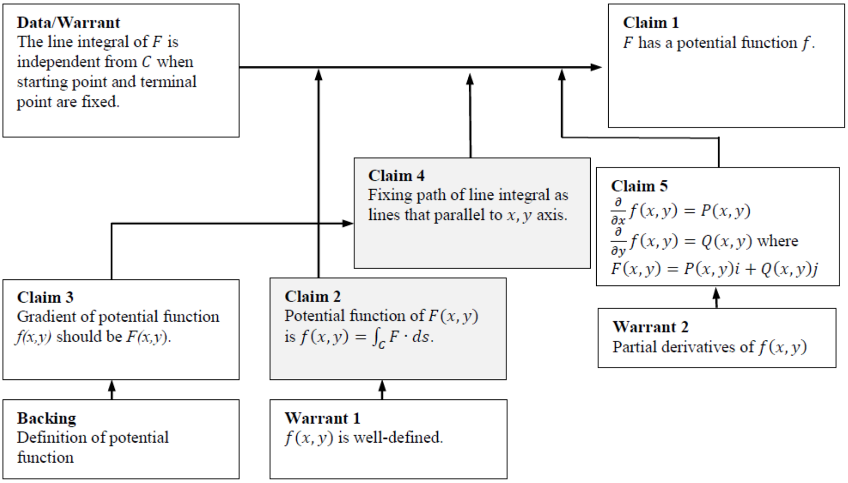

Figure 4: Compound argumentation structure in students' proof construction.

In the offline class, the students were required to prove that a vector field has a potential function (Claim 1), if the line integral of the vector field is independent from the integral path (Data/Warrant). The student arguments in this class appeared in the form of compound argumentation as expected in the HAS, but the instructor had to provide students with scaffolds to help them reach certain sub-claims (Claim 2 and 4), which are the shaded components in Figure 4. The compound argumentation structure of the whole class includes a variety of warrants for supporting the other claim that induces a new claim, while the single- or multiple-argumentation structure contains one claim with one or more warrants.

The cyclic process of revising the design principles led to the characteristics of an intervention which is effective in changing the argumentation structures in a flipped classroom. Further, it is important to study the role of instructors as well as the effectiveness of instructional tasks. When tasks are difficult for individuals to solve, small-group activities should be used to increase the students' accessibility and motivation by directing them to focus on the essential problem-solving portion. We confirmed that inquiry-provoking tasks naturally lead to communication between students and active intervention by instructors. Various forms of classroom interaction were used, including interactions within small groups, interactions between small groups, and whole-class discussions. The students were encouraged to participate in whole-class discussions after sharing opinions with each other and reaching similar degrees of understanding. The discursive role of instructors came to be more and more active as the classes progressed and changes in tasks and classroom interaction demanded instructional changes. The instructor organized classroom interaction structures for students, provided appropriate scaffolding, reorganized the tasks as needed, and lightened the students' cognitive burden.

Many undergraduate courses are run mainly with explanatory lectures given by instructors, and it is difficult for students to experience progressive mathematization developed from informal to formal mathematics. We, in turn, utilized a variety of learning possibilities in the face-to-face classes of flipped classrooms in order to develop and implement experientially real task sequences that are appropriate for a multivariable calculus course inspired by RME. Further, students were able to experience the creation of essential mathematical concepts in discussion-based classes focusing on inquiry-oriented tasks, which contributed to the improvement of student mathematical thinking. Student mathematical reasoning in proof construction activities can especially be enhanced when students are provided with appropriate instructional interventions in an FL multivariable calculus course.

Collective construction of mathematics

According to Rogoff et al. (1996), collective knowledge construction occurs when all students actively concentrate and feel responsible for their own and others' learning. When the improved interventions acquired in the FL multivariable calculus course, the students were provided with a structure through which they could participate more actively in the whole-class discussion. The instructor, who directed the students to productively construct knowledge, played the role of facilitator for discourse. In addition, the students were able to solve inquiry-oriented tasks in small-group activities and draw a conclusion from the whole-group discussion during the face-to-face classes. This encouraged students to participate responsibly in knowledge construction and learning, as was confirmed from observing how the students' argumentation gradually evolved to a more complex structure.

In the fourth week, students had to prove Young's theorem, which states that the symmetry of the second partial derivatives of a function $f(x, y)$ holds when every second partial derivative is continuous. They were required to complete their proof from the incomplete ideas suggested by one of their colleagues. During the whole-group discussion, Group 1 presented their proof, and Group 5 raised the question about the difference between Group 1' proof and the previous proof presented by the other group Instead of providing an appropriate answer to this question, Group 1 repeated their explanation of their proof and its validity. In other words, although the opinion gap between small groups needed to be narrowed in the whole-group discussion, students focused on asserting the validity of their own proofs rather than analyzing each other's findings and comparing findings across groups to foster improvement.

However, in the seventh week, students performed well in constructing their knowledge with collaboration between the groups which resulted in compound argumentation. Each sub-claim in Figure 4 was presented by a different small group, and the proof construction was then completed through whole-group discussion. This was the result of the instructor allowing various forms of interaction between students such as discussion within and between small groups prior to the whole-group discussion. Students did not hesitate to express their arguments in the discussion between small groups, in which opinions could be informally proposed and criticized. This led them to express their arguments formally and broadened the range of discussion.

EPILOGUE: HOW TO TEACH FOR FUTURE GENERATIONS

The development of mathematical competencies such as problem solving and critical thinking is important to enable students to successfully handle open-ended, real-world tasks in their future careers. Certainly, teaching goals in higher education should include fostering content knowledge and developing mathematical competencies. However, the predominant style of content-rich instruction at the tertiary education level, even when rigorously carried out, has little impact on the development of students' mathematical competencies (Holton, 2001). It seems that how to teach rather than what to teach makes a difference in student learning of higher-order thinking in mathematical competencies. The two projects discussed in this paper suggest that our inquiry-oriented approach can benefit students in undergraduate mathematics classes in several ways. It can foster students' ability to pose mathematical questions and to think critically and creatively in, with and about mathematics (Kwon et al., 2008) . It can facilitate students' development of mathematical reasoning ability (Kwon et al., 2004; Kwon, Bae, & Oh, under review). It can help students build the type of conceptual understanding that makes mathematics meaningful to them. It can positively influence their beliefs about knowing and doing mathematics (Kwon & Ju, 2004; Kwon & Ju, 2007). In light of these features, the IO-DE project and FL project may well serve as a model for those interested in exploring the prospects and possibilities of improving undergraduate mathematics education. The significance of the two projects with the associated theoretical and empirical work lie in the fact that, when coupled with careful attention to developments within mathematics itself, theoretical advances that initially started in elementary school classrooms (and which are beginning to spread to the rest of K-12) can be profitably leveraged and adapted to the university setting. Specifically, we found the constructs of social norms and socio-mathematical norms (Yackel & Cobb, 1996) useful because they offer a way of thinking about the multiple and complementary roles of argumentation as a means to conceptualize processes by which teaching mathematics for understanding can occur. Additional significance of the work of the two projects is evident in the extent to which research on these programs can contribute to advancing the work of teacher education and the professional development of mathematicians. However, regarding tertiary mathematics education, there is still great progress to be made in understanding how university teachers create and sustain classroom-learning environments in which students develop mathematical competencies through building robust relational understandings of mathematics and develop desirable dispositions and attitudes towards mathematics. For instance, we still have to resolve the notorious dilemma of an inquiry-oriented mathematics classroom for university teachers, that is, how to teach without teaching. Since students should ideally develop both content knowledge and transferable mathematical competencies, there are crucial calls for tertiary mathematics education researchers and university teachers to invest more in the development of a balanced method of education, such as incorporating more inquiry-oriented learning and teaching both of those target goals

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LEARNING MATHEMATICS BY IMITATIVE AND CREATIVE REASONING

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This paper presents an interdisciplinary research program carried out in collaboration between researchers from mathematics education, psychology and neuroscience. A large body of research has shown that imitative models of mathematics teaching and learning are both common and largely ineffective. The purpose of the program is to explore if and how teaching, including task design, based on students' creative task solution reasoning may be more effective.

INTRODUCTION

Trying to climb the mountain of mathematics seems for many students to be joyful and rewarding, but also a journey with steep cliffs, treacherous gaps, wide ravines and tricky path choices. Therefore the purpose of mathematics teaching must be to build bridges and raise staircases that take students over these hurdles, and also to take the students by their hands and lead them by the easiest paths. Or is it? An argument in favour of this type of teaching is that a mathematics teacher can quickly resolve students' immediate difficulties by proving such help. And indeed, this is what we often do. The student is grateful for avoiding the hurdle and a short-term problem is resolved. However, a large body of empirical and theoretical research shows that this is not effective for long-term learning. This paper concerns how teaching and task design may help students to learn how to understand the terrain in order to construct their own bridges and make their own path choices.

When I first came in contact with research in mathematics education two decades ago, I was teaching university mathematics. A large proportion of our students failed exams, and a majority seemed to struggle hard. Struggle can be very productive and often also necessary in order to learn mathematics. But the students' struggle was largely unproductive, it seemed that they somehow did not utilise the potential of mathematics. A key aspect of this potential is the possibility to create new knowledge through mathematical reasoning. Instead of relying only on intuition, routines, experience and empirical tests, we can often replace or complement this by reasoning in order to structure, analyse, understand and predict many phenomena. The research program *Learning mathematics by Imitative and Creative Reasoning* (LICR) has during the last years addressed this persistent problem that ineffective rote learning by imitative reasoning dominates in mathematics classrooms (Hiebert, 2003, Boesen et al., 2014). A suggestion for a solution to this problem, proposed by the Theory of Didactical Situations (Brousseau, 1997), has been operationalized by the LICR research group at Umea University by designing tasks and teaching that give students the responsibility to construct their own reasoning. Relations between four variables are analysed: 1) task

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properties, 2) teaching, 3) students' task solving reasoning and 4) learning outcomes. All results show so far that learning by construction of solutions is more effective than learning by imitation (Jonsson et al., 2014; Karlsson et al., 2015). This paper will present a background for the LICR program, the research framework, and some examples of studies.

BACKGROUND

The goal of mathematics education: developing students' competence

A main aim of the last three decades of educational reform is to help students acquire richer mathematical competence, i.e. the ability to understand, judge, do, and use mathematics. Basic competencies not developed by rote learning alone include problem solving ability (a problem is a task where the solver does not know a solution method in advance), reasoning ability (to justify choices and conclusions) and understanding. The internationally influential reform-oriented frameworks defining mathematical competence (e.g. NCTM, 2000; Kilpatrick et al., 2001; Niss, 2003) have influenced Swedish official mathematics policy documents since 1994 (Boesen et al., 2014). The notion of understanding is returned to below.

Towards more effective teaching and learning of mathematics

One of the most persistent challenges of mathematics education is the replacement of dominating teaching models based on imitation of task solutions. There are "massive amounts of converging data" (Hiebert, 2003, p. 12) showing that such teaching models fail to promote students' learning effectively and instead fosters mathematics students that try to follow rote learnt rules and task solution methods "like robots with poor memories" (ibid. p. 12). It is hardly reasonable to expect that students should attain in-depth understanding in all aspects of mathematics education. And rote learning is not only negative; it can reduce the demands on our working memory (Baddeley 2010) and set free cognitive resources to be used for more advanced problem solving. However, if most learning is done by rote and imitation, central mathematical competencies are not developed (Hiebert, 2003). Reviewing the literature on effects of traditional classroom practice, Hiebert (2003) concludes that students have more opportunities to learn facts and simple procedures than to engage in more complex processes, and achievement data indicate that students are indeed learning simple facts and calculation procedures but not how to find solution methods by themselves or engage in other mathematical processes. Similar opportunities to learn were found in a Swedish large-scale study including observations of 200 mathematics classrooms and interviews with the teachers (Boesen et al., 2014).

Research have addressed various attempts towards richer mathematics classroom practice (Cobb et al., 2003; Hiebert & Grouws, 2007; Schoenfeld, 2007; Stein et al. 2008). In order to obtain desirable opportunities to learn, a review by Niss (2007) suggests that students need to engage in activities where they have to 'struggle' (in a productive sense) with important mathematics, but that there is a delicate balance in order to prevent these struggles from being obstacles rather than promoters of learning.

Also many cognitive and brain imaging studies show that more active levels of processing promote deeper learning which is superior to shallow processing based on superficial features (e.g. Ekuni et al., 2011). However, when it comes to proposals for more effective teaching Hiebert & Grouws (2007) conclude in a research review that we are far from a coherent and systematic knowledge base that documents robust links between teaching and learning outcomes. Little is known about how to actually translate this abstract idea of 'struggle' into design of specific artefacts (e.g. textbooks) and activities useful in teaching, and about the mechanisms that link such teaching to learning outcomes (Niss, 2007).

RESEARCH FRAMEWORK

The Theory of Didactical Situations (TDS) proposes an alternative to imitation

In order to design teaching and tasks for a constructive struggle we operationalize Brousseau's (1997) Theory of Didactical Situations in Mathematics (TDS), which provides a basis for designing more effective teaching where students are given responsibility for active creation of knowledge. Students' temporary incomplete or faulty conceptions are in TDS not seen as failures but are often inevitable and constitutive of knowledge formation processes. However, the teacher may try to overcome the students' obstacles and force learning by devolving less of the task to the student. Telling the student that an standard method exists relieves her of the responsibility for her intellectual work and prevents her from struggling with the task. The teacher may try to help by providing a complete solution procedure, but then the student avoids to deal with the obstacle that can therefore become insurmountable.

TDS further explains why learning by imitation of algorithms is ineffective. Algorithm is broadly defined to include all pre-specified task-solving procedures, e.g. rules and template examples. An algorithm is a finite sequence of executable instructions that allows one to solve a given class of tasks. It can be determined in advance. The n:th transition does not depend on any circumstance that was unforeseen in the (n-1):th transition - not on finding new information, any new decision, any interpretation, or thus on any meaning that one could attribute to the transitions. Therefore, the execution of an algorithm has high reliability and speed, which is the strength of using an algorithm when the purpose is only to produce a task solution. However, if the purpose is to learn, the fact that an algorithm is (normally) executed without considering it's meaning implies that this kind of reasoning may lead to rote learning.

The aim of TDS is the design of situations allowing for the construction of knowledge by the learner. One central aspect of TDS is the devolution of problems. The student has to take responsibility for a part of the problem solving process. The teacher's task is to arrange a suitable didactic situation in the form of a problem, in such a way that if the student solves it then the student will obtain the desired new knowledge. From the point when the student accepts the problem as her own to the moment when she produces her answer, the teacher refrains from interfering and suggesting how to solve the task. This part of the didactic situation is called an adidactical situation, where the student must construct the solution and the teacher must therefore arrange the devolution of a good problem rather than describe what the student is supposed to learn.

Task solving reasoning affects learning

A series of studies resulting in a research framework (Lithner, 2008) suggest that a key factor affecting learning outcomes is whether students engage in imitative or creative reasoning. The purpose of this framework is not to characterise all aspects of reasoning, but to capture two variants found in empirical data: Imitative Algorithmic reasoning (AR) consists of an attempt to solve a task by applying a given or memorised algorithm. Both laboratory and classroom studies show that students often use superficial AR, which is a major obstacle when it comes to learning and using mathematics (Lithner, 2000; 2003; 2008; 2011; Boesen, et al., 2010). Teaching, textbooks and assessments mainly promote rote learning in the sense that algorithms are provided by teachers and textbooks, and that most practice and test tasks can be solved by AR (Lithner, 2004; Palm et al., 2011; Bergqvist & Lithner, 2012; Boesen et al., 2014). Judging from the research survey by Hiebert (2003) this is common also outside Sweden. For example, it was found in a study of common textbooks from 12 nations in 5 continents (Jäder et al., 2015).

Opportunities for students to create knowledge in line with TDS were found to be rare in teaching, textbooks and tests. When applied students were able to make better progress with Creative Mathematically founded Reasoning (Lithner, 2008). Empirical studies of the distinctions between AR and students' own constructions of solutions defined CMR as fulfilling the following criteria: i) Creativity; a (to the reasoner) new reasoning sequence is created, or a forgotten one is re-created (Silver, 1997). ii) Plausibility; there are arguments supporting the strategy choice and implementation explaining why the conclusions are true or plausible (Pólya, 1954). iii) Anchoring; the arguments are anchored in the intrinsic mathematical properties of the components of the reasoning (Lithner, 2008). Literature reviews find two main uses of 'creativity' (Sriraman et al., 2013): The extraordinary creativity of geniuses and the everyday creativity that "can be fostered broadly in the general school population" (Silver, 1997, p. 75). It is the latter meaning of creativity that is used in this framework, i.e. the creation of mathematical task solutions that are original to the individual who creates them. There are indications that ordinary students can learn by figuring out central aspects of school algebra themselves (Caspi & Sfard, 2012).

Formative assessment can guide teacher interaction

An important obligation for the teacher is to help when the student fails to solve the problem by herself. The easiest way in a short perspective is to tell the student how to do by providing a solution algorithm, but although it is a common approach it contradicts TDS and empirical evidence shows that it is not effective for long-term learning (Schoenfeld, 1985; Hiebert, 2003; Stein et al. 2008; Bergqvist & Lithner, 2012; Boesen et al., 2014; Jonsson et al., 2014). Another way, that in several research reviews have been shown to be one of the most effective ways of enhancing student learning, is to use formative assessment (Black & Wiliam, 1998; Hattie, 2009). The defining characteristic of formative assessment is that evidence of student thinking is elicited and used to modify teaching to better meet student learning needs. For formative assessment to be most effective teachers' probing questions need to elicit student responses that are interpretable in terms of conclusions about their reasoning. The teacher feedback given to the students need to be based on these conclusions, and targeted to support a successful use of the desired type of reasoning. The feedback should also be provided with a second aim of motivating the students to sustain their engagement in the task solving process and actually use the feedback to solve the task. An important determinant of student motivation is the expectancy of succeeding in their task solving, and the most important determinant of such expectancy beliefs is experience of prior success (Wigfield & Eccles, 2000). Thus, the characteristics of the teacher feedback need both include a part that helps students to further develop their reasoning, and a part that helps them to acknowledge their current achievement on the task.

THE CONCEPTUAL STRUCTURE OF THE LICR PROGRAM

Fundamental hypotheses

The theoretical TDS construct *devolution of problem* is operationalized through design of mathematical tasks that are intended to give students opportunity and responsibility to construct new knowledge through CMR. It is hypothesised that CMR will better than AR promote procedural fluency and understanding. According to TDS and the AR/CMR framework a theoretical argument in favour of this hypothesis is that if a solution method is not known in advance the student needs to construct a solution by CMR, and since this normally cannot be done by pure guesswork the student needs to consider the intrinsic mathematical meaning of the task's components in order to solve it.

Another hypothesis is that formative assessment is particularly suitable to meet the challenge of designing teaching that supports CMR. The theoretical argument behind this hypothesis is that according to TDS teaching shall support the student's own construction of the solution, but still without revealing so much of the solution that the task collapses from a CMR task into an AR task. Such teaching supporting CMR requires that detailed information about the student's thinking is elicited and used to modify teaching to better meet student learning needs, which is the defining characteristic of formative assessment. In contrast, in teaching supporting only AR this is not necessary since the main teaching strategy is just to provide algorithmic solution templates irrespectively of what the student's thinking is.

These hypotheses are reflected in the objects of study (design propositions) below.

A model relating task properties, reasoning, interaction and learning

To operationalize TDS for design of tasks and teaching we propose a model inspired by but not identical to Stein's et al. (1996) framework for relations between mathematical tasks and learning. The aim is to clarify the focus of the LICR research program, not to include all aspects of learning (Figure 1).

1) A goal of education is to develop the *student's mathematical competence*. 2) The student's prior competence will affect what *task solving reasoning* she will carry out. Conversely, the student's reasoning will affect what is learnt by the task. 3) The student's reasoning is affected by *task properties*, designed/selected by the teacher to engage the student in an activity leading to the learning goal. 4) *The teacher's interaction with the student* may support the student's reasoning.

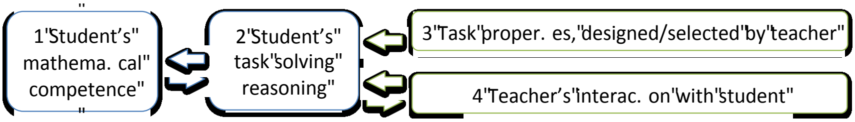

Figure 1. Student components (1 and 2) and teacher components (3 and 4)

<u>1 Students' mathematical competence</u>

Since mathematics learning is complex (Niss, 2007) it is not methodologically reasonable to analyse competence in ways that are both all-encompassing and specific at the same time. This research focuses on two central aspects of competence: students' *understanding of why* specific solution procedures are suitable and their ability *how to use* these procedures. Mathematical students spend most of the study time with tasks (e.g. Boesen et al, 2014), and solution procedures are important (Kilpatrick et al., 2001) but under-researched in mathematics education (Star, 2005). Mathematical understanding is often defined in terms of networks, representations and connections. Relating to the NCTM (2000) standards for *representations* of abstract and real mathematical entities and *connections* between representations, the LICR program developed a definition of mathematical understanding that does not aim to find a universally agreed-upon definition but a restricted one that is functional for the purposes of this study:

• Procedural understanding in task solving is defined as the ability to mathematically justify the key representations and connections of the procedures used in the strategy choices and implementations.

This definition can be extended to other aspects of understanding than procedures, but this is not within the scope of this paper. Thus procedural understanding is knowing *why* a solution procedure is suitable for a specific task. In order to define knowing *how* to solve a task we slightly modify a definition by Kilpatrick et al. (2001) to better relate to the definition of understanding above:

• Procedural fluency in task solving is defined as the skill to choose and implement procedures flexibly, accurately, efficiently and appropriately.

This research addresses how students can learn procedural understanding and procedural fluency *through* problem solving and reasoning.

2 Student's task solving reasoning

As described in the research framework above, a student's task solving reasoning will affect what kind of competence that is developed. Conversely, a student's previous competence will affect what kind of reasoning the student can carry out. For example, it is not possible to find the area of a triangle without sufficient knowledge about area and triangles.

3 Task properties, designed/selected by the teacher

The student's reasoning is affected by the *properties of the task*. A central tenet of TDS is that the teacher arranges a task-solving situation, in which the students can learn by creating a solution to a task that is designed to elicit the intended learning. In this research this is operationalized by the method of Hypothetical Learning Trajectory (Clements et al., 2004), which starts by establishing the student's prior competence in relation to the desired learning goal. Then a developmental sequence is anticipated, i.e. the student's progression through knowledge levels from the initial state to the learning goal. Finally, a set of tasks that are supposed to take the student through the developmental sequence is designed.

4 Student-teacher interaction, formative assessment

Students will not always by themselves be able to create the intended learning needed to solve a task. When this happens the teacher has the important role of facilitating the students' possibilities to take the next step in their learning. In order to follow the formative assessment approach the teacher needs to be able to quickly gather information about students' thinking and from this information deduce, or support the students to deduce, which specific mathematical difficulties that is hindering the students. In addition, the teacher needs to be able to use this deduction to provide feedback that facilitates a productive struggle for the students.

Methodology: Design research

Although there are some insights concerning how to teach mathematics to provide good opportunities for students to learn, reviews show that complete answers are far beyond reach (Niss, 2007). Teaching matters, but it is methodologically difficult to document that the desirable learning outcomes result from teaching rather than from other variables in play (ibid.). In addition, the impact of educational research on educational practice has been questioned both from the outside and from within the mathematics education research field (Schoenfeld, 2007). Methodological approaches combining educational development and scientific research are therefore being advocated, and design research is a methodology proposed by researchers in mathematics education (e.g. Cobb et al., 2003; Schoenfeld, 2007). In contrast to most research methodologies, the theoretical products of design experiments have the potential for rapid pay-off for practice since they are empirically evaluated principles for development of tasks and teaching (Cobb et al., 2003).

Propositions and principles are key notions in design research (McKenney & Reeves, 2012). Design propositions provide guidance on how to achieve the long-range goal. Based on theoretical understanding, empirical findings, and local expertise, design propositions suggest what a design should look like, and delineate how and why it should have certain characteristics. Design propositions serve practical goals of research by helping to sharpen the focus of an intervention and provide solid grounds upon which design choices can be made. They serve theoretical goals of design research by providing starting points for the theoretical framework that is used to focus empirical testing conducted on or through the intervention. During cyclic processes of design and formative evaluation, the design propositions are revised and transformed into research results in form of design principles that are theoretical insights of a prescriptive nature that recommend how to address a specific class of issues. A design proposition or principle contains a goal, claims about how to reach the goal and (empirical and/or theoretical) arguments supporting the claims (van den Akker, 2010).

Objects of study: Design propositions

The learning goal for each proposition below is procedural fluency and procedural understanding. The propositions contain claims (in italics) about how task or teaching design may promote AR/CMR and how this may promote procedural fluency/understanding (see the relations in Figure 1). Each claim is followed by a supportive argument. Due to the complexity of interaction between the components in Figure 1, only briefs summaries of the actual propositions are presented below.

<u>Task design propositions</u>

• *The presence of task solution templates lead to AR, which promotes short-term procedural fluency but neither long-term procedural fluency nor any procedural understanding.*

• *If the conceptual and creative challenges are suitable then absence of solution templates leads to CMR, which promotes long-term procedural fluency and procedural understanding.*

These propositions are theoretically supported by TDS and the reasoning framework, and partially empirically supported in the literature (e.g. Hiebert, 2003; Jonsson et al., 2014).

<u>Teaching design propositions</u>

• *Providing a solution template or helping the student to search for one promotes AR.*

• *To promote CMR, first let the student try to construct her own solution. If this fails, then in line with characteristics of formative assessment assess the student's task specific difficulties and provide feedback that supports her possibility and responsibility to construct the solution.*

These propositions are in line with TDS and have some general empirical support in surveys (Hiebert, 2003; Niss, 2007) and also indications from specific studies on how teacher interaction affects reasoning (Bergqvist & Lithner, 2012). Diagnosis and feedback supporting CMR can range from general problem solving support (Schoenfeld, 1985) to more content-oriented (Black & Wiliam, 1998), but it largely remains to clarify how to specifically support CMR.

EXAMPLES OF LICR STUDIES

Several LICR studies are analysing the design principles from various interdisciplinary approaches. Some examples of such studies are presented here.

A quasi-experiment on learning outcomes from AR and CMR tasks

Empirical design research may be in the format of laboratory pre-stages to classroom design (Schoenfeld, 2007). This particular approach of applying design research in classical experiments has rarely been reported before. There is a lack of controlled studies in education, only 3% of papers published in leading mathematics education journals reported experimental studies (Alcock 2013). In order to make use of the reduced complexity (compared to a classroom) and to focus on as few variables as possible, this laboratory study by Jonsson et al. (2014) had a quasi-experimental design. Students were divided in two groups and worked alone with tasks presented on a computer screen without teacher-peer interaction. The AR group received training through 14 task sets with laboratory versions of a design that is common in school (Figure 2): A context (mathematical or applied), a given solution method (in this case a formula), an example how to apply the method and a set of questions that can be solved by the method.

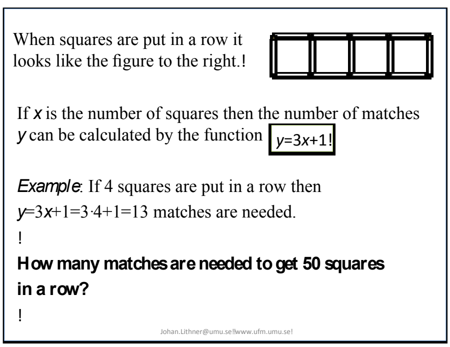

Figure 2, task with solution method.

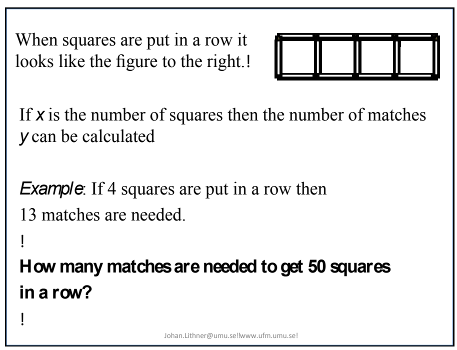

Figure 3, task without solution method

The CMR group practiced by similar tasks with the only difference that no solution procedures were provided, thus CMR was required to solve the tasks (Figure 3). This design represents the operationalization of TDS. As expected, since they had strictly more information in their tasks including procedural solution methods, the AR group outperformed the CMR group during practice (Composite practice, Figure 4).

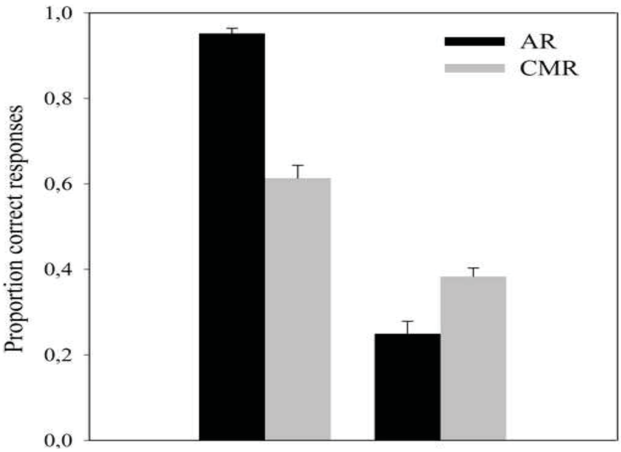

Figure 5 Composite practice Composite test

Each student practiced approximately half an hour at one occasion, which is a short intervention. One week later, students from both groups took identical tests with various questions related to the 14 sets of practice tasks. For example, given the same illustration as in Figure 2 and 3 students were asked to find the number of matches needed to get 30 squares in a row. As shown in Figure 5 the CMR group outperformed the AR group in the test. When analysing predictors of test performance it was found that cognitive proficiency was more important for the AR group than for the CMR group, an indication contradicting the common belief that tasks requiring creative reasoning is more suitable for high-performing students.

Although in line with TDS and the design principles, this result is not self-evident. For example, the reason that we as teachers normally provide our students with AR solution templates is probably that we believe that we help students to learn this way. In addition, since the AR group had strictly more information they hade more solution possibilities, including CMR, to choose from. Although this study contains hypotheses based on theoretical arguments in favour of CMR it does not provide any empirical evidence *why* CMR led to better learning than AR. Therefore further studies explores questions concerning how this result can be understood, and some examples of such studies are presented below. Other ongoing LICR studies are presently exploring variations of thi quasi-experimental design, including the role of procedural understanding, variants of CMR/AR task designs and the addition of teacher interaction based on formative assessment.

Eye-tracking indicating task solution strategies

In an ongoing study corneal eye reflection technique was used to investigate the attentional processes associated with mathematical reasoning. The information in each practice task (Figure 2 and 3) was partitioned into four areas of interest: Illustration, Formula ("If *x* is the number…"), Example and Question. The number and time of fixations in each of the areas were recorded. The basic assumption in this method is that there is a correlation between the information in the areas focused by the participant and her thinking processes. For example, it was hypothesised that participants given CMR practice tasks would need to focus on the Illustration and the Question in order to construct the solution method, and have little use of the information in the other two areas. Preliminary analyses indicate that this may be confirmed, but more data is presently collected in order to test significance levels. It was also hypothesised that participants given AR tasks would focus on the Formula and/or the Example since they contained solution method information, which was confirmed. Even though the AR practice tasks contain strictly more information than the CMR tasks the former leads to less effective learning, and a reason indicated by this study is that the AR participants' attention is only on application of the given algorithm and not on trying to consider other aspects of the task.

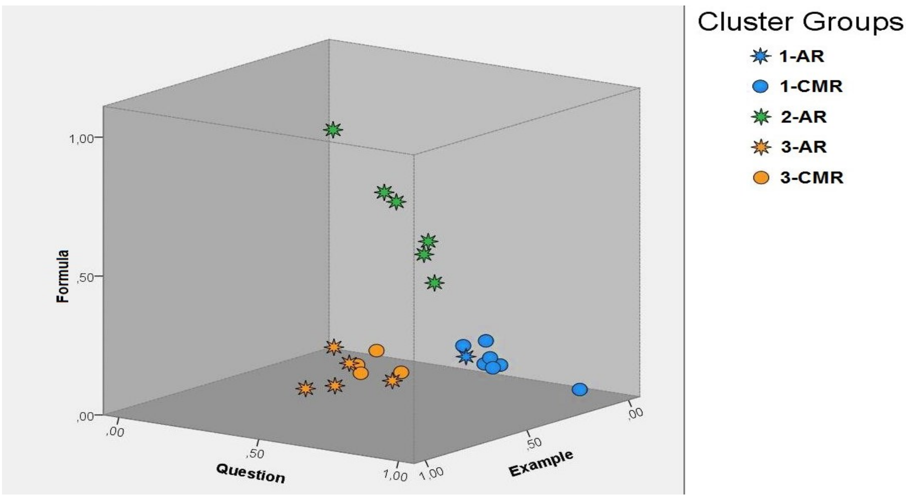

Figure 6 Three cluster groups with respect to task areas of interests

A preliminary ANCOVA cluster-group analysis on a first set of data indicates that there are not two cluster groups as expected (i.e. CMR and AR), but instead three clusters in Figure 6. Cluster 1 (blue) focused mainly on the question, Cluster 2 (green) on the formula and Cluster 3 (orange) on the example and question. For some reason, test performance was more related to cluster than to group (AR/CMR) and ongoing analyses are trying to reveal why. The 1-CMR group significantly outperformed all the other groups, including 2-CMR. The single 1-AR participant performed as the 1-CMR participants both with respect to eye attention and post test performance. Perhaps this is an empirical example of the uncommon but theoretically self-evident possibility for an AR participant to construct and/or analyse a solution method even though this is not necessary since a solution algorithm is available.

Collaborative creative reasoning supported by ICT

Alongside with earlier presented theoretical arguments and empirical evidence that students who are engaged in creative reasoning may learn better than students who are provided with procedures to imitate, there are several studies emphasizing the value of collaborative work. Students' verbalization of mathematical concepts to engage in dialog has been shown to be beneficial to enriching their conceptualizations (Hoffkamp, 2011) or establishing mathematical meaning (White et al., 2012). However, there are studies pointing out obstacles to work with non-routine problems without supporting activities (Stein et al., 2008) and issues with students' tendency to cooperate, dividing work amongst themselves rather than collaborating and sharing understanding (Roschelle & Teasley, 1994).

Research provides various methods for supporting students in developing conceptual understandings as well as collaborative work. One of the suggested methods is the use of dynamic software that allows students to visualize functions and their representations (Rakes et al., 2010), as well as distribute their collaborative problem solving process (Stahl et al., 2006). The idea of considering the appropriate support for student engagement in collaborative problem solving and creative reasoning combined with the proposition that technology may support these activities brought us to look closer into questions about the students' way of using the software for creative reasoning, collaboration, problem solving and handling mistakes.

A LICR study by Granberg & Olsson (2015) aimed to develop insights into how GeoGebra could support collaboration and creative reasoning in problem solving. The results showed that GeoGebra supported collaboration and creative reasoning by providing students with a shared working space and feedback that became the subject for students' creative reasoning. The students' collaborative activities that aimed toward sharing their reasoning with one another enhanced their creative reasoning. Ongoing LICR studies further explore the relations between students´ productive struggle, students' utilization of feedback from dynamic software and their reasoning during their collaborative ICT-supported problem solving.

fMRI studies on brain activity related to AR and CMR

73 participants were given similar practice and test tasks as described above and underwent fMRI (functional magnetic resonance imaging) during the test, see Karlsson et al. (2015) for details and additional references. In line with earlier results, the CMR group outperformed the AR group in post-tests. Participants from the CMR group showed lower brain activity in angular gyrus than participants from the AR group (Figure 5), possibly reflecting reduced demands on verbal memory. The angular gyrus is a region often implicated in imaging studies on mental arithmetic (Zamarian et al., 2009), and is also implicated in models of memory-related internally directed attention and in relation to semantic information processing more broadly as part of a semantic control network. One interpretation is that one week after practice the CMR

participants could more easily access their memory of a (complete or partial) solution method.

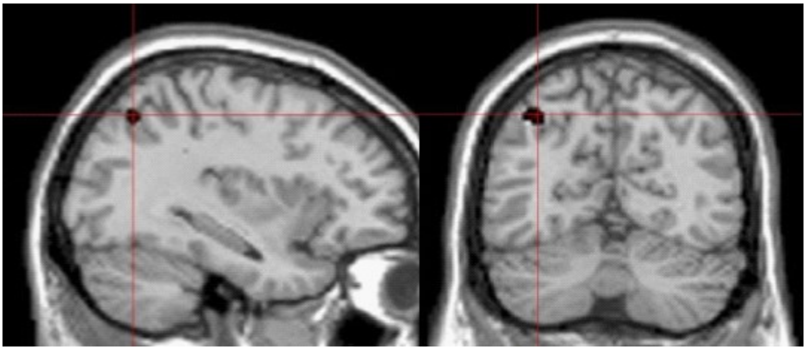

Figure 5 The AR group activated relatively more left angular gyrus during post-test

The second region with lower activity for CMR compared to AR was left precentral cortex/Brodmann area 6, a region that has often been implied in neuroimaging of cognitive functions as especially related to working memory (Cabeza & Nyberg, 2000). Precentral cortex is also commonly implicated in imaging studies of mental arithmetic in relation to untrained or more complex tasks compared to trained or less complex tasks. Tentatively, this implies that CMR participants needed to engage working memory processes at test to a lower degree than AR participants.

One tentative hypothesis discussed by the researchers before this brain imaging study was that perhaps the CMR group outperformed the AR group because the CMR participants had some kind of higher brain activity during post tests. Summarising the results it turned out that opposite hypotheses were confirmed, i.e. that those who learnt by creative reasoning somehow could use their mental resources more economically and still perform better. It is of course difficult to make direct inferences from the Theory of Didactical Situations to brain activity, but it seems that teaching through the principle of devolution of problem led to some kind of better memory encoding, so that CMR participants more easily remembered the solution methods or better understood what they have learnt and therefore more easily could reconstruct forgotten solution methods.

An ongoing LICR study is presently gathering new fMRI data comparing brain activity from students practicing with analytical (relational) and non-analytical (associative) mathematical tasks. In collaboration with neuroscientists and psychologists, the LICR program has received grants for several new fMRI studies. A final comment in relation to the PME theme, is that the purpose of this kind of interdisciplinary bridge-building is to explore forms of theoretical and empirical evidence that are still not so common within the field of mathematics education.

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LEARNING THROUGH ACTIVITY: ANALYZING AND PROMOTING MATHEMATICS CONCEPTUAL LEARNING

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The Learning Through Activity research program focuses on the processes by which individuals learn mathematical concepts and how mathematics educators can make use of these learning processes to foster particular concepts. In this paper, I give an overview of our work on the nature of mathematical concepts, our emerging elaboration of reflective abstraction, and the instructional approach we use in our one-on-one teaching experiments. I also provide three brief examples meant to provide images of reflective abstraction and how reflective abstraction can be studied in teaching experiment sessions.

The international mathematics education community has coalesced around the idea that conceptual understanding is a primary goal of mathematics instruction (Godino, 1996). At the same time, the mathematics education research literature is replete with evidence of students' difficulties with many important mathematical concepts. This raises important questions.

What do we mean by "conceptual understanding," in mathematics in general and within particular mathematical domains?

How do students develop understanding of mathematical concepts?

How can mathematics educators foster learning of particular mathematical concepts?

In this article, I discuss the work of our research program, Learning Through Activity (LTA)[1], which has focused on these three questions.

[1] The work that was foundational to the LTA program began during collaboration between Ron Tzur and me that began almost twenty-five years ago. My first funded project dedicated specifically to this work began 11 years ago.

In the LTA research program, we make use of multiple learning theories, using each to do different work (Simon, 2009). In particular, we have made use of constructs of sociocultural theory, the emergent perspective, and constructivism. The work that I will describe here makes use of and contributes to constructivist theory. Whereas, sociocultural theory has contributed greatly to understanding of the social, historical nature and basis for learning and the means of mediation involved, it has been largely mute on what Vygotsky (1978) called "internalization" of knowledge by individuals. Bereiter (1985) wrote, "How does internalization take place? It is evident from Luria's first-hand account (1979) of Vygotsky and his group that they recognized this as a problem yet to be solved" (p. 206).

The basic premise of the LTA research program is the following. *The scientific basis for mathematics instruction rests in part on understanding the process by which individuals construct mathematical concepts. Useful models of conceptual learning can provide the bases for the development of pedagogical approaches that engage the conceptual learning system of the learners in ways that are likely to result in the construction of particular concepts.* In other words, the better we understand learning, the better job we are likely to do teaching.

WHAT IS A MATHEMATICAL CONCEPT?

For the most part the idea of a mathematical concept has remained vague in mathematics education discourse. This is because explaining what we mean by a concept usually degenerates into the use of equally ill-defined terms such as *understanding*. In the LTA project, we have attempted to characterize the term *mathematical concept* in ways that make it a useful construct in mathematics education research.

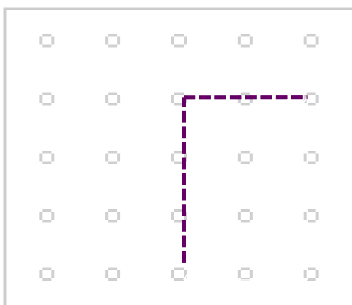
Understanding the nature of mathematical concepts and understanding how they are learned are necessarily intertwined. Each has implications for the other. It is probably clear that investigating how mathematical concepts are learned requires a specification of what we mean by mathematical concepts. However, I start this discussion by distinguishing between learning processes that produce mathematical concepts and those that do not.

In Simon (2006), I contrasted *reflective abstraction* and *empirical learning process*. Further, I argued that mathematical concepts are always the result of reflective abstraction and never the result of an empirical learning process.[2] In this section, I highlight the difference in these two learning processes. Our work on the mechanism of reflective abstraction is discussed briefly in a later section.

In an empirical learning process, students introduce inputs and observe a pattern in the outputs. For example, if students multiplied pairs of odd numbers and observed that the results were odd, they could come to believe that the product of two odd numbers is odd. However, from this activity, they would not come to know the logical necessity of that relationship. Knowing *that* something is true is not a mathematical concept. A mathematical concept involves knowledge of the logical necessity of an idea. Piaget's (2001) construct of *reflective abstraction* provided some foundational ideas about how knowledge of logical necessity is constructed.

Let's consider a classroom example from one of our earlier projects (Heinz et al, 2000). Ivy was a sixth-grade teacher (students 11-12 years of age), who was committed to students learning mathematics with understanding. In the episode, described here, she was beginning the teaching of the formula for the area of a triangle. Following is an outline of her lesson as it unfolded:

1. Students worked in groups to find the area of a given right triangle on a geoboard.
2. The class discussed their strategies. Completing the rectangle (pictured below) was a common strategy. Students used prior knowledge of the area of the rectangle, and then cut the area in half to find the area of the right triangle.



[2] These claims should be understood as a theoretical distinction used for explanation and prediction. They are researcher constructs and not claims about an objective reality.

Figure 1. Completing the rectangle in geoboard task

3. Students worked in groups to find areas of other right triangles on their geoboards. Following Ivy's instructions, they recorded measures of the base, height, and area for each triangle in a chart.
4. Ivy convened the class and recorded in a large chart the base, height, and area of the triangles the students measured.
5. Ivy then directed,

> Look at how these numbers are in this chart with our areas . . . and see if you can figure out a pattern that you can use every time using the numbers [measures of base and height] to come up with the area. ... There is something that you can do to these [measures of] the bases and the heights to get the area. (Heinz et al., 2000, p. 94)

Ivy's encouragement of students to find a pattern in the "numbers" is a promotion of an empirical learning process. In her lesson, she treated the geoboard work as if it was a black box that turned inputs (measures of the legs of the triangles) into outputs (measures of the areas) for the purpose of finding a pattern in the numbers. Finding a pattern in this way does not result in knowing the logical necessity of the relationships expressed in the formula for the area of a triangle.

At this point, I anticipate the following objection, "Denoting patterns is a key aspect of doing mathematics." Mathematics has been called "the science of patterns" (Steen, 1988; Devlin, 1996). We do not want to eliminate attention to patterns in mathematics classrooms. However, the discussion here is about explaining the process of learning mathematical concepts, not of doing mathematics more broadly. Empirical identification of a pattern never, in itself, results in conceptual understanding. It may be the trigger for some other activity that results in such learning. Identifying patterns empirically is neither necessary nor sufficient for conceptual learning. Further, in some situations it might not even be appropriate. In our example, looking for a pattern among the legs of a triangle and the area is neither optimal for learning nor does it provide a useful model of mathematics.

Consider now my modification of Ivy's lesson. I begin the lesson as Ivy did (Steps 1 and 2 are identical) in order to highlight particular contrasts in students' opportunities to learn, even though more effective lessons could be designed for this subject matter. The sequence follows.

1. Students worked in groups to find the area of a given right triangle on a geoboard.
2. The class discussed their strategies. Completing the rectangle (pictured below) was a common strategy.
3. Students work in groups to find areas of other right triangles on their geoboards. *No recording chart*.
4. Students are given a ruler and asked to find the area of right triangles drawn on plain paper (legs not parallel to sides of paper).

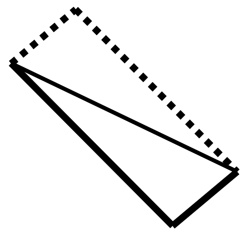


Figure 2. Completing the rectangle in paper-and-pencil task.

5. Students are given the measurement of the legs of right triangles involving larger numbers for the dimensions. They are asked to find the area of each triangle (without drawing) by mentally running through the process they did on paper.
6. Students are asked to write a generalization (algorithm) for how to calculate the area of a right triangle given the measures of the sides.

What do we see in this example? If we accept that the lesson sequence can promote reinvention of an algorithm for the area of a right triangle, we can see that it does so *without* employing an empirically generated pattern. Rather the learning based on completing the rectangles can lead to knowledge of the logical necessity of the formula for the area of a right triangle. Students do not need to focus on a pattern, because they can abstract from their informal activity, that is, they come to anticipate the related rectangle that will have twice the area of the triangle in question.

Having contrasted reflective abstraction and empirical learning processes, I now address more directly the question, "What is a mathematical concept?" A concept is a researcher construct used to specify particular learning goals and to specify what learners have come to know. *A mathematical concept is a researcher's articulation of (intended or inferred) student knowledge of the logical necessity of a particular mathematical phenomenon*. This characterization emphasizes that it is an articulation by the researcher, not the student[3], but that it aims to describe what the student knows or will know. An implication of this characterization is that the articulation of the student's knowledge of the logical necessity *must be in relation to anticipated or documented prior knowledge of the student*. In the first two examples of reflective abstraction below, I provide examples of articulation of mathematical concepts.

[3] Students may or may not be able to articulate the concept.

STUDYING CONCEPTUAL LEARNING

Much of the important research in mathematics education over the last 30 years has been described as "research on mathematics learning." Given that this language is already widespread, it is difficult to denote the differences between our research program and research that has been labelled "research on mathematics learning." There are two key distinctions. First, most of the work subsumed under this heading has focused on students' conceptual steps or a hierarchy of conceptual steps (e.g., Steffe, 2003; Cobb, McClain, & Gravemeijer, 2003). Whereas this work has been important in the field and foundational to our work, our focus is different. Our focus is on understanding the subtle shifts in thinking that account for the transition from one step to the next. Understanding these shifts can be the basis for improving instruction. This is challenging work. Hershkowitz, Schwartz, and Dreyfus (2001, p. 197) pointed out,

> In spite of the animated theoretical debate that has taken place on the nature of abstraction, little experimental research is available. … We surmise that the lack of experimental evidence is due to the difficulty of observing the processes of abstraction (as opposed to the products, for which there is more evidence).

The second distinction is that when learning is explained, it is often explained using broad concepts such as reflective abstraction, generalizing assimilation, or negotiation of meaning. For us, these are not sufficiently nuanced to be useful for instructional design. Siegler (1996, p. 223) argued, "Standard labels … are more promissory notes, telling us that we really should work on this some time, than serious mechanistic accounts."

STUDYING REFLECTIVE ABSTRACTION: THE BASIS OF LTA

Piaget (1980) indicated that reflective abstraction is the basis of all logico–mathematical structures. The LTA research program began with the central idea of reflective abstraction *–individuals construct mathematical concepts by abstracting from their activity*. This was a compelling idea, because it helped us make sense of our observations and experiences in mathematics education. It explains how a child develops a concept of number by engaging in counting activities with a parent. It explains how students in our modification of the triangle area lesson (described above) could reinvent the formula for the area of a right triangle. However, Piaget's construct of reflective abstraction was not sufficiently elaborated to undergird important changes in mathematics pedagogy. DiSessa and Cobb (2004, P 81) wrote:

> Piaget's theory is powerful and continues to be an important source of insight. However, it was not developed with the intention of informing design and is inadequate, by itself, to do so deeply and effectively.

As LTA researchers, we set out to study how students' develop mathematical concepts on the basis of their mathematical activity and how to promote such learning. For this purpose we developed a specialized teaching experiment methodology. This methodology was explicated and warranted in full in Simon et al. (2010). I summarize briefly here.

The LTA teaching experiment was developed from the teaching experiment methodology discussed in Steffe and Thompson (2000). The significant modifications of the teaching sessions are of two types:

1. To focus on the students' abstracting from their activity, we conduct the teaching experiments with one student at a time. The researcher/teacher refrains from giving ideas, demonstrating strategies, asking leading questions, and giving hints.
2. Task sequences (discussed below) are used for all sessions. However, the researcher/instructor has prerogative to make changes in the sequence in response to the students' work and verbalizations.

The figure indicates how the specialized teaching experiments contribute to building knowledge of both the students' conceptual learning processes and an instructional methodology that promotes such learning. That is the researchers use their best current instructional approach to foster learning. The analysis of the learning then informs modifications of the instructional approach, and the cycle continues throughout the teaching experiment.

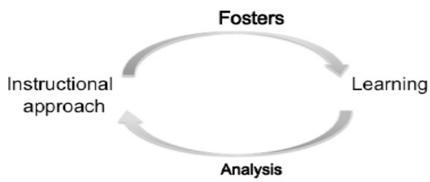


Figure 3. Co-emergence in teaching experiment

IMAGES OF LEARNING THROUGH ACTIVITY

What does it mean to promote and study reflective abstraction? In this section, I provide examples intended to provide images of the process of reflective abstraction. I begin with discussion of an activity that could foster an understanding of the product of two odd numbers. As mentioned above, accumulating empirical results from multiplying pairs of odd numbers does not lead to a mathematical concept.

Example 1: Product of Two Odd Numbers

Consider a class in which students work with sets of poker chips. The class defines a set as having an even number of chips, if all of the chips can be paired. An odd number is when one of the chips cannot be paired. They then are given tasks to determine whether the products of particular pairs of odd numbers are odd or even. They are to do so by using the chips and *without* determining the product (i.e., showing whether all the chips can be paired). Consider the figure below showing their activity for 3x5.

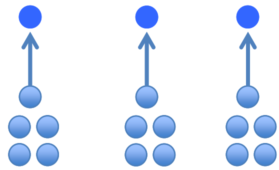


Figure 4. Chip representation of 3x5.

Each of the sets is organized to show that all elements except one are paired. To determine the product's parity, the students must attempt to pair the unpaired elements (one from each set). For some students, this activity alone may lead to the abstraction. However, to foster the abstraction for a greater number of students, the instructional plan goes further. After the students have done a number of these chip tasks, they are asked to talk through the chip activity *without* using the chips, what we call a "mental run" (Simon, et al, 2010). This additional phase promotes reflection. That is, it requires them to anticipate the number of chips to be paired based on their prior activity (not based on the result they see displayed on the desk). Student thinking might be something like, "Since I have 7 sets of chips and they are all odd, there will be 7 chips that need to be paired. But I know I cannot pair 7 chips, so the product is odd." Besides anticipating the number of unpaired chips, the student may also beginning to see that the number of paired chips does not require their attention. This can be intentionally developed through the task sequence. If the last task was to do a mental run of 7x13, it could be followed by a mental run of 7x29. Here they might respond, "Same thing! We would still have 7 single chips. It does not matter how big the set is. Each odd set will have one single chip."

In this example, we can "see" the students abstracting from their activity. They have a consistent activity that involves removing one chip from each set and trying to pair it. After engaging in this activity sufficiently, they are able to anticipate the number of single chips. During the mental runs, their mental activity is focused on the single chips. The subsequent tasks promote abstraction of the invariance of the number of single chips across variation in the odd number of chips in a set (while the number of sets remains constant).

We might articulate the goal abstraction as follows:

> The product of two odd numbers is odd, because you would have an odd number of sets with an odd number in each set. Each set would have one unpaired element. So with an odd number of sets, there are an odd number of unpaired elements, so the unpaired elements could not be all paired. Therefore, the product must be odd.

Example 2: Area of a Triangle

If we look back at my modification of Ivy's lesson, we can see parallels with Example 1. First the students are engaged with an activity: Represent the right triangle; complete the rectangle; determine the area of the rectangle, divide by two. The full mental activity needed to make the abstraction does not occur until the students are engaged in the pencil-and-paper tasks. In the pencil-and-paper tasks, they need to make the appropriate rectangle without the aid of the geoboard grid. Once they have gotten competent drawing the rectangle and determining the size of the rectangle based on the measurements of the legs of the triangle, they are asked to do a mental run. In the mental run, they anticipate the relationship between the area of the rectangle and the measures of the triangle. Once again, the mental run requires that they anticipate the result of their activity (e.g., the dimensions of the rectangle) as opposed to seeing the results in the constructed diagram.

We expect that the students would abstract:

> The area of the right triangle is one half or the product of the legs, because a rectangle can be drawn such that each leg of the triangle is a side of the rectangle. That rectangle's area (leg one x leg two) would be twice the area of the right triangle.

Example 3: A Unit Fraction of a Unit Fraction.[4]

Here I use an example from our MARN[5] teaching experiments. Kylie, a 4th grade student (age 10) was confronted with the following tasks in the Java Bars Computer environment (Biddlecomb & Olive, 2000). In our use of JavaBars, quantities were represented by the lengths of rectangular bars. The bars could be partitioned, a part of the bar pulled out, and a bar or a part of a bar iterated,[6] resulting in a composite rectangle.

Task 1: This [bar] is one-third of a unit. Make a bar that is one-sixth of a unit.

Kylie made it clear that she did not know how to "cut up" the bar on the screen. She then iterated the third three times to make the unit and then cut the first third in half. She stated immediately that one of the small pieces was one-sixth.

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[4] Hackenberg and Tillema, (2009, p. 2) refer to this concept as recursive partitioning, which they defined as "partitioning a partition in service of a non-partitioning goal, such as determining the size of 1/3 of 1/5 of one yard in relation to the whole yard."

[5] The Measurement Approach to Rational Number (MARN) Project (2010-2016) is supported by the National Science Foundation (DRL-1020154). The opinions expressed do not necessarily reflect the views of the Foundation.

[6] The MARN researchers modified Java Bars by creating an "iterate" button. We are grateful to Frank Iannucci for programming this modification.

Task 2: This is one-fifth of a unit. Make one-tenth of a unit.

Kylie used the same process. She iterated the one-fifth 5 times to make the whole and then partitioned the first fifth into two subparts. She reported, “Here, you have one-tenth of a unit.”

Task 3: This is one-third of a unit. Make one-ninth of a unit

This time Kylie *immediately* divided the one-third bar into three pieces (without iterating to make the whole).

K: One of those is one-ninth.

R: How do you know?

K: How many times does three go into nine? ... Three times. And it's one third! So, three times three is nine, and one of -- if you cut it up into thirds again. That is, um. ... And you take one, it would be ... one-third. … But that's really one-ninth of a unit.

Kylie seemed to indicate that she thought about what number of parts would iterate three times to make 9 parts (in the unit).

Task 4: This is one-fifth of a unit. Make one-twentieth of a unit

She immediately cut the fifth into four pieces. She went on to complete two more tasks in this way.

Let us look at how Kylie developed an abstraction based on her activity. In the first two tasks, Kylie had no way to think about partitioning a unit fraction $1/n$ to make the fraction $1/mn$. However Kylie had knowledge that allowed her to make the requested fractional part. She understood that $1/n$ of a unit is a part that can be iterated n times to make a whole unit. She also knew that she could partition a unit to make any unit fraction. She used this knowledge to iterate the original part, one-third, three times to make the unit. Once she had made the unit, she knew that she needed to partition it into six parts. Because she had a bar that was already partitioned into three parts, she was faced with a subtask which she could solve using her whole-number partitive division concept.

This activity that Kylie used for Task 1 eventually led to the abstraction. Kylie used the activity in the second task as well. The activity consisted of a sequence of actions, iterating the part to make the whole and using partitive division to determine the number of subparts per part. A coordination of these actions (Piaget, 2001) allowed her to know immediately that she could produce $1/mn$ from $1/n$ by partitioning $1/n$ into m parts. She could justify this relationship when asked by discussing the activity through which it developed. In Task 3, she no longer employed the sequence. She had developed a *new* action that was at a higher level than the sequence from which it was built and allowed her *to know the result at once*. She did not need to mentally generate the unit *prior* to partitioning the part. She was “seeing” the situation through a new concept, a concept that was built from the concepts that made up the activity sequence she had been using.

ELABORATING THE CONSTRUCT OF REFLECTIVE ABSTRACTION FOR MATHEMATICS PEDAGOGY

Understanding how students construct mathematical concepts involves understanding how concepts are built on prior concepts. Piaget (2001) explained reflective abstraction as a *coordination of actions*. We have elaborated this idea to better explain the hierarchical construction of mathematical concepts. We consider a concept to consist of a goal and an action that can be enacted to achieve the goal.

In the first step in the construction of a new concept, the learner *sets a new goal* (usually in response to a novel task) and *creates a new activity*. An *activity* is a sequence of actions called on to accomplish the goal. In Example 3, the activity consisted of iterating the part to make the unit and determining the number of partitions per part. No learning had occurred to this point. Both the setting of the goal and the calling upon the actions were accomplished using available knowledge.

An important point in our elaboration is that the actions called upon are components of prior concepts. That is, each action is called upon based on a subgoal, the goal of a prior concept. In our example, the first subgoal was to produce the unit, for which she called on her concept of 1/n as a part that iterates n times to make the unit. We therefore reinterpret a coordination of actions as really being a *coordination of concepts*. It is the coordination of concepts that explains how concepts are built upon prior concepts.

The coordination of concepts transforms the activity (sequence of actions) into a single, higher-level action. That is, the learner no longer needs to go through the sequence of actions; they can anticipate the result of that sequence. For Kylie, beginning in Task 3, she was able to know immediately how to partition the partial unit. She no longer needed to first iterate the part and then determine the number of partitions.

This elaboration of reflective abstraction serves our instructional approach and analysis in two ways. First, it explains how the student's activity contributes to the generation of the new concept. Second, it specifies the process of building a concept on prior concepts. In Simon, Placa, and Avitzur (in press), we elaborated the construct of reflective abstraction more fully.

THE LTA APPROACH TO THE DESIGN OF TASK SEQUENCES

The elaboration of reflective abstraction and the LTA instructional approach have co-emerged through the LTA teaching experiments. The LTA instructional approach is an approach to directly fostering particular concepts (Simon, 2013). Thus, it is different from engendering perturbation (disequilibrium), an approach that *provokes* a need for change, but does not contribute directly to the nature of the change (learning). The rationale behind the LTA instructional approach is the following: *If a concept can be thought of as the result of reflective abstraction, that is, an abstraction derived from particular activity, then it should be possible to engineer a sequence of tasks that encourages that activity and promotes the abstraction from that activity.* There may be multiple designs that can foster a particular concept.

The first two steps in our design are similar to the first two steps in most instructional design that is aimed at conceptual learning. We assess student understanding and articulate a learning goal (a concept, as discussed above) for the students relative to their current knowledge. It is after these first two steps that our approach diverges from many others.

Our third step is to specify an *activity that students currently have available* that can be the basis for the abstraction specified in the learning goal. The fourth step is to design a task sequence and related hypothesized learning process. The task sequence must both elicit the intended student activity and lead to the intended abstraction on the part of the students. The hypothesized learning process must account for not only the overt activity of the students, but also the mental activities that are expected to accompany those overt activities. I will not focus on steps beyond step 4 (e.g., symbolizing, introducing vocabulary, discussing justification), because, like the first two steps, they are common to many approaches.

CONCLUSION

In this concluding section, I discuss three implications of the LTA research reviewed above:

1. Learning can be studied directly and to productive ends.
2. Learning can be promoted in ways that do not depend on novel problem solving on the part of students.
3. Engineering learning opportunities that allow students to build abstractions upon their mathematical activity (engendering the requisite activity) may increase equitable access to high-quality conceptual learning.

Before discussing these three issues, I offer a contrast that figures prominently in all three implications.

Contrasting a Problem Solving Approach and the LTA Approach to Instruction

One of the common approaches to instruction, which represents recent emphases on conceptual understanding, mathematical reasoning, and justification, is a problem-solving approach. In a problem solving approach, new mathematics is presented by engaging students in tasks that require construction of the mathematical concept to be learned. The tasks are intended to be legitimate problems in the sense that the students do not have a solution strategy at the outset. Problem solving is an important and critical part of mathematics, and problem-solving lessons are an essential part of good mathematics pedagogy. However, the disadvantage of problem solving lessons is that the success of any individual student or small group of students is uncertain and often improbable. Many students are unable to make the problem-solving *leap*, and therefore must learn about the concept by hearing the reports of more advanced students.

The LTA approach is *not* meant to replace problem-solving lessons, it is meant to complement it, particularly for difficult concepts. LTA instructional sequences are designed to engage students in the particular activities that will lead them to making the relevant abstraction. Of course, no method is guaranteed. However, the LTA approach helps the student build what they need to construct the concept.

A disclaimer is in order here. Work needs to be done to figure out effective ways of making use of the LTA approach in classrooms.

Studying Learning Directly

As discussed, it is easier and more common to study the steps involved in learning a concept than it is in studying how students make the transition between steps. I have also argued the importance for mathematics pedagogy of understanding these transitions – the learning process. What is it about our research that allows us to gather useful data for studying the learning process?

First, the LTA design of instructional sequences gives us a way to study learning. Because the lesson design is an attempt to "engineer" the learning of the concept, the teaching experiment has a hypothesis-testing component with respect to the learning process for the particular concept. Cobb and Steffe (1983) emphasized that teaching experiments allow researchers to test their hypotheses with respect to their models of the students' current mathematics. The LTA approach allows for testing of the researchers' model of the learning process, a process of making a particular reflective abstraction. The hypothetical learning trajectory (HLT), produced by the researchers, which includes the task sequence (Simon, 1995), can be understood as the researcher's model of a learning process that leads to a reflective abstraction producing the particular concept. The enactment of the HLT with the students allows that hypothesis to be tested and modified.

Second, the LTA instructional approach engenders the abstraction *during* the teaching experiment sessions. In conventional teaching experiments, the researcher may observe during a couple sessions that the students do not have the concept in question. In the third session, the student gives evidence of having the new concept. Whereas, the work in previous sessions likely contributed to the change, it is difficult in such situations to explain the change in any detail. Steffe (1991) referred to this phenomenon as a "metamorphic accommodation." He explained, "A metamorphic accommodation of a scheme leads to a modification of the scheme that occurs independently but not in any particular application of the scheme" (p. 38). Because of the engineering involved in the LTA approach, we are often able to observe the shifts in students' thinking during the teaching experiment sessions, allowing us to make inferences about the learning process from these data.

Third, the LTA teaching-experiment methodology provides a particular tool for studying the learning process. Because it focuses on *one* student, the researchers can collect a consistent data stream reflecting the student's activity. Because the student (ideally) never receives input from the researcher or another student, the stream of data are not interrupted by periods in which the student is outwardly passive while listening to others.

A Complement to Problem Solving Instruction

The LTA approach to instruction offers an additional approach for teaching difficult concepts and for working with students who are less prepared for the mathematics being taught. It is a way of building up the concepts being taught. Students who "receive" the new idea prematurely -- before they are ready to make the abstraction – may have little choice but to try to retain the idea by rote.

Implications of the LTA Approach for Equity

Students who enter a class weaker conceptually than some of their classmates often are not the ones who make the conceptual advance through their own problem solving. They then must try to grasp the concepts through the explanation of their more able peers. It seems reasonable to consider that the students who solve the novel problems successfully and those that are unsuccessful, and must listen to the explanation of others, have different opportunities for learning and participating in the classroom. To the extent to which the LTA approach helps students build up the activity basis for the concept, we postulate that supplementing instruction with an LTA approach could reduce inequity in classroom learning opportunities.

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PLENARY PANEL

"GROUPING STUDENTS BY ATTAINMENT IS ESSENTIAL FOR THEIR LEARNING OF MATHEMATICS": A DEBATE

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The PME 39 (2015) plenary panel session in 2015 takes the form of a debate. The two members of the affirmative team, Miriam Amit and Kai-Lin Yang, will argue in favour of the claim that "Grouping students by attainment is essential for their learning of mathematics", while the negative team, Jinfa Cai and David Reid, will argue against the proposition. I, Helen Forgasz, am chairing the debate.

INTRODUCTION

> Researchers have struggled for decades to find answers to questions about ability grouping. Does anyone benefit from it? Who benefits most? Does grouping harm anyone? How? How much? Why? Reviewers of the research still disagree about the answers. For every reviewer who has concluded that grouping is helpful, another has concluded that it is harmful. (Kulik, 1992, p. vii)

The topic to be debated, "Grouping students by attainment is essential for their learning of mathematics" is indeed a contentious one. Debate has raged for years whether grouping students by attainment (ie. achievement) [7] is the best way to teach mathematics and for students to learn mathematics. The practice of grouping students by attainment is a common practice in many jurisdictions and can take multiple forms:

> In some countries, children are sent into different types of schools at a relatively early stage of their education, for example, vocational versus academic schools; this is usually decided by choice or by recommendation.

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[7] Note that the title does not refer to grouping by *mathematical ability*. It is important to reflect on whether it is even possible to measure mathematical ability. Mathematics tests do not measure ability; what they do measure is which mathematics questions students can correctly answer and whether they can employ appropriate mathematical procedures to reach a correct solution at the time the test was administered. It has been demonstrated that among students who do not provide correct answers, about one third of the errors are due to reading problems and not with whether the students know or can do the mathematics (Newman, XXXX).

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 1, pp. 65-68. Hobart, Australia: PME.

In countries where there is only one type of post-primary school, grouping by mathematics attainment at various grade levels is also common – in some countries (e.g., UK), grouping by attainment is also practiced at the primary/elementary level; mixed groups are often retained for other subjects.

In other schools, mainly primary/elementary schools, within class grouping by achievement takes place.

It is fascinating that it is teachers of mathematics who believe more strongly than teachers of other disciplines that grouping by achievement is not only appropriate but preferable. In 2010, I (Forgasz, 2010) conducted a small study with 46 post-primary mathematics teachers (grades 7-12) from different schools in Victoria, Australia. I found that 80% of the teachers reported that streaming (the Australian term for grouping by achievement, also frequently termed *ability grouping*) was practiced in their schools; 75% of these teachers agreed with the school policy. Also 75% of the teachers from schools without streaming did not agree with the school policy. In summary, about 75% of all the teachers supported streaming for mathematics. When asked how the groups were formed, mathematics test results were reported to be the most common means used to assign students into streams (e.g., for high achievers, mainstream classes, or for low achievers). In Australia, Clarke and Clarke (2008) claimed that:

> Following our time as classroom teachers, we have had the opportunity to work with teachers across the early and middle years of schooling in every Australian state and territory, in professional development settings and in classrooms. Our observations, conversations with teachers and students, and our reading of the research literature have convinced us that a major impediment to the mathematical learning of students and their beliefs about themselves as mathematical thinkers is the widespread practice of ability grouping in mathematics. (p. 31)

Passions run high whenever the topic of achievement grouping for mathematics is raised. The research evidence is equivocal about the effects on student learning of mathematics in mixed or attainment grouped classes, and about the effectiveness of the pedagogical approaches adopted by teachers in the various streams. The quality of teachers assigned to the classes of high- and low-achievers has arisen as an issue, as has the labelling and impact on future life opportunities of those students in the low-achieving groups.

Mathematics education researchers concerned about equity, gender issues, socio-economic factors, or racial/ethnic considerations highlight the inherent inequities arising from such grouping practices. Who is found amongst the high-achieving groups and who in the low-achieving groups? Research reveals that certain groups in some societies are disadvantaged; is this disadvantage reinforced through attainment grouping? Mathematics is recognised as the critical filter to a range of post-secondary studies and future career options. So, who benefits and who loses out as a result of grouping by mathematics attainment is an important consideration. Mathematics achievement is also frequently equated with general capability and intelligence. Do teachers (or policy makers) have the right to decide on, or constrain, an individual's future potential based on mathematics test results?

On the other hand, researchers concerned with promoting high mathematical achievement argue that gifted/talented mathematics students are held back in classes of students of mixed attainment levels – is this discriminatory? Mathematics teachers, it is also argued, are more effective when students are grouped homogenously. They can focus on the needs of the group of students they have – be they high, mainstream, or low achievers – and tailor their pedagogy accordingly. Can this, and does this, happen while still addressing the requirement to engage students in the full mathematics curriculum that they are expected to encounter and learn at each grade level? While it has been argued that grouping can be flexible with students moving up or down attainment groups depending on how their mathematical competencies change. Is it possible for a low-attaining student to catch up if moved into a mainstream group if the content/curriculum for the various groups has been modified to address students' needs?

And so the debate continues, as it will at the PME 39 Plenary Panel session in 2015.

HOW THE SESSION WILL RUN

The debate proposed for the PME 39 Plenary Panel session will comprise a Chairperson (Helen Forgasz) and four presenters/debaters; two will be on the affirmative team arguing in favour of the topic (Miriam Amit and Kai-Lin Yang), and the other two will be on the negative team arguing against the topic (Jinfa Cai and David Reid).

While there is the clear intention for the Plenary Panel debate to be informative and scholarly, you can expect some light-hearted exchange from time to time. A little humour can be quite persuasive.

It needs to be recognised that the words/terms included in the topic of the debate can play a critical role in that the two teams' interpretations of them serve to frame the arguments put forward.

Rules of the debate

As "Madame Chair", I will open proceedings. I will outline the rationale for the topic and describe the procedure and rules of the debate.

Each speaker will have a maximum of 10 minutes to speak. There will be a warning bell at 9.5 minutes; the speaker will be asked to sit down at 10 minutes. The summation talks will have a 5 minute time limit, with a warning bell at 4.5 minutes.

The opening speaker is from the affirmative team. Defining terms is often a part of the opening speaker's task followed by the opening arguments for that team.

Next is the first speaker for the negative; this speaker begins with a brief rebuttal of the affirmative speaker's argument, including a comment on the definitions and whether the negative team accepts the definitions or has alternative interpretations of the terms. This person then proceeds to open the negative team’s argument.

Then the second speaker for the affirmative begins with a brief rebuttal of the negative team’s opening argument and then continues with the remaining arguments of the affirmative team. Next, the second negative speaker repeats this process ie. brief rebuttal followed by continuing the negative team's argument.

To close the debate the first affirmative speaker has 5 minutes to rebut the negative team's argument and summarise the affirmative team’s case; no new arguments are allowed. Finally, the first speaker from the negative team does the same; 5 minutes to rebut followed by summary of the negative team’s case and no new material allowed.

I will then invite questions and comments from the floor for about 15 to 20 minutes. We will then take a vote from the floor to determine the winning team.

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THE ARGUMENT IN FAVOUR OF ABILITY GROUPS

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INTRODUCTION

The issue of whether or not students should be grouped based on their ability has been highlighted and debated in mathematics education for decades. One the one hand, a considerable number of researchers, policy makers, administrative staffs and school teachers are concerned with the equity of educational opportunity, and suggest maintaining a normal class grouping arrangement in schools. They claim that teaching with mixed-ability grouping offers low-attainment students opportunities to learn from the answers and explanations of faster learning students (Whitburn, 2001). On the other hand, it is also widely accepted that grouping students by ability is a suitable organizational arrangement, especially when considering the differences in students with respect to their prior knowledge, learning trajectories, and pace of learning. Ability grouping settings provide a milieu where students can make progress according to their competences. Faster learners can maintain their interest and motivation, since they are not required to wait for low-attainment students to catch up (Hallam & Toutounji, 1996). We agree with the argument for ability grouping, either in a within-class or a between-class format, and suggest that it is a better organizational arrangement for the teaching and learning of mathematics. Between-class ability grouping refers in our context to the arrangement by which similar-ability students have mathematics lessons together but can stay in a normal class grouping during the remaining school time.

The philosophical, psychological and sociological foundation for our support of ability grouping for mathematics teaching and learning is rooted in the differentiation of student learning articulated by the constructivism paradigm, for example, von Glaserferld (1989). According to this theory, each student is unique, interacting with his or her environment and constructing his or her own mathematics knowledge. In this regard, the ability grouping arrangement affords more learning opportunities for students to engage in, provides challenging mathematics activities suitable for varying levels and - consequently - optimizes student learning outcomes. The question of how to optimize learning outcomes for each individual student is the core upon which we will frame four arguments that explain why organizational arrangement by ability grouping is the better choice. These arguments are: equality of learning opportunity, effectiveness of learning, efficiency of teaching, and reduced educational cost. Each is more fully elaborated below.

THE CASE FOR ABILITY GROUPING

Our first argument is that ability grouping provides real equality of learning opportunities in education. For different levels of students, the requirements of what

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and how they need to learn are different (Wang & Walberg, 1985). In the ability-grouping environment, all learners can be encouraged to do their best by means of appropriate scaffoldings, due to the smaller variation of learning time and performance (Helwig et al., 1999; Kolikant, & Broza, 2011). Understanding, mutual respect and cooperation can be more easily developed between peers in the classroom due to their similar attainment levels. Ability grouping can thus have the effect of making peers less competitive and more cooperative, reducing mathematically disabled identities (Heyd-Metzuyanim, 2013).

The literature on tracking in some countries may lead one to assume that ability grouping reinforces social class inequalities (e.g. Hallinan, 1994). However, studies on the effects of ability grouping on equality of educational opportunities were confounded by other critical factors, such as the function of family education and resources, national examinations, curricular materials, and the criteria for tracking. As Oakset et al. (1992) stated, the effects of tracking can be better understood when both school environment and societal context are taken into account. When class ability level is controlled, grouping is found to have no significant effects on equality of educational opportunities (Betts & Shkolnik, 2000).

Although there have been studies suggesting that ability grouping increases the effects of family background on student achievement and intensifies inequality in achievement while having little effect on average achievement (e.g. Brunello, & Checchi, 2007; Schütz, Ursprung, & Wößmann, 2008), those studies used cross-national data, which cannot control for different educational and societal conditions in different countries. Although critics may claim that early school tracking is bad for equal opportunity because it reinforces the role of early childhood education in the quality and quantity of accumulated human capital (Brunello & Checchi, 2007), tracking for secondary education can reduce the influence of family background on student academic performance (Broaded, 1997; Cheung & Rudowicz, 2003; Gamoran, 2000). And though there have been studies suggesting that ability grouping increases dispersion in student achievements (e.g. Brunello, & Checchi, 2007; Schütz, Ursprung, & Wößmann, 2008), the magnified gap was far more prominently influenced by the quality of instruction rather than the by the ability grouping itself.

Our second argument is that between-class or within-class ability grouping undoubtedly generates better learning outcomes than no grouping. Meta-analysis has generally revealed that homogeneous ability groups achieve more than the heterogeneous group in studies of both within-class (Lou et al., 1996) and between-class (Kulik & Kulik, 1982) settings. Negative attitudes toward learning mathematics did not increase due to ability grouping (Ireson, Hallam, & Plewis, 2001), and reducing heterogeneity in the classroom did not have an adverse affect on the self-esteem of those students in the lowest achieving groups (Begle, 1975; Kulik, 1985). Tieso (2003) further noted the positive effect of grouping on low attainers' self-esteem, ascribing it to their opportunity to interact with the teacher and others in the classroom without the somewhat intimidating presence of high attainers.

In Israel, research on gifted seventh and eighth graders (n=561) who participated in homogenous classes in the Kidumatica program showed no impact on affective aspects such as self-esteem and confidence, and a positive impact on motivation and attitudes toward mathematics. It was also shown to be very beneficial on cognitive aspects such as non-routine problem solving, creativity and critical thinking, when compared to the same ability in students that where in mixed classes (Amit & Neria, 2008; Amit & Gilat, 2012). In Taiwan, a quasi-experimental study (Liu, 2007) showed that eighth graders (n=72) in between-class ability grouping performed better in their mathematics achievement and attitudes towards mathematics than eighth graders (n=72) who were not grouped. Moreover, the students grouped in the higher level preferred to take on different types of challenging problems, while the students in the lower level preferred to practice basic exercises repeatedly.

Studies found that the high-attainer group is more likely to be taught by more experienced teachers with better reputations, while the low-attainer group is more likely to encounter less experienced teachers (Gamoran, 2004; Oakes et al., 1992). This may account for why some studies showed the disadvantage of ability-grouping for low-attainers, and suggests that the instruction given to low-attainers may often be less suitable than the instruction given to high-attainers. If the instruction given to the low-attainer group can be effectively geared toward their needs, the disadvantage will be reduced to insignificance.

Thirdly, we argue that ability grouping improves the effectiveness and efficiency of teachers. The key to the quality of learning in the classroom is the teacher. From an administrative point of view, it is much harder for teachers to tailor the pace and content of instruction to students' needs in a heterogeneous ability grouping class than in a homogeneous one. Some educators may assume that teachers can provide differential instruction for students of most backgrounds, but this is by no means an easy feat. Any learning for a sequence of lessons comprises three inter-related components: (a) the learning objective; (b) the learning activities; and (c) the teacher's prediction of how students construct their understanding (Simon, 1995). High-quality instruction designed to meet students' needs should take these components into account. In an ideal situation, teachers must plan various and changing hypothetical learning trajectories suitable for their students' different zones of proximal development. However research has shown that teachers cannot really provide a learning environment suitable for every student in the mixed-ability class, particularly for low attainers (Gal, Lin, & Ying, 2009). Teachers focus on high-stakes testing and so they tend to teach the high-ability or above-average-ability students in class. As a result, low-ability students are ignored and keep silent in the mixed-ability class.

The variation of students' hypothetical learning trajectories is larger in the heterogeneous ability-grouping class than in homogeneous ability-grouping class. Teaching in the homogeneous ability-grouping class is therefore more challenging than teaching in the heterogeneous ability-grouping class. When educational reform aims at weakening between-school tracking of senior high schools, mathematics teachers turn to the between-class grouping environment. Because of the high-stakes testing, the same curriculum standard must be reached in all the different groups. The main difference between them lies in the sequence of the learning tasks and the distribution of task difficulty. This variation makes it easier for teachers to provide suitable learning trajectories for students of different academic levels.

Our fourth and final argument is that ability grouping can reduce the need for remedial courses. From an economic point of view, students are more likely to be provided with appropriate learning trajectories in homogeneous than heterogeneous ability grouping classes. Time, effort and money spent on additional remedial classes can be saved, and diverted instead toward assisting teachers in planning appropriate hypothetical learning trajectories and providing students with suitable learning activities. It has been acknowledged that utilization of opportunities for learning provided by the teacher is related to students' ability (Sorenson & Hallinan, 1986).

In international studies on students' performance in mathematics, Taiwan had above-average levels as well as standard deviations of student performance in mathematics. However, around 20% of elementary students in Taiwan require remedial instruction (Taiwan Ministry of Education, 2006a). The Taiwan Government launched the Hand-in-Hand After-School Care Program and allocated substantial funding to provide underachieving students with after-school remedial courses (Taiwan Ministry of Education, 2006b). Despite their achievements, however, Taiwanese students showed much lower attitude scores toward mathematics in comparison to international average (OECD, 2014; Mullis, Martin, Foy, & Arora , 2012). If students can be provided with appropriate instruction, their attitude toward mathematics will improve.

FINAL WORDS

In conclusion, teaching mathematics without ability grouping does not necessarily mean educational equality, and ability grouping can realize educational equality if different groups of students are provided with equal educational resources. Next, ability grouping is beneficial to both cognition and affect for the learning of mathematics. No direct evidence has been found to refute the benefits of ability grouping in this respect. Moreover, introducing high standards and accountability into lower-attaining groups could increase low-attainers' effectiveness (Gamoran, 2009). Third, heterogeneous ability-grouping settings require greater amounts of resources and support to aid teachers in their efforts to teach students of different abilities at once (Hallinan, 1994). Junior high school teachers have been shown to ignore lower-attainers in the mixed-ability class, and senior high school teachers feel frustrated in mixed-ability classes and attempt to generate within-class ability grouping (Gal, Lin, & Ying, 2009; Tsai, & Chen, 2013). Lastly, students in mixed-ability classes are more likely to require extra mathematics classes. In mathematics, it is important that students understand certain concepts before moving on to more complex concepts. For students who need a slower pace of instruction, ability grouping can be used to provide teachers with fewer students and more focused instruction.

Based on these four arguments, we suggest that rejecting ability grouping would be highly unfavorable. Instead, we must evaluate the equity and effectiveness of ability grouping in an effort to determine how it can be improved. We hope you will agree that the case for ability grouping is compelling, and that it should be adopted in all schools.

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IS GROUPING REALLY ESSENTIAL?

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In this debate, we are arguing against the proposition that: "Grouping students by attainment is essential for their learning of mathematics." We begin our argument by observing that the proposition as phrased is a very strong statement that contradicts all historical and everyday evidence. We then argue against a weaker interpretation of the proposition. First we provide theoretical arguments to refute the proposition and then we provide empirical evidence to show that grouping students by attainment is not essential, or even useful, for students' learning of mathematics.

REFUTING THE STRONG INTERPRETATION

The word "essential" means "absolutely necessary". If grouping students by attainment is essential for their learning of mathematics, then it should not be possible to learn mathematics in contexts where students are not grouped by attainment. However, a moment of reflection will reveal that many people have learned mathematics in many different contexts, at least some of which did not involve grouping students by attainment. Were the children in 19th century one-room schools, comprising all the children from the community in a single room with a single teacher, grouped by attainment? Of course not! And yet they learned mathematics. If we take the word "essential" seriously, then the proposition is patently false.

In addition to the inappropriate use of the strong word "essential," the statement implies that it is beneficial for all students at all grade levels. Intuitively, is it really important to group students starting from Kindergarten? Is it really beneficial for all groups of students within the same grade levels? In a review, Slavin (1990) concluded that grouping by attainment is detrimental to students in the low and middle achieving groups that consist of largely low income and minority students. The low-achieving

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students are often given the least qualified teachers and high-achieving students receive the best teachers.

REFUTING THE WEAK INTERPRETATION

In the interests of prolonging the debate, we are willing to consider instead the weaker proposition, grouping students by attainment is helpful for their learning of mathematics. However, even given this weaker interpretation, we argue that the proposition is still false.

It should be indicated that in this article, we recognize two approaches to grouping: (1) having different schools for students of different attainment, and (2) grouping students within schools into 'sets' or 'streams'. In our discussion and debating, we do not particularly distinguish one grouping approach from the other.

THEORETICAL ARGUMENTS REFUTING THE PROPOSITION

Learning Theories. We first draw on the constructivist and sociocultural perspectives of mathematics learning to argue again the proposition (Cobb, 1994). According to the constructivist perspective of mathematics learning, students actively participate in the process of knowledge construction, therefore, making sense of mathematics in their own terms. In other words, students become active participants in the creation of knowledge rather than passive receivers of rules and procedures. As students solve problems, they can use any approach they can think of, draw on any piece of knowledge they have learned, and justify their ideas in ways they feel are convincing. In fact, the students' own exploration of the problem is an essential component in their learning. According to the sociocultural perspective of mathematics learning, mathematics is learned through social interactions, meaning negotiation, and reaching shared understanding. Such activities help students clarify their ideas and acquire different perspectives of the concept or idea they are learning.

We agree with Cobb's view that "mathematical learning should be viewed as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society" (Cobb, 1994, p. 13). In classroom, we not only need to value students' personal interests, but also connect them to ideas and traditions growing out of centuries of mathematical exploration and invention (Ball, 1993). In grouping students by attainment, students are grouped according to their perceived ability or achievement levels. Students are placed in high, middle, or low groups in an effort to provide them with a level of curriculum and instruction that is appropriate to their attainment levels. Because of the relatively uniformed groups, the students in the group are less likely to provide diverse solutions strategies to mathematical problems, as well as the thinking processes. Therefore, students lack significant opportunities to interact with each other, challenge each other, and take advantage of the gift of diversity.

All too often, students view problems as only being able to be solved in one "right" way. With heterogeneous group of students, each student can contribute to various solution strategies, so that students can compare and contrast these solution strategies and then increase their flexibility and confidence in problem solving (Cai, 2003). For example, students can use multiple ways to solve the following problem:

> Given the two job offers below, determine the better-paying summer job. Justify your answer.
>
> Offer 1: At Timmy's Tacos you will earn \$4.50 an hour. However, you will be required to purchase a uniform for \$45.00. You will be expected to work 20 hours each week.
>
> Offer 2: At Kelly's Car Wash you will earn \$3.50 an hour. No special attire is required. You must agree to work 20 hours each week.
>
> SOLUTION 1: In a 20 hr. week, Offer 1 will pay \$4.50 × 20 = \$90.00. Offer 2 will pay \$3.50 × 20 = \$70.00. Since the difference is \$20 per week and the uniform for Offer 1 costs \$45.00, it will take (\$45.00/ \$20/week =) 2.25 weeks to pay for the uniform and break even. If you keep the job for three weeks or more, you should take offer 1.
>
> SOLUTION 2: At Timmy's you make \$1.00 more for each hour of work. After 45 hours of work, you'd make \$45 more at Timmy's than Kelly's. This extra money would pay for the uniform. From that point on, you'd make \$1 more an hour at Timmy's than Kelly's.
>
> SOLUTION 3: Let x be the number of weeks you intend to work. The total amount for Offer 1 = $90x - 45$ and the total amount for Offer 2 = $70x$. If $90x - 45 = 70x$, then $x = 2.25$. So if you work less than 3 weeks, you should take Offer 2, otherwise take Offer 1.
>
> SOLUTION 4: Let x be the number of weeks you intend to work, y_1 be the total amount for Offer 1 after working x weeks, and y_2 be the total amount for Offer 2 after working x weeks. Therefore, $y_1 = 90x - 45$ and $y_2 = 70x$. Using a graphing calculator to graph them, you will see they intersect at (2.25, 157.5). From the graph, you will see that if you have the job for three weeks or more, you take Offer 1.
>
> SOLUTION 5: Construct a table to show the amount of income for Offers 1 and 2 for one week, two weeks, three weeks, etc., . . . and then compare the information from the table to determine which offer you will take.
>
> SOLUTION 6: Let x be the number of weeks you intend to work. The total amount for Offer 1 = $4.5 \times 20x - 45$ and the total amount for Offer 2 = $3.5 \times 20x$. If $(90x - 45) < 70x$, then $x < 2.25$. So if you work less than 3 weeks, you should take Offer 2, otherwise take Offer 1.

Students can focus on one aspect of the relationship to solve this problem. Collectively, students generated a number of solutions. The teacher asked students to present and discuss each of their solutions to the whole class. The solutions (above) all highlight how the total amount of earnings for each offer is related to the payment for each hour and the expense required for taking the offer. However, the total amount of earnings for each offer is represented differently in these solutions. A discussion of these solutions can quickly reveal the advantages of asking students to explore alternative ways of solving the problem. Solutions 1 and 2 are based on the comparisons of the amount of money paid for both offers in a certain period of time. Solution 1 used the fact that Offer 1 pays \$20 per week more than Offer 2, but Solution 2 used the fact that

Offer 1 pays $1 per hour more than Offer 2. Then students calculate the time needed to pay for the uniform and find the break-even point, and decide which is the better offer according to the length of time one may be able to work. In Solutions 3, 4, and 5, students constructed equations, graphs and a table, respectively, to find the break-even point. By comparing and reflecting on these solutions, teachers helped students build new mathematical knowledge through problem solving. In particular, students would learn how the rate of payment and initial cost was reflected on the linear relation in each offer. Students would also better understand the concepts of slope (rate of change) and intercept (initial state) in a linear relation. In Solution 6, students set up an inequality to decide when Offer 2 would begin to earn more than Offer 1.

Fixed Mindset Theory. Dweck (2006) has conducted a series of studies to explore the nature and impact of mindset. She proposed that some students have fixed mindset beliefs and the other have growth mindset beliefs. Those with fixed mindsets would believe that they are smart or are not smart. These students often choose less challenging tasks to aim for success and easily give up on challenging tasks (Boaler, 2013). On the other hand, students with growth mindsets believe that thinking skills and "smartness" can grow with effort and hard work. These students are much more likely to choose challenging tasks and display resilience in the face of failure.

Grouping by attainment artificially differentiates smart students from not so smart students. Such grouping sends messages to low- and middle-achieving groups of students that they are not so smart. Consequently, fixed mindsets have negative impacts on the subsequent learning of students in low- and middle-achieving groups. Grouping by attainment can also create fixed mindsets for administrators. Instructional methods tend to be more engaging, reflective, and challenging for high-achieving students, whereas for low achieving students the emphasis is on good behaviour and menial skills (Slavin, 1990). Once resource allocation and class sizes are similar for schools with or without grouping, there is little or no differential effect of grouping for high-achieving, average, or low-achieving students (Bretts & Shkolnik, 2000).

EMPIRICAL EVIDENCE REFUTING THE PROPOSITION

In this section, we present two aspects of evidence to refute the proposition. The first aspect of evidence is from studies directly examining the effect of grouping on students' learning. The second aspect of the evidence comes from international studies in mathematics.

Evidence from research on grouping. Several researchers have conducted reviews of research to investigate the effect of grouping on students' learning (Alexander, 2010; Boaler, 2013; Kulik & Kulik, 1989; Slavin, 1987, 1990). Using the *best evidence syntheses*, Slavin conducted comprehensive reviews of research on ability grouping in elementary schools (1987) and in secondary schools (1990). In general, Slavin found that ability grouping has no effects, that is, grouped and ungrouped schools produce about the same level of achievement, and neither high, nor low, nor average groups obtain any special benefit or suffer a particular loss due to grouping. According to

Slavin (1990), effects that appeared in some studies resulted from random or systematic errors of measurement.

Linchevski and Kutscher (1998) reported three studies in Israel. In the first two studies, the impact of grouping on student achievement was investigated. Their third study investigated the effect of participating in workshops on teachers' attitudes towards teaching in mixed ability classrooms. They found that students of low-achieving and average-achieving students performed at higher levels when taught in mixed ability classes, and high-achieving students performed at the same level as those taught in the same ability classes. In addition, Linchevski and Kutscher (1998) found that teachers with more seniority demonstrated more positive attitudes toward student learning in a heterogeneous class than those of new teachers. Their participation in the project had a positive effect on attitudes towards teaching in mixed ability classrooms.

The UK is one of the few countries that has used ability grouping in schools. In 2006, the Cambridge Primary Review was launched as a fully independent review into the condition and future of schooling. The review included the impact of grouping and found that "there are no consistent effects of structured ability grouping, such as setting, on attainment, although there can be detrimental effects on social and personal outcomes for some children" (Alexander, 2010, p. 290).

Boaler (2013) takes the position that grouping by attainment is not desirable. She situated her discussion in the theory of mindset and on her own research. Drawn on scientific research evidence, Boaler pointed out that the brain has incredible potential to grow and change. Research clearly provides evidence that the growth mindset views have powerful impacts on students' attainment. Ability grouping is the kind of practice based on notions of a fixed mindset and ability levels which limit students' attainment and increase inequality. In fact, "[s]tudents are well aware of ability grouping practices at whatever age they happen and they take a very clear message from such practices — some students are clever and some are not" (p. 147). She has also cited her own research and reviewed research from others to support her position.

For example, one of the studies Boaler (2013) reviewed is that conducted by Burris, Heubert, and Levin (2006), who used a quasi-experimental design using multiple student learning outcome measures, including standardized tests and advanced course taking. Burris et al. (2006) found that initial high achievers' performance is not hurt if (a) curriculum is held constant and (b) the heterogeneity of initial achievement levels in the class expands. They also found that detracking would be effective if all students receive the high-track curriculum. They also found that the students from de-tracked classes took more advanced classes, pass rates were significantly higher and students passed exams a year earlier than the average in New York State.

Evidence from international studies in mathematics. In this section we will consider separately the two approaches to grouping: having different schools for students of different attainment, and grouping students within schools into 'sets' or 'streams'. Our primary source is the 2012 PISA assessment, as reported in OECD (2013).

What does PISA say about grouping students within schools? "Systems that group students, within schools, for all classes based on their ability tend to have lower performance across all participating countries and economies, after accounting for per capita GDP" (OECD, 2013, p. 36). Within the top scoring OECD countries, there are some, Korea, the Netherlands, Canada and Australia, where almost all mathematics classes are streamed. But there are also others, Japan, Finland, Poland, Germany and Austria, where streaming does not occur in a third or more of classes (see Table 1).

In addition to poorer performance for everyone, within school grouping also increases the gap related to social background:

> The impact of the socio-economic status of students and/or schools on performance is stronger in school systems that sort students into different tracks, where students are grouped into different tracks at an early age, where more students attend vocational programmes, where more students attend academically selective schools, or where more students attend schools that transfer low-performing students or students with behaviour problems to another school. (OECE, 2013, p. 36)

How about having different schools for students of different attainment? This seems not to result in statistically significantly poorer performance, but among the top scoring OECD countries, only about half have different schools based on attainment. And grouping students in different schools does perpetuate social inequities. "Between-school horizontal stratification is negatively related to equity in education opportunities" (OECD, 2013, p. 36). In addition "Students in comprehensive school systems – those that do not separate students into different schools according to their performance, such as the systems in Australia, Canada, Iceland, New Zealand, the United Kingdom and the United States – tend to regard learning mathematics as important for their later life, regardless of the system's overall performance." (p. 72)

	PISA 2012 Math score	Percentage of math classes *without* ability grouping	Number of school types for 15-year-olds	First age of selection in the education system	Percentage starting at modal age	Percentage of students who have repeated a grade	Percentage of 15-year-olds at modal grade
Korea	553.8	9.9	3	14	55.5	3.6	93.8
Japan	**536.4**	**36.9**	**2**	**15**	**100.0**	**0.0**	**100.0**
Switzerland	530.9	15.0	4	16	44.2	19.9	60.6
Netherlands	**523.0**	**6.4**	**7**	**12**	**71.6**	**27.6**	**49.2**
Estonia	520.5	10.9	1	15	80.3	3.5	75.4
Finland	**518.8**	**35.5**	**1**	**16**	**69.9**	**3.8**	**85.0**
Canada	518.1	7.1	1	16	49.9	8.0	84.6
Poland	**517.5**	**42.4**	**1**	**14**	**99.5**	**4.2**	**94.9**
Belgium	514.5	20.6	4	12	70.3	36.1	59.5
Germany	**513.5**	**31.9**	**4**	**10**	**70.1**	**20.3**	**51.9**
Austria	505.5	71.9	4	10	73.6	11.9	51.0
Australia	**504.2**	**1.6**	**1**	**16**	**58.4**	**7.5**	**70.0**

TABLE 1: Statistics relevant to grouping and PISA results for selected high scoring OECD countries (Statistics highlighted in yellow support the contention that grouping helps learning; those in green support the contention that absence of grouping helps learning)

Another finding from PISA is that if you are going to group students by attainment, you should at least wait as long as possible to do so. The age at which stratification begins is closely associated with the impact of socio-economic status on performance. This may be because in systems that stratify students early, students might be selected more than once before the age of 15. But also, when students are older, more information on individual students is available, and decisions on selecting and sorting students into certain tracks are thus better informed. So the sorting by attainment is more based on school performance. When the sorting is done at a young age, parents with more advantaged socio-economic status may be in a better position to promote their children's chances than disadvantaged parents.

Our opponents might now wish to claim they didn't mean this sort of 'horizontal' grouping at all. They meant something that is fundamental to almost all school systems, something that they believe we cannot argue against, grouping into grades. Surely it makes sense to put kids into grades, and to have the upper grades for those with higher attainment, that is those who have passed the lower grades.This suggests that schools with more 'vertical stratification' (more kids of different ages in each grade) might do better. Instead of being grouped by age, they are grouped by attainment, with those not ready to start school waiting, and those having difficulty repeating a grade. What does PISA say?

Where is the most variation in the age at which children start schooling? In other words, in what places do their parents try to have children go to school when they are ready, rather than the decision being solely based on age? Among the OECD countries, it is in Ireland and Canada. In Japan and Poland, on the other hand, all students start primary school within a two-year window, the least amount of variation. All four of these extreme cases are above the OECD average.

The start of schooling might be too early to accurately judge 'attainment'. But during schooling, some systems make adjustments, by having students who do poorly repeat a grade, another way of grouping by attainment rather than age. Which countries do this?

> In Japan, Malaysia and Norway, no 15-year-old student reported to have repeated a grade, … In contrast, between 20% and 29% of students in France, the Netherlands, Peru, Chile and Germany had repeated a grade at least once; between 30% and 39% of students in Tunisia, Uruguay, Argentina, Belgium, Brazil, Luxembourg, Portugal, Costa Rica and Spain had repeated a grade at least once; and in Macao-China and Colombia over 40% of students had repeated a grade at least once. (OECD, 2013, p. 73)

Macao-China and Japan have radically different levels of grouping by attainment in this way, but both are high scorers on PISA. And most of the countries that group by attainment in this way are among the low scorers. Among high scorers, Korea and Finland also have less than 4% of their students repeating a grade.

In addition, the OECD notes that this way of grouping by attainment perpetuates social inequality: "grade repetition is negatively related to equity in education: systems where more students repeat a grade tend to show a stronger impact of students' socio-economic status on their performance." (p. 73)

What if we don't mean grouping students by attainment is essential for the mathematics learning of all children, but only for some? Surely the ones who go to the best schools do better, even if their peers in the other school do not. True, "schools that select students for admittance based on students' academic performance tend to show better school average performance, even after accounting for the socio-economic status and demographic background of students and schools and various other school characteristics, on average across OECD countries" (OECD, 2013, p. 36). However, this is only good for those schools, not for the school system as a whole.

> A school system's performance overall is not better if it has a greater proportion of academically selective schools. In fact, in systems with more academically selective schools, the impact of the socio-economic status of students and schools on student performance is stronger. (OECD, 2013, p. 38).

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RESEARCH FORUM

RF01 *Interweaving mathematics education and cognitive neuroscience*

Coordinators: Ron Tzur & Roza Leikin

RESEARCH FORUM

INTERWEAVING MATHEMATICS EDUCATION AND COGNITIVE NEUROSCIENCE

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This Research Forum (RF) will examine recent, interdisciplinary efforts to interweave maths education and cognitive neuroscience into empirical studies that can (a) inform both disciplines of neural bases that underlie maths thinking and learning, (b) enrich research in maths education with insights gained through cognitive neuroscience, and (c) lay foundations for cross-discipline research agenda. Empirical studies conducted collaboratively by maths educators and cognitive neuroscientists have just recently begun. Thus, this RF will examine a range of research questions, conceptual frameworks, methodologies, and findings that provide participants, already active or interested in doing so, with various lenses through which to conceptualize affordances and constraints of "CogNeuroMathEd" research.

INTRODUCTION

Ron Tzur, Roza Leikin

In recent years, exciting, new research directions have been emerging, in which mathematics educators have actively engaged in collaborative, interdisciplinary efforts to study with cognitive neuroscientists the neural bases of mathematical thinking and learning (De Smedt & Verschaffel, 2010). Underlying these efforts, which started quite independently in different parts of the world, has been mathematics educators' attentiveness to a rapidly growing body of cognitive neuroscience research that focused on mathematically related phenomena (Campbell, 2006; Campbell & Leikin, 2012). While fascinating, until about 5 years ago work in cognitive neuroscience involved only a handful of mathematics educators (Leikin, Waisman, & Leikin, 2013).

Consequently, by-and-large the research problems and mathematical issues addressed, the conceptual frameworks used (if any), and the implications of cognitive neuroscience studies seemed quite foreign or negligible to the mathematics education community.

As mathematics educators, we were intrigued by the immense potential that cognitive neuroscience research may have for our field. To realize this potential, however, required we engage in the intellectually demanding, painstaking process of learning about a variety of cognitive neuroscience research designs and instrumentations, such as EEG, ERP, fMRI, eye tracking, and fNIRS (Baars & Ramsøy, 2007). We also had to relate affordances and constraints of certain methodologies to properties of neural activities that can be studied. For example, for the analysis of cognitive processes fMRI provides fine spatial resolution whereas event-related potentials (ERPs) measured by EEG provide fine temporal resolution.

Not surprisingly, initial cross-disciplinary efforts seemed to be unbalanced in terms of the contribution made by the collaborators, with maths educators assuming the subordinate role (De Smedt & Verschaffel, 2010). Gradually, however, studies have been designed and conducted collaboratively with maths educators serving a leading role. This role pertained to the framing of research questions, including selection of higher maths contents and ability levels, such as fractions, algebra, geometry (Leikin, Waisman, Shaul, & Leikin, 2014; Obersteiner et al., 2014; Tzur & Depue, 2014b; Waisman, Leikin, Shaul, & Leikin, 2014) than the typically rudimentary contents found in cognitive neuroscience research (e.g., early number knowledge, additive reasoning). This role also pertained to the explicit use of conceptual frameworks postulated in maths education to account for cognitive mechanisms/stages involved in thinking and learning (Tzur, 2011). Equally important, maths educators assumed a leading role in the collaborative, meticulous research design process, expanding cognitive neuroscience research to more populations of interest (e.g., gifted students, teachers) and devising tasks for testing conceptually significant hypotheses. The latter involved, for example, decomposing tasks into components that, to a maths educator, would involve distinct aspects of reasoning and thus also differentiated neural processes (Leikin et al., 2013; Tzur & Depue, 2014a). Others re-examined issues of early mathematical understandings to provide new insights into number processing in the brain, as well as into mathematical strategies used by university students based on tracking their eye movements (Beitlich et al., 2014; Obersteiner et al., 2014). Some of these studies will be presented and discussed at the RF.

The underlying premise of the proposed RF is that the aforementioned work is but a rudimentary step, a “drop-in-the-bucket” of interdisciplinary research that seems to hold a great promise for both fields. For maths educators, this promise can be summarized as an opportunity to delve into what, to date, could at best be considered as a “black box” that makes mathematical experiences possible. Accordingly, our main goal is to engender an open, inviting dialogue within the PME community, for further sharing insights gained, challenges faced, and excitements experienced—as well as to raise issues that can lay foundations for future research agenda. The next section presents the key questions to be addressed in this Research Forum.

Key Questions Addressed in RF

- What is the relevance of cognitive neuroscience studies to maths education? Why should maths educators care about neural bases of mathematical thinking and learning?
- How may maths educators contribute to, and guide, framing of research problems, questions, and foci for interdisciplinary, CogNeuroMathEd studies so issues of importance to maths education are addressed?
- How do different methodologies currently used in cognitive neuroscience afford, and constrain, research design and potential findings/implications for maths education?
- What other essential issues/questions do math educators have that should inform the emerging interdisciplinary research agenda?

EYE TRACKING AS A METHOD FOR IDENTIFYING MATHEMATICAL STRATEGIES

Jana T. Beitlich, Andreas Obersteiner

Analysing strategy use in mathematical tasks is a methodological challenge. Recently, eye tracking has been used successfully in an increasing number of studies. The purpose of this article is to initiate a discussion on the use of eye tracking as a method for identifying strategy use in mathematical tasks, and to provide an overview of selected studies in the mathematics education literature that used eye tracking to assess strategy use in a variety of mathematical tasks. Finally, we discuss limitations and advantages of using eye tracking and provide suggestions for further research.

THE METHOD OF EYE TRACKING IN MATHEMATICAL TASKS

Identifying strategy use on mathematical tasks is an important issue in mathematics education research that can be addressed by a variety of methods. One approach is to ask participants about their strategies retrospectively or to let them think aloud while working on a given task. A drawback of this method is that it is not very objective and reliable, and verbalizing their strategies can be very challenging for participants, especially for children (e.g., Ericsson & Simon, 1980). Another approach involves analysing response times in a computerized experiment. However, although patterns of response times have been very informative in some cases, such as using external representations (Obersteiner, Reiss, Ufer, Luwel, & Verschaffel, 2014), it is certainly possible that different strategies require the same amount of time, which limits the distinctions that can be made based on response times alone. Recently, attempts have been made to apply neuroscience in educational contexts, and brain imaging methods have been used to measure brain activation patterns during mathematical problem solving. Although these methods have successfully been used to identify differences in cognitive strategies that were not detectable on the behavioural level (e.g., Sohn et al., 2004), these methods are quite invasive and restrictive, which limits their applicability in educational contexts (see Obersteiner et al., 2010).

A method that is more objective than verbal reports, more informative than response time measures, and less invasive than brain imaging, is eye tracking. Eye movements have been analysed for a long time and initially emerged from the field of reading. In the late 1800s, the method of eye tracking was difficult to realize and not very comfortable for the participants (for an early description see e.g., Huey, 1898). Some years later, Dodge and Cline (1901) described a more comfortable method of recording eye movements using the reflection of the cornea. In principle, this method is used until today, albeit with substantial improvements that are continuously being made (see Holmqvist et al., 2011).

Nowadays, two types of eye trackers are commonly used for eye tracking research. One type is the *static eye tracker*. Here, an infrared illumination and an eye video camera are installed in front of the participant. Common models are tower-mounted eye trackers (the participant's head lays, for example, on a chin rest and is therefore fixed) and remote eye trackers (the participant can move to a certain degree as they sit freely in front of the eye tracking device). The second type is a *head-mounted eye tracker*. Here, the illumination and the camera are installed on the participant's head (e.g., commonly used eye-tracking glasses). Which type of eye-tracker is best for a study depends on the research questions to be addressed.

There are two assumptions underlying the idea of analysing eye movements. The first, *immediacy assumption* states that processing of information takes place immediately. The second, *eye-mind assumption* states that people are mainly processing the information at which they are looking (Just & Carpenter, 1980). Although this strong version of the eye-mind assumption may not hold true in general and thus should be assumed in a weaker form (e.g., Underwood & Everatt, 1992), it seems reasonable to use the assumption as an underlying paradigm for analysing eye movements, in particular when participants have to give immediate responses to visually presented stimuli.

STUDIES USING EYE TRACKING FOR IDENTIFYING STRATEGIES

An increasing number of studies used eye tracking for identifying mathematical strategies. In the following, we describe some of these studies, using different types of mathematical tasks as examples.

Schneider et al. (2008) examined how eye movements of children in grades 1 to 3 reflect their use of the *number line*. For this purpose, the children saw on a computer screen a number line from 0 to 100 without any marking other than the starting point and the endpoint. They had to find the correct position of a given number on the number line. After 4 s, a marker was shown on the screen and the children had to decide if this marker is put at the correct place for the respective number. As a measure for their analysis the authors used the percentage of correct fixations (with an error margin of 10%). The findings concerning the children's use of strategies replicated the patterns found in previous studies without eye tracking in the way that even these young children used the midpoint strategy and used frequently the counting-up strategy. In a study by Sullivan et al. (2011), adults were asked to position numbers on a *number line*. The participants had to mouse-click on a computer screen on the position of an aurally presented number on a number line with the markers 0 and 1000. To analyse strategy use, the authors used total fixation duration and total number of fixations. The participants were able to translate relatively quickly between the numerical information and the spatial position of the numbers. The authors also drew the conclusion that the estimation process is dynamic. Furthermore, the eye tracking data showed that the estimation is biased by a proportional-reasoning strategy to determine the correct positions of the numbers on the number line.

In a study to investigate *calculation* strategies by Green, Lemaire, and Dufau (2007), younger and older adults had to add two three-digit numbers presented on a computer screen. Before, they were told how to use different calculation strategies. The tasks were presented in a choice or no-choice condition, in which the participants were allowed to choose their own strategy or had to use a specific strategy, respectively. Besides other measures, the authors analysed fixation durations and number of fixations. The eye movement data validated the use of the strategies that the participants were asked to use (no-choice condition) or reported to have used (choice condition). Furthermore, the eye movements implied that the younger adults distinguished more successfully between the strategies than the older adults.

Obersteiner et al. (2014) investigated strategies for *comparing the numerical values of fractions* in a small sample of mathematically experienced adults. The participants had to choose the larger of two fractions that were simultaneously presented on a computer screen. The authors analysed the fixation times on the fraction numerators and denominators. In line with previous analyses of reaction times, they found systematic differences depending on the types of fraction pairs: When the fraction pairs had common denominators, the participants focused more on the numerators than on the denominators, and vice versa when the fraction pairs had common numerators, suggesting that for these items with common components, the participants used a componential strategy with little reliance on the fraction magnitudes. When the fraction pairs did not have common components, there were no systematic differences in fixation times between numerators and denominators, and the fixations alternated more strongly between the numerator and denominator of each fraction, suggesting that for these types of fraction pairs, the participants used holistic comparison strategies to a larger extent than componential strategies.

Huber, Moeller, and Nuerk (2014) also studied *fraction comparison* strategies in a larger sample of adults. As a measure they used the number of fixations on the fraction numerators and denominators. In addition to the influence of problem types (common components, no common components), the authors were interested in the influence of the experimental design. They found that the participants used componential comparison strategies for fraction pairs with common components to a larger extent when these items were presented in a block separated from fraction pairs without common components, allowing the participants to have clear expectations about the items within each block. When all items were presented in randomized order within the same block, the indicators of componential strategies were less pronounced.

Inglis and Alcock (2012) asked first-year undergraduate students and academic mathematicians to *evaluate mathematical proofs* on a computer screen. The authors analysed dwell times (i.e., total looking times), mean fixation durations, the time at which participants first fixated on each line of each proof, and saccades. The results revealed that the students spent proportionately more time on the formulas (compared with the non-formula parts of the proof) than did the mathematicians. Furthermore, the mathematicians shifted their attention back and forth between the lines of the proof more often than the students, suggesting that the mathematicians spent more effort on searching for between-line warrants than the students. In a study by Beitlich et al. (2014), the authors wanted to know whether and how adults with high expertise in mathematics looked at a picture presented together with a *mathematical proof*, when the participants were asked to read the proof to comprehend it. By analysing fixation times and fixation times per pixel, they found that the participants mostly spent more time on the text parts of the proofs than on the picture. Furthermore, the order of the fixations was taken into account to show that the participants alternated between the text and the picture during reading the proofs.

DISCUSSION

As the examples discussed above have shown, eye tracking is a method that can be used successfully to identify mathematical strategies in different types of mathematical tasks while overcoming drawbacks of other behavioural measures or neuroscience measures described in the first part of the article. Eye tracking also has certain constraints. For example, there can be measurement errors, or some of the detected fixations can be completely irrelevant to the specific task, producing noise in the collected data. More fundamentally, the underlying paradigms, namely the immediacy and eye-mind assumptions, need to be reconsidered in a specific experimental setting. Furthermore, though the method can make visible where participants look at and how their attention is distributed, it cannot explain, *why* people focus on specific pieces of information and *how successful* they are in processing this information. To take advantage of the benefits of different methods for identifying mathematical strategies and to overcome their disadvantages at the same time, it seems reasonable to use a combination of these methods. For example, eye tracking in combination with verbal reports could be reasonable to validate the data, and combining eye tracking with brain imaging methods could allow important information on the specific brain circuitry activated when processing specific pieces of information. This way, it should be possible to get further insights into the strategies used when solving mathematical tasks.

UNDERSTANDING NUMBER PROCESSING: THE NEUROSCIENCE PERSPECTIVE IN MATHEMATICS EDUCATION

Andreas Obersteiner, Kristina Reiss

Brain imaging techniques have enabled researchers to generate pictures of the brain at work. Is such kind of research relevant to mathematics education? We discuss this question, using the example of number processing. While a large number of studies have identified a neural network that seems to be responsible for processing very small whole numbers, research on working with more complex numbers, such as fractions, and on performing more complex arithmetic, is very rare. We conclude that combining various research methods and promoting closer collaborations between researchers from neuroscience and mathematics education could increase the relevance of neuroscience studies for mathematics education.

NUMBER PROCESSING

Proficiency in working with numbers is one of the most fundamental mathematical skills. The question of how children can acquire basic numerical skills has always been at the core of mathematics education. At the same time, understanding how numbers are processed mentally has been an issue of cognitive psychology. In the last two decades, this question has further been addressed by neuroscience research. The advancement of brain imaging techniques has enabled researchers to observe brain activities during a variety of cognitive tasks including the mental processing of numbers. The time thus seems ripe to discuss whether these studies can contribute to our better understanding of how children can gain knowledge of numbers and arithmetic. There have been controversies about the more general question of whether neuroscience studies have any potential to inform education (De Smedt & Verschaffel, 2010; Schumacher, 2007; Sigman, Peña, Goldin, & Ribeiro, 2014). The purpose of this paper is to contribute to this discussion using the example of number processing. We first provide a brief overview of brain imaging methods that have been used in most neuroscience studies on number processing, and then review selected studies that investigated brain activities during number processing. Finally, we discuss the relevance of these studies from a mathematics education perspective.

BRAIN IMAGING METHODS

Brain imaging studies have often used *functional magnetic resonance imaging (fMRI)*. When a brain area is activated, blood flow in that area increases and so does the concentration of oxygen-rich blood relative to oxygen-poor blood. fMRI relies on the fact that the magnetic properties of blood depend on its oxygenation. This allows distinguishing brain areas that are more or less activated. A limitation for practical use of this method is that participants have to lie in a scanner without body movements. For that reason, mathematical tasks participants have to solve during the fMRI session require giving responses by key presses.

Another frequently used method is *electroencephalography (EEG)*. This method measures the electrical activity of brain cells by electrodes that are placed on the scalp. Compared to fMRI, EEG is less restrictive, as participants can sit in an upright position during an EEG session (see picture in Waisman, Leikin, & Leikin's article below). Another advantage of EEG is its high temporal resolution, which allows detecting changes in electrical activity in a range of milliseconds.

A more recent brain imaging is *near-infrared spectroscopy (NIRS)*. This method relies on the fact that the blood absorbs infrared light depending on its oxygenation. Electrodes that emit and detect near-infrared light are placed on the scalp of participants. The amount of light that is reflected allows conclusions concerning the concentration of blood in a specific brain area.

THE NEURAL CORRELATES OF NUMBER PROCESSING

This section provides a review of brain imaging studies that addressed questions of how numbers are processed in the brain. While the large majority of studies have focused on processing of whole numbers, more recent studies have included processing of fractions.

Processing of whole numbers

Research of the last twenty years has shown that there is no single brain area for mathematics or numbers. Rather, a network of cortical, parietal and frontal brain areas seems to be involved in number processing. Within this network, distinct brain areas are responsible for specific aspects of number processing (Dehaene, Piazza, Pinel, & Cohen, 2003). The intraparietal sulcus (IPS) is thought to play a key role, because it is activated during representation of numerical magnitudes. When individuals compare the numerical values of two numbers, this brain region has been found to be particularly sensitive to the numerical distances between these numbers (Ansari, Garcia, Lucas, Hamon, & Dhital, 2005; Pinel, Dehaene, Rivière, & Le Bihan, 2001). The IPS is also the key region for estimation of numerical quantities and approximation of calculation results (Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004). Another important area is the left gyrus angularis, which is connected to language areas. It is important for processing automatized number facts that are stored in verbal formats. A study by Grabner, Ansari, Koschutnig, Reishofer, Ebner, and Neuper (2009) found that the left gyrus angularis was particularly activated when the participants solved arithmetic tasks by fact retrieval rather than other calculation strategies.

While these and many other studies contributed enormously to our understanding of very basic processing of numbers, their foci might limit the potential to inform mathematics education. Few efforts have been made to address questions that are more relevant to teaching and learning of mathematics. For example, Dresler et al. (2009) studied fourth- and eighth-grade students who solved two-digit addition problems that were presented in numerical format or in the form of simple word problems. In order to increase the validity of their results, these authors used NIRS and studied the children in their familiar environment within the school building. While they could confirm that parietal brain activation was higher when the students had to calculate rather than just read the mathematical problems, they found that brain activation patterns were similar in both presentation formats and in both age groups.

Processing of fractions

Little research has been carried out to detect the neural basis of non-natural numbers such as fractions. In an initial study, Schmithorst and Brown (2004) found that the brain areas that are active during processing of fractions are similar to those active during whole number processing (however, see Tzur's article below for different findings). Ischebeck, Schocke, and Delazer (2009) addressed the question of whether fractions are generally processed holistically, that is, depending on their magnitudes, or componentially, that is, as two separate whole numbers. They found that when participants had to compare the numerical values of two fractions, their brain activation in the IPS depended on the numerical distance between the fractions more than on the numerical distance between the fraction components (i.e., the numerators and the denominators). A recent study by Barraza, Gómez, Oyarzún, and Dartnell (2014) supported the idea that people are able to mentally represent the numerical values of fractions. These results could be relevant to mathematics education, as a holistic understanding of fractions is an important learning goal.

DISCUSSION

Brain imaging studies have used different methods to identify the neural correlates of number processing. A key conclusion is that a network of brain areas is involved in number processing, in which the IPS plays a fundamental role for representing magnitude information, and the left gyrus angularis plays a key role for representing verbally stored number facts.

Are these findings relevant to mathematics education? Given that brain imaging has a relatively short history, it seems too soon to give a definite answer. What is interesting so far is that the brain seems to be sensitive to numerical magnitudes. Developing a sound understanding of the magnitudes of number symbols is one of the most important goals of early mathematics instruction. If children fail to develop such understanding, they will always struggle while dealing with number symbols. As an example, being able to quickly recognize that 12/13 + 7/8 is approximately 2 requires immediate access to the magnitude information represented by the fraction symbols. A study by Carpenter, Corbitt, Kepner, Lindquist, and Reys (1981) has shown that for the majority of American eighth-graders, such recognition is not self-evident. The finding that it is in principle possible to activate mental representations of fraction magnitudes suggests that such magnitude representations could be enhanced in children through targeted instruction. A question for further research is to explore under which conditions such kind of instruction is most effective, and there are first studies addressing this question (Gabriel, Coché, Szucs, Carette, Rey, & Content, 2012). Furthermore, research could address the question of whether mentally representing the numerical magnitudes of numbers is possible for all kinds of numbers, including more difficult fractions or irrational numbers. For brain imaging research to be informative to education, it is certainly helpful to establish collaborations between neuroscience and education researchers who are equally involved from the onset of a particular study (Pincham et al., 2014).

Brain imaging methods have important limitations. For research in mathematics education, methods that are non invasive and less restrictive seem to be more adoptable. A method that is not detecting brain activation and yet comes closer to psychological processes than paper and pencil tests is eye tracking. Eye movements on well-designed stimuli are believed to correspond closely to the information that is processed. For example, Obersteiner et al. (2014) showed that eye movements could be a valuable method to detect strategy use when people compare the numerical values of fractions. Combining modern research methods allows taking different perspectives on particular tasks. By choosing educationally relevant tasks, neuroscience and educational researchers could gradually build a bridge between their disciplines.

ERP STUDY OF DISTINCT BRAIN ACTIVITIES IN DIFFERENT ABILITY GROUPS AND DIFFERENT MATH PROBLEMS

Ilana Waisman, Mark Leikin, Roza Leikin

This paper describes two research reports on ERP studies to be presented at the Research Forum: One focuses on brain activity of students with distinct mathematical abilities and the other on brain activity associated with different mathematical problems. The two reports are based on one large-scale study, and present interwoven analyses and interrelated results. Specifically, they address (a) brain activity in different ability groups through the lens of solving learned-based and insight-based problems and (b) differences in electrical potentials evoked by distinct types of mathematical tests analysed through the lens of the effects of general giftedness (G factor) and excellence in school mathematics (EM factor). We find that different mathematical abilities are reflected in differences in both the strength and the scalp

distribution of ERPs. These differences are task-dependent, i.e., they are associated with the level of insight imbedded in the tasks solution.

Applying brain research to the study of mathematical processing seems to be of timely importance. On the one hand, this research can lead to better understanding of the nature of mathematical abilities, and on the other hand, brain research can validate and advance understanding the cognitive processes involved in mathematical thinking and problem solving. Mathematical ability is a complex construct that involves a broad range of general cognitive skills, including perception, attention, and memory along with special mathematical skills including numerical cognition, grasping formal mathematical structures, logical reasoning, mathematical transformation, mathematical modelling, and mathematical generalisation (Krutetskii, 1976).

Lack of precise definitions of mathematical ability and insufficient development of methods for the assessment and evaluation of mathematical abilities in general, and high mathematical abilities in particular, complicate working with mathematically talented individuals. To solve this problem, we suggest also employing data and methods of neurocognitive research.

Previous research

Researchers have made an effort to apply neurocognitive methods to the development of educational theory and practice (De Smedt, Ansari, Grabner, Hannula, Schneider, & Vershaffel, 2010; O'Boyle, 2005). Several studies investigated the neurophysiologic basis of giftedness in general, and mathematical giftedness in particular. For example, some studies demonstrated that the brains of the mathematically gifted show enhanced development and activation of the right hemisphere (Prescott, Gavrilescu, Cunnington, O'Boyle, & Egan, 2010). Another characteristic of mathematically gifted individuals is enhanced brain connectivity and the well-orchestrated and coordinated activation of task-appropriate regions in both hemispheres (O'Boyle, 2005). The *Neural Efficiency Hypothesis* links human intelligence and strength of brain activation; it asserts that while performing cognitive tasks brighter individuals display lower brain activation than do less bright counterparts (e.g., Neubauer & Fink, 2009).

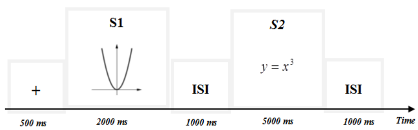
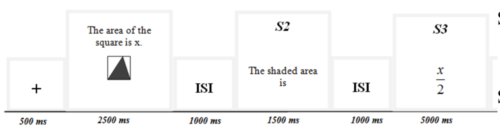
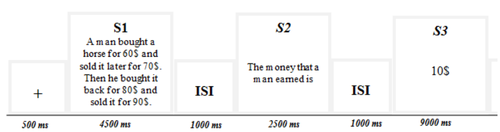
A considerable body of research has been conducted towards understanding of the neural foundation of mathematical cognition (for example, see Dehaene, Piazza, Pinel, & Cohen, 2003). Some studies have focused on the brain regions associated with mathematical processing, for example the fronto-parietal network (Arsalidou & Taylor, 2011). Additionally there is evidence that the difficulty-level of mathematical problems raises the complexity of the neuro-cognitive mechanisms involved in solving the problems (Zamarian, Ischebeck, Delazer, 2009). In this regard, insight-based problems represent one of the most complex types of problem-solving tasks in general, and of mathematical problems in particular. Solving insight-based problems is based on using existing knowledge, however processing them is seemingly characterized by different neurocognitive mechanisms (e.g., Bowden & Jung-Beeman, 2003).

THE STUDY

Study sample

In this paper, we present partial data obtained in a large-scale study directed at the development of a multidimensional characterisation of mathematical giftedness (Leikin, Waisman & Leikin, 2013; Waisman, Leikin, Shaul & Leikin, 2014; Leikin, Paz-Baruch & Leikin, 2013; Leikin & Lev, 2013). The sampling procedure in this study (described in details in Waisman, et. al, 2014) was based on two characteristics corresponding to the study goals: *general giftedness* (G) and *excellence in mathematics* (EM). This distinction was unique for the study and the two characteristics were considered orthogonal ones. There were four major groups of participants: those identified as both generally gifted and excelling in mathematics (G-EM group); those identified as generally gifted who do not excel in mathematics (G-NEM group); those who excel in mathematics but are not identified as generally gifted (NG-EM group); and those identified as neither generally gifted nor excelling in mathematics (NG-NEM group). Overall, 200 students were included in the study sample.

Tests

<table>
<tr><td>Test</td><td></td><td></td></tr>
<tr><td>Function-related Test</td><td>
</td><td>S1 Introducing a situation;
S2 Answer verification</td></tr>
<tr><td>Area-related Test</td><td>
</td><td>S1 Introducing a situation
S2 Question presentation
S3 Answer verification</td></tr>
<tr><td>Insight-based Test</td><td>
</td><td></td></tr>
</table>

+ Fixation Cross; ISI – Inter Stimulus Interval

Figure 2.1 Examples of the item design in the three tests selected for this paper

Six batteries of tests were designed to examine brain activity associated with mathematical problem solving in different topics of school mathematics (e.g., functions, area, geometrical properties, logical inferences) as well as with solving insight-based problems. The data presented here pertain to three tests to support evidence-bases for the two major research goals: the ERP characterisation of distinct mathematical abilities and the drawing of ERP distinctions between the brain activity associated with different mathematical problems. Figure 1.1 presents examples of items included in the three tests reported herein: function-based test, area-based test (which are learning-based tests), and insight-based test.

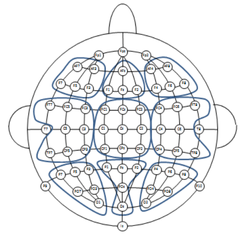
The division of the task into three stages was based on Polya's (1973) theory of problem-solving strategies, three of which -- understanding the task conditions, understanding the question, and verification of results -- constituted stages of the task design in the study. The fourth (main) strategy - performing a solution - was analysed through ERP measures.

ERP Technique

The ERP (Event-Related Brain Potentials) technique offers a high temporal resolution in the course of problem solving due to a precise reflection of perceptive and cognitive mechanisms. ERPs are electrophysiological measures reflecting changes in the electrical activity of the central nervous system related to external stimuli or cognitive processes occurring in the brain (See details in Waisman, et. al, 2014)

Electrophysiological data analysis

ERPs were analysed offline using the Brain Vision Analyzer software (Brain-products). ERPs were Zero Phase Shift filtered offline (bandpass: 0.53 Hz–30 Hz) and referenced to the common average of all electrodes. Epochs with amplitude changes exceeding ±80 μV on any channel were rejected. Ocular artefacts were corrected using the Gratton, Coles & Donchin (1983) method. The ERP waveforms were time-locked to the onsets of S1, S2 and S3. For the learning-based test, the averaged epoch for ERP, including a 200 ms pre-trigger baseline, was 1000 ms, 1500 ms, 3000 ms for S1, S2 and S3; for the insight-based test, 2200 ms, 2500 ms, and 3200 ms for S1, S2 and S3. Only correct answers were averaged. The resulting data were baseline-corrected, and grand wave was calculated for each stage. Each condition resulted in about 40 trials for each test.



We report here on the findings related to the late electrical potentials recorded at each of the problem-solving stages (300-900 ms for the function-related test and 250-900 ms for the area-related and the insight-based tests). The scalp surface was divided into nine electrode sites: anterior left (AL), anterior middle (AM), anterior right (AR), central left (CL), central middle (CM), central right (CR), posterior left (PL), posterior middle (PM), posterior right (PR).

At each of the nine electrode sites, the mean amplitude was found as an average of mean amplitude at each single electrode within the site during three time frames, which we determined experimentally for each test at each stage (e.g. 250/300-500, 500-700, 700-900 ms for the insight-based test)

Data analysis and statistics

Between-group differences on all measures were examined with ANOVA for G factor and EM factor with consequent pair-wise comparisons (G vs. NG in EM and NEM groups, and EM vs. NEM in G and NG groups separately). For pair-wise comparisons *p*-values were adjusted for multiple comparisons according to the Bonferroni adjustment. To examine hemispheric differences, repeated measures ANOVA was performed on the ERP mean amplitude considering the two within-subjects factors: Caudality (anterior, central and posterior) and Laterality (left, middle and right) as within-subjects factors. For all analyses, *p* value was corrected for deviation from sphericity according to the Greenhouse Geisser method.

We performed a qualitative comparison of the problem-solving performance on the two tests in the four groups of participants after the statistical analysis.

RESULTS AND DISCUSSION

This section presents two topics to be presented at the Research Forum: (a) ERP measures as indicators of distinct mathematical abilities and (b) ERP measures as indicators of distinct brain activities in different math problems. Table 1.1 presents all significant effects of G and EM factors found in the three tests in our study.

Table 1.1 Significant Effects in ERPs Associated With Three Tests in Our Study

Significant Effects	Location	Function-related	Area-related	Insight
G	PO electrodes			S1: 700-900 ms*
Caudality × Laterality × G			S3: 500-700 ms*	S3: 700-900 ms*
Caudality × G				S1: 700-900 ms*
Laterality × G		S1: 300-500 ms* 500-700 ms*		
	Anterior		S3: 700-900 ms*	
	Posterior	S1: 300-500 ms*	S3: 500-700 ms*	S3: 500-700 ms* 700-900 ms**

	PO electrodes			S2: 500-700 ms^{+} 700-900 ms* S3: 500-700 ms* 700-900 ms*
EM	Anterior		S1: 700-900 ms** S2: 700-900 ms* S3: 700-900 ms*	S1: 700-900 ms*
	PO electrodes		S2: 700-900 ms*	
Laterality × EM		S1: 300-500 ms*	S1: 500-700 ms*	
	Posterior	S1: 300-500 ms*	S1: 500-700 ms**	
	PO electrodes		S3: 700-900 ms*	
G × EM		S1: 300-500 ms* S1: 500-700 ms**	S3: 700-900 ms*	
	Anterior	S1: 500-700 ms**		
	Posterior	S1: 300-500 ms* S1: 500-700 ms***	S1: 500-700 ms* S3: 700-900 ms*	
	PO electrodes	S1: 300-500 ms* 500-700 ms**	S3: 700-900 ms*	
Caudality × Laterality × G × EM		S1: 500-700 ms*	S1: 250-500 ms*	
Caudality ×G × EM			S3: 700-900 ms*	
Laterality × G × EM			S3: 700-900 ms*	
	Anterior		S1: 250-500 ms* S3: 700-900 ms*	
	Posterior	S1: 700-900 ms**	S1: 250-500 ms* S3: 500-700 ms*	
	PO electrodes		S3: 500-700 ms*	

$^{*}p \leq .05$, $^{**}p \leq .01$, $^{***}p \leq .001$

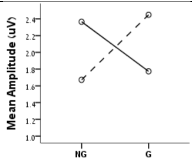
Differences in ERPs related to distinct mathematical abilities

The data in Table 1.1 demonstrate that different mathematical abilities are reflected in differences both of the strength and the scalp distribution of ERPs. We find that G and EM factors have different effects on ERPs in all the tests. For example, the differences between the effects of G and EM factors can be seen in the main significant effects that these factors had on the ERPs associated with solving area-related problems. The EM factor had main significant effect only on the strength of the ERPs during time intervals of 700-900 ms at S1, S2, S3. An additional example for the differences in ERPs associated with different levels of mathematical abilities can be seen in the significant interactions between G and EM factors revealed when students were solving learning-based tasks (function-related and area-related tests).

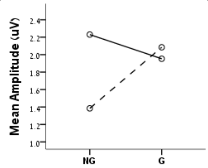
Differences in ERPs related to distinct mathematical tests

The second major finding of our study demonstrates that between-group differences are task-dependent, i.e., they are associated with the level of insight imbedded in the task's solution. That is, G and EM factors, as well as their combinations, influence brain activation patterns differently in different tasks. A significant neuro-efficiency effect in tasks with a low level of insight is revealed in the G-EM group while insight embedded in the task reduces the significance of the effect.

Function-based test	Area-based test

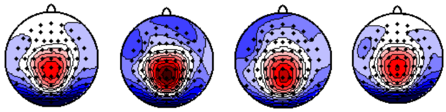


$F(1, 71) = 12.101^{***}$ η_p^2=.146



$F(1,66)$ =4.567* η_p^2=.065

G-EM NG-EM G-NEM NG-NEM



G-EM NG-EM G-NEM NG-NEM

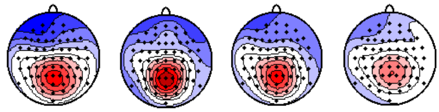
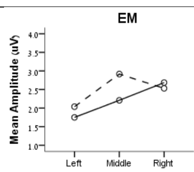
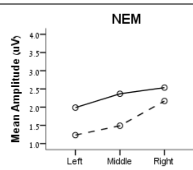


Figure 1.2 Significant interactions between G and EM factors in learning-based tests in the posterior regions at S1 during 500-700 ms time interval

The effects of G factor on electrical potentials found in EM and NEM students were different. The strength of electrical potentials among G-EM students was lower than the strength of electrical potentials of NG-EM students. While the accuracy of G-EM and G-NEM students on learning-based tests are similar, the electrical potentials of G-NEM students are higher than those of the G-EM students. That is, neural efficiency within the EM group was displayed only in combination with the G factor.

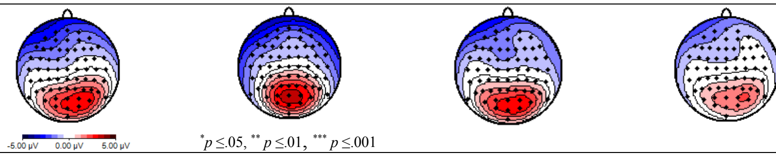
Additionally both G and EM factors and their combinations are reflected differently in the scalp distribution of the ERPs (see Figure 1.2). For example, these differences can be seen in the significant interactions between Laterality, G, and EM factors that were revealed in the area-related tests in the posterior regions at S1 during 250-500 ms time interval (Figure 1.3). During this time interval, mean amplitudes of NG-NEM students were the lowest among the four groups of participants at all the electrode sites. Mean amplitudes of the G-EM students were similar to the mean amplitudes of the G-NEM students at the central electrode sites, lower than the mean amplitudes of G-NEM students at the AM, AR, PL and PM electrode sites, and higher than the mean amplitudes of G-NEM students at AL and PR electrode sites. Pair-wise comparison demonstrated significant differences between the mean amplitudes of G-NEM and NG-NEM students and of NG-EM and NG-NEM students at the AR electrode site (Figure 1.4).





$F(2,132) = 3.399*$ $\eta_p^2 = .049$

G-EM NG-EM G-NEM NG-NEM



$^{*}p \leq .05$, $^{**}p \leq .01$, $^{***}p \leq .001$

Figure 1.3 Significant interactions between Laterality, G, and EM factors in area-related test in the Posterior regions at S1 during 250-500 ms

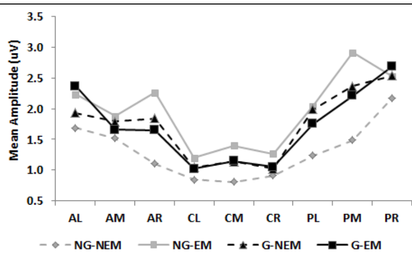
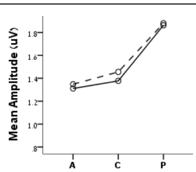
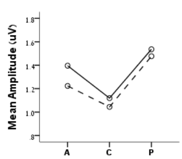
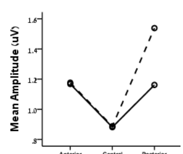


Figure 1.4 Significant interactions of Caudality and Laterality with G and EM factors in area related test at S1during 250-500 ms

For example, G factor had a main effect on ERPs in PO electrodes associated only with solving insight-based problems at S1 during 700-900 ms interval (note: 700-900 ms interval at S1 can be considered as advanced processing of problem conditions). Additionally, at this stage (during 700-900 ms time at S1) a significant interaction of Caudality with G factor appeared (Figure 1.5). At the anterior and central electrode sites, the mean amplitude of G was similar to that of NG, whereas at the posterior electrode sites, the amplitudes were significantly lower for G than for NG participants [F (1, 65) = 4.770, p < .05, η_p^2 = .068] and were accompanied with significant differences at the PL electrode sites [F (1, 65) = 7.095, p < .01, η_p^2 = .098].

Function-related test	Area-related test	Insight-based test
		
$F(1.498, 106.328) = .072$	$F(1.566, 103.377) = .332$	$F(1.766, 114.765) = 5.505^{**}$ $\eta_p^2 = .078$
G NG	G NG	G NG

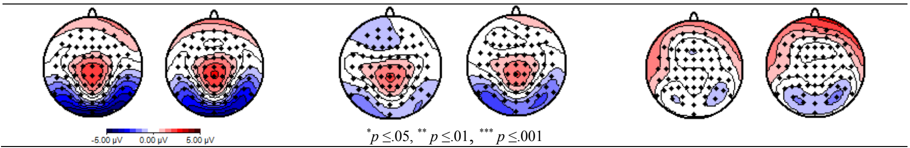


Figure 1.5 Interaction between Caudality and G factor at S1 during 700-900 ms

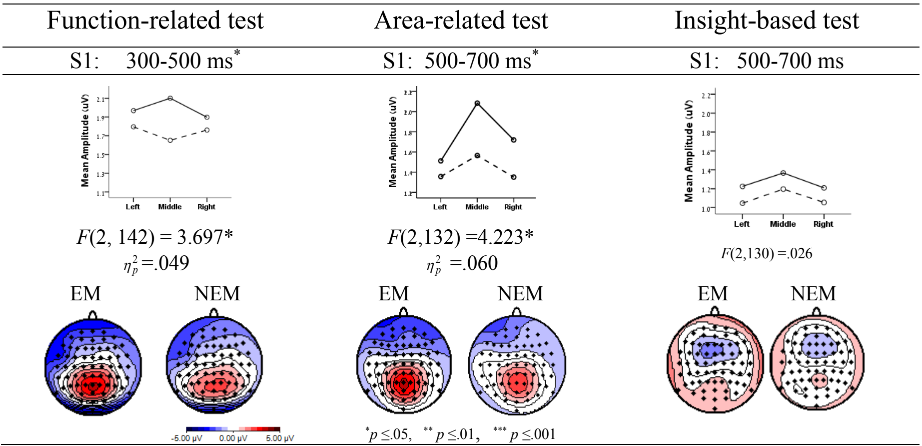


Figure 1.6 Interaction between Laterality and EM factor

The lower amplitude evoked by G students than by NG students discovered in this study, is a manifestation of the “neural efficiency effect”. Additionally, all the effects related to G factor when solving insight-based problems were revealed at the "late potentials" in the intervals of 500-700 ms at S2 and S3 and at the intervals of 700-900 ms at all the stages of the task solution S1, S2 and S3. In contrast, on the function-related task the differences among the four experimental groups were indicated only at the first stage (S1).

While the G factor appeared to affect mainly ERPs associated with solving insight-based tasks, the EM factor affected mainly ERPs associated with solving learning-based problems. In the case of the area-related task, analysis of brain potentials revealed a main effect of the EM factor at S1 (introducing a situation stage) and at S2 (the question presentation stage). Surprisingly, participants excelling in mathematics (EM) had a higher overall mean activity than their non-excelling counterparts.

The significant interaction between Laterality and the EM factor at S1 appeared at the interval of 300-500 ms for the function-related task and in the interval of 500-700 ms for the area-related test (Figure 1.6). During this interval, ERPs evoked in the middle regions were the strongest for EM students and the lowest for NEM students. The interaction between the Laterality and the EM factor appeared to be non-significant for the insight based test. Thus, we argue that solving learning-based and insight-based problems is related to different cognitive processes as reflected in different patterns of brain activation.

DISCUSSION

Participants in our study solved three types of mathematical problems of varying complexity levels. The area-related and function-related tasks were based on school curriculum and are taught in high school classes. In contrast, insight-based problems were not part of school curriculum and are considered to be relatively difficult to solve. These problems are usually unfamiliar to solvers and require the high cognitive effort associated with figuring out novel ways for solving a problem, even though the solver has previously learned knowledge for their solution.

These differences in the nature of the three types of problems integrated in our study (learning-based vs. insight-based) were expressed in separate main effects of the EM or G factors. The EM factor was mainly related to the tasks based on school curriculum (area- and function- related problems). In turn, the G factor appears to be significant for solving insight-based problems.

The most efficient brain functioning, accompanied by highest behavioural performance, was observed in the G-EM group. The most prominent differences between the G-EM and the other three groups were shown on the learning based tests while embedding insight into the task reduces the significance of the effect. As in all tests, the mean amplitude of G-NEM participants does not significantly differ from that of NG-EM participants. The G-factor compensating for the lack of EM as exhibited in the G-NEM group appears in all tests; however, on the insight-based test this compensation was the most prominent. Notably, NG-NEM students forgo investing cortical effort since they might have already reached their limit.

To sum up, the following major findings seem to be new and significant to our understanding of the nature of mathematical ability:

- The EM factor has significant main effects mainly in tasks that require implementation of knowledge familiar to students from school mathematics;
- The G factor has significant main effects in insight-based problems, which are not part of the school mathematics curriculum and, thus, require original mathematical reasoning;
- Mathematical performance in gifted students who excel in mathematics (G-EM students) has specific characteristics, including neuro-efficiency of brain activation;
- Students who excel in mathematics but are not identified as gifted (NG-EM students) exhibit high electrical potentials along with a high level of problem-solving accuracy (e.g. Leikin, Waisman & Leikin, 2013). We argue that students who excel in mathematics but are not identified as generally gifted invest high level of cognitive attempt in order to achieve their mathematical excellence. At the same time, our

findings demonstrate that general giftedness is not a necessary condition for excellence in school mathematics and students can develop their problem-solving expertise independently of the G factor.

- Effects of the G and the EM factors are task-dependent - they are associated with the level of insight imbedded in the tasks solution as reflected in the strength and the scalp distribution of the electrical potentials at the different stages of solving mathematical problems.

We will discuss the instructional implications of these findings at the Research Forum.

fMRI STUDY OF FRACTION PROCESSING IN ADULT BRAINS

Ron Tzur

This paper uses empirical findings from a preliminary, functional magnetic resonance imaging (fMRI) study to illustrate ways for addressing the key issues of this RF: (a) cognitive neuroscience research relevance to maths education, (b) maths educators' potential contributions to interdisciplinary, CogNeuroMathEd studies, (c) methodological affordances and constraints (e.g., of the widespread fMRI), and (d) questions for future research in an emerging, interdisciplinary agenda.

BACKGROUNG AND CONCEPTUAL FRAMEWORK

This preliminary study focused on how task design, rooted in a constructivist perspective (Piaget, 1971), may impact brain processing when adults compare numbers. It examined a *vital inversion in quantitative comparisons*—from whole numbers (**WN**; e.g., 7>2) to unit fractions (**FR**; e.g., 1/2>1/7). One main purpose was to understand how a *solely conceptual* intervention for teaching adults who already learned the "inverse rule" might impact reaction time (**RT**) and error rate (**ER**) when comparing WN and FR. A second purpose was to distinguish brain circuitry activated for processing such comparisons, as well as components that make up a comparison task (e.g., a numeral vs. the ">" symbol). To these ends, I collaborated with cognitive neuroscientists who were instrumental in guiding task design so it fits with both obtaining behavioural measures of performance and fMRI specs/protocols. The maths education steering of this collaborative effort drew on recent calls for extending unidirectional, neuroscience-to-education studies (Westermann et al., 2007) into cross-disciplinary, reciprocal scholarship (De Smedt & Verschaffel, 2010). Specifically, the choice of mathematical domain, research problem/questions, and task design were all rooted in the work of maths educators. Accordingly, this preliminary study allowed a combined focus on WN or FR comparisons, which cognitive neuroscience researchers have previously studied separately.

Substantial research has focused on WN capacities of the brain. Dehaene (1997) proposed a triple-coding model, in which Arabic numerals implicated in low-level visual regions are linked to number words in language regions (lingual gyrus, perysylvian cortex) and to analog magnitudes processed in a third region (the Intraparietal Sulcus, IPS) in a "number-line" manner. Only a few studies examined brain processing of fractions (Bonato et al., 2007; Ischebeck et al., 2009; Jacob & Nieder, 2009; Obersteiner et al., 2014; Siegler et al., 2013), including a study that indicated the triple-coding model in how adults solve challenging tasks such as 2/3-1/4 (Schmithorst & Brown, 2004). The present study focused on conjoining maths education and cognitive neuroscience work to distinguish regions activated in the brain when adults process WN or FR comparisons (as well as the ">" symbol).

To provide such a conjoined 'window', this study drew on von Glasersfeld's (1995) three-part notion of a scheme: a *situation* into which a person assimilates information that triggers her goal (e.g., determine which of two numbers is larger), an *activity* for accomplishing that goal (e.g., inverting "9-is-larger-than-6" to solve a True/False task such as "1/9 > 1/6 ?"), and an expected *result* (e.g., determining which number is larger and why it must be the case). Specifically, this study used Simon et al.'s (2004) reflection on *activity-effect relationship* (*Ref*AER*) framework, which elaborated on scheme theory by proposing *anticipation of such relationship* as a lens to delineate "conception"—a dyad comprising the last two parts of a scheme. The key construct of anticipation, developed via observational studies, has been supported by neuroimaging studies (Schacter et al., 2012). Importantly, the *Ref*AER* framework explicitly distinguishes operations the mind carries out (e.g., ordering numbers) from objects on which it operates (e.g., WN or FR). This distinction guided task design, so assimilation of "cues" would be triggered by only one of two possible symbols (number *or* operation) before an entire number-comparison task is presented.

METHODOLOGY

Twenty-one participants, ages 23-36, took a pre-test comprised of 4 "Runs," each including 90, four-step number comparisons (randomized). Step A (1 sec) presented a numeral or an operation (e.g., 3, 1/3, >, or =). Step B (1 sec) combined two symbols (e.g., 1/3>, 3=). Step C then presented the entire task (e.g., 1/3>1/8?, 3>8?), which the participant could solve (within 2.5 sec) by pressing a right (True) or a left (False) key. Step D (0.5-sec) presented three dots to separate between consecutive tasks (called ITI). Immediately following the pre-test, each participant received a video recorded teaching episode (~50 minutes), consisting of solving problems to promote conceptual understanding of unit fractions as multiplicative magnitudes (for details see Tzur & Depue, 2014). Each episode was concluded with a discussion of *why* a smaller denominator implied a larger unit fraction *for any FR*, but *no practice* of such comparisons took place.

Each participant then took an immediate post-test like the pre-test described above. A second post-test was taken 2-4 months later during fMRI scanning (Fig. 1). To increase fMRI signal, the 4 Runs in the second post-test eliminated Step B above and included 140 two-step tasks. In the MR, subjects' reaction time (RT) and error rate (RT) were measured based on a button press in the right hand (True) or the left (False). Roughly 90% of tasks involved a true ">" comparison and constituted the experimental part of the test; the other, 10% of control tasks involved "=" and false ">". Each run was organized in a hybrid-block design, including random-length sequences of like-comparisons (e.g., 1/3>1/8, 1/7>1/2, **8=8**, 5>3, 9>7, 4>3, 6>4, etc.). The impact of each independent variable (number type, Step A cue, testing occasion) on the two dependent variables (RT, ER) was calculated using ANOVA.


Figure 1. Before entering the MR

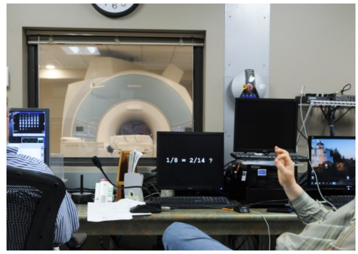


Figure 1-b. fMRI session – control room

RESULTS

Improvements in Adults' Reaction Time (RT) for Processing WN and FR

The average error rate (ER) in both occasions (pre/post) and for both number types (WN/FR) was very low (3-4%), indicating no distinguishable results for this variable. Average reaction time (RT), however, indicated statistically significant improvement ($p<.001$). Data in the chart below indicate average RT in milliseconds (ms) for each type of task design, with statistically significant pre->post improvement ($p<.001$) in comparing both FR (as expected) and WN (unexpected). Data also show a *cue X number-type interaction*: comparing FR when a ">" preceded a number took longer than when a number preceded ">" ($p<.05$), whereas no such difference was found for WN. These results lend support to task design that explicitly distinguishes among inferred parts of a scheme, as RT for recognizing and processing a mental object to be operated on seems effected by how a "situation" is identified in the person's mind.

Table 2.1: Average reaction time (RT, ms) for comparing WN or FR

	Pre		Post	
	Cue: >	Cue: Number	Cue: >	Cue: Number
FR	1208	1144 (-64 = -5.3%)	923	901 (-22 = -2.4%)
WN	925	949 (+24 = 2.6%)	757	763 (+6 = 0.8%)

Brain Circuitry Activated to Process Numbers (WN, FR) and Operation (>)

fMRI analysis shows that some adult brain regions were activated more for WN than FR comparisons (Fig. 2.1a) and some regions more for FR than WN comparisons (Fig. 2.1b). WN comparisons were implicated more in (A) the Hippocampus (Long-Term retrieval) and in (B) the Medial Frontal and Anterior Pole (abstract retrieval). FR comparisons showed *substantially greater activation* in (A) the bilateral IPS and Angular Gyrus (numerical judgments – likely needed for denominators) and the Ventral Visual Processing Stream (object-based visual processing – likely needed for reading the three-part fraction symbols), in (B) the Dorsal Fronto-Parietal control network (engaged in attention-demanding tasks – likely needed for order inversion), in (C) the Ventral-Frontal working memory network & Pulvinar (visual object attention/selection), and (D) the Supplementary Motor Area (SMA, interestingly requiring greater activation to prepare FR responses). These results suggest that brain circuitry used by adults in comparing FR involves higher activation in some areas used also for WN (e.g., IPS), along with additional brain regions.

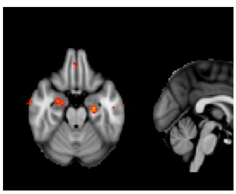


Figure 2.1a. WN > FR

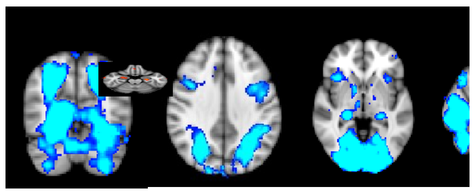


Figure 2.1b. FR > WN

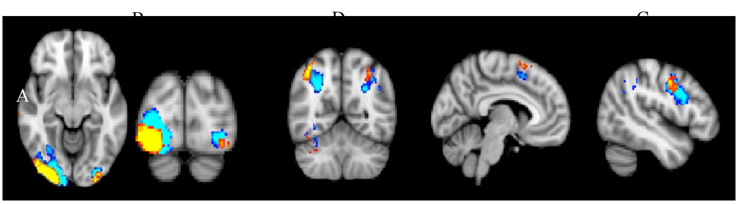


Figure 2.2. Numeral (WN, FR) Activation Larger than Operation (“>”)

Further analysis of fMRI data (Fig. 2.2) during Step A only (“cue” period) showed brain regions activated more for processing a numeral (WN – shown in yellow/red color, or FR – shown in blue color) than for a symbol of the operation (“>”). Essentially, when a participant whose goal was to determine which of two numbers is larger was looking at just a symbol of number on the computer screen, brain activation in four regions was significantly higher than when she or he was looking at the “>” symbol. These regions included: (A) the Ventral Visual processing stream/cortex (typical of object-based, visual processing mostly in the right hemisphere); (B) the IPS and Angular Gyrus (numerical judgments); (C) the SMA (preparing for response), and (D) Posterior Dorsolateral PFC (attention-demanding tasks). In all, both number types activate some similar circuitry much more than the symbol of operation, whereas processing FR does so to a greater extent than WN (when each is compared to processing “>”).

DISCUSSION

This paper presented three main findings from a study of how adult brains process WN or FR comparisons: (a) impact of an intervention on shortening RT, (b) brain regions activated more for comparing FR than WN, and (c) more for processing numerals than a symbol of a comparison operation (“>”). Combined, these findings help illustrate how the four key questions of the Research Forum may be addressed.

The first key question pertains to the relevance of cognitive neuroscience research for mathematic education. This study illustrates such relevance in demonstrating the differentiated circuitry involved in comparing WN or FR, which suggests FR is not just a simple extension of WN. Rather, the brain and mind have to build circuitry dedicated to FR, which requires greater cognitive load to make sense of and solve fractional tasks. It also illustrates relevance to pedagogical questions such as conceptual vs. procedural understandings; here, a conceptual intervention effected improved procedural mastery (-12% in RT for FR, -9% for WN).

The second question pertains to contributions of maths educators to the emergence of a CogNeuroMathEd interdisciplinary domain (De Smedt & Verschaffel, 2010). Like Leikin et al.’s (2014) ERP studies, this study illustrates how maths educators’ choices of research problems, conceptual frameworks, and task design could help framing collaborative future research. Specifically, the milestone shift from WN to FR comparisons, its analysis through the constructivist scheme theory, and the design of task components that correspond to postulated mental operations and units—were all unique maths education contributions.

The third question pertains to affordances and constraints placed on studying *brain-related problems of significance to maths education*. This is a challenging issue, because it requires maths educators to delve into the details of preparing for, collecting, and analysing cognitive neuroscience data. For example, in this fMRI study I realized that one can neither (a) interview participants of their solutions due to noise and head stability requirements nor (b) measure short-time intervals (50-200 milliseconds) in which different components of a task seem to be processed. Using ERP would allow such high temporal resolution—albeit at the expense of fine spatial resolution fMRI provides.

The fourth question pertains to other research problems of interest to RF participants. To me, questions that arose include the possibility to test certain hypotheses about the mental realm by identifying neural bases of, say, prompt-dependent and prompt-independent stages (Tzur & Simon, 2004) in the construction of a scheme. A host of other problems could focus on impact of pedagogical interventions (see Siegler, 2009), including the surprising performance improvement with WN comparisons.

DISCUSSION AND CONCLUDING REMARKS

Roza Leikin, Ron Tzur

Our goal in this Research Forum is to raise awareness of the relevance of cognitive neuroscience research methodologies to mathematics education. We provide a number of examples for the added value that research can bring to the understanding of mathematical thinking and learning. We also suggest several ways in which mathematics educators can contribute to, and guide, framing of research problems, questions, and foci for interdisciplinary, CogNeuroMathEd studies. We wonder how different methodologies currently used in cognitive neuroscience afford, and constrain, research design and potential findings/implications for maths education.

Indeed, this effort is evolving in different parts of the world. Three research groups − from the University of Colorado Denver (USA), from the Technische Universität München (Germany), and from the University of Haifa (Israel) participate in our RF. The contributions to the RF vary in their approaches: Beitlich, Obersteiner, and Reiss provide two meta-analyses of previous studies, while the studies by Waisman, Leikin & Leikin and by Tzur supported claims through empirical findings.

Neuro-cognitive methodologies and mathematical content: The RF addresses the variety of neuro-cognitive methodologies employed in studying different mathematical contents: Beitlich and Obersteiner review and analyse eye tracking methodologies used to identify problem solving strategies (e.g., comparing the numerical values of fractions, or reading and understanding of the written proofs). Obersteiner and Reiss explain the fMRI, EEG, and NIRS techniques, and review some brain imaging studies that focus on cognitive processing related to fractions. Waisman, Leikin & Leikin used the Event-Related Potentials (ERP -- EEG-based) methodology to analyse problem-solving processing related to relatively advanced mathematical topics (e.g., functions and area). Tzur employs fMRI methodology, with pre- and post-tests, to examine brain activation associated with fraction processing compared with processing of whole numbers, as well as to analyse brain correlates of the development of the number sense.

Analysis of neural correlates of mathematical processing in this RF are related to the localization of brain activation (Tzur), to the topographical maps that depict electrical potentials associated with solving mathematical problems, and to the strength of the electrical potentials with fine time resolution (Waisman, Leikin & Leikin). Waisman, Leikin & Leikin add the dimension of mathematical abilities.

Mathematical strategies and neuro-cognitive research: There is a dual role of mathematical strategies in neuro-cognitive research. Beitlich and Obersteiner demonstrate that mathematical strategies can be investigated using eye tracking, for example, to analyse use of the number line or to investigate strategies for comparing the numerical values of fractions. At the same time, understanding of mathematical strategies should be reflected in the design of the research tools. In Tzur's study, which examines brain processing when adults compare numbers, the task design is rooted in a constructivist perspective (Piaget, 1971), and in von Glasersfeld's (1995) three-part notion of a scheme: a *situation*, an *activity* and an expected *result*. The division of the task designed for ERP procedure into three stages in the study of Waisman, Leikin & Leikin corresponded to Polya's (1973) problem-solving strategies understanding the task conditions, understanding the question and verification of results. They argue that the fourth (main) strategy - performing a solution – can be analysed through the lens of ERP measures.

All the contributors at this RF stress the importance of collaboration of neuroscientists and mathematics educators and of the interdisciplinary research agenda directed at deepening of our understanding of mathematical thinking, learning and understanding.

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DISCUSSION GROUPS

DG01: CONNECTIONS BETWEEN VALUING AND VALUES: RETHINKING DATA GENERATING METHODS

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Australian Catholic Uni., Monash Uni., Uni. of Melbourne & Stockholm Uni.

What would PME colleagues find if they looked at some of our research in a different way? In particular are there new methods we could use in the future? One possibility is using role-playing as a data-generating context. In this DG we explore values and valuing, and then with colleagues perform and evaluate a role-play to this end.

VALUES, VALUING AND OBSERVABLE BEHAVOUR

We have suspected for many years now that some of the methods we have used, e.g. questionnaires, have not got us as far as we would have liked (Clarkson, Seah, Bishop & FitzSimons, 2000). Nevertheless these approaches have given us the chance to collect data, find some interesting patterns, and deepen our thinking with regards mathematical values in the mathematics classroom. What's more, we have noted the importance, yet again, that we should not separate methodology from theory.

In considering where we have moved to, we have also wondered that instead of our long-term focus on values, whether this should shift and/or be enlarged to focus on **valuing**. This contemplated change has helped us think again more about learners/ learning, rather than just teachers and curriculum, but also the complexity/coherence that exists in these links. The fundamental questions that this move has suggested are:

- How can you tell if someone is valuing something, and what if that something is a specified value?
- What does it feel like, in the moment, to be valuing a specific identified mathematical value, and what are the behaviours that are seen in that moment?

These suggest that a more nuanced approach to observing students, and also teachers (as learners) is needed, looking for systematic patterns of valuing particular behaviours. The behaviours that such students are privileging would be within a context of values linked to mathematics per se; that is when the students are learning mathematics, such as the six mathematics values which include *rationalism*, *progress* and *control* (Bishop, 1988), but may also include what we have earlier identified as values associated with their learning of mathematics (e.g. *fluency*, *understanding*).

ROLE-PLAYING AS A DATA GENERATING

A relatively new idea for us, which we have used for some years in some informal PD spaces, is to use role-play for observations of valuing. We note that role-play has been used in various forms, although not always called that (e.g. micro teaching) for teacher education and for student learning (Belova, et al, 2013; Zazkis et al, 2013). We are experimenting with role-play to explore whether it can be used as a method of data collection. In such a role-play situation, some participants act out the roles assigned to

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them (teacher or students) according to a set of values that defines the individual within the roles. The rest of the participants usually remain as observer researchers. We wish to create a context that evokes behaviour that can be reflected upon and analysed by all participants.

To this end, Day 1 of this DG will focus on an open discussion aimed at distinguishing between mathematical values, and between the valuing behaviour associated with each of these values. Building on our earlier PD scenarios, but now focusing on researching valuing, during day 2 of the DG we will set up a “role-play classroom” with “teacher” and “students”, with the students basing their classroom valuing behaviours on descriptions of mathematical values they are given.

Thus we will explore within the DG a new methodology in this research area. Foundational to the ideas for this DG are two fundamental questions: can one learn mathematical values by initially role-playing them? and, thinking of role-play as method, does this allow research observers to see what are the observable valuing traits? Hence this DG does not explore more effective ways to teach, or help students learn. Rather we will explore with colleagues, be they playing a role as student or teacher, or they are playing a role of observer / researcher in that they observe a ‘student’ who has been given a particular role of valuing a specific mathematical value, whether this can give us insights into the behaviours we should be focusing on when conducting research into mathematical values and valuing in classrooms. We suspect that in having to think through just what is the valuing behaviour that a specific mathematical value evokes, in having to inhabit the feelings that goes with this behaviour, and playing out that valuing behaviour to an audience, will give both the player and observer a much deeper appreciation and understanding of what they are dealing with when mounting research investigations concentrating on valuing specific mathematical values.

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DG02: INTEGRATING PEDAGOGICAL AND MATHEMATICAL LEARNING IN PRE-SERVICE TEACHER EDUCATION

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This Discussion Group will explore international perspectives on integrating pedagogical and mathematical learning in pre-service teacher education. In Australia, as in many other countries, pre-service teacher education programs are structured so that future teachers of mathematics learn the *content* they will teach by taking courses in the university's school of mathematics, while they learn *how to teach this content* by taking content-specific pedagogy courses in the school of education. Such program structures provide few opportunities to interweave content and pedagogy in ways that help develop professional knowledge for teaching (cf. Baumert et al., 2010).

Mathematicians and mathematics educators are members of distinct, but related, communities of practice. Connecting the communities is essential to achieving a seamless, meaningful and rigorous academic preparation for pre-service teachers of mathematics. Wenger (1998) writes of *boundary encounters* as potential ways of connecting communities. Boundary encounters are events that give people a sense of how meaning is negotiated within another practice. Wenger suggests that if "a boundary encounter…becomes established and provides an ongoing forum for mutual engagement, then a practice is likely to start emerging" (p. 114). Such *boundary practices* then become a longer term way of connecting communities. While boundary practices might evolve spontaneously, they can also be facilitated by *brokering*, a role that involves translating, coordinating, and aligning the perspectives of different communities (Bouwma-Gearhart et al., 2012).

This Discussion Group will engage participants with ideas being developed in an Australian multi-university project – *Inspiring Mathematics and Science in Teacher Education* – that is developing interdisciplinary approaches to mathematics pre-service teacher education. We will begin the first session with a synopsis of the conceptual framework that guides the project. Small groups of participants will then discuss the following questions:

1. What are some of the effective (and partially effective) ways in which experts in mathematics and mathematics education collaborate on developing and delivering pre-service teacher education programs?

2. In forms of collaboration identified in part 1, what boundary encounters, brokers, and boundary objects seem to play role in making these collaborations possible and effective? Which aspects of institutional context appear essential for emergence and continuation of the collaborations?

In the second session we will first provide an example of a proactive approach to boundary crossing. We introduce an activity developed for a dual purpose of pre-service teacher education and developing shared means of communication for mathematics and mathematics education communities. The aim of the pre-service teacher education activity is to engage students in making sense of a pedagogically meaningful distinction while working on and analysing mathematically valuable tasks. In our case, the "pedagogically meaningful distinction" focuses on proficiency strands that are part of the Australian Curriculum: Mathematics, specifically *fluency*, *understanding*, *problem solving*, and *reasoning*. The aim of the boundary crossing activity is to provide a context in which mathematics colleagues would be encouraged to engage with pedagogical content in order to strengthen the mathematical elements of the task.

New small groups will then be asked to

1. Discuss the potential of developing shared means of communication about mathematical teaching and learning.

2. Identify additional sets of "pedagogically meaningful distinctions" that both constitute important learning content (globally or in a specific country) for pre-service teachers and could be explored by pre-service teachers through analysis of mathematical tasks.

Groups will report their responses at the end of each session and research-worthy conjectures will be recorded.

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WORKING SESSIONS

WS01: INTERNATIONAL INTEGER CURRICULUM COMPARISON

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A discussion group met and discussed the current state of research in the domain of integers at the joint PME 38 and PME-NA 36 meetings (Bofferding, Wessman-Enzinger, Gallardo, Salinas, & Peled, 2014). During these meetings productive discussion revolved around what it meant to understand integers. Additionally, the organisers presented a literature review of the research on integers from all of the PME and PME-NA proceedings, which the group discussed. At the conclusion of this meeting, the group expressed interest in investigating integers further together by pursuing an international curriculum comparison study. This working group aims to begin this curriculum comparison study of integers.

WORKING SESSION GOALS

In one article from the literature review of the PME and PME-NA proceedings, Lindbland and Marton (2004) report on the utility of comparison studies and suggest that we need to better determine and compare what opportunities students have to learn the concepts presented in various international tests used for comparisons (e.g., OECD-PISA, IEA-TIMSS, LPS). A table they reproduce from a TIMSS-1999 report suggests that students have little opportunity to work with integers. This working session is intended to provide participants a space to share and begin a study that investigates differences in integer curricula across countries. With participants sharing expertise on integers from their home country, the study will provide insight into the ways that integers are introduced and learned internationally. At the conclusion of the study we intend to submit a manuscript to *Educational Studies in Mathematics*.

THEORETICAL BACKGROUND

Within the previously mentioned literature review, it was noted that there were differences in age groups targeted, contexts highlighted, and didactical models used. Further, studies focused on either integer concepts (e.g., order, value), operations (e.g., addition, subtraction, multiplication), or algebraic uses (e.g., simplifying expressions with negatives). When students first learn about negative integers, they must make sense of the multiple meaning of the negative signs and distinguish between magnitude and directed magnitude, a difficult process that can result in several synthetic (or transition) conceptions (Bofferding, 2014). Students also use a variety of conceptual models to reason about integer addition and subtraction situations. Each model involves interpreting the quantities and zero in a different way (e.g., in the bookkeeping model, zero represents not having a gain or loss) (Wessman-Enzinger & Mooney, 2014). We will begin the curricular analysis by looking for how the curricula address these integer concepts and models.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 1, pp. 127-128. Hobart, Australia: PME.

SESSSION 1

After the coordinators provide some background to this study and share goals, participants will discuss their country's standards or well-utilized curriculum. It will be expected that participants bring their country's standards with integers and integers operations. Similarly, participants should bring copies (e.g., digital or otherwise) of current curricula that introduce integers and integer operations to students. Participants will work in small groups discussing their country's standards or curricula with others. The coordinators will share a Google spreadsheet of a template with initial categorizations for the curriculum study. The initial categorization was developed from the aforementioned literature review and integer concepts. Participants will suggest modifications to the spreadsheet during this session. The first session will conclude with participants working on recording at what age negative integers are introduced, contexts used, typical didactical models employed, information about standards, and descriptions of typical curricula used to support integer learning (e.g., philosophy of the curricula, length of units, use of visuals, etc.).

SESSSION 2

Coordinators will provide a de-briefing of the previous day. After de-briefing, participants will resume work with the shared Google spreadsheet. Half-way through the session, we will discuss themes present within spreadsheet. Discussion will transition to the next moves that should be taken on the integer comparison study. The session will conclude with establishing next directions for the group, ways to continue communication internationally, and possibly generating an outline of the manuscript.

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WS02: MATHEMATICAL DISCOURSE THAT BREAKS BARRIERS AND CREATES SPACE FOR MARGINALISED STUDENTS

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THEORETICAL BACKGROUND

Classroom discourse in mathematics classrooms and who gets to participate in it has received substantial attention by educators, and researchers in the past two decades. A common thread in the discussion of classroom mathematical discourse is recognition that how students participate in communicating their mathematical reasoning directly influences their mathematical identity and disposition towards doing and using mathematics (Hunter & Anthony, 2011). There is a close relationship between who gains access to classroom mathematical discourse practices and who is able to participate in the mathematics classroom (Civil & Planas, 2004). Put simply, participation in classroom mathematical discourse practices has the potential to offer all students enhanced opportunities to learn mathematics. Our focus in this working group is placed on the different ways in which marginalised learners are provided with space to equitably access the mathematical discourse.

AIM

This working session builds on the Discussion Group 3 (DG3) on Mathematical Discourse that breaks barriers and creates space for marginalised students at PME 38 (Civil, Herbel-Eisenmann, Hunter, Wagner, 2014) conference. The Discussion Group aimed to bring together researchers in this area of research to explore the topic and to develop a research agenda for future work in this field. It was clear from DG3 that there was significant interest in the topic and that the participants of the discussion group were planning and undertaking significant evidence-led practices which supported marginalised students' equitable access to the mathematical discourse and practices. Participants at PME 38 DG3 expressed interest in a follow-up session at PME39 conference. This Working Session (WS) is intended to provide space for some participants to develop a research agenda and for others to discuss their research projects and develop collaborations towards contributing chapters for an edited book which explores how barriers to the discourse have been identified and removed for different groups of marginalised students across a range of countries. Additionally, researchers with an emerging interest in the topic of this WS will have an opportunity to gain understandings of the relevant conceptual frameworks and the types of research being undertaken in the field.

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 1, pp. 129-130. Hobart, Australia: PME.

WORKING SESSION STRUCTURES

In the first session international researchers from Australia, India, New Zealand, Spain, and the United States will provide brief 5 to 7 minute overviews of their research. Following each presentation participants will be invited to respond within a plenary context. Themed subgroups will then be formed to develop a proposed outline for chapters for an edited book. Potential subgroups include classroom-based studies, a focus on theoretical framing, and implications for professional development and pre-service teacher education programmes. The second session will be informed by the outcomes of the first session but it is anticipated that the participants will work in small subgroups to develop an agenda for the edited book, including a proposal for chapters and possible chapter outlines.

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WS03: THE BUILDING AND RESEARCH OF THINKING CLASSROOMS

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Gaye Williams — Deakin University

Our 2013 Discussion Group (DG) introduced the PME community to the notion of a *thinking classroom*. Our activities at that meeting were centred on the identification of salient characteristics of thinking classrooms, as well as the co-construction of a definition of a thinking classroom:

> a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together, and constructing knowledge and understanding through activity and discussion. It is a space wherein the teacher not only fosters thinking but also expects it, both implicitly and explicitly (Liljedahl, in press).

This emergent definition of a thinking classroom intersects with research on mathematical thinking, classroom norms, notions of a didactic contract, the emerging understandings of studenting (Fenstermacher, 1994; Liljedahl & Allan, 2013), knowledge for teaching (Schulman, 1986), and activity theory.

At the end of the DG at PME 37, the participants suggested that they were not ready to shift to a Working Session (WS). As such, at PME 38, we offered a DG again – this time with a focus on *researching thinking classrooms*. A number of categories of researchable questions emerged from this DG, including: a) What type of content (e.g., tasks etc.) promote a Thinking Classroom? b) What are the tools (including competencies) that enable teachers to transition to a *Thinking Classroom*? c) How do teachers initiate and sustain Thinking Classrooms? d) How does thinking stop in a classroom and why does it stop? e) What techniques give the most engagement? F) Given a Thinking Classroom, what are the outcomes? At the end of the DG at PME 38 one international group of researchers excitedly shared that they were ready to undertake a project together. Others declared that they are ready for a WS to develop their research design. In response, for PME 39, we will run a WS on *The Building and Researching of Thinking Classrooms* and hopefully hear of preliminary work underway and / or further research planned.

The construct examined 'research on thinking classrooms' will be informed by research designs previously employed to study such topics as cognitive, social, affective, emotional, and psychological aspects of student learning, and teachers' practices and beliefs. These include for example: videoing and reflecting on own classroom practice (Lampert, 2001), video-stimulated student interviews (Williams, 2014), and study of student emotions through emoticons (Ainley, 2010). This WS will capitalize on these designs as teams conceptualize their designs, ready to enact their projects around thinking classrooms. The goal is that this WS will inform a subsequent WS and later a Research Forum.

Session 1: An overview of 2013 / 2014 DG findings (5 Mins) is followed by a brief presentation of research into building thinking classrooms (Liljedahl, 5 Mins). Information on cutting edge technology in the *Learning Interaction Classroom* <http://education.unimelb.edu.au/news_and_activities/news/news_articles/launch_of_the_science_of_learning_research_classroom> is briefly introduced (Williams, 5 Mins). Participants brainstorm ways to utilize this classroom and in doing so illuminate their research interests (20 Mins). These ideas are shared in the WS, before research participants self-select into teams based on research interests (15 Mins). Participants will work in small groups to refine their research topic, formulate their research question, and begin to develop their research design that may or may not use the *Learning Interaction Classroom* (30 Mins). Teams briefly share progress to date (10 Mins). Homework: find a relevant paper.

Session 2: Team members share and discuss papers found, and relate these to their research design (20 Mins). They develop a poster or PowerPoint slide to capture their progress to date (e.g., research topic, research question, theoretical framing, research design, relevant literature) (10 Mins), and briefly share this with WS participants (20 Mins) before teams continue with developing their own project drawing on ideas presented where useful to them (25 Mins). Finally, we brainstorm ways to form and sustain international research collaborations about researching thinking classrooms, and the feasibility of a WS to examine research progress in 2016 (15 minutes).

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WS04: RESEARCH AROUND THE INTERSECTION OF MATH AND SPECIAL EDUCATION

Yan Ping Xin, Coordinator Helen Thouless, Assistant Coordinator

Purdue University University of Roehampton

This working group has been focused on developing a research agenda to explore pedagogical approaches for fostering conceptual knowledge of mathematics in students with special needs. The work is rooted in a twofold premise: (a) students with special needs are capable of developing and need to develop conceptual understanding and mathematical reasoning skills, and (b) special education instruction, assessment, and research needs to transition towards this focus.

THEORETICAL BACKGROUND

Recent reform in math education calls for inquiry-based problem solving, learner-centred discovery, students being able to explain their mathematical reasoning to others and follow others' reasoning, and students being able to construct their conceptual understanding. However, existing literature in the field of Special Education show that direct instruction and explicit instruction are the main forms of instruction for students with special needs (Gersten et al., 2009). This was supported by a meta-analysis study (Kroesbergen & Van Luit, 2003) that analysed 58 studies of mathematics interventions for elementary students with special needs, which concluded that reform-based mediated instruction (which requires students to discover and develop their own math skills, with the assistance of a teacher) was less effective than direct instruction. These analyses indicate that the recent changes in mathematics education "do not lead to better performance for students with special needs" (pp. 111-112). Further, recent research studies that examined the response of low achieving students to reform-based mathematics instruction "suggested that both the organization and task demands of the reform classrooms presented verbal and social challenges to low achievers" (Baxter, Woodward, & Olson, 2001). Low-achieving students "seemed to disappear during whole class discussions" (p. 545). The tension between general and special educators over the two different pedagogies in mathematics instruction involving students with LDM is far from relieved.

Since 2008, members from this working group have been collaborating on two research projects that integrated research-based practices from mathematics education and special education. One project focused on nurturing multiplicative reasoning of elementary students with LDM; the other project documented learning trajectories of elementary school children with LDM as they come to understand fractions as quantities. The preliminary collaborative work has demonstrated promising learning outcomes of students with LDM. There is a need to expand such collaboration that integrates constructivist pedagogy from mathematics education and explicit strategy instruction from special education to best serve students with LDM. Specifically, the goal of this working session is to deepen the conversation and understanding between the researchers and professionals in mathematics education and special education through further collaborative work as well as disseminating such collaborative work.

HISTORY AND PLAN FOR THE WORKING SESSION (WS)

This WS will build upon the past three WSs conducted by this collaborative working group that was created in 2012 at PME-NA34. Our first WS focused on the definition and identification of mathematics learning disabilities; the second WS (at PME-NA35 in 2013) showcased a collaborative work between the members from math education and Special education. The third WS was held in PME38/PME-NA36 in 2014, where the group considered producing a special issue to highlight collaborative work among mathematics education and special education scholars/practitioners. A proposal has been drafted and potential contributions have been identified. Currently, there are negotiations with a special education journal outlet to publish this work that will address the research around the intersection of mathematics and special education. One important goal of this year's WS is to finalize the topics/themes of this special issue. The proposed WS will provide the working group members with a unique opportunity to accomplish the set goal—the publication of this ground-breaking special issue. The WS will brainstorm future collaborative work agenda.

Session 1	Session 2
Briefly share past discussions Brain storm how to organize and put together this body of work to reflect a cohesive theme or multiple themes to be published in the special issue As only a few papers can be included in this special issue, the WS will further identify potential outlets for publishing the work resulting from this working group	Participants will be divided into sub-groups that share common interests. Articulate future research questions that the sub-group would like to address through collaborative work Identify potential funding sources for collaborative grant proposals Share progress and commitments from small group discussion Plan for future meeting agenda

Table 1: Goals and activities for the working session

References

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SEMINAR

SEMINAR: REVIEWING FOR PME – A PRIMER FOR (NEW) REVIEWERS

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IPN Kiel, IPN Kiel, Tel-Aviv University

GOAL OF THE SEMINAR

This seminar is intended to provide information about the PME review process and give the opportunity to gain first experiences in providing a high-quality review. The seminar aims especially at the needs of new reviewers, although experienced reviewers are highly welcome in order to facilitate knowledge transition within the PME community. Note that PME members with two accepted Research Reports in the past five years or three accepted Research Reports in the past 10 years are eligible to be a PME reviewer. The seminar includes an introduction in the intention and purpose of reviewing from a more general perspective (McKnight et al., 2000; APA, 2009), but also details aspects of the PME review practices. Participants will have opportunities to work with authentic examples from the PME review processes of the last years – provided we find authors that are willing to share their contributions with the review they received. Acknowledging the diversity within the PME community in the review process will be an important aspect of the seminar.

GOALS FOR THE PARTICIPANTS

Having participated in the seminar, the participants will

1. know about reviewing as an aspect of scientific quality management
2. know about the most important differences in reviewing procedures for journals and conferences as well as different types of contributions, especially in the PME context
3. be able to differentiate the specific review categories of PME
4. be able to identify aspects of quality for a review
5. be sensible to aspects of fair, constructive, and inclusive reviews

EXPECTED BENEFIT FOR PME AS A COMMUNITY

PME – as a scientific community – will benefit from the seminar as

- it is expected to improve the knowledge of (new) reviewers about the review process
- it is expected to smoothen (new) reviewers difficulties in composing high-quality reviews

2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 1, pp. 137-138. Hobart, Australia: PME.

METHODS

The seminar will last 90 minutes. It will start with a brief presentation focusing on learning goal 1 and 2. A first group work phase will focus on the specifics of PME reviews and thus contributing to the learning goals 3 and 4. A second group work phase will focus in particular on the aspects of fair, constructive, and inclusive reviews (learning goal 5). Experienced reviewers, who are willing to share their knowledge, are invited to serve as group mentors during the working phase.

APPLICATION

If you are willing to share a former contribution of yourself TOGETHER with the reviews you received as authentic examples for the group work phase, please contact Anke Lindmeier at lindmeier@ipn.uni-kiel.de as soon as possible.

References

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Note: Seminars are intended to provide specific courses for the professional development of PME members. This workshop format was introduced 2008 at PME 32 in Morelia but never used so far.

SHORT ORAL COMMUNICATIONS

ESTABLISHING A STEM EDUCATION LEARNING COMMUNITY ACROSS SECONDARY SCHOOL AND UNIVERSITY BOUNDARIES

Judy Anderson — The University of Sydney

Kathryn Holmes — The University of Newcastle

Amidst calls for a greater focus on STEM education in schools, attention is inevitably drawn to the quality of teaching and to appropriate means of supporting the teaching workforce so that more young people are engaged in STEM subjects. This short oral describes the development and implementation of a STEM Teacher Enrichment Academy at a metropolitan university, in conjunction with teachers from a variety of school systems. The findings draw on survey and interview data from the 62 teacher participants from 13 secondary schools and three STEM mentors as they progress through the Academy program, working towards the establishment of a professional learning community for enhancing STEM teaching in schools.

This communication reports on progress made toward the establishment of a STEM teacher professional learning community involving secondary school and university educators. Drawing on the notion of a professional learning community (Stoll, Bolam, McMahon, Wallace, & Thomas, 2006) we outline how a newly established STEM Teacher Enrichment Academy is making progress towards supporting teachers in "developing the agency and autonomy to drive their own professional growth and become stewards of their profession" (Gillespie, 2015, p.38). We draw on interview and survey data from 62 teachers and three teacher mentors as they progress through the program, which includes residential and in-school components. The participants were also supported via an interactive online platform designed to build and sustain the professional learning community.

Data analyses indicate a strength of the initial program was the inclusion of up to six teachers from each participating school – two mathematics, two science and one or two technology teachers – to encourage cross-disciplinary communities of practice. However this did not occur in all school settings and was highly dependent on school structures, and support from other staff members and principals. The online community was successful when actively promoted by the STEM mentors but did not lead to the development of an overall community of practice. The results from this first trial has lead to refinements in the program towards a second trial later in 2015.

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CHALLENGING ABILITY GROUPING IN NEW ZEALAND PRIMARY MATHEMATICS CLASSES

Glenda Anthony and Roberta Hunter

Massey University

Challenging socially embedded practices associated with ability grouping in mathematics classrooms requires coherence from research evidence, policy directions, and professional learning support. Internationally, converging research evidence suggests that benefits which accrue from ability grouping serve only very high achievers at best, with a negative impact—as evaluated across achievement and affective/social measures—for the majority of students (Sullivan, 2015). Moreover, OECD findings (Schleicher, 2014) report that "stratification is negatively related to systems' overall performance" (p. 104). Despite these findings, and concerns that lower stream classes/groups are more likely to be assigned less capable mathematics teachers, and experience low work expectations and more disruptive working climates, the practice of ability grouping is New Zealand primary mathematics classes is currently expanding to include cross-class grouping or setting (Golds, 2014).

Grouping practices in New Zealand reflect a policy led on the use of ability grouping both within the national Numeracy Professional Development Project and as part of the National Standards assessment implementation. Lack of research and policy coherence, combined with a relative absence of teaching exemplars and prior experience in mixed ability approaches within the primary sector, makes changing grouping practices particularly challenging. In this presentation, we draw on a survey involving 90 primary teachers, who have a mathematics leadership role in raising mathematics achievement levels within their respective schools, to look at how current grouping practices came into being—who promoted them, and the reasons these teachers gave for justifying or challenging existing practice. In light of fact that 40% of these teachers expressed uneasiness with their school's current grouping practices, we examine teacher dilemmas associated with changes to practice. We also explore change possibilities related to a range of initiatives being implemented at the policy and professional development level within New Zealand.

References

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2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 1, p. 142. Hobart, Australia: PME

EXAMINING PRESERVICE TEACHERS' TECHNOLOGICAL PEDAGOGICAL STATISTICAL KNOWLEDGE VIA TINKERPLOTS

Berna Aygün, Neslihan Uzun, Ercan Atasoy

Recep Tayyip Erdoğan Üniversitesi, Faculty of Education

Statistics is a vital part of the school curriculum, whereby all students are expected to collect, organise, analyse, interpret and communicate information. Students should be provided opportunity to (1) formulate questions that can be addressed with data and collect, organize and display relevant data to answer them, (2) select and use appropriate statistical methods to analyzed data, (3) develop and evaluate inferences and predications that are based on data (Shaughnessy, 2007). With developing technology, visualisation and simulations tools (such as Tinkerplots) can help learners do interactive exploratory data analysis and to understand statistical concepts and methods deeply. The increasing importance of technology in statistical education, and teachers' technology knowledge of how to use in classroom is added to teachers' professional knowledge. Lee and Hollebrands (2011) proposed a Technological Pedagogical Statistical Knowledge (TPSK) framework that characterised the aspects of knowledge needed to teach statistics with technology.

Considering the importance of pre-service mathematics teachers' technological pedagogical statistical knowledge, the purpose of this study is to determine how pre-service mathematics teachers construct technological pedagogical statistical knowledge with Tinkerplots. In line with the nature and the purpose of this research, the present study uses the method of qualitative research to reveal pre-service teachers' TPSK. In a case study, researchers analysed four pre-service teachers' lesson plans and observed their integration of technology into a statistics lesson. Following the observations, interviews were also conducted. Themes emerged from the analysis, and were further analysed using TPSK as a framework to identify key features. Findings indicated that pre-service teachers can develop learning activities which provide an opportunity to improve statistical knowledge and thinking, however, they could not use these activities effectively. The students' statistical explorations were limited and primarily teacher-centered rather than student-centered.

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ADMINISTRATORS' MATHEMATICAL NOTICING: SUPPORTING TEACHERS' INSTRUCTION

Cory A. Bennett — Idaho State University, United States

Julie M. Amador — University of Idaho, United States

This study focused on, and is theoretically framed around, developing administrators understanding of professional noticing (van Es, 2011) as a means to initiate professional conversations around mathematics instruction. For administrators, noticing requires gathering specific evidence of students' mathematical thinking to initiate non-evaluative feedback through professional conversations to support teachers' instructional practices. Thus, the research questions are: 1) to what extent does a multi-tiered professional development system support administrators' ability to notice, and 2) how does this system influence the instructional feedback they provide in mathematics?

Thirty-four administrators and other teacher leaders, 23 from an urban district and 11 from a rural district, were purposefully chosen to participate. Participants engaged in a multi-tiered professional development structure, including workshops and school-based observations of teachers. First, a two-day workshop focused on learning the structures of noticing by analysing videos of primary and secondary mathematics teaching, followed by independent practice with participants' teachers. Next, follow-up sessions, based on independent practice, were held 1-3 months later to discuss revised understandings and next steps. For the final component, researcher supported school-based observations were conducted in classrooms at the participants' schools. Data included reflections, audio recordings, of conversations and field notes. All data were analysed using the learning to notice framework (van Es, 2011).

Findings indicate that 24 participants improved in their ability to notice shifting from a pedagogical focus to specific evidence of students' mathematical thinking; participants were able to maintain this focus when assessed at subsequent workshops. Participants indicated that school-based sessions were the most helpful as they were able to get immediate feedback and felt more confident moving forward with the professional conversations on mathematics instruction. On-going professional development supports with scaffolds, such as those used in this project, show promise in developing administrators' ability to notice and thus facilitate professional conversations around mathematics instruction.

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2015. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 1, p. 144. Hobart, Australia: PME

BALANCING EDUCATIVE AND DIRECTIVE GUIDANCE IN TEACHER GUIDES IN THREE TEACHING CULTURES

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Many teachers use teacher guides when preparing their mathematics lessons, and the ways authors of textbook series communicate with teachers is therefore a central aspect when linking the intended curriculum to the real world in the classroom (Valverde et al., 2002). Teacher guides are designed to support teachers in their work. We analysed the balance between two different types of support: *directive guidance,* where the teacher guide tells the teacher (or the students) what to do or say, and *educative guidance,* where the teacher guide informs the teacher on important educational aspects. Using Davis and Krajcik's (2005) design principles we identified three approaches to educative support: *Design transparency*, which communicates the intent with suggested activities, *Anticipating student thinking*, which indicates student understanding or likely misconceptions, and *Explaining mathematical ideas*, which describes important mathematical concepts.

In this study, we explore the balance between directive and educative guidance in textbook series from three cultural regions. The goal of the study is to understand differences in this balance in relation to differences in the educational culture.

The sample included 72 lessons from teacher guides from six different textbook series, two from the U.S., two from Flanders (Belgium) and two from Sweden. 12 lessons (school year 3, 4 and 5) from each textbook series were analysed using a coding scheme based on the design principles of Davis and Krajcik (2005).

Our results show that the balance between educative and directive guidance differ across the three regions. The Flanders lessons are much more directive then educative. The US lessons are quite balanced, as is one of the Swedish teacher guides. In the U.S., there is a strong commitment to student collaboration and some of the educative features in U.S. guides may be aimed at supporting this less directive role. The second Swedish guide is more educative than directive. This coincides with our understanding of the approaches to teaching in the three regions. In Flanders and the US teachers play a clear directive role, while the Swedish classroom is centred on the students textbook with the teacher as a facilitator. In the presentation more data on our results will be presented along with a discussion of possible implications.

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COMPARING NEGATIVE INTEGERS: ISSUES OF LANGUAGE

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Making number comparisons is an important prerequisite for later mathematical learning. According to Case's (1996) number theory, children coordinate their understanding of whole number order and values by about six years of age. Therefore, they understand that each subsequent number in the counting sequence is more than the previous, which corresponds to an increase in the quantity of a set. When students learn about negative integers, they must extend the number sequence to less than zero and continue to interpret numbers to the left on the number line as smaller, even though the numerals (ignoring the negative signs) appear to increase or get "higher" (Bofferding, 2014). Because students might focus primarily on order *or* quantity when making negative integer comparisons, we explore the following research question as part of a larger study: What is the role of language (specifically question phrasing) on children's interpretations of negative integer values?

We report on 47 first graders who completed all phases of the larger study (see Bofferding & Hoffman, 2014); this analysis focuses on four sets of integer comparisons from the follow-up (n=28): "Which integer is…" (a) closer to 10, (b) farther from 10, (c) higher, (d) more. Sets were counted as correct if students solved over half of them correctly. Performance on the "higher" and "more" sets were significantly correlated (r=.52, t=2.44), suggesting students associated "higher" with quantity. Performance on the ordered-based "closer to 10" and "farther from 10" sets were also significantly correlated (r=.69, t=3.82). Furthermore, students did better on the order-based comparisons than on the quantity-based comparisons. These results suggest students do not necessarily coordinate their understanding of integer order with integer values when learning about negatives. Thus, instruction should explicitly foster this coordination with greater attention paid to the phrasing used.

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ANALYSING DISCOURSE IN WHOLE CLASS INTERACTION: SOME INSIGHTS ON THE LEARNING OF MATHEMATICS

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Our work is placed within the tradition of design experiments in mathematics education research (Cobb, Confrey, diSessa, Lehrer & Schauble, 2013), with a focus on the exploration of mathematics learning in situations of social interaction (Goos, 2004). To provide evidence of learning, there is a complementary focus on the analysis of classroom discourse (Sfard, 2001). The reported experiment consisted of three lessons in a class of thirty 12 year-old students. The teacher was given geometry problems to be worked out with problem-based materials. For each lesson, the students read the problem, discussed approaches and strategies in small groups, participated in the whole group interaction guided by the teacher and, to finish, were asked to revise what they had written at the beginning. In order to examine the discursive production of mathematics learning in whole group, we drew on classroom instances of alternate participation between students and teacher regarding the resolution of the problem.

For the identification of mathematically thematic episodes in whole class discussion, we worked with the video of the lesson and its transcript. In the context of each episode, we assumed that certain actions of collective argumentation promote favourable conditions for the learning of mathematics. At this point of the research, there were many back and forth movements oriented to inductively detect some repeated actions initiated by either the teacher or the students. It was found, for example, that the teacher shifted registers from informal to formal language when a student had introduced an idea without precise technical vocabulary. More generally, it was proven the role and relevance of particular actions of collective argumentation in the development of conceptual mathematical knowledge. What remains to be done is to analyse the role of materials on the emergence and exploitation of specific actions.

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STATISTICAL CONTENT KNOWLEDGE FOR TEACHING: VARIABILITY AND DIGITAL TOOLS

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Our work with elementary preservice teachers (PSTs) focuses on developing their statistical content knowledge for teaching (SKT) while making use of statistical digital tools. A suggested framework for describing SKT outlines two main facets, that of common and specialized statistical knowledge, as well as considering mathematical and non-mathematical aspects of such statistical knowledge (Groth, 2012). Further, we referred to the Guidelines for Assessment and Instruction in Statistics Education (GAISE) report (Franklin et al., 2005) for information on levels of statistical understanding appropriate for the PSTs.

The context of the work is a probability and statistics course designed for future elementary/middle schoolteachers. As part of an ongoing design experiment, the redesign of existing tasks, the incorporation of new tasks, and the sequencing of topics are conducted in order to study which tasks may best facilitate the development of the PSTs' statistical knowledge for teaching. Some of the newly created tasks evolve around the use of the digital tool TinkerPlots®, statistical software designed for data analysis and probability explorations for upper elementary and middle school students. Our chief data sources for this discussion will be in-class assessments, instructor lesson notes, and post-class interviews.

In our initial analysis we found evidence the PSTs progressed in their SKT regarding ideas of variability, specifically that of making sense of the mean absolute and standard deviation measures. The TinkerPlots® tool provided PSTs with several ways of thinking about spread in general as they engaged in activities that made use of dynamic features in constructing dot plots, box plots and divider tools. We attribute some growth in understanding of these measures to using simple geometric approaches, e.g. drawing line segments to represent a distance from the mean of a data set, to make sense of the numerical values describing variation. Tool features permit the PSTs to physically draw on the graphical display, providing a tactile sense for the measures. We see such activities as an important part of developing the specialized content knowledge that PSTs need when teaching statistics to children.

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DISCURSIVE ACTS OF POWER: A CRITICAL ANALYSIS OF SINGLE-SEX AND COEDUCATIONAL MATHEMATICS CLASSES

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Single-sex instruction in U.S. public schools has only emerged within the past nine years as an option for schooling. In this study, we explore the flow of power instantiated through discursive acts of teachers and students and how this flow may be similar and/or different based on type of classroom, an all-girls, an all-boys, or in coeducational mathematics classrooms. Classroom discourse is defined in this study as ways individuals use language to say, to do, and to be things (Gee, 2011), specifically within a mathematics classroom. The positioning of students through classroom discourses may influence how they perceive themselves as doers of mathematics (Bishop, 2012), as well as their engagement (or lack thereof) within the classroom (Kelly, 2007).

We use a multiple case study to examine the flows of power in the form of classroom discourse by critically analysing the minute discursive acts of teachers and students in the three different classroom types (i.e., all-girls, all-boys, & coeducational). Participants include two male teachers, who each taught an all-girls, an all-boys, and a coeducational mathematics class in the same day. Ten instructional sessions for each of the three class types were recorded. We employed both a thematic analysis as well as an analysis of discursive moves or actions taken by teachers and students to participate in or influence the discourse in the classroom.

We are uncovering three discourse patterns for the classroom discourses: (1) teacher talk dominating discourse, (2) subtle variations in teacher expectations of student behaviour and potential, (3) tendencies towards deficit perspectives of students in the all girls' classes. These patterns point to the potential for teachers to communicate underlying biases and inclinations through subtle and nuanced discourse, which are often not apparent on other measures of classroom environment. Supporting evidence and additional results will be further discussed in the presentation.

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THE EFFECTS OF WORK ETHICS AND ATTRIBUTIONS TO FAILURE ON MATHEMATICAL LITERACY PERORMANCE: A STUDY OF LEARNING CHARACTERISTICS OF ACADEMIC RESILIENT STUDENTS IN SHANGHAI, SINGAPORE, HONG KONG, TAIWAN AND KOREA

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Many students from ESCS-disadvantaged homes participating in PISA 2012 were classified as academic resilient (called *disadvantaged high-achiever (DHA)*. These students were able to beat the odds against them and to advance in mathematical literacy attainment (Cheung, Sit, Soh, Ieong, & Mak, 2014). Drawing data from the PISA 2012, this study sought to examine the similarities and differences in two learning mathematics characteristics (i.e. mathematics work ethics, and attributions to failure in mathematics) amongst students of the five top-performing Asian economies. Examples of good work ethics include working hard and paying attention in lessons. Examples of undesirable attribution of failures include referring to bad luck or poor teacher instruction. From the international comparative education perspective, these students were classified across economies as DHA with reference to their disadvantaged counterparts respectively. Percentages of DHA students of Shanghai, Singapore, Hong Kong, Taiwan and Korea in their 15-year-old populations were 19.2%, 15.1%, 18.1%, 12.3% and 12.7% respectively. Logistic regression was carried out for the DHA versus non-DHA student classification, as a function of the demographic and two selected mathematics learning characteristics. The results of the logistic regression analysis showed that the variables *gender*, *immigration status*, *family structure*, *years of attending kindergarten*, *grade repetition*, *mathematics work ethics* and *attributions to failure in math*ematics variables were able to predict whether a student of comparable disadvantaged home background is more likely to be classified as DHA or not. Specifically for the two mathematics learning variables, *mathematics work ethics* and *attributions to failures in mathematics* are predictive for *all* the five top-performing East Asian economies in PISA 2012. The findings are important to shed light on the principles and methods of mathematics education so as to help the low-achievers, whether ESCS-advantaged or disadvantaged, to advance to higher level of mathematical literacy attainment.

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DEVELOPMENT AND PRACTICE OF THE INSTRUCTIONAL MATERIALS TO ALLEVIATE MATH ANXIETY BASED ON CEREBRAL NERVE PHYSIOLOGY

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An example of research into math anxiety as measured through the Electro Encephalo Graph (EEG) during execution of the arithmetic, is provided by Colome, Nucez-Pena, & Suarez-Pellicioni (2013). In their study, the researchers analysed the differences of people who were evaluated as high mathematics anxiety (HMA) and low mathematics anxiety (LMA) when solving arithmetic problems by EEG. Our study explored the possible ways that may reduce the math anxiety of students, in the aspect of the teaching and learning of mathematics, based on the brain physiology.

The participants were 40 undergraduate students attending one university. They consisted of 20 students who were enrolled in the Department of Natural Science and 20 in the department of Humanities and Society. Each participant completed the Mathematics Anxiety Scale for Students (MASS) before taking the mathematical tasks which had been developed to measure EEG of students. After the data were collected, we analysed them statistically using SPSS after using the brain wave analysing program, ERP. The students were divided into the low mathematics anxiety (LMA) group and the HMA group based on 3.0 on a 5 point Likert scale. As a result of the analysis of the EEG brain waves of both groups, the HMA group was found to record higher amplitude in brain waves, leading us to conclude that the HMA group was using more working memory to solve the same task than the LMA group.Accordingly, based on the results of MASS and EEG, we developed three treatment programs to reduce math anxiety (William, 1998). The first is non-psychological treatment programs as a category of math-dominated interventions; the second is psychological treatment programs as a category of anxiety management training; the third is complex treatment programs integrated with the first and the second method. The programs will be revised through the pilot study to determine the effectiveness when applied to students in middle and high school. We expect to alleviate math anxiety of students by the developed treatment programs.

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7^{TH} GRADE STUDENTS' AND MATHEMATICS TEACHERS' SOLUTIONS IN ALGEBRAIC EXPRESSIONS - (8+4)÷(4-3)·2 = 24 (OR 6)

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The syntactic rules of algebra operate on two levels (Kirschner, 1989). The first level of the rules consists of working with parsing symbols such as parentheses, brackets and braces. The second level consists of the rules about the order of operations. The omission of the multiplication symbol in algebraic expressions may be a source of errors in the order of operations (see Bush, & Karp, 2013). In this presentation we focus on how 7th grade students and mathematics teachers handled the omitted multiplication symbols and the rules for the order of operations.

The student sample consisted of 121 students (58 boys, 63 girls) whose ages were between 13 and 15. The teacher sample comprised 21 mathematics teachers who participated in a professional development program on the assessment of mathematical knowledge. The algebra test used consisted of 28 items and had an appropriate reliability (Cronbach's $\alpha = .84$).

On the item $(8+4) \div (4-3) \cdot 2$, students outperformed math teachers (93% vs. 76% performance rate). The item on whether the $15x \div 3x$ expression is equivalent with $15 \cdot x \div 3 \cdot x$ resulted in 17% performance rate among students, and 52% among teachers. In several cases the omitted multiplication symbol was handled as a grouping symbol.

"Students and teachers should be taught to write parentheses to provide clarity and avoid ambiguities as mathematicians do" (Barbeau, 2008 p. 381). It is recommended that Textbooks and teacher training programs should make the role of the omitted multiplication symbol more explicit.

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PEER TALK AND HELPING ACTIVITY IN MATHEMATICAL PROBLEM-SOLVING GROUPS

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This exploratory study investigated the interactions during group work between two classes of similar mathematical attainment working in different classroom structures. Students in one of the classes (11-12 year olds) were taught group work skills to enable them to work in groups on open-ended mathematical problems and students in the other class (12-13 year olds) worked as 'mathematicians' in self-selecting friendship groups, on similar open-ended mathematical problems without any direct teaching of skills for group work. The study was underpinned by socio-constructivist arguments about the importance of social and language interactions for cognitive mathematical development (Morgan, Craig, Schuette and Wagner, 2014), the Vygotskian (1978) model of a 'zone of proximal development', and recent work on small group interactions in mathematics classrooms (e.g., Hunter and Anthony, 2014).

Qualitative analyses of audio-recordings identified thirty distinct types of talk/activity from peer talk within groups in both classes. Findings also indicated a greater degree of 'helping' activity within groups in the class that were taught group work skills (with the exception of one group which is explored more fully in this session). In groups in the class that were not taught group skills, mutually-derived solutions to problems reflecting the inputs of each group member were more evident. Quantitative comparisons are made about the frequency of occurrence of each of the thirty types of talk within groups across both of the classes, and suggestions are offered for reasons why specific frequencies occur in each of the groups/classes.

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STUDENTS' INDIVIDUAL CONTRIBUTIONS TO EXPLAINING PRACTICES

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Theoretical background. This presentation reports a small comparative case study that aims at grasping qualitative differences between students' participation in explaining practices. For this purpose, two theoretical approaches are combined: On the one hand, the project refers to an interactionist perspective (cf. Cobb & Bauersfeld, 1995) and conceptualises explaining as a mathematical practice that is regulated by shared social and sociomathematical norms. In this view, learning to explain means to increasingly participate in these interactively established practices. On the other hand, an additional epistemic perspective allows investigating the mathematical core of explanations by locating given or demanded explanations in the so called epistemic matrix (Prediger & Erath, 2014). Altogether, explaining is conceptualised as practices of navigating through different epistemic fields (Prediger & Erath, 2014).

Research questions. The observation that not all students participate is deepened and systematized by the following question: How can the differences in students' participation in the classroom explaining practices be grasped by means of the developed framework?

Methods. The sample comprised three students (all male) from a higher tracked grade 5 class with similar background but contrasting ways of participation in whole class discussions. 12 lessons were videotaped, 8 at the beginning and 4 in the middle of the year.

Results. The three boys' different ways of participation were reconstructed by the means of the framework with different profiles in the epistemic matrix. A connection between the students' linguistic resources and their profiles can be observed and is an issue of further research. Notwithstanding the methodological limit of only observing 12 lessons over half a year, a main and surprising result was the stability of the ways of participation shown by the students.

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‘HALF OF SOMETHING’: HOW STUDENTS TALK ABOUT RATIONALS

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Italian national standard evaluation highlights some common students’ difficulties in dealing with rational numbers. An example is the question “Do 4/8 and 0.5 indicate the same quantity?” which was released in Italian national test as a multiple choice question. Just 53% of students selected the right option. In order to investigate students’ answers and arguments, we re-administered it as an open question to a sample of 231 students from two different Italian cities.

Similar to the national results, half of the students gave a correct answer. The analysis of the answers shows that students approach the question in different ways. Students who refer to “the half of something” reach the correct answer in almost all cases, while the usage of iconographic representation is rarely related to success. Many students convert the fraction to decimal or vice versa, the first kind of conversion is more frequent but results to be less effective. We focus on the first kind of answer because it is the most successful. A paradigmatic example is showed below:

Sono entrambi la **metà** di qualcosa per esempio 0,5 è la metà di 1 che è unità, 4/8 è la metà di 8/8 che è un'unità [They are both half of something, for example 0.5 is the half of 1 which is a unit, 4/8 is the half of 8/8 which is a unit.]

We notice that the student uses the word ‘metà’ which in Italian means ‘half’ but it strictly belongs to the *colloquial register* (Morgan, 1998). In the Italian mathematical register, ‘one half’ is ‘un mezzo’. The word ‘metà’ usually refers to one of the two parts obtained from cutting a concrete object. This word evokes the part-whole conception of rational numbers (Behr, Lesh, Post, & Silver, 1983).

In this context the usage of colloquial register proves to be more suitable than the manipulation of mathematical symbols. Although many authors claim that the colloquial register can interfere with learning (Bardelle & Ferrari, 2011), this result suggests that colloquial words can also support it.

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A STUDY ON STATISTICAL INQUIRY PROCESS INVOLVING MATHEMATICS EDUCATIONAL SIGNIFICANCE

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In an information-oriented society, statistics education comes under the spotlight in mathematics education. Because statistics itself has high versatility, the contents of statistics are addressed within various content-based subjects. Thus, there are some studies putting into doubt the assumption that statistics education is a part of mathematics education. It has been argued that statistics is not mathematics, and as a result there is no consideration of statistics education from the viewpoint of mathematics education. Therefore, the purpose of this study is to clarify the significance of statistics education within mathematics education and to find out suggestions for grasping a variety of students' inquiries from this significance.

To achieve this purpose, the author focuses on the viewpoint that mathematics is science of patterns because this is an adjusted viewpoint of mathematics in students at all levels. In addition, the author considers the implication which this viewpoint of mathematics contributes to mathematics education, and then discusses the significance of statistics education within mathematics education using this implication.

As a result, this implication is to conduct the objectification of methods. This is a qualitative jump of mathematical thinking that transforms from the method in order to inquire into an object to next inquiring object. Furthermore, the author proposes the statistical inquiry cycle as the objectification of methods (Fig. 1) according to Wild & Pfannkuch's PPDAC (Problem – Plan – Data – Analysis – Conclusion) cycle (1999). Moreover, the author constructs the framework for grasping the statistical inquiry process as the objectification of methods according to Pirie & Kieren's transcendent recursive model (1994).

Object	Phenomenon	Problem 1	Plan 1	Data 1	Analysis 1	Conclusion1	Phenomenon	…
Method	Problem 1	Plan 1	Data 1	Analysis 1	Conclusion1	Phenomenon	Problem2	…

Fig.1. The statistical inquiry cycle as the objectification of methods

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MEASURING PRESERVICE TEACHERS' GROWTH OF MATHEMATICAL KNOWLEDGE FOR TEACHING

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Research has shown that a teacher's mathematical knowledge for teaching (MKT) has a significant impact on student progress (Hill, Rowan, & Ball, 2005; Gilbert et al., 2010), which immediately suggests the possibility that improving preservice teacher's MKT may result in improved student learning. However, little research has been done measuring the level and growth of MKT in preservice teachers. This paper reports on a study of this growth using scores from the University of Michigan's Learning Mathematics for Teaching (LMT) instrument.

Schools are increasingly challenged to increase student achievement in the core subject area of mathematics. The purpose of this study was to increase the field's understandings of how this knowledge evolves during the preservice experience. The domain of MKT should be understood as encompassing not only common mathematical knowledge, but also the distinct subject matter knowledge that supports teaching. For example, why and how specific mathematical procedures work, how best to define a mathematical term for a particular grade level, and the types of errors students are likely to make with particular content. (Hill, et al., 2008)

Study participants were in a graduate licensure program at a public university in New England. Across three years and a total of eight classes, 139 preservice elementary teacher candidates took a pre-test on the first class day and the post-test during the last class. Class sizes ranged from 10 to 22 students. As an aggregate group, they showed a mean improvement of 15% from pre- to post-test, with the averages by class ranging from 37% to 0%. However, given the significant variation of the individual student scores, with one student increasing 73% and another student decreasing 21%, making inferences about causality would be premature.

As the connections between teaching effectiveness and MKT are better understood as a result of this research, teacher educators will be better equipped to prepare preservice teacher candidates to become effective teachers.

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A QUALITATIVE VIEW ON FIRST YEAR UNIVERSITY STUDENTS' ASSESSMENT PREPARATION

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There is evidence to suggest a high impact of assessment methods on students' assessment preparation strategies (e.g. Iannone & Simpson, 2014). This encourages the quest for assessment methods which foster deeper learning and understanding. In this regard the present study aims at giving an insight into the following questions:

Which learning strategies for assessment preparation do students report after their first closed book examination? Which consequences do these learning strategies have?

Empirical data to investigate these questions are taken from interviews which have been led with five pre-service teachers and two undergraduates with a major in mathematics which attended their second semester at university at that time. All these students had passed the closed book assessment of a (definition-theorem-proof based) analysis I course, which took place approximately two months before the interviews.

A first analysis of the data identifies two typical approaches to assessment preparation: (A1) to retrace and comprehend the structure of the lecture content or (A2) to gather, recapitulate and work through potential types of exam tasks. Usually these approaches occur in mixed form. There is a wide variety of materials (e.g. lecture notes, exercise tasks, tasks from previous exams, books, video lectures, Wikipedia, Google etc.) being used. The extent and purpose these materials are used varies according to the respective approach. The time spend on the assessment preparation is linked to the approaches in the sense that students who spend less time on the preparation often (need to) focus on tasks, while students who spend more time on the preparation often plan to do both, recapitulate the lecture content first and eventually lack the time to work through tasks. The exam performance of the reviewed students indicates that a certain amount of both approaches is necessary for good grades while none of the approaches outperforms the other. However, students who mostly rely on approach (A1) appeared to be more content and confident in terms of their learning progress than students who mostly rely on approach (A2).

Approach (A2) seems to be aligned to written examinations. However, it is not obvious whether in case of e.g. an oral assessment this approach would decline or be modified e.g. into gathering potential types of oral exam tasks.

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STRENGTHENING MATH LEARNING DISPOSITIONS

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The goal of the research was to understand whether (and if so how) participation in after school mathematics clubs, focused on active learner participation and sense making, could shift mathematics learning dispositions. A socio-constructivist perspective of learning as a social activity in which learners actively construct meaning and Kilpatrick et al.'s (2001) five-stranded definition of mathematical proficiency guided both the research and the club intervention. The 5th strand of productive disposition, "refers to the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (p.131). Kilpatrick et al. note that developing a productive disposition requires frequent opportunity for sense making. Classroom-based research in South Africa points to teaching foregrounding ritual participation, passive listening and little access to sense making (Hoadley, 2012). After school mathematics clubs were introduced as part of the South African Numeracy Chair project to support the development of mathematical proficiency with a particular focus on sense making.

The methodology combined qualitative and quantitative research methods across learners in clubs. This paper draws on qualitative data of one case study after school math club with six learners from four different Grade 3 classes who met weekly during 2012 at an after school care centre. Methods included club session observations, learner mathematics interviews and dispositional interviews (all video recorded and transcribed) and teacher questionnaires. This paper focuses on questionnaire data from the four teachers who taught mathematics to the six Grade 3 learners in the case study club (facilitated by myself). Questions asked teachers to comment on possible influences of club participation on the learners in their classes. Teacher comments pointed to learners shifting ways of participating and understanding and to increased: enjoyment (3/4 teachers referred to this), willingness to discuss methods (2/4 teachers), willingness to try math problems without fear of being wrong (2/4 teachers) and to 'increased confidence' (3/4 teachers). This data cohered with data of learners in the club thus indicating that clubs can provide opportunities for development of increasingly productive mathematics learning dispositions.

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INVESTIGATING STUDENTS' POSITIVE RELATIONSHIPS WITH MATHEMATICS

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For decades, mathematics education research has addressed the topic of students' emotional relationships with mathematics through studies of "attitude" and "disposition". However, this research tends to focus on students who have negative relationships with the subject area and often places students in a deficit model (Brahier & Speer, 2011). In this presentation, we will discuss a group of students that is under-examined in the literature – those who report having a positive relationship with mathematics (i.e., students who claim to like mathematics and/or feel confident in their abilities) – and identify factors that contribute to those feelings.

This presentation will focus on a portion of a large-scale, ongoing study that seeks to understand students' lived experiences learning mathematics in Canadian schools. The study is framed by enactivist theory, which emphasizes the interrelationship of cognition and emotion in learning and troubles the positioning of self and identity as static, individual phenomena (Varela, Thompson, & Rosch, 1991). As per our narrative inquiry methodological framework (Clandinin, 2007), the study's data include artistic renderings, autobiographical interviews, and written mathematics autobiographies. This paper draws on a dataset comprised of 94 autobiographical interviews with Kindergarten to Grade 9 students in the Canadian province of Alberta. Analysis included emergent and thematic coding of both the interview transcripts and associated drawings related to questions about participants' feelings about doing mathematics.

Our findings show that many of the younger students were unable to articulate the reasons for their positive feelings, but when offered, such reasons tended to relate to external validation, such as feeling "smart" and getting good grades, since mathematics was easy for them. In some cases, parental support was an important contributing factor to the participants' positive feelings. Interestingly, notions of mathematics as being intrinsically enjoyable were rarely provided by any age group. Our presentation will include a critical analysis of examples from the interviews and a discussion of how our findings might be applied to help increase the proportion of students who develop positive relationships with mathematics.

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STRATEGIES WITH PATTERNS IN EARLY CHILDHHOOD

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In this study a group of first and second grade elementary school students, eight boys and seven girls, were observed while working with muggle stones. The children worked in pairs which were formed in order to mix ages and abilities, and together they performed a variety of tasks related to repetitive and growing pattern sequences. Previous investigations in this area show that even young children are capable of identifying patterns which are repeated, and of describing and documenting structures. In addition studies by Warren et al. (2005) show that even elementary school children are able to classify patterns by analyzing their exact structures.

The method of videography was chosen for data collection. In order to textualize the video data an observation protocol was customized. In this way both the sound and the visual image could be reviewed in order to achieve an integral description encompassing both aspects (Dinkelaker & Herrle, 2009). The interpretation of the video data was supported by using a selection of individual images which clearly and visually emphasized the central moments of interaction (Moritz, 2010). The fact that the investigation was divided into different phases according to the tasks being undertaken, offered the possibility of doing an additional segmentation analysis. The thereby leading question was: Are 7-8 year old children able to recognize, re-create and continue repetitive and growing pattern sequences and if so, what strategies do they utilize?

The results show that repetitive pattern sequences can be recognized, copied and continued. With growing or complex repetitive pattern sequences, there were occasional problems. Recognizable solution strategies were as follows: 1. Distinguish the colour of the stones; 2. Focus on the positional relationship of the stones; 3. Repeat aloud the pattern sequence. In solving complex pattern tasks, the following procedures were observed: 1. The more complex the pattern the more was communicated; 2. The analysis of the positional relationships lasted longer; 3. The strategies employed when solving simple patterns were transferred to the complex patterns. The results provide teachers with a differentiated approach to recognizing and continuing repetitive growing patterns and pattern sequences in the school beginners' phase.

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HELP OR HINDRANCE? THE USE OF IPADS IN MATHEMATICS TEACHING AND LEARNING.

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The rise in the use of technological tools in society, including calculators and computers, has influenced the utilisation of these devices in classrooms. Instruments such as iPads have become common features of both primary and secondary classrooms throughout much of Australia. Although there is no clear-cut evidence that the use of technology is beneficial to students' learning outcomes, there are moves to encourage – and in some cases mandate – the integration of information and communication technologies (ICTs) into school education (Goos, 2009). Even with policy and curriculum changes, it is ultimately the role of teachers to implement the proposed recommendations.

A case study was undertaken to determine the impact of the use of iPads in the teaching and learning of mathematics and literacy in a range of year levels at one school, with a particular focus on gender differences in students' attitudes to the use of iPads in both subject areas. Participants consisted of teachers, students and parents from a non-government co-educational F-12 school located in Victoria, Australia. As part of a larger case study, interviews were undertaken with teachers, exploring their views on teaching with iPads. Students were also surveyed to obtain their views on learning with iPads. The findings guided subsequent classroom observations. Six mathematics classes were observed with the focus being student and teacher use of iPads.

As stated by Ward and Parr (2011), research suggests that teachers will enact policy when it is consistent with their values and beliefs. Although most teachers appeared to hold positive views about the use of technology in terms of their teaching practice, very little iPad use was observed; that is, teacher use of technology in the classroom was not always consistent with previously reported beliefs. In addition, use of iPads by students often supported the survey finding that students considered the device to be a distraction. In the presentation, issues and concerns related to the teaching and learning of mathematics with iPads will be discussed in further detail.

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HOW DO STUDENTS CONSTRUCT NEW MATHEMATICAL KNOWLEDGE DURING PEER INSTRUCTION – A CASE STUDY

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Mazur (1997) recommends the use of multiple choice questions (mc questions) in lectures as follows: The mc question is presented, students vote initially, discuss their vote for a few minutes with their neighbours (Peer Instruction or PI), and then re-vote before the correct answer and reasoning are presented. However some lecturers skip the PI, arguing that discussion with peers can only be successful if someone initially knows the correct answer and reasoning. They argue that in undergraduate courses, too few students have the required knowledge. In a survey, Smith et al. (2009) found 47% of the 328 undergraduate students felt it unnecessary that someone in the group initially know the correct answer. How can they find the correct answer and reasoning if no one explains it? A deep insight into the discussion process can be one step to find out. I want to reveal how students work together during PI and in which way they are able to construct new mathematical knowledge, especially if no one knows the correct answer. I recorded and transcribed the PI of different groups initiated by different mc questions. In my presentation I will focus on a small group whose discussion was initiated by a mc question with the target of training to distinguish and understand the meanings of "for all…there exists" (VE) and "there exist…for all" (AE) on a specific mathematical example. The learning progress of the conversation was interpreted and analysed with Steinbring's (2005) epistemology oriented methodology.

Although no one in the group had the correct answer and reasoning at the beginning, the group worked together cooperatively and could give a correct answer and reasoning after six minutes of discussing. Beside the exchange of different ideas, one major support for the learning process was the famous gradual generation of thought through talk. I will explain in detail how the students worked together and outline the learning progress. I will present some interpretations of the mathematical expressions and explain how they were eliminated. The case study shows how students are able to detect and correct errors during PI and how powerful PI can be to support students' learning processes. Further research is needed to identify ideal conditions for support.

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JOURNAL WRITING IN A MATHEMATICS METHODS COURSE: PRE-SERVICE TEACHERS' THOUGHTS, NEEDS, AND CONCERNS IN TAIWAN

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Mathematics writing has been used as an assessment tool. It allows instructors to get deeper understanding of students' ideas and views regarding mathematics learning, and modify classroom activities to meet their needs (Danielson, 2010; McCormick, 2010; Seto & Meel, 2006). The purpose of this study was to investigate pre-service teachers' thoughts, concerns, and needs concerning learning to teach mathematics, via journal writing.

Data were collected from 37 pre-service teachers enrolled in one 2 credit-hour elementary mathematics methods course during 2014 spring semester, taught by the author of this study. Most of the participants had no prior mathematics writing experience. The pre-service teachers were asked to write a mathematics journal each week. The foci could be their thoughts about activities in class, reflections on their past experiences, mathematics questions, mathematics teaching problems, questions for the instructor, etc. The instructor read preservice teachers' journal each week, and responded to their concerns frequently. A total of 550 journal entries were analysed using content analysis.

Results included the types of content (i.e., CK, PK, PCK) and the proportion of each type, and how the information was written in the journals. Initial results indicated that those who had experienced mathematics journal writing had more positive attitudes toward mathematics journals, compare to those who did not. Some pre-service teachers expressed that they would try to use mathematics writing as an assessment tool in their own classes. Other findings will also be reported.

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THE INTERPLAY BETWEEN DIAGRAM CHARACTERISTICS AND GEOMETRIC PROPERTIES GENERATED BY STRAIGHTEDGE-AND-COMPASS CONSTRUCTION

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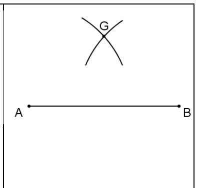
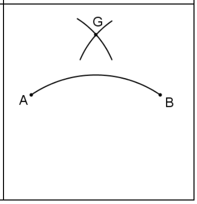
Duval (1995) distinguished four types of diagram apprehensions: perceptual, sequential, discursive, and operative. To function as a geometric diagram, according to Duval, requires activation of perceptual apprehension and at least one of the other three types. Activation of diagram apprehensions, especially the transformation among them, involves high cognitive demand which usually cause students difficulties. For example, it may be easy to identify a diagram by its appearance (perceptual apprehension) but challenging for students to recognize geometric properties (discursive apprehension) embedded in the diagram. The transformation between sequential and discursive apprehensions may be even more demanding.

Let A, B be centres and apply a proper measure to draw arcs by compass which intersect at point G. Please identify which segments in the diagram are of the same lengths.	
Let A, B be centres and apply a proper measure to draw arcs by compass which intersect at point G. Please identify which segments in the diagram are of the same lengths.	

This study examined the transformation among types of apprehensions generated by straightedge-and-compass construction. For example, drawing an arc by a certain measure can make it true that lengths from any point on the arc to the centre are equidistant. Particularly, we focused on investigating the extent to which diagram characteristics play a role in influencing students' recognition of the geometric properties created because of the straightedge and compass tools. To this end, we identified four diagram characteristics including radius, arc, centre of a circle and basis for drawing arcs; and designed pairs of survey items accordingly. The paired items on the right are an example demonstrating the way we examine whether the basis as a straight line or a curve for drawing arcs by compass makes a difference for students in recognizing AG=BG. About 300 8th grade students and 300 9th grade students answered the survey items. Analyses of students' responses on survey items show that diagram characteristics significantly influence students' performance in identifying the geometric properties created by straightedge-and-compass construction.

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MATHEMATICAL PROBLEM SOLVING AND USE OF EXAMPLES BY UNDERGRADUATE STUDENTS

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In mathematics education, requesting that students generate examples is particularly valuable. Zazkis and Leikin (2007) suggested that asking students to produce examples ‘‘provides a ‘window’ into a learner’s mind’’, and ‘‘mirror their conceptions of mathematical objects involved in an example generation task’’. This study deal with the activity of example generation as a *problem solving* activity, using a theoretical tool that was taken from Schoenfeld’s (1992) structuring of problem solving behaviour. This point of view makes the study of producing examples and the underlying problem solving processes meaningful.

The participants comprised 15 first-year undergraduate students at a university of technology in Taiwan who previously completed courses of derivative and definite integral. The questionnaire contained three false mathematical statements regarding differentiation and integration. The students were asked to determine the accuracy of the mathematical statements and justify their answers. The primary data sources were the written responses to the questionnaire and clinical interviews. The protocols were analysed based on the four categories of resources, heuristics, control, and belief.

The findings suggest that the factors the participant failed to solves problems include: mathematical intuition and prototype example hindered the constructing of counterexamples, too much focus on symbol representations, limited resources limit the emergence of self-control. Example generation is a complex problem solving activity. The study of processes of example generation reveals aspects that are important for reflection on both cognitive and metacognitive aspects of mathematical thinking. The richness and complexity of these processes has also been emphasized herein using the notions of prototypes, concept image, concept definition, visualization, intuition, example space, and belief. The insights into problem solving processes that are presented in this study will be helping in enabling researchers and teachers to regard example generation as a problem solving activity to be exploited in the construction of mathematical concepts. In the presentation further results will be discussed in detail.

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TEACHING IN CULTURALLY RESPONSIVE WAYS TO ACHIEVE EQUITABLE OUTCOMES IN MATHEMATICS

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A disproportionally large number of Pasifika students achieve well below that of their Asian and European fellow students in mathematics in New Zealand. Current research contends that attending to the cultural perspectives of such diverse learners is essential if they are to learn and achieve equitable outcomes. We know that the lived reality of the everyday life of Pasifika learners differs in significant ways from the cultural capital of the dominant middle class cultural groups reflected in many schooling practices in New Zealand. To this direction the current study draws on the theoretical framing of culture and cultural capital proposed by Bourdieu (1977) and adopts Gay's (2010) model of culturally responsive teaching as a way forward to address some of the inequities which emerge through the institutionalised practices. In particular the paper focuses on what happens when the language and culture of Pasifika students is drawn on to enact pedagogical practices which provide teachers with opportunities to teach mathematics in equitable ways. Particularly, the research project aims to explore the effect on Pasifika students' relationship with mathematics when teachers use Pasifika focused culturally responsive pedagogy.

The sample is comprised of 12 teachers and 345 students aged between 9 and 12 years. 97% of these students are of Pasifika ethnicity. Data included video recorded observations and teacher and student interviews. For three years the teachers had been involved in developing learning communities which celebrated student diversity as a strength and inclusion of cultural perspectives a required commitment. The findings indicated that when teachers intentionally engage home cultures as learning tools they empower students in multiple ways. The findings illustrated that using the Pasifika students' everyday cultural experiences and language in mathematical activity and connecting to the home and cultural values to shape prosocial behaviour explicitly supported them to construct positive identities as Pasifika learners. They also showed that their construction of positive mathematical dispositions was also supported. In summary, it is evident that when teachers seriously consider the cultural capital of students and adopt culturally responsive pedagogy it has a significant impact on their relationship with mathematics. In the presentation further results will be presented.

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ANALYSIS ON BEHAVIOURS OF USING CALCULATOR BASED ON DEVELOPMENTAL STAGE OF PROPORTIONAL REASONING OF GIFTED ELEMENTARY STUDENTS

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This study analysed eight 5th Grade gifted students' behaviour of using calculators based on the analysis of qualitative data in a class on direct proportion. Pretesting with questionnaire had been made to verify students' developmental stages of proportional reasoning, and the stage was categorized according to Baxter & Junker (2001). After the lesson, a transcript of the recording of the class was analysed with Guin & Trouche's (1999) model of calculator type. According to the result, types of behaviour of using calculator varied with each developmental stage of proportional reasoning and each type of the behaviour affected students' development of proportional reasoning differently. The result and implication are as follows.

First, types of behaviour of using calculator varied with each developmental stage of proportional reasoning, and those behaviours have different influence to the development of proportional reasoning. Therefore, the teacher should recognize that each student has different way of using technology and find the appropriate way for developing the proportional reasoning ability with each behaviour.

Secondly, this study has found that the knowledge of proportional reasoning is the important variable for determining the behaviours of using calculators in the process of solving problems by gifted students, and it was different from the process found Guin & Trouche's (1999) research which studied regular students. According to this result, calculators may play a role as an information tool with help from peers when the students are in small groups.

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AN EXPLORATORY STUDY OF PEER FEEDBACK DURING COLLABORATIVE MATHEMATICAL PROBLEM SOLVING

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Peer feedback is an inherent feature of classroom collaborative learning. Students, working in pairs or small groups, invariably turn to their peers for feedback when they work on a non-routine mathematical task. The feedback given is often unstructured and may positively or negatively influence students' learning as they complete the task. Hattie and Timperley (2007) in their model for feedback practice identified four possible uses of feedback, namely related to the task, process, self-regulation and personal.

In this study, we explored the function of peer feedback that prospective mathematics teachers (PMTs) offered to one another whilst engaged in collaborative mathematical problem solving. Six PMTs participated in the study. They worked in pairs and solved a non-routine problem collaboratively. The small group size allowed us to maximize the interaction between the PMTs. The dialogical discourse of the pairs was audio-recorded.

The main source of data for the study was audio transcriptions of the dialogical discourses. The transcriptions were coded in a step-wise manner. The transcriptions were analysed interpretively, with a focus on the peer feedback. Polya's (1945) four stages of problem solving, *understand the problem (UP)*, *make a plan (MP)*, *carry out the plan (CP)* and *look back on your work (LB)* were used to provide thematic boundaries and served as the units of meaning for segmenting the discourse for analysis. Specifically each PMTs pair's transcribed dialogue was/were coded by identifying an exchange usually starting with a PMT question or new idea related to the problem and followed by the other PMT's comment that led to feedback which may extend, modify or start a different line of thought. The data was coded by both the researchers independently and meetings between the two researchers followed to arrive at consensus for disagreements.

The data analysis resulted in a few findings. Firstly, only three of Polya's four stages of problem solving, *UP*, *MP* and *CP*, were utilised as units of meaning to segment the discourse for analysis. Secondly, the peer feedback moves comprising *Initiate*, *Response*, and *Feedback* focussed mainly on the task and process levels. Lastly, in all the three pairs of PMTs, the interaction patterns were different.

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AN INVESTIGATION OF THE CHARACTERISTICS OF FUNCTIONAL THINKING IN ELEMENTARY STUDENTS

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Functional thinking plays an important role in promoting students' understanding of early algebra (Blanton, Levi, Crites, & Dougherty, 2011; Carraher & Schliemann, 2007). In spite of the importance, most previous studies dealing with algebraic thinking have focused more on generalised arithmetic than on functional thinking. Some studies demonstrate that even young children are able to think functionally (Carraher et al., 2007) but these studies are limited to cases with small numbers of children.

Given this background, this study examined Korean students' functional thinking to explore instructional implications to promote their thinking through regular mathematics lessons. For this purpose, a functional thinking ability test was devised, dealing with both recognising and representing functional relationships with 30 items. It was then conducted with a total of 1254 students (405 fourth graders, 420 fifth graders, and 429 sixth graders) of randomly sampled schools across the country.

The results of this study showed that the students were good at solving the tasks dealing with additional relation and direct proportion relation, but not at inverse proportion relation. As for recognising functional relationships, students were able to find the correspondence values when near numbers were given, but had difficulties when the number were further apart because they did not recognise the relationships between two quantities. As for representing functional relationships, students were better at expressing functional situations in words than in equations. A noticeable finding was that students tended to confuse independent variables with dependent ones, and that they had difficulties in understanding functional relationships in figure patterns. These results were related to what and how students had learned with regard to functional thinking. On the basis of the results, specific instructional implications are suggested and discussed in terms of task design in curricular materials, as well as the recognition, representation, and employment of functional relationships in mathematics lessons.

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DEVELOPMENT AND APPLICATION OF SENIOR MATHEMATICS MATERALS USING SMART DEVICES

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This research aims to find out how effectively an educational program, which is a mathematical study program that involves continuous learning, can relieve mild cognition impairment of the elderly when it is implemented. Ahn, Lim, Lee, and Kim (2011), expected that when a cognition program is done by hand on a smart tool, it would effectively improve cognitive function based on the belief that the decrease in memory of elderly is caused by decrease in concentration which is affected by decrease in work concentration (Jansen & Keller, 1998). Moreover, playing computer games had a positive effect on the elderly's self-control. The self-control lessened their depressive mood but increased their satisfaction in life, showing that computer games can have an effect on mental health of the elderly through their self-control, reducing their depression.

The study is planned that, in the first year, an educational program with 10 levels (with 5 sub-levels) will be developed, and in the following year, the results and the effectiveness will be verified through interviews, questionnaires and brain wave examination. The program consists of ten phases, all of which are made up of games and daily-life situations that require an immediate response.

This research foresees that the elderly could recognise mathematics relevant to their lives, may show an increased interest in math-study, leading to a positive effect on their sense of competence and self-direction (FitzSimons, 2011). By means of these affective and cognitive effects, a reduction may occur in the elderly's depression and mild cognition impairment, and if we take into account that mild cognition impairment partially progresses onto dementia, math may be utilised as a method to prevent and cure dementia and work as a non-medical treatment.

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RELATIONSHIP BETWEEN CONCEPTUAL CONNECTION AND TYPES OF QUESTIONS THAT CAUSED CONTEXT-DEPENDENCY BETWEEN SCIENCE AND MATHS

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Science and Mathematics are taught separately in many countries, including Zambia. Students are expected to construct the meaningful knowledge by connecting science and mathematics. However, when students answered the same types of questions in the two different contexts of science and mathematics they got different answers. This is known as context-dependency. Although types of questions that caused context-dependency between science and mathematics have been recognised, little is known about relationship between students' characteristics and types of questions that cause context-dependency. Based on the above background, the objective of this study is to examine the relationship between students' conceptual connection and types of questions that caused context-dependency between science and mathematics.

One hundred and sixty-one students in Grade 12 at highs school level in Zambia were chosen for this study. Their academic performance was average among high schools in Zambia. The research was conducted by using the same two types of tests about functions providing different contexts between science and mathematics. The instruments were developed based on previous studies (e.g. Kosaka, 2013). The two tests were conducted on different dates in order to avoid any influence of the first test. A week later, 50 students from the 161 students were chosen for examining the conceptual connection. Eleven keywords related to the topic of function were chosen from students' science and mathematics textbooks. Then students were asked to create a conceptual map, based on the connections they made among the chosen keywords. The questions that caused context-dependency and students were classified by using students' response patterns. In each classification, the relationship between types of question and students' conceptual connection were discussed.

The result showed that the questions and students were classified into three categories based on the students' response pattern. In each category, types of questions that caused context-dependency and students' conceptual connection were different. It is concluded that as the students got different answers in the same two types of questions in the different contexts that they don't have enough conceptual connections between science and mathematics.

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EPISTEMIC FUNCTIONS OF GESTURES: RESULTS FROM AN EMPIRICAL STUDY

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When students construct mathematical knowledge with peers, the social interaction is shaped in a multimodal way by verbal, written and gestural expression and the use of artefacts. Recent studies have tended to focus mainly on what students make explicit in speech, neglecting the gestures' role as implicit modes of expression in social interaction. Therefore, this study aimed at investigating the role of gestures in social processes of constructing mathematical knowledge in order to enrich an existing epistemic action model (GCSt-model, Bikner-Ahsbahs, 2005) by a semiotic perspective. Since this model was based on speech act analysis, the epistemic role of gestures is added by asking 'How can gestures contribute to acting epistemically in social interaction?'

Three pairs of high-achieving students were videotaped, each working on three tasks related to three different mathematical fields. The epistemic processes have first been reconstructed based on the verbal utterances. Following an interpretation of the gestures in relation to other semiotic resources within developing semiotic bundles (Arzarello, 2006), epistemic functions of gestures were identified according to the research question. Based on the gesture analysis, the epistemic processes were reconstructed once more, assuming that utterances are not only verbal, but multimodal.

Gestures fulfill an ***epistemic function*** when affecting the accomplishment of an epistemic action. I identified epistemic functions of gestures in two categories: (i) Four *forming*-functions are related to the representation of the mathematical object involved in an epistemic action: Fulfilling a forming-function, a gesture can create visual access to this object and thereby encourage epistemic actions. (ii) Six *performing*-functions are directly related to the accomplishment of an epistemic action: The effect of the visual access created by using gesture is directed on performing an epistemic action.

The study has shown that the use of gestures constitutes a non-negligible part of social epistemic processes and provides indications for a possible didactic role of gestures in the mathematics classroom. For example, the students' use of gestures can disclose a potential for structure-seeing as central epistemic action. Furthermore, epistemic functions of gestures can be used to catalyze social epistemic processes.

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LESSON STUDY AS A VEHICLE FOR IMPROVING PRESERVICE TEACHERS' SKILLS IN CRITIQUING MATH LESSONS

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Lesson Study, a professional development approach which originated in Japan, is becoming increasingly popular internationally (Cajkler et al., 2013; Isoda, Stephens, Ohara, & Miyakawa, 2007). Through Lesson Study, teachers improve their ability to reflect on their teaching practices and critique mathematics lessons (Isoda et al., 2007). Employing Lesson Study and PSTs' ability to reflect on and critique mathematics lessons as dual theoretical lenses, this study focused on how six PSTs improved their abilities to critique mathematics lessons through the Lesson Study approach in the field experience. The research questions guiding this study were: (1) How do PSTs critique mathematics lessons in the earlier and later Lesson Study meetings? (2) Are the changes in PSTs' skills in critiquing mathematics lessons statistically significant? To address the research questions, videos of six junior elementary PSTs' mathematics lesson were analysed in terms of contents, subjects, rationales, levels of critique, and levels of suggestions by using a grounded theory methodology. Then frequency analysis and significance tests were applied to the coded data.

Findings suggested that the later videos of the six PSTs' Lesson Study critiques of mathematics lessons demonstrated significant improvements in these five aspects. (1) Contents: PSTs focused more on students' reasoning and instructional strategies (2) Subjects: PSTs focused more on critiquing students or the interactions between students and teachers. (3) Rationales: PSTs depended more on their own experiences and knowledge from university coursework rather than on facilitator or host teachers' comments. (4) Levels of critique: PSTs evaluated and interpreted mathematics lessons from various viewpoints beyond describing what they observed. (5) Levels of suggestions: the frequency and the levels of suggested alternatives in their critiques increased. Implications for developing PSTs' expertise in critiquing mathematical lessons in teacher education programs are discussed.

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EXPLORING FIFTH-GRADE TEACHERS' DIFFICULTY IN DEVELOPING ARITHMETIC PROBLEMS FOR CREATIVITY

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Today's world requires creative thinkers (Fryer, 1996; Jeffrey, 2001). Studies have been conducted in different domains to promote students' creativity (Craft, 2002; Pollard & Filer, 1996). Sriraman (2009) indicated that problem solving contributes to students' development of creativity in mathematics classes and it is important for teachers to develop mathematics problems that could develop students' creativity. Little, however, is known about teachers' difficulty, if any, in developing mathematics problems for creativity. Such information is essential because it could improve mathematics teaching for creativity and promote students' creativity.

The purpose of this study was to explore fifth-grade teachers' difficulty, if any, in developing arithmetic problems for creativity. The participants of the study comprised ten fifth-grade teachers. They were asked to develop arithmetic problems that could be used to develop fifth-grade students' creativity. Based on Sriraman (2009), non-routineness, heuristics and explorations-demanding were used to assess the developed arithmetic problems. Interviews were conducted with the teachers after they developed the problems. To analyse the data, the developed problems and the interview protocols were analysed separately and collectively to identify the teachers' difficulty in developing arithmetic problems for creativity. The findings of the study showed that explorations–demanding appeared the most difficult for the teachers to meet when developing the problems. The results of the study suggested that enhancing fifth-grade teachers' problem solving experience through explorations might contribute to their ability in developing arithmetic problems for creativity.

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THE ROLE OF CAS IN TRANSFORMING MATHEMATICAL REPRESENTATION

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This study examined mathematics classes using the CAS (Computer Algebra Systems) targeted for Korean high school first grade students. We examined what kinds of transformation of representations were used and what role CAS plays in the transformation of representations according to mathematics subject contents in this classroom. This study analysed 15 mathematics lessons during one month and the focus of analysis was on the classroom teacher. When analysing the transformations among representations this study mainly used theoretical frameworks such as the transparent and opaque representation described by Lesh, Behr and Post (1987) and the descriptive and depictive representation of Kosslyn (1994). According to the results of this study, CAS technology affected the transformation of representations used in high school mathematics classes and this transformation of representations improved the students' thinking and understanding of the mathematical concepts. CAS technology also provided the opportunity to create the representation of individual student. Such results of this study suggest the importance of CAS technology's new role in transforming of representations and they offer the chance to reconsider the fact that CAS technology could be used to improve students' ability of transforming representations at the mathematics class.

In the beginning of class using CAS, it provided the dynamic and visual depictive representation which was couldn't be found in the textbook. However, the teacher was trying to change it as the descriptive representation in the curriculum after all. Furthermore, even though the CAS might be helpful when it was used for the change between the depictive and descriptive representation, it is not helpful for all the change from the opaque representations of a mathematical concept to the transparent representations that the teacher means to teach. In other words, even though CAS may provide more chances for the representational change, the change to the transparent representation with definite concept still needs to be guided by the teacher's explanation. Thus, the transparent representation and the opaque representation are the types that appear in the process of teaching and learning, not the dichotomous classification of the external representation.

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IMPACT OF STATISTICAL GRAPHS ON THE READING COMPREHENSION OF INTERACTION EFFECTS

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Interaction effect is one of the important aspects of statistical literacy. While examining an interaction effect, it is necessary to identify the relationship between variables, which requires scientific thinking (Zohar, 1995). Reading comprehension of interaction effects is the process of integrating information in the text and graph, especially by reading a graph appropriately (Carpenter & Shah, 1998). Consequently, this study aimed to explore the impact of statistical graphs on comprehending the interaction effect.

The Eyelink 2000 was used to collect data from 60 adults. Experimental materials were "text only" and "text with graph" descriptions of an interaction effect. Additionally, a researcher-made test of interaction effect was used. ANOVA and sequential analysis were used to deal with research data. Results showed that the reading comprehension of the text with graph group ($M = 15.06$, $SD = 2.11$) was significantly better than the text-only group ($M = 12.55$, $SD = 2.67$) [$F(1,59) = 10.46$, $p < .01$]. The reading sequence of text and graph describing an interaction effect has been shown in Fig 1.

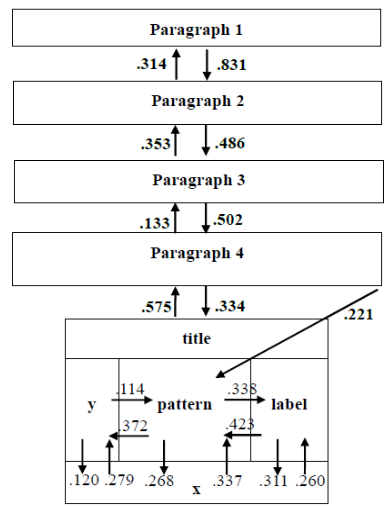

Fig 1 Reading sequence of the text and graph describing an interaction effect

It is suggested that a statistical graph functions as a spatial and semantic representation for readers. This helps them map surficial or propositional representations, fostering construction of mental models, which enhances them achieve reading comprehension efficiently. Additionally, the recognition of an axis and central pattern facilitated the confirmation and mapping of information in the graph unto propositional representations.

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RECONCILING OPPOSING FINDINGS IN THE INVESTIGTION OF METACOGNITION IN PROBLEM SOLVING

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Not many studies on metacognition used a multi-method design. While the survey inventory and retrospective self-report of the problem solving processes are familiar instruments used in research on metacognition, they are often used singly and researchers have expressed concerns in their reliability. This study explores and contrasts the use of these two approaches in data collection.

The paper reports a study that aims to examine the metacognitive strategies students employed during mathematical problem solving. Categories of metacognitive strategies at each phase of Pólya's four phases were adapted from a few related literature (e.g. Garofalo & Lester, 1985). The sample comprises 783 Secondary One students (age 13 years old) in Singapore. They completed both a survey inventory and a problem-solving test comprising 4 mathematics problems with retrospective self-report of the processes involved in each of the problems.

The survey inventory, with a 5-point Likert scale, consists of statements that describe various metacognitive strategies involved during problem solving. These statements were adapted and modified from a number of survey inventories (e.g. Wong, 1989). In the retrospective self-report, the metacognitive strategies identified were coded based on the statements from the survey inventory so as to facilitate triangulation.

Data from two sources were analysed: the mean value of the frequency of codes for each phase in the self-report and the mean value of each phase in the survey. From the survey, the highest mean frequency of student usage of metacognitive strategies is at Phase 4 ($M = 3.71$, $SD = .71$) while the lowest is at Phase 1 ($M = 3.46$, $SD = .59$). However, the results ran contrary to that from the retrospective self-report. The highest mean frequency of student usage of metacognitive strategies is at Phase 1 ($M = 2.14$) while the lowest is at Phase 4 ($M = 0.08$). In the presentation, the implication on the use of both data collection instruments in context will be discussed in detail.

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TRACKING THE PROFESSIONAL LEARNING OF PRIMARY TEACHERS' MATHEMATICAL REASONING

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Whilst every professional development program aspires to impact participants, measurement of their learning is problematic. Goldsmith et al. (2014) broadly defined teachers' learning as "changes in knowledge, changes in practise and changes in dispositions or beliefs that could plausibly influence knowledge or practice" (p.7). In this presentation we explain how we used the Primary Teachers' Perceptions of Mathematical Reasoning Framework (Herbert, 2014) to track shifts in primary teachers' knowledge and understanding of mathematics reasoning. Research has shown that many primary school teachers have limited understanding of the reasoning proficiency (Clarke, Clarke, & Sullivan, 2012). A total of 26 primary school teachers from four schools in Victoria, Australia and one school in British Columbia, Canada participated in an extended professional development program involving the use of demonstration lessons on mathematical reasoning in the primary mathematics classroom. The participants observed two demonstration lessons, attended pre- and post-demonstration lesson group discussions and taught each lesson in their own classroom. Three rounds of interviews were carried out providing data for analysis. The results showed that there were three main groups of teachers: 1)Teachers who had pre-established reasoning knowledge and understanding, 2)Teachers who demonstrated a shift in reasoning knowledge and understanding from less to more complex during the project and 3)Teachers who did not demonstrate shifts in reasoning understanding or knowledge. The shifts in the teachers were attributed to the following factors: a) opportunity to observe someone else teach the demonstration lessons b) participation in post-lesson discussions, and c) the teaching of modified lesson plans by teachers. Being able to track changes in teachers' perceptions of reasoning enables us to reflect on improvements to our professional development program to support teachers' knowledge and understanding of reasoning.

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TIMORESE GENERAL SECONDARY MATHEMATICS CURRICULUM: STUDENTS' PERSPECTIVES ON ITS IMPLEMENTATION

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In the aftermath of the 2002 Independence, Timorese education planning focused on the challenges of rebuilding schools, changing from an Indonesian curriculum into a new one, which required changing the language of instruction to Portuguese, and recruiting skilled teachers (Shah, 2012). A new curriculum for general secondary education (GSE) was implemented in 2012, which involved the development of didactic resources and teacher training - key elements in curriculum implementation (Tuwei, 2013). In 2014, the Timor Project [supported by the Portuguese Foundation for Science and Technology (FCT) (PTDC/MHC-CED/5065/2012), by the COMPETE Programme and by the European Regional Development Fund (FEDER)] was initiated with the aim of evaluating the impact of restructuring the GSE curriculum, to identify future considerations for improvement (Albergaria Almeida, Martinho & Cabrita, 2014).

This paper presents partial results from this ongoing project and focuses on students' perspectives regarding the implementation of the new Mathematics curriculum. Ninety-nine students from nine GSE schools participated in focus group discussions. Content analysis was used to analyse data pertaining to school conditions and infrastructures, textbooks and teachers' competences. Results showed that schools are ill-equipped, underfunded and lack (basic/manipulative) learning materials. The shortage of qualified teachers often resulted in inadequate service distribution and non-compliance with the number of weekly hours allocated to mathematics. Textbooks are clear and well organised, but language used demands a proficiency in Portuguese most students do not have, and they generally do not detail the resolution of tasks proposed. Teachers lack linguistic competences and continue to rely on textbooks from the Indonesian curriculum.

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MATHEMATICAL FORESIGHT: AN INTEGRAL PART OF AUTHENTIC MATHEMATICAL ACTIVITY

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Being able to "see ahead" in a task, by forming an image of a desired end goal and picking an orientation that is likely to reach that goal, has long been identified as a hallmark of expert-like behaviour. Mathematicians, from Poincaré (1900) to Pólya (1980), have identified this ability as intuition. Mathematics education researchers have attempted to place this ability on a more rigorous footing by casting it as a form of strategic knowledge. Though strategic knowledge has not been well defined in the literature, Weber (2001) presents a working characterization of this type of knowledge, based on (Greeno, 1973): "…heuristic guidelines that [effective problem solvers] can use to recall actions that are likely to be useful or to choose which action to apply among several alternatives." (p. 111)

We propose mathematical foresight as a way of conceptualising the particular ability to divine the end state of a mathematical task and a trajectory that leads from the present to that state. The notion of foresight is more rigorous than the gut feeling of intuition and less constrained than the construct of strategic knowledge, though the relationship with both is apparent.

To justify our conceptualisation of mathematical foresight as an object worthy of study, we present highlights from interviews we conducted with practicing mathematicians. All identified foresight as an integral step in approaching known or novel mathematical situations. Additionally, we present observations of student work that we interpret as exhibiting mathematical foresight. We conclude our presentation by speculating about how students' mathematical foresight might be further observed and how, as educators, we can construct experiences that develop our students' mathematical foresight.

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DEVELOPMENT AND VALIDATION OF INDICATORS OF SECONDARY MATHEMATICS TEACHERS' POSITIVE DISPOSITIONS TOWARD PROBLEM SOLVING

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Current practice recognizes mathematics teachers' positive dispositions toward problem solving and students' productive dispositions toward problem solving as significant, beneficial factors in teaching and learning. This report describes the application of Wilkerson and Lang's (2007) Disposition Assessment Aligned with Teacher Standards (DAATS) model to develop indicators of teachers' positive dispositions toward problem solving and an assessment framework to distinguish among teachers who exhibit more positive (or negative) dispositions toward problem solving. This study produced four major results. First, participants reached consensus on the purpose, use and content of an assessment framework. Second, participants generated five broad indicators of positive disposition toward problem solving: (1) The teacher values worthwhile and mathematical rigorous problem solving; (2) The teacher values diversity of students' explanations, ideas and observations about mathematical problems; (3) The teacher values a risk-free problem-solving environment that ensures success of all students; (4) The teacher values the use of a variety of tools including technology to solve problems; and (5) The teacher believes that students must demonstrate a productive disposition toward problem solving (i.e., confidence, interest, appreciation, enjoyment, and perseverance). Third, participants' generated typical teaching behaviours for each indicator at each of the five taxonomic levels of the Affective Domain (Krathwohl, Bloom, & Masia, 1956). Fourth, participants developed an assessment framework that correlated indicators with methods for measuring teacher affect (i.e., disposition) at varying levels of inference. The results of the study also suggest items and instrumentation for assessing teachers' dispositions toward problem solving. An assessment framework of this type could serve the accreditation, certification, professional-development and research goals of individuals working in the field of mathematics education.

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MATHEMATICAL CERTAINTIES: OBSTACLES OR DRIVERS FOR COMPREHENSION?

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Several studies have acknowledged that certainty and other epistemic states, such as doubt, have a bearing on the mathematical comprehension that arises in educational settings. The purpose of the research is to contribute evidence that supports the conjecture according to which epistemic states experienced by students with respect to mathematical proposals can act to curb progress of some of their knowledge, but at the same time acting as a driver in the development of other knowledge. In order to suggest the possible epistemic states experienced by students around mathematics proposals, as well as their understanding, the Toulmin Model was applied and theoretico-methodological instruments were designed as of the successive triangulation among social categories (empirical data), those of the interpreters themselves and those of other authors (Bertely, 2000). The qualitative research outlined focuses on an interpretative-type case study and said research was carried out in a Diploma Program where the purpose was to strengthen training of people who teach algebra to adults. The Diploma Program was given remotely using the Moodle platform, so interaction among the students and a tutor was recorded. The tutor proposed and guided the activities related to the study themes. A fragment of the interaction was chosen for this report. In that fragment, the tutor took a teaching proposal associated with Model 3UV (Ursini, Escareño, Montes & Trigueros, 2005) as reference, presented a problem situation, explicitly asked the students for the two unknowns involved in the problem, and asked them for the equation system that solved it. However, Jeymi –the case study, concluded with certainty that the problem only involved one unknown on the basis of an incorrect warrant (according to Model 3UV), according to which if there are two unknown related data, then there is one unknown quantity. As such, the student's certainty with respect to the existence of one single unknown acted as an obstacle for her being able to come up with a system of equations, albeit it also acted as a driver in her ability to identify the relationship among the variables. The foregoing shows that certainty of a falsehood is a constructive element for certain areas of mathematics activity, yet at the same time it is an impediment for progressing in others. The foregoing contributes evidence to sustain the conjecture raised.

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TEACHING PRIMARY MATHEMATICS THROUGH MULTIPLE METAPHORS AND MULTIMODALITIES

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This paper explores the development of a conceptual framework using a socio-cultural (Vygotsky, 1933) and social semiotic lens (Lemke, 1990) incorporated into a research study in Western Australia. Sfard's (2008) description of a realisation tree to learn abstract mathematics underpins the framework. The conceptual framework also outlines how semiotic resources are an important learning tool to be incorporated in these learning experiences (Lemke, 2002) and that multiple metaphors and multimodality are central components of mathematical semiotic resources. Lakoff and Nunez (2000) argue that there are four main metaphors used when teaching number and arithmetic. It is becoming apparent that a single metaphor is not sufficient in order to develop a rich understanding of mathematics and consequently multiple metaphors are required. An effective teaching process also often uses more than one modality, such as gestures, visual and verbal (Sfard, 2008). Through engaging in multimodal and multi- metaphoric experiences, the child develops an understanding of the mathematical object (Ernest, 2010; Sfard, 2008).

Research involved a case study of a Year 2 teacher and six Year 2 students to investigate the question: How are multiple metaphors and multimodalities used by the teacher and children when interacting in the mathematical learning tasks designed to teach computational strategies? The collaborative study took place over two terms in the teacher's classroom with multiple data collection methods including the filming of mathematical learning experiences. In order to answer the research question the data analysis will involve both inductive and deductive methods. It is expected that the data will reveal how students engage with a combination of modalities and metaphors so that they may begin to understand the commonality between these and therefore begin to develop a robust mathematical understanding of Year 2 computational strategies.

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UNDERSTANDING FUTURE TEACHERS' INTENDED INSTRUCTION AND ASSESSMENT OF SLOPE

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This case study extends prior research and accepted theories of understanding to consider how future secondary mathematics teachers understand and plan to teach slope. Using the components of slope (*constant ratio*, *determining property*, *behaviour indicator*, *trigonometry*, and *calculus*) outlined in the Slope Conceptualisation Network (Nagle & Moore-Russo, 2013), the researchers study how three future teachers explained their own conceptions of slope and how they addressed slope in their planned lessons and assessments. The researchers used 2x2 descriptive matrices for each slope component, to focus on whether the future teachers employed a conceptual or procedural emphasis (Rittle-Johnson, Siegler, & Alibali, 2001) and if their approaches were visual or analytic (Zazkis, Dubinsky, & Dautermann, 1996). In short, this study considered the following research questions:

- How do future teachers describe slope (when asked about what it is, how it is represented, and how it is used)?
- Do future teachers' descriptions of slope align with their intended instruction?

The cases under investigation involved three future secondary mathematics teachers who had been recognised by their peers as submitting the "best" plans for lessons assessments in their methods course. The data used for the study included: responses to a paper-and-pencil questionnaire, unit and lesson plans designed to introduce slope, and the related assessments included in the unit plans.

Results suggest that future teachers: (a) may not have a connected understanding of slope and (b) may not always align their initial individual descriptions of slope with their intended instruction and assessment.

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WHAT DOES IT MEAN TO SAY A TASK IS "DIFFICULT"? PLAIN QUESTION, COMPLEX ANSWERS, SPECIFIC FINDINGS

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This paper revisits an ever central problem: What does it mean to say a mathematical task is difficult? As simple as the question seems, the answers are complex. However, some specific empirical findings do exist.

The first, but obvious answer to this question is: Difficult is what only a few can do, and easy is what many can. This is the logic of the "big studies", and it is right as long as one realises the limitations, and makes the necessary differentiations. There are aspects to be considered when thinking about "difficult", and some not so well-known findings that these aspects indeed play a role. "Being difficult" depends on:

the modes of thinking mathematically:
There is no universal way of making a task more difficult. There are different features in procedural tasks vs. conceptual thinking tasks. Findings come from item analyses of the PISA-Study in Germany.

the traditions and cultures of teaching:
This aspect is neglected, as long as one copes with tests only in the horse-race-way. On the level of the items however, striking differences occur (Neubrand, 2013).

the inertia of changing systems:
What a reform movement intends is not promptly mirrored in the related tests, as a detailed analysis of the solving processes of students' in high-stakes tests reveals.

the implementation of the task in the classroom:
A task can change its character when embedded in the class. Teachers' professional knowledge is a contributing factor, as case studies show.

the sensibility during the construction of tasks:
Tasks are sensible to smallest changes in the text and the numbers used. However, only a precise inspection of the inner implications of a task can disclose these effects.

A compendium of the various features of mathematical tasks, and the relation to the question of difficulty, should be on the agenda of mathematics education research.

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WHY DO UNDERGRADUATES CHOOSE TO SERVE AS MATHEMATICS MENTORS/TUTORS?

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Current research examines how out-of school learning environments such as near-peer mentoring/tutoring is helping underachieving students meet grade-level standards in mathematics. Despite the lack of pedagogical training by mentors/tutors, there are reports that they are able to help underachieving students in learning mathematics (Walker, 2012). However, not much is known about why near-peers choose to serve as mentors/tutors. One reason noted in the literature is that mentors/tutors ('mentutors') make use of their own learning experiences by bringing in their entire autobiographies such as, their values, beliefs, attitudes, hang-ups, and hopes in supporting their mentees/tutees ('mentutees') (Nieto, 2003). We extended Nieto's (2003) framework to examine reasons why two undergraduates decided to serve in a cascading mathematics mentoring and tutoring program known as Prepare2Nspire (P2N).

The following research questions guided the study: 1) what is the nature of mentutors' mathematics learning experiences, 2) how do mentutors' beliefs, hang-ups, values, and hopes influence their mentoring/tutoring practices? The two cases were selected using criterion sampling informed by mathematics learning experiences. A multiple case study design was used to look both within-and-across-cases. A semi-structured interview lasting about 30 minutes was conducted with each of the mentutors. Their responses were treated as narratives and coded for themes using a deductive approach informed by Nieto's framework of autobiographies.

The statement by Nieto (2003) that those who teach do not leave their values, beliefs, and hopes behind them at home before setting foot into their classrooms seems to hold in the case of these mentutors. Results show that these mentutors seemed hung-up on their bad experiences in mathematics which influenced their mentoring and tutoring practices. Valuing the importance of learning communities to one's learning, the mentutors joined the program in order to build such support systems to help mentutees. They, therefore, went the extra mile to adapt some of the learning materials prepared by the program staff to better serve their mentutees' needs and knowledge gap. They held high expectations for all their mentutees with the belief that with the necessary support they can succeed. Subsequent studies will look at how mentutors supported the development of new mathematical identities for their mentutees.

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PROSPECTIVE HIGH SCHOOL TEACHERS' USE OF A DYNAMIC GEOMETRY SYSTEM TO COMPREHEND AND REASON ABOUT FUNCTION ESSENTIALS

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The aim of this study was to analyse and characterise ways in which prospective high school teachers rely on a Dynamic Geometry System's affordances to represent and reason about mathematical tasks in a problem-solving environment. Research questions that framed and guided the development of the study include:

What ways of reasoning do prospective high school teachers construct and exhibit when they systematically use a dynamic geometry system to represent, explore and solve mathematical tasks?

To what extent do the participants reconcile visual or empirical results and conjectures with the search for and presentation of mathematical arguments?

Eight students who have completed a Bachelor of Science in Mathematics participated in a problem-solving course during one semester. The activities and tasks included problems addressing big ideas and essential understanding of functions such as the function concept, co-variation and rate of change, family of functions and the role in modelling, combining and transforming functions, and multiple representations of functions. The conceptual framework incorporates two related fields: A problem solving approach that emphasizes learners' goals, orientations and the decision making process involved in dealing with mathematical tasks; the process involved in learners' transformation of an artefact into an instrument or tool to solve mathematical problems (Santos-Trigo, 2014). In this context, we focus on analysing and discussing what the participants showed when they worked on a task that involves making connections between linear and quadratic functions. The GeoGebra's affordances were essential for participants to make sense of the task statement that led them to think of task parameters (slope and y-intersect) in terms of sliders. So, they reasoned about the problem via a dynamic representation that provided useful information to explore possible patterns and relationships between the linear functions and its product. In this process, the participants formulated and pursued a set of conjectures which were validated via analytic and geometric arguments.

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A CULTURAL PERSPECTIVE ON VISUOSPATIAL REASONING: AN OVERVIEW OF 40 YEARS OF RESEARCH

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By studying a large number of mathematical activities in Papua New Guinea (PNG), it became evident that people were visuospatially reasoning. Literature, research reports, questionnaires, and observations of activities described by participants and schooled family members showed visuospatial reasoning influenced decision-making. Ethnomathematics in PNG is strong (GLEC, 2008). The research question was to find out how people thought in mathematical activities. Besides my own lived experiences in many villages and earlier classroom research, the study drew on over 10 years of ethnomathematics projects at the University of Goroka, PNG, questionnaire data about measurement and space, and over 15 in-depth interviews often in villages, covering a wide range of practices in a wide range of ecologies and 350 cultural groups.

From these grounded-theory studies, it emerged that cultural identity was often an affective trigger for teachers to identify with mathematical thinking when given the task of reporting on mathematics in culture (Owens, 2014). One aspect of this cultural mathematical thinking was the degree of visuospatial reasoning occurring in practice. Visuospatial reasoning is apparent in designs distinct to different cultures including continuous lines, symmetries, similarities, and angle sizes occurring in carvings, string bags, painted faces and bodies and woven walls. Visual equivalents to ratios are regularly used for determining angles and associated lengths associated with, for example, the slopes of roofs; areas of grass or plant size are related visuospatially with the size of roofs or lengths of rope or the products for which it is used; lengths are used to discuss volumes of pigs, for example; aspects of canoe shapes are visualised; and visuospatial imagery is used even for creating right-angled and equilateral triangles. Floor areas are associated with ratios rather than measures (Owens, 2015).

Cultural practices, spiritual links, relationships, shared activities, and ecology were referred to by the participants and literature. School experiences should harness the strengths of visuospatial reasoning and heed these related aspects.

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ANALYSIS OF MATHEMATICAL PROCESSES IN ELEMENTARY MATHEMATICS CURRICULA

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Mathematical processes as well as content are significant because they play a crucial role in the quality of students' mathematical learning. Given this, most mathematical curricula specify not merely what students are expected to learn through school mathematics, but also how they would learn it. Recent international studies of mathematical achievement such as TIMSS and PISA, tend to assess mathematical processes. The cognitive domains of the mathematics assessment framework for TIMSS 2011 include *reasoning* which goes beyond *knowing* and *applying* to encompass non-routine problems and complex situations (Mullis et al., 2009). In a similar vein, the analytical framework for PISA 2012, elaborates the relationship between mathematical processes and fundamental mathematical capabilities (OECD, 2013). Although these frameworks reflect the significance of mathematical processes, they differ in specifying what such processes are.

This study analysed mathematical processes elaborated in the mathematics curricula of Korea, China, Japan, and the U.S. Ten mathematical processes were extracted: (a) learning of concepts, principles, laws, and skills; (b) problem solving; (c) reasoning; (d) communication; (e) representation; (f) connections; (g) creativity; (h) character-building; (i) self-directed learning; and (j) positive attitude towards mathematics. This study specified the meaning of such processes and their sub-domains, noticing similarities and differences among the curricula. For instance, the mathematics curricula of China and Korea emphasise mathematical creativity, whereas those of Japan and the U.S. do not. More importantly, mathematical creativity is regarded as a generic concept of problem-solving, communication, and reasoning in the Korean curriculum, whereas it is related to self-directed learning in the Chinese curriculum. These similarities and differences urge us to contemplate what students are intended to learn in school mathematics. Subtle but meaningful differences are expected to provoke discussions among mathematics educators and to promote a more elaborated conceptualization of each mathematical process.

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A STRUCTURAL ABSTRACTION POINT OF VIEW ON A STUDENT'S PARTIAL (RE-) CONSTRUCTION OF THE LIMIT CONCEPT OF A SEQUENCE

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Consistently finding that students do not grasp the whole meaning of a particular mathematical concept highlights the need to pay particular attention to partial constructions of the concept rather than trying to draw a sharp line between whether an individual has (or has not) constructed a mathematical concept. The dual nature of the structural abstraction framework (Scheiner, 2013) provides appropriate lenses for investigating students' partial constructions, as it dually considers the (mental) structuring of meaningful aspects and the underlying structure of specific objects, and the restructuring and expanding of the knowledge system. Structural abstraction aims at complementarity (rather than similarity) of diverse aspects that creates conceptual unity among objects falling under a particular concept. This presentation uses the theoretical lenses of the structural abstraction framework to revisit the partial (re-) construction of the limit concept of a sequence by a mathematics major student (Pinto, 1998). The student's visual/verbal representations of a convergent sequence remain related to a descending curve and are coherent with his partial written reconstruction of the formal definition of limit. Although recognizing and describing properties, he fails complementarizing as he is unable to restructure them, not as observed properties, but as a formal definition. The phenomena appear related to how the representations of the concepts are inserted in his mathematical discourse – as representations for restructuring and expanding the knowledge system. In the case under study, many statements are interpreted as self-evident, and, although meaningful for him, are conflicting with the formal definition. The complementary power of the formal definition (Scheiner & Pinto, 2014) is overshadowed, hampering the structural abstraction process as a whole.

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STUDENTS' EMERGING REASONING ABOUT LARGE-SCALE DATA TABLES

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The open data movement makes unprecedentedly available a large-scale of authentic data sets on a wide range of social important topics. Competent use of large-scale data predominantly requires comprehension of tables and other visual data representations. In this paper, we examined tabular understanding of a group of thirty-two 15-year-old students in Cyprus, while they were investigating tables of real data published by EUROSTAT, the statistical office of the European Union. Participants investigated trends in real data displayed in tables, and reflected on the components of population change in several EU countries, including natural increase of population, net overseas migration, and total population growth. Our analysis of the participants' reasoning indicate that four levels of data-table comprehension, in line with Shaughnessy's (2007) framework for smaller data, emerged:

- *Level 1 (reading data):* Simply reading data displayed in a two-way table either horizontally or vertically to answer explicit questions for which the obvious answer is in the data-table, without making any judgements with regard to comparing any variations in growth rates among different countries.
- *Level 2 (reading within data):* Making comparisons of data between different countries and within a country for different years. Attending to one or more relevant aspects of the data, but failing to integrate the aspects in their context.
- *Level 3 (reading beyond the data):* Interpreting and attempting to contextualise the data by providing qualitative interpretations of possible social, historical, environmental, economic or political factors that might have impacted on the variation in data values.
- *Level 4 (reading behind the data):* Looking for possible causes of trends observed in the data by connecting the information provided in a table to its context. For example, being able to identify which indicators of population change in Europe (e.g. number of live births, crude rate of population change, immigration, emigration, population by citizenship-foreigners) correspond to European countries badly affected by the economic crisis.

Our findings provide a 4-level theoretical framework for investigating learners' understanding of large scale data-tables that can be used by statistics teachers as a guide for helping students enhance their understanding of large scale data-tables.

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ANALYSIS OF TEACHER MATHEMATICAL KNOWLEDGE FOR TEACHING CONNECTION WITH THEIR BELIEF ABOUT SCHOOL MATHEMATICS

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Teacher mathematical knowledge for teaching (MKT) (Ball et al., 2008) is one of the most promising perspectives for developing teachers and improving student learning. However, it is strongly criticised as it does not acknowledge teacher beliefs. Researchers note that teacher beliefs about the nature of mathematics and the teaching and learning of mathematics are strongly related to teaching and learning. Beswick (2012) claims that teachers regard school mathematics differently from mathematics as a discipline, and based upon the claim, she proposed a framework combining beliefs about the nature of mathematics as the discipline and the school mathematics. This research aims to analyse secondary school teacher MKT connecting with their belief about the nature of school mathematics with a particular reference to the plane shape concept in secondary geometry which is one of the problematic areas of secondary mathematics in context where the research is carried out. Teacher MKT is measured by a paper-and-pencil test; in order to dig into the teacher responses, based on results of the test, teachers are interviewed. Teacher belief is investigated using a questionnaire with Likert format items and followed by a structured interview to support analysis of the connection between MKT and the belief.

The sample comprises 34 secondary school teachers in Mongolia where mathematics content is highly emphasized in school mathematics and teacher education. Data are analysed using combination of canonical correlation analysis and thematic analysis. Findings indicate that in general, Mongolian secondary school teachers' MKT is characterised by the common content knowledge and knowledge of curriculum and content, yet lacked pedagogical content knowledge of geometry. Mongolian teachers' MKT is consistently connected with their belief that "school mathematics is part of a body of hierarchical interconnected knowledge which forms the basis on which some will learn higher level mathematics" (Beswick, 2012). It also identified that Mongolian secondary school teacher MKT was strongly influenced by secondary mathematics curriculum, thus it is wise to study further how context is related to teacher MKT.

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PRESERVICE SECONDARY SCHOOL MATHEMATICS TEACHER EFFICACY: FOUR MINI-CASE STUDIES

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What narrative emerges from an in-depth exploration of what four preservice mathematics teachers' (PMT) think contributed to their teacher efficacy as they look back on their teacher preparation experiences? What insight comes from exploring potential meaning through a lens of teacher efficacy (Tschannen-Moran & Woolfolk Hoy, 2001), locus of control (Guskey & Passaro, 1994), teacher concerns, and orientation to teaching? How do their words and stories draw attention to a difference between self-efficacy and teacher efficacy? Do they lament about their concerns of survival in the classroom; do they privilege their mathematical knowledge?

Four Preservice Mathematics Teachers (PMTs) became a mini-case study within a larger study of 36 PMTs' teacher efficacy and the contributing aspects of the teacher preparation program. A two-phase sequential mixed methods study was used. In the first phase, all participants completed a questionnaire with two efficacy scales (one for teacher efficacy [te] and one for locus of control [loc]) and short answer questions. Program document collection and interviews occurred in the second phase. The larger study suggested teacher concern and teacher orientation align with teacher efficacy in a complex manner that is not linearly related but nested, and the distinct contexts of practicum and teacher preparation courses as learning spaces were important to the PMTs' sense of what they learned and to changes to their teacher efficacy.

Four participants were selected for more in-depth study because of the interplay between high and low *te* and *loc* values. The four mini-case study PMTs offered insight into the complexities of the sources of *te* and perceptions of professional practice. PMT_1 expressed high *te* and high internal *loc* with concerns of the impact on student learning and orientations of caring for students' emotional well-being to inform pedagogical decisions. PMT_4 expressed low *te* and high external *loc* with survival concerns and orientations that privileged knowing mathematics in a highly control-focused and custodial classroom practice. PMT_2 and PMT_3 offered perspectives of high and low *te* (respectively) but different expressions of higher external *loc*. As they discussed their professional practice and the contributions to their *te* it became apparent how important and influential personal context was.

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THE IMPACT OF AN ELEMENTARY MATHEMATICS SPECIALISTS PROGRAM: EXAMINING STAGES OF TEACHER LEADERSHIP

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In 2010 the Association of Mathematics Teacher Educators (AMTE) developed the Standards for Elementary Mathematics Specialists as a guide and support for states to develop their own Elementary Mathematics Specialists (EMS) programs and certification standards. (AMTE, 2013) To date, more than half the states in the U.S. have, or are in the process of developing, EMS programs. AMTE's vision included the development of strong mathematical content knowledge for elementary teaching, pedagogical knowledge for teaching, and leadership skills needed for coaching and mentoring. In 2008 the National Council of Supervisors of Mathematics (NCSM) developed a framework for leadership to provide a vision of what "ought to be." The NCSM called on mathematics educators and teachers of mathematics to not settle with the current state of mathematics in the U.S. In response to these calls for reform, this project focused on the development of leadership skills of teachers seeking certification as an Elementary Mathematics Specialist program in one state in the U.S.

Sixteen experienced elementary teachers completed 18 hours of required graduate coursework specifically developed for EMS certification along with a 30 hour field experience. Qualitative and quantitative data were collected both prior to and following the completion of the program requirements focused on the participants' notions of leadership and their leadership skills, abilities and activities. The NCSM PRIME Leadership framework of leadership of self, leadership of others, and leadership in the extended community were used as categories for data analysis.

The findings of this study revealed that while the majority of the participants began the program exhibiting qualities of leadership of self, few had beliefs about leadership and leadership activities that extended to the third category, leadership in the extended community. The findings further revealed that the program was effective in moving participants along a continuum of leadership, such that most participants now exhibit, both in their beliefs and in their leadership actions, leadership in the extended community.

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COLLABORATIVELY PLANNING AN INQUIRY-BASED MATHEMATICS LESSON WITH PRE-SERVICE TEACHERS

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There are many components that need to be integrated into an elementary mathematics methods course in order to improve pre-service teachers' (PSTs) mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). One aspect is developing the PSTs' skills in planning inquiry lessons that encourage students to justify, explore and articulate their ideas. This project was an attempt to integrate several meaningful components in an elementary mathematics methods course while modelling the lesson planning process, which involved developing higher-level objectives and activities. Using a modified lesson-study approach, the research intervention involved first planning a lesson as a whole methods class, and then allowing the PSTs to observe the lesson taught to children. This project sought to examine how PSTs develop in their abilities to create higher-level objectives and activities to address specific mathematics standards, and to characterise the PSTs' objectives and activity ideas before and after the intervention.

Data were collected with two sections (32 students) of the elementary mathematics methods course. A questionnaire was given before and after the intervention activity. The data for this project were analysed qualitatively using open coding. The data for the questionnaires were coded using the framework of the "Cognitive Rigor Matrix" (CRM) (Hess, Carlock, Jones, & Walkup, 2009).

The majority of the pre-intervention responses were in the upper left corner of the CRM. This concentration of responses suggests that the PSTs were predominantly thinking of memorisation and procedural tasks that do not require higher-level thinking. After the intervention, the responses reached further into the CRM in a direction that is characterised by applications, explanations, and justifications.

Based on the results, the intervention had a positive and beneficial impact on the PSTs' mathematical knowledge for teaching and their ability to develop higher-level activities and objectives. Specific results and implications will be shared.

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TO GIVE OR NOT TO GIVE: DILEMMA THAT UNDERLIES AN EXISTENCE PROOF

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Using elements of Sáenz-Ludlow and Zellweger's (2012) proposal, based on Peirce's triadic Sign theory, for interpreting meaning-making in the classroom and teacher semiotic mediation, our research centred on pre-service mathematics teacher's meaning-making of the procedure needed to prove a theorem, specifically, an existence theorem. To prove the existence of a geometric object that satisfies two or more properties, meaning-making implies recognising the necessity of constructing an object that fulfils some of the properties that implicitly contain elements that assure the other needed conditions. Underlying the appropriate use of this procedure is understanding that definitions carry with them only potential existence; that the procedure, proposed by students, of Randomly Choosing a geometric element and then Forcing it to satisfy a particular Property (RCFP) is not valid; that starting the proof with an object that has some of the required properties is acceptable if each given property can be justified with elements of the theoretical system on hand. For example, to prove the existence of the midpoint of a segment, students pick any point between its endpoints, and then force it to be equidistant from them, lacking, for this, the corresponding theoretical warrant (RCFP). The proof requires proceeding exactly the opposite way: first theoretically establishing there is a point that is equidistant from the endpoints, property from which being a point between them emerges in a valid way.

As research has shown, difficulties to learn to prove are many and of different nature. But we have not found mention of RCFP in the literature. Selden and Selden (2011) organize the difficulties into distinct structural topics related to proving. Within their category *use of logic to construct proofs, understand the statement's structure, and use quantifiers*, they mention two difficulties that are somewhat related to the one we want to expose in this article: knowing (i) when to pick an element that belongs to a certain set or when to construct it, and (ii) how to prove a theorem whose thesis is a disjunction. Our analysis led us to determine what could be the cause of RCFP.

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FACTORS CONTRIBUTING TO CHILDREN'S FINANCIAL PROBLEM-SOLVING AND DECISION-MAKING

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This short communication reports findings from a research project exploring the potential of open-ended mathematical problems situated in realistic contexts (termed "financial dilemmas"). The research aimed to reveal insights into the nature of 10-12 year old students' developing financial literacy, the assumption being that such insights would have implications for the teaching of "Money and financial mathematics". The research questions were:

What are 10-12 year students learning about money from their parents?

Do these understandings influence financial problem-solving and decision-making? If so, how?

Eight parents and their Year 6 children were interviewed to assemble case studies of within-family financial literacy teaching and learning. The parent interview involved an open conversation about money and family life so as to uncover the parents' financial attitudes, values and behaviour. The student interview explored the children's observations about and experiences with money through a series of financial dilemmas. Data included audio recordings of the interviews and students' mathematical working. The theory of planned behaviour (Ajzen, 1991) was used to guide data collection and analysis. Within-family data sources were matched, analysed, and categorised with a view to understanding the students' family background, the nature of the social understandings about money they brought to school from home, and the ways in which these seemed influence their financial problem-solving and decision-making.

The findings highlighted the significant role of parents as their children's first financial literacy educators. The findings also confirmed the contribution of sociocultural and psychological factors to children's financial problem-solving and decision-making. While the participants were from different family and socioeconomic backgrounds, the importance of working to earn an income, living within one's means, and saving for the future were values that they shared. These values were so powerful that the student participants tended to rely on them when tackling financial dilemmas, often avoiding engaging in mathematical calculation or reasoning.

The findings were used to inform the development of an educational intervention designed to elevate the perceived importance of mathematical knowledge and skills to informed financial problem-solving and decision-making.

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DISCUSSING STUDENTS' THINKING AND LEARNING IN POST-LESSON DISCUSSION IN LESSON STUDY

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Lesson study by a group of teachers is a particular form of professional activity that includes planning, implementing, and discussing actual lessons to improve teaching and learning (Fernandez & Yoshida, 2004). One of the key elements in lesson study cycles is a post-lesson discussion, which provides teachers with opportunities for sharing what they observed and for learning how to improve their teaching. This study aims to describe the key features and shifts in the focus of the post-lesson discussion.

In a two-year project with one public elementary school in Japan, an entire group of twenty-eight teachers in the school participated in the cycles of lesson study. During the post-lesson discussion, teachers in groups were first invited to write their thoughts and comments on the lessons they observed. The comments were then structured and shared by the teachers themselves. In this way, teachers' "voices" were better captured than the typical oral post-lesson discussion where experienced teachers' comments often dominated. In the current study, five cycles of lessons study that included planning and implementing research lessons were conducted and the data from all the post-lesson discussions were collected and analysed. Data also included video records of lessons and their transcriptions.

A framework to examine the proficiency in teaching mathematics (Schoenfeld & Kilpatrick, 2008) was used as a conceptual framework for this study. The framework was used for qualitatively analysing the discourse during the post-lesson discussion in the series of lesson study. The result of analyses revealed that the focus of post-lesson discussions shifted as the cycle of lesson study proceeded, and that the focus of topics discussed by teachers during the post-lesson discussion became more focused on "knowing students as thinkers" and "knowing students as learners". The shifts were identified as revealing changes in teachers' views on students' thinking and learning, along with their accumulated experiences of participating in lesson study.

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SUPPORTING SELF-CONCEPT AND INTEREST IN STATISTICS

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Motivational variables such as (academic) self-concept and interest relate to learning and achievement and hence are considered to be important goals of schooling (Bong & Skaalvik, 2003; Krapp, 2005). Self-concept expresses an individuals' confidence in the own competencies in a particular domain such as mathematics. Thus, it is mainly built by former competency experiences related to this domain (Bong & Skaalvik, 2003). Interest is defined as a specific person-object relationship that is accompanied by a high valuing and positive feelings (Krapp, e.g. 2005). The experience of competency, autonomy and social relatedness are crucial for developing interest.

This study focuses at fostering domain-specific self-concept and interest through a statistics-related intervention. During four lessons, students worked in pairs on learner-centred material including dealing with tables, diagrams and numerical summaries. Whenever they needed, help from flash cards and sample solutions was available. After each topic, students received additional individual feedback about their work. Hence, the intervention was designed to have an impact on both students' skills and motivational variables as it provided the experience of competency, autonomy and social learning.

We analysed data from 450 students (212 girls) participating in the intervention and 53 students (21 girls) of a baseline group (all grade 8). The students responded to the items of established scales for mathematics- and statistics-related self-concept and interest. Confirmatory factor analysis revealed that the students perceived mathematics and statistics differently with regard to these motivational variables. Whereas mathematics-related self-concept and interest remained relatively stable, both scales related to statistics significantly increased in the course of the intervention with a small to a medium effect size (self-concept: $d = 0.25$; interest: $d = 0.22$). Hence, in line with theoretical considerations, it appears that the intervention enabling the experience of competency, autonomy and social relations in the specific domain of statistics was adequate to foster specifically statistics-related self-concept and interest. This finding may also help educators in developing learning material to support these variables.

NB: This study is supported by research funds of Ludwigsburg University of Education.

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LIVING IN THE FORMAL WORLD OF MATHEMATICAL THINKING

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This case study examined a mathematician's thought processes while teaching Abstract Algebra for two consecutive semesters. The study was motivated by Dreyfus's vision that "one place to look for ideas on how to find ways to improve students' understandings is the mind of the working mathematician" (1991, p. 29). The theoretical framework employed here is based on Tall's (2013) three world model of embodied, symbolic and formal worlds of mathematical thinking. The overarching aim of this study is to investigate how mathematicians live and function in the formal world of mathematical thinking and at the same time communicate their knowledge to their students. In Tall's view, "formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used" (2013, p. 18).

The research team consisted of two mathematicians and two mathematics educators. The daily journal entries of one of the mathematicians, along with the transcripts from the audio recordings of the weekly research meetings reflecting on his journals, were coded using a standard open-coding scheme. In addition, the team examined journal entries of a student in the mathematician's class. The main themes emerging from the data were: (a) pedagogical challenges of communicating the greatness of a concept (e.g. Galois Theory) to a novice, (b) difficulties of teaching very abstract concepts (e.g. Tensor products) which are hard to explain or break down, (c) having a dynamical class while still being traditional, (d) mediating the disconnect between desire for mathematical elegance and the struggles of a student learning difficult material.

This paper focuses on which worlds of mathematical thinking the mathematician was accessing the most, as well as a comparison of the mathematician and his student's journals, four days prior to the proof of an elegant theorem. We found that the mathematician's approach agreed with the historical way of many examples first, followed by an attempt to make sense of it all and revealing the main theorem. Once accomplished, the mathematician revisited the old and new examples, to obtain more clarity. Thus, the sequence of embodied (but imperfectly), followed by formal and then embodied was the route that the mathematician took in teaching this theorem.

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MATHEMATICAL LANGUAGE USE IN ONLINE INTERACTIONS: A BAKHTINIAN LENS

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The role of language as social semiotic has been extensively acknowledged within teaching and learning research. In this study we analyse students' use of language during collaborative mathematical problem solving in a Computer Supported Collaborative Learning (CSCL) environment in order to identify their use of language at the stage of transition from informal to formal mathematics.

A Bakhtinian lens, as interpreted by Barwell (2012), was applied to dialogue generated within the CSCL environment. This placed a focus on four tenets applicable within mathematics dialogue: language is dialogic; language precedes us; tensions exist between unitary language and heteroglossia; and language is not unidirectional.

This study, involving 54 Grade 5 students (10 – 12 year olds), took place over ten weeks. These students were allocated to 10 mixed ability groups and collaborated online to investigate nine mathematical problems. Students' work in the CSCL environment enabled language to be tracked at a level that is not normally possible in the classroom. While analysis of their online dialogue showed evidence of the vocabulary of their teacher, students' use of mathematical words increased in both density and variety over the course of this study. This suggests that they were becoming more confident in their use of this language. We conjecture that the opportunity to privately revisit, trial, experiment with, and engage in mathematical dialogue in this online environment contributed to this result.

Analysis of the data supports our contention that Grade 5 provides a 'bridging' point in student mathematical language development. No longer are students only required to develop and utilise the language of basic place value and the four operations, communicating their understanding of more sophisticated concepts requires the acquisition of aspects of formal mathematical language. Students demonstrated an emerging competence with such language through appropriating familiar informal language in combination with the newly introduced formal vocabulary. This mixture of language allowed students to reason and communicate their emerging understandings. As students struggled to appropriate the unified formal mathematical register, they utilised elements of both this formal mathematical communication together with aspects of their everyday register. We describe this amalgamation as the 'transitional mathematical register'.

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TEACHING MATERIALS FOR TEACHING MATHEMATICS TO NON-FIRST-LANGUAGE LEARNERS IN NON-CLIL SITUATIONS

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INTRODUCTION

Pupils from minority cultures and/or those with a migrant background encounter even more difficulties than their native classmates in acquiring fundamental maths skills. A team of seven European partners in the EU-funded project M³EaL investigates the situation, asks about teachers' needs and strategies, and develops teaching materials.

THEORETICAL BACKGROUND

A number of mathematics teaching and training materials are available for teachers in CLIL courses. In CLIL, students voluntarily choose to have part of their subject teaching in a non-first-language (often by a teacher whose first language is also not the language of instruction). However, many teachers and students are confronted with a different situation when students from minority cultures and/or those with a migrant background are in regular courses where the language of instruction is not the first language for these students. While there is a lot of support for language courses in such situations, there is very little suitable support for mathematics courses, though several studies (e.g. Barton, Barwell and Setati 2007, Norén 2010, Ulovec et al. 2013) show the need for such support. The M³EaL-team set out to investigate the situation and come up with teaching and training materials to support teachers and their minority/migrant students in mathematics courses.

METHODOLOGY AND RESULTS

The team developed a questionnaire to find out experiences and needs of teachers teaching mathematics in multicultural situations. This questionnaire was distributed in six European countries and then analysed (Ulovec et al. 2013). The analysis results were then used to develop suitable teaching materials that are now piloted in classrooms. The Short Oral will briefly discuss the analysis results, present the teaching materials, and summarize the results from the piloting phase.

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PROSPECTIVE TEACHERS' GEOMETRIC DISCOURSE ABOUT SIMILAR AND CONGRUENT POLYGONS

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Similarity is an important topic connecting many other mathematical concepts such as spatial reasoning, ratio and proportion. The classification of similar and congruent figures illustrates not only what students understand about ratio and proportion and what properties of shapes they perceive, it also indicates their understanding of how those properties can be measured and compared. For some teachers, similarity is still a difficult topic to teach because they received little training in their teacher preparation program. Moreover, prospective teachers' (PSTs) geometric thinking in the context of similarity and congruence are rarely investigated in recent research. Therefore, this research project aims to explore prospective teachers' geometric discourse in the context of similar polygons. The research questions are:

What are PSTs' routines for identifying similar and congruent polygons?
What are PSTs' narratives of similarity and congruence?

Sfard's (2008) discursive framework is served as an analytical tool to investigate PSTs' geometric discourse. In particular, she proposed that mathematical discourse differ one from another in four features: *Word use*, mathematical vocabularies and their use; *Routines* are repetitive patterns characteristic of the given mathematical discourse; *Visual mediators* are mathematical symbolic artefacts related for particular communication; *Narratives*, any text, spoken or written, which are descriptive of objects and relations between them. Six PSTs participated in the study and they were enrolled in a mid-west university teacher preparation program in the US. During a 30-minute one-on-one interview, the PSTs were each asked to complete tasks including identifying congruent and similar triangles and hexagons among other geometric shapes (n=50), and to substantiate their claims and to explain their strategies. All interviews were recorded and transcribed for analysis.

Using Sfard's framework, it sheds a light on PSTs' use of words such as *similar*, *same, congruent, proportion*, *scale* and what they mean when those words are used in the context of describing similar and congruent polygons; it is also a useful aid to determine PSTs' strategies of identifying and classifying congruent and similar triangles and hexagons. In this presentation, more detailed analysis and findings will be discussed and shared.

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DEVELOPING LESSON PACKAGES FOR TEACHING CALCULUS USING A HISTORICAL PERSPECTIVE

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In this case study, we explored the professional collaboration of four mathematics teachers in a professional learning community (PLC) (see DuFour, DuFour, & Eaker, 2008) to develop a lesson package infused with the history of calculus for students in secondary 3 (Grade 9) preparing for the O-Level course in Additional Mathematics. The teachers in the PLC team had experience in teaching mathematics ranging from three to 25 years at the secondary level.

To design the lesson packages, the teachers in the PLC team discussed some ideas based on suggestions from Tzanakis and Arcavi (2000) and also sought some feedback from the students through a pre-study questionnaire on their preferences about the ways in which the historical development could be infused in the learning of calculus. Based on the content to be taught, a total of six teaching episodes, spanning ten teaching lessons, were developed. Using a lesson study approach, one of the teachers first taught the lesson package. It was videotaped then shared with other members of the PLC. It was refined and then another member of the team taught the second round of lessons. This helped the PLC team to polish further the lesson packages. The final version of the lesson packages was then shared with the rest of the teachers in the mathematics department. It took a total of three months to develop the lesson packages using the weekly protected team time built into the teachers' timetable.

Three sets of semi-structured interviews were conducted with the teachers in the PLC team: a pre-study interview, an interview after completing three episodes and a final interview after completing all six episodes. The data showed that initial concerns about time issues for completing the syllabus and lack of any existing model for teaching using the historical development of calculus led to a gradual interest and belief in the effectiveness of the lesson packages. Data collected from the students are not reported here.

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EXAMPLE-BASED LEARNING: HOW DOES PRESENTATION ORDER AFFECT LEARNING AND PERCEPTION OF COGNITIVE LOAD

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Learning by examples is one of the most common ways that we learn. Research on cognitive load theory also suggests that example-based learning is more effective than problem-based learning for beginning or novice learners (Renkl, 2005). According to cognitive load theory, given the lack of sufficient schema, beginning and novice learners benefit from studying worked examples. However, it was also suggested that adding practice items might enhance example-based learning (Tso, et. al., 2011). The question is how to arrange the order of the example and practice items. Four types of presentation orders were examined in this study: (1) paired example-problem (PEP) (2) paired problem-example (PPE) (3) blocked example-problem (BEP) (4) adaptive, in which a worked-example was presented first followed by a problem. What the participants got for the next item depended on their performance on the first problem. If the answer was correct, they would proceed to solving problems, otherwise they would proceed to see the next problem as a worked-example. This study also examined whether presentation order affected the participants' perception of cognitive load and efficiency in learning. The content was multimedia material for learning the concept of height and the height of plane figures (parallelogram, triangle and trapezoid), which was developed with PowerPoint and Visual Basic for Applications. The participants were fourth graders in an elementary school in Taiwan. The results showed that the adaptive version was superior to the PEP version in the immediate posttest and it also outperformed the other three versions in the delayed posttest. In addition, the adaptive version used significantly less time to learn. The participants in the PEP version perceived least cognitive load, while those in the BEP version perceived highest cognitive load. In terms of efficiency, overall the adaptive version showed best efficiency in learning while the PPE version the lowest though it did not reach a statistical significant difference.

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A FRAMEWORK FOR EXAMINING HOW TEACHERS MEDIATE TEACHING IN THAI PRIMARY MATHEMATICS CLASSROOM

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Studies have shown the importance of improving teacher and student interaction as well as the quality of mathematics communication in order to improve mathematics classroom instruction (Walshaw & Anthony, 2008). However, limited research is available on how teaching strategies can be used to increase teacher-student communication and interaction (Walshaw & Anthony, 2008). This study seeks to develop a framework to guide Thai primary school mathematics teachers' use of mediation strategies to achieve quality teacher-student interaction and communication. A rubric to assess the quality of teaching practice will also be developed in this study.

The framework for this study is grounded from the literature related to the characteristics of high quality mathematics instruction through classroom discourse as well as the Mediational Intervention for Sensitising Caregiver (MISC) framework. The following eight components from literature on mediation strategies were adapted for the framework: questioning, expanding, wait time, giving feedback, focusing, mediation of meaning, planning, and selecting example. From the eight components, five levels of quality of using these strategies were developed. The processes of validating the framework involved feedback from eight mathematics educators and four experienced primary mathematics teachers in Thailand. Rubrics were also used to score videos of Thai primary mathematics teacher's teaching practice. Following each trial round, the framework and the rubrics were refined based on the comments and feedback from the expert panel and educators. The framework and the rubric were also trialled by the researcher and two mathematics educators in Thailand to check for inter-reliability.

In the presentation, the framework, its development, and the rubrics will be discussed in detail.

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DEVELOPING YOUNG CHILDREN'S UNDERSTANDING OF PLACE VALUE

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The challenge of learning about place value is evident when students in the middle grades show limited understanding of two-digit numbers (Ross, 1989). A focus on multiplication and quotitive division problems is not usually the way most teachers approach the teaching of place value with young children. We found in a two-year study with five- and six-year-olds that these problems led to children developing foundational ideas about place value. This is contrary to Thompson's (2000) view that place value is often introduced to young children too early. It also raises questions about national and international curricula that introduce place value before multiplication and division.

Two series of 12 focused lessons were taught to four junior classes (five- to seven-year-olds) using multiplication and quotitive division problems. The problems began with groups of two, five, and then ten, and involved familiar groupings such as pairs of socks, fingers in a glove, and 10 eggs in an egg carton. Children solved quotitive division problems with two-digit numbers and began to make links between the groups of tens and the 'tens' digit, as well as between the singletons (leftovers) and the 'ones' digit.

The results from selected assessment tasks based on Ross's (1989) theoretical place-value framework were used to select the children who had made the greatest gains on ten-structured tasks. The purpose was to identify exactly what baseline knowledge contributed to the children's high levels of achievement. We found that these children initially had stronger knowledge of number facts, number word sequences, counting strategies (e.g. counting on and back), skip counting, and use of derived or known facts to solve problems. They had a better understanding of the underlying pattern and structure of groups of ten. These findings support Mulligan and Mitchelmore's (2009) extensive work on children's ability to abstract and generalise from the underlying structure of a mathematical concept.

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POSTER PRESENTATIONS

TEACHERS' VIEWS ON LOW ACHIEVING STUDENTS IN ABILITY GROUPED CLASSES

Monica Baker

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The debate around grouping mathematics classes by "ability" has been ongoing for many decades. This practice, called streaming in Australia, has been shown to have negative effects for low achieving students, both in terms of their learning environment (Zevenbergen, 2001) and their achievement (Lamb & Fullarton, 2002). However, streaming remains prevalent in Victorian secondary schools and little research has examined stakeholders' rationales for streaming.

This paper reports on data from semi-structured interviews conducted as part of a project which aims to examine stakeholders' rationales for streaming, and how rationale impacts classroom practice. Four teachers of streamed year 9 mathematics classes from a government secondary school in metropolitan Melbourne were interviewed about their views on streaming, the issues surrounding streaming, and what aspects of their practice would be different if they were teaching a different stream. The data were analysed using some codes that arose from the literature and some grounded theory techniques.

The results showed that teachers often referred to low achieving students when providing a rationale for streaming, and held views on these students that influenced their views on streaming and their plans for teaching low streams. For example, teachers held ideas that low achieving students could only learn procedures, and not concepts, and goals for low achieving students were often around engagement rather than learning. Teachers described feeling frustrated at their inability to help low achieving students. Some saw the low stream as a way to help such students, while others worried that its main purpose was to remove this cohort from the mainstream. These findings suggest that teachers' views on low achieving students influence both their practice in a low stream classroom, and may support their view that streaming is helpful to this group.

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FOSTERING CREATIVE MATHEMATICAL THINKING IN ELECTRONIC MATHEMATICS BOOKS (C-BOOKS)

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and the MC-squared project team

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The MC squared project (http://www.mc2-project.eu) aims to design and develop a new genre of authorable e-book, which the project calls the 'c-book' (c for creative), extending e-book technologies to include diverse interactive components, learning analytics and collective design. As a research lens, literature from communities of interest (CoI) is used (Fischer, 2001). The project aims to harness the structure of a CoI to stimulate social creativity (SC) and creative mathematical thinking (CMT). In the UK CoI we are treating CMT in problem-posing and solving as entailing the indicators fluency, flexibility, originality, elaboration, and usefulness (e.g. Silver, 1997).

In cycle 2 of the project, one of the research questions was: "how do five c-books authored by the CoI, demonstrate the elements of our CMT definition?" To address this question the CoI produced five c-books: one on transformations of graphs, one on planetary orbits, one on mathematics for biology, one on numbers and one on generalization. The c-books differed greatly in number of pages and number of interactive elements (widgets). After their conception, an evaluation instrument was used to evaluate the CMT potential. The instrument consisted of a template incorporating the creativity indicators mentioned above.

The evaluation showed that most c-books had a mix of open and closed elements, sequenced in an intentional way to facilitate learning. Two out of five c-books (planets, and mathematics for biology) had a particular *multi-disciplinary* focus, while the other three stayed more in the realm of mathematics. It could be seen that the open or closed character of a c-book was mainly determined by the overarching learning objectives of the c-book. However, in all cases, the CoI still viewed them as *creative* products. In other words, *creativity* according to the CoI's definition is not a simple case of creating open or closed tasks but a carefully-designed sequence of 'pages' and tasks that together potentially induce creativity.The poster presents several visual examples of the five c-books and how they may, or may not, promote CMT.

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TEACHING AND LEARNING MATHEMATCS: THE VOICE OF TIMORESE TEACHERS AND STUDENTS

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Teachers' and students' beliefs (Leder, Pehkonen & Torner, 2003) have significant influence over teaching practices – from 'traditional' to 'exploratory' (Allwright, 2003) - and learning 'styles' and 'approaches' adopted (Sims & Sims, 2006). In order for the investment made on the restructuring of Timorese general secondary education curriculum not to be compromised, it is of utmost importance that teachers' and students' perspectives are identified so that the implementation of the new curriculum can be improved in a timely manner. This is one of the main aims of the ongoing *Timor Project* (supported by the Portuguese Foundation FCT (PTDC/MHC-CED/5065/2012), by the COMPETE Programme and by the European Regional Development Fund (FEDER)). Within its scope, teachers and students from public and private schools were surveyed. The main quantitative results from the questionnaires showed that teachers seemed to consider teaching as a process of fostering students' knowledge building, more than a process of reproducing and applying it; to value knowledge transmission and questioning students as their main teaching practices; to use formative and summative evaluation and resort to tests and their correction as common evaluation practices. Regarding students, they seemed to value knowledge transmission, questioning the teacher, solving tasks and discussing their resolution as learning activities. In general, they preferred working alone to groups and selected textbooks, dictionaries and other didactic books as the most important resources for their learning. Their study habits involved doing homework assignments and summarising information from text or notebooks; learning was considered to be a process of knowledge creation, but also of memorisation. These perspectives seem to be in line with a *traditional* way of teaching, with *passive* and *sequential* styles and with a *superficial* approach to learning that is urgent to reverse.

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SECONDARY TEACHERS' PERSPECTIVES OF MATHEMATICS KNOWLEDGE FOR TEACHING

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This study reports on the first year of a five-year nationally funded project supporting change in the mathematical culture of a Grades 2–12 school. It focuses on secondary teachers' perspectives of mathematics knowledge for teaching [MKT] related to their planning to teach. MKT is described in the literature from different theoretical perspectives that include categories of knowledge and ways of knowing and acting. While some categories were developed from elementary teachers' practice, secondary teachers have not received such attention. Thus, this study adds to our understanding of MKT resulting from learning experiences designed to help secondary teachers to further develop it and their classroom experiences. Theoretically, the teachers' perspectives are considered through the lenses of practical knowledge and personal meaning (Polanyi, 1958) from these experiences. The nine participants, covering Grades 7–12, were part of a group of mathematics teachers from their school who participated in an intervention consisting of a four-course, yearlong graduate certificate program. The courses, developed and taught by members of the research team, engaged them in topics that included theory and hands-on collaborative activities involving students' mathematical understanding, mathematics concept study (Davis & Renert, 2014), task design, and problem solving.

Data sources included: (i) the teachers' reflective journals on their views of MKT at the secondary level, the MKT they used in planning lessons for different mathematics courses of their choice, and the changes they noticed in their MKT; (ii) the teachers' unit plans for the mathematics courses; and (iii) field notes of the teachers' discussions of MKT. Data analysis involved identifying themes that emerged from the data based on recurring features of the teachers' thinking and use of MKT with a focus on what they explicitly considered on their own from their participation in the intervention. Findings indicated major shifts in the teachers' perspectives of MKT for planning teaching, from knowing how to break down the content to make it easy to transmit to students and for students to acquire, to a perspective that included knowledge of: connections and nested relationships among curriculum topics, students' misconceptions, engaging mathema- tical tasks/activities, what worked or did not work for students in past teaching of the topic, and the mathematics content including its history, metaphors, and meanings.

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MEASUREMENT CURRICULUM EXPECTATIONS OF YOUNG CHILDREN IN AUSTRALIA AND GERMANY

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Independently our research projects have found children are capable of more than our national curricula specify. To lay the foundations of future collaborative research we investigated the question: How do our early measurement curricula compare?

Our purpose was to: describe the intended curriculum for measurement for primary school students aged 5-8 years in Australia and Germany; focus on the measurement of length and mass due to their contrasts conceptually and pedagogically; and consider the intended curriculum in light of research.

A document analysis of the relevant curricula for 5-8 year-olds was undertaken (Mayring, 2010). Comparison between Australian and German early measurement curriculum is not simple: children begin school at different ages; German curricula are specified by each state with variation between states; in Germany, a national standard is first defined at Year 4, whereas in Australia national standards apply from Year 1. The length curriculum is specifically described in each country, however, mass measurement less so in the German curricula. There are differences between the two nations in the use of formal measures and the statement of measurement principles. In Australia the conceptual principles of measurement are underplayed. Prominent in German curricula is the development of a mathematically sound concept of a specific magnitude together with a focus on building mental models for formal units as a precondition for estimating.

In the light of our research findings, curriculum standards from both countries seem to underestimate the learning potential for children of 5-8 years of age. Perhaps if the curriculum statements were more specific about the ways in which, in learning to measure, young children build concepts of unit, number and composite numbers, teachers would find more reasons to use measurement to contextualise number.

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A STUDY ON HOW PRE-SERVICE ELEMENTARY SCHOOL TEACHERS EMPLOY MATHEMATICAL REPRESENTATION

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The objective of this study was to explore the purposes for which pre-service teachers employ representation in mathematics class, the difficulties they encounter in the use of representation, and their actual performance in the use of various representations to modify the curricula of teacher training programs. The subjects of this study comprised 40 pre-service teachers in an undergraduate program at an education university on public funding. We collected research data via a questionnaire survey and adopted quantitative and qualitative approaches to analyze the data. The findings of this study were as follows. 1. The situations in which pre-service teachers used representation depended on the characteristics of the teaching materials, the needs and comprehension skills of the students, communication and expression, and the design requirements of the curricula. 2. The functions that representation serves in teaching include promoting thinking and conversion, teaching mathematical concepts, communicating and expressing concepts, and increasing learning motivation. 3. The difficulties in the use of representation included difficulty in comprehension, not knowing how to employ representation, and the differences among individual students. 4. The response strategies to difficulties in the use of representation included promoting the conversion of representation, teaching according to the ability of the student, seeking expert support, and considering the nature of the materials. 5. The subjects found the actual employment of representation in the problem-solving process to be difficult because their own previous habits and experience in problem solving affected their use and conversion of representations, their own mathematical knowledge and teaching methods influenced which representations they chose and how they used them, and they did not fully understand the functions of representation and when to apply it. These reasons prevented them from creating scenarios that could reasonably support their formulas. Finally, we put forward suggestions to educators on how to adjust the curricula of teaching training programs.

DEVELOPING YOUNG CHILDREN'S IDENTIFICATION OF MATHEMATICAL PATTERNS

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Preschool curriculum recommend teachers should teach young children pattern reasoning in their contexts (NCTM, 2000). Clements and Sarama (2009)reported that 6-year-old children could translate and identify the ABBABB pattern according to the ABB core unit. If assisted by an adult with pattern exploration based on several types of connections with real life, between different topics of mathematics and even with other curricular areas (Orton, 1999), the children could address the challenge of confused patterns such as ABBCA, and simultaneously describe, extend, and represent patterns. The purpose of this research was to explore whether or not 6-year-old children could identify complex repeat patterns and growth patterns through instruction.

The methodology employed empirical study. Subjects were recruited from two classrooms of 60 kindergarteners in Taiwan, using an experiment and a control group. The two groups had the same mathematics textbook, which included content on patterns. To determine the invention effectiveness of the subjects, the researchers developed seven pattern tasks for two groups before and after the invention. The tasks comprised three types of repeat pattern, namely three ABC, one ABB, and two complex AABCABBCABCC patterns, which were developed using different hands-on materials with pictures (e.g., a worm, a train, and linking clouds) as well as a growth pattern. The subjects responded to the tasks by completing the missing element of each pattern. The data were analysed using t tests and interpreted using quality data such as observation and video records.

The results showed that the experiment group could identify complex repeat and growth patterns after the invention tasks. The study group's performance level (M = 7.00) was higher than that of the control group (M= 5.10) for the pattern tasks. The t test showed a satisfactory score of 14.62. The study determined that 6-year-old children can effectively perform tasks and identify and predict complex repeat patterns. They could also predict and identify complex, large core units with an AABCABBCABCC pattern after receiving pattern teaching.

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USING DIALOGIC TALK TO FOSTER MATHEMATICS LEARNING IN INTERACTIVE GROUPS

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This poster illustrates the capacity of dialogic interactions to foster students' learning. Dialogic talk promotes a kind of interaction susceptible of verification. Learning emerges from this kind of interaction, since children must be (mathematically) competent looking for valid and true mathematical arguments.

Learning is a social process mediated by interactions (Zack & Graves, 2001). It arises when individuals share their answers regarding a particular task (problem, activity, exercise, etc.). The hypothesis discussed in this poster is that not all types of interactions may lead students towards learning and understanding. Drawing on learning theories, three different types of interaction will be displayed in the poster (TI model). I suggest that only dialogic interaction will lead students towards a real process of learning mathematics.

I discuss that hypothesis drawing on an episode of interaction conducted in a lesson about converting improper fractions into mixed ones. The context is an elementary school, placed in a suburban area in Catalonia, working as a Learning Community. The teacher organized a lesson using *interactive groups* (IG) to organize the classroom (Valls & Kyriakides, 2013). The poster includes relevant quotes from participants. Data collected comes from a 5-year-research project funded by the Spanish Ministry of Education (2011-2016). Discourse analysis is used for data analysis, drawing on the TI model.

Evidence suggested that students using valid and true arguments to justify their statements helped others to clarify the mathematical objects, while consolidating their own understanding. On the other hand, students not arguing, or using "power claims" to impose their point of view, obstruct and even prevent peers' learning. This result leads us to think that maybe we need to organise the classroom in such a way to promote dialogic interactions among participants.

Valls, R. & Kyriakides, L. (2013). The power of interactive groups: How diversity of adults volunteering in classroom groups can promote inclusion and success for children of vulnerable minority ethnic populations. *Cambridge Journal of Education, 43*(1), 17-33.

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DRAWING ON DELIBERATIVE DIALOGUE TO OVERCOME TEACHERS' SOCIAL REPRESENTATIONS REGARDING FAMILY INVOLVEMENT IN MATH PRACTICES

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This poster introduces two of the main results obtained in the RTD project Curriculum Development for Middle and High School Teachers based on Competence (EDU2012-32644): (a) transformation of teachers' social representations and (b) research evidence-based criteria to help teachers re-designing mathematical tasks in order to facilitate families' engagement in teaching and learning process. Previous research suggests the importance of collaborations between family and school to improve students' performance (Díez-Palomar, 2015). This is the case of interactive groups developed within the learning communities (Valls & Kyriakides, 2013), which draw on family involvement and evidence-based educative actions.

We developed a six-step design process creating mathematical tasks to include family members within the learning process in order to challenge (a sample of) 3 groups of 24-26 and 25 in-service and pre-service teachers, who were exposed to situations designed to provoke teachers' critical thinking, using deliberative dialogue. We collected video data and recorded observations. The analysis was conducted using the 'Ontosemiotic' approach (Font, Planas y Godino, 2010). Some selected excerpts will be displayed and discussed in the poster.

After three years, we found evidence suggesting that teachers transform their social representations regarding family involvement. We obtained a set of criteria (displayed in the poster) to include effectively families in the teaching practice. We conclude that teacher-training programmes should include competences to help future teachers to deal with potential conflicts when performing their work as teachers of mathematics, based on a "didactical analysis" evidence-based.

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IDENTIFYING STUDENTS' DIFFICULTIES IN UNDERSTANDING LINEAR EQUATIONS WITH TWO UNKNOWNS

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According to Sfard and Linchevski(1994), the nature and the growth of algebraic thinking was analysed from an epistemological perspective supported by historical observations. Eventually, its development was presented as a sequence of even more advanced transitions from operational to structural outlook. The focus was on two crucial transitions. The first was from the purely operational algebra to the structural algebra of an unknown. The second was from here to the functional algebra of a variable. Some difficulties experienced by the learner at both these junctions were illustrated with much empirical data.

However previous studies only pointed out part of the difficulties of algebraic learning. This study focused on *the versatility* of students at the situation in which students must change the liner equation of "equality" into the liner equation of "functional relationship". This study was a part of the basic research for identifying students' other difficulty in the situation to change the outlook of letters in the equation; from "an unknown" to "a variable". The purpose of this study was to identify students' difficulties in the case of the linear equation with two unknowns and the linear function.

For this purpose, this study analysed a *hypothetical learning trajectory* (Simon, 1995) of linear equations with two unknowns and conducted a qualitative research based on data collected through a survey. The ninth grade students (N=154) were given a set of assessment tasks about the graph of a linear equation with two unknowns, and the distinction between linear equations with two unknowns and equations of linear function. In order to analyse data from the viewpoint of functional approach, some categories were made by identifying skills and concepts for functional approach. The results showed two difficulties as follows; (1) 26% of the students recognised the graph of the linear equation with two unknowns as a discrete graph. (2) 37% of the students could not flexibly change the outlook of linear equations. Namely, the students distinguished the linear equation with two unknowns and the linear function by the form of expression (i.e. explicit function or implicit function).

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DEVELOPING PROBLEM SOLVING ABILITIES BY LEARNING PROBLEM SOLVING STRATEGIES

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The research presented in this poster is about students' development of problem solving abilities. More specifically we look at how problem solving ability is affected by the learning of problem solving strategies in mathematics.

We conducted a one-year long intervention study with an experimental group of 16 and 17 year old students in a Swedish upper secondary school, who participated in specially designed lessons about problem solving strategies. The aim was to help them to identify problem solving strategies and to experience strategy thinking. The lessons fitted with regular teaching, did not alter the mathematical content but included aspects of strategy thinking.

We used activities designed using the three following principles based on variation theory:

(1) Let the problem solving strategy vary and keep the task invariant

(2) Let the task vary and keep the problem solving strategy invariant

(3) Let both the task and the strategy vary and allow students to evaluate the effectiveness of different strategies for different tasks (Fülöp, 2015*)*.

We measured the effect of the intervention by comparing the development of the experimental group with a control group, from a pre-test to a post-test. We studied their problem solving ability but also their mathematical knowledge in general.

A one sided t-test shows a significantly better development in the problem solving ability in the experimental group compared ($p= 0.038$; $p<0.05$). The results also showed that the experimental group developed at least as well as the control in general mathematical knowledge.

The poster presentation also includes a discussion on the concept of problem solving strategies and its relation to two closely related concepts, method and algorithm.

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CHILDREN'S DIFFICULTIES WHEN LEARNING FRACTIONS

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Fractions can be difficult to learn. Understanding difficulties in the early learning of fractions is crucial as they can affect opportunities for further engagement in mathematics and science. Various hypotheses have been proposed to explain these difficulties: fractions can denote different concepts (Behr, Lesh, Post & Silver, 1983); the understanding of fractions requires a conceptual reorganisation with regard to natural numbers (Stafylidou & Vosniadou, 2004); and using fractions involves the articulation of conceptual knowledge with complex manipulation of procedures (Byrnes & Wasik, 1991). This study aims to investigate the difficulties encountered by primary school children when they learn fractions, which may stem from a lack of conceptual understanding of fractions.

Taking these issues into account led us to design a written test in which two main components were considered: (1) Conceptual knowledge was assessed through tests about the different meanings of fractions (part-whole, proportion, magnitude); (2) Procedural knowledge was assessed through simplification tasks and operations on fractions. A total of 439 Grade 4, 5 and 6 children were tested, and the results showed large differences between categories. By Grade 6, children had a good understanding of the concepts of part-whole (M=77%), proportion (M=85%), and the procedure of simplification (M=71%), but not of magnitude (M=54%) or arithmetical operations (M=53%). Our analyses of teaching practices indicated a stronger focus on procedures than on concepts, which are not sufficient in the absence of conceptual understanding.

The major implication of our findings for teaching would be to shift the emphasis away from procedural knowledge and towards conceptual knowledge, focusing on magnitude. With a shift in focus to a more concept-based teaching approach, the teaching of fractions should become more efficient and successful, laying a firmer foundation than the current regime, and opening up higher mathematical concepts.

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MIDDLE-SCHOOLERS' SPACING PATTERNS FOR WRITING ARITHMETIC EXPRESSIONS

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Computing the value of an arithmetic expression like 2+3×5 is a simple task, but it turns out to be surprisingly difficult for sizable numbers of students even in university (Pappanastos, Hall, & Honan, 2002). The spacing pattern with which an arithmetic expression is written affects how adults compute its value (Landy & Goldstone, 2010). For instance, computing 2 + 3×5 is easier than computing 2+3 × 5. Moreover, many children compute arithmetic expressions systematically from left to right, and the presentation of expressions with a spacing pattern consistent with the correct order of operations may help them (Gómez, Benavides-Varela, Picciano, Semenza, & Dartnell, 2014). Altogether, these findings show that visual information can influence the application of the correct order of operations.

Mathematically competent adults spontaneously write arithmetic expressions with a spacing pattern that is consistent with this order (Landy & Goldstone, 2007). Here, we present data from 5th grade children (N = 22) showing that the consistent spacing pattern found in adults' writing is absent at this age. Children wrote down 24 auditorily-presented arithmetic expressions. Spacing measurements revealed that expressions starting with multiplication (e.g. 7×3+6) had uniform spacing, but those starting with addition (e.g. 2+8×5) had larger spaces around the multiplication ($t(21) = 4.5$, $p < .0001$). This shows children do not spontaneously write in a manner consistent with the order of operations, and that consistent writing should be introduced by the educator (be it in an implicit or explicit way) in order to benefit from it. [This research was funded by the Chilean programs CONICYT Basal (grant FB0003) and CONICYT PAI/Academia (grant 79130029), the Italian Ministry of Health (grant F-2009-1530973), and the NEURAT project, Università di Padova.]

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DIFFERENCES IN CHOICE OF OPERATION FOR WORD PROBLEMS INVOLVING MULTIPLICATION OR DIVISION BY FRACTION

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The misconceptions "multiplication makes bigger" and "division makes smaller" have been discussed at length in the context of choosing an operation for a multiplication or division word problem involving non-negative rational numbers. Researchers have shown that subjects choose division for multiplication word problems involving a multiplier less than 1, because the answer to the problem is expected to be less than the multiplicand and "division makes smaller". In a similar fashion, subjects choose multiplication for division word problems with a divisor less than 1, because the answer to the problem is expected to be greater than the dividend and "multiplication makes bigger". However, as in Prediger (2011), in the present study we found additional explanations for the choice of operation, some of which are unique to fractions.

We asked 213 sixth-grade students and 267 eighth-grade students to write a mathematical expression for a word problem involving multiplication or division by a fraction, using the numbers that appeared in the problem. For each student, we examined the arithmetic operations that were chosen in pairs of word problems containing the same numbers. For example, one might expect the same operation to be chosen (correct or incorrect) in the following pair of multiplication word problems: (1) The price of a meter of fabric is 30 shekels. What is the price of 3/5 of a meter of fabric? (2) Rina has 30 shekels. She bought a pencil case with 3/5 of her money. How much did the pencil case cost? We found that 21% of the sixth-grade students and 6% of the eighth-grade students wrote a multiplication expression for Problem (1) and a division expression for Problem (2), even though both have a result less than 30. In addition, 11% of the sixth-grade students and 16% of the eighth-grade students wrote a (correct) multiplication expression or an (incorrect) division expression for Problem (1) and a subtraction expression for Problem (2).

We found connections between characteristics of a word problem, a student's choice of operation, and explanations that he/she gave in interviews. Our results indicated that among these students there were intermediate levels of understanding of the meaning of multiplication and division by a fraction.

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MORE THAN NUMERICAL COMPETENCE? DIMENSIONALITY OF KINDERGARTENERS' MATHEMATICAL COMPETENCE

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Research on mathematical competence of kindergarten children is often limited to numerical competencies, especially in psychological studies. Sometimes tests also contain a few items on measurement or patterns, which usually have a strong numerical character. In contrast, in mathematics education research there has been an increasing interest in other areas like "space and shape". Despite that theoretical diversity of areas of mathematics, Clements, Sarama, and Liu (2008) conclude from previous research that kindergarteners' knowledge in different mathematical domains grows in parallel. Following this assumption, they developed a test that covers different mathematical areas (sets, numbers, operations, measurement, patterns, space, shape), but measures mathematical competence as 1-dimensional construct. However, the question on the dimensionality of kindergarteners' mathematical competence and its development is still not answered. Accordingly, we started a longitudinal study to address this question.

We report results of the first time of measurement of a longitudinal study with 207 children (M = 48.5 months, SD = 7.6; 50.5% boys). We assessed mathematical competence with the "Kieler Kindergartentest (KiKi)". The version for young children covers "sets, numbers and operations" (15 items), "change and relationships" (4 items) and "space and shape" (7 items). The test was administered by trained assistants in individual interviews of max. 30 min. In a subsample (n = 171) we also assessed working memory. In multidimensional IRT-models, the 3-dimensional model showed a much better model fit for the data on mathematical competence than the 1-dimensional model. Since the 3-dimensional model indicated a strong correlation (.97) between the dimensions "change and relationships" and "space and shape" we considered also a 2-dimensional model. Here, a $\chi 2$ difference test showed that the 3-dimensional structure is superior. Additional analyses indicated that all three dimensions are separable from the domain-independent working memory abilities. In total, with the KiKi it is possible to measure separate dimensions of mathematical competence of children already at the age of four. The longitudinal study allows following the individual development of children in the different dimensions and to investigate whether a multidimensional approach is promising.

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A STUDY ON IMPROVEMENT OF STUDENTS' COVARIATIONAL REASONING ABILITY BY THE GEOGEBRA APPLET

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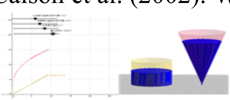
Covariational reasoning has been received considerable attentions in mathematics education literature as a way of thinking about functions (Calson et al., 2002). It was suggested that the concept of function should be introduced as covariation and students should be provided with more experiences to explore the concept of rate of change in earlier grades (Confrey & Smith, 1995). However, students appeared to have difficulty sketching accurately increasing and decreasing rate for dynamic function situations. In this paper, we investigate students' representing level for dynamic situations and analyse effects of Geogebra applet to improve students' covariational reasoning ability.

Three 7th graders of a Korean middle school who had already learned the concepts and graphs of function y=ax, y= a/x (a: constant) were chosen and asked to sketch graphs of the water height filling in the cylinder, cone, and reverse cone shaped bottles as a function of the time. We then identified students' level of covariational reasoning ability according to the covariation framework developed by Calson et al. (2002). We developed a Geogebra applet to help students construct the graphs of functional relationship between two continuous variables –time and water height filling in the bottles (Figure) and applied it in 3 hours of lessons. After the lessons, we check their level of covariational reasoning ability for the sphere- and gourd-shaped bottles.



The result revealed that in the pre-test, the first student sketched the wrong graph of the water height in cylinder which demonstrated covariational reasoning ability in Level 3(Quantitative coordination). But he went up to the Level 5 (Instantaneous rate) in sphere shaped bottle item after the lessons. The second and third students sketched the wrong graph of the water height in cone which demonstrated covariational reasoning ability in Level 2 (Direction) and Level 4 (Average rate) respectively. But they went up to the Level 4 and Level 5 respectively in sphere shaped bottle item after the lessons. In conclusion, students appeared to be weak in accurately representing concave up or concave down for dynamic function situations. However, technology like Geogebra applet is helpful to improve students' covariational reasoning ability in various dynamic situations.

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A STUDY OF MATHEMATICS MATERIALS DEVELOPMENT FOR KOREAN SENIOR

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Over 11 percent of the Korean population was over age 65 in 2010 and this figure is projected to be over 20 percent in 2026, so that Korea will be a super-aged society (Statistics Korea, 2011). However, Korean society still seems to espouse negative attitudes toward elderly learners as targets of education. The importance of education for the elderly is increasing every day; the development and dissemination of materials that promote physical and mental health may be necessary to improve the lives of the elderly (Ko, 2010).

The aim of this study is to develop materials to improve numeracy, mathematical thinking, and the application of these abilities to daily life. The developed materials are organized in stepwise levels as follows: Step 1 (Numbers and Counting), Step 2 (Single-digit Addition and Subtraction), Step 3 (Addition and Subtraction), Step 4 (Principles and Applications of Multiplication), Step 5 (Principles and Applications of Division), Step 6 (Analysis and Presentation of Data), Step 7 (Analysis and Presentation of Data in Real-life Contexts), Step 8 (Finding Rules), Step 9 (Number of Cases), Step 10 (Applications of the Number of Cases).

Lately, the education field has become open to change, in an attempt to take advantage of smart devices. However, these changes are only being applied to the education of young students. In order to overcome this situation, we have developed content with specific consideration of the abilities and aptitudes of the elderly. This content should be able to be accessed for the purposes of senior learning. Therefore, we plan to disseminate the developed mathematics materials in the form of a smart mathematic application for seniors.

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ON THE NATURE OF REPRESENTATIONAL GESTURES AS GROUNDED IN THE MATHEMATICAL TASK

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In the social process of mathematical knowledge construction, students develop their own representational gestures used as part of their language on a way to understanding (Krause, to appear). *But where does a shared meaning of gestures come from when the knowledge on the object they refer to just develops?* This question has been posed in a study on the role of gestures in social learning processes. The study was framed within the theories of social constructivism and embodied cognition, taking a multimodal perspective. Knowledge is considered to be constructed in social interaction by individuals, based on their (bodily) experiences with the world. The meaning of gestures was reconstructed within developing *semiotic bundles* (Arzarello, 2006): Taking into account synchronic and diachronic relationships between gestures, speech and inscriptions allows analysing the interplay of gestures with other semiotic resources, as well as reconstructing how shared meaning of gestures develops over time. I traced how the students developed *associated gestures*. These become associated with a mathematical object, establishing a situationally conventionalized representation of it.

As part of the empirical study, one pair of students was observed while solving three tasks (developed in the GIF-project "Effective knowledge construction in interest-dense situations") that differ in the mathematical subject they deal with: a geometric-algebraic task on the parabola as geometrical locus, an arithmetic-analytic task on a continued fraction and its limit, and a task on logical reasoning. A comparison of the students' gestures revealed that the origin of meaning differs between the tasks, but is similar for different gestures established for the same task: For the geometric-algebraic task, the associated gestures developed by the students refer to graphical representations while for the arithmetic-analytic task, the gestures are rather developed as grounded in physical actions, also in a metaphorical way. The students did not develop associated gestures for the task on logical reasoning but used metaphorical gestures not explicitly conventionalized, but grounded in their bodily experience with the world. Understanding the students' gestures may help getting a better comprehension of their conceptualization of a mathematical idea. Furthermore, the nature of gestures as grounded in the mathematical subject rather than the concrete task may provide a beneficial didactical approach, worth being tested in upcoming studies.

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CONCEPTIONS OF TEACHING AND LEARNING FOR KOREAN SECONDARY MATHEMATICS PRESERVICE TEAHCERS

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Recently researchers in the “Teacher Education and Development Study: Learning to Teach Mathematics (TEDS-M)” performed international comparative studies on teacher education for primary and lower secondary mathematics teachers. One of their interests to compare teacher education programs is teacher beliefs that are regarded crucial for applying teacher knowledge in teaching practices (Wang & Hsieh, 2013). To understand preservice teachers and their education programs, this study investigates beliefs, in particular, conceptions of teaching and learning for Korean secondary preservice teachers in mathematics.

This study used the instrument, Teaching and Learning Conceptions Questionnaire (TLCQ), developed by Chan & Elliot (2004). The 30-item TLCQ was translated into Korean and reviewed by two professors specializing in English and education. A survey was conducted for 86 secondary mathematics preservice teachers in a teacher education program of one university in Korea. In the university, the teacher education program was open not only to undergraduate education majors but also to education major graduate students. Among the 86 participants, there were 46 undergraduate juniors, 33 undergraduate seniors with mathematics education majors, and 7 graduate students with mathematics education majors.

The results of this study showed that the participant secondary mathematics preservice teachers strongly agreed with the constructivist perspectives. Some items in the conceptions of teaching and learning for juniors were different from seniors’ ones. The seniors responded more positively in the questions related to the traditionalist view. The juniors took more courses of education and liberal arts than mathematics content courses, whereas the seniors took content focused courses mostly in the earlier year that the data collected. These results indicate that preservice teachers’ beliefs are influenced by their teacher education programs as one of the educational culture.

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A STUDY ON CAUSES OF UNDERACHIEVEMENT IN MATHEMATICS BY BIG DATA ANALYSIS OF KOREAN STUDENTS

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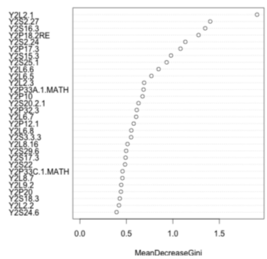
The PISA 2012 results revealed that Korea ranked in the top five for mathematics (OECD, 2014). Yet only 60% of Korean students reported being happy at school, the lowest average in the OECD. Many secondary students lose their interest and self-confidence so that they give up learning mathematics. This becomes a big social problem to be solved in Korea. In this paper, we investigate when underachievers give up learning mathematics and what causes them to give up by big data analysis.

The data used in this paper are from the “Korean Education Longitudinal Study 2005” which had surveyed the state of the secondary school education from 2005 to 2010. This data set contains total of 746 variables for each of 6,908 sampled students in four area of assessment: students' overall educational state, self-evaluated affective attitude state, teacher-evaluated student’s affective attitude state, and their parents' state.

Research findings showed that most of middle school underachievers gave up learning Mathematics in the 8th grade. This result was also supported by the analysis of the self-evaluated affective attitude scores and teacher-evaluated affective attitude scores. Causes of underachievement were analysed by using "random forests" developed by Breiman (2001) which is one of the big data analysis methods to select the important variables of affecting underachievers. The results revealed that 10 major causes which included: the lack of immersion (Y2L2.1), weak will of studying (Y2S2.27), lack of studying hours in cram school (Y2S16.3), low educational expenditure by parents (Y2P18.2RE), and so on (Figure). Among 10 causes, eight are responsible for students and two for parents.



In conclusion, it is required that the contents of mathematics curriculum should be lessened and the level of difficulty of mathematics text books should be lowered specially in 8th grade. Schools should also help underachievers with overcoming difficulties they suffer.

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DEVELOPMENT OF ON-LINE MATHEMATICAL READING SYSTEM AND ANALYSIS ON STUDENTS' PERFORMANCE

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The purpose of this study was to develop the internet system of mathematical reading and investigate pupils' performance. Mathematical reading played an important role in the mathematics learning process and such issue was related to mathematics achievement and mathematics literacy. Therefore, it was prospective to establish the internet system with the intension of making the instruction and assessment of mathematical reading feasible. Theoretical foundation of this study was based on two perspectives. One was to use an efficient learning technology with respect of the learning technology acceptance to improve mathematical reading comprehension. The other was to investigate on three components as to mathematical reading, in which the three cognitive components were general reading comprehension, prior knowledge of mathematics, and specific skills of mathematics (McKenna & Robinson, 2009). With regard to mathematical reading, little was known about the performance and difference of students on these three components. Therefore, this study was to develop the internet system for mathematical reading texts and reading assessment in order to help students improve their reading comprehension. Mathematical texts for reading and its reading assessment were established on the internet system after a pilot test was confirmed the reliability and validity of the assessment. The study examined 478 sixth-grade students' response in participating the activity of mathematical reading built in the internet system. To investigate the nature of learners' reading performance, each participant read two articles related to the concept of number and quantity and then took the reading assessment based on three cognitive components. Clustering of participants in accordance with reading assessment data was conducted so as to realize their performance and difference. The results addressed the internet system for mathematical reading and assessment was feasible for pupils. Additionally, all participants could be classified into three clusters and there existed significant difference in the determinants of three cognitive components. Results of this study also indicated that each cluster revealed its features of mathematical reading. The findings suggested that remedial instruction for mathematics reading could be conducted based on these features. Further investigations on possible reasons to influence the performance of mathematical reading were prospective.

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WHY IS IT SO? – ELICITING PRECURSORS OF MATHEMATICAL REASONING IN KINDERGARTEN

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Mathematical reasoning is a standard of school mathematics (e.g. NCTM, 2000) and, hence, a goal of mathematics education. Besides a narrow understanding of mathematical reasoning as mathematical proof, a broader understanding includes precursory skills, e.g. making connections, investigating conjectures, or comprehending arguments. Whereas mathematical instruction – encompassing mathematical reasoning – starts with preK-level (at the age of 5) in some traditions, in others (including Germany), mathematical instruction is not common before the start of elementary mathematics education at first grade. Hence, it is an open question if and to what extent children at the transition to first grade have the disposition to use these mathematical reasoning abilities. However, even if early mathematical instruction is common, the development of these abilities is not yet understood in detail (for an overview, see Ginsburg et al., 2008). Accordingly, this study aims at modeling and measuring precursors of mathematical argumentation abilities of children in their last year of kindergarten (age 5-6) with the foci: What precursors of mathematical argumentation can be identified across mathematical content areas? Can these precursors be distinguished from mathematical knowledge, general cognitive skills, and effects of maturing?

Interview-based tests were developed for mathematical reasoning and mathematical knowledge and administered to $N = 120$ children (age $M = 5.2$ yrs, $SD = 0.5$). The results based on $N = 75$ complete data sets show that mathematical reasoning is interrelated with mathematical knowledge ($r = .58^{**}$) as well as working memory capacity as an indicator of general cognitive skills ($r = .29^{*}$), but not with the age ($r = .04$ n.s.). Regression analyses show that mathematical knowledge is the only significant predictor explaining 34% of the variance. Hence, mathematical reasoning abilities were found to be distinct from related abilities. In the following studies, the development of these abilities during the last year of kindergarten will be investigated in order to evaluate its variability during the transition to first grade.

The poster will focus on the framework for precursors of mathematical reasoning and present sample items in detail. Quantitative information will be presented in more detail as a basis for the discussion of the general importance of these findings.

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EFFECTS OF USING DYNAMIC FIGURES ON LEARNING GEOMETRY PROOFS

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Understanding geometry proofs involves a cognitive process of substantial complexity for most middle school students. Assisting students in understanding and constructing mathematical proofs is a crucial research topic in mathematics education. For example, students can discover and understand these dynamic procedures and the implicit properties of mathematical statements by exploring in a dynamic mathematics environment (de Villiers, 1998; Sinclair & Robutti, 2013). Moreno-Armella, Hegedus, and Kaput (2008) suggested that new forms of symbolic thinking can be modelled through the evolutionary transition from static to dynamic inscriptions. This study investigated the effects of learning geometry proofs by using dynamic figures for eighth grade students. The study employed the proof of Thales' theorem as a reading text, and adopted a mixed method to collect data on 114 eighth grade students' learning performance and their cognitive load. Subsequently, 24 students were randomly chosen according to their mathematics achievements and interviewed to evaluate their thoughts and feelings. The main results showed that: (1) The students exhibited superior performance and a low cognitive load when they read geometry proofs with static figures. (2) According to the responses of 24 students, 12 and 9 students (50% and 37.5%) respectively stated that learning geometry proofs with dynamic figures and static figures would entail superior learning effects. In particular, most high-achieving students preferred static figures, but moderate- and low-achieving students preferred dynamic figures. (3) The students indicated that static figures stimulated their active thinking, and writing could reinforce their impressions and enhance their learning experience. Concurrently, dynamic figures triggered the students' motivation and were easier to understand than static figures, because the illustrations were presented stepwise.

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CAN SCHOOL SYSTEMS ATTAIN HIGHLY BOTH COGNITIVELY AND AFFECTIVELY? A STUDY OF MATHEMATICS LITERACY PERFORMANCE AND INTERESTS OF THE HIGH-PERFORMING ECONOMIES IN PISA 2012

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Many school systems aspire to attain highly not only cognitively but also affectively. That is, the students will achieve well academically and have enjoyment in their learning. Unfortunately, because of the examination-led school curriculum and inappropriate instructional methods, many school systems find that high attainment cognitively and affectively is not attainable. This is confirmed in the findings of past international sampled surveys. This study examines the relationship between cognitive and affective factors in the latest PISA 2012 survey (OECD, 2013a, 2013b) of which there were 65 participatory economies around the globe. This survey included scaled mathematical literacy performance measures and anchored mathematics interest measures.

At the turn of the new century, many economies such as Singapore and Shanghai are actively revising their mathematics curriculum standards to have dual foci on both cognitive and affective outcomes of schooling. The percentages of responses to the 5-point Likert scale of the four attitude items in the PISA 2012 mathematics interest scale of the top six high-performing economies (i.e. Shanghai, Singapore, Hong Kong, Taiwan, Korea and Macao) were analyzed. The findings reveal that both Singaporean and Shanghai students do have a very high interest in their mathematics learning. For instance, in Singapore, 78% of the 15-year-olds are interested in the things they learn in mathematics. In Shanghai, this percentage is 60%. In addition, it was found that in the two top-performing economies Singapore and Shanghai the sampled students attain highly both cognitively and affectively. Hong Kong, Taiwan, Korea and Macao, the other four high-performing economies in PISA 2016, demonstrate a balance in school mathematics instruction so that their students exhibit a positive interest in their mathematics learning. It is proposed that Shanghai and Singapore model mathematics teaching that may be used by other educational systems to improve mathematics education in the new century.

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DEVELOPMENT AND VALIDATION OF INDICATORS OF SECONDARY MATHEMATICS TEACHERS' POSITIVE DISPOSITIONS TOWARD PROBLEM SOLVING

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Current practice recognises mathematics teachers' positive dispositions toward problem solving and students' productive dispositions toward problem solving as significant, beneficial factors in teaching and learning. This report describes the application of Wilkerson and Lang's (2007) Disposition Assessment Aligned with Teacher Standards (DAATS) model to develop indicators of teachers' positive dispositions toward problem solving and an assessment framework to distinguish among teachers who exhibit more positive (or negative) dispositions toward problem solving.

This study produced four major results. First, participants reached consensus on the purpose, use and content of an assessment framework. Second, participants generated five broad indicators of positive disposition toward problem solving: (1) The teacher values worthwhile and mathematical rigorous problem solving; (2) The teacher values diversity of students' explanations, ideas and observations about mathematical problems; (3) The teacher values a risk-free problem-solving environment that ensures success of all students; (4) The teacher values the use of a variety of tools including technology to solve problems; and (5) The teacher believes that students must demonstrate a productive disposition toward problem solving (i.e., confidence, interest, appreciation, enjoyment, and perseverance). Third, participants' generated typical teaching behaviours for each indicator at each of the five taxonomic levels of the Affective Domain (Krathwohl, Bloom, & Masia, 1956). Fourth, participants developed an assessment framework that correlated indicators with methods for measuring teacher affect (i.e., disposition) at varying levels of inference. The results of the study also suggest items and instrumentation for assessing teachers' dispositions toward problem solving.

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STUDENTS' DISCUSSION ON SAMPLE SURVEY BASED ON AN ANIMATION STORY

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Recently in Japan, the importance of learning statistics has been emphasized. In the latest version of the national curriculum in Japan, contents related statistics are enriched. In addition, the aim of learning statistics has been changed. In the past curriculum, skills (for example making histograms, calculating averages, and so on.) were more emphasised. In contrast, in the present curriculum, solving problems which exists in our daily life or society applying statistics is considered more important. However, it is difficult for mathematics teachers in Japan to change their mind along with curriculum improvement. Therefore, we need support tools for teachers. Then, we have developed an animation story which can be used in classes of sample survey.

The CTGV (1990) developed video based stories called "The Jasper Series." The videos were developed based on a concept of the anchored instruction. The concept of the anchored instruction is very important for Japanese class. However, in Japan there are some difficulties in using the Jasper video series, that is, the contexts were somewhat different from Japan especially in terms of culture. In addition, the concepts used in the videos were not so sufficient corresponding to the national curriculum.

Therefore, we developed a new animation video focused on sample survey with simulated data. In the story, two students stand as a candidate for student council president. The other two students (Makoto and Egawa) perform election exit surveys. As the result, the two students concluded different results because of difference in sampling. By the simulated data, we can read the sample size in the survey by Makoto is greater than by Egawa. However, the survey by Egawa was intentional. The problem is which students could predict the tendency of the election more appropriately.

The class practices with the animation video have been held twice for grade 3 of junior high school. In the first practice, students could discuss about the problem. However, there were some students who express opinions not based on data, for example, based on students' experiences and looks of the characters in the animation.

Then, the second practice, we improved the simulated data. That is, students can easily read the tendency of voting by students' grades. As the result, we concluded that the number of opinions based on data increased.

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TRANSITION SCHOOL – UNIVERSITY: MEASURING MATHEMATICS FRESHMEN'S ACADEMIC BUOYANCY

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With mathematics freshmen, high dropouts are to be observed. Dropout does not seem to be only due to lacking knowledge but also due to unfavourable affective factors. Academic buoyancy may therefore be an important factor to be considered when examining dropouts. Academic buoyancy describes students' ability to cope with everyday setbacks, challenges, and pressures in a learning context that is "students' everyday academic resilience" (cf. Martin & Marsh, 2008, p. 53). We adapted academic buoyancy to the university context for mathematics freshmen and developed a questionnaire with the goal to assess this construct.

The questionnaire contained 11 Likert-type items addressing setbacks related to obligatory mathematics exercises, as these are the most pressing instances in the first semester (e.g., "If I don't manage to solve a math problem in less than three attempts, I resign.", "I don't mind to puzzle over a complex math problem for a whole afternoon or even longer."). Each item contained a 7-point Likert-scale (1 = strongly disagree, 7 = strongly agree). To check construct validity, we administered a Big Five personality scale. The sample comprises 100 mathematics and informatics freshmen (68% male, mean age 20.7) in their first week at university. For both groups a mathematics lecture and related exercises are obligatory in their first semester.

Based on an exploratory factor analysis two items were removed from the academic buoyancy scale (the items addressed the consequence of quitting the studies while the others did not). The remaining nine items showed a good reliability ($\alpha = .89$) and the distribution was fine ($M = 4.61$, $SD = 1.13$). As desired from a theoretical perspective, there was a medium correlation with the Big Five factor "conscientiousness" ($r = .42$, $p < .001$) but no correlation with the other four Big Five factors. Accordingly, we successfully developed a reliable instrument measuring academic buoyancy for mathematics freshmen and we have first evidence for convergent and discriminant validity. We think that this instrument is useful to shed further light on the crucial phase of the first semesters within mathematics studies. Currently, we collect data on content validity (expert rating) and prognostic validity (students' examination results) to finalize the quality check of the instrument.

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EVALUATION OF A SUPPORT PROGRAM FOR AT-RISK STUDENTS IN ARITHMETIC IN FIRST AND SECOND GRADE: THE ROLE OF TEACHERS' QUALIFICATION

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In Germany, at the end of primary education (grade 4), about 20% of the students do not achieve an adequate level of mathematical competencies. To address this problem, the support program MMS ("Mathe macht stark") which aims to support at-risk students in arithmetic in first and second grade, was implemented. MMS combines tasks for an individual diagnosis of learning problems with suitable learning material and is accompanied by teacher training sessions. The program begins in grade one, based on empirical evidence that basic arithmetical competences in pre-school and grade 1/2 substantially influence students' school-based competence development in mathematics (e.g., Aunola, Leskinen, Lerkkanen, & Nurmi, 2004).

To investigate the effectiveness of the MMS program, the implementation was evaluated in a quasi-experimental design with a control group and two implementation conditions. In the first condition, teachers were trained and received the program material. The second condition is similar to the first, but supplemented with two additional teaching lessons for the schools per week. We report on the first results of the first grade evaluation with 793 students. We used data from a mathematics test which were adjusted by variables on cognitive learning prerequisites. Since teacher qualification have substantial influence on students learning gains (Mullis, Martin, Foy & Arora, 2012), we also analysed differences between certified and non-certified mathematics teachers.

ANCOVAS for the whole sample and also for a subsample of students with low arithmetical prerequisites revealed no significant effects for the group variable (control group, implementation groups). For the whole sample, the analysis provided a small negative effect of the MMS program for students taught by non-certified mathematics teachers. Different possible explanations for these unexpected results will be presented in the poster.

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RECIPROCAL TEACHING ON MATHEMATICAL WORD PROBLEM LEARNING IN A REMOTE DISTRICT IN TAIWAN

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Students who solve a math word problem without completely reading may lead to incorrect presumptions regarding what the problem is asking. Therefore, the reciprocal teaching (RT) method, proposed by Palincsar and Brown (1984), was introduced to help students comprehend the meaning of mathematical word problems (e.g., van Garderen, 2004). Modifying their design of RT, five steps are used in this study; that is, predicting, questioning, clarifying, summarizing and solving.

This study aims to examine the effect of RT on mathematical word problems learning. The research questions are: (1) Does the intervention have a positive influence on students' mathematics performance and reading comprehension of word problems? (2) What is the students' perception of RT method? Therefore, the quasi-experiment and semi-structured interviews were conducted in this study. The participants were 25 6th graders from 4 different elementary schools in a remote district in southern Taiwan.

In order to answer the research questions, the author developed a mathematical word problem comprehension and performance test. Each word problem in this test consisted of five questions, which were designed to examine different abilities: the interpretation of sentences, the understanding of the problems' goal, the integration of sentences and reasoning, solution planning and solution execution. The purpose of the former three questions was to examine the ability of word problem comprehension; the others were designed to examine the students' mathematics performance. The t-test was used to compare participants' post-test scores to their pre-test scores.

The findings indicate that students' abilities of word problem reading comprehension and the accuracy of problem solving are significantly increased after the intervention. The results of the interviews suggest that students benefit greatly from RT method, because it provides more time to question and clarify the meaning of problems and the hidden information in word problems, to summarize a definitive solution plan and solve the problems. In addition, they are interested in predicting the goal of an unfinished word problem and taking turns to be a leader in the group.

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AN ANALYSIS OF CONNECTING PRACTICE IN ELEMENTARY MATHEMATICS INSTRUCTION

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Productive mathematics discussion is essential for high-quality mathematics instruction. Among five practices for orchestrating such discussions proposed by Smith and Stein (2011), the connecting practice plays a crucial role in providing students with an opportunity to learn the mathematical constructs embedded in various solution methods to a given task. However, even competent teachers often experience a difficulty in connecting students' solution methods so that they regard the connecting practice as the most difficult to be implemented (Pang & Kim, 2013).

Given this background, this study investigated in what ways elementary school teachers implemented the connecting practice. Three sixth-grade teachers who are familiar with five practices participated in the study. Nine lessons were videotaped and transcribed for an analysis. Students' worksheets as well as the teachers' lesson plans were collected. For this paper, three specific lessons were closely examined when the same task was employed across the classrooms. The task was about making a reasonable decision with different sizes and prices of pizza.

The findings of this study indicated that the teachers implemented the connecting practice in a similar way by (a) initiating classroom discussion with students' common misunderstanding, (b) employing appropriate instructional strategies, (c) making full use of various visual representations, and (d) making some connections between students' solution methods and key mathematical ideas. However, subtle but significant differences were revealed by a closer examination of the lessons. For instance, Teacher A frequently posed questions to induce the specific response she had expected, and Teacher B tended to use teacher-directed and short-answer questions. In contrast, Teacher C urged students to figure out essential mathematical ideas for themselves by comparing and contrasting multiple solution methods. Our proposed poster illustrates these differences as well as similarities with representative excerpts. It also displays students' various solution methods and their connections implemented differently across the classrooms. As such, this poster is expected to provoke discussions towards better implementation of the connecting practice.

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PROFESSIONAL LITERACIES FOR THE PRESERVICE MATHEMATICS TEACHER

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Multiple "literacies" (see Ahmed, 2011) or "professional literacies" such as media literacy, and those of subject content knowledge, assessment and evaluation, leadership, learning disabilities, and pedagogy, to name a few, provide support for and a foundation to all teachers' professional discourse. Mathematics teachers' professional discourse requires a complex interplay of this plurality of literacies. For preservice mathematics teachers (PMTs), mathematical literacy may be arguably considered a necessary and underlying connecting literacy to their professional literacies. Thus no matter the length of a teacher preparation program, it is a relatively short amount of time compared to the length of a teaching career for exposure to and learning about the professional discourse of the mathematics teacher.

In conjunction with the literacies required of educators, the literacies that students use and learn are evolving, creating an additional impact on the literacies that teachers need. Learning to teach then becomes less about absorbing particular bits of information and more about acquiring specific literacies that will endure this evolution in communication practices (Pilgrim & Bledsoe, 2011).

The purpose of this study is to explore the nature of PMTs' professional literacies and the relationship with mathematical literacy, and PMTs' acquisition of these literacies. A phenomenological stance is being employed within the context of a preservice secondary school mathematics teaching and learning course. Data are being acquired from three questionnaires exploring PMTs' recall and understanding of professional literacies, and a focus group discussing professional practice scenarios.

Available data and analysis will be used to create an in-depth understanding of PMTs' perceptions and knowledge of professional literacies, and what these literacies mean to PMTs' development of their professional practice. Changes in PMTs' knowledge of these literacies may be tracked over the duration of the teacher preparation program, for example, from incorrect usage to buzz words to authentic expression. Understanding how these literacies are acquired and then hearing how PMTs perceive using these literacies in classroom scenarios may provide insight into what PMTs are learning, as well as into teacher preparation course design changes.

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CHILDREN´S USE OF STRATEGIES IN ESTIMATING LENGTH AND CAPACITY

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In recent years, measurement estimation became part of the mathematics curricula in Germany. At the same time it became obvious that little is known about the abilities of children in this field, especially on the primary level.

Since most research in measurement estimation is focused on lengths and mainly on older students (Sowder 1992), we focused on younger children and included lengths as well as another measurement, area. Our tasks were constructed with reference to Bright's (1976) typology of requests in estimating length. In each measurement area and four types of requests five tasks were constructed, so overall 40 estimation tasks were presented. 46 (27 ♀; 19 ♂) 4th-graders from different schools solved these tasks in individual interviews which lasted about 15-20 minutes and were videotaped.

Although strategies in estimating lengths are labeled differently by different authors (Hildreth 1983; Joram et al. 2005; Siegel et al. 1982), using *benchmarks* as reference points, *iterate mentally a unit* or *decompose and recompose* the to-be-estimated object mentally can be extracted from most studies.

The results of our study show that 4th-graders can make use of a variety of different strategies to estimate lengths and capacities. All strategies known from literature could be observed in our data as well. We could specify all of them into different subgroups as well as describe some more. The strategies to estimate capacities are mainly the same as those to estimate lengths. The poster will show in detail the tasks and the categorical system of strategies, which in some sense is depending on the measurement area as well as on the task environment.

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THE EFFECTS OF WORK ETHICS AND ATTRIBUTIONS TO FAILURE ON MATHEMATICAL LITERACY PERORMANCE: A STUDY OF LEARNING CHARACTERISTICS OF ESCS-ADVANTAGED LOW-ACHIEVERS IN SHANGHAI, SINGAPORE, HONG KONG, TAIWAN AND KOREA

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Some students from ESCS-advantaged homes participating in PISA 2012 were classified as *academic slackening* (called *advantaged low-achiever*, the ALA in this study). These students have not attained as highly in their academic studies as might have been predicted from their advantaged home background (Ieong, Cheung, Sit, & Mak, 2014). Drawing data from PISA 2012, this study examined the similarities and differences in two selected learning mathematics characteristics (i.e. *mathematics work ethics* and *attributions to failure in mathematics*) amongst students of the five top-performing Asian economies: Shanghai, Singapore, Hong Kong, Taiwan and Korea (OECD, 2013). Examples of good work ethics include studying hard and avoiding distractions in lessons. Examples of undesirable attribution of failures are reasons referring to difficult curricular materials or poor teacher explanations. From the international comparative education perspective, these students were classified as ALA with reference to their advantaged counterparts respectively.

Logistic regression was carried out for the ALA versus non-ALA student classification, as a function of the demographic and the two selected mathematics learning characteristics. The results of the logistic regression analysis showed that *some* of the following variables like family and academic background (i.e. *gender*, *family structure*, *years of attending kindergarten*, *grade repetition*), as well as the *mathematics work ethics* and *attributions to failure in math*ematics variables, are able to predict whether a student is more likely to be classified as ALA or not. Specifically, *mathematics work ethics* is predictive for Taiwan and Korea, whereas *attribution to failure in mathematics* is predictive for Singapore, Hong Kong and Taiwan. The findings are important to shed light on the principles and methods of mathematics education so as to help the ESCS-advantaged low-achievers to capitalize on their opportunities and potential to advance to a higher level of mathematical literacy attainment.

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THE IMPACT OF ADAPTED TEACHER COACHING REFERRING TO LEARNING DIFFICULITIES RELATED TO FUNCTIONS

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Reasoning with functions is an important goal of mathematics education. Findings of empirical studies indicate that several learning difficulties are common in this field and that teachers often are not aware of them (e.g. Hadjidemetriou & Williams, 2002). Nitsch (in press) concludes from class differences referring to learning difficulties with linear and quadratic functions that teachers can counteract such learning difficulties through their teaching. Even if these findings suggest that teachers' professional development (TPD) focusing on such typical learning difficulties may enhance teachers' pedagogical content knowledge (PCK) and students' learning in this field, there is no empirical evidence about these interdependencies, yet.

The research goal of this study in developmental stage is to promote teachers' PCK and students' achievement concerning (learning difficulties in) reasoning with functions, particularly related to linear functions. Therefore, this project develops and evaluates coachings for teachers implementing components of TDP that have shown to be effective in prior research (Lipowsky, 2013: e.g. domain-specificity, long-term nature, feedback). To gain empirical evidence about the effectiveness of these coachings, 60 teachers are randomly assigned to a control group or to one of two treatment groups. Both treatments convey PCK about learning difficulties related to reasoning with functions and ways to overcome them, whereas only one treatment additionally coaches teachers in giving supportive feedback to students exhibiting such learning difficulties. These teachers as well as their students pass pre-, post-, and follow-up-tests: Specific test instruments evaluating teachers' PCK are developed in this project. Students' competency tests can be adapted from the instrument conceived by Nitsch (in press). Multilevel analyses of the data also include relevant covariates such as motivational variables or cognitive abilities.

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LEARNING FRACTIONS WITHOUT EQUIPARTITION

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Fractions are a well-researched content area. Yet, student learning of fractions remains problematic. We outline a novel path to initial fraction learning and document its promise. Our work derives from Freudenthal's (1983) analysis of the fraction concept in that we regard *comparing*, instead of *fracturing*, as the primary activity from which students are expected to make sense of fractions. The path we outline is a product of a classroom design experiment conducted with a class of 14 fourth grade pupils. In this path, we identify two successive images of fractions that the students came to develop and indicate how their emergence was supported.

The first image involves picturing unitary fractions as numbers that account for the size of entities that are physically independent from the reference unit. The size of these independent entities is determined by the number of times they would have to be iterated to make as much as one reference unit. For instance, a rod would be 1/3 as long as a stick when by iterating it three times, it would render a length identical to the length of the stick. The *inverse order relation of unitary fractions* is part of this image: The more times a rod fits on the reference unit, the smaller it has to be.

The second image entails the reciprocal relation. When students develop this image, it becomes clear that three iterations of the span of a rod that is 1/3 as long as a stick would render a length identical to the length of the stick. Relying on this image, students become capable of correctly judging a fraction as representing a size smaller than, as big as, or bigger than the reference unit (e.g., $2/3<1$; $3/3=1$; $4/3>1$)

Our data indicate that the learning path, which excludes scenarios based on equipartition from initial fraction instruction, succeeds in circumventing what Norton and Hackenberg (2010) consider to be serious developmental hurdles in fraction learning. From the very start, students came to see fractions as numbers that quantify the relative size of an attribute (i.e., the length span of a rod). They also viewed unitary fractions as capable of being iterated unrestrictedly. Thus, they did not need to overcome the image in which the iteration of a unit fraction is considered to be restricted by the unit whole (e.g., a third can only be iterated three times).

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AN INTERNATIONAL COMPARISON OF THE RELATIONSHIP BETWEEN MATHEMATICS AND READING ACHIEVEMENT: FOCUSING ON PISA2003 AND PISA2012

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This study was undertaken to better understand the relationship between mathematics and reading achievement through a secondary analysis of PISA2003 and PISA2012. The search for this relationship has been a focus in mathematics education research (e.g., Máire & John 2009). In PISA, the mathematics and reading literacy are examined continuously every three years from the year 2000. The relationship between mathematics and reading can be analysed including its secular changes.

In this study, the data of PISA2003 and PISA2012 are analysed from the view point of international comparison. The countries which have higher achievement in mathematics tend to also have higher achievement in reading. However, mathematical and reading literacy in PISA are different types of psychological construct. Therefore, the correlation between mathematics and reading achievement is identified by means of participating countries' average score, but even so the relationship between two of them at student-level within a country remains incompletely understood. In order to make this point clear, we will focus not only on the indexes of reflecting the levels of mathematics and reading achievement of each country, but also the indexes of reflecting the relationship between two of them at student-level within a country, such as correlation coefficient and regression coefficient and so on.

In this analysis, multilevel regression analysis is mainly used to look at country-level and student-level in order to clarify the degree of the relationship between mathematics and reading achievement at student-level among participating countries. As a result, the higher the countries have scored in mathematics and reading achievement, the stronger the relationship between mathematics and reading at student-level. The results comprehensively suggest that there should be an international trend that reading has the effect of enhancing mathematics achievement. In addition, since we have gotten comparable result in PISA2003 and PISA2012, this trend has a certain amount of robustness.

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THE CONSISTENCY EFFECT ON COMPARE PROBLEMS: AN EYE MOVEMENT STUDY IN PRIMARY SCHOOLS

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Students have difficulty in solving arithmetic problems containing a relational term that is inconsistent with the required arithmetic operation, leading to more reversal errors. Previous studies have showed that high-accuracy undergraduates took more time for inconsistent than consistent problems, but low-accuracy undergraduates didn't. The present study aimed to examine elementary students' performance and eye movements on two-step compare problems. Twenty-nine sixth graders were recruited from two schools in Taipei. Four consistent items and 4 inconsistent items were used. Participants' eye movements during reading were collected by the Eyelink 1000. They were categorized to 3 types based on the solution pattern and their read comprehension tests were equal. Good solvers (n = 8) committed 2 or less errors on the 8 items. Reversal solvers (n = 8) committed 3 or more errors, most of which were reversal errors. Poor solvers (n = 8) committed 3 or more errors, most of which were not reversal errors. Figure 1 shows the accuracy and total fixation duration (TFD) per item, for the 3 types of students. Good and reversal solvers solved consistent items more successfully than did the poor solvers, $F(2, 21) = 4.90$, $p = .018$. Additionally, good solvers solved inconsistent problems more successfully than did the reversal and poor solvers, $F(2, 21) = 15.76$, $p = .000$. The TFD of consistent items was marginally shorter for the good solvers, than that for the inconsistent items, $t(7) = -2.18$, $p = .066$, while it was the same for both types of items for the reversal and poor solvers. The results showed that good solvers possessed schema for compare problems, took more time to process inconsistent problems, and solved the problems more successfully. Reversal error solvers possessed basic mathematics skills, but perceived inconsistent problems as consistent problems. Poor solvers exhibited worse mathematics skills than their peers, committed more errors on both types of problems, and took the same time to solve them.

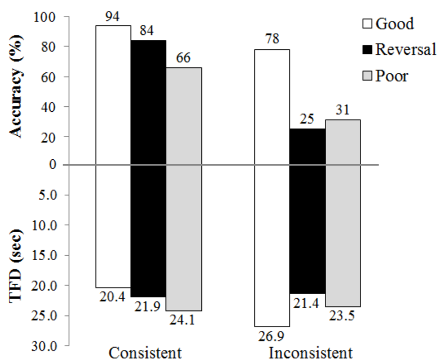

Fig.1 Accuracy and total fixation duration

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GROWTH IN COGNITIVE DOMAIN OF GEOMETRY: A 3-YEAR LONGITUDINAL STUDY

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The majority of research in mathematic growth focuses on content domain, such as number or geometry (Anderman, Gimbert, Connell, & Riegel, 2014). Research into mathematics growth in cognitive domains can offer information for teachers on how to enhance instruction. The Van Hiele model is widely used to describe students' learning levels in geometry from a general perspective. However, there is very limited study on growth in cognitive domain of geometry. Further, most studies use standardised scores from different scales to chart growth. The use of standardised scores may raise a problem that measures of growth did not have a substantive meaning with reference to mathematics (Mok, McInerney, Zhu, & Or, 2015).

The purpose of this study was to investigate students' longitudinal growth in detail in the cognitive domain of geometry. The research questions are: 1) What is the relationship between initial achievement and growth rate in the domain (concept understanding, procedure knowledge, and problem solving)? 2) Is there any growth subtype in geometry development? The sample comprises 452 students with three waves (Grade 4, 5, and 6). Three tests with anchor items were developed. A common measurement scale across grade levels was constructed by vertical linking based on item response theory. The latent growth model was applied to analyse the longitudinal data and cluster analysis is used to explore the subtypes of developmental trajectories.

Based on the fit statistic, the linear latent growth models have reasonably good fit with the data. The respective growth of conceptual understanding, procedural knowledge, and problem solving in geometry among students is in a linear trend. The correlations between intercept and slope in conceptual understanding, procedural knowledge, and problem solving are -.89,-.88, and -.86 respectively. These negative values means there was evidence of compensation effect. The subtype of developmental trajectories in concept understanding of geometry include "low initial status, faster growth", "average initial status, stable growth", and "high initial status, static growth".

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AN INTELLIGENT TUTOR-ASSISTED MATHEMATICS INTERVENTION PROGRAM FOR STUDENTS WITH LDM

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Students whose math performance was ranked at or below the 25 percentile are often considered at risk for learning disabilities or for having learning difficulties in mathematics (LDM). The most recent National Assessment of Educational Progress results show that score gains were seen in mathematics at grades four and eight for higher performing students, but there were no significant changes over the same period for lower performing students at the 10th and 25th percentiles (NAEP, 2013). That means the gap between students with LDM and their normal achieving peers is getting wider. There is a need to explore potential intervention support to facilitate concept development of students with LDM so they become an independent problem solver. Constructivist-oriented learning theory promotes an inquiry-based instructional environment where the learners assume an active role in their efforts to make sense of mathematics. However, there is a lack of literature on measuring how students with LDM grow from less matured (need a lot of prompting) to a more independent problem solver in a constructivist learning environment.

As the outcome of a collaborative work that integrates research-based practices from mathematics education and special education, the researchers in this study have developed an intelligent tutor, PGBM-COMPS, that integrates constructivist mathematics pedagogy, model-based problem solving, and intelligent tutoring techniques to promote multiplicative reasoning and problem solving. Using a multiple baseline design (MBD, Kazdin, 1982) across four 4th grade students with LDM, the purpose of this study was to explore the impact of the PGBM-COMPS intelligent tutor on multiplicative concept development and problem solving of students with LDM. Unlike uncontrolled case studies, intervention effects in MBD can be demonstrated by introducing the intervention to different participants at different points in time. If each baseline changes when the intervention is introduced, the effects can be attributed to the intervention (Kazdin, 1982). In this MBD study, each student's progress in concept development (measured by the number of prompts needed to solve the problem correctly) was recorded from session to session, and it was compared to the accuracy of his/her problem solving monitored by a multiplicative reasoning/problem solving test throughout the program. The results will be shared and discussed at the session.

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THE CONCEPT OF UNITS AND ITS RELATION TO THE UNIT ITERATION

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Children entering school have already acquired competences regarding measuring length. The following research investigates different aspects of a conceptual knowledge about measuring as well as children's competencies at pre-school age concerning this knowledge. Three main aspects of a measuring-concept can be identified regarding different empirical studies. Although the aspects are mentioned with slightly different notions the following aspects can be described: (1) choosing a unit, (2) decomposing the length into equal parts (procedural activity: unit iteration), (3) counting the units (e.g. Benz, Peter-Koop & Grüßing 2015, p. 234f).

In order to investigate the measuring-concept of young children the interrelation between these three aspects will be analysed. 40 children in the age of 4-6 years were asked to conduct an indirect comparison in different tasks (1) without any tool, (2) with different tools (standardised and non-standardised media) (3) with many tools of equal lengths. Through analysing the videotaped interviews, different categories were generated to describe the children's processes (Kluge 2002) in order to draw conclusions about the children's conceptual understanding of units.

The analysis revealed that many children are able to use unit iteration in a correct way without any gaps when they are offered many real objects of the same kind as "units" and also counted their "units". When these children are offered only one object of each standardised or non-standardised shorter tool, however, they either count with some kind of imaginary units while holding the tool in their hands, or they decompose the length into different parts, without producing equal partial lengths. The results can be interpreted in various ways. On the one hand, it could be inferred that children at this age have no technical skills to find a solution when there are not many "units" available. On the other hand this could be seen as an indicator that although children are capable of conducting unit iteration with many tools correctly on a procedural level, having an idea about units seems to be another part of a conceptual knowledge. This may lead to the conclusion that a conceptual knowledge of units is quite demanding which could be explained through further empirical and theoretical analysis.

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