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Research Reports

Gom - Pan

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IS THERE A NATURAL NUMBER BIAS WHEN COMPARING FRACTIONS WITHOUT COMMON COMPONENTS? A META-ANALYSIS

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The natural number bias (NNB) refers to the detrimental effect that knowledge about natural numbers might have on performance in reasoning about fractions and rational numbers. Some studies have, however, documented conflicting results about the presence of biased reasoning in fraction comparison tasks where the fractions to be compared have no common components. Here, we report a meta-analysis of five datasets where we specifically looked at these violations of the NNB predictions. Results show consistent departures from the NNB across datasets, suggesting the need for a richer understanding of the cognitive underpinnings of fraction comparison. Such rich understanding requires approaching fraction comparison as a problem that learners face using specific, sometimes spontaneously devised, strategies.

INTRODUCTION

Rational numbers constitute a pivotal mathematical content of the elementary and/or middle school curricula. Much research has been devoted to document the many misconceptions that children and adults display when learning and reasoning about rational numbers, as well as searching for the educational and cognitive bases of these misconceptions (e.g., Clarke & Roche, 2009; Vamvakoussi, Van Dooren, & Verschaffel, 2012). A common framework for understanding many of these misconceptions is called *whole number bias* or *natural number bias* (hereafter, NNB). Ni and Zhou (2005) pointed that the actual cause for such bias is not clear so far, proposing three possible accounts differing in terms of the involvement of cognitive and educational mechanisms. One interpretation of the NNB account is that learners—not just children—fail to understand rational number concepts and operations because, when dealing with rational numbers, they resort to the concepts and operations that correspond to natural numbers. Common examples are: (a) stating that $4/7 < 5/7$ but that $1/4 < 1/5$, in line with the fact that $4 < 5$; (b) computing operations such as $1/2 + 1/3$ on a component-by-component basis, leading to an incorrect result of $2/5$; (c) claiming that there are only three other fractions between 1/7 and 5/7. These mistakes have been amply documented by many researchers (e.g., Clarke & Roche, 2009; DeWolf & Vosniadou, 2011; Vamvakoussi et al., 2012).

Recently, we presented data from a fraction comparison task administered to Chilean children from 5th to 7th grade (Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014). In addition to the presence of a strong NNB (the overall accuracy gap between congruent and incongruent items was about 37%), we observed another relevant pattern, namely that congruent items without common components (e.g., 5/7 vs. 1/3) had very low, or even negative, correlations with overall test performance. Moreover, high achievers performed substantially lower in these items than in all other item types (58% vs. 80%). This implies, in particular, that when fraction pairs had no common components, these children performed better in incongruent than in congruent items, against the predictions of the NNB account. Nonetheless, ours was not the first result in the literature pointing in this direction. Obersteiner, Van Dooren, Van Hoof, and Verschaffel (2013) presented a fraction comparison test to expert mathematicians and observed that their response times showed an advantage for congruent items in fraction pairs with a common component (congruent items were answered about 250 ms more quickly) but that this difference reversed when fractions lacked a common component (congruent items were answered about 570 ms more slowly). DeWolf and Vosniadou (in press) tested skilled young adults from the US and Greece with fraction pairs without common components, and observed opposite effects in both populations: US participants behaved in agreement with the NNB, whereas Greek participants erred significantly more in congruent items and showed no response time difference due to congruency. These conflicting results raise the question about what results can be generalised across different samples and what others are specific to a given age group, country, or level of expertise.

In this work we provide a deeper assessment of the predictions of the NNB, focusing on the specific case of fraction comparison and on the issue of whether a reversal of the congruency effect occurs when considering fraction pairs without common components. We do so by putting together a group of previously published and novel datasets from fraction comparison tasks spanning children and adults. A consistent reversal over a variety of these populations would strongly indicate the need for a revision of the NNB framework, at least for the case of fraction comparison. Solving items without common components (no-CC) takes systematically longer than solving items with a common component (CC), suggesting that learners engage in strategic reasoning when solving the latter item type. A strong reversed bias for no-CC items might thus reveal the kind of strategies learners are using to deal with fractions.

Throughout this report and in line with previous research (e.g., Gómez et al., 2014; Obersteiner et al., 2013; Vamvakoussi et al., 2012), we classify fraction pairs according to two dimensions: the presence or absence of common components, and the congruity relation between the natural numbers composing the fractions and the fraction magnitudes. *Congruent* items are those in which the largest fraction has the largest numerator and denominator; *incongruent* items are those in which the largest fraction has the smallest numerator and denominator; and *neutral* items are those in which the largest fraction has the largest numerator and the smallest denominator. Table 1 shows the five possible item types stemming from these dimensions, and exemplars for each type. According to the NNB, in congruent items the magnitude information of numerators, denominators, and fractions all align, making these items the easiest to answer. In contrast, incongruent items portray an information conflict between numerators and denominators on the one hand and fractions on the other, making them the hardest items to answer. Finally, numerators and denominators in neutral items point in opposite directions and are thus supposed not to strongly affect judgements about the fractions themselves.

	Congruent	Incongruent	Neutral
With a common component	16/21 < 20/21	4/17 > 4/39	-
Without common components	17/21 > 9/14	11/25 < 8/13	7/16 > 5/29

Table 1: Examples for each item type in the fraction comparison tasks analysed (taken from Obersteiner et al., 2013).

METHODS

Datasets

For this meta-analysis, we considered five datasets from fraction comparison tasks. (A) Gómez et al.'s (2014) data from 450 Chilean children from 5th to 7th grade; (B) Van Eeckhoudt's (2013) data from 62 Belgian 6th grade children; (C) Van den Brande's (2014) data from 91 Belgian undergrads from Educational Sciences; and (D) a dataset from an ongoing project in the University of Chile, led by the first author of this report. This dataset comprises data from 49 Chilean undergrads from a variety of University courses. (E) Obersteiner et al.'s (2013) data from 46 expert mathematicians.

For all datasets, fraction pairs were presented on a computer screen and participants selected the largest fraction of each pair by using the keyboard. Children in dataset A were given a maximum of 10 s to respond to each item with an on-screen clock showing the time left for answering, whereas all other participants had no time limit.

Data, items, and item types

All datasets consist of accuracies and response times to fraction comparison items. Dataset A includes 24 fraction pairs divided into congruent/incongruent and with/without common components (see Gómez et al., 2014, for the full list of items). Datasets B, C, and E comprise 90 fraction pairs: 36 pairs having a common component and divided into congruent/incongruent, and 54 pairs with no common components and divided into congruent/neutral/incongruent pairs (see Obersteiner et al., 2013, for the full list of items). Dataset D uses a subset of 40 items extracted from Obersteiner et al.'s (2013) list, equally divided into congruent/incongruent and with/without common components (and no neutral items). Given the diversity of item types across studies, we focused analysis on congruent and incongruent items only.

Data analysis

As noted by Gómez et al. (2014), dataset A displays a high number of children presenting extremely biased accuracies, with all congruent items answered correctly and all incongruent items incorrectly. Given this marked NNB pattern present in some children, we separated dataset A into low (A-low, n = 291) and high (A-high, n = 159) achievers depending on whether overall accuracy for comparing fractions was below or above the average value of 59.3%, respectively.

For the response time analysis, we computed median response times per participant and item type (considering only correctly answered items).

We used a Mahalanobis distance (MD) criterion in order to discard participants behaving in a very different way with respect to the others within each dataset. In brief, the MD measures the distance of each participant to the centroid of the whole sample, taking into account the differences in variability for each dimension. In this case, we represented each participant by his/her four accuracy values for each item type. Separately for each dataset, means and covariance matrices were estimated in this 4-dimensional space and used to compute MDs. Participants with a MD higher than 11.34 (99% percentile of a chi-square distribution with 3 degrees of freedom) were discarded from further analysis. This criterion led to the exclusion of 26 participants (9%) in dataset A-low, nine (6%) in dataset A-high, two (3%) in dataset B, nine (10%) in dataset C, four (8%) in dataset D, and three (7%) in dataset E.

RESULTS

Accuracy

Table 2 shows accuracy values for all datasets and item types. We analysed the interaction of congruency and presence/absence of common components (CC) by means of logistic regressions with these two factors as fixed factors and participants as a random factor, and computed congruency effects with separate regressions for items with and without common components (Table 3).

When comparing CC fraction pairs, the different samples show high variability. Children in the A-low dataset show a strikingly large congruency gap, confirming that their answers were strongly influenced by a NNB. All other datasets display a much smaller congruency effect for CC items, and this effect seems to have no consistent direction. Statistical significance varies importantly from one dataset to the other, as well. These results stand in sharp contrast with those of no-CC fraction pairs, where apart from the A-low dataset, all samples showed a negative congruency effect. This supports that responses of children in the A-low sample were almost entirely driven by congruency, regardless of the presence or absence of common components. Nonetheless, all other samples displayed higher accuracy for no-CC incongruent items and this effect was highly statistically significant except for the case of expert mathematicians. Experts performed very close to ceiling levels for all item types (28 out of the 43 experts erred in no more than 2 items), so any congruency effect in their accuracy has, at best, a very small magnitude.

Response times

Table 4 shows response times for all datasets and item types. We analysed the interaction of congruency and presence/absence of common components by means of linear regressions with these two factors as fixed factors and participants as a random factor, and computed congruency effects with separate regressions for items with and without common components (Table 5). Given the very low numbers of correct answers for incongruent items of children in the A-low sample, we analysed response times only for the other 5 datasets.

Dataset	With a common component		Without common components	
	Congruent	Incongruent	Congruent	Incongruent
A-low (children)	81.1%	14.9%	84.2%	14.5%
A-high (children)	92.1%	89.1%	56.5%	87.7%
B (children)	70.7%	79.8%	53.4%	75.0%
C (undergrads)	93.8%	93.0%	82.5%	92.5%
D (undergrads)	93.8%	96.4%	77.3%	84.7%
E (experts)	97.9%	98.8%	95.2%	96.0%

Table 2. Accuracies for each dataset and item type.

Dataset	Congruency × CC interaction	Congruency effect (CC items)	Congruency effect (no-CC items)
A-low (children)	-0.26 (p = .06)	3.22 (p < .0001)	3.48 (p < .0001)
A-high (children)	2.20 (p < .0001)	0.36 (p = .03)	-1.83 (p < .0001)
B (children)	0.57 (p = .0001)	-0.70 (p < .0001)	-1.09 (p < .0001)
C (undergrads)	1.16 (p < .0001)	0.14 (p = .39)	-1.02 (p < .0001)
D (undergrads)	-0.09 (p = .81)	-0.64 (p = .06)	-0.50 (p = .004)
E (experts)	-0.29 (p = .54)	-0.48 (p = .24)	-0.19 (p = .45)

Table 3. Interaction between congruency and presence/absence of common components and congruency effects for CC and no-CC items separately, obtained from logistic regressions on accuracies. A positive congruency effect means that congruent items were answered more accurately than incongruent items.

Three out of the 5 considered samples showed statistically significant congruency effects in response times to CC fraction pairs, in the direction predicted by the NNB account (incongruent slower than congruent). Dataset D also showed an effect in the same direction, although statistical significance was not reached. Congruency effects for no-CC fraction pairs, instead, were highly consistent and systematically positive, with statistically significant differences present in all samples excepting D. Further support for the presence of a differential congruency effect for CC and no-CC items is provided by the regression interaction terms, which were statistically significant for all datasets.

This pattern adds to the evidence for a reversed congruency effect across experimental samples: answering a no-CC incongruent item correctly takes a median 20% less time (A-high: 20%, B: 21%, C: 22%, D: 10%, E: 19%) than answering correctly a no-CC congruent item.

Dataset	With a common component		Without common components	
	Congruent	Incongruent	Congruent	Incongruent
A-high (children)	2855	2914	3822	3074
B (children)	2723	2947	3513	2761
C (undergrads)	1992	2292	3480	2698
D (undergrads)	2968	3757	7430	6684
E (experts)	1746	2075	4805	3912

Table 4. Response times (in milliseconds) for each dataset and item type.

Dataset	Congruency × CC interaction	Congruency effect (CC items)	Congruency effect (no-CC items)
A-high (children)	-694 ($p < .0001$)	0 ($p = .99$)	662 ($p < .0001$)
B (children)	-1624 ($p < .0001$)	-591 ($p < .0001$)	913 ($p < .0001$)
C (undergrads)	-1235 ($p < .0001$)	-388 ($p < .0001$)	849 ($p < .0001$)
D (undergrads)	-1536 ($p = .03$)	-410 ($p = .19$)	1083 ($p = .09$)
E (experts)	-1138 ($p = .0001$)	-374 ($p < .0001$)	800 ($p = .002$)

Table 5. Interaction between congruency and presence/absence of common components and congruency effects for CC and no-CC items separately, obtained from linear regressions on response times. A negative congruency effect means that congruent items were answered more quickly than incongruent items.

DISCUSSION

We have presented a meta-analysis assessing the prevalence of a NNB across a range of experimental datasets with children and adults. Our results support the presence of a NNB when learners compare fraction pairs with a common component, but point in the opposite direction for fraction pairs without common components.

Natural Number Bias as a data pattern

Results show that learners' performance in CC fraction pairs aligns with the prediction that incongruent items are more difficult in terms of their response times but not necessarily in accuracy. In contrast, for no-CC fraction pairs the data systematically points in the opposite direction to the NNB account: congruent items are answered in general less accurately and more slowly than incongruent items. The only exception for this is dataset A-low, where children seem to endorse a view of fractions as two independent numbers (the initial explanatory framework identified by Stafylidou & Vosniadou, 2004) and thus respond in a way highly aligned with the NNB. In this sense, DeWolf and Vosniadou's (in press) sample of USA undergrads, whose answers to no-CC items followed the NNB, seem to be the exception rather than the rule.

Natural Number Bias as a cognitive mechanism

Ni and Zhou (2005) identified as one of the possible sources for the NNB an innate disposition towards natural numbers. Such disposition would act at a cognitive level, actually causing the struggle learners face when facing rational numbers. Other researchers have proposed that the NNB is a form of intuitive reasoning (e.g., Vamvakoussi et al., 2012), possibly due to the extensive practice and familiarity with natural numbers that children acquire before learning fractions. A general cognitive mechanism or disposition towards natural numbers, however, fails to explain why congruency impairs performance in comparison of CC fraction pairs but not of no-CC fraction pairs. One must then consider that different cognitive mechanisms might underlie performance for CC and for no-CC items: whereas the former might involve cognitive interference mechanisms (Meert, Grégoire, & Noël, 2010), the latter probably engages complex strategies (e.g., benchmarking) and thus the origin of the reversed congruency effect depends on the actual strategies used. In particular, learners who have already succeeded in understanding the basic concepts of fractions might use different strategies for CC and for no-CC items. CC items can be quickly dealt with by componential reasoning and without resorting to fraction magnitudes. No-CC items, instead, might be treated in heuristic ways such as that proposed in Gómez et al. (2014), namely choosing as the largest fraction the one with the smallest denominator. This heuristic answers correctly all no-CC incongruent items and incorrectly all no-CC congruent items, and it is probably appropriate for fraction pairs whose numerators are close to each other.

Stafylidou and Vosniadou (2004) showed three explanatory frameworks used by children in dealing with problems about fractions: (a) fractions as two independent numbers, (b) fractions as parts of a whole, and (c) fractions as a relation between numerator and denominator. A pure NNB account strongly identifies with the first framework, and coincidently we observe that children with the lowest level of understanding of fractions presented the most componential, NNB-aligned behavior. In contrast, learners who have reached the second and third frameworks are likely to engage in strategic thinking, making their responses more diverse and complex. Most probably, learners choose for each comparison problem whether to respond heuristically or to estimate the magnitude of the fractions involved. Further studies should carefully look at how children and adults solve no-CC items, in order to discover the spontaneous strategies that they employ, and the appropriateness of these strategies for the development of a mathematically sound concept of rational number.

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CONCEPTUALISING AND ENACTING NUMERACY ACROSS THE CURRICULUM

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Numeracy refers to the use of mathematics in non-mathematical contexts. In this paper two approaches to conceptualising numeracy across the whole school curriculum are identified: one based on interdisciplinary inquiry and the other on embedding numeracy into each school subject. The latter approach informed a systematic audit of resources available to Australian teachers for understanding and enacting numeracy across the curriculum. It was found that few resources addressed the need for teachers to recognise and take advantage of the numeracy learning demands and opportunities within the subjects they teach.

BACKGROUND

Numeracy is a term used to identify knowledge, skills and practices related to the use of mathematics in non-mathematical contexts and, in particular, to the use of mathematics in work, home and civic life. Steen (2001) identified seven dimensions of numeracy (using the term quantitative literacy): confidence with mathematics; appreciation of the nature and history of mathematics and its significance for understanding issues in the public realm; logical thinking and decision-making; use of mathematics to solve practical everyday problems in different contexts; number sense and symbol sense; reasoning with data; and the ability to draw on a range of prerequisite mathematical knowledge and tools. Increasing international focus on numeracy, as part of schooling and beyond, is evident in the emergence of testing regimes such as the Programme for International Student Assessment (PISA) and the Programme for the International Assessment of Adult Competencies (PIAAC).

In Australia, the notion of numeracy as an important goal for schooling was confirmed through a national numeracy review (Council of Australian Governments, 2008), which also promoted the view that the development of students' numeracy requires a cross-curricular commitment by schools and systems. This review recommended that:

> …all systems and schools recognise that, while mathematics can be taught in the context of mathematics lessons, the development of numeracy requires experience in the use of mathematics beyond the mathematics classroom, and hence requires an across the curriculum commitment. (p. 7)

Further, numeracy has been identified as one of seven General Capabilities embedded in the Australian Curriculum – the first ever nationally mandated curriculum in this country. Numeracy is described within each school subject's curriculum document via the following statement:

> Students become numerate as they develop the knowledge and skills to use mathematics confidently across all learning areas at school and in their lives more broadly. Numeracy involves students in recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully. (Australian Curriculum, Assessment and Reporting Authority, 2014a, p. 13)

A commitment to developing numeracy across the curriculum is also evident in the Australian Professional Standards for Teachers, a set of statements that specify the professional knowledge, professional practices, and professional engagement required of effective teachers (Australian Institute for Teaching and School Leadership, 2014a). Standard 2 states that teachers should "Know the content and how to teach it", and one of the focus areas elaborating on this statement relates to knowledge of literacy and numeracy strategies. Thus proficient teachers should be able to "apply knowledge and understanding of effective teaching strategies to support students' literacy and numeracy development".

However, apart from these curriculum mandates and professional standards statements, Australian teachers are provided with little guidance in understanding and enacting numeracy across the curriculum. This paper reports on preliminary findings of a research project that aims to provide such guidance. The project builds on our previous research, which has developed a methodology for auditing the numeracy demands of the school curriculum and a professional development approach for supporting teachers' planning and pedagogical decision-making in relation to numeracy across the curriculum (Goos, Dole, & Geiger, 2012; Goos, Geiger, & Dole, 2014). This current project will add a new dimension to our previous work by translating what we have learned about teachers' planning, including how they design numeracy tasks, into a more general design framework that teachers can use to adapt existing resources or to create their own.

The first stage of the project, reported in this paper, involved conducting a literature review of "good practice" in teaching of numeracy in schools and an audit of existing resources for teaching numeracy across the curriculum. The following research questions informed this stage of the project:

1. How can numeracy across the curriculum be conceptualised?

2. To what extent do existing resources available to Australian teachers support their understanding and enactment of numeracy across the curriculum?

The first research question is addressed in the next section, which summarises the findings of our literature review. The subsequent section addresses the second question by presenting the methodology and outcomes of the resources audit.

CONCEPTUALISING NUMERACY ACROSS THE CURRICULUM

The literature review revealed that research into numeracy across the curriculum falls into two broad categories: (1) interdisciplinary inquiry that combines mathematics with one or more disciplines into a single program, and (2) identifying the distinctive numeracy demands and opportunities in school subjects other than mathematics.

Interdisciplinary inquiry

Interdisciplinary inquiry refers to tasks, teaching programs or approaches to instruction that connect two or more academic disciplines. While some researchers argue that integrating teaching and learning across disciplines offers greater possibilities for engaging adolescent learners (e.g., Venville, Wallace, Rennie, & Malone, 2002), this approach brings with it challenges that educational institutions often struggle to address when attempting to move away from existing discipline-based approaches. These challenges include the structure of schooling, much of which is designed to protect disciplinary interests, and factors such as discipline-based teacher training, assessment, and parental preferences for a traditional discipline-based curriculum that contribute to maintaining the status quo. Because of these limitations, some people argue against integration and assert that ideas like numeracy should be considered "educational by-product[s] … [that results from] … studying mathematics, physics, chemistry, biology, business studies and various other subjects in which numbers and mathematics concepts find application" (Lee, 2009, p. 218).

Numeracy demands and opportunities in subjects other than mathematics

Numeracy can also be addressed across the curriculum by attending to numeracy demands and opportunities as they emerge when teaching subjects other than mathematics. This does not mean that teachers in other subjects should be required to be expert teachers of mathematics. It does mean that teachers need to be familiar with the inherent numeracy demands of their subject, can recognise a numeracy opportunity when it arises, and have the disposition and pedagogical skill to take advantage of such opportunities. Studies have demonstrated that such opportunities arise in a wide range of subjects, such as science (Quinnell, Thompson, & LeBard, 2013), economics (O'Neill & Flynn, 2013), and the social sciences (Lake, 2002). These subjects not only demand quantitative skills but also offer opportunities to develop critical thinking and active citizenship as important elements of numeracy.

Hogan (2000) argues that being numerate requires three types of knowledge:

Mathematical – understanding of mathematical ideas and techniques
Contextual – capacity to link and use mathematics in life situations
Strategic – identification of key features of a problem in order to make an appropriate choice of mathematics relevant to a situation and recognise the limitations of results.

This framework was used as the foundation for a project that investigated the demands and opportunities in teaching numeracy across the curriculum (Thornton & Hogan, 2003). The findings suggested that teachers can plan for numeracy teaching provided such activity is prioritised and that a numeracy-oriented approach to teaching across the curriculum enriches students' learning in other curriculum areas.

In a series of research and development projects, Goos and colleagues investigated the effectiveness of a teacher professional learning program aimed at enhancing numeracy teaching across a range of school subjects, including history, science, English, health and physical education, and studies of society and environment. This program was based on a multi-faceted model of numeracy that represents a synthesis of research related to effective numeracy practice. The model, which was constructed as an accessible instrument for the purpose of teachers' planning and reflection, incorporates the dimensions of *contexts*, *mathematical knowledge*, *tools*, and *dispositions*, embedded in a *critical orientation* to using mathematics. These are summarised in Figure 1. This model has been used to identify the numeracy demands of non-mathematics subjects in the Australian Curriculum, investigate teachers' understanding of numeracy, and analyse teachers' capacity to recognise and take advantage of numeracy opportunities in the subjects they teach (Goos, Geiger, & Dole, 2011, 2014; Goos, Dole, & Geiger, 2012).

Mathematical knowledge	Mathematical concepts and skills; problem solving strategies; estimation capacities.
Contexts	Capacity to use mathematical knowledge in a range of contexts, both within schools and beyond school settings
Dispositions	Confidence and willingness to use mathematical approaches to engage with life-related tasks; preparedness to make flexible and adaptive use of mathematical knowledge.
Tools	Use of material (models, measuring instruments), representational (symbol systems, graphs, maps, diagrams, drawings, tables) and digital (computers, software, calculators, internet) tools to mediate and shape thinking
Critical orientation	Use of mathematical information to: make decisions and judgements; add support to arguments; challenge an argument or position.

Figure 1. Elements of the numeracy model developed by Goos and colleagues

RESOURCES THAT SUPPORT NUMERACY ACROSS THE CURRICULUM

Audit Methodology

Because the Australian Curriculum maintains strong boundaries between subjects rather than promoting interdisciplinary inquiry, the framework for the resource audit was aligned with the second conceptualisation of numeracy described above – based on identifying the numeracy demands and opportunities in subjects other than mathematics. We were interested in ways in which existing resources supported teachers' *understanding* and *enactment* of numeracy across the curriculum, and so we constructed an audit framework that captured these qualities. The framework consists of statements sourced from the Numeracy Standards for Graduate Teachers published by the Board of Teacher Registration (2005). Although these Numeracy Standards pre-date the Australian Professional Standards for Teachers (AITSL, 2014a), they have a similar organisational structure in describing Professional Knowledge, Practice and

Engagement/Attributes but with explicit reference to numeracy. The Numeracy Standards comprise 22 statements, four of which were selected for the audit framework because they refer to understanding (Professional Knowledge) and enactment (Professional Practice) of numeracy across the curriculum (Figure 2). For the purposes of the audit, they were preceded by the sentence stem "How might this resource help teachers to …?"

Professional Knowledge
PK1: Understand the meaning of numeracy within their curriculum areas.
PK2: Recognise numeracy learning opportunities and demands within curriculum areas.

Professional Practice: Planning
PPP: Take advantage of numeracy learning opportunities within their curriculum context.

Professional Practice: Teaching
PPT: Demonstrate effective teaching strategies for integrating numeracy learning within their own curriculum context.

Figure 2. Framework for resource audit

We limited our search for numeracy resources to those that are (1) readily accessible to Australian teachers and (2) endorsed or produced by the authorities responsible for the Australian Curriculum or the Australian Professional Standards for Teachers, or by teacher professional associations. As a result, we searched the following sources:

1. the numeracy statements for all non-mathematics subjects in the Australian Curriculum: the Arts, English, Science, History, Geography, Economics and Business, Civics and Citizenship, Health and Physical Education, and Technology (ACARA, 2014b);

2. the *Illustrations of Practice* that accompany the Australian Professional Standards for Teachers – an online professional development package comprising video clips of classrooms, teacher interviews, and discussion questions (AITSL, 2014b);

3. the government-endorsed repository of digital resources mapped to the Australian Curriculum and available via Scootle (http://www.scootle.edu.au);

4. teacher professional journals in mathematics and non-mathematics subjects.

Preliminary Results

The first source of numeracy resources was the numeracy statements in each of the Australian Curriculum documents. These statements could help teachers to understand the meaning of numeracy within their curriculum area (PK1). For example, in Geography, the numeracy statement explains that students "investigate…the effects of location and distance, spatial distributions and the organisation and management of space within places".

The second source of numeracy resources was found to provide little assistance in understanding and enacting numeracy across the curriculum. Only two of the 325 *Illustrations of Practice* were related to numeracy, and only one of these (titled

Embedding mathematics in everything, see Figure 3) connected mathematics to non-mathematical contexts – but in the form of extra-curricular activities rather than other school subjects. Because this resource illustrates a particular teacher's planning practices as well as his understanding of numeracy and demonstration of effective teaching strategies, it might help teachers develop professional knowledge and practice in all of the ways identified in the audit framework (PK1, PK2, PPP, PPT).

This teacher works closely with other staff to link mathematics learning to students' experiences. He encourages a collaborative, inquiry-based approach to teaching mathematics, modelling the use of questioning to encourage the use of problem solving with other staff and students. An activity that allows for mathematical investigation, is facilitated by a parent who has an engineering background. The parent visits the school to teach students how to design and construct see-saws using Lego.

Figure 3. Summary of *Embedding mathematics in everything*

For the third source, a search of Scootle using the term "numeracy" returned 235 resources, almost all of which were related to the teaching of mathematics rather than numeracy across the curriculum. Seventeen numeracy resources were identified, all of which were judged to have the potential to help teachers understand the meaning of numeracy within a particular curriculum area (PK1) and, if implemented as directed, to help teachers demonstrate effective teaching strategies for integrating numeracy learning in this curriculum context (PPT). For example, a unit of work in the science curriculum on plants, included activities involving measurement of plant growth, development of a scale for a cross section diagram, and the collection and representation of data in tables and graphs.

The fourth source of numeracy resources was teacher professional journals. A search of 17 journals aimed at teachers of science, English, mathematics, computing, health and physical education, English as a second language, modern languages, geography, art, history, and music, as well as more general journals focusing on early childhood or middle years education, found only 15 articles on the teaching of numeracy across the curriculum. Eleven of these were published in mathematics teacher journals, which are unlikely to be read by teachers of other subjects looking for help in understanding (PK1 and PK2) and enacting (PPP and PPT) numeracy in their own curriculum contexts.

DISCUSSION AND CONCLUSION

Numeracy has been a national educational priority in Australia for over a decade and remains on the international educational agenda because numerate citizens are able to participate and function more fully in society. Thus, numeracy must be seen as a basic right to be fostered through schooling and beyond. The concept of numeracy across the curriculum, however, is relatively new and so research into how best to promote numeracy capabilities is only beginning to emerge. Two approaches are evident in the literature. One is based on interdisciplinary inquiry that aims to integrate mathematics with other subjects (e.g., Venville et al., 2002), and the other leaves the separate disciplines intact and instead encourages teachers to identify subject-specific numeracy demands and opportunities (e.g., Goos et al., 2014). Both approaches have their challenges. However, it seems that the latter approach would be more feasible for teachers to implement because it avoids the well-documented problems of curriculum integration.

An audit of existing resources available to Australian teachers found very few resources to support teachers' understanding and enactment of numeracy across the curriculum. Most resources that were found did offer some explanation or examples that could enhance teachers' understanding of the meaning of numeracy in their own curriculum context, and many also provided "ready-made" activities for integrating numeracy into the teaching of subjects other than mathematics. However, almost none addressed the need for teachers to recognise and take advantage of the numeracy learning demands and opportunities within the subjects they teach as part of their curriculum planning and pedagogical practice.

While we cannot claim that our numeracy resource audit identified every resource available to Australian teachers, its findings highlighted important gaps. In particular, it seems unlikely that teachers will be able to embed numeracy across the school curriculum without structured assistance in learning how to "see" the numeracy demands and opportunities in all the subjects they might teach. To address this gap, the next stage of our research will translate the numeracy model we developed in previous studies (Figure 1) into a design framework to support teachers in selecting, adapting, and creating resources for embedding numeracy across the curriculum.

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PROCEDURAL AND CONCEPTUAL KNOWLEDGE IN CALCULUS BEFORE ENTERING THE UNIVERSITY: A COMPARATIVE ANALYSIS OF DIFFERENT DEGREE COURSES

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The role of mathematics as a subject in tertiary education differs enormously among various degree courses. For Natural Sciences, mathematics is an important tool for every student. In economics, its role depends on the areas of interest, whereas in physics and mathematics, it is in the core of the study programmes. In all of these courses, a particular emphasis is put on calculus. A testing instrument is presented for students' procedural and conceptual knowledge in calculus at the end of their school careers, based on German common core standards covering three areas of procedural knowledge and one area of conceptual knowledge. In a survey with 1134 students of different degree programmes, students' knowledge is compared. Finally, it is investigated as to which dimension best describes competencies in calculus.

THE CHALLENGES IN UNIVERSITY DEGREE COURSES

The increase in the numbers of students who successfully complete degree courses in the STEM academic disciplines of science, technology, engineering, and mathematics is a declared goal in many countries. The widespread efforts to attract more students to start their careers go along with problems of those who have opted to do so. Recent studies have made it clear that dropouts remain as a significant problem in different countries. See, for instance, Chen (2012) for the case of colleges in the United States and Dieter (2011), who examines degree courses in mathematics in Germany.

The changes in school mathematics over the last two decades have not brought many changes to the problem that mathematics remains a challenge in all degree courses at colleges and universities. Hoyles, Newman, and Noss (2001) even claim that the shift towards utilitarian mathematics makes the situation rather more difficult. This also involves the area of calculus (Ganter, 2000), which is traditionally important in the beginning of tertiary education in mathematics because sciences and the economy make frequent use of it. Lately, studies have indicated that motivational aspects are crucial for successful completion of calculus courses at colleges (Pyzdrowski, 2013). Interestingly, interventional studies at Colorado State University (Pilgrim, 2010) have shown no significant differences concerning epistemological beliefs using the Modified Indiana Mathematical Belief Scales, between the students who passed the calculus exams easily, and participants in an intervention course for those who were at risk of failing the exams.

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PROCEDURAL AND CONCEPTUAL KNOWLEDGE IN CALCULUS

As it is a broad field, knowledge in calculus has to be built up slowly and over a long period of time. Various learning theories describe the cumulative nature of building up knowledge in calculus, and it is not an easy endeavour to compare results that were obtained in different theoretical settings. Among the various systems for describing knowledge in calculus, the distinction between procedural and conceptual knowledge (Hiebert, 1986) is tried and tested in calculus (See e.g., Porter & Masingila, 2000).

Procedural knowledge is defined as action sequences for solving problems, whereas conceptual knowledge aims at "explicit or implicit understanding of the principles that govern a domain and of the interrelations between pieces of knowledge in a domain" (Rittle-Johnson & Alibali, 1999, p. 175). Star and Stylianides (2013) argue theoretically that there has to be a gap between procedural and conceptual knowledge. It is one of our aims in this study to better understand to what extent procedural and conceptual knowledge differ in calculus even before entering the universities.

ACHIEVEMENTS IN CALCULUS AT SCHOOL

For the understanding of the design of this study, please note that the data were gathered at a university in Germany, where calculus is compulsory for all high school students. Education standards in Germany established a consensus among the conference of ministers of education of the federal states with the aim to improve school education. One aim of the standards was to provide a theoretical framework that allows students to gain competencies that can be measured empirically (Ehmke, Leiß, Blum, & Prenzel, 2006).

National standards for students to pass final school exams and qualify for entering universities (Kultusministerkonferenz, 2012) are the bases for the core curricula in the federal states. The standards distinguish between comprehensive mathematical competencies (arguing mathematically, mathematical problem solving, mathematical modelling, using mathematical representations, and being in command of symbolic, formal, and technical elements of mathematics) and content-related competencies, following the "guiding ideas" of "algorithm and number", "measuring", "room and shape", "functional relations", and "data and chance". The core curriculum of the state of Lower Saxony (Niedersächsisches Kultusministerium, 2009), in which about 70% of all participants of this survey have passed their final school exams, is quite compatible with this system.

Since calculus plays an important role in universities, often in special courses, this project aims at looking at different degree courses in a much more detailed way. Our longitudinal study on different areas of mathematics (Halverscheid & Pustelnik, 2013) compared competencies of students of physics and mathematics on entering the university and their exam results in the first courses. This project concentrates on the calculus and aims at considering both procedural knowledge and conceptual knowledge. For this aim, the competencies named in the federal core curriculum are considered in four different areas. It should be stressed that the underlying theories for mathematical competencies on the one hand and the difference between conceptual and procedural knowledge on the other hand are not exchangeable. What we did was to classify the competencies according to three areas of procedural knowledge and one area of conceptual knowledge as shown in Table 1.

	Area	Competencies according to the federal core curriculum. Students…
T	Procedural knowledge on calculating derivatives	… compute derivatives for the following classical functions with the rules of sums, products, factors, and composition: polynomials, sin, $\sqrt{}$, exp, and compositions of these
S	Procedural knowledge on curve sketching	…determine slopes of tangents to graphs …search extremal points and inflexion points with derivatives …use derivatives to discuss monotonicity and curvature, investigate extremal points, and analyse functions defined by sections
I	Procedural knowledge on integration	…compute integrals with the help of antiderivatives of polynomials, sin, $\sqrt{}$, exp, $x \mapsto x^z, z \in \mathbb{Z}$, …reconstruct graphs of a function from that of its derivative and vice versa …interpret the integral as an area and reconstructed stock …illustrate the main theorem of calculus for the graph, the function, and its derivative
C	Conceptual knowledge on differential calculus	…use pre-concepts of limits for differentiation and integration … use different classes of functions and compositions of them to describe functional phenomena and to solve inner- and outer-mathematical problems …describe and interpret rates of growth functionally …explain rates of growth …interpret derivatives as rates of growth …employ models of limited and logistic growth

Table 1: Grouping of competencies according to the national standards (in Germany)

RESEARCH QUESTIONS

On the one hand, we expected higher conceptual abilities of students in Physics and Mathematics. On the other hand, these are the subjects with no limited access at all, i. e. everyone with a successful final school exam may enrol in physics and mathematics.

The area of techniques should not be too difficult for either of the degree courses. In all of the courses, many students should be able to answer many questions correctly, and the differences between the degree courses should be smaller than the differences between the areas.

For the problem of how to deal with the difficulties of students in their first academic year, it would be important to know how heterogeneous the groups of the degree courses are.

The following questions served as guiding lines for this research project:

To what extent do the attendants of the degree courses enter the university with different prerequisites concerning calculus?

Can differences within a single degree course be detected?

Are there characteristic differences between the areas of procedural and conceptual knowledge? Is it possible to develop a high standard of conceptual knowledge in calculus while having less elaborate procedural knowledge?

TEST DESIGN

For each of the areas (T), (S), (I), and (C), 15 items were constructed in such a way that to every competence at least two items correspond.

To illustrate the test design, we give a couple of examples for the listed competencies. In area (T), the item

"A function is given by $f(x) = \sin(a \cdot x + b)$. Compute its derivative. Mark the correct answer:

☐ $f'(x) = \sin(a)$ ☐ $f'(x) = \cos(a)$ ☐ $f'(x) = a \cdot \sin(a \cdot x + b)$

☐ $f'(x) = a \cdot \cos(a \cdot x + b)$"

is relevant for the competence to "compute integrals with the help of antiderivates of sin".

The following item refers to the first competence in area (S), "procedural knowledge on curve sketching".

"The function defined by $f(x) = x^2$ has the derivative $f'(x) = 2x$, which assertions on the monotonicity properties of f hold? Mark the correct answers.

☐ f is strictly monotonically increasing on all of $\mathbb{R}$.

☐ f is strictly monotonically increasing on. $\mathbb{R}_0^+$.

☐ f is strictly monotonically decreasing on all of $\mathbb{R}$.

☐ f is strictly monotonically decreasing on $\mathbb{R}_0^-$.

☐ f is monotonically increasing on the interval, $[-1; 1]$.

☐ f is monotonically increasing the interval, $[1; 2]$."

The following item concerns the first competence in area (I).

"Compute the integral $\int_{-1}^{3} \frac{1}{2} x^2 dx$ and mark the correct result.

☐ $\frac{4}{5}$ ☐ $\frac{5}{6}$ ☐ $\frac{6}{5}$ ☐ $\frac{14}{5}$ ☐ $\frac{7}{2}$ ☐ $\frac{14}{3}$."

Finally, consider the following example for conceptual knowledge, area (C), for the competence to "describe and interpret rates of growth functionally":

"At a and b, the graph of a function f has a horizontal secant. Which of the following assertions on f is true?

☐ f is on the interval constant.

☐ f is on the interval monotonically increasing or monotonically decreasing.

☐ f is a linear function.

☐ None of the above holds."

PARTICIPATING STUDENTS AND RELEVANT DEGREE COURSES

In this university, mathematics is compulsory in the degree courses of Natural sciences, Agriculture, Forestry, Economy, Computer Science, and of course, Mathematics itself. For the Natural sciences, Economy, Computer Science, and Mathematics, this involves both degree courses with one major and degree courses for teacher education with two subjects of equal weight. To ease the transition from school to tertiary education, a system of preparatory courses is offered to students in four clusters. The corresponding degree courses are listed here jointly with the numbers of participants in the tests. Economy: 494 participants; Physics, Computer Science and Mathematics: 195 participants; Geology and Biology: 131 participants; Agriculture and Forestry: 314 participants.

METHODOLOGY

For each of the test sections a one-dimensional Rasch analysis was conducted, so every person was assigned one parameter per section. To gain questionnaires satisfying the Rasch model, some of the items had to be eliminated. Finally, the section on Calculating derivatives contains eight items, the section on Curve sketching contains eleven items, the section on Integration contains six items, and the section on Conceptual knowledge on differential calculus contains seven items.

Since abilities for a person answering every item correctly or answering every item incorrectly cannot be estimated, the number of persons per test section had to be reduced. Some data has also been excluded due to some participants not filling out all of the questionnaire. Overall between 413 and 896 participants are part of the analysis.

RESULTS

To answer the first research question the mean values of person parameters were calculated for each preparation course, which can be seen in Table 2. The four Rasch models were scaled such that the mean values for participants of the degree courses in Agriculture and Forestry were 0. On the whole, the use of IRT methods has led to convincing results. However, the personal parameter estimation for Agriculture /

Forestry should considered carefully in the areas (I) and (C), where 90 % of the students did not give any correct answer at all.

While the differences between Mathematics and the other three courses are highly significant ($p<0.01$) for all of the test sections, the differences between Economics and Geology/ Biology are not significant in all cases ($p>0.05$). The differences between Agriculture/Forestry and Economics respectively Geology/ Biology are also significant for two sections: Calculating derivatives and Curve sketching. The variances of the four courses can also be seen in Table 2.

	Economy	Mathematics, Computer Science, Physics	Geology, Biology	Agriculture, Forestry
Integration	0.25/ 0.81	1.29/ 1.52	0.38/ 0.65	0/ 0.48
Curve sketching	0.32/ 0.90	2.22/ 1.33	0.34/ 1.13	0/ 0.78
Calculating derivatives	0.79/ 1.61	2.47/ 0.77	0.75/ 2.00	0/ 1.67
Conceptual knowledge	-0.04/ 0.75	1.54/ 1.66	0.16/ 0.86	0/ 0.79

Table 2: Mean Values and Variances of person parameters for each test section

The effect size of these differences is strong in comparing Mathematics and the three other degree courses for all sections (d>0.8). The two significant differences between Agriculture/ Forestry and Economics and Geology/ Biology are of small size (d>0.4).

Whereas the variances are the highest for Mathematics in three of the areas, they are the smallest in Calculating derivatives. The ratio of variances differs from 1 for area (T) and area (I) is highly significant ($p<0.001$) for differences between Mathematics and the other three degree courses, whereas other differences are not significant.

To investigate the reason of the small variance in Calculating derivatives for students of Computer Science, Physics, and Mathematics, we look at the quartiles of the distribution of the person parameters. About 25% of these students answered every item in the Calcuating derivatives test correctly, and were not estimated by the IRT method. Half of the remaining students had only one item wrong. So the small variance seems to be due to a ceiling effect.

DISCUSSION

With respect to the first two research questions, we can see that the students in the Computer Science, Physics, and Mathematics degree courses show by far the best results in every area of Calculus. This group of students possesses the highest mean values in every section. The students of Economy and Geology and Biology have mean values being nearly the same for every section and students in Agriculture and Forestry have the lowest values besides conceptual knowledge.

The other courses show results with smaller differences. While students in Agriculture and Forestry have the weakest results in three out of four areas, the differences have only small effect sizes. No significant differences between students in Economy and Geology/ Biology can be established.

Finally, solid conceptual knowledge occurs only in exceptional cases apart from in Computer Science, Physics, and Mathematics. And even in that group, there is a dichotomy between those with good conceptual knowledge and those who answer only a small part of these questions. In their empirical study on children's conceptual understanding of mathematical equivalence, Rittle-Johnson & Alibali's (1999) findings, "suggest that conceptual knowledge may have a greater influence on procedural knowledge than the reverse". One might see the results of this survey as supportive of this claim for the case of calculus in as far as those with a strong conceptual knowledge also did very well on the procedural knowledge of calculus.

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AN ATTEMPT TO INVESTIGATE THE USE OF MATHEMATICS IN PHYSICS CLASSROOMS

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We outline a framework to study the use of mathematics in physics classrooms. The framework focuses on the relations made between Reality, Theoretical models and Mathematics. In this paper the analyses of one teacher and her 3rd year classes at secondary school are presented. The results show that phenomena in reality are often used as a short prelude to put focus on the relationship theoretical model and mathematics. Mathematics is generally used in an instrumental way to handle various formulas without further insight or discussion of the related models or their relation to reality. There is a lack of varied communication with a structural use of mathematics, i.e., mathematics used to support reasoning in relation to a theoretical model, highlighting the meaning of concepts and models in the studied classrooms.

INTRODUCTION AND FRAMEWORK

Mathematics is an inherent part of theories in physics and used to analyse and make sense of real-world phenomena. The ability to use mathematics to argue for results within the framework of models is central in physics (e.g. Uhden, Karam, Pietrocola, & Pospiech, 2012). However, various studies point to students' problems in transferring mathematical knowledge to new and applied situations (e.g. Karam, 2014; Kaiser & Sriraman, 2006; Michelsen, 2006). This study adds to the line of research on the use of mathematics in physics classrooms. We have developed an analytical framework to analyse the relations made between the three entities *Reality*, *Theoretical models* and *Mathematics*, during classroom communication (for a more detailed account of the framework see Hansson, Hansson, Juter & Redfors, in press). *Reality* refers to objects or phenomena (or observations of them) in the real world. *Theoretical models* refer to theoretical models in Physics and concepts related to them. The models could be mathematically or qualitatively formulated. *Mathematics* refers to mathematical concepts, theorems, representations, mathematical reasoning and methods. The aim is to apply the developed framework in the analysis of physics instruction to identify different focuses in the classroom communication during different instances of a lesson or in different kinds of instructional situations.

In figure 1 the relations between the three entities Reality, Theoretical models and Mathematics are represented by the triangle's three sides in the form of bidirectional links 1, 2 and 3. The first type of link (1, in Figure 1) represents relations made between Reality and Theoretical models. We know from previous research that such relations are important in physics instruction (e.g. Lederman, 2007).

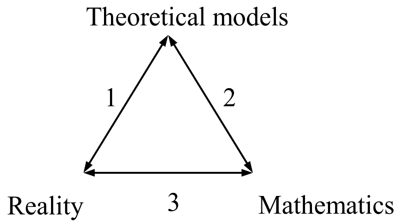

Figure 1: The links between Reality – Theoretical models – Mathematics

The second type of link (2, in figure 1) represents relations made between Theoretical models and Mathematics. In the classroom communication we look for when a theoretical model is described in mathematical terms, or when a problem is transferred from a physics problem, to a mathematical problem (e.g. manipulation of formulae, solving equations or constructing graphs). This link can be made in structural or technical ways (Karam, 2014; Uhden et al., 2012). Structural in relation to link 2 means that mathematics is used to support reasoning in relation to a theoretical model, while a technical use of mathematics is characterized by manipulations of formulae without discussing theoretical meaning, or when searching for the correct formula using a "plug and chug" approach to problem solving. Related dichotomies, or dualities, of technical and structural use of mathematics are instrumental and relational understanding (Skemp, 1976) or conceptual and procedural knowledge (Hiebert & Lefevre, 1986) in mathematics education.

The third type of link (3, in Figure 1) depicts relations made between Reality and Mathematics. This could happen when observations are discussed in mathematical terms (without contextualisation of physics concepts), for instance when referring to experiences, e.g. it hurts more and more in the ears when diving deeper and deeper. Other examples can be various quantifications during lab work, e.g. measurements of angles, time or distances, or when a real world phenomenon is related to a mathematical object, e.g. the slope of a hill is related to a right-angled triangle.

Analysing the communication through looking for these three different relations made (links 1-3) is not to say that they are independent of each other. The fact that we categorise a statement as link 3 (relations made between Reality and Mathematics) does not mean that observations are not theory-laden per se. Only that the theoretical model is not made explicit during the communication. In the same way do links made between theoretical models and mathematics (link 2) not say that there is no Reality, only that it is not explicitly included in the communication. The framework should be viewed as a way to analyse what students and teachers say and do during different parts of a physics lesson.

METHOD AND PROCEDURE

We have done observations of one mathematics and physics teacher and different physics classes taught by her of which two 3rd year classes are presented in this paper.

The intention was to follow the normal practice of physics instruction at upper secondary school. The students studied at the science respectively technology programs in an ordinary upper-secondary school in Sweden. One 3rd year class (11 students) studied optics and atomic physics (diffraction gratings and spectroscopy) and the physics content for the other 3rd year class (7 students) was electric fields (movement of charged particles).

The two classes have been observed during sequences of lectures, problem solving in groups, and lab work, two lessons (40-100 min each) per class. The classes were observed during lecture and problem solving respectively lecture and lab work. During problem solving sessions and lab work students worked in small groups (2-4 students). During the lectures we video recorded using a camera focused on the teacher and the whiteboard, and other cameras focused on the students. During problem solving sessions and lab sessions we video recorded the work of selected student groups.

We have through the use of video recording analysed the communication during lectures and student-centred work. The data is analysed from a perspective where we deductively identify relations between Reality, Theoretical models, and Mathematics communicated by teachers and students. During the analysis a multi-step process was used: watching a video sequence in its entirety, identifying major events within the sequence, transcribing the interactions (words and actions) and identifying the links made in the communication.

RESULTS

Lecture

The lecture about electric fields was 40 minutes long and began with the teacher conducting a demonstration of electric field strength. A detailed account of the distribution of links of type 1, 2 and 3 is presented in Table 1.

Time (min)	Activity	Link 1, R–TM	Link 2, TM–M	Link 3, M–R
0-5	Demonstration of electric field strength in a parallel-plate capacitor.	There is a spark between the plates because the capacitor wants to equalize the charge, explains the teacher.		The teacher points out that the spark occurs more often and with less required voltage when the distance between the plates decreases.
5-10	The teacher calculates the field strength required for a spark between	The teacher uses data from the demonstration in the calculations.	The teacher discusses mental arithmetic and decimal adjustment.	A student measures the distance between the parallel-plates

	the plates on the whiteboard.			in the demonstration experiment.
10-18	Discussion: “What would happen if you put an electron in an electric field?”	Reasoning about what forces acts upon the electron.	The teacher writes down formulae that are relevant in this context on the whiteboard.	
18-33	The teacher calculates the deflection of an electron projected into an electric field. Comments on similar textbook problems.		Focus on finding suitable formulas and formula manipulation.	
33-40	The teacher calculates the deflection angle of an electron (which is traveling through an electric field) when it leaves the electric field.		Focus on formula manipulation. Decompose the motion: Accelerate in y and constant in x. Refers to similar problems in mathematics.	
40-41	Discussion: “What use could we have with all this?”	Students mentioning applications from the textbook, inkjet printer etc.		
41-43	The teacher puts the end of a fluorescent lamp to the capacitor to make the tube flicker.			

Table 1: Relations made between Reality (R), Theoretical Models (TM) and Mathematics (M) during the lesson about electric fields

To summarize, the lecture (see table 1) begins with a demonstration of a real phenomenon, sparks in a parallel-plate, with links of both type 1 and 3, before discussions on how to calculate field strength and thus link 2 began. How to do calculations link 2 (technical) was then a dominant part of the lecture, with explicit references of how to use methods from mathematics to solve the problems. Discussions related to link 1 was present, but often intended as a prelude to a problem with following calculations and formulas related to link 2.

An example of link 2 technical is when the teacher drew a parallel-plate capacitor on the whiteboard (horizontal, with negative plate up and positive down) and asked “what would happen if you put an electron in the electric field” noticing that an electron would accelerate downward as it travels horizontally between the plates. She then draws attention to that they can use the formulas $F=ma$ and $E=F/Q$ and focuses on formulae manipulation to determine F.

Examples of link 2 structural are more infrequent, but one example is when calculating the deflection angle α of electrons passing through an electric field. The teacher started drawing a triangle (see figure 2) from which the calculations were performed, and later when the problem was solved she continued to draw a parallelogram of forces acting on the electron (see figure 2) commenting, “this is how it should be (how it is supposed to be drawn)” to illustrate how the calculations support the reasoning of the theoretical model.

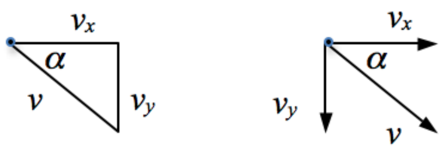

Figure 2: The teacher’s drawing on the whiteboard

Problem solving session

During the problem solving session (80 minutes long) the students were assigned to work with problems about electric fields in their textbook. The students’ problem solving was focused on formula manipulation and what formulae they needed to solve a problem. The students also frequently consulted the Answer section of their textbook. Their reasoning involved the theoretical model and the mathematics required to solve problems related to the model (link 2, technical), while infrequently contemplating on or discussing their experiences of a phenomenon in reality. However, in one group of three students the communication changed when they were working with the following problem:

The voltage between the ground and a high-voltage cable is 100 kV. How high above the ground must the cable at least be in order not to risk a flash from the cable to the ground? It requires 3 MV/m to ionize the air.

To solve the problem the students may use the relation $E = U/d$ to form the inequality $\frac{100000}{3000000} = \frac{1}{30} < d$. Thus, the cable should not be closer than 0.033m to the ground.

The three students, Adam, Pete and Eric, were reading the problem in their textbooks. Pete introduced the variable x and set up the equation $3000000 = \frac{100000}{x}$. Adam commented "The equation should not be satisfied" and agreed with Pete "d should be larger than x" (link 2, structural). Pete entered 100000/3000000 in his calculator and concluded "It should be larger than 3 cm, 3.3 cm". However, both Adam and Eric questioned the calculations, and Pete became uncertain if his calculation was correct, as described below:

Adam: What! Only?

Pete: Have I divided the wrong way?

Eric: You must have. 3 cm for a large high-voltage cable, Pete! [Laughing]

The students assumed that Pete had made a calculation error. Adam performed the same calculations on this calculator and comments "What! Stop, this is not true at all!" The answer obviously conflicts with the students' experiences of how high-voltage cables are located high above the ground (link 1). Notice that the students' conceptions of the quotient 100000/3000000 was one reason for their reactions, they all acted surprised by the small decimal number it represents. The students followed a procedural approach in their calculations rather than a conceptual understanding of the quotient. The teacher listened to the group at a distance and walked up to the group, and started asking questions:

Teacher: Why is it [the cable] so high up then?

Adam and Peter took the teacher's question as evidence that they solved the problem correctly and found practical reasons for the placement of cables, e.g., that they are not in the way of car traffic (link 1). However, Eric did not agree with their calculations and continued discussing the equation $3000000 = 100000/x$ with Pete:

Eric: You have divided in the wrong direction.

Pete: No, here they are. [Showing his calculations] You move over this one and then that one.

Eric: What! Why not just move directly and multiply it? No, that's not possible.

Eric was the one most hesitant about how to solve the equation and discussed the calculations with Pete. The solution to the problem was contrary to his personal experiences from reality that made him doubt the accuracy of the calculations.

One may notice that none of the students questioned the accuracy of the theoretical model or the application of the relation $E = U/d$ to the problem (link 1), but questioned their calculations and mathematical reasoning (link 2, technical). Adam and Pete seemed to rule out the possibility of a calculation error after their conversation with the teacher. However, Eric was unwilling to accept the solution to the problem and consulted the answers section of the textbook. After this he accepted the answer and thought that safety issues might be the reason for the cables located high off the ground (link 1).

Lab-work

One of the 3rd year classes did lab-work during the study. It was preceded by a lecture (40 minutes long) where the teacher explained interference in a double slit and derived the grating equation $dsin\theta_k = k\lambda$, and solved problems to demonstrate how to use the equation. Although link 1 was present in a demonstration of interference during lecture, the focus was to obtain formulas and to discuss how to use them to solve textbook problems (link 2, technical).

At the lab work lesson (100 minutes long) the teacher wrote the grating formula on the whiteboard and pointed out that it is the wavelength λ the students were going to calculate (link 2, structural) during the lab and then handed out instructions and pre-printed tables to fill in with data from the lab (order of spectrum k, and left and right angel on the spectroscope to determine θ_k). The students worked in groups of two or three.

The teacher was circulating between different groups and helped them get started, in some cases the apparatus required some adjustments to work as intended. The light was then turned off for easier reading the spectroscope and the students started to fill in the pre-printed tables with data (order of line (spectrum) and angles, link 3) for the different colours they detected, using hand lights to fill in the tables. This was a slow process, with some technical difficulties, that took most of the time before all groups were done. The students then calculated the wavelengths using the grating formula (link 2, technical), also taking the mean value to compensate measurement errors, to determine what kind of gas there where in the groups' different discharge lamps (link 1).

CONCLUSIONS

The results show that in the studied cases the bulk of the discussion in the classroom is concerning the relation between theoretical models and mathematics (link 2, Figure 1). It is also shown that when such relations are made, the emphasis is often on technical use of mathematics. Links of type 2 that instead emphasise structural use are not frequent. Neither are links of type 1, which communicate how the theoretical models can be understood in relation to reality. Surprisingly this result seems to hold true also for teacher led lectures. In the lab work situations the main focus were on collecting data through measuring (link 3), followed by using the collected values in different formulae (link 2, technical).

This result adds to our understanding of the role of mathematics in the physics classroom. The conspicuous focus on manipulations of formulae adds to the understanding of why teachers view poor mathematics skills as a big problem in the physics classroom and a hindrance for learning (e.g. Karam, 2014; Uhden et al., 2012). When the communication has the focus as in the studied cases this makes perfect sense.

Proficiency in solving standard physics problems (which often means a technical use of mathematics) does not imply conceptual proficiency, which is demonstrated by the students' communication as depicted with the three students working with the high-voltage problems during the problem solving session. The students' actual capacity to understand the physical concepts remains hidden until the communication in the classroom becomes more varied. This would mean a decreased emphasis on formula manipulation when relations are made between theoretical models and mathematics (technical use of mathematics), so that more emphasis instead could be made on the meaning of concepts and models, that is, a structural use of mathematics. This would also mean an increased emphasis on the relation between theoretical models/concepts and reality, e.g. on how theoretical models could be used to describe and predict real world phenomena and events, link 1 in Figure 1. It also supports results from both science and mathematics education research (cf. Kaiser & Sriraman 2006; Michelsen 2006; Uhden et al. 2012) about the importance of mathematical modelling in physics teaching.

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POINTING IN AN UNDERGRADUATE ABSTRACT ALGEBRA LECTURE: INTERFACE BETWEEN SPEAKING AND WRITING

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The typical teaching format of undergraduate mathematics classrooms is the lecture, which commonly involves a large amount of talking and writing on the board. In addition to speaking and writing, professors use their hands to communicate. In this paper, we focus specifically on the use of the hand (and fingers) to point. We focus on acts of pointing by a professor in a third-year group theory lecture. These acts of pointing are classified into the following categories: touches, holds, points, sweeps, shakes, and waves. We analyse the function of these kinds of pointing and argue that they form a central component in communication, particularly in terms of (1) bringing mathematical objects into being, (2) relating these objects to each other and (3) connecting the spoken with the written and drawn.

INTRODUCTION

Consider for a moment a typical upper year undergraduate mathematics classroom. Perhaps you pictured a professor near the front of the room, writing on a blackboard and talking. It is probably safe to say that this is the dominant mode of undergraduate mathematics teaching: the lecture. While many view this mode as straightforwardly well-understood, we are interested in the communicative richness that lies beyond (and perhaps between) writing and talking. In particular, we focus on the act of pointing. Pointing is a kind of gesture that has received much less attention in mathematics education research, which tends to focus on what McNeill (1992) calls iconic or metaphorical gestures (see Arzarello et al., 2009; Edwards, 2009). Even in the literature on teachers use of gesture, pointing has not figured prominently, particularly in research that focuses on how teachers make use of student gestures (see Singer & Goldin-Meadow, 2005; Valenzeno et al., 2003). Similarly, Núñez's (2003) study focused on the metaphorical gestures that an undergraduate lecturer used. Our research goal is to study the way pointing is used in a lecture-style classroom in order to better understand its communicative function. We anticipate that pointing might play a particularly important role in a mathematics lecture because of the nature of the objects being discussed and the arguments being made.

THEORETICAL FRAMEWORK

Child development psychologists agree that pointing is one of the earliest communicative human actions. Infants begin to point at about 11-12 months (Bates, 1979). One of the pre-eminent figures in infant development defines pointing as follows: "Pointing is a deictic gesture used to reorient the attention of another person so that an object becomes the shared focus for attention" (Butterworth, 2003, p. 9). The

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focus in this study will be on "pointing" as a deictic gesture that establishes joint attention on some object.

The first major survey of deixis in language is due to Bühler (1934/1990). He drew attention to the distinction between two classes of words, corresponding to the two basic operations of pointing and naming: (1) demonstratives and other deictic expressions ("*Zeigwörter*") and (2) "naming words" ("*Nennwörter*"). He developed a two-field theory of communication: the "deictic field" consisting of the specific physical and verbal context of a speech event and the "symbolic field" consisting of the "synsemantic environment" of words, phrases and other lexical knowledge possessed by the speaker and hearer. Bühler postulates the existence of an "*origo*", an origin for a frame of reference of the body of the speaker. He also distinguishes three axes along which these usages of deictic expressions can place the object that the speaker is referring to, the spatial, temporal and personal, which he refers to briefly as Here/Now/I. For example, an object or event that the speaker is referring to could be "here" or "there", it could be "this" or "that", it could be "these or those"; the event is "now" or "then"; a person referred to might be "I' or "you". In this study, spatial deictics will be of greatest interest.

DATA COLLECTION

A lecture in a third-year undergraduate mathematics course at a mid-sized University in Western Canada was video-recorded. The video camera was focussed exclusively on the lecturer (and not the students). In this paper, we focus on the 7th lecture in a course of 37 lectures because it is in some ways a representative lecture: it contains important definitions, theorems, proofs, and diagrams, many of which were used or referenced throughout the remainder of the course. The lesson lasted approximately 50 minutes and was conducted in a lecture style. A transcript was made, which consists of 6846 words.

It became clear from watching the lecture that at any given moment in the lecture there was a present topic at hand, a present context within which an argument was developing, a present object about which some explanation was being profferred; a present locus of attention and focus of interest within which acts of pointing were occurring. The transcript was therefore divided into pieces ("stanzas"), each of which constituted a deictic field, where transitions were determined by a judgment made as to whether a new topic had been raised and the old one dropped. This also served to improve the readability of the transcript.

Many of the 72 transitions (between 73 stanzas) are obvious, and reasonable observers would agree that they constitute a change in the deictic field: beginning the definition of a subgroup (4), beginning each of the 4 Cayley diagrams drawn (8, 15, 16, 18), beginning each of the three subgroup tests (23, 37, 45), beginning the proofs of each of these tests (25, 39, 48), ending the proofs of each of these tests (33 "we're done", 41, 59 "our proof is complete"), beginning an exercise (62), ending the exercise (70 "therefore we're done."), beginning two brief definitions to be looked at in detail next time (71, 73). On a smaller scale, other transitions are clear. He opened the class with a transparency that is divided into three paragraphs; these are stanzas 1, 2, 3. His proof of the first subgroup test had four parts: that the subset is associative, that it contains the identity, that it contains inverses, and that it is closed under multiplication: these are begun in stanzas 25, 26, 30 and 31. After he had finished his proof of the first subgroup test, the next three stanzas (34, 35, 36) treat three separate comments or remarks that he wanted to make about the test and its proof. After he had finished the proof of the third subgroup test, he made a comment about the proof (60), and drew a picture to help illustrate the main idea in the proof (61). Of the 72 transitions identified in the transcript, 30 have been mentioned.

We kept a time-series record of time spent writing and not writing. A four column table was created: Begins writing, Time writing, Ends writing, Time not writing. Two rows are shown below:

384	2	386	2
388	13	401	5

meaning that at the 384 second mark of the lecture the professor began writing, did so for 2 seconds, stopped at the 386 second mark, did not write for 2 seconds, began writing again at the 388 second mark for 13 seconds, and so on. The shortest unit of time used was 1 second, and all times were rounded off to a multiple of 1 second. There was no occasion in the lecture when the professor pointed at the same moment that he was writing. This meant that in the "gaps" identified in column 4, all of the acts of pointing must occur, which makes the analysis easier.

All of the acts of pointing were noted, along with the words used during each act. When an act of pointing was identified, it was viewed again at a slower speed, usually 50% speed, a few times. After this pass through the video, we watched only the gaps in the writing that had been identified above, and watched for the acts of pointing during these gaps. This second look at the video helped capture a few quicker pointing gestures that had been missed in the first scan, as well as improve our understanding of what the gesture was pointing at, and why.

DATA ANALYSIS

Two hundred and eighty-two acts of pointing were found in this lecture. For a lecture 3023 seconds long, this is about one every 10 or 11 seconds. Eight hundred and eighty-seven seconds of the lecture is spent in writing, and 2136 seconds not writing. During time not writing, then, an act of pointing occurs about one in every seven and a half seconds. By looking at the actual motion of the hand/fingers in each pointing gesture, we identified four main categories of pointing acts: *touches* (78 examples), *points* (52), *sweeps* (52) and *holds* (42).

Touches are acts of pointing when the professor touches the board briefly. These account for more than a quarter of all acts of pointing. In a proof, for example, the professor touched three items in quick succession: the i, the minus sign and the j in the exponent of an expression (Figure 1(a)). Here it is not the theorem that this term is in that is being referred to, nor the proof of that theorem, nor the section of that proof, nor the specific equation that this term is in that is being referred to, nor the expression itself, nor even the exponent: but three individual terms within the exponent.

We infer three important functions of the touches. First, they enable a stronger degree of precision—not precision in the sense of a numerical accuracy, but precision in the sense of precise reference. When touching the board, the professor was indicating exactly *this* item and no other. Mathematical expressions can have items within them that are spatially close but semantically very different. Indeed, given the hierarchical structure of contexts that mathematical objects can live in, the precision of pointing may not seem surprising.

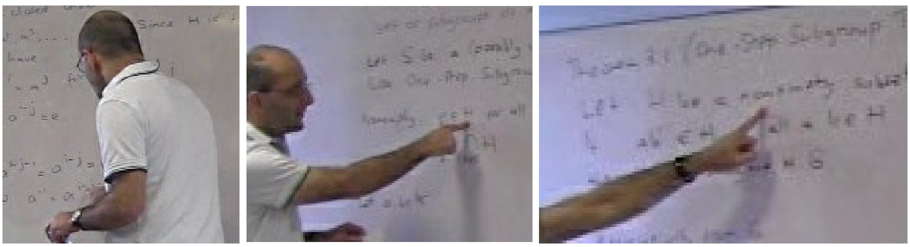

Figure 1(a): Touching a minus sign; (b) Holding a subgroup H; (c) Holding the word "nonempty" – a key condition in the hypothesis of the theorem.

Second, it indicates a confidence on the part of the speaker. He is not in these examples vaguely referring to a fuzzy region that he can later, if challenged, fudge his way out of: he is committing himself, moment by moment, with every touch, to referring to exactly this object and no other. Third, it indicates a close engagement with the subject matter, with the matter at hand. He is literally touching the objects of concern over and over again, dozens of times as he speaks. There is a warmth here, a sense of getting as close as possible to the unfolding logic and pattern of the mathematical structures.

Holds are acts of pointing when the professor touches the board, and holds it (Figure 1(b), 1(c)). Holds and touches were not difficult to distinguish. Some holds were quite long, lasting more than five seconds. Sometimes holds occurred during "commentary" phases, where he held a statement or an expression, and made a more general remark. However, they also occurred during faster-paced argumentation, in the thick of touches and points and sweeps, holding fast to one central expression for a few seconds to reorient the main theme, before flitting away to other business. One in seven acts of pointing are occasions when the professor holds his hands or a finger on the board for longer than a second or so.

Holds might be thought of as indicating a strengthening of purpose, an indication of heavier pointing. Their function might be to draw students attention more explicitly to a particular place on the whiteboard, and thus to underline its significance. Holds can also be seen as encapsulating longer stretches of talk, thus providing a context or reference for that extended talk.

The two other major categories are Points (Figures 2(a), (b)) and Sweeps (Figure 2(c); sweeps are almost by definition difficult to capture in a picture).

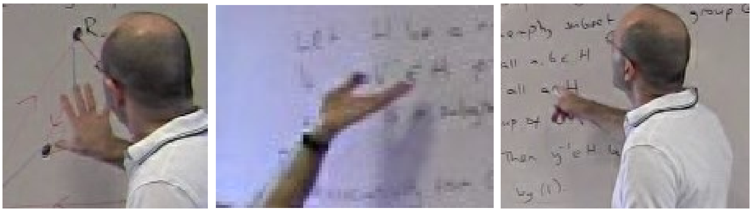

Figure 2(a) Hand pointing at Cayley diagram, palm towards board; (b) Hand pointing at membership relation, palm up; (c) Sweep upwards through three lines of proof.

Points and sweeps form a natural pair: points are discrete, static; sweeps are continuous, dynamic. Points indicate individual items, sweeps indicate a line or a path - often a segment of a proof, or an equation, or a curve in a diagram. Points happen more quickly than touches or holds. The vast majority of points occurred very close to the board. We can conclude then that the "skin depth" of the board in a mathematics lecture is small: 120 of the 282 acts of pointing occur directly on the board, and many of another 52 occur within one or two centimetres. Points often occur in clusters; when they do, their function is to help the students maintain two or three objects within their attention, nearly simultaneously, in the order that the professor has chosen.

Sweeps come in many varieties: from left to right; from left to right and then back; up and down; from left to right repeated twice, or multiple times; circular. Unlike points, which can progress in emphasis to touches and holds, sweeps never progressed to touching the board, likely because of the danger of smearing the written text. Therefore repeating the sweep is the likeliest option for increased emphasis. With sweeps, the viewer is being asked to consider a process, and to view it come alive, ever so briefly. Then this process will be talked about. It is on the boundary of points and sweeps that we can see the well-known nominalization/reification habit of the mathematician (see Sfard, 2008). An equation or a statement might be swept, or pointed at, depending of the level of the analysis being conducted at that time, or the level of sophistication of the class at the time of the discussion.

Consider the double point, which occurs three times in the lecture. Two examples, occurring in quick succession, are pictured here (Figures 3 (a), (b)). Here, by using two fingers, the professor asks the student to focus their attention on exactly two items at exactly the same moment. It is not that one item is to be considered before the other, or logically prior to the other, but that the structure of the situation demands that both be considered on an equal level. In each of the figures, the two marked points together constitute a subgroup of a larger group that is represented by its Cayley diagram. This is reflected in the act of pointing. The same pointing gesture from Figure 4 (a) was held in the hand and carried over to the other part of the board and used as in Figure 4(b). This is a concrete depiction of the fact that the two subgroups, though sitting in different groups, are isomorphic as groups.

Here is what is said during the pointing:

17.8 so __these two alone would__ be a subgroup
__just like those two were__

The demonstratives 'these' and 'those' obtain their precise reference together with the act of pointing. When not pointing, such demonstratives obtain their reference by a combination of context and guesswork. It is interesting that there is more information in the act of pointing than in the language; the words say that each set of two marked points is a subgroup; the double point gesture, held in the hands from one subgroup to the other makes it clear that these subgroups are isomorphic.

One of the important functions of pointing is to indicate that two mathematical objects are the same, or isomorphic. Another important function of pointing is to highlight differences and distinctions, in as close to simultaneous visual awareness as possible. Examples of such differences are legion: this map goes from the first group to the second, the other map goes the other way; this was the forward implication, now we are doing the backward implication; whereas the red arrow does this, the blue arrow does that; multiplying by g does one thing, multiplying by g inverse takes us back; and so on. Each of these differences is highlighted particularly vividly by acts of pointing that mirror the symmetries of the sentences spoken. Instead of words like "just like" in the excerpt above, we would see words like "on the one hand… on the other hand', for example.

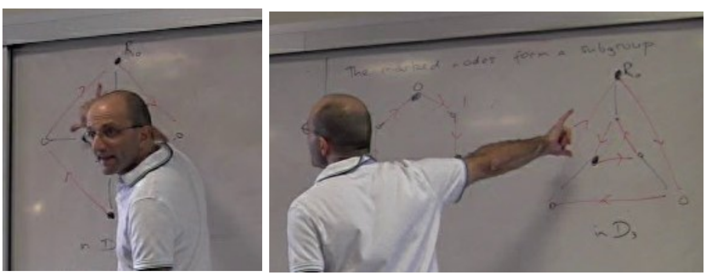

Figure 3(a) Double pointing at the subgroup Z2 sitting inside D4; (b) Double pointing at the subgroup Z2 sitting inside D3.

Two more categories of pointing gestures are *Shakes* (5 examples) and *Waves* (5 examples). Both are made with the hand, with the palm facing the board. In a shake the hand moves side to side a few times quickly. In a wave the hand vaguely indicates a region and ends by breaking a little at the wrist, with the palm going towards the board. They are pointing gestures that in the first case include an element of imprecision, and in the second, an element of disdain; in both there is a feeling of "arms-length" to the subject being discussed or the object being referred to: "This sort of thing"; "we'll do this better or more exactly later"; "I'm not entirely sure what precisely to refer to" might be verbal translations of shakes and waves. They have been separated because the hand motion looks different, but there are insufficient examples to ascertain whether both categories function differently.

DISCUSSION

As evidenced by the data presented above, pointing seems to act as a significant interface between speech and written text. First, the written text appears *on* the board, and speech can happen *near* the board or further *away* from the board. Pointing happens in a membrane parallel to the board and stretching out away from it. We might call this the *skin effect* of the board in a lecture (named by analogy to the skin effect in conductors). Acts of pointing occur here at an interface in a concrete, physical, spatial sense.

Second, while the speech may be about all sorts of matters, usually mathematical, quite often the speech pertains to material that is about to be written, is being written at the time of speech, has already been written on the board and is still visible, or had been written on the board at some point in the course but is now gone from sight. Pointing to text that has already been written is a major component of talking about that text, and in addition, pointing to text that has already been written is a major component of helping the watcher/listener understand and appreciate the written text that is about to happen. Pointing during speech can, by focusing attention on the written, modify what is being written and will be written, and sometimes can help catch errors or omissions in what has already been written. From this perspective, pointing as a verb serves as an active intermediary between writing and speaking, helping to improve each action and helping to coordinate the two actions.

The importance of pointing, as a way of connecting and relating, as Bühler (1990) described below, is certainly not restricted to mathematics:

> Anaphora makes it possible to make insertions of all kinds [into the chain of speech] without losing sight of the overall course, and to make a smaller or larger jump over intervening points in order to draw what has already been or what is yet to come into consideration along with what is now being named. Overall it is an exceptionally multifarious means of connecting and relating, and largely compensates the limitations imposed by the psychological law that the words in the flow of speech can only be produced in a chain one after the other. (p. 444)

However, what Bühler writes here of anaphora is true of the acts of pointing considered in this study and arguably even more true in mathematical communication where the things being connected may not be shared as discursive objects and where relating, especially in proofs, is the primary goal of communication. A student who misses a class, "gets the notes", and even goes to the trouble of asking a friend to make an audio-recording of the lecture, has still only captured two of the three major verbs in the typical mathematics lecture. The student has missed, and can only try to guess at a reconstruction of, all the acts of pointing which connected and related in a nonlinear manner all sorts of individually referred-to objects, and which fashioned these objects into constructed argument after constructed argument. The lecturer acts as a conductor of attention, carefully and frequently deploying his or her acts of pointing, and the students in the room engage in the performance of attending to this and then that. Wherever there is a serious engagement with mathematical explanation or exposition, or a staging and reliving of a line of mathematical argumentation, we expect that pointing will play a central role. It would be interesting in future work to determine whether the frequency and nature of pointing change in other lectures, to understand what these changes depend on, and to interview lecturers to learn what they can articulate about pointing that they have noticed in themselves or others.

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THE "SENSO-MATH" PRESCHOOL PROGRAM: SUCCESSFUL COOPERATION BETWEEN MATHEMATICS FACILITATORS AND PRESCHOOL TEACHERS

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This paper presents a quantitative and qualitative study of the innovative "Senso-Math Preschool" program and the reactions of both the facilitators, who underwent a special training program, and the preschool teachers in whose classes the program was implemented. The goal of the program is to enhance mathematical development in preschool children through the intervention of trained facilitators who bring the adjunct program into preschools. The results indicated a positive change in the attitudes of the facilitators to their professional calling and, after an adjustment period, a positive attitude overall regarding the facilitators' contribution to mathematics education in the preschool as evidenced by the significant relationships that developed between the facilitators and the preschool teachers.

INTRODUCTION AND THEORETICAL BACKGROUND

Today's preschool teachers are expected to have sufficient knowledge to teach early mathematics in preschools. This is in keeping with the modern trend worldwide that advocates familiarity with mathematical concepts already in preschool so as to build mathematical readiness for formal school. Research has affirmed that children at preschool age are able to understand concrete mathematical processes—and sometimes even abstract ones—and the earlier children acquire mathematical experience, the more it contributes to their future development and abilities in the field (Baroody, 2000). It also enhances their cognitive abilities in general. Fostering such exposure requires a teacher to have professional knowledge. However, preschool teachers are not always prepared for such a task, considering the minimal (or lack of) training they may have received in college in the realm of preschool teaching in general, and preschool mathematics in particular. Without sufficient, appropriate knowledge, the teacher, already overburdened with a multitude of responsibilities, is hard-pressed to teach mathematics effectively in preschools.

Furthermore, if the preschool teacher feels unequipped for the role of teaching mathematics, she may not present mathematics in a cheerful and pleasing manner to the children. This might affect the children's attitudes towards the subject, since the attitude of the teacher is one of the main factors that influence students' attitudes toward mathematics (Phlippou & Christou, 1998), because teachers serve as role models for their students (Charalambous, Panaoura, & Philippou, 2009). Recent studies in various countries point to the difficulties that preschool teachers have regarding mathematics in general. Such personal, negative feelings may increase feelings of ineptitude regarding teaching the subject, especially if the teachers have not

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received adequate training for the role (Guo & Justice, Sawyer, & Tompkins, 2011; Ilany, Almog, Ben-Yehuda, & Rosenthal, in press).

These studies led to the conclusion that there is a need to construct appropriate adjunct programs to encourage and empower preschool teachers to teach mathematics. The programs should be built upon new pedagogical principles regarding the development of quantitative, creative and analytic understanding; teaching for the purpose of building thought and understanding; and encouraging mathematical discourse and meta-cognitive thought processes (NCTM, 2000). The “Senso-Math Preschool” (SMP) program (Hassidov & Ilany, 2014) is based on precisely these principles.

THE “SENSO-MATH PRESCHOOL” PROGRAM

SMP is unique in that it promotes the professional mathematical knowledge of the preschool teacher by introducing professional facilitators who model methods by which mathematics can be taught to pre-schoolers. These facilitators come to the preschool and work with the children in small groups. The teacher observes and becomes a partner in the process. By providing guidance and assistance in presenting mathematics to the pre-schoolers, the facilitators reduce the responsibility placed on preschool teachers to achieve the above-mentioned goals.

To implement the program, 30 SMP educational units were created. Special materials were designed that provide accessories and materials to present varied, graduated exercises that follow the already-established preschool curriculum. The mathematical concepts are taught combining sensory and motoric activities in a way that is appealing and engaging for children, using their day-to-day experiences as a basis for their learning. Kits were produced for the facilitators, and also for the individual children, who can take the material home to share with their parents.

In each preschool, learning groups were assembled according to the recommendation of the teacher. The facilitator worked with the children once or twice a week, either in their formal preschool or in the afternoon, within the framework of enrichment classes, and for a period of 40 minutes each time for ages four to six, and 30 minutes for ages 3 to 4 years. Activities were held in groups of up to 10 children. The pedagogical and mathematical rational of the “Senso-Math” teaching kits were initially tested in 20 preschools. The results of the initial program were validated through observation, data collection, and accompanying research, then were revised to further enrich the curriculum framework. After final approval by the Israel Ministry of Education, several hundred preschools were chosen to integrate the program in facilitated preschool mathematical education.

A parallel goal of the program is to provide career opportunities for women who receive specialised training in teaching math to pre-schoolers, allowing them to become integrated into the teaching regimens of preschools and to use their acquired knowledge and skills to contribute to the field of education.

This present study examined the facilitators' attitudes regarding 1) their appreciation of the necessity of teaching mathematical principles already in preschool; 2) their self-confidence vis a vis teaching mathematics; 3) their responsibility towards developing their own careers; and 4) the extent that the program gave them the proper tools and support for their job in the preschool.

In addition, we assessed (through interviews) the preschool teachers' acceptance and appreciation of the SMP program, and how it affected their mathematics teaching in their classes.

FACILITATOR TRAINING

Qualified women (see below) studied in a 128 hour academic program over a course of 20 meetings. Training was on two levels: 40 hours on the natural integration of facilitators into the preschool, 88 hours on mathematical content in early childhood education, and practical work administering and teaching SMP in preschools (with one-on-one mentoring, as required). After completing the course, participants were qualified to work as independent teachers in the field.

The specially designed training kit was introduced during the course to familiarise the participants with its contents and demonstrate how to use it as an activity center.

THE RESEARCH POPULATION

The initial pilot training program included 500 women from different socio-economic sectors in Israel who had the appropriate education for teaching preschool: generally, they had a certificate from a college for teachers (or for preschool teachers) or were graduates of academic institution. The average number of years of education was 14.5 years.

Of those 500, a sample of 49 who had a background in preschool education and enough mathematical orientation to provide them with the ability to work as mathematical facilitators in preschools were chosen for this study (details in Hassidov, 2014).

METHOD

Research methods were quantitative and qualitative. Data were collected via a 22-item questionnaire written by the researchers. Questions were designed to examine the attitudes of the participants concerning teaching and learning mathematics in preschool, and the training that they had undergone. Respondents were asked to rate the statements from 1 (not at all) to 5 (a great extent). Negative statements were marked. In addition, participants were asked if they would recommend the program to a friend (an indication or overall satisfaction). Afterward, participants underwent semi-structured interviews to qualify the opinions presented in the questionnaire. The participants—and the preschool teachers involved—were interviewed after completing the training to describe their experience with the program.

RESULTS

The overall results (see Table 1) show that the 49 participants viewed the course favourably and considered it a viable career alternative. Table 1 shows a summary of their attitudes (listed in Table 2) regarding mathematics in preschool, how the course developed their professional confidence and their self-confidence in teaching mathematics, and their satisfaction with the program.

Attitudes of participants	Attitude to the study of mathematics	Developing professional confidence	Developing self-confidence	Evaluation of the program	Would you recommend to a friend?
Average	4.34	3.46	2.19	4.29	4.04
(sd)	(.40)	(.56)	(.90)	(.64)	(1.19)

Table 1: Attitudes of participants towards various facets of the SMP

Table 2 presents a more detailed picture of the participants' responses. The 22 statements are in separate categories and the number of responses for each level (from 5—to a great extent, to 1—not at all) are listed along with the average rating.

#	Statement	5	4	3	2	1	Av.
	Attitude regarding the study of mathematics:						
1	It is important that children start learning mathematics in preschool.	33	13	2	--	--	4.65
2	Children of preschool age can learn mathematics.	29	15	4	--	--	4.52
3	If the basics of mathematics are learned before first grade, the child will develop a positive attitude towards the subject.	27	20	2	--	--	4.51
4	If the basics of mathematics are learned before first grade, the child will develop a positive attitude towards the subject.	27	20	2	--	--	4.51
5	Anyone can learn mathematics.	18	18	11	--	--	4.15
6	I see my future in teaching children mathematics.	19	15	13	1	--	4.08
7	Anyone can enjoy learning mathematics.	16	14	14	4	--	3.88
	Development of professional confidence						
8	Anyone who aspires to succeed can do so at any age.	20	18	9	1	1	4.40
9	Unemployed women should be concerned about their professional development.	2	11	10	10	16	4.12
10	Teaching mathematics in preschool requires readiness, knowledge, and professional maturity.	6	11	13	11	7	3.79

11	The training encouraged me to start teaching mathematics in preschool.	6	10	15	7	4	3.67
12	I feel I can incorporate the SMP into the preschool.	14	17	11	2	3	3.57
13	The training gave me professional confidence.	23	20	4	--	--	3.17
14	The training encouraged me to pursue my professional aspirations.	11	16	13	2	3	2.96
15	I am considering making mathematics teaching my main profession.	8	15	19	3	1	2.45
	Statements about self-confidence in teaching mathematics:						
16	The training gave me confidence to teach mathematics.	6	10	15	7	4	3.17
17	Had I not participated in the SMP, I would not have confidence to teach mathematics*.	1	2	9	5	29	1.72
	Statements about satisfaction with the program:						
18	The SMP facilitators' kit was a valuable aid for facilitating mathematics in the preschool.	17	15	4	1	--	4.30
19	The SMP activity pages were valuable for teaching mathematics in the preschool.	19	14	9	1	--	4.19
20	The course was conducted professionally.	20	14	7	1	1	4.19
21	The training gave me tools to facilitate mathematics in preschool.	19	14	11	--	--	4.18
22	The training gave me tools to teach mathematics in preschool.	19	18	7	1	1	4.15

*The overall low score to this question was likely due to it being presented as a negative statement.

Table 2: Responses to the questionnaire

The attitudes of the participants regarding the study of mathematics were, on the whole, favourable: the average ratings for all the statements are in the vicinity of 4 and above. The statement that received the highest average rating was the one regarding the importance of having children learn mathematics already in preschool (4.65, average). The statement that received the lowest rating was "Anyone can enjoy learning mathematics," but was still above 4. The results indicate that the facilitators thought it was important for children to learn mathematics as early as preschool, and that this would help the children develop a positive attitude to the subject (4.51).

Regarding professional confidence, the statement that received the highest average score was "Anyone who aspires to succeed can do it at any age." Moreover, while the answers indicate that the training was successful (Statements 11, 12, and 13), statement 15, "I am considering making mathematics teaching my main profession," received the lowest score. So although the results indicate that training encouraged facilitators to begin teaching mathematics in preschool, and that they realised that teaching preschool requires readiness, knowledge and professional maturity (Statement 10), they did not necessarily show readiness to continue in this area.

Regarding the participants' satisfaction with the SMP and the course, the results (the average of all statements being over 4, with a clear majority of answers at levels 5 or 4) indicate that the SMP was considered very valuable for teaching mathematics in preschool. Participants indicated satisfaction with the tools, kit and activity pages, as well as with the professional way in which the course was conducted.

Other aspects of our study examined the participants' satisfaction with the course correlated with factors such as number of years of education and age, or the number of children in their preschool groups. Results can be found in Hassidov and Ilany (2014).

SUMMARY OF QUALITATIVE RESULTS

A year after the first SMP pilot program, a large proportion of the 500 participants (75%) had been integrated into preschools as facilitators. In interviews conducted at this time, their experiences regarding their integration into the preschool system were recorded. The qualitative results support, strengthen, and clarify the quantitative data.

The general impression was that the facilitators created a rich, diverse environment for learning mathematics in the preschool, and that teachers benefitted from the presence of a professional colleague who came once or twice weekly to take responsibility for mathematics instruction. The facilitator brought learning materials for the children, and the teacher received guidance as to how to continue the experience during the week. The facilitators reported that the teachers observed their activities with the children and repeated them during the course of the week. One of the facilitators reported that, as a result of her activity, the teacher's policy regarding mathematics changed: "The teacher told me that since I had begun coming to the preschool … she has begun integrating daily mathematical activities into her program."

However, reports indicated that the relationship between facilitator and teacher did not always start out smoothly. Some of the preschool teachers seemed to feel threatened: "Why do I need someone else—an 'expert' in mathematics—to come every week? What can they show that I can't?" Some teachers initially suspected that the facilitators represented the Department of Education and were there to check on the teacher's ability to teach math. This may be because the program was overseen by Ministry of Education inspectors or, perhaps the teachers suspected that the facilitators were initiating a program to increase the teachers' burdens regarding teaching mathematics and did not realise that the facilitators were there to lighten their load, not to increase it.

Thus, at the beginning of the program the teachers tended to be uncooperative and did not give the facilitators freedom to present the program as they wished. For example, one of the facilitator reported:

> I was anxiously looking forward to working in a preschool ... I had a good feeling that I could contribute and cooperate with the teacher … However, I seemed to be received with some suspicion and a feeling of uncooperativeness on the side of the teacher … She couldn't find an appropriate spot where I could work … or she would say that the children were involved in some other activity … Each time I got started, I would hear something that made me feel that she wasn't happy to have me in the class.

However, after a settling-in period (the facilitators received guidance on dealing with the teachers) the situation changed. She continued:

> The course trainer accompanied me and gave me some advice, and I understood the fears the teacher had regarding the situation … After a number of weeks during which I had been teaching and the teacher observing, things started to change for the better. At the end of the year, the teacher asked me to explain the models to her, and she asked me to help her prepare a mathematics program for next year. … She also asked me to come to the parents-teachers' night to update the parents on what we had done in the class, and to explain how items to aid learning mathematics had been incorporated into the play area.

One preschool teacher who taught 4-6 year olds reported: "After a number of lessons during which I had not been asked to do anything, I understood how much simply observing her and her way of working with the materials contributed to me." She continued: "After the facilitator left the preschool, I, myself, used the teaching materials that she had left, and I saw how easy it was to teach the children with them."

A third teacher reported:

> After a number of weeks … I started to look forward to her arrival. Believe me, I stood at the door and waited for her. I had to tell her what had happened the day before when I taught the children about pattern: the children told me that Nadine's socks are also a pattern because they have stripes—blue, red, green, blue, red, green. Then all the children started looking for patterns on the clothes of their classmates. It made me so happy!

Another preschool teacher said: "I never had anyone with whom I could discuss how to teach mathematics in my classes; now, I have someone to talk to every week and I can consult with her. My supervisor is aware of this too."

With time, the instruction of mathematics in the preschools entailed full cooperation between teacher and facilitator. The facilitator taught the children mathematics once or twice a week and the teacher observed the activity. The teacher continued the facilitator's activities during the week, to reinforce the teaching for the children.

CONCLUSIONS

This study indicates that the SMP program contributed greatly to both those who studied to be facilitators as a way of professional development, and to the heavily burdened preschool teachers who do not have enough knowledge or training to adequately provide mathematical instruction in their classes. The presence of facilitators in the preschool transformed the subject of mathematics into one that is interesting, fascinating and challenging, and central to the daily schedule. The preschool teacher came to realise that teaching preschool mathematics is an area that requires professional training. As one of the facilitators reported:

> At first, the teacher objected to having me in her preschool teaching mathematic. After several months, though, we were collaborating nicely and she told me that she now realised that teaching preschool mathematics is important and requires professional training, something that I, the facilitator, received and that she lacks.

The "Senso-Math Preschool" program provides an answer to a definite need in today's educational system where mathematics must be taught to children at a young age in order to prepare them for their mathematics studies in grade school.

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EFFECTS OF INSTRUCTION ON STRATEGY TYPES CHOSEN BY GERMAN 3RD-GRADERS FOR MULTI-DIGIT ADDITION AND SUBTRACTION TASKS: AN EXPERIMENTAL STUDY

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In an experimental study, we implemented two instructional approaches to teach 73 3rd graders from 17 school classes adaptive strategy use. The explicit approach encompassed the explicit teaching and practicing of selected strategies, whereas the problem-solving approach emphasised the analysis of task characteristics and the individual generation of strategies. Results from post- and follow-up tests after the intensive one-week intervention did not yield significant differences between the two approaches in the efficency and in the accuracy of the applied strategies. In this contribution we report an additional analysis of the data examining the types of strategies the students chose. Although both groups used efficient strategies, it turned out that they differed significantly in the types of strategies they chose.

INTRODUCTION

Adaptive strategy use in arithmetic, i.e., solving computation tasks efficiently by flexibly choosing an "advantageous" strategy, is considered as an important aspect of mathematics education. Although the standard (written) algorithms for the basic arithmetic operations still play a prominent role in arithmetic education, in many countries text books and primary school curricula also address students' competence to adequately use different strategies for solving arithmetic tasks. However, as empirical studies repeatedly revealed, the acquisition of such an adaptive expertise is quite challenging and empirical findings indicate unsatisfactory results for primary school students (e.g. Heinze, Marschick, & Lipowsky, 2009; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009). Accordingly, specific instructional approaches are discussed to organise effective learning opportunities to support students. These approaches are based on different learning theories and follow different assumptions about the acquisition of adaptive expertise. However, there are hardly empirical studies on the comparison of these instructional approaches for students' adaptive strategy use.

THEORETICAL BACKGROUND AND EMPIRICAL FINDINGS

Strategy types and adaptive strategy use

For an empirical examination of students' strategies it is necessary to choose a category framework to make the observed strategies accessible for a deeper analysis. Arithmetic computation strategies for multi-digit addition and subtraction can be categorised in various ways (see an overview in Threlfall, 2002, pp. 33ff.). In prominent German mathematics education books the categorisation in Table 1 is described. It distinguishes five main types of strategies for addition and subtraction problems, each type covers several strategies. For example, the jump strategy type encompasses jump strategies

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which successively add the hundreds, tens and units of the second summand or the other way round the units, tens and hundreds of the second summand or the two strategies which analogously decompose the first summand. The types jump and split strategy encompass universal strategies which can be applied for all addition and subtraction problems. [It is an open discussion how to deal with the split strategies in case of subtraction problems with regrouping. Some of the German textbooks introduce a split strategy but avoid the notation of intermediate (negative) results.] The strategies of the other three types are advantageous only for specific problems and cannot be applied efficiently in general. All these strategy types are idealised strategy types in the sense that children obviously are quite creative and generate strategies of further types, especially by combining two or more strategies of different types (e.g., Selter, 2001).

Jump strategy	Split strategy	Compensation strategy	Simplifying strategy	Indirect addition*
123 + 456 = *579*	123 + 456 = *579*	527 + 398 = *925*	527 + 398 = *925*	701 – 698 = *3*
123 + 400 = 523	100 + 400 = 500	527 + 400 = 927	525 + 400 = 925	698 + 3 = 701
523 + 50 = 573	20 + 50 = 70	927 – 2 = 925		
573 + 6 = 579	3 + 6 = 9			

Table 1: Idealised types of computation strategies with examples.
[*The indirect addition strategy is for subtraction problems only.]

As in our previous research, we describe students' competence for an adaptive strategy use by the efficiency of the applied strategy for a given task (Grüßing, Schwabe, Heinze, & Lipowsky, 2013). Here, we take into account two perspectives: For a student solving a given arithmetic task, one can check (1) from a mathematical perspective which strategy (or strategies) need(s) the smallest number of solution steps and (2) from a psychological perspective how much cognitive effort different solution steps require, which obviously depends on the knowledge and skills the individual has acquired so far (probably biased by affective variables like self-efficacy). Based on these criteria, we can define normatively which strategies are considered as efficient for a student solving a given arithmetic task and which are not. This norm is not restricted only to the properties of a given task but as in other studies like Klein, Beishuizen, and Treffers (1998) takes into account knowledge and skills of the considered student. Accordingly, in our research with 3rd graders, we first identify the range of strategies which can be expected by the group of students under investigation (i.e., strategy repertoire in the sense of declarative knowledge as well as the fluent and accurate application of these strategies with low cognitive effort in the sense of procedural knowledge). Then for these strategies we analyze how they fit to the characteristics of a given task and, thus, provide a short solution. However, it has to be mentioned that there might be other influental factors beyond these criteria. For example, Verschaffel, Luwel, Torbeyns, and Van Dooren (2009) suggest the context (in the sense of socio-mathematical norms) as possible factor when a teacher in her/his class implicitly conveys a reference framework which favors specific strategies. This problem is addressed in our sampling procedure by selecting only a few students from each class and, thus, reducing the influence of shared socio-mathematical norms.

Teaching adaptive strategy use

Empirical findings repeatedly revealed a low proficiency of primary school students in adaptive strategy use. In particular, many students have one or two favorite strategies – mostly one for addition and a different one for subtraction (in Germany: jump strategy for subtraction, split strategy for addition, Heinze et al., 2009). Moreover, most students solely use the standard algorithms after they have been introduced (e.g., Selter, 2001). Based on these results the question arises how to teach the adaptive strategy use to students.

In the literature, we find the traditional approach and so-called reform-based approaches (e.g., Verschaffel et al., 2009). In the traditional approach firstly only one strategy – in general, the jump strategy – is taught to and practiced by the students so that it can be applied accurately as a routine procedure. After that sometimes other strategies are mentioned in a sense that there exist helpful „computation tricks" for specific tasks. The reform-based approaches can be divided in two quite different types which we denote as explicit approach and problem-solving approach (see Heinze et al., 2009 for details). In the *explicit approach* firstly students invent their own strategies in an introductory phase. After that the teacher structures and reduces the diversity of invented strategies to a set of main strategies (cf. Table 1) which are successively practiced by the students. Finally, the adaptive strategy use is emphasised through solving tasks and discussing different solutions. An example for this explicit approach is the realistic program design as implemented in the study by Klein et al. (1998).

In contrast to the explicit approach, the *problem-solving approach* does not follow the idea of selecting a strategy from an individual strategy repertoire (cf. Threlfall, 2002). There are no official strategies introduced or named by the teachers. Students consider each arithmetic task as a new problem and generate a specific solution strategy for this problem (based on their knowledge and experience and on the task characteristics). Hence, students get many opportunities to analyze task characteristics, to solve problems and to discuss the efficiency of the students' solution strategies. Accordingly, they can accumulate knowledge on task characteristics and on skills in applying and judging individual strategies so that they will optimise their adaptive strategy use step by step.

Currently, we do not have much empirical evidence for the effectiveness of these instructional approaches. The one-year quasi-experimental study of Klein et al. (1998) indicates an advantage of the explicit approach in comparison to the traditional approach. Heinze et al. (2009) report that 3rd-graders taught by textbooks following the explicit or the problem-solving approach outperform 3rd-graders taught by textbooks following the traditional approach. Moreover, this study and also the findings of Torbeyns, De Smedt, Ghesquière, and Verschaffel (2009) indicate that high achieving students can also reach a high level of adaptive expertise when they are taught by the traditional approach. Grüßing et al. (2013) report a controlled experimental study comparing idealised implementations of the explicit and the problem-solving approach (see 3.1 for the design). Their results suggest that there are no significant differences in the short term and long term effects of both reform-oriented approaches on the competence of adaptive and accurate strategy use.

RESEARCH QUESTION AND METHODOLOGY

Although there exists only a small number of empirical studies on instructional approaches teaching the adaptive strategy use, it seems that reform-oriented approaches are more beneficial than the traditional approach. Interestingly, our results in Grüßing et al. (2013) indicate no difference in the effectiveness of the explicit and the problem-solving approach. Since the approaches have quite different theoretical assumptions about the acquisition of adaptive expertise and since they strongly differ in the derived teaching activities in the mathematics classroom, we conducted a further fine grained analysis of the data to answer the following research questions:

Do the children of both groups

- differ in their choice of specific strategies after the intervention (i.e. in the posttest and the follow-up tests)?
- develop differently during and after the intervention?

Sample, design and instruments

This section presents the main information of the experimental study as it was already described in Grüßing et al. (2013). The sample of the study comprised 79 randomly chosen 3rd-graders (9-10 years old) from 17 classes of German primary schools from which we included 73 in this additional analysis. Six students were excluded because already in the pretest they used almost exclusively the efficient compensation strategy or the dominant written algorithms (i.e., their pretest results showed that they were more than 6 months ahead of the grade 3 curriculum, possibly due to out of school support). In a first step, the 73 children were randomly allocated to one of the two instructional approaches and after that the groups were parallelised according to general cognitive abilities, general mathematics achievement and socio-economic status.

The intervention was organised as a one-week course at our research institute during fall holidays. The overall intervention time was equivalent to 16 schools lessons (45 min) and accompanied by breaks for playing games and lunch. The lessons were taught by two trained research assistants following detailed teaching scripts of the explicit and the problem-solving approach (a short overview is given in Table 2). Expert ratings confirmed that teaching scripts and material mirrored the two approaches and that the comparison is fair. To limit the group size, we had two student groups for each approach (one group was taught in the first and one in the second holiday week). To control for teacher effects, both teachers taught each approach once.

Day	Explicit approach	Problem-solving approach
1	Repetition of numbers up to 1000 and introduction of small group discussions	
2	Discovery & practice of jump and split strategy, small group discussions of individual solutions	Distance of given numbers, decomposing numbers, categorising tasks in easy, smart[1] and other tasks
3	Discovery & practice of indirect addition, compensation & simplifying	Categorising tasks, generation of easy and smart tasks
	Solving tasks and comparing solutions in small group discussions	
4	Repetition of all strategies	Categorising tasks and discussing individual criteria for categorisation
	Solving tasks and comparing solutions in small group discussions	
5	Post-tests and interviews[2], closing session	

1. "Easy tasks" can be solved immediately (e.g., 150 + 230), "smart tasks" easily by a specific strategy (e.g., 329 + 141). Obviously, the allocation of tasks depends on the individual.
2. We also conducted interviews which are not discussed in this paper.

Table 2: Content of the one-week holiday course for both approaches

Data for adaptive strategy use was collected by trained university assistants with a pre-test 2 weeks before the intervention (T1), an immediate post-test (T2) and two follow-up tests after 3 (T3) and 8 months (T4). The test T4 was administered after the students learned standard algorithms for addition and subtraction. Each test consisted of 8 multi-digit addition and subtraction tasks suggesting specific strategies as efficient solutions (e.g. compensation, simplifying, etc., see Table 1). The tests were linked by anchor items: consecutive tests had 6 common items and 4 anchor items were used in all tests (403-396, 1000-991, 398+441, 502+399).

The item solutions were categorised by the strategies the students used for their solution. We started with a fine-grained system of 21 strategy categories which we retrieved from the literature and from theoretical analysis supplemented by some "bottom-up" strategy categories which frequently occurred in the student solutions. The category system included the main strategy types from Table 1 (e.g. jump strategy, split strategy, compensation strategy etc.) with several subcategories (e.g. jump strategy starting with units). For each test, the allocation of student solutions to categories was conducted independently by two trained research assistants (all Cohen's $\kappa > .70$) followed by a consensual agreement in case of different ratings.

For answering the research questions, we applied Chi-squared tests for homogeneity. This statistical test allows determining whether the distribution of the chosen strategies is significantly related to the group variable, i.e. whether the distribution of observed strategies in the explicit group and in the problem-solving group are similar or not. Due to mathematical prerequisites for the statistical Chi-squared tests, we had to merge categories in a senseful way to avoid too many small cell frequencies. We finally used 11 strategy categories for the first and 7 categories for the second research question.

RESULTS

Effects of the teaching approaches on the chosen strategies

To analyse the effects of the one-week intervention, we compare the chosen strategies of the children in the explicit group with those of the children in the problem-solving group separatley for the pretest (T1), the posttest (T2) and the follow-up tests (T3, T4). The categories and the frequencies of the chosen strategies in each group in each test are presented in Table 3. In the pretest the two groups did not differ significantly whereas in all other tests we found significant differences with moderate effect sizes. Concerning the specific efficient strategies for the test items, we can observe that immediately after the intervention the explicit group preferred strategies of the types indirect addition and simplifying whereas the problem-solving group preferred strategies of the type compensation. In the follow-up tests after three and eight months, the preference for the indirect addition and simplifying strategies in the explicit group is lost whereas the problem-solving group still keeps stable in the preference of the compensation type strategies.

Change of preferred strategies over time

For the second research question, we analyzed the development of the strategy distribution in both groups separately. Due to space limitations we cannot present the table in this contribution. The analysis is based on the four anchor items so that we can compare the three time intervals. For both groups we found significant differences between two consecutive tests except the interval T2-T3 in the problem-solving group. The associated effect sizes indicated that – as expected – in both groups striking changes occurred during the intervention phase T1–T2 (Cramér's V is .54 for the explicit and .46 for the problem-solving group) and in the phase T3-T4 when the dominant standard algorithms are taught in the regular mathematics classroom (Cramér's V is .45 for the explicit and .46 for the problem-solving group). Remarkable is that during the three months after the intervention (T2–T3), the students of the problem-solving approach remained comparatively stable in their strategy choice whereas in the explicit group the use of specific strategies trained in the intervention (indirect addition, compensation, simplifying) decreased.

Frequencies	T1 (pretest)	T2 (posttest)	T3 (after 3 months)	T4 (after 8 months)

	Explicit	Problem solving	Explicit	Problem solving	Explicit	Problem solving	Explicit	Problem solving
Written algorithms	5	4	3	11	20	10	118	103
Split strategy	32	21	55	9	59	14	31	9
Short split	7	7	1	1	1	4	6	7
Jump strategy	101	109	42	42	48	38	4	11
Short jump	43	51	23	49	30	69	9	29
Combination split & jump	40	42	12	39	13	11	19	6
Indirect addition	5	4	61	26	20	24	9	11
Compensation[1]	5	6	29	55	35	65	23	75
Simplifying[1]			45	18	16	12	21	21
Purely mental	11	19	14	17	30	10	19	11
Not assignable	26	8	7	3	7	3	3	1
Total[2]	275	271	292	270	279	260	262	275
χ^2	$\chi^2(9, N = 546) = 15.27$		$\chi^2(10, N = 562) = 96.19$		$\chi^2(10, N = 539) = 70.52$		$\chi^2(10, N = 537) = 58.04$	
p	.084		< .001		< .001		< .001	
Cramér's *V*[3]	.17		.41		.36		.33	

[1] For T1 compensation and simplifying were merged to avoid too many low cell frequencies

[2] Sample N = 584 (73 students times 8 items for each test) was reduced by missings (single items not processed or single students did not participate in one test); the subsample of 63 students which participated in all four tests yields similar results.

[3] Effect size Cramér's *V*: < .3 weak relation, .3-.5 moderate relation, >.5 strong relation between the variables

Table 3: Comparisons of the strategy distributions of the two groups at T1-T4

DISCUSSION

The results give further insight into the relation between instructional characteristics and the strategy choice of students. As mentioned, the two instructional approaches which follow different educational philosophies have similar positive effects on students' competence to find efficient solutions for given arithmetics tasks (Grüßing et al., 2013). However, the effects of the instructional approaches are quite different if we take a qualitative perspective. After the intervention (T2), students of the explicit group use more frequently the demanding specific strategies (categories "simplifying" and "indirect addition") which were explicitly taught. However, the frequency of these strategies decreases in the following three months, perhaps, because they were learned only superficially. In contrast, students of the problem-solving group use more frequently and stable self-invented strategies after the intervention (short jump, compensation, combined strategies). As expected (cf. Selter, 2001), at T4 the dominant written algorithms are the main strategy type for both groups (45% in explicit, 37% in the problem-solving group). Nevertheless, eight months after the intervention students of the problem-solving group still choose frequently (self-invented) compensation strategies.

Summarising the findings, it seems that, firstly, the availability of an individually acquired strategy repertoire is more sustainable if the strategies are self-invented by the students. Secondly, it turns out that important strategies like indirect addition or simplifying are quite demanding and many children cannot invent such strategies on their own so that an adequate support is needed.

Acknowledgements

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AN ACTIVITY THEORY ANALYSIS OF GROUP WORK IN MATHEMATICAL MODELLING

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In this paper we analyse the activity of a group of engineering undergraduate students while working on a mathematical modelling task. Using Cultural-Historical Activity Theory as analytical framework, we focus our attention on their social interactions to understand how these mediate the collective sense making of the group and determine in great part the outcome of the activity. We conclude that a key factor to students' mathematical learning in collaborative tasks is the quality of peer interactions which stems from students' competences, such as communicative and inter-personal skills.

BACKGROUND

Many lecturers and researchers agree that the development of problem-solving and mathematical modelling skills is an important aspect of the education of undergraduates studying Science, Technology, Engineering and Mathematics (STEM). For example, in their seminal paper, Blum and Niss (1991) presented several arguments in favour of including aspects of modelling and problem-solving in mathematics instruction, amongst which are the development of skills and attitudes such as open-mindedness, self-reliance, confidence and critical thinking.

In many cases, pedagogical implementations of mathematical modelling and problem-solving are accompanied by collaborative group work. The potential benefits of team collaboration to student learning have been well documented (see for example Laal & Ghodsi 2012). In particular, several studies of collaborative work in mathematics suggest it can help the students' process of modelling and problem-solving, and hence contribute positively to their learning, but that this relation is complex and still not well understood. For example, Lowrie (2011) reports on the tensions between collaborative learning and the use of "genuine" artefacts in problem-solving mathematical tasks. And Clark et al. (2014) suggest that effective group work collaboration depends on the type of problem choice that can elicit certain behaviours and the establishment of a "group synergy" that can lead to increased group interaction and activity.

However, most research on collaborative group work in mathematical problem-solving and modelling has been done from a cognitive perspective, e.g. comparing the characteristics and behaviours of novice and expert modellers/problem-solvers and focusing mainly on the heuristics of the process (i.e. selection of key variables and appropriate assumptions to the problem, construction of relations between variables, etc.). For instance, Paterson and Watt (2014) describe how third year undergraduate students working in groups failed to solve a mathematical problem because 'they ignored explicit constrains, over-generalised earlier examples and left a number of erroneous assumptions unchallenged' (p. 19). Further, Soon et al. (2011) describe first

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year undergraduate students' difficulties with mathematical modelling as an inability to connect "real life contexts" and "mathematical representations".

Of the few studies that take a socio-cultural perspective, Goos et al. (2002) is a significant example. Taking a Vygotskian perspective, these authors reconceptualise metacognition (or "learning to learn") as a social practice, and they conclude that:

> the interplay between transactive challenges and metacognitive decisions was significant in creating zones of proximal development that shaped problem solving outcomes, since challenges eliciting clarification and justification of strategies stimulated further monitoring that led to errors being noticed or fruitful strategies being endorsed (p. 218).

However, in this paper we will argue that there are other relevant issues that remain insufficiently explored in the literature and that may have a significant effect on the outcome of small group collaborative mathematical activity. The aim of this paper, therefore, is to investigate how the social interactions that occur in small group collaborative work affect the outcome of a mathematical modelling task, and particularly how these interactions contribute to mathematical sense making.

To this aim we observed students in a one semester second year undergraduate mathematics for engineering course at an English research-intensive university. A feature of this course is the use of mathematical modelling tasks as a complement to traditional style lectures; in order to solve these tasks, students work in small groups (4 -5 members). In this paper, we selected a one-hour episode in which the students failed to produce a mathematically correct solution to the task. The analysis of this particular episode allowed us to gain important insights into issues that we will argue are significant in shaping collaborative mathematical activity. The research question guiding our analysis was: How do social interactions in a small group collaborative work influence the students' mathematical sense making and the outcome of the activity?

THEORETICAL FRAMEWORK AND METHODOLOGY

In order to study how social interactions influence an activity such as small group collaborative work, we found Cultural-Historical Activity Theory (CHAT), as described by Engeström (1987), a helpful analytical framework. CHAT's well-known "triangle" (Figure 1) draws attention to the complexity of (social) factors mediating human activity.

Human activity, in this case collaborative learning in a mathematical modelling task, is our unit of analysis; subjects engage in this object-oriented activity with the purpose of obtaining an outcome (e.g. learn something, solve a problem). The community, the rules and the division of labour represent the social/collective elements of the activity, which interact between them and, along with the tools, mediate the activity.

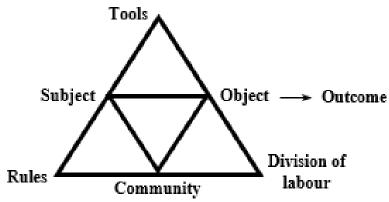


Figure 1. The structure of a human activity system (Engeström 1987).

The episode analysed in this paper was audio-recorded and transcribed. The second author, who observed the episode, also took notes that were included in the transcript. The transcript was then coded separately by the two authors according to the elements in the CHAT triangle and the results compared. Meanings were negotiated between the authors (there were no major discrepancies) and shared interpretations were achieved.

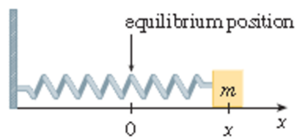
CHAT ANALYSIS OF COLLABORATIVE GROUP WORK IN A MATHEMATICAL MODELLING TASK

The activity

The activity described in this paper was the modelling task shown in Figure 2.

There is an object of mass m attached to the end of a spring as in the illustration. **Hooke's law** states that the force exerted over a spring that is stretched x units from its resting position (equilibrium) is proportional to the distance x.

Use Newton's second law of motion (Force is equal to mass times acceleration) to model this type of motion (known as simple harmonic motion) with a second order differential equation. Then, solve the equation.



Now suppose that a spring with a 3 kg mass is held stretched 0.6 m beyond its equilibrium position by a force of 20 N. If the spring begins at its equilibrium position (when t=0 what is x?) but a push gives it an initial velocity of 1.2 m/s (when t=0 what is the velocity?), find the position of the mass after t seconds.

Figure 2. The modelling task

The topic of the task, on harmonic motion, was second order Ordinary Differential Equations (ODE) and students were asked to work in groups (of their own selection) to solve the problem and write a brief report of their solution to be handed in at the end of the tutorial session (1 hour). Students had already covered how to solve second order ODEs in previous lectures and also during their first year course. Students were reminded of the modelling cycle (Blum & Borromeo-Ferri, 2009), and the lecturer handed out a sheet on "effective group work" and explained it to the group. The task was not formally assessed; the lecturer introduced it as "preparation" for their coursework, and offered to give constructive feedback to each group the following week.

The subjects of the activity

The group that we followed started with 4 members (all pseudonyms): (1) Steve, who described himself as not confident with mathematics. He usually does not attend lectures; (2) Mike, who attends most lectures, said he was "catching up" with the topic and brought with him a textbook on engineering mathematics. He said he usually revises only before the exam; (3) Hank, who said he was confident with the material, attends all lectures and usually spends 30 minutes ("but no more") revising problems seen during lectures. He always brings with him his lectures notes; and, (4) Tom, who attends lectures regularly but does not feel confident with mathematics.

After 10 and 15 minutes, respectively, two other students joined the group (Steve sent a mobile text to them asking to come and help him with the task): (5) George, who said he found the material difficult and that he is not confident with mathematics in general; and, (6) Alan, who described himself in similar terms as George but usually helps Steve with his mathematics. Both do not attend lectures regularly.

It can be noticed from the description of the members of this team that only Hank feels confident and indeed he takes the lead throughout most of the activity. However, contrary to common descriptions found in the literature where one team member (usually a "high achiever") dominates and determines the group's ideas, in this case important ideas came from various group members who took the lead at different points in time. In fact, it was observed that throughout the activity all members of the group went "in and out" of the mathematical problem, sometimes taking a "back seat" and then "coming back" to contribute to the discussion, sometimes forming sub-groups to discuss unrelated issues (e.g. sports activities).

The object of the activity

It can be assumed that, for the subjects of the activity, their object was to solve the mathematical problem and, as a result, produce a group report. Members of the group ratified this objective at different times. For example, after George joined the group, Steve and Hank said:

Steve: So what do we actually have to do?

Hank: Make a report (and Mike proceeds to read the task).

The community and the tools

In this case, the community was formed of the members of the group but it was not a "community of practice" in the sense of Wenger (1998). This "community" was formed spontaneously and for the purpose of solving this one task, and then dissolved. Members did not share any history in relation to the practice (although they have a history of knowing each other in other practices), or any shared discourse associated with mathematical collaborative work. There were no "master" and "apprentices".

Due to the informal and spontaneous nature of this community, there was an attempt to build shared meanings and understandings and to collaborate in achieving the object of activity. This interactivity and willingness to build consensus made this activity genuinely collaborative (as opposed, for example, to a co-operative task). There was also a constant questioning and challenging of the work throughout the task, e.g. Mike: 'Are you sure it's supposed to be an H there?' or Hank: 'Think. Any ideas people?' or George: 'What am I doing here?' Alan: 'All right, then you divide it by x'. George: 'Oh, yeah, yeah'. Throughout the activity, peers took different roles as the group tried to make sense of the problem. For instance, around 15 minutes into the task the group had been struggling in defining variables and constants but had come to a shared agreement that Newton's second law of motion ($F = ma$) could be differentiated into $\frac{dF}{dx} = m$ (F differentiated with respect to x, m is a constant and a is a variable). However, differentiating again (to obtain a second order differential equation) would yield a zero! At that point Hank, unable to come up with a constructive idea, took a back seat to read from the workbook and Steve and George "took over" the task. George then came with an idea:

George:	I was thinking that, I was thinking I must put, like, because when F is the same in both, so we must put $kx = ma$ and then, take it from there.
Steve:	Yeah. But you know that k is obviously a constant.
George:	Yeah, k is a constant.
Steve:	And mass is a constant, therefore you've got a and F.
George:	Sure. So wait, and isn't force not the same in both of these things, so, would a not equal to k?

While George's first idea could have eventually resulted in a satisfactory solution, the interaction with Steve resulted in an equation which contained mathematical errors: $m = kx$. Taking then his peers' idea as a resource, Hank continued to elaborate on the problem by trying to manipulate their expression, eventually getting stuck again.

Our interpretation of the type of interactions like the one described above is that peers' ideas proposed at each moment in time become the "tools" that the group use in their process of sense making. Also, the relations in the group change and evolve (peers take different roles) and, as a consequence, the meaning making process can take unexpected directions according to the tools at hand. It would be simplistic to explain this group's difficulties as *only* a matter of learning to distinguish between variables and constants or to establish sound relations between variables. How the members of the group (the community) interact between them and with the "tools" available at the time mediates the outcome of the activity in fundamental ways, shaping what is learnt individually and collectively.

The rules and the division of labour

The rules of the activity (how members of the group interact in order to achieve the outcome of the task) can be explicit but often also implicit. This adds to the complexity of the activity, as members interpret these rules in different ways. For example, while some group members were adamant that they should not use external help to solve the problem, others seem to be more willing to use external resources to advance their solution. This also impacts on the tools available to the group:

Steve: Literally, I'm just going to go on Google and google this to see if there is an equation for it.

Hank: Half the fun though.

Mike: No, you can't do that.

It is also important how different group members perceive their peers and are perceived by the group. This has an effect on whose ideas are considered valuable or worthy of taking into account. For instance Tom, who had not been participating in the discussions until then, suggested the course's workbook might help find a solution to the task. But this suggestion was followed by hesitation:

Tom: But, I don't know, from these two pages I don't know where to go from there.

As a result, the group dismissed his comment and went back to work on a previous idea. Indeed, in the division of labour some declare themselves "out" from the beginning (e.g. Steve: 'I'm pretty useless to be honest (…) I told him to come down because this isn't my thing'), even though they might change their "status" as the activity evolves and they feel they can contribute to the task. Newcomers (e.g. lecturer, new team members) disturb the division of labour by introducing new ideas, which are taken in differently by group members that react (or not) to this new information.

Furthermore, new ideas (externally sourced or internally proposed) are of little use if these are not specific, that is, if there is not a "connection" between the new idea and the current group's understanding of the problem (i.e. does it make sense in relation to the group's thinking of the problem at the time?). As seen above in the case of Tom, his idea of searching the workbook for help was discarded because it was not clear to the group how it could advance the solution. His hesitation ("I don't know where to go from there") meant that his suggestion was not worth considering. Also, his status in the group (as not having participated fully until then) meant that his voice was less heard. Similarly, suggestions of external help (e.g. Google) are rejected because the rules of the group are that this kind of help constitutes "cheating" or does not help someone "learn" and that some of these sources are not always reliable (a belief expressed in another occasion), hence having a lesser status.

In a few occasions, ideas that could have steered the group's thinking in the right direction did not materialise because they were either communicated without confidence or not clear enough to connect with the group's current thinking. For example, around 30 minutes into the task, Hank identified that a key variable in the problem is the distance (x), but he was unable to identify the variable with respect to which it varies ($\frac{dx}{?}$). This hesitation meant that the idea was discarded and the group had gone back to a previous state where they tried to differentiate k (a constant!) with respect to the acceleration ($\frac{dk}{da}$), getting a zero after differentiating twice.

Even if a new idea comes from an authority (e.g. lecturer) it still has to connect to the current thinking of the group in order to be useful, or in other words, to become a "tool" for sense making. For example, a couple of minutes after the previous episode, the lecturer approached the table and gave the group the following information:

Lecturer: The acceleration, that's the derivative of the velocity, velocity is the derivative of the distance.

The current thinking of the group was that they should, somehow, differentiate an expression twice to get a second order differential equation. They did not realise that the first expression $kx = ma$ was already a second order ODE and that they just had to "spell out" the acceleration as a second derivative of the distance w.r.t. time: $\frac{d^2x}{dt^2}$. After this intervention by the lecturer, the group's struggles continued without approaching a satisfactory solution. In jest, Hank said: 'This is going to be a long hour, isn't it?'

Conclusions

Our research question was: How do social interactions in a small group collaborative work influence the students' mathematical sense making and the outcome of the activity? We have tried to answer this question from a CHAT perspective:

The composition of the community (with their members' individual histories of previous and present engagement with mathematics), the rules (explicit and implicit) and the division of labour (which influences whose ideas are valuable or not) shape in unique ways the social interactions that occur in a group activity. These interactions determine the tools that are available to the group, which in turn mediate the sense making process and influence the outcome of the activity.

In our particular community, there was no one who was mathematically confident enough to challenge the shared meanings constructed by the group, even though they all had studied second order ODEs before. While some research studies (Patterson & Watt, 2014; Goos et al., 2002) suggest the challenging of ideas as an important factor in the success of problem-solving collaborative group work (something like "playing the devil's advocate"), our data suggests that this is not always enough. In our group there was challenging but students were unable to respond to the challenge or express their peers' contributions in a way that could shift the collective meanings towards more productive thinking that could positively achieve the object of the activity. Even when potentially fruitful ideas were introduced into the activity, these were not presented or communicated in a way that could have been connected with the group's sense making process at the time, that is, the group could not transform the ideas into effective tools for the achievement of the outcome, and therefore were lost, side-tracked or discarded. Hence, the meanings produced by the group were mathematically incorrect and the object of the activity was not achieved.

What could have helped this group achieve a better outcome, and so what can we learn from this analysis? We believe the key factors in achieving a positive outcome reside in the quality of interactions that are produced during collaborative work, and hence, in the shared construction of effective tools that can contribute to a sense making process that results in a mathematically correct outcome. However, the skills necessary for these interactions to occur are not normally an explicit or even implicit part of school/university mathematics curricula or assessment. Competences such as communicative skills (e.g. active listening, reflection, effective speaking) or inter-personal skills (e.g. negotiation, assertiveness) are not normally associated with mathematics, a subject that remains largely individualistic. We believe therefore that the explicit teaching of these skills within more socially-oriented mathematical pedagogies can benefit students' mathematical learning, engagement and achievement.

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PROBLEM SOLVING WITH STRATEGY KEYS – A STUDY TO IDENTIFY USER TYPES

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In this study 10 students aged 7 to 10 have been videotaped and interviewed while solving mathematical problems and using so called strategy keys. In a way, these keys are prompts giving heuristics as hints. We investigated if and how the use of strategy keys influences problem solving processes. As a result, three user types could be identified: The use of strategy keys is (1) essential, (2) helpful, or (3) not necessary but never distracting or disturbing.

BACKGROUND

Theoretical Background

Problem solving is "[t]he process that lies at the heart of all mathematical activity" (Lester 2001, p. 570) and therefore, is one of the key competencies students should acquire during their school life (cf. NCTM 2000). Problem solving means "engaging in a task for which the solution method is not known in advance." (NCTM 2000, p. 52) Someren, Barnard and Sandberg (1994, p. 44) characterise problem solving "as a cognitive process that is goal directed and requires effort and concentration of attention". Moreover, they state that a "solution is not found directly in a single step but via intermediate reasoning steps […]" (p. 44). Our understanding of a problem corresponds with Schoenfeld, who defines a problem "as a task that is difficult for the individual who is trying to solve it" (1985, p.74). and, "if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem".

Concerning this, the inevitable question is how to become a successful problem solver? A lot of research deals with this topic and tries to identify necessary factors for students to become successful problem solvers. Two of the most important factors that contribute to this endeavour are heuristics and metacognition (cf. Schoenfeld 1985; 1992). However, most attempts to develop trainings for the use of either heuristics or metacognition show only moderate results. These trainings take a lot of time – which teachers are not willing to spend easily – and their results are mediocre, at best (cf. Schoenfeld 1992). The participants show only moderately better scores than members of control groups at specifically designed tests; and these training outcomes are rarely transferable to unrelated problems (cf. Schoenfeld 1992; Hembree 1992; Mevarech & Kramarski 1997).

Origin of this Project and Preliminary Work

After all the research on problem solving and heuristics, there is still no guaranteed way to solving a problem or to even becoming a good problem solver. This is the starting point of the strategy keys. Strategy keys were originally devised in a PhD

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project by Philipp (2013) and further developed during a case study in the project "Fit for Maths" (Herold, Barzel & Ehret, 2013). These keys are supposed to be another, more practical approach to helping students become better problem solvers. Thus, strategy keys are an alternative to explicit, time-consuming heuristic trainings with unknown effectiveness. They are used as prompts - similar to aid cards - that do not need previous training, a specific introduction, or special attention by the teacher (cf. Bannert 2009). Hence, students have access to them when solving mathematical problems. Additionally, they get hints as well as new stimuli that might cause an alternative perspective on the problem and the solving process. The project *Problem Solving with Strategy Keys*, which is reported in this article, investigates the impact, use, efficiency, and usefulness of strategy keys in the process of problem solving.

Research Questions

Based on the findings of the two former projects, experience shows that the strategy keys have a high potential to successfully influence students' problem solving processes and to indirectly teach general problem solving strategies (heuristics). These assumptions are now being investigated in this project. The main goal of this study is to identify user types when dealing with the strategy keys. The following questions will be discussed and analysed qualitatively using students' processes.

1) How do students work with the strategy keys?
2) How does the use of strategy keys influence the problem solving process?

DESIGN OF THE STUDY AND METHODOLOGY

Selection of the students

Due to practical and economical reasons we videotaped maths students at the age of 7 to 10 years (3rd and 4th graders). These students voluntarily attend an after school course called *Mathe für schlaue Füchse (Maths for Clever Foxes)* which takes place at the University of Duisburg-Essen regularly. In this course, students solve mathematical problems and learn about some historical aspects of mathematics. Taking this into account, we assume that these children are willing to learn mathematics, that they are highly motivated and are therefore suitable for our study. However, as an ability test is not carried out to enter the course, we cannot say anything about these students' actual mathematical knowledge or competencies.

In total ten children aged 7 to 10 years participated in the study (6 boys and 4 girls). In this article, we focus on three of them to show typical processes concerning the use of strategy keys.

Selection of the tasks (6 in total)

For this research, tasks were selected as being non-routine problems for students of grades 3 to 7. Additionally, those tasks ought to meet the following criteria:

The task should be open concerning the amount of possible ways to get to a solution, not necessarily the amount of different solutions. (cf. Schoenfeld 1985 & 2011)

To solve the task, the required prior knowledge should be as limited as possible. This way, solutions from grade 3 upwards are possible.

At least two of the selected strategies keys, i.e. heuristics, should possibly be helpful to solve the task (see Table 1).

This way, we selected six tasks of different mathematical areas. The two following arithmetical problems are the most frequently chosen ones and show interesting processes. This is the reason why both will be discussed in this article.

Task	*7 Gates:* A man picks apples. On his way to town, he has to pass seven gates. At each gate stands a guardian claiming half of the apples and one apple extra. At the end, the man has only one apple left. How many apples did he have at the beginning? (Bruder et al. 2005: 7)	*Farm:* On a farm is an open-air enclosure for chickens. In this enclosure also live rabbits. Jens stands by the fence and counts 20 animals with 70 legs in total. How many chickens are there? (Collet 2009)
Useful Strategies (Heuristics)	Trying systematically, Working forwards, Table, Equations, Looking for patterns, Working backwards, (Looking for analogies)	Trying systematically, Working forwards, Table, Equations

Table 1: Selected Tasks and useful heuristics

Task "7 Gates": 6 of 10 students tried to solve the problem "7 Gates". Two students (Alwin and Rik) encountered suitable strategies that could have led to the correct solution. As this task was particularly difficult for all the children, we can see especially interesting processes. However, as none of the students – even those using the strategy keys – solved the problem correctly, we assume this task to be better suited for older students.

Task "Farm": 9 of 10 students tried to solve the task "Farm", 6 of the 9 successfully. This task was identified as problem for 7-graders (cf. Collet 2009) and seems to be suitable even for primary school children. Nevertheless, the number of strategies (i.e. heuristics) to solve the task is limited in that age and will increase in higher grades.

Each child solved one to four problems. Hence, we videotaped and analysed 27 processes in total. This article focuses on six of those processes.

Selection of the strategy keys

The strategy keys were selected from the pool developed in "Fit for Maths" (cf. Herold et al. 2013) and were adapted for this study. Each key aims at a certain heuristic that is thought to be useful when solving problems. The keys have been worded in a way that is easily accessible and understandable for children without a background in heuristics or problem solving. For this reason, the keys need little interpretation and can be used without being explained beforehand. Some of the keys' formulations resemble heuristic questions by Pólya (2004) whereas other keys are not that clearly related to a heuristic from the literature (e.g. Pólya 2004; Bruder & Collet 2011).

The following eight keys are used in this study (the original German is in italics):

Draw a picture. *(Male ein Bild.)*
Make a table. *(Erstelle eine Tabelle.)*
Work from behind. *(Arbeite von hinten.)*
Find an example. *(Finde ein Beispiel.)*
Look for a rule. *(Suche nach einer Regel.)*
Read the task. *(Lies die Aufgabe.)*
Use different colours. *(Verwende verschiedene Farben.)*
Start with a small number. *(Beginne mit einer kleinen Zahl.)*

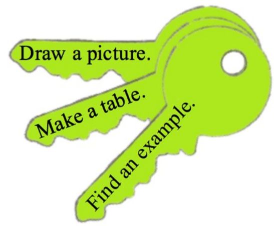

Figure 1: A bundle of strategy keys

The intention of providing eight different keys was to allow as many different prompts and therefore heuristics as possible. This way, children could choose from a pool of heuristics and apply their existing knowledge in a possibly new context.

Selection of the methodological instruments

In order to investigate the influence of the strategy keys on problem solving processes, students need to work with tasks and to actually use the keys. To gain as much information as possible about the students' thinking we decided to carry out *task-based interviews* (cf. Goldin 2000). Each participant of the study could choose his/her favourite task and was then videotaped while working on the selected tasks.

First, the interviewer introduced the strategy keys that stayed at the table at all times. Second, the student was encouraged to *think aloud* and to explain his/her approaches and actions during the problem solving process (cf. Maher et al. 2014). Whenever the interviewer had difficulties in understanding the student's thoughts, she asked and, in this way, interrupted the process with questions like: What are you thinking right now? How do you know? How did you do it? (cf. Philipp 2013). After finishing the process, the student was asked to reflect on the key usage and to explain his/her thoughts altogether when the process was still in mind (cf. Someren et al. 1994).

Analysis of the Data

All processes were coded using the framework by Schoenfeld (1985) as operationalised by Rott (2011). Following Schoenfeld, the processes were divided into episodes: namely Reading, Analysis, Exploration, Planning, Implementation, and Verification. Additionally and more importantly for this study, the use of strategy keys in the course of the processes has been coded.

RESULTS

Results concerning the Problem Solving Processes

In the following, six processes have been chosen for an analysis in this article because they illustrate typical dealings with the strategy keys.

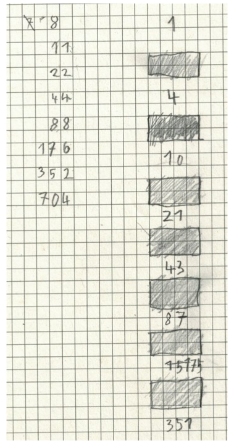

Figure 2: Rik's notation (7 Gates)

Alwin (7 yrs, 3rd grade, "7 Gates"): Alwin reads and analyses the problem. He explains how to pass the first (i.e. the last) gate (times 2 plus 1) – he works backwards. He then writes down the amount of apples when passing the last gate. At the end, his result is 255 apples which is one of the regular answers (but not the correct one). For his whole process, including the explanation to the interviewer, he only needs 3:30 min and does not use any strategy keys.

Rik (9 yrs, 4th grade, "7 Gates"): Rik reads the task, thinks about 7 times half apples and about the amount of apples at the beginning. He comes to the conclusion that it must be 8 apples. However, he is not sure and rereads the problem. He continues using different starting numbers and assumes that dividing a number by 2 (e.g. 70) means that from that point on every guardian would get 35 apples. Now, he doubts the solvability of the task. At that point, the interviewer offers the strategy keys (4 min have passed) and Rik chooses "Work from behind". He now starts with the example 11 and passes the 7 gates beginning at the last one. Rik explains that this is a random example, but goes back to his original idea and tries to find starting numbers again. After 9 minutes he looks at the keys again, does not name one but starts drawing the gates as blocks. We assume he used the "Draw a picture" key. When he finished his drawing, he starts at the "last" gate with one apple and adds one and doubles it. At the 4th gate, he changes his pattern. Now, he doubles first and then adds another apple. His final result is 351 apples. Eventually, he verifies and explains his result without finding his mistake. His process takes 16:30 min.

Carolin (8 yrs, 4th grade, "7 Gates"): Carolin reads the problem twice and then analyses it. As there are 7 gates, she assumes 8 apples as a start. Then she tries 100 as a start and recognises that this is not the correct way. She re-reads the task and asks whether or not half apples would be allowed. After 3 min without a sustainable approach, she chooses the strategy key "Start with a small number". Shortly after, she writes down her first example and increases the amount of apples systematically. After 8 minutes Carolin chooses the key "Draw a picture". This gives her the idea to draw 7 gates with the guardians in them. She keeps on trying different numbers and keys (Work from behind, Look for a rule) until she goes back to the task. Just now, she understands that she has one apple left and starts at the last gate going backwards. Her final result is 136 apples: double at each gate and 8 more for each gate. She used four keys and said that each of it gave her a new, helping idea. It took her 13 minutes to solve the task in her way.

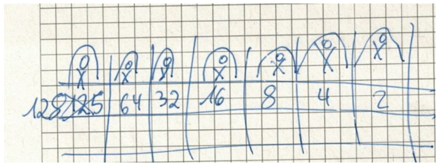

Figure 3: Carolin's notation (7 Gates)

Alwin (7 yrs, 3rd grade, "Farm"): Alwin reads the problem. He then analyses it saying things like "20 animals with 70 legs" or "rabbits have 4 legs, chicken 2". After a short while, he suddenly comes up with the correct solution. Again, he only needs 3 minutes for the entire process and does not use any strategy keys.

Rik (9 yrs, 4th grade, "Farm"): Rik reads the problem and analyses it thinking about the number of solutions, the number of legs and the relation between the number of legs and of animals. He then tries to divide 70 legs by 4 but rejects this idea. Shortly after, he decides on 15 chickens and finds more examples. After 4 minutes he rereads the task and explores more examples. After 5:30 min the interviewer offers the keys and Rik chooses "Make a table". Now, he plans to create a table and draws it. Using this table, he systematically varies the number of animals to reach the correct number of legs. After using the key he only needs 3 minutes to successfully solve the problem. In total, his process took 9 minutes, probably accelerated by the strategy key.

Carolin (8 yrs, 4th grade, "Farm"): Carolin reads and analyses the task. She thinks about necessary information to answer the task, about the number of legs and even about the formulation of the task. Then, she explores the problem dividing 70 by the number of legs and animals and gets stuck. After 7 minutes she decides to use a key and chooses "Make a table" and "Work from behind". Now, she creates a table and explains it. Suddenly, she comes up with the idea of 10 rabbits and 10 chickens. Shortly after, she systematically tries to add a rabbit and to subtract a chicken. She understands that adding a rabbit and subtracting a chicken means to add two more legs. Five minutes after choosing both keys she found the correct solution. In the interview she said that the table helped her to try systematically and the key "work backwards" gave her the idea to use the amount of animals first and then the amount of legs. In total, her solving process including the short interview took 22 minutes.

Analysing all 27 processes, we contrasted different examples and summarised similar ones (cf. Kelle & Kluge 2010). Eventually, we identified three different types of strategy key users which are illustrated by the described processes above.

- Strategy key usage is *essential*. (e.g. Carolin – Farm, Rik – 7 Gates): This user type is not able to solve the task without the use of the strategy key(s); the keys significantly improve the process' progress.
- Strategy key usage is *helpful*. (e.g. Carolin – 7 Gates, Rik – Farm): This user type manages to accelerate the solving process by using the keys.
- Strategy key usage is *not necessary*. (e.g. Alwin – 7 Gates, Farm): This user type encounters barriers but has heuristics available to overcome those barriers without the help of strategy keys.

So far, we did not encounter a fourth type in which the keys distracted a student, worsened or hindered the solving process in any way. But, of course, these types only include students who actually understood the task and could somehow solve it. The students who do not understand the problem, would not find help in the keys either.

Another finding was the positive reaction of the students towards the keys. It seems as if they liked using the keys. We think this is due to the simple and plausible design.

DISCUSSION AND CONCLUSIONS

The use of heuristics is an integral part of (successful) problem solving. However, heuristic trainings are often time consuming and/or of limited success. Thus, we developed an alternative approach to introduce students to the use of heuristics without prior training: heuristic aid cards in the form of strategy keys. Our results verified the expected potential of these keys. Answering the first research question, we observed a very intuitive and natural dealing with the strategy keys. In answering our second research question, we identified three types of strategy key users, assuming that a student understands the task. Sometimes, the keys were not needed at all, but in most cases, they helped the students to better approach a problem without ever distracting or disturbing them. We believe the keys to also be useful in higher grades and with other tasks. They could build up a kind of "tool box" helping use heuristics flexibly.

Methodologically speaking, students worked individually. However, in the classroom it is much more realistic for students to solve problems interactively. This is why data will be collected from pairs, small groups and whole classrooms in future research.

As some of the young students had difficulties in using the prompts and in effectively including them into their problem solving process, and as our user types might not yet be complete, we will test the keys as well as the tasks again with older students, i.e. grades 5 to 7.

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CHILDREN'S PERFORMANCE IN ESTIMATING THE MEASUREMENTS OF DAILY OBJECTS

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This study examined children's ability to estimate measurements of large-size daily life objects. The data were collected using estimation tasks and interviews from 81 children, each in a fourth- (n = 21), fifth- (n = 32), or sixth-grade (n = 28) class at local elementary schools in cities in Northern Taiwan. There were significant differences in the children's estimation performance among grade levels. The relationships between the levels of children's estimation performance and strategy use were significant in the instances of area and length estimations. In addition, no relationships were observed between the levels of estimation performance and children's perspectives on the support of the provided measuring units for the various estimations.

INTRODUCTION

Understanding how to estimate measurements accurately by using effective strategies is a crucial competency for solving problems requiring the estimation of quantities in school mathematics and daily life (Joram, Subrahmanyam, & Gelman, 1998). Previous studies have suggested that several factors may influence estimators' performance in estimating measurements, such as the size of the to-be-estimated (TBE) objects, types of measurements being estimated (e.g., length or area or volume) and measuring units (MUs) (Chan, 2001; Forrester, Latham, & Shire, 1990) and grade level (Huang, 2014; Montague & Van Garderen, 2003). However, how grade level affects children's ability to estimate the measurements of various objects, specifically those with a large area or length, in daily life remains unclear.

Previous studies have discussed strategies used in estimating measurements (Chan, 2001; Forrester et al., 1988), including how estimation accuracy relates to strategy use (Siegel, Goldsmith, & Madson, 1982). Montague and Van Garderen (2003) suggested that children with varying levels of mathematical achievement exhibit different levels of performance in using strategies for estimating. However, insufficient research has examined the relationship between children's estimation performance and strategy use. Chan (2001) reported that children tended to change their initial estimation strategies to using an MU after being provided the MUs. However, few studies have addressed the perspectives of children regarding the support or non-support of the provided MUs in estimating and how estimation performance levels are related to children's perspectives of provided MUs.

The current study explored Grade 4-6 children's estimation ability and strategies used for estimating, as well as their perspectives on the support (or non-support) of the provided MUs. Particularly, the study focused on children's performance in solving problems involving estimating the lengths, areas, and volumes of daily objects. This

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study addressed three questions:

What are the differences among grade levels in the ability of children to solve estimation problems?

What is the relationship between strategy use and estimation performance levels in the problems involving volume, area, and length estimations?

What is the relationship between children's perspectives on the support (or non-support) of provided MUs and estimation performance levels in problems involving volume, area, and length estimations?

THEORETICAL FRAMEWORK

Mathematical Knowledge and Thinking Involved in Measurement Estimation

Measurement estimation involves the mental process of providing a gross estimate for a measurement problem without using measuring instruments (Joram et al., 1998). Estimating measurements requires knowledge regarding measurements and the ability to use appropriate strategies to obtain a close estimate (Siegel et al, 1982). In addition, measurement experiences in everyday application situations, such as activities in which students engage in estimating various measurements of daily objects, have been recommended as a crucial component of measurement education (Huang, in press; Taiwan Ministry of Education [TME], 2010).

Length, area, and volume measurements, which involve concepts of spatial structuring, require similar cognitive characteristics—the unit-covering principle and unit iteration (Huang, in press). An approach that involves estimating the length (or distance) of an object, namely, applying the unit-covering principle and mentally repeating units to estimate a one-dimensional measure, is effective for estimating an object in two dimensions (area) or three dimensions (volume) (Forrester et al., 1990).

Relationships Among Grade, Size of TBE Objects, and Estimation Performance

Forrester, Latham, and Shire (1990) and Joram, et al. (1998) have suggested that students' estimation abilities increase as the students advance to higher grade levels because the additional knowledge in real measurement and experiences in estimating measurements obtained through instructional activities facilitate the development of estimation abilities. By contrast, other studies (e.g., Huang, 2014; Montague & Van Garderen, 2003) have indicated that students in higher grade levels (e.g., Grades 6 or 8) did not necessarily perform better at estimating measures than students who were in a lower grade level (e.g., Grades 4 or 5). Although grade level has been considered a factor that may affect students' estimation performance (Forrester et al., 1990; Joram et al., 1998), whether grade level consistently plays a crucial role in the estimation ability of children who are at Grades 5 and 6 requires further investigations.

In addition to grade level, children's estimation performance depends on the levels of problem difficulty (Siegel et al., 1982). For example, estimating area measurements seems more challenging than estimating length measurements (Huang, 2014). Forrester et al. (1998) observed that, in area and volume estimation tasks, the size of

TBE objects influenced children's estimation performance; however, the size of TBE objects did not seem to affect children's performance in estimating length.

Estimation Strategy Use and Its Relationships with Estimation Performance

Previous studies (e.g., Chan, 2001; Crites, 1992; Forrester et al., 1990) have classified the strategies that children frequently use for estimating measurements into eight categories: (a) Looking, (b) Benchmark comparisons, (c) Standard units, (d) Mental rulers, (e) Prior knowledge (or experiences), (f) Guessing, (g) Multiple strategies, (h) Ambiguous answer. Looking involves using only the naked eye (perception). The benchmark comparison strategy entails employing objects that are readily available as references, such as objects in a classroom or body parts. Standard units is the successive mental application of standard units with unit iteration; mental rulers is successive mental application of a mental image of a nonstandard unit with unit iteration (e.g., using a mental image of 5 cm or 10 cm). Prior knowledge (or experience) involves employing knowledge or information gained by the estimators in previous experiences with using measurements (e.g., by estimating linear dimensions and then using volume (or area) formulas). A guessing estimate represents a gross estimate (Joram, et al., 1998). Guessing without thinking properly involves a conjecture. Using multiple strategies involves two or more strategies. For example, an estimator employs multiple strategies may combine the use of body parts with standard units. An ambiguous answer is a response with vague descriptions of a strategy.

Regarding the relationship between children's strategy use and their levels of estimation ability, Crites (1992) reported that skilled estimators were inclined to use "high-order strategies" such as a benchmark or multiple benchmarks, whereas less skilled estimators were more likely to use guessing or not provide an estimate (p. 609).

Children's Perspectives on the Support of the Provided MUs for Estimating

Bright (1979) indicated that estimating measurements is more difficult when the TBE object or MU is absent because absence is likely to lead to more mental operations such as imaging the length of a centimetre. In addition, Chan (2001) reported that children changed their estimation strategies after being provided MUs for estimating measurements. For example, the students initially used their hands, but used an MU (i.e., 1 m^2) for estimating the area of a blackboard (5.18 m^2) after it was provided.

The provided MU seemed to improve children's performance in area estimations in Chan's (2001) study; however, information about the perspectives of children regarding the support of the provided MUs in estimating various measurements and how these perspectives relate to the level of estimation performance remains unclear.

METHODOLOGY

Participants and Instruments

The sample consisted of 81 children (43 boys and 38 girls), each in a fourth- (n = 21), fifth- (n = 32), or sixth-grade (n = 28) class at local elementary schools in cities in Northern Taiwan. A multiple-choice and a filling-in versions of an eight-item estimation task were developed, respectively. The TBE objects involved in the eight estimation questions in Task A (the multiple-choice version) and Task B (the filling-in version) were identical, including two length questions, three area questions, and three volume questions. Specifically, an example of an item from the multiple-choice version was "About how many cubic centimetres is this Tetra-Pak box? (①226-276 cm^3; ②276-314 cm^3; ③188-226 cm^3);" an example of an item from the filling-in version was "About how many cubic centimetres is this Tetra-Pak box? ____ cm^3." Most of the TBE objects described in the problems were visually presented to the participants by using real objects. These TBE objects were familiar to the children. The difference between the two tasks was that the MUs in Task A were described but were physically absent. By contrast, the MUs in Task B were described and physically presented. Task B was implemented after Task A was completed by the participants.

To understand children's use of estimation strategies, in Task A, an item requiring a short written description of estimation methods used for estimating was provided with each estimation question. Similarly, to understand children's perspectives on the provided MUs, in Task B, an item requiring a short description of the support or non-support of the provided MUs were provided with each estimation problem. Moreover, one-on-one interviews were conducted to collect information about children's estimation strategies and their perspectives on the provided MUs.

In this study, children's strategies and perspectives on the provided MUs were focused on the data obtained through the interviews and the participants' responses to the three estimation questions. The three estimation questions involved estimating the volume of a Tetra-Pak box (250 cm^3), and area of a plastic bag (1.08 m^2), and length of a rope (5.6 m), with the MUs provided for estimating the volume, area, and length measurements being a 1-cm^3 cube, 1-m^2 plastic sheet, and 1-m rope, respectively.

Scoring, Estimation Strategies, and Perceptions of the Support of the Given MUs

In the study, an "accurate" estimate was defined as being within ± 10% of the actual value, as described by Coburn (1987), and was scored 2 points. An "acceptable" estimate was defined as being between +10% and +25% or between -10% and -25% of the actual value and was scored 1 point. If an estimate was greater than +25% or lower than -25% of the actual value, then a score of "0" was allocated. The total possible score in Task A was 16 points and so was Task B.

For classifying the children's estimation strategies, the eight strategies that children frequently used for estimating measurements (see Theoretical Framework section) were adopted. Moreover, according to Chan (2001), children were able to express their viewpoints on "support" or "non-support" of the provided MUs. In the study, the support category included three subcategories: (a) "facilitating improvement of estimation accuracy," (b) "facilitating recognition of the MUs," and (c) "serving as references for making comparisons." The non-support category included three subcategories: (i) "using prior experiences" (i.e., estimators believe that their prior experiences are adequate for estimation), (ii) "inconvenient for making direct comparisons" (i.e., the MUs do not facilitate making direct comparisons because these MUs are too large or small), (iii) "irrelevant responses" (i.e., responses that are unrelated to support or non-support, such as "I use a grid" or "I draw").

To compare the estimation performance of the children in various grade levels, the total average scores in Tasks A and B were calculated and served as the scores of the overall estimation performance. In addition, the numbers and types of strategies that the children in various grades used for estimating were coded. The accumulative frequency for each strategy used was calculated. To examine whether differences existed in strategy use between the good and poor estimators, children who scored in the upper 25% of the score distribution on the estimation tasks were classified as "good" estimators, whereas children who scored in the bottom 25% of the score distribution on the estimation tasks were defined as "poor" estimators in the study.

The written answers for the estimation questions of 32 children were independently scored by two raters. The inter-rater agreement on the scores of the estimation questions was 100%. In addition, Kappa analyses were administered to test the reliability of the coding of 28 children's estimation strategies and their perspectives of the support of the provided MUs. The results of the Kappa analysis of the coding of the estimation strategies was assessed at .72, $p < .01$, whereas the Kappa analysis of coding of children's perspectives on the support of the provided MUs was .81, $p < .01$.

RESULTS

Estimation performance. The means of the total averaged scores and standard deviations of the children's performance in measurement estimation for the fourth, fifth, and sixth grades were 7.86 (1.63), 9.44 (2.12), and 9.59 (1.99), respectively. A one-way ANOVA was conducted to examine whether grade levels affected children's estimation performance. The results indicated significant differences in the children's estimation performance among the three grade levels, $F\ (2, 78) = 5.59$, $p < .01$, $\eta^2 = .13$. Schéffe post-hoc tests, used to analyse the differences in grade levels, indicated that both the six-grade group and fifth-grade groups outperformed the fourth-grade group. No differences were observed between the fifth-grade and sixth-grade groups.

Nineteen children who scored in the upper 25% of the score distribution in the estimation tasks, (i.e., 2 fourth-graders, 10 fifth-graders, and 7 sixth-graders), were categorised as good estimators ($M = 11.95$, $SD = .85$). Seventeen children who scored in the bottom 25% of the distribution, (i.e., 7 fourth-graders, 6 fifth-graders, and 4 sixth-graders), were classified as poor estimators ($M = 6.47$, $SD = .67$).

Table 1 shows the frequency at which each strategy was used for estimating the volume, area, and length by the good and poor groups. The strategies were classified into eight categories. The good estimators were more likely to use multiple strategies, which entails combining two or three types of strategies. Specifically, they combined looking with mental rulers, or use looking with standard units and previous knowledge (e.g., the previous experience in measuring the length of a meter). By contrast, few of the poor estimators adopted multiple strategies. They preferred combining looking with benchmark comparisons (e.g., body parts or known objects as references).

Moreover, the children frequently estimated the lengths of the three- (or two-) dimensional TBE objects, and then used volume (or area) formulas to calculated the measurements while estimating volume (or area) measurements based on the interview data, though the formulas were not written in the answers for solving the questions.

Good and poor Estimators	Strategy used															
	Looking		Benchmark Comparison		Standard Units		Metal Rulers		Prior Knowledge		Guessing		Ambiguous answer		Multiple Strategies	
	f	%	f	%	f	%	f	%	f	%	f	%	f	%	f	%
Volume estimation																
Poor estimators (n = 17)	3	17.6%	7	41.1%	1	5.9%	1	5.9%	1	5.9%		-	2	11.8%	2	11.8%
Good estimators (n = 19)		-	8	42.0%		-		-	4	21.1%	1	5.3%		-	6	31.6%
Area estimation																
Poor estimators (n = 17)	2	11.8%	9	52.8%		-	1	5.9%	2	11.8%	2	11.8%	1	5.9%		-
Good estimators (n = 19)	1	5.3%	2	10.5%	2	10.5%	1	5.3%	2	10.5%	1	5.3%	1	5.3%	9	47.3%
Length estimation																
Poor estimators (n = 17)	2	11.8%	8	47.0%	1	5.9%		-		-	2	11.8%	3	17.6%	1	5.9%
Good estimators (n = 19)	1	5.3%	5	26.3%		-	9	47.3%		-	1	5.3%		-	3	15.8%

Table 1. The frequency and percentage of strategy use by groups of estimators

Chi-square tests were used to examine the relationship between estimation strategy use and estimation performance levels in various cases. The results showed that significant relationships between strategy use and performance levels existed in the area estimation case, $\chi^2(7, N = 36) = 16.06, p < .05$, and the length estimation case, $\chi^2(5, N = 36) = 11.68, p < .05$. No relationships between strategy use and performance levels were observed in the volume estimation case, $\chi^2(7, N = 36) = 11.79, p = .11$.

Perspectives of groups regarding the provided MUs. The children in the good group (70.6%) and poor group (60.1%) were prone to express that the provided MUs aided them in their estimation tasks. The children in the two groups reported support perspectives pertaining to all three subcategories in the volume and length cases: (a) “facilitating improvement of estimating accuracy,” (b) “facilitating recognition of the measuring units,” and (c) “serving as references for making direct comparisons.” In the area estimation case, only Subcategories (a) and (c) were reported by the two groups. Generally, the good group was likely to report Subcategory (c), whereas the poor group tended to report Subcategory (a).The children in the two groups reported non-support perspectives pertaining to all the three subcategories in both the volume and length estimation cases: (i) “using prior experiences” (e.g., “I know the length of my opened arms, and it is adequate for estimating objects”) and (ii) “inconvenient for making direct comparisons” and (iii) “irrelevant responses.” In the area estimation case, the two groups reported only Subcategories (ii) and (iii). Generally, more children in the poor group than in the good group reported Subcategory (iii), “irrelevant responses.”

Regarding the relationship between perspectives on provided MUs and estimation performance levels, the results of Chi-square tests revealed that no relationships were found between the perspectives and estimation performance levels in the volume estimation case, $\chi^2(5, N = 36) = 3.40, p = .64$, in the area estimation case, $\chi^2(4, N = 36) = 6.82$, $p = .15$, or in the length estimation case, $\chi^2(5, N = 36) = 6.21, p = .29$.

DISCUSSION AND IMPLICATIONS FOR MATHEMATICS EDUCATION

The results of this study showed that grade levels were related to differences in the measurement estimation ability of the children. Overall, the fifth- and six- graders had higher competence in estimating measurements of daily objects than did the fourth-graders. However, the fifth- and sixth-grade groups performed equally well in estimating measurements of various daily objects. The results partially supported those of Siegel et al. (1982) that the sixth-graders provided more accurate estimates than did the second to fifth-graders. The partial results that were inconsistent with those of Siegel's study may have resulted from differences in problem contexts (e.g., TBE objects and MUs used) and the curricula and instruction of school mathematics (Huang, 2014; Forrester et al., 1990). Although the fifth- and sixth-graders exhibited comparable performance levels when solving the estimation tasks, they successfully completed approximately 60% of the estimation problems. The poor group completed only approximately 40% of the estimation problems successfully based on the mean of the total averaged scores of their estimation performance. The results imply that more instruction and practices for enhancing students' estimation abilities is required.

The evidence indicates that benchmark comparison was the most frequently used strategy. This may be because the curriculum and instruction of school mathematics suggest using body parts as benchmarks (Huang, 2014; TME, 2010). In addition, the findings that good estimators tended to adopt multiple strategies and mental rulers more frequently than the poor estimators seem to support Crites' (1992) perspective that estimation ability is positively correlated with the use of sophisticated strategies. Additionally, compared with the TBE objects in the area and length cases (i.e., the 1.08 m^2 plastic bag and the 5.6 m rope), the children were more familiar with the TBE object in the volume case (i.e., the 250-cm^3 Tetra-Pak box), according to the interviews. The lack of a difference in strategies used for estimating the volume measurement between the two groups may have resulted from the children being more familiar with the TBE object. In future studies, researchers should examine how levels of familiarity with TBE objects affect children's estimation performance.

Finally, the children in the good and poor groups were inclined to report that the provided MUs assisted them in estimating various measurements. This result supports the assertion of Bright (1979) and Joram et al. (1998) that providing MUs may facilitate reducing cognitive challenges in estimating measurements, such as recalling mental images of a unit of measure. Although no relationships were observed between children's estimation performance levels and their perspectives of the support or non-support of the provided MUs in various estimation cases, the poor group was more likely to report irrelevant responses than the good group. Students' development of estimation skills relies upon teacher's instruction (Huang, in press). Providing sufficient experiences of actual measurement and guiding students to obtain reasonable estimation answers by means of effective estimation strategies is crucial for fostering children's ability to estimate measurements.

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MEASURING TEACHER AWARENESS OF CHILDREN'S UNDERSTANDING OF EQUIVALENCE

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Children's understanding of the equals sign has been widely studied and identified as an important issue for thinking flexibly about arithmetic and learning algebra. Experiences in primary mathematics lessons impact significantly on understanding, but relatively few studies have investigated primary teachers' awareness of children's understanding of equivalence. One reason may be that while instruments that have been carefully validated exist for measuring children's understanding of the equals sign, no such instrument is available for evaluating teacher awareness. We analyse the performance of a questionnaire administered to 197 primary teachers in New Zealand and the United Kingdom and identify how individual items are likely to elicit different teacher responses.

INTRODUCTION

There has been an increased focus on the teaching and learning of algebraic reasoning in recent years (Kaput, 2008; Knuth, Stephens, McNeil, & Alibali, 2006). An outcome of this focus is a growing consensus between researchers and educators that algebra can be introduced at a much younger age with a focus on the integration of teaching and learning arithmetic and algebra in classrooms (e.g., Department for Education and Employment, 1999; Ministry of Education, 2007). Essential to effective teaching and learning is developing understanding of the equals sign as a representation of an equivalence relationship, and using this understanding to work flexibly with numbers and expressions.

Previous studies (e.g., Freiman & Lee, 2004; Knuth et al., 2006) have identified three types of student responses to problems involving equivalence. These responses reflect an *operational* view in which the equals sign is an indicator for a numerical result, a *sameness* view in which the equals sign indicates the same value is on each side, and a *relational* view in which arithmetic and algebraic relationships are exploited to reduce computational burden when establishing equivalence. For example, consider the problem 9 + 6 = __ + 5. Students adhering to an operational view may put 15 in the blank space and those adhering to a sameness view may put 10. Those adhering to a relational view, which subsumes sameness, may put 10 by noticing that the solution must be one more than nine and avoiding further computation.

Previous studies (e.g., McNeil & Alibali, 2005) have shown that the operational view is dominant and resistant to change and this leads to inflexible thinking about equivalence, arithmetic, and algebra (Knuth et al., 2006). Helping children overcome or avoid operational thinking requires careful planning and teaching of mathematics

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lessons (e.g. Li, Ding, Capraro & Capraro, 2008). However, while student misconceptions have been widely studied and findings replicated across different contexts, there is less research available on primary teachers' knowledge of typical student misconceptions regarding the equals sign. It is therefore not well understood how aware primary teachers are that children view the equals sign in varied ways, or the impact that teacher awareness might have on student conceptions. One reason this important area is under-researched may be that whereas carefully validated instruments exist for assessing student knowledge (Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011), there is no standardised way of evaluating teacher awareness of student knowledge. In this paper we analyse the performance of items designed to evaluate teacher awareness of their students' conceptions of equivalence.

METHODOLOGY

Questionnaire design

As a starting point to developing an instrument to analyse teacher understanding of student conceptions of equivalence, we turned to studies by Zhang and M. Stephens (2013) and A. Stephens (2006). Zhang and M. Stephens (2013) analysed teachers' responses to students' solution strategies to two missing-number equations, similar to item three in Table One. The authors reported that the teachers responded in a variety of ways, suggesting the two items were appropriate for eliciting teacher knowledge. However, the results are likely to be partly dependent on the particular tasks presented to teachers, as has been reported to be the case for children (Rittle-Johnson et al., 2011). Similarly, A. Stephens (2006) investigated teacher trainees' responses to five equivalence items, one of which was a definitional item while four were equation items. A. Stephens (2006) reported that the number of trainee teachers providing relational responses varied across the five items. Two of the items (similar to items four and seven in Table One) elicited relational responses from 80% of participants, a further two items (similar to item one and item two in Table 1) elicited relational responses from about two thirds of the participants, and one item (item eight in Table 1) elicited relational responses from about half of the participants. The study reported here seeks to build on this work by identifying the performance of different items, and the range of responses they elicit, for the case of practising primary teachers.

We adapted the items used by Zhang and M. Stephens (2013) and A. Stephens (2006) to construct an instrument comprising of eight items in total. This included one definitional item (item one) which asked participants to "list possible student responses to the question 'what does the equals sign (=) mean?'" A further seven items involved equations (see Table 1) and were adapted from the examples discussed in the above paragraph. For these seven items, the participants were asked to identify "what answers would you expect students to give and what strategies might they have used to get those answers?"

Item	Prompt	Item	Prompt
2	15 + 8 = _ + 10	5	8 + *f* = 8 + 6 + 4
3	24 - _ = 21 - 15	6	8 + 2 + *h* = 10 + 6
4	37 + 15 = 52 is true.	7	99 + 87 = 98 + 86 + *p*
	Is 37 + 15 + 8 = 52 + 8 true or false?	8	The solution to 2*n* + 15 = 31 is 8. What is the solution to 2*n* + 15 - 9 = 31 - 9?

Table 1: The equation items administered to teachers.

A key difference between our approach and that used by Zhang and M. Stephens (2013) and A. Stephens (2006) is that we did not provide participants with sample student solutions. This was done to avoid prompting specific responses. Our instrument was therefore expected to produce relatively fewer relational responses overall than that used by A. Stephens (2006). Moreover, based on the findings of A. Stephens (2006) we expected participants to provide fewer relational responses for item eight than for the other items. We also expected participants would provide fewer relational responses for item one than for the other items. Although A. Stephens (2006) found two thirds of trainee teachers provided relational responses to a similar definitional item, studies with children have found that most do not respond with relational definitions or examples of equivalence. For items two to seven, the study was exploratory and no firm predictions about their performance were made.

Participants

Participants in the study consisted of 197 primary teachers. Forty-nine of the teachers were in the United Kingdom and all of these teachers completed the questionnaire online. One hundred and forty-eight of the teachers were in New Zealand of which ten completed the questionnaire online and the remaining 138 completed a paper version of the questionnaire.

Code	Solution and explanation	
Operational	23	Added the 2 numbers without considering the 10 on the RHS.
Sameness	13	Understood the concept of balancing the equation.
Relational	13	10 is 2 more than 8 so you need to take 2 away from 15.

Table 2: Examples of coding of responses for Item 2 (15 + 8 = _ + 10).

Coding procedure

Responses to all items were coded as operational, sameness and/or relational. Responses coded operational were those that referred to student misconceptions related to the equals sign. Responses coded sameness referred to students understanding the need to maintain equivalence on either side of the equals sign. Responses coded relational referred to students drawing on arithmetic and algebraic relationships to maintain equivalence. Examples of codes for item two are shown in Table 2.

The codes were not exclusive and a given response by a participant could reflect none, one, two or all three of the codes. Therefore for each item there were three binary codes producing a total of 24 responses per participant (eight questions × three codes). The coding was undertaken independently by two researchers and the initial inter-rater agreement was 78.7%. Following the initial coding, meetings were held in which disagreements were resolved.

Analysis

To explore the performance of the eight items for measuring teachers' awareness of student understanding of the equals sign, we undertook two procedures. First, we assessed the internal consistency of the codes by calculating Cronbach's alpha for each (operational, sameness, relational). Typically, values of Cronbach's alpha > .7 are considered to reflect acceptable internal consistency.

Second, we subjected the codes across the eight questions to Rasch modelling (Bond & Fox, 2007) as is common for investigating the performance of an instrument. For the case of traditional test data, the Rasch model aligns item 'difficulties' and participant 'abilities' onto a single scale, as described below. For the case of the data presented here, which arises from researcher codes rather than traditional test scores, we refer instead to conception 'difficulties' for each item and participant 'awareness'. Rasch modelling was undertaken using the eRm package for *R* statistical software (Hatzinger & Mair 2007). To interpret the outcomes of the Rasch modelling procedure we used the outcomes to construct a Wright Map.

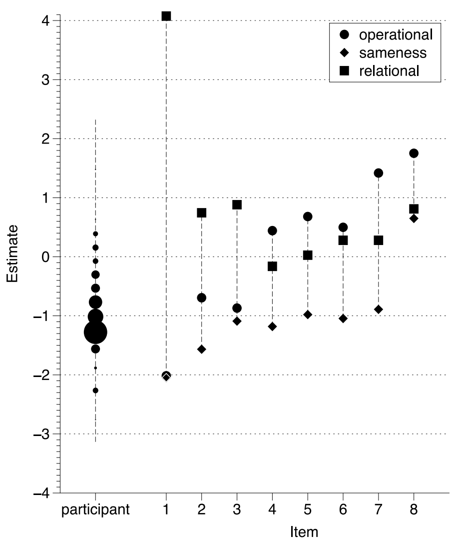

Figure 1: Wright Map of the outcomes of the Rasch modelling process. Participant 'awareness' estimates are shown on the left where the size of the dot indicates the number of participants at each level. 'Difficulty' estimates of the three conceptions are shown for items one to eight.

RESULTS

Internal consistency. Cronbach's alpha for the operational, sameness, and relational coding was .79, .68 and .72 respectively. These figures suggest acceptable internal consistency and overall performance of the instrument, although the internal consistency of the sameness coding was marginal.

Rasch modelling. The outcome of the modelling procedure is the Wright Map shown in Figure One. The Wright Map displays conception difficulties and participant awareness as *z*-scores for each item on a single scale, enabling direct comparison. The Rasch model frames outcomes in terms of the probability that a given participant will provide a given response to each item. This enables the analyst to make inferences between a participant's estimated awareness and their likely responses to items (Andrich, 1988). Accordingly, our discussion of the results is framed in terms of the probabilities of teachers providing responses coded as operational, sameness, and relational for each item. Where a conception difficulty is lower/higher on the Wright Map than a given participant's awareness this indicates that the participant had a more/less than 50% chance of being coded as aware of that conception for that item. For example, for item two most participants were likely to provide a response coded as sameness, unlikely to provide a response coded as operational, and very unlikely to provide a response coded as relational.

All item responses were more likely to be coded as sameness than operational or relational. The definitional item and open number sentence responses were more likely to be coded as sameness than relational. Conversely, for the true/false arithmetic item and algebraic item, responses were more likely to be coded as relational than sameness. Most participants (indicated by the largest dots on the participant scale) were likely to provide responses coded as sameness with a probability of about 50% for all the items except the definitional item one and one open number sentence item two. Similarly, most participants were unlikely to provide responses coded as operational or relational for all the items, bar the definitional item for which most participants were likely to provide a response coded as operational.

Scrutiny of the Wright Map suggested two items are problematic for eliciting teacher awareness of children's understanding of equivalence. For the definitional item one, almost all participants were very likely to be coded as sameness and operational, and all participants were very unlikely to be coded as relational (in practice only two out of 197 participants provided relational responses for this item). For the final item eight, almost all the participants were unlikely to be coded for any of the three conceptions. In fact 56.3% of participants provided no response that could be coded as any of the three conceptions for the final item.

DISCUSSION AND IMPLICATIONS

There have been limited studies which have addressed teacher understanding of students' mathematical thinking related to the equals sign with the exception of the studies by Zhang and M. Stephens (2013) and A. Stephens (2006). Our results suggest that while teacher awareness can be measured, there is a need to include a variety of items.

Overall we found that items two to seven performed well, suggesting that these are the most appropriate of the items investigated to explore teacher awareness of children's understanding of equivalence. To investigate whether this may be the case, we re-estimated the internal consistency of the coded data with the two poorly performing items omitted. Cronbach's alpha was found to be higher for all three conceptions when items one and eight were omitted. For the operational coding it was .81 (up from .79), for sameness it was .73 (up from .68), and for relational it was .74 (up from .72), suggesting improved internal consistency.

Equation-based items may be preferable to definitional items when examining teacher awareness. This is contrary to the finding reported by A. Stephens (2006). However, in a study investigating children's understanding by Rittle-Johnson et al. (2011), an analogous definitional item was found to perform well for eliciting sameness and operational conceptions but was less likely to elicit relational understanding than equation-based items. In our study the item did not to translate to eliciting teacher awareness due to flooring effects for the relational conception, and ceiling effects for the operational and sameness conceptions.

Similarly, the results of our study indicated poor performance of item eight, which assumes understanding that "solution" refers to the value represented by a letter. This was unsurprising as A. Stephens (2006) also found that few participants provided relational responses to item eight. The poor response rate to this question may be due in part to primary teachers viewing this type of problem as unrelated to the level in which they teach. For example, one participant wrote simply "*The Year Sixes I work with would struggle with this.*"

For items two to seven, which performed satisfactorily, the first two are more likely to elicit sameness than relational responses whereas the last four are more likely to elicit relational responses. This may be because items two and three are 'fill-the-blank' items for which relational thinking reduces computational burden, but which nevertheless can readily be solved non-relationally. Conversely, items four to seven are not 'fill-the-blank' items and perhaps prompt a substitutive view (an arithmetic expression or letter must be replaced) which has been argued to be part of relational thinking (Jones, Inglis, Gilmore, & Dowens, 2012).

CONCLUSION

We analysed the suitability of eight questionnaire items to investigate primary teacher awareness of children's understanding of equivalence. Two of the items performed poorly and are not suitable for use, including a definitional item. The remaining six items involved missing number or substitution problems and performed satisfactorily but varied in the responses elicited. We conclude that teacher awareness can be validly and reliably measured and recommend a variety of equation-based items be used.

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CHANGE IN IN-SERVICE TEACHERS' DISCOURSE DURING PRACTICE-BASED PROFESSIONAL DEVELOPMENT IN UNIVERSITY

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This paper explores in-service mathematics teacher learning in a practice-based professional development program. Through an analysis of daily journals written by an in-service teacher during practicum, we try to identify changes in his discourse over the course of a two-year Professional Degree Program newly implemented in Japanese universities. We classify the teacher's daily reflections on observed mathematics lessons and/or on taught lessons into four categories according to the teacher knowledge: empirical discourse, practical discourse, quasi-theoretical discourse, and theoretical discourse. The analysis results show a considerable change in discourse between the first and the second year.

INTRODUCTION

The forms of professional development of in-service teachers in Japan vary from those facilitated through teaching practice such as lesson study (cf. Stigler & Hiebert, 1999, ch. 7) to those having no direct relation to classroom instruction, such as university mathematics courses. The most common form would be the one-day participation in a research lesson open to colleagues organised inside or outside teacher's own school (cf. Miyakawa & Winsløw, 2013). Another relatively common form especially for 'active' teachers is long-term professional development in a university setting. The teacher is detached from his/her own school for a certain amount of time—varying from half a year to two years—and often enrols in a graduate program to obtain a master's degree.

Teachers' professional development in university is often criticised as being removed from actual practice and, consequently, incapable of allowing participants to acquire practical knowledge or skills useful for teaching. This criticism led to the creation of a new graduate program for professional development in the faculties of education of several Japanese universities in 2008. The Ministry of Education is now conducting a reform to expand this program to all faculties of education for teacher training in some years. This program, the 'Professional Degree Program', is a two-year graduate program comparable to a master's program. Field practice is emphasized, and it includes long-term practicums. A master's thesis is not required for graduation.

This program was also created in our university, and the first author was involved in it as an educator. We think it works effectively to some extent for in-service teacher training. However, it is not obvious, from the mathematics education research perspective, what kinds of learning are realised and what kinds of knowledge are acquired in different forms of professional development. Characterisation of teacher knowledge and learning is a big issue in our research area (cf. Ball, Thames, & Phelps, 2008; Margolinas, Coulange, & Bessot, 2005; Steinbring, 1998). We are, as

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mathematics teacher educators and researchers, interested in the nature of teacher learning during practice-based professional development programs, specifically during the ones organised by our university, in order to clarify the role of university and researchers (educators) in teachers' professional development.

This paper explores the nature of in-service mathematics teacher learning through the analysis of a case in which, from an educator's viewpoint, an in-service mathematics teacher certainly acquired some knowledge or learned something during our university's two-year Professional Degree Program. From the daily journals kept by students noting their reflections on lessons observed and/or taught during practicum, we identify changes in discourse over the two years of the program.

FRAMEWORK FOR TEACHER KNOWLEDGE

In this paper, we explore the evolution of teacher knowledge through changes in discourse. We examine how teacher knowledge can be characterised, in particular the knowledge in daily journals, and propose a framework to categorise discourse.

Some researchers characterise teacher knowledge by explicitly taking into account mathematics teachers' activities inside and outside the classroom. Steinbring (1998) believes that a teacher's role is not to make subject matter knowledge comprehensible to students, but to understand students' construction of personal knowledge in context and create learning environments (pp. 158-159). Based on this idea, he proposes the notion of *epistemological knowledge of mathematics in social learning settings*, which is required for the above activities. Margolinas et al. (2005) call *didactic knowledge* teacher knowledge specific to the mathematics to be taught, and characterise it according to different levels of teachers' activities. They investigate *observational didactic knowledge*, which 'grows from the teacher's observation and reflection upon students' mathematical activity in the classroom' (p. 205). Miyakawa & Winsløw (2013) also take into account teachers' activities around 'open lesson' and identify teacher knowledge from the perspective of the Anthropological Theory of the Didactic.

In our study, as data, we have teacher's written reflections (discourse) on observed lessons and/or taught lessons. Observation is about not only students' mathematical activities, as in the case in Margolinas et al. (2005), but also someone's teaching activities. We believe, therefore, that the teacher knowledge explored in our data is *observational systemic knowledge* of the teaching and learning system in classroom.

As a way to characterise the evolution of teacher knowledge during practice-based professional development, we propose the following categories that classify teachers' discourse according to the knowledge behind observation: *Empirical discourse*; *Practical discourse*; *Quasi-theoretical discourse*; *Theoretical discourse*.

Empirical discourse denotes the most naïve description of teaching and learning activities in the classroom and their reflections, made without professional knowledge of mathematics teaching. The latter three categories are for the discourse based on the professional knowledge of mathematics teachers. In Japan, mathematics teachers use some technical terms specific to mathematics teaching, which are not for students' use in the classroom. For example, 'measurement division' and 'partitive division' are technical terms for identifying different problem situations related to division. Similar terms have developed for the sake of communication among teachers, seen in teachers' national curriculum guides and textbook guides. These terms principally allow teachers to draw attention to significant facts—the nature of mathematical problems, teachers' acts, students' acts, etc.—in the complicated teaching and learning situation and apply some labels to them. We define *practical discourse*, including the above terms based on the practical knowledge mainly developed in teachers' community. We define *theoretical discourse* based on the theoretical knowledge or theory developed in mathematics education research to understand the mechanism of mathematics teaching and learning system. This distinction between practical and theoretical relies on the distinction proposed by Margolinas (1998) between *fact* and *phenomenon*. Theory is a coherent structure that provides a meaning to a *fact* that can be verified and transforms it into a *phenomenon* that can be produced by that theory and understood at the level of mechanism. Therefore criteria to distinguish theoretical discourse from practical discourse are the use of theoretical terms and the way one regards teaching and learning activities: as a fact or phenomenon. One may sometimes use theoretical terms just to label some isolated facts without taking into account the structure of theory. This is why we further dissociate *quasi-theoretical discourse* from theoretical discourse.

METHODOLOGY

We explain here the nature of the data first and then the analysis process. One Professional Degree Program year at our university consists of two semesters. The first semester is devoted to course work, which is aimed at allowing graduate students to acquire, through typical instructional cases, theoretical viewpoints on teaching and learning developed in mathematics education research. The second semester includes four months of practicum and post-practicum report writing. Since it is a two-year program, graduate students take practicum twice. Practicum is carried out as a part of a school support project, which is conducted under the supervision of university professor by a team comprising graduate students including in-service and prospective teachers, and cooperating teachers of the school. In this practicum, the cooperating teacher is not a student teacher supervisor, but a team member who aims at improving his/her own instruction in collaboration with other members. There are 150 hours of practicum a year. During practicum, graduate students visit school three days a week, and the rest of week is used for reflection on observed lessons, lessons to be taught, lesson they taught, etc. They keep a journal consisting of one-page reports on each day of the school visit. Each report includes the timetable of classes attended and the rubric 'Reflection'. The author of the daily journals we are going to analyse is an in-service teacher, Hiro, enrolled in this program and supervised by the first author of this paper. He was a mathematics teacher with eighteen years of experience in junior high school. His practicum was carried out in elementary school in both the first and second year.

In the analysis, we, the two authors, separately code Hiro's journal day by day, especially the rubric 'Reflection'. Then, we discuss the results of each coding and reach an agreement for all reports. The procedure of coding is as follows. We first identify whether there is a reflection on specific mathematics lessons observed, taught, or to be taught. Some writings are not always about a specific lesson, but might be about future project plan, schedule of units, results of interview with cooperating teacher, etc. Then, among the qualifying reports, we look for technical terms of practical discourse and theoretical terms of theoretical discourse, and count the number of reports (days) using such terms. Technical terms are those shared in the Japanese mathematics teacher community that can be found mainly in teachers' guides, but not in students' textbooks or in everyday life. We include in this category the terms locally shared in Hiro's teacher community. Theoretical terms are those that are not shared in the teacher community, but in the mathematics education research community. Among the reports using theoretical terms, we identify how these terms are used. If they are used just to label an isolated fact, such a report is classified into quasi-theoretical discourse. If the fact labelled by a theoretical term is considered in relation to other theoretical objects, this is a clue that the teacher regards a fact as a phenomenon in a coherent structure, and such a report is classified into theoretical discourse. Reports using neither practical nor theoretical terms are classified into empirical discourse. However, non-use of these terms does not necessarily imply non-use of practical or theoretical knowledge. Finding more accurate criteria is a further issue to be tackled. In this paper therefore, we will not go into the details of empirical discourse which will probably be more useful for analysing prospective teacher knowledge.

RESULTS

There are reports of 46 days in the first year and 51 days in the second year, among which 43 days and 47 days respectively are qualified for coding. Quantitative results are given in Table 1. Empirical discourse is exclusive from others and quasi-theoretical discourse and theoretical discourse are also mutually exclusive. However, practical discourse is not exclusive from the other two theoretical discourses; that is, a report might be coded as a practical and theoretical discourse at the same time.

Year	No. of days	Qualified days	Empirical discourse	Practical discourse	Quasi-theoretical	Theoretical discourse
2010	46	43	13	26	8	0
2011	51	47	12	17	16	12

Table 1: Quantitative results of teacher discourse

In Table 1, one may find a considerable change in Hiro's discourse between the first and the second year: the use of theoretical or quasi-theoretical discourse is more frequent in the second-year journal. In what follows, we provide examples of discourse, except empirical discourse, in order to show how Hiro's discourse changes over two years of program and to discuss teacher knowledge in the next section.

Practical discourse

In almost half of the reports (26 days) of the first-year journal—more frequently than in the second year—one finds technical terms of practical discourse. The following excerpt translated into English shows an example from the first-year journal.

> First year: 19 Oct. 2010
>
> For third period, I taught a grade 6 class with a graduate student, Ms. K, in the form of team teaching. The main goal of the lesson was to teach students a method for calculating 'whole number ÷ fraction'. Ms. K taught mainly until the moment of summarising the calculation method for the case of *unit fraction* as a divisor (÷ 1/2, ÷ 1/3, ÷ 1/4, ÷ 1/5). I was, as her supporter, in charge of manipulating the interactive whiteboard, distributing worksheets, and providing individual support to students based on the *Mitori* of their individual resolution. In the second half of the lesson, the teacher roles were switched. (…) Due to time shortage at the end of the lesson, students who could solve all problems comprised 77 % of the class. However, there was no student who could not solve any problem, and the rest 23 % could solve one or two problems. Based on these facts, I think these word problems of *measurement division* were easy for students to understand, and the diagram worked well as a support for students to tackle these problems. (…)

Hiro describes what happened in a grade 6 class first, and then describes students' performance on some problems with some personal comments. The terms in italics are technical terms of practical discourse. One may think the term 'unit fraction' is just a mathematical term. However, we assumed it was a technical term, because it is not a term for students that can be found in Japanese mathematics textbooks. Practical discourse is based on professional knowledge in addition to knowledge found in students' textbooks. 'Mitori' is a term used by elementary and lower secondary school teachers in Japan. It means the act of identification of students' different ideas in order to determine the flow or structure of the discussion phase (*neriage*), in which mathematical ideas are elicited and converge to the one aimed at teaching. 'Measurement division' is the term we mentioned earlier. Hiro pays attention to some specific events or facts he considers significant, in a complicated teaching and learning situation where so many different things happen. This should be allowed by means of technical terms associated with practical knowledge. Technical term labels a fact and provides a particular practical meaning to it. Without such term, one may not get what to see and what to convey to colleagues.

Quasi-theoretical discourse

While a few reports (8 days) use theoretical terms in the first year, more than half (28 of 47 qualified days) do it in the second year. Additionally, 12 of 28 days are coded as theoretical discourse. The following excerpt is an example of quasi-theoretical discourse taken from the second-year journal.

> Second year: 22 Nov. 2011
>
> In grade 6, I held a second introductory session on 'proportionality'. When asked to find the value for a *quasi-general number* in the table of correspondences, students were trying

> to find the value of y using the pattern of correspondences between concrete numbers. One could say that students were working for a *supporting contact*. Additionally, they could formulate the method as $x + 1 = y$ or $x \times 2 = y$. However, they could not arrive at the accurate formulation of the method for finding the values of y based on the idea of *Bai-hirei*. (…)

This reflection was coded as quasi-theoretical and practical discourse. Hiro uses two technical terms that label some mathematical ideas used in teaching. The first one is 'quasi-general number'. This is a term locally used in Hiro's teacher community, which means a relatively big number that requires the use of a general pattern to find the corresponding number. The second is 'Bai-hirei', the term used in Japan's teachers' community to denote a method of solving a proportionality problem without finding the quantity per unit. In addition to these terms, a theoretical term is used. It is Polya's term, 'supporting contact', which means the activity of checking whether a conjecture holds true for the general case by exploring specific cases (Polya, 1954, ch. 1). This theoretical term allows him to simplify the complicated teaching and learning situation and pull out from there a specific fact whose significance might not be perceived through technical terms of practical discourse. Hiro draws attention to students' activity of finding the value of y and labels it as 'supporting contact'. The theory usually provides a particular meaning to the identified fact, and allows understanding in relation to other facts. In Polya's theory of induction, 'supporting contact' is related to 'suggestive contact'. However, it is not clear in this report how identified students' activity relates to 'supporting contact' and how 'supporting contact' comes into being in relation to teachers' acts. It seems that the fact stays isolated and is not understood in the structure of teaching and learning system. If he described how teachers' action of 'ask[ing] to find the value for a quasi-general number' relates to students' 'suggestive contact' and 'supporting contact', then the reflection would have been coded as theoretical discourse.

Theoretical discourse

In the second year journal, theoretical discourse could be identified in 12 reports. Hiro not only describes or labels specific facts from an observed lesson, but also interprets them as a phenomenon in a coherent structure of theory.

> Second year: 26 Sep. 2011
>
> In grade 3 classes, what I liked about teaching acts of cooperating teacher are as follows.
>
> There is a scene where the teacher joins in with the student's *incomplete idea* and tries to get the discussion going. This act triggers a counter opinion from students. In this lesson, one could observe it in the scene where the teacher explained that the second biggest number is 78654321 by replacing 8 with 7 in the biggest number 87654321, in the task of making an eight-digit number with the cards from 1 to 8.
>
> (...) Behind these two points, there would be an important teaching act that related to *students' control of 'the goal level'*. I hope we will be able to intentionally use this act in future instruction.

Hiro uses two theoretical terms to label the cooperating teacher's act and students' mathematical activity, respectively. The first one is 'incomplete idea' and the second is 'students' control of "the goal level"'. The former is derived from John Dewey's idea 'indeterminate situation', which is a condition for starting inquiry (Dewey, 1938). The teacher's act of agreeing with the students' wrong answer, '78654321', is labelled as 'join[ing] in with student's incomplete idea'. The latter term is from S. Mellin-Olsen's idea of knowledge control referring to students' independence from the teacher when solving a problem in the classroom: three levels of control—tool, choice and goal—are considered (Mellin-Olsen, 1991). In this report, 'a counter opinion from students' (not from teacher) is interpreted as a starting point of 'students' control of "the goal level"', meaning students are responsible for the problem they are going to solve—the problem of finding the second biggest number has not been resolved yet. We coded this reflection as theoretical discourse, because we consider that Hiro not only pulls out these significant facts from the theoretical viewpoint, but relates them together in a structure of mathematics teaching and learning system. To understand this, we need to clarify his theoretical background. These terms are not adopted from pre-existing theories, but from the framework developed by Hiro and his team based on other theories. They extended Mellin-Olsen's idea in order to deal with Japanese lesson and integrated into it the idea of devolution of intellectual responsibility (Balacheff, 1990) and Dewey's idea in order to describe the mechanism how students establish independence in problem solving situation (Iguchi, Kuwahara & Iwasaki, 2011). In this framework, one of the conditions that provokes students' control of 'the goal level' is the teacher's act of joining in with students' incomplete ideas. Hiro therefore saw this phenomenon in the class of making some numbers with the cards from 1 to 8.

DISCUSSION AND CONCLUSION

Through the analysis of Hiro's daily journals, one could find technical and theoretical terms in both the first and second years. These terms belongs to teachers' professional knowledge that allows teacher to simplify the complicated teaching and learning situation and pulls out some significant facts. In particular, the theoretical terms allowed him first to identify the facts whose significance have not been perceived previously—as he learnt these terms in the university—, and second to understand them in relation to other facts from theoretical perspective. We consider that the knowledge at stake in Hiro's theoretical discourse is a kind of professional knowledge that is comparable to *epistemological knowledge of mathematics in social learning settings* (Steinbring, 1998). It enables teachers to 'become aware of the specific epistemological status of the students' mathematical knowledge' (p. 159). In Japan, this kind of knowledge is needed for experienced teachers who play a leadership role for instructional improvement in the teachers' community. For example, in lesson study, the role of leading teachers is not to criticise observed lessons but to understand what happens at the deeper level of structure of mathematics teaching and learning system and communicate this to their colleagues. One may also see here the crucial role of university and researchers for in-service teacher professional development.

Regarding Hiro's more frequent use of theoretical discourse in the second year, what is remarkable is that most of the theoretical terms relate to the mechanism of knowledge control and intellectual responsibility. This is because Hiro and his team developed their own framework. They analysed data collected in the first year practicum with this framework, and even wrote a research paper at the end of the first year (Iguchi et al., 2011). Therefore, in order to place theories at teachers' disposal, it would not be enough to provide course work to learn them and long-term practicum.

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APPROACHES TO TEACHING MATHEMATICS AND THEIR RELATION TO STUDENTS' MATHEMATICAL MEANING MAKING

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This paper addresses theory and practice in the teaching of mathematics in university lectures and tutorials that is designed to promote students' mathematical meaning-making. It draws on areas of theory and five research studies to characterise teaching, to illustrate how theory can be used to analyse teaching and to relate meaning making in teachers' teaching and students' mathematics. It begins to develop a body of research on teaching practice and its development at university level, identify key areas of knowledge and to present questions for further research.

INTRODUCTION

This paper is a theoretical/philosophical paper dealing with *mathematics teaching at university level* and *associated meanings*. It is linked closely to teaching practice through several studies that have sought to illuminate practice at this level. The focus is the relationship between teaching approaches, the meanings that students make of the mathematics taught, the ways in which teachers gain access to student meanings and ways in which teaching can focus on creation of meaning.

Kilpatrick, Hoyles and Skovsmose (2005) write "Teachers of mathematics must deal with questions of meaning, sense making, and communication if their students are to be proficient learners and users of mathematics" (p. vii). They ask the questions:

> "How can meanings for teachers and didacticians be developed?" and
>
> "How can teachers best explore the meanings which students have constructed?"(p.16)

Skovsmose writes further "… for students to ascribe meanings to concepts that have to be learned, it is essential to *provide meaning* to the educational situation in which the students are involved (p. 85, our italics). Artigue, Batanero & Kent (2007) suggest that learning at this level is seen as enculturation in advanced mathematical practices, while Ben-Zvi & Arcavi (2001) write that making meaning in mathematics is a process of "socialisation" into the culture and values of "doing mathematics" (p. 36). The studies to which we refer below address the above questions, taking into account the full sociocultural context of learning and teaching as far as is possible. In terms of what we mean by 'teaching', we follow Pring (2000 and 2004) who claims:

> An action might be described as 'teaching' if, first, *it aims to bring about learning,* second, it *takes account of where the learner is at*, and, third *it has regard for the nature of what has to be learnt*. (2000, p. 23). For example, the teaching of a particular concept in mathematics can be understood *only within a broader picture of what it means to think mathematically*, and its significance and value can be understood *only within the wider*

2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 97-104. Hobart, Australia: PME.

evaluation of the mathematics programme. (Pring, 2000) and [not attending to this] is *to accept a limited and impoverished understanding of teaching* (2004, p. 22), (our italics).

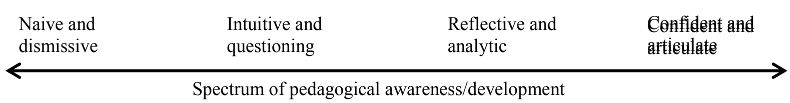
Further, we seek to redress the deficiency expressed by Speer, Smith and Horvath (2010) who write “very little research has focused directly on teaching practice [at university level] – what teachers do and think daily, in class and out, as they perform their teaching work” (p. 99). We refer to five studies, below, which arise from an in-depth focus on specific examples of teaching practice with qualitative analyses which reveal aspects of teaching, capture the intentions and reflective thinking of the teacher and subject outcomes to critical scrutiny through rigorous analysis. They expand on existing theory and introduce new theory, illuminate the nature of teaching at this level and its relations to students’ meaning-making, raise and elaborate on issues that arise in the practice of teaching and offer insights that can be of relevance and significance more generally. In doing so, they begin to theorise this (relatively) new area of study into the university teaching of mathematics.

STUDIES HIGHLIGHTING CHARACTERISTICS OF TEACHING

We exemplify the practices, to which we refer, through references to research which is taking or has taken a sociocultural approach, seeking to address the full context of learning and teaching as far as is possible. This has involved the use of qualitative approaches to data collection including participant observation of teaching-learning events and interviews or conversations with the teacher in each case. Analysis is grounded and interpretative with care taken to justify interpretations in relation to the wider context of the events. These studies have drawn variously on existing theory and in some cases have developed theory through the research. We sketch briefly some of the significant findings in each case (with reference to publications which provide greater detail: some in earlier PME proceedings) and follow with a theoretical synthesis and questions for further research.

1. Characterising pedagogy in mathematics small group tutorial teaching

Nardi, Jaworski & Hegedus (2005) report a qualitative study in the UK of the teaching of six mathematicians in *small group tutorials* over one university term (8 weeks). Analyses led to a characterization of teaching approaches from the perspectives of the tutors. All tutors recognized students’ difficulties and dealt with them in differing ways, episodes from which were seen to fit into or between four pedagogic characterizations: *Naive and Dismissive; Intuitive and Questioning; Reflective and Analytic; and Confident and Articulate*; the whole being characterized as a *Spectrum of Pedagogical Awareness*. This spectrum offers a theoretical



perspective on the links between mathematics and pedagogy, and the knowledge of the teachers in working for the meaning-making of their students. In parallel, teaching episodes were analysed using a theoretical construct “The Teaching Triad” (Jaworski, 1994; 2003), consisting of 3 domains – *Management of Learning* (ML), *Sensitivity to Students* (SS) and *Mathematical Challenge* (MC). Findings showed *largely* that ML involved teachers in showing and explaining mathematics; SS involved ensuring the student was made aware of the *correct* mathematics; while MC left it up to the student to go away and make sense of the mathematics presented to them. There were of course exceptions: for example, one tutor frequently invited a student to go to the board and explain his or her solution to a given problem. He explained his own actions in relation to the student’s activity at the board:

> I do promise to help; or will help . . . they actually know I’ll start them off. They won’t just be stood at the board and me twiddling my thumbs. I might, after a few seconds, like 30 seconds, or something like that, or perhaps even less if they’re looking panicky, I would suggest, er, “Well, OK, write down, what’s the first line? What’s it mean to say that?” (Jaworski, 2003, p. 89)

For example, the following text is recorded from a dialogue about *group theory* in which a student (S2) writes at the board and the Tutor (T) supports him:

4	S2	If you’ve got h and l in H, then this one tells us that $g_1h=k_1l$ (pause) for some l in H [S2 and tutor say this together] and same sort of thing for the twos, [he writes $g_2h=k_2l$]. (pause)
5	T	… call it h' and l', [h-prime and l-prime] well, call it l', you might want the same h as ... (Jaworski 2003, p.84)

This kind of teaching was seen to show SS in both affective and cognitive domains as the tutor helped the student to feel supported and encouraged in the given task. Moreover MC was seen in the tutor’s requirement for students to express their own mathematical meanings publicly. This kind of activity was characterized in the spectrum as “Reflective and Analytic”, whereas the more common forms of teaching activity, mentioned above, were seen to be largely in the domain of “Intuitive and Questioning”, with some at the level of “Naive and Dismissive” (Nardi et al, 2005).

2. Characterising teaching in mathematics lecturing

An ongoing study in Greece, of teaching in *mathematics lectures* in two University Mathematics Departments, involves observation of lectures and interviews with six lecturers who are research mathematicians. Analysis identifies a lecturer’s teaching goals, actions and characteristics, allowing insights into how the actual teaching practice in this context takes students’ needs into account. The *Teaching Triad* is used as an analytical frame to characterise teaching episodes and to interpret the identified scheme of teaching actions, goals and characteristics (e.g. Petropoulou et al, 2015). Sensitivity to Students and Mathematical Challenge are identified through lecturers’ main *goals*; are inherent in the nature of their teaching *actions* and are reflected in basic characteristics of their teaching. For example, one lecturer aims to stimulate students affectively by drawing on students’ experiences and encouraging their engagement in the lecture through interaction, which is for him an important element of the learning process. He challenges students mathematically, encouraging mathematical investigation by posing open questions and making connections by exemplifying basic mathematical ideas in dialogue with students. For example,

L: Now, can you hypothesize, when a series may converge? You can base on the series S1 and S2. [*an open, and challenging question*]

$$S1 = \sum_{k=0}^{+\infty} x^k = \frac{1}{1-x}, \quad |x|<1 \qquad S2=\sum_{k=1}^{+\infty}(-1)^k$$

S: The base [of the x^k] in the first case [S1] is a positive number. [*student meaning*]

L: Not necessarily. If x is a negative number the sign [of x^k] changes. …

Dialogue allows the lecturer access to students' meanings, to which he can then respond. Analyses are showing "An uneasy balance" between *Sensitivity to Students* and *Mathematical Challenge* in this teaching, reflected in a tension between lecturer's intentions to include students' thinking in the activity of a lecture while at the same time presenting mathematical meanings in a rigorous form. The ways in which this tension is addressed in teaching episodes can be seen to fit with differing positions on the *Spectrum of Pedagogical Awareness.*

3. Characterising one lecturer's teaching of linear algebra

A UK study of the teaching of linear algebra involved a small community of inquiry of two mathematics educators and one mathematician (the lecturer) in which *teaching in lectures* was explored in depth and characterised. The focus was centrally on the thinking and actions of the lecturer with the three members engaging deeply with ideas from linear algebra, approaches to teaching and the engagement/meaning-making of students. The teacher talked extensively about the mathematics and the ways in which he would (and did) work with students on this mathematics, reflecting particularly after a lecture in which he had given time to students to work on a task and observed their work. For example, after a lecture on subspaces, he reflected:

> At some points I realised I need to find different ways of phrasing the questions in order to make them more accessible. One example of that was the introductory example on subspaces, where I had asked the students to find solutions to a homogeneous equation system with unknown coefficient matrix, given that they know a couple of solutions that I've given them. That was one question where I saw quite clearly that some of the students found it very easy, and some of the students didn't have the slightest idea even if they tried. (Thomas, 2012, p. 117)

Central to analysis was the lecturer's articulation of teaching goals and his realization of the goals in his day to day teaching. The study showed episodes of teaching and talking about teaching in the realm of 'reflective and analytic' as the lecturer expressed his goals for teaching for the benefit of his co-researchers. We noted particularly two modes of reflection which we called 'expository' (in which the lecturer expressed the mathematical meanings he wanted students to make) and 'didactic' (in which he articulated his goals for teaching and the associated actions he would, and did, take in his lecturers) (Jaworski, Treffert-Thomas & Bartsch, 2009). The study provides important insights into the teaching of linear algebra. (Thomas, 2012.)

4. Knowledge in mathematics teaching in small group tutorials

An ongoing UK study of university teaching in small group tutorials is exploring tutors' knowledge in teaching. Initially 26 tutorials were observed from 26 different tutors, including both mathematicians and mathematics educators. Subsequently tutorials of three tutors, over one semester, are being studied in depth. The focus of analysis is on teachers' knowledge for teaching in mathematical, didactical and pedagogic domains, drawing on a grounded analytical approach and a range of theoretical positions in the literature (e.g., Teaching Triad; Knowledge Quartet – Rowland, Huckstep & Thwaites, 2005). One key concept emerging is tutors' use of mathematical examples: dialogue between tutor and students provides insights into students' meaning-making and tutors' adaptation of teaching to students' thinking as they see it (Mali, Biza and Jaworski, 2014).

The study seeks to identify characteristics of university mathematics teaching (an example of which is a tutor's *use of a generic set of examples* for students' meaning making of a mathematical concept) where a characteristic of teaching constitutes *a pattern of teaching* identified repeatedly in the data. The characteristics as a whole form an image of a tutors' teaching practice and, in a finer layer of analysis, each one of them is distilled into tools and strategies for teaching. For example, in one tutorial, a generic set of monotonic functions on intervals was used for the students' appreciation of the property that every monotonic function is injective. This set formed a mathematical object coded as 'tool for teaching', and included the functions

$f(x) = x^2$ on $[0, +\infty]$; $f(x) = \sin(x)$ on $[-\pi/2, \pi/2]$, $[\pi/2, 3\pi/2]$;
$f(x) = \log_a(x)$ for $a>1$, $0<a<1$; and $f(x) = x$.

A strategy for teaching is a process consisting of the mode of tool use and the associated decision making. In this tutorial, the tutor attempted to make the connection between a concrete set of examples and the abstract mathematical concept of monotonicity and its properties. The strategy here was the use of different mathematical representations to foster students' meaning making of mathematics through making connections within mathematics. The layers of characterization, tools and strategies form the basis of a theoretical construct to capture knowledge in teaching (Mali et al, 2014). The particular tools and strategies used by the tutor comprise the tutor's pedagogy; an example of a tool-strategy pair was given above.

5. Developing mathematics teaching to address students' meaning making

In this study, one university tutor designed teaching approaches to focus on the mathematical meanings made by her students in a small-group tutorial, and adjusted the approaches to relate to meanings discerned. The students were first years, in a joint degree in Mathematics and Sport Science, and were relatively weak in mathematics. The study involved a small community of inquiry of two researchers, one being also the tutor; as the tutor designed teaching, worked with students and reflected on learning outcomes, her co-researcher gathered data and acted as a sounding board for tutor reflections. A fine-grained analysis was made of dialogue from the tutorials, using the Teaching Triad as a tool (Jaworski & Didis, 2014).

It was clear from the beginning of work with her five students that the students were unaccustomed to speaking their mathematics or explaining concepts, so it was hard to discern students' meanings. Various approaches were seen to address these issues. For example, one tutorial was spent entirely on the definition of 'limit'. After realising that students were unfamiliar with quantifiers, and unaccustomed to reading mathematics aloud, the tutor first established meanings of symbols and then requested students to read parts of the definition, sometimes independently and sometimes chanting as a group. These strategies led to each student being able to read the definition in a meaningful way which gave some insight into their understanding of the limit concept. However, at the end of the tutorial, one student expressed frustration with not having tackled any problems in the tutorial. The tutor learned from such observations and adapted her practice for future tutorials.

A question established during this study was "How can we foster student expression of mathematical meanings in relation to the teaching experienced?" We recognised that the tutor was a learner, developing teaching through a design/action/reflection approach. In terms of the spectrum she might be seen as shifting between 'intuitive and questioning' and 'reflective and analytic' as reflection and analysis enabled her to became more aware of the meanings and perceptions of her student.

DISCUSSION

The papers referenced in each case above provide details of research which explores and characterises the university teaching of mathematics in both lectures and small group tutorials. In each case the research has studied the ways in which the teachers (lecturers and tutors) constructed teaching to enable students to make sense of the mathematics, to make mathematical meanings. The studies have been interested in teachers' thinking related to didactics and pedagogy – how they design tasks and use strategies in teaching, and how they interact with students in meaningful ways (or not). We see teachers identifying students' difficulties and using strategies to deal with difficulty; using examples to aid students' meaning-making; relating goals for student comprehension to strategic action in teaching sessions, and designing teaching actions to cope with tensions between sensitivity to students and the rigorous mathematics desired by the tutor.

Two theoretical constructs, particularly, have been used to make sense of teaching approaches: the *spectrum of pedagogical awareness* captures the degree to which teachers engage consciously with didactic and pedagogic issues and design teaching to engage students and enable them to create meaning; *the teaching triad* identifies aspects of sensitivity towards students and associated mathematical challenge within an overall management of the learning environment. Where challenge can be seen as offered with due sensitivity, mathematical rigour might seem to be in question. This is an important issue to explore further. Relationships between the spectrum and the triad also require further exploration.

These studies are embedded in a sociocultural perspective, thus seeking to consider the holistic nature of the teaching/learning process rather than fragmented images of the tutor's practices and students' meaning. Recognition of goals and tools is related to Vygotskian theory, in particular the mediational triangle in which tools are seen to mediate the achievement of the object of activity, and goals are related to actions in activity (e.g., Leont'ev, 1979). For instance, Study 4 conceptualises *tools for teaching* – the *generic set of examples* is seen as a tool which mediates students' understanding of the concept of *injectivity*. Analysis here is linking tool use, and the strategies teachers are seen to use, to *knowledge in teaching*, discerned through in-depth conversations with each tutor. In Study 3, *goals* are the teachers' intentions for teaching, instantiated in his *teaching actions* in a lecture and theorised in terms of Leont'ev's theory of activity (Leont'ev 1979; Thomas, 2012). In Study 5, teaching was developed to respond to students' meaning making and perceived learning needs, and to support their meaning-making, thus demonstrating forms of *sensitivity to students* which recognises their particular context (e.g., sport-science students, rather than mainstream mathematics). These studies together start to define a field of research in which the following elements are significant:

1. Emergence of theoretical constructs: provides ways of categorising teaching and comparing teaching processes and events.
2. Different ways of characterising teaching (related to theoretical constructs): allow others to question their practices and develop knowledge in the field.
3. Methodological approaches and methods: allow in-depth study of teaching practice within a sociocultural frame with attention to details of context.
4. Recognition of *teaching intentions, goals*, and *actions*, and identification of *tools in teaching*, provides the beginnings of a 'tool box' (Nardi et al, 2005) for teaching which can be developed through scrutiny and critique ◦
5. Meaning making in teaching, related to students' mathematical meanings: opens up ways of linking teaching with learning in very initial and tentative approaches which can be developed further through scrutiny and critique.

Finally we recognise key questions for further research:

I. In what ways can teaching be characterised so that university teachers can gain relevant insights to teaching processes and develop teaching?

II. How can theories of teaching be employed to aid the design and development of teaching?

III. How do/can meanings in teaching relate to students' mathematical meanings to enable students to gain deeper conceptual meaning in mathematics?

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MATHEMATICAL PROBLEM SOLVING ONLINE: OPPORTUNITIES FOR PARTICIPATION AND ASSESSMENT

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Gibson's notion of affordances is used to analyse the opportunities for collaborative mathematical problem solving utilised by a group of four Grade 5 students in a Computer Supported Collaborative Learning (CSCL) environment. Three patterns of student participation are identified which suggests that different students perceived and utilised different opportunities for action. One student, who is described as being shy, demonstrated increased participation in the CSCL. A second analysis employed a critical thinking assessment framework to ascertain whether CSCL may afford teachers an opportunity to assess student thinking. The increase in some students' participation and the potential to assess student thinking are argued to warrant the use of CSCL in primary mathematics classes.

INTRODUCTION

This study analyses the online posts of Grade 5 students engaged in collaborative mathematical problem solving in a Computer Supported Collaborative Learning (CSCL) environment. Gibson's (1979) notion of affordances was employed to investigate two aspects of the 'opportunities for action' that were acted upon by students and a potential 'opportunity for action' that CSCL may provide for teachers. As CSCL use enables peer collaboration, it has been shown to help foster the development of mathematical problem solving in the context of tertiary level mathematics education (Stahl, 2009). In the context of primary level mathematics education, there is little data that demonstrates whether a CSCL approach would have the same benefits for mathematics learning as when deployed with young adults. Given the increased use of computers in mathematics classrooms around the world, CSCL is of research interest in that it offers a method of utilizing computers which goes beyond 'skill and drill' software and games. Also, many mathematics curricula are increasing focused on developing students' mathematical problem solving skills which may require changes in pedagogy. Hence, CSCL may have the potential to supplement or enhance traditional methods of teaching mathematical problem solving while using digital technology effectively.

The first research question investigates whether CSCL affords students with opportunities to participate in collaborative mathematical problem solving. It is hypothesised that some students may be able to participate more or less in peer collaboration when that collaboration takes place online. The second research question relates to a possible affordance of CSCL for teachers. It is hypothesised that the text record of student posts generated by CSCL may afford teachers the opportunity to

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systematically assess student thinking during collaborative mathematical problem solving.

Computer Supported Collaborative Learning (CSCL) describes a method of computer use through which students are able to collaborate on mathematical problems that require problem solving skills in addition to basic mathematical procedures. The collaborative learning referred to here is described by (Dillenbourg, 1999) as learning in which peers are working more or less at the same level, while being able to perform the same action, while having a common goal, and working together. In this study of Grade 5 students working in a CSCL we have adopted Lester and Kehle's (2003) conception of Mathematical Problem Solving:

> Successful problem solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve some tension and ambiguity (i.e. lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity (p. 510).

Affordances – opportunities for action

In mathematics education, the notion of affordances are often used to assess technologies which are deployed in educational contexts (Brown & Stillman, 2014). The term affordance, originally used by Gibson (1979), refers to the opportunities for action an agent perceives in their environment. When defined by Gibson, an affordance is the product of a relationship between an agent and their environment relevant to the task at hand. Hence, an object or event (such as pencil) can afford writing if one is engaged in the task of note taking. If the task changes – the agent gets an itch in a hard to reach place – the affordance of the object may change – the pencil can then afford the opportunity to scratch the itch. Thus, affordances are dynamic properties born out of the requirements of a situation (Gibson, 1979).

Most analyses of the affordances of educational technology focus on beneficial, potential affordances of the technology (Brown & Stillman, 2014). These analyses tend to view an affordance as the property of a technology rather than as the product of an interaction between an agent and their environment. This use of the term affordance is of research interest (Brown & Stillman, 2014) and is used to test the second hypothesis in this paper, in that an opportunity for assessment of student thinking is posited as being a property of CSCL.

To investigate whether different students perceive different opportunities for participation in collaborative mathematical problem solving in a CSCL environment, the notion of affordances is used in a different way. For this analysis, affordances are viewed in terms of a relationship between individual students and their environment (Gibson, 1979). CSCL may afford different opportunities for action to different students as different students perceive and utilise CSCL in different ways. Rather than investigating a potential affordance of CSCL, this analysis investigates the affordances that were utilised by individual students. As these affordances are the product of a relationship between each student and the CSCL environment, variation in the affordances utilised may be indicative of variation in students' capacities to perceive and utilise opportunities for action (Gibson, 1979; Kirlik, 1995).

Description of the CSCL environment

The data analysed in this study was gathered from a 9-week intervention in two team-taught Grade 5 classrooms in a primary school in Victoria, Australia. Edmodo (Edmodo, 2014) was used as a platform to create the collaborative online space. In week 1, students were familiarised with the online platform. From weeks 2 to 7, the second author spent one hour per week reviewing student solutions from the previous week and introducing students to the next open-ended mathematical problem. Students then worked in small groups (typically 4 students) in the CSCL to solve the week's problem. Groupings were 'mixed ability' compared to the usual 'ability groups' used by the classroom teacher. Students were not given class time to work on each problem beyond the initial hour, thus most of the students' engagement with the problems occurred at home. The CSCL was asynchronous, meaning that students did not need to be logged in at the same time. Student interaction in the CSCL took the form of message board posts which could also include uploaded files (such as Word documents and Excel spreadsheets).

Can CSCL afford opportunities to assess students' thinking during collaborative mathematical problem solving?

While many have argued that critical thinking in mathematical problem solving is important (Facione, 2013), practical methods of assessing students' mathematical problem solving are less developed than assessments of students' mathematical procedural knowledge (Yeh, 2001). Perkins and Murphy (2006) suggested that asynchronous online discussion may yield data which could be used to assess critical thinking. Student activity in CSCL produces a text record of students' discourse. It is hypothesised that this text record has the potential to afford opportunities for assessment which do not exist in face-to-face classroom environments. Classroom environments may require a teacher to monitor multiple groups of students working simultaneously and whose interaction is largely oral (Webb, 1991). In contrast CSCL text provides a written record of all between-student interactions that may be accessed by a teacher, group by group, without such time pressure and distractions. Perkins and Murphy (2006) developed a model for identifying engagement in critical thinking designed for use with text records of online discussion. In this study Perkins and Murphy's model will be used to assess the text record of students' online interactions to ascertain whether CSCL may afford teachers an opportunity to assess student thinking.

METHOD

The analysis provided in this paper presents data collected from one group of four students who participated in the CSCL described above. Records of online discussion were used as data relating to student interaction and activity in the online environment.

Student participation was analysed in terms of number of posts, proportion of the group's overall posts and average length of posts (in terms of number of words). The researchers inferred that a student perceived that CSCL afforded them an opportunity to participate in collaborative mathematical problem solving if the data showed that they made use of the affordance.

A proxy of students' ability to engage in face-to-face, small-group collaborative work has been constructed using data gathered in post-intervention interviews and by discussing student behaviour with the students' classroom teacher. Post-intervention student group interviews provided data concerning students' patterns of interaction in face-to-face, small-group discussion. Individual students' patterns of interaction in this interview were discussed with the class teacher to confirm whether the patterns observed in interviews were representative of a student's general classroom behaviour in face-to-face collaborative group work.

The assessment framework developed by Perkins and Murphy (2006) was applied to the records of 8 weeks of online discussion. The first week of the 9-week intervention was not analysed as the data gathered in this week had focused on familiarising students with the Edmodo platform used rather than mathematical problem solving.

In order to investigate whether CSCL afforded different students with different opportunities to participate in collaborative mathematical problem solving, student participation and categorisation of students' critical thinking were compared. Participation in CSCL interaction was also contrasted to the proxy of students' participation in face-to-face, small-group collaborative work. The results of the analysis of student thinking using Perkins and Murphy's (2006) framework form the basis for answering the second research question relating to the potential affordance for teachers who use CSCL.

RESULTS

Table 1 summarises student participation in the CSCL and face-to-face environments respectively. Individual students displayed different patterns of participation in each environment. Zaid led participation in both environments but the other three students' level of participation varied. Chaz and Igor made few contributions in the CSCL environment (6% and 9% of post respectively) but participated more when face-to-face (13% and 38%). Olive made a significant number of contributions in the CSCL environment (39%), and participated less when face-to-face – both in terms of number of contributions (19%) and average length of utterance. These results support our first hypothesis that individual students may perform differently in different environments as only Zaid had a similar pattern of performance across both environments. The variation in participation between environments of Olive, Chaz and Igor indicates that these students utilised different opportunities for action in the two environments. While Chaz and Igor participated more in face-to-face interaction, Olive was able to make use of a CSCL to greater effect.

Discussions with the students' classroom teacher confirmed the picture painted by this data. Olive was seen by the classroom teacher to be a shy, quiet girl who had been streamed into a 'low mathematical ability' grouping based on her performance in mathematics assessments. Zaid had been streamed into a 'high mathematical ability' group and was seen to participate regularly in classroom discussions and group work. Chaz and Igor are referred to as 'at level' by their teacher.

	Online contribution			Face-to-face contribution		
Student	Number of Posts	Proportion of Total Posts (%)	Average length of post (words)	Number of utterances	Proportion of total utterances (%)	Average length of utterance (words)
Zaid	107	46	29	27	31	35
Olive	90	39	20	16	19	19
Chaz	15	6	6	11	13	32
Igor	20	9	10	32	37	21

Table 1: Student participation in online and face-to-face environments

Perkins and Murphy's (2006) Clarification, Assessment, Inference Strategies (CAIS) framework was used to code students' online posts. Each post was treated as an utterance. Coded utterances were classified as relating to critical thinking and then broken down into the four categories suggested by Perkins and Murphy. Un-coded utterances were utterances which could not be related to critical thinking. These un-coded utterances typically related to organisational or conversation queries such as: "Please reply here" or questions such as, "is anybody online?"

	Aspect of critical thinking (%)					
Student	Assessment	Classification	Inference	Strategies	Coded utterances	Uncoded utterances
Chaz	4	8	0	2	5	10
Igor	8	3	7	6	7	13
Olive	46	73	29	29	58	32
Zaid	42	16	64	63	58	49

Table 2: Assessment of students' critical thinking

Each author coded the data independently. Of the 232 utterances coded, the authors coded 12 utterances differently from each other. After discussion, agreement was reached for coding these 12 utterances. Table 2 summarises the proportion that each student posted assessment, clarification, inference and strategies type utterances. The total number of coded and un-coded utterances for each student is also provided. Chaz and Igor made the smallest number of posts and contributed less than 10% to the group's post relating to critical thinking. The bulk of the group's critical thinking posts were made by Zaid and Olive. Zaid and Olive contributed an almost equal proportion of assessment related posts (42% and 46% respectively). Olive contributed a greater proportion of the group's classification posts (73%) while Zaid posted a greater proportion of the inference and strategies posts (64% and 63%).

DISCUSSION

Affordances utilised by students

It appears that most students' participation varied between face-to-face interaction and CSCL interaction. Of course, the task that students were engaged in within each environment also varied in this data set. Thus, while it cannot be concluded that variation in student participation is solely the product of a different environment, it seems plausible that this variation may be partly a consequence of the move to a CSCL environment. While Zaid's contribution remains consistent across both environments (he makes numerous utterances in both environments), the other three students appear to make use of opportunities for action in one environment more than the other. Chaz and Igor do not make use of opportunities to collaborate online but, Igor in particular, made many contributions in the face-to-face environment (37% of utterances). One possible interpretation of this data is that Igor had the capacity to recognise and utilise affordances in the face-to-face, group environment which he did not recognise nor utilise in the CSCL environment.

In contrast, Olive may have had the capacity to recognise and utilise affordances of CSCL which were in sharp contrast to the affordances she recognised and utilised in face-to-face environments. Although she did not contribute the fewest number of utterances to the group discussion, the utterances Olive made were generally shorter than the other students. When this is taken into consideration we can see that her overall contribution within the physical environment of the semi-structured interview was the smallest of the group. In the audio recordings of the interview, it was also difficult to hear her voice. This does not contradict the picture that her classroom teacher has of her; that she is shy and quiet, and that she only reluctantly contributes to class discussion and requires prompting by her teachers to do so. Additionally, Olive had been 'streamed' into the lowest achieving mathematics group for her daily mathematics instruction. Tests assessing her ability to perform routine mathematical skills and procedures were used to inform her placement in this group. Yet, in the CSCL environment, we can see that Olive's contribution to online learning is significant, making up almost 40% of the group's discussion. Of her posts, 64% pertain to critical thinking and she makes the most 'classification' type posts (73% of all classification posts). The authors suggest that the contrast between Igor and Olive's performance in the CSCL environment can be explained is in terms of each student's capacity to recognise and utilise affordances. Olive, a shy, soft-spoken girl working with a group of vocal boys, may lack the capacity to utilise opportunities for participation in face-to-face collaboration. The CSCL environment however, affords Olive opportunities for collaborative mathematical problem solving which do not exist for her in a face-to-face interaction. Perhaps the CSCL environment affords Olive a greater opportunity to participate because she has more time to consider her responses before she posts. It is likely that a combination of factors are at play – on the one hand, a CSCL environment may afford her opportunities; on the other hand, face-to-face environments may constrain Olive. If Webb's (1991) finding that much classroom collaborative activity is verbal applies to Olive's schooling, and Olive lacks the capacity to make use of opportunities for action in verbal interaction, then teachers may form the view that Olive is not able to perform well in collaborative mathematical problem solving. Yet Olive found her voice in the CSCL environment, and made significant and valuable contributions to her group's work, which contradicted her classification as a 'low mathematical ability' student within the class.

Further research will aim to develop an understanding of Olive's level of agency within the group. Analysis of *interaction* within the online environment is required to ascertain levels of dominance or sub-ordinance between group members.

Affordances for assessment

A simple breakdown of student participation in the CSCL environment (Table 1) enables teachers to monitor participation in collaborative mathematical problem solving which is not possible in face-to-face environments. The written record of interaction produced by the CSCL environment enabled analysis of students' critical thinking using Perkins and Murphy's (2006) assessment framework. This more detailed breakdown of students' critical thinking, takes some time to complete but could be used by a teacher to assess the posts made in one week rather than the nine weeks of coding performed by the researchers. While Igor's high level of participation in face-to-face environments might be interpreted as suggesting that he contributed significantly to his group's collaborative mathematical problem solving, analysis of the written record of his online participation suggests that his contribution was minor and approximately two thirds of his contributions do not show evidence of critical thinking. Analysis of the written record produced by the CSCL environment also provided evidence of Olive's critical thinking and problem solving skills which would not be present in a small group, face-to-face discussion.

CONCLUSION

This study analysed the affordances utilised by a group of four students' who participated in collaborative mathematical problem solving in a CSCL environment. Gibson's (1979) notion of affordances was used to analyse the opportunities for action which each student made use of. Three patterns of participation were identified. One student – identified as 'low mathematical ability' by her teacher, contributed significantly to her group's collaborative mathematical problem solving in the CSCL environment. In face-to-face, small-group settings, this student participated less. It is possible that this student was able to recognise and utilise opportunities for action in a

CSCL environment that she was unable to utilise in a face-to-face setting. Hence, the use of CSCL to teach collaborative mathematical problem solving may be warranted in terms of providing some students with opportunities to participate and have their voice heard.

The written record of student interaction produced by CSCL may also afford teachers with the opportunity to assess students' mathematical problem solving. When Perkins and Murphy's (2006) framework was applied to the data generated in the CSCL environment, students' critical thinking could be assessed. This assessment could help teachers identify students' abilities in the area of mathematical problem solving. Hence, employing CSCL in mathematics instruction in primary school may be warranted in terms of enabling teachers an opportunity to systematically assess students' critical thinking during collaborative problem solving.

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AN INVESTIGATION OF THE IMPACT OF SAMPLE QUESTIONS ON SIXTH GRADE STUDENTS' MATHEMATICAL PROBLEM POSING

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This study examined the impact of sample questions (SQ) on sixth grade Chinese students' problem posing. Problems posed by the students were classified as parallel problems to the SQ and non-parallel problems. The complexity of problems was also analysed in terms of number of steps to solve them. Students given SQ posed questions that were similar to the given SQ more frequently than students not given SQ. The first and the second problems posed by students not given SQ were at the same complexity level as the first problems posed by students given SQ. Similarly, the third problems posed by students not given SQ were at the same complexity level as the second problems posed by students given SQ. In both cases, the complexity level of problems increased.

INTRODUCTION

Problem posing (PP), as an inseparable component of problem solving (Silver, 1994), has been emphasised in mathematics curriculum standards at different educational levels around the world (e.g., Chinese Ministry of Education, 1986, 2001, 2003, 2011; National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO], 2010). Interest in incorporating PP in school mathematics instruction has grown steadily among mathematics education researchers and practitioners (Cai, Hwang, Jiang, & Silber, in press; Singer, Ellerton, & Cai, 2013). In the previous study (Cai, Jiang, Hwang, Nie, & Hu, in press), we found that there were only small percentages of PP tasks included in the textbooks in Mainland China and in the United States. In addition, the following five types of PP tasks were identified: (a) Posing a problem that matches the given arithmetic operation(s); (b) Posing variations on a question with the same mathematical relationship or structure; (c) Posing additional questions based on the given information and a sample question (SQ); (d) Posing questions based on the given information; and (e) Unconstrained problem-posing tasks. The third type involves sample questions (SQ). That is, students are provided with problem situations and SQs to be answered based on the situations, and then are asked to pose additional problems. It was found that a higher percentage of PP tasks in the textbooks used in Mainland China had SQs than in the textbooks used in United States.

In the past, researchers have usually provided problem situations without SQs and then asked students to pose problems (e.g., Cai, Moyer, Wang, Hwang, Nie, & Garber,

2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 113-120. Hobart, Australia: PME.

2013; Silver & Cai, 1996). What is the impact of sample questions on students' problem posing? This study is designed to answer this research question.

Worked-out examples are important in developing conceptual understanding of mathematical ideas (Fukawa-Connelly & Newton, 2014; Johnson, Blume, Shimizu, Graysay, & Konnova, 2014). However, if the examples are constrained to limited contexts and are over-utilised, they may hinder students from developing problem solving skills. Sample questions in problem posing can be viewed as parallel to worked-out examples in problem solving. What impact will sample questions have on students' subsequent problem posing? So far, little research has been done to explore this research question.

What roles do sample questions play in mathematical problem posing? They may help problem posers to understand the relationships among the quantities within a given context if they have tried to solve them. Therefore, will the provision of sample questions improve students' performance in problem posing compared to those who have not been given sample questions? Alternately, sample questions may limit the poser's thinking, preventing him or her from seeing other relationships among the quantities that may exist. To what extent do sample questions affect students' problem posing when they are asked to pose more problems? The current study will address these questions.

METHOD

Participants

A total of 119 sixth grade students from a primary school in a city in Central China participated in the study. Their average age was 11 years and 7 months. Fifty eight percent of them were girls.

Problem-posing Tasks

To examine students' problem posing, we developed an instrument with five PP tasks concerning rate. In this paper, we shall only report the results obtained from one of the PP tasks (see Appendix). The task was constructed in two forms, A and B. Form A had a sample question, but Form B did not. Otherwise, the two forms of the task were identical. The two forms were randomly distributed to students. About 52% of the students (62) answered Form A, and the other 48% of the students (57) answered Form B. Because Form A had a sample question, students only needed to pose two additional problems. Form B required students to pose three problems.

Data Analysis.

Problems posed by the students were analysed in four steps: (1) Determine whether the posed problem is mathematical. (2) Determine whether the posed problem is solvable. (3) Code the posed problems as SQ, parallel problems, or other. Previous studies showed that students usually pose parallel problems and their subsequent problems are usually related to their previous posed problems (Silver & Cai, 1996). The SQ in Form A is: If the older brother makes all the greeting cards, in how many days can he finish the task? Following the SQ, the following two problems may naturally come out: (Parallel Problem 1): If the younger brother makes all the greeting cards, in how many days can he finish the task alone? (Parallel Problem 2): If they work together, in how many days can they finish the task? Following these two problems, students may pose problems like: If they work alone, how many more days does the younger brother take than the older brother? If they work together and finish the task, how many cards does the older brother make? How many cards does the younger brother make? How many more cards does the older brother make than the younger brother? Since these problems did not appear as frequently as the first two, we did not include the results at this stage. (4) We examined the complexity of the posed problems by analysing the number of steps needed to answer them. Number of steps is often taken as a measure of complexity of word problems in mathematics (Zhu & Fan, 2006). We are fully aware that a problem might be able to be solved in multiple ways. To code the number of steps, we focused on the minimum number of steps to solve a problem. Below are two examples.

Example 1: If the younger brother makes all the greeting cards, in how many days can he finish the task? (The solution to this example is: $450 \div 10$. The number of steps is 1.)

Example 2: How many fewer days does it take if they work together compared to the time taken if the older brother does the job alone? (The solution to this example is: $450 \div (15+10) - 450 \div 15$. The number of steps is 4.)

RESULTS

All of the posed problems ($2\times62+3\times57 = 295$) were mathematically meaningful and over 96% of them were solvable. All of the remaining analyses were based on the solvable problems.

The Impact of the Sample Question on Posed Problems

Sixty out of 62 students answered the SQ in Form A correctly. Of the two whose answers were wrong, one chose the correct operation but made errors in computation. The other chose the wrong operations.

The data in Table 1 show the number of problems posed of each type: same as the SQ, same as one of the above 2 parallel problems, or other problems. More students who took Form A posed problems similar to the SQ than those who took Form B. For example, 71% of the students (44) taking Form A posed Parallel Problem 1, while only 47% of those taking Form B did so ($z = 2.62$, $p < 0.01$). However, the percentages of students who posed Parallel Problem 2 were not significantly different under the two conditions.

Problem Categories	With SQ		Without SQ		
	PP-1	PP-2	PP-1	PP-2	PP-3
Sample Question: If the old brother makes all the greeting cards, how many days can he finish the task?	NA	NA	22	3	3
Parallel Problem 1: If the younger brother makes all the greeting cards, how many days can he finish the task?	40	4	2	22	3
Parallel Problem 2: If they work together, how many days can they finish the task?	13	37	15	7	21
Others	7	20	15	22	28

Note: PP-1 means the first PP task. NA = Not Applicable.

Table 1: Number of posed problems same as SQ and the two parallel problems or others under the conditions with and without SQ

The sample question was a one-step question. It is not strange that students posed the analogous Parallel Problem 1 as the first posed problem. For the second posed problem, about two-thirds of the students who took Form A posed Parallel Problem 2. Including the students who posed Parallel Problem 1 for their first problem, slightly more than 80% of the students who took Form A posed Parallel Problem 2. In total, a higher percentage of students posed Parallel Problem 2 than posed Parallel Problem 1 although the difference did not reach a significant level.

It is encouraging to find that 22 students (39%) who took Form B posed the same problem as the sample question for their first problem. Did they just try to figure out the relationships between the first two givens? It is interesting to see also that exactly 22 students who took Form B posed Parallel Problem 1 for their second problem. Did they simply apply analogical thinking for this posed problem? Similarly, a higher percentage of students who took Form B posed Parallel Problem 2 than posed Parallel Problem 1 ($z = 3.08$, $p < 0.01$). This means that the students who took Form B did figure out the relationships among the three given numbers even though they were not provided with the sample question.

Numbers of Steps Needed to Answer the Problems

Table 2 shows the mean number of steps that were needed to answer the questions posed by the participants. Similar to previous studies (Silver & Cai, 1996), the complexity of the problems increased as the students posed additional problems. For the students who took Form A, the complexity of the problems increased significantly from the first to the second posed problems ($t = 6.01$, $p < 0.001$). Repeated measures ANOVA indicated the complexity of problems posed by students who took Form B increased as well ($F(1.79, 100.30) = 15.00$, $p < 0.001$).

	PP-1	PP-2	PP-3
With Sample Question (n=62)	1.37 (0.79)	2.08 (0.87)	NA
Without Sample Question (n=57)	1.54 (0.91)	1.56 (0.93)	2.35 (1.30)

Note: PP-1 means the first PP task. NA = Not Applicable.

Table 2: Mean (SD) of steps to answer problems posed to tasks with and without Sample Questions

Pair-wise comparisons indicated that the mean number of steps of the third problems posed by students taking Form B was significantly higher than for their first and second problems. However, the numbers of steps of the first and the second problems posed by students taking Form B were not significantly different. Three t-tests were also conducted to compare the number of steps of problems posed by the students who took the two forms. The difference in the numbers of steps of the first problems posed by the two groups of students was not significant. However, it was significant for the second problems ($t = 3.15$, $p < 0.01$). Although the steps of the third problems posed by students taking Form B were higher than those of the second problems posed by students taking Form A, the difference was not significant.

CONCLUSIONS AND ADDITIONAL ANALYSIS

This study found that the provision of SQs affected students' problem posing in the following two ways: (1) students provided with SQs did pose parallel problems more frequently than students not provided with SQs; (2) students provided with SQs posed second problems with more steps than students not provided with SQs. Those students who were not given SQs posed third problems with similar numbers of steps as second problems posed by the students given SQs. This suggests that when there are not SQs provided, students should be given opportunities to pose at least three problems so that they can develop a better understanding of the given context and pose more complex problems.

It was decided to use the number of steps to examine the complexity of problems posed by the students. However, there are some drawbacks to this approach. For the given three numbers, addition and subtraction involving 15 and 10 is also meaningful. For example, students may pose problems like, "How many cards will the two brothers make in 1 day if they work together?" "How many more cards does the older brother make in 1 day than the younger brother?" Students may also add further information (e.g., number of days one works), and pose a question like, "How many cards will one make after n days?" Multiplication would be used for this kind of question. All these questions are one-step problems. However, the thinking involved is different. The use of number of steps as the measure of complexity leaves out such rich information about students' thinking. Further in-depth analysis with the categories of operations involved may provide a fuller picture of students' cognitive processes involved in mathematical problem posing. During the presentation of this research report at the 2015 PME conference we will include the results from the analysis of operations involved.

SIGNIFICANCE

Despite the interest in integrating mathematical problem posing into classroom practice, our knowledge remains relatively limited about the cognitive processes involved when solvers generate their own problems and the instructional strategies that can effectively promote productive problem posing in classroom. Although we know that students and teachers are capable of posing mathematical problems, we have a considerably less fine-grained understanding of how they go about posing those mathematical problems in any given situation (Cai, Jiang, et al., in press). Some researchers have identified general strategies students may use to pose problems. Others have explored some of the variables that may have an impact on students' problem posing. However, there is not yet a study that examines the impact of sample questions on students' problem posing. This current study contributes to our understanding about the cognitive processes of problem posing from one point of view.

The more that teachers know about their students' thinking, the better equipped they are to help their students develop (Cai, 2005). However, there is much work needed to connect research-based understandings of student cognition to teachers' practice. In elementary mathematics textbooks, some problem-posing tasks have sample questions, but other problem-posing tasks do not (Cai, Jiang, et al., in press). Thus, it is important to know the impact of sample questions on students' problem posing. Therefore from this perspective, this study has the potential to help us develop instructional strategies that can effectively promote productive problem posing in classrooms.

Appendix: The Problem-Posing Task

Form A. Daddy bought materials that could be used to make 450 greeting cards. The older brother can make 15 cards in one day, whereas the younger brother can only make 10 cards in one day.

a. If the older brother makes all the greeting cards, in how many days can he finish the task?
b. Please pose two more mathematical questions that could be answered with the information presented above.

Form B. Daddy bought materials that could be used to make 450 greeting cards. The older brother can make 15 cards in one day, whereas the younger brother can only make 10 cards in one day. Please pose three mathematical questions that could be answered with the information presented above.

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RELATION BETWEEN MATHEMATICAL REASONING ABILITY AND NATIONAL FORMAL DEMANDS IN PHYSICS COURSES

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It is widely accepted that mathematical competence is of great importance when learning physics. This paper focuses on one aspects of mathematical competence, namely mathematical reasoning, and how this competency influences students' success in physics. Mathematical reasoning required to solve tasks in physics tests, within a national testing system, is separated into imitative and creative mathematical reasoning. The results show that students lacking the ability to reason creatively are more likely not to do well on national physics test, thus not fully mastering the physics curricula. It is further discussed how the high demands of creative mathematical reasoning in physics tests stand in contrast to what is known about the educational practices in mathematics and physics in upper secondary school.

INTRODUCTION

Many scholars discuss the importance of understanding how mathematics is used in physics and how students' mathematical knowledge affects their learning of physics, e.g., Basson (2002) who mentions how difficulties in learning physics not only stem from the complexity of the subject but also from insufficient mathematical knowledge, Bing (2008), in his discussion of the importance of learning the language of mathematics when studying physics, as well as Redish and Gupta (2009), who emphasise the need to understand the cognitive thinking of experts in order to teach mathematics for physics more effectively to students.

According to the Swedish National Agency for Education (2009a) a common activity in physics classes is students using physics laws and formulas to solve routine tasks. The most common homework is to read in the textbook and/or to solve various tasks posed in the book, and sometimes to memorise formulas and procedures (ibid.). Similar results are described by Doorman and Gravemeijer (2009), who notice that most of the attention in both physics and mathematics in school is paid to the manipulations of formulas instead of focusing on why the formulas work. Redish (2003) states that practice, in the meaning that students just solve various tasks, is necessary but not enough to develop a deeper understanding of the underlying physics concepts. Students must learn both *how* to use the knowledge and *when* to use it.

The impact of mathematical reasoning *on mathematical* learning has been discussed and studied from multiple perspectives. Schoenfeld (1992), for example, points out that a focus on rote mechanical skills leads to poor performance in problem solving in contrast to the performance of mathematically powerful students. Lesh and Zawojeskij (2007) discuss how emphasising low-level skills does not give the students the abilities needed for mathematical modelling or problem solving, neither to draw upon interdisciplinary knowledge. Students lacking the ability to use creative mathematical reasoning thus get stuck when confronted with novel situations, and this negatively influences their possibilities to learn (Lithner, 2008). Since mathematics is a natural part of physics, it is reasonable to assume that the ability to use mathematical reasoning is an integral part of the physics knowledge students are assumed to achieve in physics courses.

FRAMEWORK

During studies on how students engage in various kinds of mathematical activities, Lithner (2008) developed a framework for characterising students' mathematical reasoning. The framework distinguishes between *creative mathematical founded reasoning* (CR) and *imitative reasoning* (IR). To be regarded as CR the following criteria should be fulfilled: **i. Novelty**. A new reasoning sequence is created or a forgotten one is recreated. **ii. Plausibility**. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible. **iii. Mathematical foundation**. The arguments made during the reasoning process are anchored in the intrinsic mathematical properties of the components involved in the reasoning (Lithner, 2008, p. 266).

Reasoning categorised as IR fulfils: **i.** The strategy choice is founded on recalling a complete answer. **ii.** The strategy implementation consists only of writing it down (Lithner, 2008, p. 258), or **i.** The strategy choice is to recall a solution algorithm. The predicted argumentation may be of different kind, but there is no need to create a new solution. **ii.** The remaining parts of the strategy implementation are trivial for the reasoned, only a careless mistake can lead to failure (ibid. p. 259).

In the application of the framework for the analyses described in this paper, an additional category, defined in Johansson (2103), is used. This category consists of those tasks that can be solved by only using physics knowledge; and this category is called *non-mathematical reasoning* (NMR). Physics knowledge is here referred to as relations and facts that are discussed in the physics courses and not in the courses for mathematics, according to the syllabuses and textbooks, e.g. that angle of incidence equals angle of reflection.

RESEARCH QUESTIONS

There is a significant amount of educational research on the relation between the school subjects of mathematics and physics that support the necessity of different mathematical competencies when learning physics. However, no studies on what type of mathematical reasoning is required of physics students were found. As an approach to the assumption that students' ability to reason mathematically affects how they master the physics curricula, this study use a previous analysis (Johansson, 2013) of the mathematical reasoning requirements to solve tasks in physics tests together with actual students' results on the same tests.

The Swedish national physics tests are the government's way of concretising the physics curricula. Thus, the requirements of mathematical reasoning to solve tasks in national physics tests should capture the mathematical reasoning that is required to master or fully master the curricula. The goals and the subject descriptions in the Swedish curricula of what it means to know physics are quite rich and are highly in accordance with the content and cognitive domains in the TIMSS Assessment framework (Garden et al. 2006; Swedish National Agency for Education, 2009b). This alignment with TIMSS suggests that the results from this study are relevant to an international context.

By addressing the questions: *Is it possible for a student to get one of the higher grades, Pass with distinction and Pass with special distinction, without using CR?*, and *If it is possible, how common is it?*, this study examines how the universal requirement of a mathematical reasoning competency to master the physics curricula relates to a specific assessment system's formal demands, in this case Sweden's.

METHOD

The empirical data consisted of student data from eight randomly chosen Swedish national physics tests for upper secondary school, and the tasks in the tests. There are mainly two different physics courses in the Swedish upper secondary school. Physics A that is compulsory for all natural science and technology students and Physics B that is an optional continuation. The tasks had previously been categorised according to mathematical reasoning requirements (Johansson, 2013), and together the tests comprised 169 tasks. The tests, which are classified to not authorised users, and the student data were used by permission from Department of Applied Educational Science at Umeå University, the department in charge of the National Test Bank in Physics. Student data come as excel sheets, one sheet for each test. The sheets contain information about individual students' grade, whereas the grade is one of the following: *Not Pass* (IG), *Pass* (G), *Pass with distinction* (VG), and *Pass with special distinction* (MVG). Further information in the sheets are individual student's scores on each task separated in G- and VG-scores, and their total score on the tests. No names of the students are present in the sheets, instead each student has got an ID-number. The IDs are unidentifiable for anyone outside the Department of Applied Educational Science at Umeå University, so data could be considered anonymous. The number of student data for each test varies from 996 to 3666.

For each test there are certain score levels the students need to attain to get a certain grade. To get the grade MVG, students need to fulfil certain quality aspects besides the particular score level. To decide if it is possible for a student to get one of the higher grades, VG or MVG, without using any kind of CR, each test was first analysed separately. This analysis consisted in comparing the score level for each grade with the maximum scores that are possible to obtain, given that the student only has solved (partly or fully) IR- and/or NMR- tasks. The available student data did not give any information about which of the qualitative aspects required for MVG the students have fulfilled, but the data sheets included students grades, thus MVG could be included in the analyses as one of the higher grades. After analysing if it is possible at all to receive the grades VG or MVG without solving any CR-tasks, students' actual results on the categorised tasks for those particular tests are summed up. The proportion of students who only got scores from IR- and/or NMR-tasks is then graphed with respect to the different grades.

RESULTS

Table 1 shows how the scores, possible to obtain on each of the eight tests that were analysed, are distributed among the reasoning categories IR and NMR. The table also includes the levels for the grades G, VG and MVG. The notation for the scores follows the convention G/VG.

Test	Max score (G/VG)	Min required score for G	Min required score for VG	Min required score for MVG	Max scores for IR-tasks	Max scores for NMR-tasks	Max score possible without CR-tasks
Physics A May 02	43 (26/17)	12	25 (with at least 6 VG scores)	25 (with at least 12 VG scores)	12/0	3/3	18 (with 3 VG)
Physics A Dec 04	40 (23/17)	12	24 (with at least 5 VG scores)	24 (with at least 12 VG scores)	14/3	3/3	23 (with 6 VG)
Physics A May 05	38 (22/16)	12	24 (with at least 6 VG scores)	24 (with at least 12 VG scores)	12/3	8/4	27 (with 7 VG)
Physics B May 02	48 (23/25)	12	27 (with at least 7 VG scores)	27 (with at least 13 VG scores)	11/4	2/0	17 (with 4 VG)
Physics B May 03	43 (23/20)	12	25 (with at least 6 VG scores)	25 (with at least 13 VG scores)	12/8	5/1	26 (with 9 VG)
Physics B May 05	44 (22/22)	12	25 (with at least 6 VG scores)	25 (with at least 12 VG scores)	8/5	7/2	22 (with 7 VG)
Physics B Feb 06	43 (22/21)	12	25 (with at least 7 VG scores)	25 (with at least 13 VG scores)	11/7	9/9	36 (with 16 VG)
Physics B April 10	44 (24/20)	12	25 (with at least 6 VG scores)	25 (with at least 12 VG scores)	9/4	4/1	18 (with 5 VG)

Table 1: Analysis of the distribution of G- and VG-scores among IR- and NMR-tasks.

For example, for the Physics A test from May 02 is the maximum score 43, and of these scores are 26 G-scores and 17 VG-scores. To pass this particular test a student has to have at least 12 scores, it does not matter if these scores are G- or VG-scores. To get the higher grade VG, a student has to have at least 25 scores and at least 6 of these scores have to be VG-scores. To get the highest grade, MVG, a student has to have at least 25 scores and at least 12 scores of these have to be VG-scores. As mentioned above, students also have to fulfil some additional quality aspects to achieve the grade MVG. Further, for the Physics A test from May 02, if a student only solves all tasks categorised as IR, he/she can obtain at most 12 G scores. If a student only solves all tasks categorised as NMR, he/she can obtain 3 G-scores and 3 VG-scores.

Solving all IR- and NMR-tasks thus result in total 18 scores of which 3 are VG-scores. The scores for the rest of the analysed tests are presented in the same way.

In three of the eight tests (highlighted in Table 1) it is possible to get the grade VG by solving tasks not requiring any CR. In one of these tests, Physics B from February 2006, it is with respect to score level possible to obtain the grade MVG by solving only IR- and NMR-tasks. The analysis does not reveal anything about whether the requirements of the qualitative aspects for MVG are possible to fulfil by solving only these kinds of tasks.

Figure 1 illustrates the proportion of students on the three highlighted tests in Table 1 who only had solved IR- and/or NMR-tasks graphed with respect to their grades on the tests.

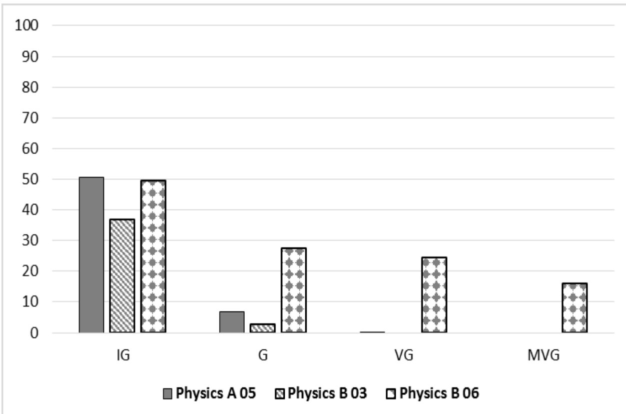

Figure 1: Proportion of students who only solved IR- and/or NMR-tasks with respect to the different grades.

It turned out that it is not frequently occurring that a student gets a higher grade than G by only solving these kinds of tasks. In the test for Physics A from 2005, only 0.17 % of the students got a higher grade; and in the Physics B test from 2003 none of the students got higher grades than G. The Physics B test from 2006 seems to be an exception though, since 25% of the students taking this test got a VG and 17% got a MVG. The analysis of how the scores are distributed among the reasoning categories for the different tests shows that the Physics B test from 2006 contains a lot more scores in the NMR category than any of the other tests (see Table 1). The total scores possible to obtain by only solving NMR-tasks are 18; nine of these are VG-scores, which is more than enough to fulfil the requirement for a VG (minimum 7 VG).

DISCUSSION

The analysis shows that it is possible to receive a higher grade than G by using only IR and NMR on three out of eight tests. When this result is compared with student data it is revealed that not using any CR, still receiving a higher grade, only occurs on one of the eight tests. This particular test, for which this occurs, is slightly different compared to the other tests with respect to how the scores are distributed among the reasoning categories (see Table 1). Further analysis of the test shows that tasks where it is possible to show the qualitative aspects required for the highest grade can be solved without using any mathematical reasoning i.e. these tasks are in the NMR category. This explains the higher frequency of students receiving the higher grades by using only IR and NMR, compared to the other tests.

The analysis of the tests furthermore shows that it is impossible to pass six of the eight tests without solving any tasks requiring mathematical reasoning. As seen in Table 1 it is only on the tests Physics A, May 05 and Physics B, Feb 06 a student can get at least the score 12, which is required to pass a test, by only solving NMR-tasks. These results strengthen the outcome from the author's previous study, which are that the ability to reason mathematically is an important competency and an integral part when taking physics tests (Johansson, 2013).

Mathematical reasoning is defined as a process to reach conclusions when solving tasks (Lithner, 2008). When students have the ability to use creative mathematical founded reasoning, they know how to argue and justify their conclusions and they can draw on previous knowledge. The result in the present study shows that CR is required to succeed on most of the physics tests. The alignment between the TIMSS framework and the Swedish policy documents suggests that this is a universal demand on upper secondary physics students. Viewing the physics tests from the National Test bank as an extension of the national curricula, one can assume that students' results on the tests are a measure of their knowledge in physics. It is well known that a focus on IR can explain some of the learning difficulties that students have in mathematics. The results above show that a focus on IR when studying physics in upper secondary school will make it hard for the students to do well on the physics tests, thus fully mastering the physics curricula. Therefore, a reasonable conclusion is that focusing on IR can hinder students' development of knowledge in physics, similar to results found about mathematics, and a creative mathematical reasoning competency can be regarded decisive.

The argumentative side of mathematics, which is a reasoning based on intrinsic properties of the components involved in the task-solving process, seems to be an inseparable part of mastering physics. All students should have the same possibilities to achieve the goals in the physics curricula. Therefore, they ought to be given the opportunity in school to develop and practice this creative mathematical reasoning competency that is required. As mentioned in the introduction, it is common in the physics classes that students solve routine tasks and focus on manipulations on formulas instead of focusing on the conceptual understanding of the underlying principles (Doorman & Gravemeijer, 2009; Swedish National Agency for Education, 2009a). Although it is the physics perspective that is discussed in the above studies, it is reasonable to assume that if there is more focus on physics procedures than on the understanding of physics concepts, there is also little focus on creative mathematical reasoning.

It is not only the physics classes that might provide students the opportunity to develop a mathematical reasoning competency, this competency is of course relevant also in the mathematics classes. According to studies about the learning environment in mathematics classes, the focus is on algorithmic procedures and the environment does not provide extensive opportunities to learn and practice different kinds of reasoning (e.g., Boesen, Lithner & Palm, 2010). During observations of classroom activities it was shown that opportunities to develop procedural competency was present in episodes corresponding to 79% of the observed time; compared to episodes involving opportunities to develop mathematical reasoning competency, which were present in 32% of the observed time (Boesen et al., 2014). Also tests have an indirect role for students learning, both as formative, when students get feedback on their solutions, and as summative, when the character of the tasks give students indications of what competences that are sufficient for handling mathematical tasks. Analyses of teacher-made mathematics tests have shown that these focused largely on imitative reasoning, in contrast to the national mathematics tests, which had a large proportion of tasks requiring creative mathematical reasoning (Palm, Boesen, & Lithner, 2011). Altogether, the above discussion shows that students are provided limited opportunities to develop the creative mathematical reasoning competency that is formally required to master the physics curricula. The importance of the relation between mathematics and physics has been known for a long time. The result from the present study, that the ability to creatively mathematically argue and reason is decisive in order to fully master the physics curricula, should have implications on how the education is organised and carried out.

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TASK DESIGN: FOSTERING SECONDARY STUDENTS' SHIFTS FROM VARIATIONAL TO COVARIATIONAL REASONING

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Covariational reasoning is essential for secondary students, yet little is known about its development. Reporting on a study with five ninth grade students (~15 years old), this research documents a student's shift from variational to covariational reasoning. Recommendations for task design include: (1) Incorporate dynamic representations that can provide students' opportunities to attend to multiple changing quantities. (2) Include nontemporal quantities from the same measure spaces. (3) Provide students engaging in variational reasoning opportunities to interact with students engaging in covariational reasoning when making sense of task situations.

Despite the pervasiveness of the concept of change in the study of mathematics and science, secondary students may not form and interpret relationships between changing quantities—engage in covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002)—when reasoning about rate of change (e.g., Lobato, Ellis, & Muñoz, 2003) or interpreting graphs (e.g., Leinhard, Zaslavsky, & Stein, 1990). If students consistently engaged in covariational reasoning, their conceptions of rate of change would be more robust (e.g., Carlson et al., 2002; Thompson, 1994). However, students may engage in variational reasoning—envisioning only one changing quantity—when interpreting situations involving multiple changing quantities (Johnson, 2013). Yet, little is known regarding how students might shift from variational to covariational reasoning.

Dynamic computer environments are useful for investigating students' reasoning about changing quantities (e.g., Kaput & Roschelle, 1999), and secondary students have demonstrated positive affect when interacting with a dynamic computer environment (SimCalc Mathworlds) that incorporated time as one of the changing quantities (Schorr & Goldin, 2008). However, few environments incorporate changing quantities such that neither is time (nontemporal quantities), for example volume and height of liquid in a filling bottle (Thompson, Byerly, & Hatfield, 2013), which can provide students opportunities to form and interpret relationships between changing quantities (Johnson, in press).

In Spring 2014, using a dynamic computer environment involving a turning Ferris wheel, I conducted a small-scale, exploratory study investigating five ninth grade students' reasoning. Employing design experiment method (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), building from tasks I designed and piloted (Johnson, 2013, in press), I investigated the following questions: How do secondary students shift from variational to covariational reasoning when interacting with dynamic computer environments that involve nontemporal changing quantities? What design aspects of mathematical tasks foster such a shift in reasoning?

THEORETICAL FRAMING: STUDYING SHIFTS IN REASONING

When studying students' shifts in reasoning, I investigate changes in the focus of students' attention. For example, when making sense of a situation involving a bottle filling with liquid, a student may shift from attending to only the changing height of the liquid (variational reasoning) to attending to both the changing height and volume of the liquid (covariational reasoning). I distinguish between a shift in a student's reasoning (a change in a student's focus of attention) and a student's learning of a new mathematical idea (a change in a student's understanding). In particular, I am not suggesting that a student who has shifted her reasoning has developed new conceptual structures, but I do argue that shifts in students' reasoning could play a role in students' learning of new mathematical ideas. For example, to come to understand rate of change as a single entity that represents a relationship between varying quantities, a student engaging in variational reasoning would need to shift to covariational reasoning.

To theoretically frame this inquiry, I coordinate constructivist and sociocultural perspectives (Cobb, 1994). Drawing on a constructivist perspective, my unit of analysis is individual students' reasoning, with reasoning referring to purposeful mental activity in which an individual could engage (Piaget, 1970). Drawing on a sociocultural perspective, I account for conditions (e.g., task design principles) that could foster shifts in students' reasoning (Cobb, 1994), explaining how tasks might be designed and small group instructional settings might be organised to provide students opportunities to shift their reasoning.

WHAT WOULD A SHIFT FROM VARIATIONAL TO COVARIATIONAL REASONING ENTAIL?

When students shift from variational to covariational reasoning, tasks or task situations that, from a student's perspective, once involved only variation (one changing quantity) now involve covariation (quantities changing together). By quantity I mean an individual's conception of a measurable attribute of an object (Thompson 1994), which is not synonymous with determining a particular amount of measure. For example, one can envision measuring the height from the ground of a Ferris wheel car without actually determining particular amounts of height.

Shifts from variational to covariational reasoning can occur within tasks, across tasks, or across task situations. By tasks I mean problems that are purposefully designed for a particular audience (Sierpinska, 2004). By task situations I mean common experiences in which students have might have engaged or which students could envision occurring (e.g., riding a Ferris wheel or "filling" shapes with area), used to unite multiple tasks. Task situations I have used include filling bottles (Johnson, in press), shapes "filling" with area (Johnson, 2013), and a turning Ferris wheel.

DESIGNING THE FERRIS WHEEL ENVIRONMENT

To provide students opportunities to form and interpret relationships between changing quantities, using Geometer's Sketchpad Software (Jackiw, 2009), I designed a dynamic computer environment that incorporated dynamically linked representations of nontemporal quantities (Figures 1 and 2). The Ferris wheel environment links an animation of a Ferris wheel and a dynamic Cartesian graph. To depict a Ferris wheel, I used a circle containing an active point, representing a car on the Ferris wheel. Represented quantities on the Ferris wheel animation (Figures 1 and 2, left) included *distance* the car travelled around the Ferris wheel (arc length, shown in Figures 1 and 2), *height* from the ground (vertical distance shown at left in Figure 1), and *width* from the center (horizontal distance shown at left in Figure 2).

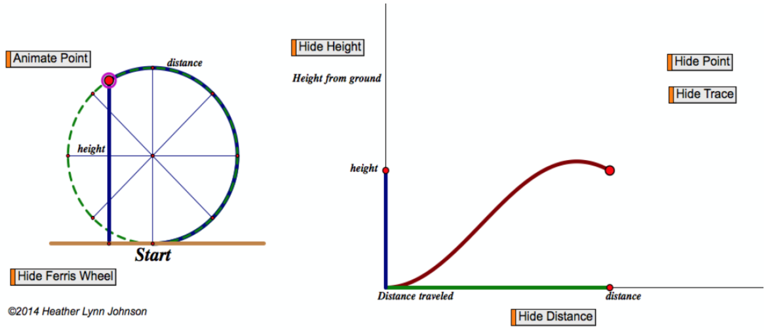

Figure 1. The Ferris wheel: Distance and height

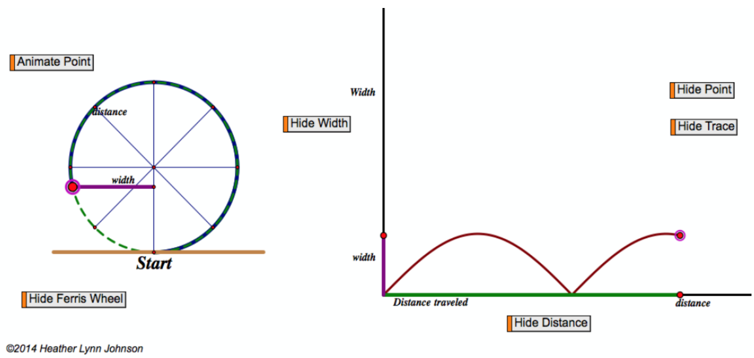

Figure 2. The Ferris wheel: Distance and width

To interact with the Ferris wheel environment, students could press Animate Point to move the car (active point) around the Ferris wheel or they could click and drag the car to control the motion. As a student moves the car around the Ferris wheel, the lengths representing distance and height (Figure 1, left) or distance and width (Figure 2, left) on the Ferris wheel animation dynamically change.

Linked to the Ferris wheel animation is a dynamic Cartesian graph containing added features not seen on typical Cartesian graphs. On each axis, a dynamic segment represents changing *distance* (green segment on horizontal axis; Figures 1 and 2), *height* (blue segment on vertical axis; Figure 1), or *width* (purple segment on vertical axis; Figure 2). Although the Cartesian graphs in Figures 1 and 2 show both the trace and the moving point, either can be shown separately. Notably, the Ferris wheel and graph can be hidden or shown to allow students to *make predictions without seeing the actual motion*—a key design feature of tasks fostering students' reasoning about changing quantities (Johnson, 2013).

RESEARCH METHODS

Researchers have described design experiments as "test-beds for innovation" (Cobb et al., 2003, p. 10). A goal of design experiment research is to develop theory that is closely tied to practice. Through this exploratory study I intended to (1) develop empirically based explanations regarding how students might shift from variational to covariational reasoning and (2) hypotheses regarding the design of tasks that might foster such a shift in reasoning.

Setting

Gutiérrez (2008) called for research that avoids focusing on gaps between groups of students from different races or socioeconomic statuses, but rather focuses on complexities within a group of students. I conducted this exploratory study at a 6-12 neighborhood school, serving primarily Mexican-American students, in a working class community in an industrial area of a large midwestern city in the United States. In 2013-14, 97.8% of students were eligible for free and reduced lunch and 96.4% of students were nonwhite. I have partnered with this neighbourhood school since 2012, having developed relationships with administrators, teachers, administrative staff, and students in the school. Although I am not from the community that the school serves, my longstanding relationship with stakeholders at the school has demonstrated my intent to collaborate in mutually beneficial ways that can support students' development of robust mathematical reasoning—a critical resource that students can carry with them beyond the bounds of a mathematics classroom or research study.

Task design and sequencing

I drew on variation theory (Marton & Booth, 1997) when designing and sequencing tasks and task situations through which students could experience differences that could provide them opportunities to change the focus of their attention. I incorporated two different task situations, the Filling Bottle and the Ferris wheel, with the Filling Bottle situation involving quantities from different measure spaces (height and volume) and the Ferris wheel situation involving quantities from the same measure space (distance and height or width). Within the Ferris wheel task situation, I incorporated different quantities (distance and height, distance and width), represented quantities on different axes of a Cartesian graph (e.g., distance represented on horizontal and vertical axes), and changed the orientation of the axes on the Cartesian graph (axes opening left rather than right).

Data Collection

I conducted a series of six clinical interviews with individuals, pairs, or trios of students. Table 1 shows the task situations, *tasks*, and represented quantities.

Interview	Task Situations/*Tasks*
1	Filling Bottle: *Volume and Height;* Cartesian Graph: vertical axis (volume), horizontal axis (height)
2	Ferris wheel: *Distance and Height;* Cartesian Graph: vertical axis (height), horizontal axis (distance)
3	Ferris wheel: *Distance and Height;* Cartesian Graph: vertical axis (height), horizontal axis (distance)
4	Ferris wheel: *Width and Height*; Cartesian Graph: vertical axis (width), horizontal axis (distance)
5	Ferris wheel: *Distance and Height*; *Width and Height*; Cartesian Graph: vertical axis (distance), horizontal axis (height and width, respectively)
6	Ferris wheel: *Distance and Height;* Cartesian Graph: vertical axis (distance), horizontal axis (height), with axes opening left Filling Bottle: *Volume and Height;* Cartesian Graph: vertical axis (volume), horizontal axis (height)

Table 1. Task Situations, by Interview

Each of the five students participating in the study was a ninth grade student (~15 years old) enrolled in an Algebra course, which was typical for ninth grade students at the school where I conducted this research. For each task in the Ferris wheel task situation, I implemented a five-part task sequence, shown in Table 2.

Part	Task Description: Ferris Wheel Task Situation
1	Predict then view how quantities change in the Ferris wheel animation.
2	Without viewing dynamic Cartesian graph, sketch a graph that represents a relationship between quantities in the Ferris wheel animation (distance and height or distance and width).
3	Predict then view how vertical and horizontal segments shown on the dynamic Cartesian graph relate to quantities in the Ferris wheel animation.
4	With the Ferris wheel hidden and only the moving horizontal and vertical segments showing on the Cartesian Graph, predict the car's location on the Ferris wheel.
5	Compare graphs sketched in Part 2 with the trace shown on the dynamic Cartesian graph.

Table 2. Description of the five parts of each task in the Ferris wheel task situation

Data analysis

Data analysis encompassed both ongoing and reflective analysis. Ongoing analysis, including reflective notes compiled after each interview, informed future interviews. I conducted multiple passes of analysis. In the first pass, I used open coding (Corbin & Strauss, 2008) to identify and describe data when students were focusing on change in one quantity (variation) or coordinating change in quantities (covariation), attending to the types of quantities, the prompts I used, and the interaction between students. In subsequent passes, I used comparative analysis, examining data when shifts in reasoning seemed likely to occur (e.g., parts of tasks that have potential to problematise the use of only one quantity to make predictions), then looking across all tasks for each student to trace shifts in students' reasoning within and across tasks.

RESULTS: A PROMISING EMPIRICAL FINDING

Prior to implementing the Ferris wheel task situation, I had not documented a student shift from variational to covariational reasoning in an empirical research study. To illustrate, I share data from Lucia and Sofia's work in Part 4 of Interview 4. I prompted Sofia to hide the Ferris wheel, choose when to stop the moving segments, then ask Lucia to predict the car's location. Lucia (Figure 3, right) predicted the car would be on the right side of the Ferris wheel, just before the width would have reached its longest amount. When prompted to explain, Lucia responded: "Cause, in the graph it's (purple segment representing width, Figure 2) like going up." Sofia (Figure 3, left) predicted the car would be on the left side of the Ferris wheel, just before the width would have reached its longest amount.

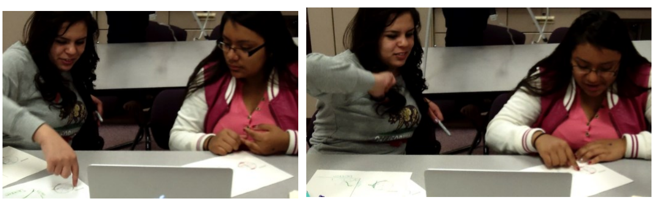

Figure 3. Sofia's prediction (left); Lucia's prediction (right)

When prompted to explain, Sofia responded: "Because the distance is really great here, and this distance (points to location Lucia predicted) is shorter." Next, I suggested we show the Ferris wheel, and after seeing the car's location, with a smile Sofia said: "See, I told you." Lucia grinned in response, moving her index finger up and down (Figure 4) and saying: "I basically focused on that (purple segment representing width, Figure 2)."

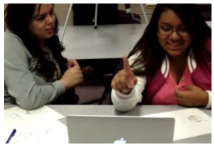

Figure 4. "I basically focused on that."

Once Lucia conceived of the task as involving multiple changing quantities, she engaged in covariational reasoning on other tasks involving the turning Ferris wheel. Specifically, she focused on how *both* the horizontal and vertical segments were changing (covariation), rather than focusing on how only one segment was changing (variation). Importantly, Lucia's shift provides empirical evidence of a student's shift from variational to covariational reasoning in a small group interview setting.

TASK DESIGN PRINCIPLES

I argue that three key design principles contributed to a student's shift in reasoning. First, incorporating *graphs with dynamic segments* (e.g., vertical and horizontal segments on graphs in Figures 1 and 2), drew students' attention to two changing quantities rather than just one. Second, incorporating *changing quantities from the same measure spaces* (e.g., height and distance) provided richer opportunities for students to attend to multiple changing quantities than did tasks incorporating changing quantities measured with different kinds of units (e.g., height and volume in the filling bottle task situation). Third, *pairing a student engaging in variational reasoning* (e.g., Lucia) *with a student engaging in covariational reasoning* (e.g., Sofia) provided students opportunities to discuss different ways in which they were making sense of the situation, thereby fostering a shift from variational to covariational reasoning (cf., Vygotsky, 1978).

IMPLICATIONS

When students engage in covariational reasoning, it expands not only their mathematical horizons, but also their ability to make sense of change in science and social science (e.g., the unemployment rate is decreasing more rapidly in 2015 than in 2014). Promoting students' covariational reasoning can support their success in algebra and open doors of opportunity that might otherwise have been closed. In fact, during the Spring 2014 study, Sofia said that working on the Ferris wheel tasks helped her to make sense of algebra problems in new, useful ways. Important, such tasks have potential to foster students' study of mathematics as an investigation of relationships between quantities rather than a pursuit of answers. The study of relationships, not the finding of answers, imbues students with mathematical power.

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LEADERSHIP: BUILDING STRONG LEARNING CULTURES IN REMOTE INDIGENOUS EDUCATION

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Indigenous students living in remote and very remote areas of Australia are most at risk of failing school mathematics. There are many reasons for this wide-spread and complex phenomenon. It is increasingly recognised that practices are needed to support mathematics learning that are both mathematically strong, pedagogically strong and are embedded in school-wide policies. Drawing on two cases from a much larger project, this paper draws on the leadership teams' rationale and descriptions of their practices. It is argued that for mathematics learning to occur, schools need to adopt a number of strategies that support quality learning in mathematics for Indigenous students. School leaders provide the vision for an enacted curriculum.

INDIGENOUS EDUCATION IN THE AUSTRALIAN CONTEXT

The project discussed herein draws on schools located in remote and very remote settings. Nationally there are recognised concerns with regard to the poor performance of remote and very remote Indigenous students in both literacy and numeracy. There is a considerable gap between Indigenous and non-Indigenous students and this gap widens as remoteness increases (Ockenden, 2014). The complexity and interaction of variables impacting on success for Indigenous learners need to be understood and addressed by schools if changes in outcomes are to occur. There are many documented reasons for this gap that often centre on patterns of attendance and engagement. Difficulties with attendance are often associated with previous negative experiences with school, relationships between teachers and students, poor perceptions of academic ability and racism (Ockenden, 2014). What is less well known in terms of success in mathematics for these schools is what works. There are many communities that are highly functional and enjoying success in many aspects of community life, including schooling. There is often a tendency in research and policy to highlight the challenges or problems with Indigenous education. This project draws on the success stories of schools that are producing positive outcomes in mathematics learning.

In remote areas of Australia, Indigenous people occupy significantly greater proportions of Indigenous people in comparison with the national figure of 3%. For example, approximately 35% of the Northern Territory population is Indigenous and as the populations move more into community and remote living, the percentages increase for local communities. Unlike mainstream settings where Indigenous people are in the minority, in remote contexts they are the dominant group. It is these schools, where there are considerable percentages of Indigenous students, that are the focus of the study. To be included in this study, schools must have at least 80% of their students being Indigenous. In most schools, this figure is closer to 100%.

2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 137-144. Hobart, Australia: PME.

There has been considerable funding allocated to projects to support mathematics programs that are interventionist in design – such as Yumi Maths, RoleMe or Quicksmart. These have been funded to nearly $13million over 4 years and are expensive innovations. Questions of impact and sustainability have yet to be determined. Similarly costs for schools to buy into these programs can be quite prohibitive; in some cases, it is as much as $80k for a two-teacher school for one year. This is a sizeable cost for a small school and may limit accessibility to the programs. The programs are largely standard mathematics programs that are no different from those found in mainstream schooling but with greater support for teachers. As Australia has a poor track record in equity achievement (McGaw, 2004), it is important to understand what works in remote settings, particularly in mathematics/numeracy. There is a growing awareness that many interventionist programs are failing to realise their potential.

There are many identified factors that impact on the quality of student learning experiences in these remote sites. Teachers are often early career teachers, young and remain in community for the length of their contract of employment – usually 2-3 years – and use the position to levy for a 'better' position in an urban setting. As neophyte teachers, usually in their first teaching position (Goos, Dole, & Geiger, 2011), the teachers are not only confronting the first year of teaching but also in a remote, isolated context working with families whose language and culture are very different from their own (Howard, Cooke, Lowe, & Perry, 2011). Increasingly, systems and employers are building programs to help support teachers in this critical period of their teaching career and in their cultural induction into remote community life. Some communities demand that prior to coming to community, teachers must be provided with cultural inductions and some language learning but this is very rare. In the past, it has been the case that some systems have recruited on the basis of the appointment being an adventure for the teacher rather than a career opportunity.

The majority of teachers are very early career teachers so the possibilities for mentoring are limited *in situ*. Furthermore, the tyranny of distance means that the provision of professional development and support is limited. This creates quite unique circumstances for the induction and on-going development of early career teachers in remote settings. The high turnover of teachers and leaders in remote communities also means that there is often a loss of knowledge as staff continually move through communities (Helmer, Harper, Lea, Wolgemuth, & Chalkiti, 2013). Building sustainability in programs is challenging with the regular turnover of staff.

What is clear, however, that despite considerable funding being allocated to interventionist programs to bring about success in mathematics learning for the most at-risk learners in Australia, there has been very little success in terms of measurable learning gains. In contrast to interventionist programs, the project cited in this paper recognises the possibility of teachers and educators working in remote contexts to bring about success. To document the work of successful practitioners working in the field, a large national study is seeking to identify elements that may be contributing to the success of Australia's First People who are most likely to perform poorly on standard measures of success in mathematics. This paper reports on elements of practice of two schools that are part of this study.

THE STUDY

The study employs an ethnographic case study approach where each site is developed into a case study report that outlines the practices adopted by the individual school. The case study is developed after a visit to the school in which three key data collection tools are employed - interviews with key personnel; observations of mathematics lessons; and document analysis of the site. The data collected at each site varies depending on the size of the school, and the focus of the site. Interviews with the leadership team provide a context of the school from their perspective and help set a rationale for the work of the school. Interviews with teachers and other school staff (such as numeracy coaches, curriculum leaders, Aboriginal staff and community members) provide perspectives and thick descriptions of the practices adopted at the school. Lesson observations provide detailed descriptions of how the practices are enacted at the lesson of the classroom. Triangulation between the 'big picture' from the leadership team, descriptions provided through the teacher interviews and observations of lessons provide the rigor to the case study reports. Data are entered into NVivo and coded. The analysis for this paper sought to explore the data from the two schools that are the focus of this paper. The queries undertaken through NVivo were varied and sought to find points of similarity and differences between the two schools. This paper draws on the interviews with the formal leadership team at two schools (three deputy principals) who were responsible for leading curriculum at two schools.

SYNOPSIS OF FINDINGS

In the following sections, I outline the practices from two schools. These two schools had developed school-wide cultural reforms that had "learning" as the focus of the reforms. The reforms differed in their focus –one on mathematics learning, the other on the culture of the school community. The practices adopted in the rollout of the two initiatives were very similar with a very strong intent for the school participants (teachers, students, community) to embark on a common journey. In both cases, the reforms were very clear in their intent and action to include all members of the school community to share in the vision of the school.

Both schools were in the same region of Australia, both were state (government) schools and both served primary and secondary school students. A summary of the schools' demographics can be seen in Table 1.

	School 2	School 1
Enrolments	418	148
Attendance rate	58%	79%
Teaching staff	38	20
Non-teaching staff	36	17

Table 1: Summary of the Two Schools' demographics

IMPETUS FOR CHANGE

Attendance and behaviour were key catalysts at both schools for developing reforms. Both of these factors were impacting on performance and the learning cultures at the two schools. School 2 was concerned that many students were not achieving levels commensurate with their age (particularly for students who attended regularly), and with the need for teachers to understand the scaffolding needed to cater for the diversity of learners in mathematics classrooms. A strong focus was to build the professional learner of teaching staff in relation to mathematics and pedagogy.

Deliah: So if a kid gets to Year 9 and has never encountered anything other than whole number at school in terms of expectations of what you're going to do, what does it take to teach them? Understand fractions? What does it take to…? Yeah, you've got to go back but going back, doing what you do in fractions with the year3 in the same way as what you would do it with the year 3 is not ok with the year 9's. The good news is, if you know what you're doing you can accelerate it very fast so they can go from next to nothing to having a good understanding because a year 9 brain is a bit different than an 8 year old one. That was the initial push. (School 2)

In contrast, School 1 had focused heavily on building a happy environment at the school where there were established expectations of behaviour. Being happy and supported at school was seen to lead to learning.

Catie: I think the other thing too … that the main thing, and it's always my belief, if the children are happy at school and they feel safe and secure and you've got a good relationship with your teacher, then the learning will happen. And it does take a long time to get to that point, but I feel like we're at that point and certainly for me, … we've been able to move past the behavioural stuff where you're just supporting people because the behaviour in the classrooms isn't conducive to learning. (School 1)

As with many remote schools, at both schools there was a relatively high turnover of staff; there were many neophyte teachers on staff, and many were not skilled in teaching mathematics to diverse learners. To achieve success in mathematics, the schools invested in targeted teachers whose role was to develop consistent approaches across all year levels and to support teachers to achieve these outcomes. Further support was also directed at supporting the Aboriginal education workers who took a number of roles including working with smaller groups of students on targeted teaching activities so that the students would be able to work in the larger classroom, supporting the teacher in class with activities, language and behaviour. In order to achieve these goals, the leadership team developed professional learning activities to enable the development of a consistent approach across the schools. In so doing, the school presented as a coherent whole school to the wider community.

School-Wide Policies and Practices

There was a sense that the schools needed some practices and policies supported by professional learning for the teachers that would bring about learning and cultural change. Both schools recognised challenges confronting families and communities but did not what this to be an excuse for providing a limited learning experience for students. While the foci of the two schools varied, there was a strong articulation on implementing mathematics curriculum that met high expectations and that there should not be any excuses for offering an impoverished or lower level curriculum.

Deliah: We talk about attendance, we talk about emotional things, we talk about trauma, and we talk about all of those things and even making the school a welcoming place and all of that. But … you are not exposing them to the curriculum that they're expected to learn at that level and yet we are marking them as if they have. That was my real thing, at the end of the first year that was my real thing... Once we got some basic little bit of maths in place in terms of some professional learning for some of those teachers, particularly the ones that were staying on. (School 2)

Similarly, the leadership team at School 1 recognised considerable community issues, but this was not an excuse for teachers to offer an impoverished learning experience.

Catie: You have to have high expectations. I think too often it's easy to go, "Well, there's all these social issues, there's things that we don't have control of out in the community, there's a lot of abuse, alcohol, domestic abuse, the violence, all of those sorts of things," and it's easy to go, "Well, that's a reason why these kids can't learn." (School 1)

There was also a strong sense that the school needed to work with the families and community. Ensuring that the families were aware of what the school required of them, in a partnership for their children's learning. Communicating with families and communities was an essential ingredient in the learning partnership.

Catie: And so if we've, we've been fairly successful in saying to our parents, "That's what we want. That's what we expect. That's what the children need to be successful here. If you can do that, then we can do this."

Both schools had adopted school-wide reforms and through a range of professional development activities had developed consistency in the approaches across the schools. This not only helped with consistency across the school, but also helped teachers support new teachers coming into the school. This change process was noted by members of both leadership teams as requiring considerable time (3-5 years) and then to be on-going to ensure that the approach remains consistent but also improved.

Quality Teachers

There is a sense that quality teachers make a difference to learning. Both schools recognised the value of quality teachers and proactively sought to employ teachers who were quality in terms of their teaching, dedication and experiences.

> Catie: …but it does mean that we've been able to pick up some really fabulous dedicated people. [We] have incredible staff here. I'm always still amazed at the amount of additional time that they're prepared to put in to make the school a nice place to be. (School 1)

There was also a strong recognition that the teachers should have strong mathematical content knowledge so that the teachers could scaffold the mathematics for the students. School 2 had a strong emphasis on content knowledge in their professional learning activities for teachers.

> Deliah: … what I really talked about is the maths content, getting the maths content on and … we were not overtly about pedagogy. An assumption of the "Getting it Right" strategy was that the specialist teachers we had to work with were already competent classroom teachers who had a repertoire of strategies, our job was, is to give them the mathematics to that sequence where it's going. (School 2)

The Getting it Right strategy is a strategy that was implemented across both schools. It meant that there was a Numeracy consultant in each school who worked with staff on the content of mathematics, particularly in terms of building and scaffolding the mathematics content for the students. For many of the teachers, content knowledge was not a strong point so considerable professional learning was targeted in this area.

Both schools recognised the high turnover of staff and their early career status meant that it was important to ensure continuity of programs and to have staff adequately skilled for the work they undertake in diverse settings. In both schools, the numeracy coordinator provided oversight of mathematics education and to ensure the professional learning for staff.

> Catie: …because of the nature of our school, we have quite a big turnover every few years, we have a lot of graduate teachers coming through so having a Getting It Right Numeracy (GIRN) person to plan and help the teachers to decipher the curriculum and exactly what they need to teach and how to teach it, I think has made a huge difference to our school. (School 1)

The curriculum coordinator (GIRN) at each school had a targeted role in terms of supporting teachers to develop mathematically strong learning experiences for the students. At the same time, assessment was integral to the process so that teachers could identify students' learning needs and target teaching to those needs. With the considerable diversity within a classroom, differentiation was commonplace but in both schools, the capacity of teachers to differentiate across a classroom was a skill that was the focus of considerable professional learning activities.

Deliah: One of the biggest things I found, because of that differentiation and so much variation within the course, the necessity to differentiate and typically what people were doing was like the same worksheet for everyone. A few kids might actually do it, a couple of kids, oh that's too easy and other kids rip it out, I'm outta here sort of thing. So finding ways of kind of differentiating and choice, one of the simplest ways is just give kids, same lesson, but give them a choice in terms of the difficulty level or the actual numbers they work with or whatever (School 2)

In terms of differentiation, there was some "streaming" but this was usually on the basis of attendance rather than achievement per se. Attendance, as noted by the participants, often correlates with behaviour. Students who attend regularly are often working at or above where their age-equivalent urban peers are working. In contrast, students whose attendance is low, poor or sporadic are often working below their age-equivalence and often create behavioural problems in classes.

Dennis: The average, our whole school attendance rate is about 76%. It's sitting on that, which is really exceptional for a remote school. The B group, who is the higher ability group, their attendance rate is around 83%. The A group, who is the lower group, their attendance rate's about 58%. So, yeah, to answer your question, absolutely, a huge correlation [between attendance and behaviour]. And these kids have missed a lot of primary school and stuff as well. So some of them even in maths now are still doing, telling the time and things like that. (School 1)

THE CHANGE PROCESS

What was clear from the interviews at both the schools was that change takes time. The elements of the changes adopted at each school of which some are noted in this paper took time to be embedded. It was also the case, that over time, the practices that were being implemented needed to be reflected upon and refined.

Catie: . .that was nearly five years ago that, you know, we first introduced it. And we've just kind of built on it every year, and everything that we do now we make a link to it. And so it's become, I mean we call it the School 1 Way ... This is just how we do things at School 1. And we actually had [consultant] back for the beginning of this year because we felt that we had, that group of people, a lot of them had moved on, and so I think there was only about maybe six or seven of us that were here originally, and so even though people had come in since then … we talked about all these things there wasn't that understanding. So having him back just reinvigorated it for us, and he helps you to generate a lot of new ideas and strategies and things. … I mean five years really has to be your minimum if you want to see any significant change in a school.

CONCLUSIONS

The leadership team at these two schools have provided a strong vision for the mathematics curriculum reforms, as well as on-going, professional learning of their teachers. The schools were very focused in their adoption of school-wide practices. What was clear was the leadership team recognised the issues that needed to be addressed and then developed school-wide approaches to address the issues in a very systematic way. Collaboration among the staff and with community was central to the process. Building a learning culture (for students and teachers) was central to both schools, as was a focus on mathematics. Both schools sought to identify the learner's needs and then build mathematics scaffolding for the students so that they could achieve expectations. High expectations were expected of learners and teachers.

Change is slow if it is to be embedded – five years at a minimum if real change is to be embedded. In remote schools where there is a high turnover over staff, with many neophyte teachers who require considerable support, building sustainable and successful cultures to support mathematics learning required the leadership team to undertake on-going professional learning for staff, and to build the skill set of the teachers. In some cases, teachers remained at the schools for longer than their nominated contracts (2-3 years) which helped to build sustainable practices.

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STORYTELLING AS A COGNITIVE TOOL FOR LEARNING THE CONDITIONAL PROBABILITY

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This research investigates how mathematical discourses develop while students learn conditional probability based on storytelling. Analysing classroom discourses through the lens of a commognitive framework reveals that storytelling may promote students' progressive mathematisation on the concept of conditional probability. As students try to resolve the commognitive conflicts caused by the story or problem, their discourses develop from the situational level to the referential level and to the general/formal level.

INTRODUCTION

Storytelling is widely used as a communication method for developing empathy with other people and for stimulating a creative thought. The use of the storytelling in mathematics classroom has been known to promote students' motivation, creative skills, and a mathematical attitude (Egan, 2005; Zazkis and Liljedahl, 2008; and Balakrishnan, 2008). However, few empirical studies have been conducted on how mathematics classes implemented based on storytelling help students learn the mathematical concept. This study explores the possibility of storytelling as a cognitive tool for learning the conditional probability by investigating how the discourses of students develop during mathematics classes on conditional probability implemented based on storytelling.

THEORETICAL BACKGROUND

Kwon et al. (2013) developed the *Model Mathematics Textbook Based on Storytelling for High School Students*. The model textbook presents progressively diverse situations and tasks borrowed from real life, in accordance with the development of a story complete with a plot while the solution reached in the process affects the plot of the story again. Therefore, all the situations presented in each chapter are organically interconnected. Active storytelling between a student and another student, students and their teacher, and students and a textbook is realized during the problem-solving process, thereby helping the students construct mathematical knowledge through the interactions enabled by the storytelling. The model textbook has much in common with the Realistic Mathematics Education (RME) in that it formulates knowledge progressively through active storytelling based on real-life cases that use non-formal solution strategies proposed by the students. Gravemeijer (1998) deconstructed the learning process in RME as a series of development processes from the situational level, to the referential level, to the general level, and to the formal level. The model at the referential level can be considered the model of the situation, whereas the model at

2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 145-152. Hobart, Australia: PME.

the general level can be considered the model for inference. The development from the previous stage to the next stage is achieved through reflection on the model in the previous stage.

Sfard (2008) considered mathematical discourse as containing mathematical words, visual mediators, routines, and endorsed narratives; and she considered mathematical learning the development of mathematical discourse, and the development of mathematical discourse as a meta-level development that creates a new discourse routine. Experience of commognitive conflicts is inevitable in the development of mathematical discourse. Commognitive conflicts are created between participants of a discourse with varying levels of academic development, or internally by an individual in the process of advancing his/her academic level. For such cases, a meta-level discourse on the discourses at the previous level is being developed. In other words, discourses are being developed at the meta-level by modifying the four elements of mathematical discourse through reflection on the discourses at the previous level and through an agreement process among the participants of the discourse towards a higher level (Sfard, 2008, 2012).

METHODOLOGY

One chapter that deals with probability in *the Model Mathematics Textbook Based on Storytelling for High School Students* was adopted as a material for the classes that lasted for two hours per day for a total of three days in July 2013. A total of 16 students volunteered for the classes and they were divided into four groups of four members each, with each group holding a group discussion and a class discussion. All the ideas shared during the class discussions were voice and video recorded under their permission.

The classes required students to investigate given tasks under diverse situations while reading an entire chapter of the story presented in the model textbook. As the classes required students to think narratively and to express their thoughts verbally, it would be appropriate to take on the commognitive framework proposed by Sfard (2008), in which the development of mathematical discourse is understood as a mathematical learning method. In view of this framework, mathematical discourse, i.e., the development process of mathematical learning, was analysed in accordance with the stages proposed in the occurrence model. The framework through which the level of the occurrence model can be determined from the commognitive perspective is presented in Table 1. The experts in the mathematics education field verified the validity of the analytical framework.

Level	Word		Visual mediator	Routine	Endorsed narratives
	Word	Change of use		Probabilistic judgment	
Situational	Probability	Passive use	Concrete, situational figures	Subjective or situational judgments	About the situation
Referential	Conditional probability	Routine driven	Mixture of mathematical symbols, concrete figures, and linguistic expressions	Numerical process but possibly inaccurate judgments and rituals	About numerical information
General	Conditional probability	Phase driven	Various representations such as mathematical symbols and Venn diagrams	Accurate numerical judgments and exploration (recall, construction, and substantiation)	About generalization of numerical information
Formal	Conditional probability	Object driven	Mathematical symbols relevant to the given context	Formal judgment, choice of proper strategies, and exploration (recall, construction, and substantiation)	About relationships between objects and symbols

Table 1: Analytical Framework of Learning Levels

RESULTS

Development to the referential level

A total of four characters-Hong, Louis, Hyeri, and the Teacher-appear in the story in the chapter on probability in the model textbook used in the classes. Hong and Hyeri are competing with each other in their love for Louis, a popular singer, and are either tangled in a conflict or a resolution. In the process, the Teacher helps them learn mathematics. The students also walk through the story, thereby learning the concept of conditional probability while investigating a task after a task tailored for each given situation.

In Situation 1, Louis experiences *the Dilemma of Monty Hall* while participating in a television quiz show. Hong participates in a research to help him win the quiz show. Hong experiences an internal conflict as to whether he should decide that it is more advantageous to switch to the door with a goat hidden behind it after the game presenter shows him the door, or to stay with his current choice. The students participating in the class also engaged in the discussion as to which option they should choose.

In our classroom discussion, a student named Eunjoo narrated that she would not change her choice of the door (endorsed narrative) because it is troublesome to go through the thought process all over again, which is subjective and situational judgement (routine). She did not opt to use the word 'probability' in the discourse (use of word) and kept using some of the descriptive illustrations presented in the textbook (visual mediator). In other words, Eunjoo's understanding of the concept of conditional probability remained at the situational level.

For the following task, a figure about *the Dilemma of Monty Hall* was presented in the textbook to help students make their choice. After Eunjoo explored how the presented figure might be interpreted mathematically, she presented her thoughts as follows.

269 Teacher: Oh, you said you would not change your choice early in our discussion.

270 Eunjoo: Ah, theoretically.

285 Teacher: So you cannot describe this verbally? Here we have a car, a goat here and a goat there, and the probability of choosing this was one-third in the beginning, which means this is one of the three and then?

286 Eunjoo: And given that the probability of choosing an empty door is two-thirds, if you choose here, then you are going to open here when you open the door, and then you will realize this is not the right choice. Then you will need to change, as a car was found there. So this adds up to one-third, and if you chose here, it adds another one-third, thereby totalling two-thirds. But if you choose to open here, it will add only one-third, so it would be advantageous for you to change your choice even if you did not choose anything in the first place.

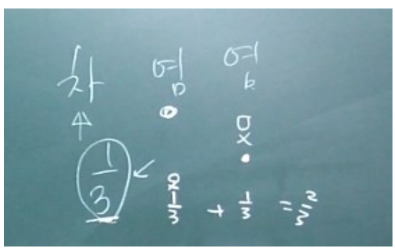

Figure 1: Visual mediator used by Eunjoo in Discourse 286

Even though Eunjoo failed to rigorously define the conditional probability, it is clear from Discourse 286 that she understood that the probability of Louis winning the prize would be two times higher when he changed his choice than otherwise if the previous event was restricted to an occasion in which she was lucky. The visual mediator was codified as shown in Figure 1 in the process of explaining the content. Though the code is not elaborate and accurate, it contains a routine in which Eunjoo made a probabilistic judgment complete with a rudimentary level of mathematical interpretation. Helped by such mathematical interpretation, she came up with an endorsed narrative of the numerical information that changing the doors is two times more advantageous for her. Therefore, it can be concluded that Eunjoo's development level is inching along from the situational level to the referential level.

The aforementioned discussion can be thought to have been triggered by the commognitive conflict created due to the difference in the students' understanding of the probability in the total event. To help them solve this conflict, the Monty Hall situation presented in the textbook was simulated under the auspices of the teacher. After all the class participants defined conditional probability, it was deduced through a factual description that it is two times more advantageous to change the choice of doors by applying the concept of conditional probability. As the teacher led the process of inducing the agreement at the referential level, it can be said that the routine was ritual. However, it still helped the students understand the need to adopt the conditional probability by allowing them to experience the commognitive conflict in a real-life situation.

Development to the General and Formal Levels

A situation was presented in the textbook in which Loius was hospitalized for an illness. There, he had to choose to be injected with hepatitis-B vaccine or not. In the process of making a choice, the textbook directs the students to interpret the contextual meaning of $P(A|B)$ and $P(A \cap B)$, where A indicates the event of Louis having an antibody and B indicates the event of the antibody test being turned out to be positive. An excerpt of the conversation of the students on this matter follows.

119 Eunjin: So this one and that one are identical?

120 Jihee: Eh, I believe so… or not.

126 Jihee: It was divided as such. No, it was divided by $P(B)$.

127 Eunjin: That is right. Look! P.. over $P(B)$ … like this.

159 Jihee: This and that one. What is that? Now add up the two to get this one (*writing* $P(A \cap B) = P(A|B) + P(B|A)$*)*.

160 Eunjin: Eh, I don't know but it looks good though.

161 Eunjin: It looks right.

162 Eunjin: Ah! Then let's try it once. Let's turn them into equations.

224 Eunjin: This one is an event wherein an antibody was discovered after the test among the right cases, whereas that one is correct and the antibody exists.

227 Jihee: To sum up, this event signifies that the test was performed accurately and the antibody was discovered.

229 Jihee: And does this one signify that the test was performed accurately when the antibody was discovered? Is this an event wherein this one… and the antibody were present at the same time?

230 Eunjin: Come to think of it, it looks right.

232 Jihee: They are identical when represented in pictures.

233 Eunjin: Identical when represented in pictures?

234 Jihee: Venn diagram

235 Eunjin: How do you represent the probability with pictures?

The students struggled to recognize and use $P(A|B)=\frac{P(A\cap B)}{P(B)}$, a mathematical definition of the conditional probability they learned through *the dilemma of Monty Hall* (126), but they failed to interpret this definition correctly given the new context of inoculation with a preventive vaccine, thereby concluding that the contextual meaning of $P(A|B)$ and $P(A\cap B)$ are identical (119). In other words, the students remained at the stage of routine driven use of conditional probability (word). $P(A|B)=\frac{P(A\cap B)}{P(B)}$ is a symbolic representation of the conditional probability and the task in which the respective contextual meaning of $P(A|B)$ and $P(A\cap B)$ are interpreted according to the given situation requires contextual understanding of the conditional probability. In this study, the students experienced a commognitive conflict caused by the tension between the symbolic representation and the contextual understanding of the conditional probability. They confirmed, while in the process of checking their hypothesis $P(A\cap B)=P(A|B)+P(B|A)$ in accordance with the mathematical definition of each probability (159-162), that the hypothesis missed the mark. In the end, they both recognized that they are required to induce semantic differences between the two given equations. Finally, they reconciled the contextual understanding with the symbolic representation through continuous reflection on the mathematical definitions (224 and 227). Furthermore, they both attempted to expand the scope of their thoughts by visualizing the conditional probability (232 and 235).

The students arrived at the endorsed narrative on their own by proving the hypothesis they established, reflecting on the definition of conditional probability, and attempting to come up with new visual mediator. The preceding observations clearly show that the students' routine was ritual in the process of their drawing up an agreement at the referential level, whereas the routine in the process of their drawing up an agreement at the general level was exploratory. In other words, the students transitioned the routine from the ritual to an exploration.

In the following stage, the students were asked to induce the multiplication theorem of the probability in the model textbook. In the process of inducing the multiplication theorem and presenting it to the class participants, Minjoo presented her perceived implications of $P(A|B)$ and $P(A\cap B)$ as follows.

583 Minjoo: So $P(A\cap B)$ signifies this part among this and that set if they are considered sets, whereas $P(A|B)$ signifies the same inner section. As we are talking about

B given the premise that it is A, it signifies the same inner section. As we assume A arbitrarily while talking about the entire set, the total field was narrowed down to A.

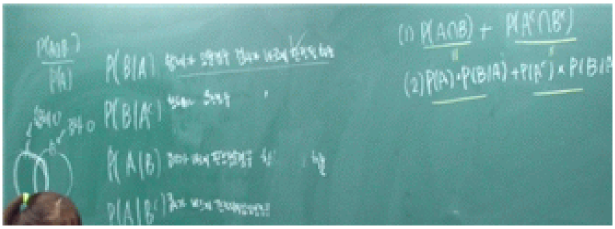

Figure 2. Visual mediators used by Minjoo in Discourse 583

584 Eunjin: Is it the same story you told earlier? Eunjin?

585 Eunok: Hmm…

588 Eunok: So we will calculate A first and among them…

589 Minjoo: We multiplied the probability that the test is positive with the probability that the antibody is present given the premise that an antibody exists.

590 Eunjin: You are repeating what you said already.

While listening to the explanation of Minjoo, Eunjin realized that Minjoo's presentation was identical to what she had already presented to the teacher, and she shared her finding with Eunok (584 and 590). Eunok seconded her opinion and reconfirmed what she said (588). It can be said that both Eunok and Eunjin understood the definition of conditional probability in the process of inducing the multiplication theorem and applying it. In other words, the students accurately perceived the significance of the terms on the conditional probability, used it without changing its significance in any contextual situation, and employed diverse visual mediators (Venn diagram, mathematical symbols, etc.) in the discussion process. The students not only successfully deduced the multiplication theorem from the formal definition of the conditional probability in the routine of probabilistic judgment but were also able to explain them in the linguistic context. Therefore, it can be concluded that the students demonstrated positive development to a level in which the general level and the formal level coexist.

CONCLUSIONS

This study was conducted to explore the development mechanism of students' learning levels, as suggested by the occurrence model, by analysing the discourses of students in storytelling-based probability classes from the commognitive perspective. The study confirmed that the students engaged in the commognitive reaction and struggled to construct their knowledge through discourses in the process of coping with the occurrence of a commognitive situation and resolving it. In other words, the students demonstrated a type of sequential development that starts from the situational level and goes further to the referential, general, and formal levels with regard to the concept of conditional probability. Therefore, it can be concluded that storytelling can be considered a useful learning aid for students in the cognitive domain.

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THE NATURE OF INTERVENTIONS IN WRITTEN AND ENACTED LESSONS

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The demand of interventions in daily lessons is high in the classroom, and curriculum programs make an effort to include resources for such interventions. Yet, there is no clear theoretical and practical guidance on daily interventions for both teacher and curriculum. This study examines interventions that are offered in written lessons from a range of elementary mathematics curriculum programs and those that teachers actually incorporate into instruction, aiming at understanding the nature of interventions embedded in daily lessons and the role of teacher and curriculum in classroom interventions. The results of the study highlight the importance of intervention resources in the curriculum and teacher role in recognising the affordances of resources to provide appropriate interventions.

INTRODUCTION

This study focuses on interventions *within* daily lessons that are designed to support students when they have difficulty understanding the instructional material or completing the assigned task. Teacher reactions to student difficulties can be based on planned or on-site decisions. In either case, these interventions provide short, prompt support situated within regular ongoing lessons along with the curriculum being used, as opposed to a long-term program segregated from daily lessons. The demand for interventions in daily lessons is high in the classroom, and curriculum programs make an effort to include resources for such interventions. Yet, there is no clear theoretical and practical guidance on daily interventions for both teacher and curriculum. This study examines interventions that are offered in a range of curriculum programs in the USA and those that teachers incorporate into instruction, in order to understand the nature of interventions embedded in daily lessons and the role of teacher and curriculum in these classroom interventions. Specific research questions are:

What kinds of interventions are available in the written lessons from a range of elementary mathematics curriculum programs?

What do teachers use among those available and in what ways?

What do teachers do when no interventions regarding observed student difficulty are available in the written lessons?

THEORETICAL PERSPECTIVES

Often, interventions are interpreted as special courses of instruction, usually with long duration, to promote important learning goals that typical classroom practice has had difficulty in supporting (Stylianides & Stylianides, 2013). These interventions are usually designed and tested through teaching experiments (e.g., Blanton, Stephens,

2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 153-160. Hobart, Australia: PME.

Knuth, Gardiner, Isler, & Kim, 2015; Thomas & Harkness, 2013), and such interventions utilise existing research and innovative approaches to redesign instruction for a particular topic and/or a specific pedagogical aim. In contrast, while steering daily instruction, teachers provide interventions moment by moment in order to accomplish lesson goals when they observe students struggling in understanding and using a particular concept to complete an assigned task or to solve a problem. Alibali, Nathan, Church, Wolfgram, Kim, and Knuth (2013) call this latter type of intervention a *micro-intervention* in that it occurs "as a lesson unfolds" at the micro level. Timely interventions are critical in enacting lessons productively, and our field needs to understand the nature of these interventions embedded in daily lessons.

There has been little research examining the nature of micro-interventions. Although they examined micro-interventions, Alibali et al.'s (2013) focus was mainly on non-verbal teacher actions in trouble spots, such as gestures. Other studies investigated some general approaches to interventions, such as student interactions and levels of mathematical content (e.g., Dekker & Elshout-Mohr, 2004). Nevertheless, previous research on interventions has not examined how teachers use curricular resources to intervene when students have difficulty with the main mathematical idea of the lesson.

Even though it is difficult to plan daily interventions since any issue can come up during instruction, there are foreseeable student struggles on the main mathematical idea of the lesson. Many curriculum programs provide anticipated difficulties students may have around the mathematical point of the lesson and suggestions for teacher actions in such occurrences. In implementing written lessons, teachers evaluate curricular resources as well as student thinking to determine appropriate teaching actions. Therefore, micro-interventions impose challenges, on both teacher and curriculum, of predicting student struggles and addressing issues productively toward learning goals. Emerging questions are: How do curriculum programs support teachers to prepare for dealing with students' difficulties in daily lessons? How do teachers use such resources in the curriculum to cope with the moments in which students need extra support? This study investigates the nature of micro-interventions around the mathematical point of the lesson and the relationship between the interventions provided in written lessons and those in enacted lessons.

METHODS

Data Sources

This study draws on data from a larger study on teachers' use of curriculum materials to design instruction in Grades 3-5 in the USA. For curriculum analysis, 15 lessons (five per grade) were randomly selected from each of five elementary mathematics curriculum programs, ranging from reform-oriented to commercially developed: (1) *Investigations in Number, Data, and Space* (INV), (2) *Everyday Mathematics* (EM), (3) *Math Trailblazers* (MTB), (4) *Math in Focus: Singapore Math* (MiF), and (5) Scott Foresman–Addison Wesley *Mathematics* (SFAW). Twenty-five teachers (five per program) were observed in two rounds of three consecutive lessons and interviewed after each round of observations. All the observed lessons were videotaped and transcribed; the interviews were also transcribed.

This study uses all the written lessons selected to see the patterns in interventions from each program. This study also uses enacted lessons and interviews from all five teachers implementing INV and one teacher per program for the other four programs who was representative of the teachers using the same program. Data from all INV teachers are used because INV is unique in providing interventions in terms of their frequency, extensiveness, and emphasis. For example, each INV lesson includes a section of "INTERVENTION" after the main student activity/task, providing anticipated student difficulty and suggested teaching actions. The other four programs include a section of intervention in varying degrees. In addition, all five programs include intervention suggestions that are embedded in the lesson guidance (besides those in the designated area). All the observed lessons and interviews of the nine selected teachers were used for analysis. The written lessons used by the nine teachers were also collected for analysis of interventions in the curriculum and for comparison of written and enacted lessons.

Data Analysis

First, I analysed the nature of interventions in the written lessons per program: their frequency, format and location, emphasis (procedural or conceptual), relationship to the mathematical point of the lesson, and extensiveness of guidance. Then, I specifically focused on the written lessons that the nine teachers enacted in order to examine written interventions and anticipate what difficulties students might have and what teachers might do in the enacted lessons.

When analysing the enacted lessons, I first identified trouble spots in each lesson where interventions are needed, by using the criteria Alibali et al. (2013) articulated: student-initiated questions, incorrect responses and statements, and lack of certainty. Since this study examines the relationship between interventions in written lessons and those in enacted lessons, however, I mainly focused on trouble spots in which students encountered difficulties understanding and applying the main mathematical idea to complete the assigned task, rather than examining every individual trouble spot revealed in a lesson. I hypothesised that written lessons provide anticipated trouble spots related to the core mathematics of the lesson along with interventions crafted. Then, I analysed how teachers reacted in these core trouble spots in each lesson and compared and contrasted each teacher's interventions during instruction with those provided in the written lessons in order to find a pattern within each teacher. When there was no specific intervention provided in the written lesson, I examined how the teachers utilised the instructional guidance (e.g., directions, representations, and mathematical explanations) in the written lessons while helping students with difficulty. In order to understand teacher intentions behind their specific intervention, I analysed teacher interview responses to questions on specific teacher actions during the observed lessons. Finally, I compared and contrasted the patterns in the nine teachers' interventions along with the written lessons they enacted.

RESULTS

Overall, interventions in the written lessons were limited in terms of specificity and comprehensiveness, and many of the micro-interventions in the enacted lessons were not productive, especially when important resources provided in the written lessons were not used. When teachers used curricular resources well, they tended to serve student needs better. When they did not, their interventions did not work well, repeating the same explanation and not moving beyond the procedural level. The same confusion and difficulty were even observed over three consecutive lessons for two of the nine teachers. The results of the study are presented in three parts: (1) overall interventions in the written lessons in the five curriculum programs, (2) teacher interventions in relation to those provided in the written lessons, and (3) teacher interventions when there were no specific interventions provided in the written lessons.

Interventions in the Written Lessons

Interventions provided in the five programs vary greatly. Whereas EM rarely provides interventions, MiF and MTB occasionally do in designated sections called, respectively, "Common Errors" and "For Struggling Learners," and "Meeting Individual Needs." INV and SFAW include interventions along with "on-going assessment" on a regular basis. MiF and SFAW tend to have interventions on procedural errors. For example, MiF includes the following guidance in one of the written lessons examined: "Students may not always write their answers in simplest form. Remind students to check that the numerator and denominator in their answer have a common factor other than 1." INV provides the most extensive guidance for intervention, including specific actions and questions to ask, and materials to use. INV interventions address student difficulty with the mathematical point of the lesson, providing conceptual support for those who need assistance in the content of the lesson. However, sometimes it is not clear when to do such interventions, or the curriculum explains only what the student may struggle with without any specific instructional suggestion. Even interventions in INV at times have limitations in addressing student struggles sufficiently, because they deal with minor issues not necessarily related to the mathematical point of the lesson.

Interventions in the Enacted Lessons

All the enacted lessons exhibited student difficulty in relation to the mathematical point of the lesson at various moments. Students expressed their difficulty or confusion in varying degrees. In some classrooms, students' difficulty was related only to procedures because that was the focus of the lesson; in others, students expressed their confusion based on the lack of conceptual understanding.

Surprisingly, the teachers who were analysed rarely used interventions provided in the written lessons. They created their own interventions regarding the mathematical points of the lesson. In some cases, teacher actions apart from curricular guidance caused student difficulty. Although INV provides the most extensive and conceptually based interventions among the five programs analysed, the teachers implementing INV did not utilise most of the interventions that could have been very effective in the trouble spots that they faced. The same trouble spots recurred since they were not handled properly. For example, one teacher emphasised key words in solving and creating multiplication and division story problems, and her students had tremendous difficulty creating their own word problems. The intervention suggestions provided in the written lessons are:

> Help students talk through the elements of a multiplication situation (two known factors and an unknown product and a division situation (product and one known factor). Write multiplication and division equations with small numbers and ask students to model the action of each with cubes. (TERC, 2008, p. 127)

This intervention guidance is further detailed with the specific script shown below, to use during intervention.

> Look at this equation, 3×4=__ (or 12÷4=__). Can you show me with cubes what this problem would look like? Can you think of a situation to write about in which you might have 3 groups of 4 things (or 12 things divided into groups of 4 or 4 groups)? (TERC, 2008, p. 128)

As seen above, the written lesson predicted that students would have difficulty distinguishing multiplication and division situations and creating story problems on their own, and provided detailed guidance to support such students. The intervention highlights the meaning of multiplication and division with a pair of related equations (i.e., 3×4=__ and 12÷4=__). In fact, in a previous lesson, students were asked to summarise division and multiplication situations in a chart by using specific terms such as *number of groups, number in each group, product,* and *equation* (see Fig. 1).

Number of Groups	Number in Each Group	Product	Equation
?	4 muffins	20	20÷4=__ or __×4=20
5packs	4 yogurt cups	?	3×4=__

Figure 1. Summary table of multiplication and division situations suggested in INV

The written lessons also include the following guidance, using the meaning of equal groups:

> Listen for student understanding of the difference between multiplication and division. For example, do the problems students make for the expression 18÷3 begin with the quantity 18 and divide it into 3 equal groups or groups of 3? Do the problems for 6×3 involve 6 groups of 3 or 3 groups 6? (TERC, 2008, p. 126)

Not using any of the extensive, specific interventions that INV provided to evoke the meaning of operations in problem contexts, the teacher repeatedly reminded students of key words they generated. In her interventions the teacher constantly stated, for example, "If it says 'in each,' it's gonna be a division problem." She also asked questions, such as, "Now remind me, what are our multiplication key words? If it's a multiplication story problem, it's gonna have what key words in it?" As a result, she lost an opportunity to highlight the characteristics of multiplication and division in relation to each other, and students continued to have difficulty creating their own multiplication and division story problems.

Teacher Actions When Specific Written Interventions Not Available

When there were no interventions provided in the written lessons or, if any, only procedural ones, teachers had difficulty providing appropriate interventions. They inaccurately assessed what students had difficulty with or what might have caused the difficulty, and they tried to tell students facts and information or repeated the same explanation they had already provided. Even when they tried to assist students with conceptual meaning, they did not go beyond the surface level and stopped pursuing a further intervention. For example, the teacher using SFAW barely brought up the notion of a typical value of a data set to address the meaning of *mean*.

Although at times no specific interventions were provided in the written lessons, some lessons included critical curricular resources, such as representations and mathematical explanations based on the meaning, which could be used effectively during interventions. I observed that teachers did not use such critical resources provided in the curriculum. For example, the teacher who enacted lessons from MiF did not use a bar model representing addition and subtraction with fractions (see Figure 2). The written lessons introduced two methods for subtracting a fraction from a whole number or a mixed number:

Method 1: $3 - \frac{4}{9} = 2\frac{9}{9} - \frac{4}{9} = 2\frac{5}{9}$

Method 2: $3 - \frac{4}{9} = \frac{27}{9} - \frac{4}{9} = \frac{23}{9} = 2\frac{5}{9}$

Figure 2. Bar model used in MiF

Students had difficulty making sense of the methods introduced by the teacher and how the two are related. In MiF there were no specific interventions regarding this difficulty other than one sentence in the guidance for the lessons: "Note: Reading the number sentences aloud may help students understand why only the numerators of the fractions are subtracted" (Kheong, Sharpe, Soon, Ramakrishnan, Wah, & Choo, 2010, p. 253). This particular intervention emphasises the meaning of fraction and fractional units, such as how many ninths are there as a result of subtraction. However, it does not help students understand why 3 needs to be renamed as 2 and 9/9, or 27/9, why both methods work, and how they are related.

As seen in Figure 2, the written lesson uses a bar model to represent 3 – 4/9 visually and conceptually—what it means to subtract 4/9 from 3, and what is left as a result of the operation. Without using the bar model, however, the teacher verbally explained renaming of 3 in different ways (e.g., 2 and 9/9, and 27/9) in order to subtract 4/9. Explaining renaming without the model kept the concept on an abstract level and students continued to have difficulty understanding similar solutions to other problems in the three observed lessons. Without the representation, her explanations did not help students see the rationale for the procedures, and many of the students chose just one of the two methods to solve other problems and were not able to relate the two methods presented by the teacher. Even when students mentioned using the model ("I can draw a picture on the board"), the teacher said, "No, that's okay. If somebody needs a picture, we will add that. I don't want to confuse anybody." The teacher strongly believed that the model would confuse students rather than helping them see why the procedure works and explained the renaming repeatedly.

DISCUSSION

This study highlights the importance of intervention resources in the curriculum and teacher role of recognising the mathematical point of the lesson and the affordances of curricular resources to use intervention resources productively and to create an appropriate one when not available in the curriculum. The latter is a critical component of teacher *pedagogical design capacity*, which Brown (2009) refers to as a teacher's ability to perceive affordances of the curriculum, make proper decisions, and follow through on plans. This study has implications for teacher education and curriculum design regarding teachers' instructional decisions, although further studies on micro-interventions are needed for theoretical and practical elaborations.

It seems that two kinds of teacher knowledge were particularly critical in the interventions in the enacted lessons: teachers' knowledge of student need (what students have difficulty with and where the difficulty comes from) and curricular knowledge (Ball, Thames, & Phelps, 2008; Choppin, 2011; Remillard, Kim, & May, under review). The teachers recognised student difficulty, but many of them failed to accurately assess the origin of the difficulty and what could be done to resolve the problem. Choppin (2011) elaborated teacher knowledge of resources that facilitate student thinking, suggesting that teachers need to recognise the affordances of resources to help students learn the content. It seems that most of the teachers analysed in this study failed to recognise the affordances of the resources in the curriculum they were using.

This study also revealed inconsistences and limitations of intervention resources available in the written lessons. Curriculum developers need to examine the way they provide intervention resources, because crafting appropriate, timely interventions is a real instructional challenge for teachers. Further research can guide the direction for providing proper resources to teachers.

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ENGAGING STATISTICS: WHY THE DIFFERENCE BETWEEN STATISTICS AND MATHEMATICS MATTERS IN TEACHING AND LEARNING STATISTICS

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This paper takes a theoretical approach in identifying the disciplinary differences between statistics and mathematics and the implications of these distinctions for teaching and learning statistics. Acknowledging and understanding the differences and intersections between statistics and mathematics is key in considering pedagogical approaches that retain the integrity of statistics as a discipline and support the development of statistical thinking. Considerations include embedding core characteristics of statistics in the types of statistical problems provided in school-based learning experiences and the affective dimensions of students' experiences with and beliefs about uncertainty.

TEACHING STATISTICS AND THE CURRICULUM

Statistics is increasingly used to add credibility to the way data are presented and to persuade through data-based arguments. Understanding statistics impacts decision making, and as a consequence, the ability to reason statistically, that is, to make sense of and reason about statistical information has become increasingly important. Too often, analysis of data is accepted as factual because statistical analysis and conclusions present complex numerical results. It is important to develop a critical orientation and attitude towards statistics and to recognise the uncertainty inherent in statistical analysis and conclusions (Whitin, 2006). The purpose of teaching statistics then should extend to an ability to critically interpret the use of statistics in real-world situations, which requires an understanding of how to apply statistical tools and the influence of context on the relationship between chance and data (Watson, 2006), including how they are quantified using mathematics. Such a purpose has implications for how school based statistics teaching develops understanding of statistics as a discipline and its relationship to the mathematics it engages.

'Statistics and Probability' is one of the three strands in the national Australian Curriculum: Mathematics (ACARA, 2014), mandating its teaching from the commencement of formal schooling, and arguably reflecting a national focus and pragmatic view of the value of developing statistical literacy in evaluation and decision making across disciplines and in everyday life (Watson & Neal, 2012). Research in statistics education has increased significantly in the last two decades (Watson & Neal, 2012), however research in teaching and learning statistics in primary school settings, particularly the early years of schooling, is under-represented. As a consequence, more understanding is needed of how 'statistics' is represented and presented in the data-based problems students are provided with and the influence this has on how students engage with statistics and statistical reasoning. If statistics is the focus, then how core statistical concepts are defined and contextualised in data-based problems has significant implications for how statistics is taught and engaged with, and what 'statistics' is actually learned.

DIFFERENTIATING STATISTICS FROM MATHEMATICS

Statistics and mathematics differ. Statistics developed when a search began for a common logic to measure and examine the consequences of uncertainty generated when working with the variation found in the world (Stigler, 2003). The result was the development and use of mathematical concepts and methods to quantify the uncertainty resulting from variation found in data (Salsburg, 2001). The relationship between statistics and mathematics is therefore both critically dependent, complementary and co-dependent, however the embedded nature of the relationship does not diminish that the two disciplines are dissimilar.

At the core of the distinction between statistics and mathematics is statistics' role in working with variation and uncertainty (Moore, 1990). Data are contextualised, varied and uncertain, and ways of thinking about and reasoning with data subsequently demands inference. Mathematics, including probability, supports statistics' management of variation when data are collected, handled and conclusions are drawn (Watson, 2006). Although variation is at the core of statistics, it is often given little emphasis in the curriculum, and hence in formal learning experiences.

Statistics reasoning, processes and conclusions also contrast with mathematics, as inductive reasoning is needed to manage the variation, uncertainty and multiplicity inherent in statistical problem solving. Inductive reasoning moves from specific to general, where real world knowledge drives forming connections in order to decide the likelihood of a statistical conclusion. Statistical reasoning necessitates interpretation, and the ability to draw inferences through induction is described as a process of making "mental connections between something that we already believe is true and something we believe connects to it in some way" (Chiasson, 2005, p. 215). This is where the critical intersection between context and the statistical problem to be solved is found, and it demands attention in statistical teaching and learning.

A real-world statistical problem supplies the context for the problem, and at the same time, engages the problem-solver's real-world knowledge of that setting as he or she finds a solution. The relationship between the real-world origin or setting (as context) of a problem and the statistical concept of 'context' creates a definitional conundrum, as each serve to define the other. Data are collected in order to solve the problem, and in doing do, "engage our knowledge of their context so that we can understand and interpret, rather than simply carry out our operations" (Moore, 1990, p. 96).

The core defining elements of context (as the source of variation) and inferential reasoning are central to conceptualising statistics. How identifiable differences between statistics and mathematics are perceived and engaged pedagogically can profoundly impact how statistics is taught and learned. Statistical teaching and learning we argue, should keep statistics, not mathematics in mind, and engage the disciplinary specific characteristics of statistics.

MATHEMATICS IN STATISTICS TEACHING

An over emphasis on mathematics and the formal nature of mathematical processes in teaching statistics can lead to mathematics predominating learning outcomes in statistics education. Snee (1988) argues that mathematics and mathematicians rarely deal with, or are comfortable with either variation or uncertainty in data. In mathematics there is often a need for just one correct answer however in statistics variation means that the answer is invariably one of multiple possibilities, questions and uncertainty (Gattuso, 2008). Too often teaching statistics is approached through mathematical calculations, ignoring statistics as a practical tool that can illuminate phenomena in a given context through engaging contextualised variation and inferential reasoning. Watson (2006) notes, that "it is the uncertainty associated with statistical variation that produces conflict with the determinism of calculating correct numerical answers" (p. 21).

The potential impact of taking a mathematical approach to handling and examining data in a statistical problem creates a conundrum not lost on Paramore (2011), who notes that "data are rarely problematic when the focus is on the right answer" (p. 74). If students associate mathematics with absolutes and certainty, variation and uncertainty are the potential sources of conceptual discomfort. If statistics teaching and learning is approached mathematically, it runs the risk of ingraining students into a way of thinking about and reasoning in statistics that bypasses, and is counterproductive to the reasoning and processes that are critical to statistics.

ENGAGING STATISTICAL REASONING AND KNOWLEDGE

Statistical problem solving engages reasoning with variation inherent in data. It includes making reasoned decisions about what attributes to measure in order to collect data, how to collect and display data and how to analyse and interpret data. As a consequence, all processes in a statistical inquiry involve attending to reasoning processes of one form or another. When inferential reasoning is engaged in these decision making processes, it both relies on and draws from the data to make judgments and focuses attention on the role evidential reasoning has in coming to a solution to a statistical problem. In statistics, inference is the statistical means by which knowledge of the data and the context move thinking and reasoning beyond the description of the immediate data to hand to the wider context in which the data have been generated.

The continual interaction between statistical knowledge, data context knowledge and knowledge of the data plays a critical role in statistical problem solving (Wild & Pfannkuch, 1999). It is here, however, that the role of connecting existing context knowledge with decision making in inductive reasoning raises issues for statistical problem solving. Issues exist because a core component of statistics is its grounding in the context data carries, and the interference of everyday knowledge people possess about the context with the use of data-based evidence. This tension interferes with the types of connections and relationships that people make when working statistically (delMas, 2004). Students have both strengths and vulnerabilities in their use of their everyday life experiences in statistical reasoning and when reaching statistical decisions. Students' real world context knowledge and beliefs are a major influence in their reasoning with data that impacts how they resolve data that contradicts that knowledge or falls outside their sphere of experience. Students are prone to be bound by the beliefs and interpretations they have developed and can take a subjective approach to problem solving that disregards the data (Nikiforidou & Pange, 2009, 2010). Reasoning from everyday knowledge can thus produce errors in statistical thinking and reasoning that are difficult to change (delMas, 2004).

The contrast with the relative certainty of mathematics is important when considering the role of everyday knowledge and the inherent characteristic of variation in statistical problem solving. Everyday knowledge includes the affective dimension of students' prior everyday experiences with variation and uncertainty, not just with chance events, but also with in the way everyday experiences with variation impact data handling in statistical problem solving. For example, the generation, selection and measurement of attributes for data production engage analysis of the sample data, as categories or objects and that analysis requires knowledge of the context for the problem to be drawn on. This means that the use of measurement in statistics differs from that in mathematics as the data context must be considered (Rossman, Chance & Medina, 2006). As a consequence, concurrently, measurement of sample data collected for categorisation and classification are subject to different types of variation (Snee, 1988) and involve decisions about variation. Those decisions involve uncertainty (Moore, 1990) and therefore, as the decisions involve working with uncertainty generated by variation, inductive reasoning must be employed in the process of attribute decisions in statistical reasoning. We suggest that engaging uncertainty through contextualised variation in statistical problem solving also pulls in an affective dimension to students' experiences with and beliefs about uncertainty generally. We are aware that acknowledging the dynamic and unpredictable nature of affective responses in how a learning task is presented are key variables in managing students' responses to encountering uncertainty in learning (Evans, 2006). Students' data based explanations draw out both contextual and statistical knowledge (Gil & Ben-Zvi, 2011), and yet the interplay between students' knowledge and beliefs, handling data and the influence of the way a learning task is presented is undervalued when considering how statistics is learned (Langrall, 2010).

How a statistical problem is contextualised in a statistical investigation therefore creates a contextual contradiction, as the context of a problem has the capacity to both motivate and mislead (Ben-Zvi, Makar, & Bakker, 2009). Students can be motivated by the data context to engage in statistical sense making when reasoning inferentially. Students' informal and personal knowledge of the data context can bring additional information and insight to data that can influence interpretation and explanation of data, justification for the use of data and conclusions drawn from data (Masnick, Klahr, & Morris, 2007). Conversely, students' data context knowledge that is potentially inconsistent or insufficient can mislead them as they consider the statistical knowledge they have from the available data. Makar, Bakker, and Ben-Zvi (2011) state that although distinguishing between statistical and context knowledge is not easily done, students must coordinate between context knowledge and statistical knowledge as they look for evidence for their reasoning in moving to a problem solution. Context in the broad sense (as problem setting and problem process) therefore has the potential to make a statistical problem more accessible and at the same time constrain it (Langrall, 2010).

STATISTICS, CONTEXT, PROBLEMS AND TASKS

The real-world context and the context of statistical problems is interdependent (Langrall, Nisbet, Mooney, & Jansem, 2011). Students engage their existing knowledge of the setting for a statistical problem and experiences of the world, including knowledge of the way data has been created, defined and measured, when they search for a problem solution (Pfannkuch, 2011). The setting of a statistical problem therefore contextualises and provides meaning for the data and so becomes the framing structure for data analysis and reasoning (Ben Zvi et al., 2009). The need for context in statistics is in direct contrast to mathematics, where the context of a problem is inevitably obscured or irrelevant to finding a solution to a problem. The juncture of context and analysis highlights the importance of the task context in statistics problems that bring problem, data, knowledge and reasoning together. Task context has been described as "the presentation of data or the way they are encountered" (Langrall et al., 2011, p. 50) and expands thinking about how a statistical problem represents and presents the statistics students are to encounter.

The presentation of a statistical problem as task context plays a critical role in how students engage a real-world and includes pedagogical decisions such as the task sequence (Pfannkuch, 2011). The multiple dimensions of the task context influence the way data are approached, engaged, analysed and interpreted by students and hence how statistical problems are reasoned and what knowledge is engaged to find a solution. In data analysis, the relationship between data context and data is described by Wild and Pfannkuch (1999) as involving an interplay or shuffling between the data and context spheres, "finding something out" and "ascertain(ing) meaning of what we have seen" (p. 336). The form of the data context for the statistical problem, that is, how the data for the problem is contextualised and the task that presents it, should be central considerations for teaching and learning statistics. In a statistical problem, the content and structure of the task facilitate and trigger statistical problem solving, and paradoxically integrate the data context. Students accordingly should encounter data in ways that support their interaction with, not on, data (Makar & Rubin, 2009).

PLANNING TO MEETING THE DISCOMFORT OF UNCERTAINTY

We argue that in addition to the statistical imperatives variation brings to statistical problems solving, task context should plan for students meeting uncertainty. By this we mean that in addition to developing statistical knowledge of the impact and role of variation inherent in the discipline, we should be mindful of students' beliefs, attitudes and emotions potentially triggered when encountering uncertainty, particularly discomfort, as compelling factors in learning (Schuck & Grootenboer, 2004). Student discomfort when dealing with variation can be an opportunity for educators to forefront the uncertainty inherent in statistical reasoning and conclusions, and to facilitate the development of a critical orientation and attitude we believe are necessary to learning statistics. Hattie's (2009) work considers influential factors in student learning, and the role of teachers' pedagogical decisions and practices with respect to framing task contexts in statistical learning should be of interest to statistics educators. Teachers should make relevant connections to the wider dimensions of students' real world experiences, particularly uncertainty, to increase engagement and deepen understanding in the application of concepts when problem solving (Pierce and Stacey, 2006). Task contexts that actively considering affective phenomenon engaged when encountering uncertainty could potentially create and strengthen cycles of positive student affect that reference both statistics as a discipline, and students' self-perceptions, which we argue are important considerations in the development of positive, critical orientations and attitudes to statistics.

CONCLUSION

This paper argues that the theoretical distinction that identifies differences between statistics and mathematics should inform statistical teaching and learning experiences in school. Statistics education should aim to develop a critical orientation and attitudes to statistics and the understanding of statistics as a discipline it would necessarily encompass. The differences between statistics and mathematics oblige educators to evaluate how tasks represent and present statistics, including how data is contextualised, handled and managed, and the ways that the use of inductive reasoning is facilitated through encounters with variation. Statistical problems should recognise and heed the affective dimension of uncertainty generated in encounters with variation that can influence student engagement, analysis and interpretation of data and the type and use of knowledge employed in finding a solution. The tight connection between data context and statistical problem is the crux of the contextual dilemma in statistics and therefore in teaching and learning statistics. Student knowledge of the central role of context, variation and inference in statistics is critical. Student knowledge of the role of their affective responses to statistical problem solving also deserves attention. More research is needed on the impact of task context and the way we genuinely engage children with statistical learning.

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SOLVING MULTIPLICATIVE WORD PROBLEMS WITH DECIMAL FRACTIONS: FOCUS ON RELATIONSHIPS OF PROPORTIONAL REASONING

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The purpose of this paper is to investigate the effects of proportional reasoning and on students' ability to solve multiplicative word problems with decimal fractions. In this study, 256 Japanese elementary school students in Grades 4, 5, and 6 were given a test involving multiplication and proportion problems. The mean scores such as multiplication and proportion problems increased in Grade5, but these are not changeable in Grade 5 and 6. The correlation coefficient of between multiplication and proportion problems increased in Grade 6, but these are not changeable in Grade 4 and 5. As a result, although conceptions such as multiplication or proportion are developed into Grade 5, the connection between multiplication and proportion is reinforced in Grade 6.

It is widely known that many students have difficulty to solve multiplicative word problems with decimal fractions. Researchers have investigated various factors that are presumed to be associated with the difficulties they have in solving multiplicative word problems (e.g. Graeber and Tanenhaus, 1993; Greer, 1987b, 1992; Mangan, 1989). For example, the familiarity of context or type of quantities involved in word problem may affect problem difficulty (e.g. Bell, Fischbein, & Greer, 1984; Bell, Greer, Grimison, & Mangan, 1989; Bell, Swan, & Taylor, 1981). Students may change the operation needed to solve the problem, depending on the specific numerical data.

To give the appropriate operation for multiplicative word problems, one need not only to understand conceptions of multiplication, but also to develop many other abilities such as proportional reasoning. Greer said "There is a need for synthesis of hitherto rather separated bodies of research on multiplication and division word problems, proportional reasoning, and rational number concepts." (Greer, 1992, p.293)

As Vergnaud (1988) also pointed out, mathematical concepts are tied to situations such as multiplication, division, fraction, ratio, proportion, and linear functions. He suggested that students develop these concepts not in isolation but in concert with each other over long periods of time through experience with a large number of situations. Therefore, researches on their ratio and proportion concepts also need to consider the other concepts that are a part of one's developing "multiplicative conceptual field".

Proportional reasoning refers to the ability to infer the value of one quantity if another quantity is changed, given that a proportional relationship exists between two

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quantities (cf. Tourniaire, 1986). In other words, if a relationship between two quantities represents as formula y=f(x), it means that f(a x)=a f(x).

For example, we consider the proportional word problem such that “7 children get 3 pizzas. How much do 21 children get pizzas?” The conditions in this problem are represented such as x=21, f(7)=3. Therefore this relationship is shown as f(21)=f(3·7)=3·f(7)=3·3=9. And, we also consider the multiplicative word problem such that “one child gets 3/7 pizza. How much do 21 children get pizzas?”. The conditions in this problem are represented such as x=21, f(1)=3/7. Therefore this relationship is shown as f(21)=f(21·1)=21·f(1)=21·3/7=9. Without assuming a proportional relationship between two quantities, it may be not possible to find the correct solution.

This paper focuses on proportional reasoning as significant factors in solving multiplicative word problems with decimal fractions. The purpose in this paper is to investigate the relationships of proportional reasoning on students' ability to solve multiplicative word problems with decimal fractions.

METHOD

Subjects

The subjects who participated in the investigation were 256 students from 3 different elementary schools in Japan. Subjects included 83 students in Grade 4 (9 or 10 years old), 83 students in Grade 5 (10 or 11 years old), and 90 students in Grade 6 (11 or 12 years old). In the Japanese curriculum, multiplication with whole numbers is introduced in Grade 2 and multiplication and division with decimal fractions in Grade 5.

Instrument

A test used in the investigation was consisted of 4 multiplication and 4 proportion problems. Table 1 shows these problems included on the test. These problems had been modified in an attempt to use terminology and notation more familiar to Japanese population.

Within the set of multiplication problems, we made items based on types of quantities involved in word problems such as integer, pure decimal fraction, and mixed decimal fraction. 4 addition and subtraction problems were also included so that students could not assume that multiplication were always the correct operation. Subjects made only an appropriate choice operation for multiplicative word problems.

Within the set of proportion problems, we made items based on the work of Lamon (1993). She identified four types of problems that are semantically distinct (well-chunked measures, part- part-whole, associated set, stretchers and shrinkers). The test items constitute on these four semantic types of problems. In the proportion problems, there are both of ‘missing value problems’ and ‘comparison problems’. But we decided to use comparison problems as proportion problems. Because, by using these problems, we would like to analysis students’ solving strategies in details.

Multiplication Problems
(1) 1 kg of oranges costs 580 yen. What is the cost of 2.4 kg? (580×2.4)
(2) There are 1.2 kg of sauce per 1 litre. One restaurant uses 7.6 litres of sauce a month. How many kgs of sauce does the restaurant use per month? (1.2×7.6)
(3) 1 litre of oil costs 600 yen. What is the cost of 0.3 litres? (600×0.3)
(4) 1m of iron pipe weighs 1.2 kg. How much does 0.8 m of iron pipe weigh? (1.2×0.8)
Proportion Problems
(1) The student is shown a subscription card from a popular magazine. It offers three plans: 1) A 6-month subscription for 3 payments of 4000 yen each; 2) A 9-month subscription for 3 payments of 6000 yen each; 3) A 12-month subscription for 3 payments of 8000 yen each. Do you get a better deal if you buy the magazine for a longer period of time? (Well-chunked measures)
(2) The student is shown pictures of two egg cartons, one containing a dozen eggs (8 white eggs and 4 brown eggs) and the other containing 1 1/2 dozen eggs (10 white eggs and 8 brown eggs). Which carton contains more brown eggs? (Part- part-whole)
(3) The student is shown a picture of 7 girls with 3 pizzas and 3 boys with 1 pizza. Who gets more pizza, the girls or the boys? (Associated set)
(4) The student is shown a picture of two trees. Tree A is 8 feet high and tree B is 10 feet high. This picture was taken 5 years ago. Today, tree A is 14 feet high and tree B is 16 feet high. Over the last five years, which tree's height has increased more? (Stretchers and shrinkers)

Table 1: Test Problems

Procedure

The test was administered to each of subjects late in November 2013. The test took approximately 1 hour. Their responses were conducted by Analysis of Variance designed in two-way layout and Correlation Analysis.

RESULTS

Correct responses

Table 2 shows mean scores of correct responses to multiplicative and proportion problems in each Grade. The mean number of correct responses to ‘multiplication problems’ was 1.13 in Grade 4, 2.80 in Grade 5, and 3.11 in Grade 6. And the mean number of correct responses to ‘proportion problems’ was 0.46 in Grade 4, 1.86 in Grade 5, and 1.99 in Grade 6.

At first, results of multiplication problems were analysed by Analysis of Variance (Grades (4,5,6) and Tasks (multiplication, proportional reasoning)). These results were shown as follows; Grades ($p<.001$), Tasks ($p<.001$), and Interaction of Grades Tasks ($p<.001$) were significant in main effects. These main effects may be qualified for a significant of the Interaction.

Secondly, 'simple main effects' were also significant. These results were shown as follows; Grade (Multiplication) ($p<.001$) and Grade (Proportional reasoning) ($p<.001$) were significant. Task (Grade 4) ($p<.001$), Task (Grade 5) ($p<.001$), and Task (Grade 6)($p<.001$) were also significant. All factors of simple main effect were significant.

Thirdly, for further detail, these results were analysed by Multiple Comparison (Ryan method). These results were shown as follows; For Multiplication, 'Grade 4 and 5($p<.001$)' and 'Grade 4 and 6($p<.001$)' were significant, but 'Grade 5 and 6' was not. Similarly, for Proportional reasoning, 'Grade 4 and 5($p<.001$)' and 'Grade 4 and 6($p<.001$)' were significant, but 'Grade 5 and 6' was not. As a result, it was said that there was a difference between 'Grade 4' and 'Grade 5 and 6'.

	Grade 4		Grade 5		Grade 6	
	Mean	SD	**Mean**	SD	Mean	SD
Multiplication	1.13	0.997	2.80	1.197	3.11	1.126
Proportion	0.46	0.668	1.86	1.280	1.99	1.230

Table 2: Mean Number of Correct Responses in Each Grade

Correlation coefficient

Table 3 show the correlation coefficient between multiplicative word problems and proportion problems in each grade. In addition, the Pearson's correlation coefficient of two variable quantities was tested statistically. The correlation coefficient between multiplication and proportion was 0.311 in Grade 4, 0.307 in Grade 5, and 0.538 in Grade 6. All correlation coefficients were significant. In particular, the correlation between multiplication and proportion are high in Grade 6 (0.538).

	Grade 4	Grade 5	Grade 6
R	0.311**	0.307**	0.538**

**p<.01

Table 3: Correlation between Multiplication and Proportion in Each Grade

Solving strategy

We decided to analysis students' solving strategies in proportion problem (2) because they made more kinds of solving strategies in this problem than in other problems. Table 4 shows some typical examples of solving strategies. And Table 5-7 show the distribution of their responses corresponding with the number of correct choice operation in multiplicative word problems. Although many students in Grade 4 use additive or additive comparison strategy, they in Grade 5 and 6 do other strategies as part-whole or part-part strategy. In all grades, students who give correct choice operation in multiplicative word problems use multiplicative strategy such as part-whole or part-part strategy. Students who do not give correct choice operation in multiplicative word problems often use additive strategy. Students in Grade 5 and 6 use an error part-whole strategy that is a multiplicative strategy. They can't understand conceptions of a unit. They take a brown egg as unit.

Strategy	Typical Examples
Part-whole/ Part-part	Part-whole: (1) 12:4=36:12, (2) 18:8=36:16, Ans. (2). Part-part: (1) 8:4=40:20, (2) 10:8=40:32, Ans. (2).
Part-whole/ Part-part (Error)	Part-whole: (1) 12÷4=3, (2)18÷8=2.25, Ans. (1). Part-part: (1) 8÷4=2, (2)10÷8=1.25, Ans. (1).
Additive Comparison	(1) White eggs:8, Brown eggs: 4, (2) White eggs: 10, Brown eggs: 8 In the condition (1), if one adds two white eggs and two brown eggs each other, the number of white eggs become 10 and the number of brown eggs do 6. , Ans. (2).
Additive	(1)8-4=4, (2)18-12=6, Ans. (2).
Unsuitable or Blank	"Counting"

Table 4: Solving Strategies in Proportion Problem (2)

Grade 4	Part-whole/ Part-part	Part-whole/ Part-part (Error)	Additive Comparison	Additive	Unsuitable or Blank
0 Correct				19.3%(16)	12.0%(10)
1 Correct			4.8%(4)	30.1%(25)	
2 Correct	1.2%(1)		2.4%(2)	18.1%(15)	2.4%(2)
2 Correct			1.2%(1)	6.0%(5)	1.2%(1)
4 Correct				1.2%(1)	
Total	1.2%(1)		8.4%(7)	1.2%(1)	

Table 5: Responses of Solving Strategy in Grade 4

Grade 5	Part-whole/ Part-part	Part-whole/ Part-part (Error)	Additive Comparison	Additive	Unsuitable or Blank
0 Correct			1.2%(1)	3.6%(3)	1.2%(1)

1 Correct	1.2%(1)	1.2%(1)		1.2%(1)	3.6%(3)
2 Correct	2.4%(2)	3.6%(3)	2.4%(2)	15.7%(13)	1.2%(1)
3 Correct	8.4%(7)		2.4%(2)	9.6%(8)	3.6%(3)
4 Correct	14.5%(12)	3.6%(3)	6.0%(5)	10.8%(9)	2.4%(2)
Total	26.5%(22)	8.4%(7)	12.0%(10)	41.0%(34)	12.0%(10)

Table 6: Responses of Solving Strategy in Grade 5

Grade 6	Part-whole/ Part-part	Part-whole/ Part-part (Error)	Additive Comparison	Additive	Unsuitable or Blank
0 Correct					3.3%(3)
1 Correct	2.2%(2)		2.2%(2)	1.1%(1)	3.3%(3)
2 Correct	2.2%(2)	1.1%(1)	2.2%(2)	4.4%(4)	2.2%(2)
3 Correct	12.2%(11)	5.6%(5)	1.1%(1)	5.6%(5)	4.4%(4)
4 Correct	34.4%(31)	3.3%(3)		8.9%(8)	
Total	51.14%(46)	10.0%(9)	5.6%(5)	20.0%(18)	13.3%(12)

Table 7: Responses of Solving Strategy in Grade 6

DISCUSSION

Correlation coefficient

On the one hand, mean scores of multiplication and proportion problems increased in Grade 5, but these are not changeable in Grade 5 and 6. The mean scores in ‘Grade 4 and 5’ and ‘Grade 4 and 6’ were significant, but ‘Grade 5 and 6’ was not.

On the other hand, the correlation coefficient of between multiplication and proportion problems increased in Grade 6, but these are not changeable in Grade 4 and 5. This result is also supported by Inhelder and Piaget (1958) that children's proportional conception would develop around age 11 or 12. In Japan, the topics such as multiplication and division with decimal fractions, rate, and ratio are taught in Grade 5, and proportion is taught in Grade 6. The teaching of these topics would help to promote the development of proportional reasoning. Therefore it is conjectured that conceptions such as multiplication or proportion are developed into Grade 5. And the connection between multiplication and proportion is reinforced in Grade 6.

Solving strategy

With respect to kinds of solving strategy in each Grade, students in Grade 4 use an additive strategy, and they in Grade 5 and 6 do a multiplicative strategy. With respect to rate of correct responses, students who don't take correct choice operation for multiplicative word problems often use an additive strategy. They who take correct choice operation for multiplicative word problems use a multiplicative strategy. Many previous researches show that students who don't understand the proportion concepts often use an additive strategy (Karplus, Pulos, & Stage, 1983; Tourniaire,1986). Students need to understand multiplicative concepts away from additive concepts to solve multiplicative word problems.

Although many students in Grade 5 or 6 use a multiplicative strategy, one of them makes a mistake to take a unit quantity. For example, in proportion problem (2), some students give an answer such as '(1) 8÷4=2, (2) 10÷8=1.15'. They take a brown egg as a unit. It is conjectured that they don't fully understand concepts of a unit. And the misconception such as "the divisor must be smaller than the dividend." affects students' solving activity (Fischbein, Nello, & Marino,1985)

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USING IPAD DIGITAL DIARIES TO INVESTIGATE ATTITUDES TOWARDS MATHEMATICS

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In this paper we report on early findings from a project in which we developed a methodology to elicit young students' thinking about mathematics. We describe the use of iPad diaries to collect data so as to better understand students' experiences of mathematics, from three economically and socially distinct schools, at two key junctures - Year 3 and 6. This paper focuses on the unique methodology we developed over three iterations and on the student attitudinal comments regarding mathematics as these give significant insights into the experiences and possibilities for mathematics education of young learners.

This project explores a methodology that enables learners to recount experiences, feelings, emotions or thoughts in relation to their mathematics learning. Modifying a 'Big Brother' methodology by using iPads, students were able to enter a neutral space set up in the school to talk freely (to the iPad) about their experiences. This method sought to elicit the experiences of young learners in ways that would allow researchers access to their "true" feelings, at least insofar as they were prepared to discuss them. Recognising that interviews or surveys, can produce biased results, the electronic diaries approach offers a more robust and reliable account of students' lived experiences (Buchwald, Schantz-Laursen, & Delmar, 2009) and provides greater opportunity for students to discuss any aspect of mathematics they chose (Di Martino & Zan, 2010; Larkin & Jorgensen, 2015).

LITERATURE REVIEW

We present here initially a snapshot of the literature on student attitudes towards mathematics, much of which suggests that secondary school students, where they have the choice, are "opting out" of mathematics (Brown, 2009). In a primary school context, students do not have this option but may be psychologically distancing themselves from engagement with mathematics (which may pre-empt the later physical withdrawal in secondary years). The literature suggests that this "opting out" is based upon negative experiences of, and attitudes towards, mathematics and that these attitudes and experiences are often associated with, shame, inadequacy, anxiety and hopelessness resulting in declined performance (Lewis, 2014). Research into beliefs, attitudes and emotions has indicated an important, and inseparable, relationship between cognitive and affective mathematical domains. Ma and Kishor (1997) suggest that "there is a cognitive component to every affective objective and an affective component to every cognitive objective" (p.26) suggesting that any investigation into reasons for non-participation in mathematics must include an examination of both domains. Although significant research into beliefs and attitudes on mathematics has been conducted with older students (See Carter, 2014), we need to investigate when

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the first signs of mathematical withdrawal occurs to determine "how the 'curiosity machine' [the student] turns into a 'mathematical idiot" (Di Martino and Zan, 2010, p. 28) and how this aversion to mathematics may be avoided or at the very least minimised.

Collecting Authentic Data on Student Attitudes

In this research we used iPads as a tool for collecting information about student attitudes. Proponents of using video research (Buchwald, Schantz-Laursen, & Delmar, 2009; Lundström, 2013; Noyes, 2004) argue that using videos enables researchers to collect data of a more profound, compelling quality than the data normally collected in interviews, surveys, or observations. One purpose of this research was to gain knowledge concerning the students' thoughts, feelings and emotions as they engaged with school mathematics and thus we relied heavily on student voice, mathematics talk regarding the context of learning mathematics and ongoing narratives regarding their experiences of mathematics. The limited literature available suggests that the use of videos encourages students' voice and the telling of personal narratives (Buchwald, et al., 2009) and that student voice is critical as it can often be problematic for adult researchers to understand the world view of students. Di Martino and Zan (2010), Lundström (2013), and Noyes (2004) each suggest that video diaries can be a means of empowering participants to speak authentically of the experience under investigation and to thereby "create representations of their own experiences" (p. 7).

In the previous cited research, the students were able to video themselves whenever they chose; however, they were not able to delete their videos. Therefore, the use of the iPads to self-record electronic diaries adds a high degree of autonomy for the students in our research as they are in complete control of the entire recording process. This means that students had control over creating a digital diary entry or not; determining what they would like to say; and then deleting the material afterwards if they were not satisfied with the result. In addition, the act of recording a diary entry demonstrates a degree of comfort in the process and a willingness to share personal narratives (Buchwald et al., 2009). This willingness to share is of particular import if we are to a) uncover more clearly student attitudes towards mathematics and b) improve the teaching of mathematics as a consequence of an increased understanding of the attitudes and emotions students bring to, or experience, whilst completing mathematical activities. From the literature the following question emerged: *What attitudes and emotions towards mathematics were reported; and are there any patterns in this self-reporting that coincide with two junctures in primary schooling?*

METHOD

Using electronic diaries as a means for identifying students' experiences in primary mathematics, we sought to develop the method using iPads. Students used either the iPad camera or AudioNote (iPad App) to create a digital diary. Students were invited to be part of the project and consent was gained prior to their involvement. Written prompts (e.g. what would I tell my mum and dad about what I did in maths today?)

were placed within a small tent which we used to create a "mathematical thinking space" (See Larkin & Jorgensen, 2015 for a detailed explanation of methodology). School A is a Queensland State School (2014 ICSEA: 1055) and involved 105 students. School B is a NSW Public School (2014 ICSEA: 970) and involved 96 students. School C is a private girls school (2014 ICSEA: 1135) and involved 67 students. [ICSEA is an index used by the Australian Curriculum and Assessment and Reporting Authority to indicate relative social dis/advantage. The national average is 1000 with each standard deviation being 100.] Three different data collection methodologies were deployed. School A and B were similar and in both these schools a shared iPad was placed in the tent where students could record their video. In School A, the lead author downloaded the videos and in school B a research assistant did so. We had ethical concerns regarding this method as students (and possibly their teachers) could view the recordings of others. In order to maximise the security of the data, in School C we used a generic email account on each of the iPads such that the students could record their diary entry, email it to a secure researcher email address, and then delete their diary (email and internet access were not permitted in Schools A and B). In future research, we will encourage schools to use the AudioNote-email methodology as this fully guarantees both the anonymity and security of the diary entries.

FINDINGS

From the number of digital diaries recorded (Table 1) it is apparent that students from school A and B recorded more diary entries than those in School C; however, this is somewhat counteracted by the fact that the entries from School C were quite lengthy – some almost five minutes long, whereas many of the entries from students in Schools A and B were much shorter – some only a couple of sentences in length.

School / Year Level	Year 2/3	Year 5/6	Total
A	76	37	113
B	65	40	105
C	20	20	40
Combined	161	97	258

Table 1. Total number of video / audio entries by School and year level.

Regardless, the data suggests that the students were very comfortable in recording a diary. We take this as evidence for the success of the iPad as a means of accessing student thoughts about mathematics.

Leximancer – Quantitative and Qualitative analysis

Leximancer was used to initially analyse the data which had been transcribed from the digital diaries. Concept and theme mapping was completed and frequency counts were generated for the entire cohort and then by individual schools (See Table 2).

Table 2: Frequency tables for entire cohort and categorised by school

Entire Cohort			School A			School B			School C		
Word	*Count*	*Rel*	*Word*	*Count*	*Rel*	*Word*	*Count*	*Rel*	*Word*	*Count*	*Rel*
maths	607	100%	maths	170	100%	maths	262	100%	maths	175	100%
fun	163	27%	fun	59	35%	fun	53	20%	fun	51	29%
feel	96	16%	easy	50	30%	feel	51	19%	teacher	40	23%
teacher	95	16%	times (multi)	39	23%	teacher	39	15%	groups	36	21%
easy	91	15%	division	37	22%	times (multi)	33	11%	feel	26	15%
times (multi)	81	13%	feel	19	11%	numbers	27	10%	fractions	24	14%
groups	70	12%	boring	18	11%	groups	25	10%	easy	19	11%
division	57	9%	hate	17	10%	difficult	22	8%	love	18	10%
difficult	48	8%	love	16	9%	easy	22	8%	probability	15	9%
numbers	46	8%	teacher	16	9%	division	18	7%	diagrams	12	6%
love	43	7%	sad	11	7%	pods	12	5%	chunking	11	6%
fractions	41	7%	numbers	10	6%				difficult	11	6%

Excluded concepts: doing; things; stuff; today; use; name; food; animals; favourite

The count is a raw score of the number of times a word was used and the relevance (rel) is calculated by dividing the frequency of a selected word by the frequency of the most often used word expressed as a percentage. Some words are used more frequently, and thus are more relevant, in particular schools. For example, the word easy is (15% relevant to the entire cohort, but respectively 30%, 8% and 11% per school). Some words appear on the overall list but, as they did not reach the 5% relevancy threshold at individual schools, do not appear on all of the separate lists (however, their frequency still contributes to the overall relevance). Some words have been excluded from analysis e.g. words such as "doing", "stuff", "things" did not contribute to any understanding of their attitude to maths; and words such as "food", "animals", "favourite" were excluded as they formed part of the prompt questions which many students read prior to answering. An important observation is that the frequency count does not provide information regarding the specific context of the comments; e.g. "I have fun doing fractions" and "I don't have fun doing fractions" contribute two counts for both fun and fractions and yet have opposite attitudinal content. Hence, in this paper we have used the frequency count to support a grounded theory approach (Strauss & Corbin, 1997) to point us in the direction of further inquiry and the generation of the themes for later discussion. Using this approach, we were able to locate the statements containing the frequently used words, identify the context in which they were used, and then use these insights to generate themes so that we can provide insights into the mathematical lifeworlds (Boylan, 2010) of these students.

FEELINGS AND ATTITUDES ASSOCIATED WITH MATHEMATICS

Due to the limited space available in this paper, we will only discuss the main themes that emerge across the three cohorts. The major themes that emerged from the data were a) various emotions regarding mathematics; b) relative ease or difficulty with mathematics; c) the influence of the teacher; and d) grouping and streaming.

Emotional Responses to Mathematics

A range of words were used by the students to describe their feelings about mathematics. Besides the obvious use of the word maths or mathematics, frequently used words include fun, feel, love, hatred, boring, sad and useful. The frequency of the word fun (Overall 27%, A 35%, B 20%, C29%) is interesting in that it was used in both positive (mainly School C) and negative (mainly School A) contexts. It is also interesting as a component of an increasingly common discourse in educational language where learning is only a consequence of, or at least greatly enhanced, when students are having fun. Investigating the validity of this discourse is beyond the scope of this paper; here we will take the students attitudes at face value and use the word association as an indication of attitude towards mathematics. Students who used the word fun in a positive context spoke largely in terms of it occurring when: a) mathematics involved activity (e.g. measurement, outside tasks, use of materials); b) involved new learning (e.g. money, Cartesian Plane, Probability, Problem Solving); c) included games (e.g. puzzles, mathletics) and d) included working with peers (pairs or groups - but not when streamed). In contrast, students who use the word in a negative context would often use a stem e.g. "Sometimes maths is fun but"… it is often boring; made them sad and frustrated; only when the maths isn't challenging; only when we are in groups; and markedly, only when it is easy. The use of a large range and frequency of highly emotive, negative words (boredom, sadness, wanting to be sick, to cry, hatred) in relation to why maths is not fun, and more broadly in their description of mathematics, is concerning. In School A, and to a decreasing extent in Schools B and C, these negative attitudes were common place at both junctures. We expected to reflect the findings from PISA and TIMMS that Australian students had positive attitudes towards mathematics - around 66% in Year 3 then dropping away to around 30% in Year 7/8 (Brown, 2009). However, in this research, many of the Year 2/3 students had already developed negative attitudes and dispositions towards mathematics and were beginning to identify that they were not Maths people e.g. "I'm just more of an English kind of person". When words such as hatred, hate, dislike, don't like were used they related to: a) mathematics as a subject; or b) elements within mathematics – e.g. fractions; or c) the method that was used to teach mathematics – e.g. worksheets, excessive copying from the board or, in School C, the practice of streaming. Some students were negative, but used less emotive terms such as frustration, confusion and annoyance. Again, these words followed the same pattern as before in relation to mathematics as a subject, specific content areas within mathematics, and methods of teaching mathematics.

Ease or difficulty with mathematics

There was a significant level of commentary from students regarding, perhaps paradoxically, the ease (Overall 15%, A 30%, B 8%, C 11%) or the difficulty (Overall 8%, A 35%, B 8%, C 6%) of mathematics. This paradox happened across the three schools but also within year levels in individual schools. Where easy was used, it related to either how they felt about mathematics in general (in most cases it made mathematics fun, but in a number of cases easy was linked to a dislike of mathematics), or related to specific content. Similarly, for the many students who referred to mathematics as difficult, this related in minor cases to specific content but more often to mathematics as a discipline. Further work is needed on the notions of easy and difficult in terms of how they become operationalised within learners' ideas of learning and progress. The various positive and negative connotations attached to both easy and difficult mathematics suggests a significant challenge for teachers to target mathematics at the appropriate learning level for each student. Although mathematics being too easy generated some negativity, by and large the stronger emotions occurred in relation to difficulties in mathematics and these difficulties generated feelings of hatred, anger, frustration, annoyance, and confusion - attitudes which seemed to manifest themselves in one of two ways; sadness or boredom. These were evident at each of the research schools, albeit less so in School C, and across both year levels. It is of significant concern to us that Year 2/3 students were reporting levels of sadness, crying, feeling sickness, or complaining of headaches when doing mathematics and is indicative of a strong physiological response to the experience of mathematics. An additional symptom, or perhaps cause of the sadness, was boredom. Reasons given for the boredom included: the overreliance on worksheets; significant levels of copying work from the board; lack of adequate instruction; repeatedly completing work they already knew how to do; and work that was very easy for them.

The impact of the teacher

The word teacher (Overall 16%, A 16%, B 15%, C 23%) was, like the use of the word fun, used in positive and negative contexts. In terms of positives, many of the students commented that their teacher was very helpful; that they were very influential when they needed to learn new things; that they loved mathematics because of their teacher; and, particularly in School C, mathematics was positive when they had their normal class teacher. When the word teacher was related negatively to mathematics it was in terms of: over-reliance on students copying work down from the board or on worksheets; teachers' attitudes including shouting, not spending enough time with individual students, or expecting that students should be able to do the work; incorrect or inadequate teaching – e.g. teacher confusion between positive / negative integers on a number line (indicating cardinality) and positive / negative numerals in the Cartesian plane (indicating location) and the use of incorrect mathematical language such as "plusses", "takeaways", "minuses, and "times"; and when they were taught mathematics by someone other than their normal class teacher.

Groupings and Streaming in Mathematics

The high frequency of the word group (Overall 12%, A <5%, B 10%, C 21%) was reflective of some initiatives that were being trialled in Schools B and C. School B was using mathematics groups with their Year 2/3 students (largely seen as a positive by students) and School C had recently introduced streaming for their Year 3 and Year 6 students (where a mixed but generally negative response) was forthcoming from the students. In this paper group work is that which occurs within the home classroom and streaming is ability groupings with different teachers. Positives regarding group work (including paired work and streaming) across the three schools were: the value of getting to know other students from a social cohesion perspective as well as the peer-support they provide; the opportunity to learn at their appropriate level; support of the teachers in small group scenarios; and the likelihood of games and activities being higher with group work. The negatives were: streaming in School C because they were identified as not being good at mathematics; having to work with a teacher who did not know them as learners; likelihood of distraction and off task behaviour; and group dynamics issues such as not being listened to, being made fun off and lack of co-operation. Issues around group work contributed significantly to the number of hate, dislike, sad comments noted earlier. The issue of mixed, but often negative, student attitudes towards group work within classrooms (and the more predominant negative commentary when the groups were formed according to ability and taught by a second teacher) has clear implications for schools considering the employment of specialist teachers of mathematics in the early and middle years as, at least as evidenced in this research, may be counterproductive in the long run in terms of student attitude towards mathematics.

IMPLICATIONS

Although there was a range of emotional responses to mathematics, including that mathematics is fun and that teachers are supportive, what we found alarming was the strong negative reports from the students across the three schools. As our data indicates, there are many areas for concern here. If students are developing negative attitudes towards mathematics in primary school, as appears to be the case for many students in this study, we suggest that this is a strong indicator for later withdrawal from mathematics in secondary school when this becomes an option for them. From this stage in the research process, we now have some very clear issues that have been raised by the students. The process made possible through Leximancer has highlighted areas that are needed to be explored in greater detail. The analysis has highlighted the salient concerns and positives articulated by the students. The concommitment analysis across the diversity of schools (based on social background in particular) allows further scrutiny of potential differences between the three sites. We now have a robust basis from which to undertake a rich analysis of the data. As is commonly noted as a criticism of grounded theory, the categories identified by the researcher/s may be based on personal bias. The use of Leximancer eliminates much of this bias. It has allowed the creation of categories across our three schools, but also has highlighted the differences between the schools. We are able to move confidently into a much richer analysis of the data using grounded theory knowing that the categories that have been identified through Leximancer have an empirical basis to them. **Acknowledgement:** *This project was funded through a Griffith AEL research grant scheme. We also acknowledge the contribution of Associate Professor Peter Gates (Nottingham University) to the writing*

of this paper. An article based on School A data is currently in press (IJSME) and we refer to some of those findings in this paper.

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ASSESSING THE MATHEMATICAL CREATIVITY OF PRE-SERVICE TAIWANESE TEACHERS

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In this paper, we report preliminary testing results of a 5-item, open-ended, multiple-response instrument designed for assessing mathematical creativity of pre-service teachers in Taiwan. Participants included 38 pre-service elementary teachers enrolled in a mathematics course in a Taiwan's teacher preparation program. The categorised responses and corresponding written examples for one sample item are presented. Participants' performance on three cognitive dimensions of divergent production: fluency, flexibility, originality, as well as the overall mathematical creativity for each item are also discussed.

INTRODUCTION

Mathematical creativity has long been an interest of many researchers in the context of school mathematics because of its connections to mathematical ability, mathematical achievement, problem solving and problem posing. There is a consensus that mathematical creativity should be the focus of school mathematics and can be developed through instructional tasks that focus on problem solving and problem posing (Haylock, 1997; Silver, 1997), and pre-service teachers should be provided with such tasks to develop their own mathematical creativity (Vale, Primentel, Cabrita, Barbosa, & Fonseca, 2012).

However, despite of this consensus, only a handful of studies have examined mathematical creativity in the context of teacher education. Leung and Silver (1997) examined the relationship among pre-service elementary teachers' ability to pose arithmetic problems, their mathematics knowledge, and their mathematical creativity as measured by the verbal subtest of the Torrance Test of Creative Thinking (TTCT-V). They found that the pre-service elementary teachers in their study were able to pose plausible arithmetic problems and that more than half of the posed problems required more than one step for their solutions. They also found that pre-service teachers' performance on problem solving was related to their mathematics knowledge but not to their verbal creativity. Recent studies have focused on pre-service teacher's conceptions of mathematical creativity (Bolden, Harries & Newton, 2009) or tasks that have the potential to assess or develop mathematical creativity (Shriki, 2010; Vale, Pimentel, Cabrita, Barbosa, & Fonseca, 2012). For example, Bolden, Harries and Newton (2009) found that pre-service elementary teachers in U.K. held a narrow view about mathematical creativity and had difficulty developing an activity and assessment that could be used to support student development of mathematical creativity in their future classrooms. However, all of these previous studies examined pre-service

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elementary teachers' mathematical creativity in single context: for example, problem solving or problem posing. In this paper, we present the results of a study that extended these previous studies by examining the mathematical creativity of pre-service elementary teachers in Taiwan through three different types of non-conventional tasks: problem solving, problem posing, and redefinition. Both types of problem solving and posing tasks consisted of geometrical and non-geometrical tasks while the type of redefinition task only consists of a non-geometrical task.

THEORETICAL FRAMEWORK

When surveying psychological literature, Silver (1997) identified two distinct views of creativity. The first one, the genius view of creativity, treats creativity as a rare mental trait and not likely to be influenced by instruction. The second view suggests that creativity is closely related to "deep, flexible knowledge in content domains" (p. 75) and is subject to instructional influence. This latter view encourages the development and implementation of creativity-enriched instructional activities that may benefit a broader range of students.

Haylock (1987; 1997) identified two main constructs for creativity: overcoming fixation and divergent production. Overcoming fixation is often a critical step during problem solving activity when a solution requires students to break away from the pre-established mindsets to consider alternative paths. Typically, this is assessed through one to one interviews in which the researchers can pose a series of questions to assess the rigidity of students' reasoning. As a contrast, divergent production is frequently assessed through paper and pencil format. Haylock (1997) identified three different types of task that have been used to assess this construct: problem solving, problem posing and redefinition. A redefinition task requires students to "re-define the elements of a situation in terms of their mathematical attributes" (p. 72). An example of such task given by Haylock is to find all the things that are the same for numbers 16 and 36.

The *Torrance Test of Creative Thinking* (TTCT) is a validated, normed and widely used instrument for evaluating general creativity. Since it was first published in 1966, it has been re-normed five times with a total number of 272,599 participants ranging from kindergarteners to adults (Kim, 2006). Three key dimensions of creativity: fluency, flexibility and originality (referred as novelty by some researchers, e.g., Silver, 1997 and Leikin & Lev, 2007), commonly used by mathematics education researchers to assess the mathematical creativity, have come from the design of TTCT. Roy (2011) explained fluency as the "number of correct responses", flexibility as "the number of categories of the responses," and originality as "the statistical infrequency of the responses" (p. 72).

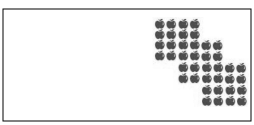
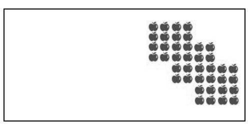
METHDOLOGY

Participants

Thirty-eight Taiwanese pre-service teachers enrolled in a basic college mathematics course took this test as part of the in-class activity. The duration of the assessment was 50 minutes.

Instrument

Previous studies have found that nature of the task (conventional vs. non-conventional) had an impact on student performance. Leikin and Lev (2007) found that the gifted students performed similarly to their non-gifted counterparts on the conventional task but outperformed them on the non-conventional task. Van Harpen and Sriraman (2013) also noted that the geometrical task used in their study was less likely to create cultural biases. Based on the theoretical framework and literature reviews, a five-item mathematical creativity instrument was developed and implemented. The instrument consists of two problem solving tasks (#2 and #4), two problem posing tasks (#1 and #5), and one redefinition task (#3). Each pair of the problem solving and problem posing tasks contains a geometrically-based one (#4 and #5). Each task contains 10 response spaces. A reminder of "Please watch out for the time distribution" was listed after each question. Figure 1 shows the five items.

Item #	Task
#1	Pose as many word problems as possible that can be solved by using one division sign only (no plus sign, minus sign or multiplication sign is allowed). Try to make these word problems as different as you can. Question #1: Question #2: …...
#2	Use different ways to calculate the number of apples inside each box below. Try to make these ways as different as possible. For each way, please illustrate the process of thinking in the box and then state the corresponding calculation steps. (adapted from Vale et. al., 2012).   …...
#3	Write out the common mathematics attributes shared by the numbers 16 and 36. Try to make the attributes as different as possible. (adapted from Haylock, 1997). Common attribute #1: Common attribute #2: …...

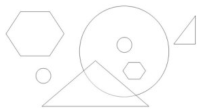
#4	Draw different quadrilaterals in the boxes below. Try to make these quadrilaterals as different as possible. ......
#5	Use the figure below to generate mathematics problems. Try to make the problems as different as possible.  Question #1: Question #2: …...

Figure 1: The five items

The validity of the instrument was established using the six criteria suggested by Haylock (1997) for an effective task in revealing creativity and differentiating their levels. That is, all these five items allow the participants to: (1) show a range of different responses; (2) provide the possible number of such responses larger than 20; (3) have consistent interpretations; (4) contain several obvious solutions; (5) afford unique solutions; and (6) demonstrate mathematical importance and creativity in those unique solutions.

Data Analysis

The calculation of the mathematical creativity score was based on the assumption that all three cognitive dimensions made equal contributions. The score was established through a multi-step process described below.

Calculation of the fluency score for each task. Each participant response, if judged to be valid, receives 1 point. The total number of valid responses is the score for that item.

Calculation of the flexibility score for each task. First, identify the number of different categories of response each participant generated (flex). Second, the flexibility score for each item is the fraction of flex/Maxflex, where Maxflex is the total number of different categories of response generated by all participants.

Calculation of the originality score for each task. First, for each category, calculate the percentage of all participants who gave that variety. These percentages are a_1, a_2, …. $a_{maxflex}$. Second, for each participant who has given n categories, he or she has a corresponding set of percentage b_1, b_2,….b_n, which is a subset of the { a_1, a_2, …. $a_{maxflex}$ }.The originality score for this participant on this item is calculated using the formula:

$$\frac{(1-b_1)+(1-b_2)+...+(1-b_n)}{(1-a_1)+(1-a_2)+......+(1-a_{\max flex})}$$

Calculation of the mathematical creativity score for each participant. First, let A/B/C be the participant's fluency/flexibility/originality preliminary score. Use mean= 50 and standard deviation 10 to obtain a T-score to get the standardised individual scores. Second, each participant's mathematical creativity score for each item is the mean score of those three individual scores. Third, each participant's final fluency, flexibility and originality cores are the means of their respective T-scores on all items. Lastly, each participant's mathematical creativity score is the mean of their final fluency, flexibility and originality scores.

Two trained mathematics education graduate students scored all the papers independently. For each of the five tasks, they first generated the coding categories of possible response. They next coded an appropriate category for each valid response. In the end they then assigned fluency score (the number of valid responses) and flexibility score (the number of categories used) for each participant on that task. Ten randomly selected papers for each task were scored twice independently to calculate the inter-rater reliability. The inter-rater reliability ranged from 0.94 (division task in Item #1) to 0.99 (apple task in Item #2) across items.

SELECTED PRELIMINARY RESULTS

An Example of Possible Categories of Responses: Apple Task

Based on the written responses collected from the participants, five primary categories of responses comprising (a) partition, (b) move, (c) supplement, (d) overlap, and (e) mixture were categorised. They can be further broken down into 9 sub-categories of responses. Table 1 below gives the frequency of each sub-category used by all 38 participants. According to Table 1, the use of partition category was popular. Over 60% of responses from each participant were completed through using partition as its primary response.

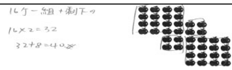
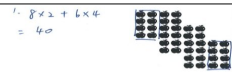
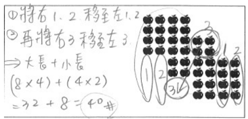
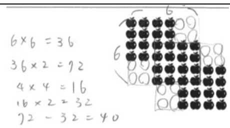
Table 1: Pre-service elementary teachers' performance on the apple task in Item #2 (Total number of valid responses = 334)

Category	Single Partition	Double Partitions	Multiple Partitions	Single Move	Double Moves
# of usage	88	142	14	21	1
(% of usage)	(26.35%)	(42.51%)	(4.19%)	(6.29%)	(0.30%)

	Single Supplement	Double Supplements	Overlap	Mixture
# of usage	36	3	28	1
(% of usage)	(10.37%)	(0.90%)	(8.38%)	(0.30%)

Table 2 shows examples for the most popular sub-category: Double Partitions, and for the two most unique ones: Double Moves and Mixture.

Table 2: Selected sub-categories with examples

Category	Example(s)	
Double Partitions	 Partition by 16 and 8 .	 Partition by 8 and 6.
Double Moves		Moves two 4×1arrays in #1 and #2 spots on the right bottom and a 2×2 array on the top right in #3 to the left bottom side to form 8×4 and 4×2 arrays.
Mixture		In each of two partitions, supplement two sets of 4 apples to form a 6×6 array. Then, subtract 4×4 overlapping and 4 supplemented sets of 4 apples.

Pre-Service Elementary Teachers' Performance on the Apple Task

Based on the categories discussed earlier, pre-service elementary teachers' performance on the apple task can be summarised in Tables 3 and 4 below.

Table 3: The distribution of responses by pre-service elementary teachers (n=38)

# of response	1	2	3	4	5
# (%) of participants	0 (0.00%)	0 (0.00%)	0 (0.00%)	1 (2.63%)	0(0.00%)
# of response	6	7	8	9	10
# (%) of participants	4(10.53%)	6(15.79%)	1(2.63%)	4(10.53%)	22(57.89%)

The average number of valid response generated by these participants is 8.79, which indicates a fairly good fluency.

Table 4: The distribution of sub-categories by pre-service elementary teachers (n=38)

# of sub-category	1	2	3	4	5
# (%) of participants	0 (0.00%)	5 (13.16%)	11 (28.95%)	10 (26.32%)	10 (23.32%)
# of category	6	7	8	9	10
# (%) of participants	2(5.26%)	0 (0.00%)	0 (0.00%)	0 (0.00%)	0 (0.00%)

The average number of sub-categories used by participants was 3.82, which indicates that the majority of them were able to utilise more complex strategies that required spatially manipulating the physical arrangements of apples.

Pre-Service Elementary Teachers' Performance across Five Tasks

Table 5 provides a preliminary comparison of the T-score and standard deviation of the fluency, flexibility, originality and mathematical creativity across five tasks.

Table 5: Pre-service teachers' performance across five tasks

Task	#1 posing non-geometric	#2 solving non-geometric	#3 redefinition non-geometric	#4 solving geometric	#5 posing geometric
Fluency	46.47(11.08)	53.57(6.93)	47.87(9.95)	55.40(5.07)	46.69(11.98)
Flexibility	40.95(4.57)	47.30(5.33)	53.85(8.17)	62.32(6.66)	45.57(8.22)
Originality	41.70(4.12)	44.47(5.46)	53.12(8.58)	61.64(9.5)	49.07(6.91)
Creativity	43.04(4.87)	48.45(4.50)	51.61(8.06)	59.79(5.94)	47.11(8.62)

Generally speaking, participants earned higher scores on problem solving than problem posing tasks, and on geometric than non-geometric tasks within those problems types in the overall mathematical creativity.

DICUSSION

Preliminary results of pre-service elementary teachers' performance indicate that the instrument—for assessing mathematical creativity via problem solving, problem posing and redefinition tasks—was effective in illuminating the fluency, flexibility and originality in their thinking.

The results of this study also add to the literature on pre-service teachers' mathematical knowledge in general and mathematical creativity specifically. The tasks used in this study can serve as a springboard for potential ideas that can be used for students (and teachers alike) at all levels to develop their mathematical creativity. This study found that pre-service teachers adopted the partition category more often than the other primary categories. We suspect that the use of this category is less cognitively demanding and more intuitive compared to the other categories. For instance, the use of partition category does not need to consider interpenetrating elements (Krutetskii, 1976) like the use of overlap category; nor does it involve deconstructive reasoning (Rivera, 2009) needed for using move or supplement category. This study also found that pre-service teachers performed better in problem solving than problem posing tasks. Since problem posing demands more verbal processing than problem solving, it is reasonable to question whether or not verbal ability impacts mathematical creativity.

In sum, it is interesting to explore how deconstructive reasoning, interpenetrating elements, verbal processing, or any other potential factor(s) are related to mathematical creativity in future studies. We are in the process of administrating the instrument to samples of different populations that include gifted and non-gifted elementary students, in-service teachers, and pre-service teachers in the USA so we can illuminate the nature of mathematical creativity across different populations.

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TEACHER TENSION: IMPORTANT CONSIDERATIONS FOR UNDERSTANDING TEACHERS' ACTIONS, INTENTIONS, AND PROFESSIONAL GROWTH NEEDS

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Mathematics teachers do not come to their professional growth opportunities as blank slates. They come to them carrying a complex array of tensions that affect their intentions and actions as a teacher, and are often the very reason that they are seeking out professional growth opportunities. In this article we explore some of these tensions in the form of dichotomous pairs of forces that emerge out of, and act on, mathematics teachers' experiences. Results indicate that, unlike prior research on tensions, teachers do not simply manage these opposing forces, but also work at, and seek help in, resolving them. This extension has important implications for our work, as a research field, in the crafting and delivery of professional development opportunities for mathematics teachers.

INTRODUCTION AND THEORETICAL BACKGROUND

Teachers do not approach their professional learning as blank slates. They come to it with a complex collection of experiences (as students, future teachers and teachers) and of wants and needs and use professional development opportunities as resources to satisfy those wants and needs (Liljedahl, 2014) in the light of their previous experiences. Often, what teachers want are answers to the tensions that they are experiencing in their daily practice – tensions between what they want to do and what they have time to do, tensions between what they believe to be important and what they are being pushed to do. These tensions are, themselves, complex collections of opposing forces of wants and needs that complicate the decision making processes of teachers. For a field that places at its core the mathematics education of students, it is important to better understand these tensions and the role they play in the decisions that teachers make, the lessons they deliver, and the answers they seek through their professional growth. In this article we explore a framework for looking closer at these tensions and offer an extension of this framework that will allow us to better understand the wants and needs of mathematics teachers.

Tensions, often expressed as "dilemmas", have been recognised as an integral part of teaching practice dating since the early 1980's. Berlak and Berlak (1981) examined the complex and sometimes contradictory behaviours of teachers in responding to the curriculum within socio-cultural contexts. Building on this work, Lampert (1985) emphasised the personal and practical aspects of dilemmas. For her, tensions can be

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understood as problems to be managed, rather than solved. As such, Lampert (1985) characterises teachers as "dilemma managers", who find ways to cope with conflict between equally undesirable (or desirable but incompatible) options without necessarily coming to a resolution. For Lampert (1985), the ongoing internal struggles presented by the tensions arise from, and contribute to, the developing identity of the teacher, and as such have value in themselves. In contrast to other approaches to understanding the practice of teaching, from Lampert's (1985) perspective the admission that some of the conflicts encountered in teaching are not resolvable is not a weakness.

Adler (2001) also takes the view that dilemmas in teaching are often managed rather than solved. As well, she agrees with Lampert (1985) that their instances arise in the context of teaching, and that the recognition and management of the dilemmas is tied to personal biography. However, along with these considerations, she integrates the Berlaks' (1981) socio-cultural perspective, which emphasises the importance of the wider context, beyond the classroom situation.

Building upon the work of Adler (2001) and Lampert (1985), Berry (2007a, 2007b) utilized the notion of tension as a framework for both doing and understanding her research. By looking at tensions as dichotomous forces acting on a teacher, Berry (2007a) conducted a self-study of her efforts to improve her practice in her new role as a teacher educator. She found that the notion of tensions captured well the feelings of internal turmoil experienced by teacher educators as they found themselves pulled in different directions by competing pedagogical demands in their work and the difficulties they experienced as they learned to recognize and manage these demands (Berry, 2007b). The result was twelve tensions, expressed as dichotomous forces, that "capture the sense of conflicting purpose and ambiguity held within each" (Berry, 2007b, p. 120):

1. *Telling and growth*: between informing and creating opportunities to reflect and self-direct or between acknowledging prospective teachers' needs and concerns and challenging them to grow.

2. *Confidence and uncertainty*: between making explicit the complexities and messiness of teaching and helping prospective teachers feel confident to progress or between exposing vulnerability as a teacher educator and maintaining prospective teachers' confidence in the teacher educator as a leader.

3. *Action and intent*: between working towards a particular ideal and jeopardising that ideal by the approach chosen to attain it.

4. *Safety and challenge*: between a constructive learning experience and an uncomfortable learning experience.

5. *Valuing and reconstructing experience*: between helping students recognise the 'authority of their experience' and helping them to see that there is more to teaching than simply acquiring experience.

6. *Planning and being responsive*: between planning for learning and responding to learning opportunities as they arise in practice (Berry, 2007b, p. 32-33).

Although developed within the context of teacher education, Berry's (2007a, 2007b) structuring of tensions as dichotomous forces appeals to us in our consideration of teacher tensions. As such, our research question is aimed at identifying comparable pairs of forces within the lived experiences of mathematics teachers. Although we are ultimately interested in developing a more detailed understanding of how these tensions are born within teachers and teacher practice, as well as how they affect their decision making and teaching actions, for the purposes of this study we are focusing only on emerging a set of tensions, expressed as dichotomous forces.

METHODOLOGY

The data from this study comes from our collective experiences as teachers, teacher educators, professional development facilitators, and researchers. Through each of these roles we have encountered thousands of teachers and collected endless data. Because of space considerations it is not possible to describe the varied and various contexts and methodologies from which our data is drawn. As such, we have chosen, instead, to present our data in the form of an amalgam – a fictionalized aggregate of four different cases drawn from our collective data sets.

This amalgamation of cases into one single case is not new. For example, Piaget (1923/2001) built his developmental stages of a single, fictional child, from the disparate observations of his own children, each at different stages of development. In the context of mathematics education, Lerron and Hazzan (1997), and more recently Zazkis and Koichu (2014), have used the methods of virtual or fictional inner monologues as a way to tell a more complete and aggregated story than any one set of data could.

The cases from which this amalgam was constructed were each, in themselves, carefully developed through methods of narrative inquiry (Di Martino & Sabena, 2010), ethnographic study (Liljedahl, 2014), mathematical biographies (Andrá et al, 2010), or case based research (Pezzia & Di Martino, 2011). Because our various research programs were focused on different aspects of teaching and teachers, from teacher beliefs to teacher professional development, each case tells a portion of a story with the amalgam telling the whole story – the story of Janet.

THE STORY OF JANET

As an elementary school student, Janet was good at mathematics. When she moved onto lower secondary school, however, her marks began to slide and by the time she was in upper secondary school she was barely passing her mathematics classes. The rest of her marks, however, were good enough to give her entry to university. Janet had known all her life that she wanted to be an elementary school teacher. Knowing that this would require her to take more mathematics courses, coupled with her recently emerged low self-efficacy around mathematics, almost dissuaded her from following this dream. But she endured and finished her undergraduate degree and gained entry into the teacher education program.

During teacher education Janet learned philosophies and methods of teaching mathematics that allowed her to see that mathematics doesn't have to be the way she had experienced it as a student. It could be taught through activity, with a focus on building understanding through collaborative problem solving. This gave her hope that she could become the type of teacher that did not drive students to fear mathematics.

During her practicum Janet was paired with a teacher that was more traditional in her views and practices. Although willing to let Janet build the types of classroom that she wanted, the practicum teacher was also quick to criticize Janet's teaching for its non-conformity to the traditional values that she held. Janet understood the importance of pleasing her sponsor and so she shifted to a teaching model based on transmission of information and practice of rudimentary skills.

Janet knew that this was not the kind of teacher she would be once she had her own classroom. She was playing a part – a part that would get her through the program and into a job. She played this part very well and was one of the few student teachers who were immediately given her own classroom after graduation.

Janet was now a grade 7 teacher. The school where she worked had a strong sense of teamwork among the staff. In mathematics, each grade had a team leader who picked the textbook, identified and created resources and tests, and who sequenced and paced out the curriculum. The particular teacher in charge of the grade 7 mathematics curriculum was unhappy with the new textbooks that were being written and had, instead, opted to use a series of workbooks to guide the students through the curriculum. These workbooks were very traditional in nature, requiring Janet to give brief lectures on how to do a skill and then the students would practice this skill in the workbook. These were then to be checked for completeness every day. Janet did not like these workbooks, but as a beginning teacher, felt that she was too novice to complain. So, she followed along with the system set out for her.

Janet's first two years of teaching were unbelievably busy. She was quickly named as the curriculum coordinator for grade 7 language arts and this took a lot of her time. Although still not happy about the mathematics program, she did appreciate the little effort and time it took her to deliver the mathematics lessons.

In her 3rd year of teaching Janet began to take stock of her mathematics teaching. She had two students who were really struggling and she could see their frustration and anxiety building. She began to make small changes in the way she taught. She would let students work in pairs on their exercises and would occasionally have little warm-up puzzles at the beginning of mathematics class. Janet also decided to have the students do a project on a famous mathematician. Other than this, however, her teaching remained much the same. She still relied heavily on the transmission model and the students worked out of the workbooks for the majority of class.

Janet knew that the changes she had made were not enough. The light in the two girls' eyes, although briefly illuminated during the project, were continuing to dim. She needed to do more to change her teaching. So, over the next two years she sought out some of the professional development opportunities offered within her school district. She attended workshops on teaching mathematics through literature, formative assessment, and technology. With each of these she made small changes in her teaching. Her students were not much impressed with her new use of literature and hated the journaling she was trying to get them to use. She endured for three months, but in the end abandoned these efforts. Her principal was really keen on her interest in technology and supported her efforts to bring this into her classroom. The student liked this too, but in the end it had little to no impact on her mathematics teaching.

Janet then attended the first of a number of sessions on teaching mathematics through problem solving. During the first session, in which the facilitator immersed the teachers in a number of problem solving activities, Janet immediately felt at home. This was the same experience she had had during her teacher education program – the experience that had given her so much hope for the type of teacher she could become.

The next day she implemented one of these problem solving activities. The students did not put up any resistance. They were working with their friends and they were used to this. However, they were not as effective in working on their own as Janet had hoped. But she persisted and, with the help of the ongoing professional development sessions, became more skilled at facilitating such an environment. Over the course of the next month she began to teach with and through collaborative problem solving more and more. At first, everything was fine, but after about two weeks she began to get questions from some of students' parents about when she would be going back to teaching mathematics the "normal way" and one of her students was suddenly transferred out of her class. At about the same time she also began to see resistance from some of her students. But Janet believed in what she was doing and was seeing some positive effects in some of her students. So she persisted.

ANALYSIS AND DISCUSSION

In what follows we present an analysis of the case of Janet in the form of dichotomous forces present in the Janet's lived experience as a mathematics teacher. In so doing, we begin with some of Berry's tensions (2007a, 2007b), but then extend this framework to include other pairs of dichotomous forces present, not only present in the case of Janet, but also recurrent in our experiences with teachers.

Confidence – Uncertainty

Before entering a teacher education program, Janet was confident that she would like to become a teacher. Her recent performance in mathematics, however, made her uncertain to the point that her bad marks "almost dissuaded her from following this dream". But Janet's desire to become an elementary school teacher and her confidence that she would be a good teacher, were stronger than her uncertainty, and she persisted.

We see here an emergence of the dichotomy confidence-uncertainty, and a way that Janet coped with this tension through persistence.

Intent – Action

During her teacher education program Janet learned new ways of teaching that she liked, and in the practicum she took action – enacting with her students teaching methods she had recently learned. But, a conflict emerged between her practicum teacher's traditional ways of teaching and Janet's more progressive and innovative efforts. Janet deals with this tension in a very pragmatic way. Since her intent is to become a teacher, rather than challenging the practicum teacher, she plays along with her practicum teacher's wishes. Not taking action is not a resolution of this conflict, but a way to cope with it. This tension between intent and action is something that emerges over and over in the next few years of Janet's experiences as a mathematics teacher.

Tradition – Innovation

The aforementioned tension between intent and action, resolved (or postponed) through inaction is centred around another tension – a tension between the traditional wishes of Janet's practicum teacher and Janet's desire to be more innovative. This tension comes up again when she is a grade 7 teacher and she is stuck between wanting to enact her own teaching program and following along with the school adopted workbooks. Again, Janet does not resolve this tension, but takes the safe position in consideration of the social and collegial environment of her new school.

Safety – Challenge

Her preference for safety also characterizes her early time as the language arts coordinator when she "appreciates the little effort" required to teach mathematics. In order to change this situation, something needs to happen – and it does.

Janet noticed two students whose frustration and anxiety towards mathematics were beginning to build. Janet recognized these feelings from when she was a secondary school student. She recalled how her negative relationship towards mathematics was attributed to her teachers and their ways of teaching. So, Janet decided that she had to do something. So, she made little changes. In the dichotomy of safety-challenge, Janet is still on the "safety" side because the little changes she made do not challenge her way of teaching and she can still largely rely "heavily on the transmission model".

Valuing – Reconstructing Experience

Then Janet makes some major changes by, once again, valuing her experience from her teacher education program. The dichotomy valuing-reconstructing experience emerges as the tension between Janet's recollections of the, *then*, impact of her experience as a student teacher is pitted against the reality that there is more to teaching than simply acquiring experience – and that Janet, *now*, needs to reconstruct this experience for her students. This stress on "valuing" leads her to give up, and not to pursue her effort to change her practice, even if her principal was supportive.

Telling – Growth

Janet entered the teaching profession aware that the transmission model of teaching was not in the best interests of the majority of her students. Her desire was to provide mathematical experiences for her students that were unlike her own. She felt her role should be to create opportunities for students to construct their own knowledge rather than lecturing and explaining. Yet, a tension surfaces during her practicum when she finds herself succumbing to traditional teaching methods. Her later attempts to incorporate aspects of collaborative learning heightened the tension as she saw how positively her students responded.

Conforming – Personal Convictions

Teachers often feel a great deal of pressure to conform to the norms and standards of their school, their mentors, and their grade partners. This is especially true for beginning teachers. Tension can emerge when abiding by the norms conflicts with personal pedagogical beliefs. Janet experienced this tension twice. The first was when her practicum teacher criticizes her non-traditional approach. She then experienced it again when she discovered she was required to follow her team leader's mathematics workbook policy. In both of these instances Janet felt the tension between the need to conform and personal convictions. Initially compliant, the resulting tension, in the end, becomes the impetus for her to seek out professional development opportunities.

Time – Results

Eventually Janet began to teach with and through collaborative problem solving. She had ambitious teaching goal and she encountered her first significant teaching failure when one of her students was suddenly transferred to another class. This failure could have persuaded Janet to backtrack on her educational choices and develop a didactical approach where the results were less significant but more immediate. But Janet persisted with her didactical choice and decided to give more time to herself and to her students.

DISCUSSION AND CONCLUSION

These are but a few of the tensions experienced by Janet. Many of these, like time-results and safety-challenge, are connected and interrelated. Some of these tensions can be recast as other pairings. For example, in the case of Janet, conforming-personal conviction can also be seen as a tension between her as a novice, and her mentor teacher and colleagues as being experienced.

Janet is a fictional person, but the tensions she experienced are those experienced by the four real teachers that the case of Janet is built upon, as well as the countless teachers that have experienced, or are currently experiencing, similar dichotomous forces pulling on their intentions and their actions.

Lampert (1985) and Adler (2001) would characterize Janet as managing these tensions. And they would be correct – for a time. As much as Janet does initially manage her tensions by choosing safe and conforming paths, our results show that eventually these managed tensions move her to try to seek resolution. Janet starts to make changes in her teaching, she begins to seek out professional development opportunities. These changes create new tensions for Janet – tensions that she is learning to deal with through persistence. In the end, persistence turns out not to be a management strategy, but a resolution strategy.

Much of who Janet is as a mathematics teacher, like all mathematics teachers, is shaped by the tensions she is experiencing. Regardless of whether teachers manage these tensions or try to resolve them, better understanding of these tensions would allow us, as mathematics education researchers, to better understand the intentions and actions of mathematics teachers – and to better respond to their needs in the crafting and delivery of professional development opportunities.

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FACTORS THAT ASSIST PRE-SERVICE TEACHERS TO DEVELOP MATHEMATICAL CONTENT KNOWLEDGE DURING PRACTICUM EXPERIENCES

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Teachers' mathematical content knowledge (MCK) is crucial and determines how they will teach. Identifying factors that develop pre-service teachers' (PSTs) MCK will assist them to extend their knowledge for teaching. This paper reports on factors that assisted two PSTs to develop MCK during coursework and practicum experiences, but also highlights limitations of the learning experiences intended to extend MCK. The results indicate that factors such as learner and teacher identity, sustained engagement and quality of teaching experiences should be considered when designing improvements to PSTs' practicum experiences.

INTRODUCTION

Australian standards state that teachers should know the content they teach (Australian Institute for Teaching and School Leadership (AITSL), 2011) because the knowledge a teacher brings to the classroom is fundamental as it underpins the decisions they make during teaching (Rowland, Turner, Thwaites, & Huckstep, 2009). Many studies have refined our understanding of knowledge needed for mathematics teaching and different categories of MCK (e.g. Ma, 1999; Rowland et al., 2009) however, further studies are required that focus on how PSTs learn knowledge for teaching (Anthony, Beswick, & Ell, 2012). Little is understood about contributing factors that assist PSTs to develop their MCK during practicum experiences.

LITERATURE REVIEW

The process of becoming a teacher can be influenced by pre-program identity such as prior beliefs about mathematics teaching (Anthony et al., 2012) or self-efficacy, shaping how PSTs imagine mathematics is taught (Walshaw, 2008) and their response to opportunities to learn (Tatto, Lerman, & Novotná, 2009). Pre-program identity may include a reliance on procedural methods developed at school before commencing teacher education (Ponte & Chapman, 2008) or beliefs such as "mathematics is difficult" or "mathematics is all about one answer" (Liljedahl, 2005, p. 1).

Primary teacher education programs should aim to assist PSTs to create interest in and passion for learning, including the knowledge needed for teaching. Southwell, White and Klien (2004) concluded there is evidence that PSTs can change their beliefs about mathematics during their program but it is not clear what factors might influence change and its sustainability. Other research has identified a need for further understanding of how PSTs use knowledge in practice with students (Anthony et al.,

2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 201-208. Hobart, Australia: PME.

2012). Butterfield and Chinnappan (2010) concluded that although PSTs can build on their MCK, they have difficulty in *transforming* this knowledge when designing tasks for students.

Practicum experiences assist PSTs to develop their pedagogical content knowledge and have potential to develop MCK. Butterfield (2012) reported programs that immerse PSTs in practicum experiences assist with development of their MCK. McDonough and Sexton (2011) suggested shared school-based teaching experiences between PSTs, university lecturers and practising teachers can provide opportunities for PSTs to develop their teaching skills.

Others have explored PSTs' development of identity as a means of fostering knowledge. Walshaw (2008) recognised past educational experiences as a student, teacher education programs including coursework, and practicum experiences as factors contributing to teacher identity. From a research perspective, frameworks of teacher knowledge can assist with deepening our understanding of the different categories used to describe MCK (Bobis, Higgins, Cavanagh, & Roche, 2012). For example, Ma (1999) described accomplished teachers as having profound understanding of fundamental mathematics (PUFM) showing breadth, depth, connectedness and thoroughness of MCK. The Knowledge Quartet framework identified four categories of MCK: *foundation* knowledge what teachers know and believe, *transformation* choice of examples and representations, *connection* between procedures and decisions about sequence and *contingency* responding to student questions (Rowland et al., 2009). Ensuring PSTs have the opportunities to develop categories of MCK is important for primary mathematics teaching.

METHOD

Setting and program structure

This paper reports on two PSTs' practicum experiences and opportunities to develop MCK for primary mathematics teaching during a four-year teacher education program. The participants, Lisa and Rose (pseudonyms) were chosen because their program experiences identified factors that contributed to development of their MCK and findings could be compared with the larger study (Livy, 2014). They were enrolled in a Bachelor of Education (Preparatory–Year 12) program that combined learning at university with partnership-based teaching experiences in schools in which the practice of learning to teach is combined with theory. On graduating, PSTs have the qualifications to teach primary school students (aged 5-12), including primary mathematics, and two secondary specialisation subjects. The PSTs completed three core primary mathematics education units during first and second year (Units 1A, 2A and 2B), with an option to undertake a fourth (Unit 1B) to assist them to extend their MCK. PSTs were also required to pass a Mathematical Competency Skills and Knowledge (MCSK) test. They had opportunities to develop their knowledge for teaching in primary schools (first, second and fourth year); and teaching of their secondary discipline studies in a secondary school (third year). Due to space limitations the results have been confined to practicum placement and fourth-year teaching.

Selection of participants

Lisa and Rose were selected from 17 PSTs who agreed to take part in a four-year longitudinal study of the development of PSTs' MCK (Livy, 2014). Both identified difficulties with their MCK during the program and completed Unit 1B to extend their MCK. Rose was able to demonstrate her MCK and passed the MCSK test at the end of first-year during Unit 1B. Conversely, it was not until the beginning of third year after several attempts, that Lisa passed her MCSK test and Unit 1B. These characteristics were representative of more than half of their cohort (N=300). The findings aim to assist future PSTs and inform program design.

Data collection and analysis

Data were collected at different times and in different situations throughout the four years of Lisa and Rose's program. An ethnographical design was chosen and included four methods of data collection: questionnaires, observations, interviews, and analysis of documents (McMillan, 2004). Data collection consisted of an initial questionnaire (Year 2); lesson observations when practising teaching a primary mathematics lesson (Year 2 and 4); one-on-one interviews (Year 2, 3 and 4); artefacts and documents from coursework; and practicum experiences (Year 1, 2 and 4). Data collection, management, and analysis occurred simultaneously and included content analysis, reducing the data by using coding (Simminoff & Jacoby, 2008) as well as triangulation of the data (McMillan, 2004) to identify factors that developed MCK.

RESULTS AND DISCUSSION

PSTs' opportunities to learn MCK were influenced by program structure and approaches including learner and teacher identity, sustained engagement and quality of teaching and learning experiences.

Learner Identity

Learner identity was inferred from the way in which PSTs might extend their MCK and was shaped by program choices, identity as learners of mathematics and self-efficacy. In fourth-year Lisa and Rose, were in composite Year 3/4 classes at different schools and their lesson preparation provided evidence of their learner identity and categories of MCK they demonstrated.

When planning Lisa described how she met with her mentor teacher (classroom teacher) to discuss ideas before the lesson, then planned an activity at home for the following day but did not, or was not required to, write lesson plans. When observed teaching by the researcher during fourth-year, Lisa taught a measurement lesson.

> When you add up perimeter you need to add the distance all the way around… If I have this square and it is two centimetres high and four centimetres wide what is the equation I use to write the area?

Lisa relied on procedural knowledge when explaining the perimeter and area of rectangles rather than considering how to assist student understanding. She was not yet demonstrating *foundation* knowledge (Rowland et al., 2009) because she needed to extend her knowledge of mathematical terms and did not consider the *connections* within her lesson that would assist her students to make sense of the mathematics.

Rose taught geometry and properties of triangles. During an interview with the researcher, Rose explained that before the lesson she checked her MCK before teaching a topic by looking up terms referring to coursework notes, the internet or by asking her mentor teacher to check her lesson plan. Rose was a dedicated learner who believed in the importance of planning and knowing what and how she would present when teaching primary mathematics lessons.

Rose introduced her lesson by asking the students to brainstorm what was similar and different about a set of laminated triangles she had made. Next they sorted and labelled the triangles into three groups whilst discussing the properties for scalene, isosceles or equilateral triangles. When teaching, Rose's development of her MCK was revealed by her choice of question types, including open-ended questions to discuss differences and counter examples of triangles demonstrating *foundation* knowledge.

> Rose: How do you describe a triangle?
>
> Student: Three sides.
>
> Rose: Anything else?
>
> Student: Three angles.
>
> Rose: What is the same or different about the triangles? [showed a set of laminated triangles that represented isosceles, scalene and equilateral]
>
> Student: That triangle has an obtuse angle.
>
> Rose: What other angles can you see?

An appropriate range of examples and representations of triangles were suitable for extending students' mathematical understanding. She also made *connections* by choosing similar tasks to assist students to identify the differences, similarities and properties of triangles during the lesson. Rose was able to rely on her MCK when teaching because of her developing teacher identity and thoroughness when preparing lessons, including revision of her *foundation* knowledge before teaching.

Teacher identity

Teacher identity is "assuming the values and norms of the profession" (Ponte & Chapman, 2008, p. 241) and learner identity was also influential in forming PSTs teacher identity and readiness to teach. The PSTs had the opportunity to develop their teacher identity when observing their mentor teacher teaching mathematics lessons. For their practicum teaching they were in different primary schools for first, second and fourth year (Table 1).

Name	First-year primary school n=20	Second-year primary school n=32	Third-year secondary school n=42	Fourth-year primary school n =50
Lisa	Year 1 and 2	Year 3	No mathematics	Year 3 and 4
Rose	Year 1 and 2	Year 5 and 6	No mathematics	Year 3 and 4

Table 1: Distribution of PSTs practicum days and number of days (N=144) in schools

Lisa and Rose experienced 102 days in primary schools under the supervision of their mentor teacher. The different experiences and teaching situations would have assisted Lisa and Rose to reflect and consider their own professional identity. Others agree that learning together develops effective teaching (McDonough & Sexton, 2011) and personal experiences as well as practicum experiences construct teacher identity (Walshaw, 2008).

Sustained Engagement

Sustained engagement, such as the distribution and number of days of practicum experiences (Table 1) were another factor that influenced PSTs' MCK. The PSTs at this university experienced more days in primary schools than their counterparts in other countries (Tatto, Schwille, Senk, Ingvarson et al., 2012) and program accreditation minimum of 45 days, providing more practical opportunities to extend their teacher MCK.

Opportunities to develop *depth* and *breadth* (Ma, 1999) of MCK were identified by comparing the distribution and year levels PSTs experienced in schools (Table 1). Lisa did not experience Year 5 and 6, limiting opportunities to extend her *breadth* of MCK in the upper years. Rose experienced *depth* of teaching experience ranging from Year 1 to Year 6, maximising her opportunity to extend her MCK across more year levels. The program structure did not ensure Lisa continued to sustain and extend her MCK and should be reviewed for future PSTs when considering opportunities to develop *breadth* and *depth* of MCK of all PSTs.

Quality of teaching and learning experiences

The expectations and quality of learning experiences provided by mentor teachers when PSTs were planning and preparing lesson plans, was another factor that assisted development of PSTs' MCK. During an interview with Rose, she said that her mentor teachers sometimes suggested resources such as websites or books, assisting her to plan and check her MCK. Rose also had a mentor teacher who checked her lesson plans providing further opportunity to improve her MCK or *foundation* knowledge such as use of mathematical language, *connections* within or across lesson.

The PSTs had different mentor teachers and experiences for each year of their program which impacted upon their learning of MCK. PSTs such as Lisa who had an ad hoc approach to planning before teaching and limited assistance or expectations from her mentor teacher, had difficulties relying on different categories of her MCK when teaching in fourth-year. Furthermore mentor teachers would have been influential in assisting PSTs to develop, focus on and extend different categories of MCK because of the large number of days experienced in schools. A limitation of the study was that mentor teachers were not interviewed.

CONCLUSION

Walshaw (2008) suggests that PSTs enter the program bringing ideas or assumptions of what teaching should be and these ideas can compete with theories presented during program experiences. Lisa and Rose's practicum experiences varied, and either limited or extended their MCK. However their experiences did identify factors that were important for developing different categories of their MCK. Furthermore it is important to consider these factors and implement changes in program structures to ensure sustained growth of MCK for all PSTs.

By fourth year Lisa was not yet demonstrating *foundation* knowledge because she had difficulty relying on her MCK and used rules without fully demonstrating why the rule worked when teaching. Rose was a more accomplished PST who could rely on her MCK after revising mathematical concepts before teaching.

While limited conclusions can be drawn from two case studies, factors that restricted Lisa's opportunities to develop her MCK were:

- Limited learner identity and engagement;
- Practicum teaching not experienced across different year levels during her program, particularly upper primary levels;
- Ad hoc planning and little preparation before teaching during her practicum teaching experience with limited assistance from her mentor teacher;

Factors that assisted Rose to develop her MCK during the program were:

- Careful planning and preparation before teaching during her practicum teaching experiences assisted by her mentor teacher;
- Program structure, sustained opportunities to develop breadth and depth
- Mentor teachers that promoted opportunities to develop MCK
- Development of *foundation* knowledge, learner and teacher identity

Ensuring that future PSTs entering teacher education identify the importance of knowing mathematics for primary teaching, and seek opportunities to learn MCK during practicum teaching across different year levels with guidance from their mentor teacher, may assist with improving the quality of future PSTs' knowledge for teaching mathematics. Factors influencing PSTs' development of MCK and *foundation* knowledge include their learner and teacher identity, sustained engagement for each year of the program and quality of teaching and learning experiences.

Although this study focussed on two PSTS, the findings reported here were consistent with the study reporting on the larger cohort of participants (Livy, 2014) and therefore should be considered when designing improvements to PST education. The findings should also be validated against other primary teacher education programs that do not include a secondary structure. Further studies may report on communication between mentor teachers and PSTs to identify how mentor teachers can assist PSTs to develop different categories of their MCK.

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THE MISSING LINK? – SCHOOL-RELATED CONTENT KNOWLEDGE OF PRE-SERVICE MATHEMATICS TEACHERS

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Professional knowledge is seen as a core component of teacher expertise. Hereby, domain-specific knowledge is usually modelled as content (CK) and pedagogical content knowledge (PCK). To date, the development of this knowledge during teacher education is so far not investigated comprehensively. The article focuses on a refined model of domain-specific teacher knowledge for that purpose that adds a school-related content knowledge (SRCK) as a specific applied mathematical knowledge for teaching. The article reports the development of an instrument to assess the professional knowledge. A study with N=505 pre-service teachers results in reliable and sufficiently separable scales for CK, SRCK, and PCK. SRCK seems to play an intermediary role between CK and PCK. The measures will be used to investigate the longitudinal knowledge development during teacher education. Practical implications are discussed.

INTRODUCTION AND THEORETICAL BACKGROUND

Professional knowledge of teachers is considered one core component of teacher expertise. From a domain-specific perspective, content knowledge (CK) and pedagogical content knowledge (PCK, Shulman, 1986; Baumert et al., 2010) are important aspects of this professional knowledge. Recent studies indicate that professional knowledge contributes to instructional quality and to student progress (Krauss et al., 2008; Kersting, 2010; Hill, Schilling, & Ball, 2005; Hill et al., 2008). Consequently, there is broad consensus that teachers' professional knowledge is a key goal of teacher education.

Nevertheless, the development of teacher expertise is still not comprehensively understood. Especially, there is a lack of research on the growth of teacher professional knowledge during initial teacher preparation. The project *KeiLa – Development of Professional Competence in University-based Teacher Education* aims to describe longitudinally the development of teacher knowledge from a broad perspective, including amongst others individual characteristics and learning opportunities across different domains (educational psychology, mathematics, biology, physics, chemistry). This interdisciplinary approach seems suited, as in several countries including Germany teachers major in two subjects and university-based teacher education includes education in educational psychology (cf. Lohse-Bossenz, Kunina-Habenicht, & Kunter, 2013).

One of the main challenges for research focusing on longitudinal effects of teacher education lies in the assessment of subject-specific knowledge. Although for mathematics, a few standardized tests of components of this knowledge were already

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developed, it can still be considered an emerging field, especially if a longitudinal perspective is taken. Existing approaches still differ widely, so that we conducted a study with preparing character (KiL – *Measuring the professional knowledge of preservice mathematics and science teachers*, Kleickmann et al., 2013) to develop instruments for the assessment of domain-specific professional knowledge. In this article, we focus on the mathematical part of the KiL-study. Therefore, we 1) review the state of research on (pre-service) teachers' content and pedagogical content knowledge (CK, PCK), 2) argue for the need of a complementing new construct of school-related content knowledge (SRCK), 3) report on the psychometric quality of the developed KiL-tests for pre-service teachers on CK, PCK, and SRCK and 4) present findings on the structure of professional knowledge as a whole and its components. Although the study is conducted in Germany, the focus on areas of domain-specific knowledge and its acquisition is seen as fundamental for mathematics teacher education in general.

The constructs of content knowledge and pedagogical content knowledge

Advancing the research of Shulman (1986), empirical studies were undertaken to operationalize the constructs of content knowledge (CK) and pedagogical content knowledge (PCK) for mathematics teachers. First investigations focused on the separability of the different domain-specific knowledge components as well as their importance for teaching quality and student learning. However, empirical studies could not completely answer the important questions concerning the structure of mathematics teachers' knowledge. In some studies for example CK and PCK are highly correlated (Hill et al., 2004, 2005; Krauss et al., 2008; Blömeke, Kaiser, & Lehmann, 2008). However, it is not always clear if this correlation is caused by the underlying conceptualizations, the different operationalisations or if it mirrors the nature of the investigated cognitive structures. For example, CK is often intended to mirror mathematics knowledge acquired through formal teacher education. Despite of this, most conceptualizations are predominantly focused on mathematical school content, even for teachers of academic track schools that receive a profound academic education in mathematics in most countries (Baumert et al., 2010; see also Tatto et al., 2012 for the structure of mathematics teacher education in 17 countries). In analogy, PCK is intended to mirror a kind of knowledge very specific for teaching mathematics. But operationalisations show that the delineation of PCK from analytical mathematical competences can be subtle (Buchholtz, Kaiser, & Blömeke, 2014).

Accordingly, one can ask if CK and PCK and the relation between these constructs are fully understood. Moreover, the existing approaches are not fully aligned with the aims of formal teacher educations. Thus, they are not suited to trace the effects of formal teacher education. Consequently, in the KiL study we furthered the conceptualizations of pre-service mathematics teachers' domain-specific knowledge to account for the depth and breadth of demands of mathematics teacher education.

In the KiL conceptualization, CK is conceptualized as academic mathematical knowledge, as expected to be acquired through formal teacher education. This mathematical knowledge is – in respect to content, precision, and notation – clearly beyond school mathematics. Students in mathematical study programs without aiming at a teaching license would also be expected to acquire this knowledge. Thus, this CK conceptualization refers to the original idea of Shulman (1986) who expected the "subject matter understanding of the teacher [to] be at least equal to that of his or her lay colleagues, the mere subject matter major" (p. 9). However, in line with modern teacher education programs, we would not expect a secondary teacher to complete a full mathematics major, but to have profound basic mathematical knowledge on the level of an introducing lecture in each major area of mathematics (e.g. analysis, algebra, geometry, applied mathematics) and further advanced knowledge in at least one major area with relevance for school mathematics. However, it is important to understand that our conceptualization of content knowledge is not restricted to elementary mathematics from a higher viewpoint (Klein, 1908).

Pedagogical content knowledge (PCK) refers to the knowledge about the instruction of specific mathematical topics. In KiL, we follow the suggestions of Baumert and colleagues (2010) and subsume knowledge of instructional strategies for a certain topic, knowledge about student cognitions, e.g. typical student misconceptions of a topic, and knowledge about the learning potential of specific mathematical tasks (Baumert et al., 2010). In other approaches, items were used to operationalize PCK that have a predominant mathematical demand (or could be solved by mathematical means, e.g. a mathematical argumentation; cf. Buchholtz, Kaiser, & Blömeke, 2013). But if PCK is understood as the knowledge "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (Shulman, 1986, p. 9, emphasis in original), we suggest to understand the conceptualization of PCK more rigorously. PCK then has to be a genuine and specific kind of knowledge about instruction, so it is per se knowledge about the teaching and/or learning of a certain topic and should clearly relate to student thinking. This, of course, has consequences for an operationalization of PCK, where mere mathematical problems should be avoided.

School-related content knowledge as a special kind of applied content knowledge

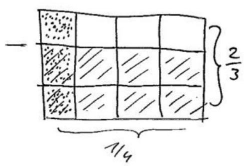
Using these conceptualizations of CK and PCK, we see the need for a complementing construct we call school-related content knowledge (SRCK). First, as neither CK nor PCK include knowledge about mathematical school contents and their curricular alignment, SRCK should encompass this knowledge. Curricular knowledge is commonly understood to be neither genuine PCK nor CK and some conceptualizations of teacher knowledge account for that knowledge explicitly (Shulman, 1987; Hill et al., 2005). But beyond, the sequencing of contents in specific curricula should inform instructional decisions. To solve such instructional problems, a cross-cutting subject-specific knowledge is needed: Answering questions of implications of curricular decisions needs specific knowledge about learning these topics as well as profound knowledge about the underlying connections that are caused by the deep mathematical structures, hence a knowledge intertwining content and pedagogical content knowledge. Here, two sub-facets can be identified: Teachers need to know how the topics of school mathematics are rooted in the mathematical structures and, vice versa, how mathematical structures can be reduced for teaching purposes (cf. "unpacking mathematics", Ball & Bass, 2003). As an example for the first facet, only the profound understanding of limits enables teachers to understand repeating decimals, especially the (non-trivial) validity of $0.\overline{9} = 1$. As an example for the other facet, the academic way of constructing real numbers via Cauchy sequences or Dedekind cuts is not suited for school mathematics. However, a profound mathematical knowledge helps connecting e.g. Cauchy sequences to the way irrational numbers are approximated with the help of nested intervals, a standard way to estimate the size of the square root of 2 at school. To sum up, we understand SRCK knowledge as a very special kind of application of mathematical knowledge for the teaching purpose. These ideas are informed by early reflections on the profession of mathematics teachers and the relation between academic mathematics and school contents (cf. meta-mathematics, e.g. Fletcher, 1975, Dörfler & McLone, 1986; cf. mathematical background theory, e.g. Vollrath, 1988).

Thus, we decided to conceptualize school-related content knowledge (SRCK) as a kind of applied mathematical knowledge for teaching that should be important to enable teachers to transform academic mathematical knowledge (CK) into knowledge for teaching mathematics at school and relate school mathematics to the structure of the discipline. It is questionable whether SRCK as an applied knowledge can be learned on its own. It seems that SRCK is deeply rooted in academic CK. At the moment, we do not see a well-defined place for the systematic development of this kind of knowledge in German teacher education programs. All the more, we see the need to investigate the development of this theoretically important knowledge area for pre-service teachers of mathematics.

INVESTIGATING DOMAIN-SPECIFIC PROFESSIONAL KNOWLEDGE OF PRE-SERVICE MATHEMATICS TEACHERS

In order to comprehensively assess pre-service mathematics teachers' domain-specific knowledge, we distinguish in our studies between the three dimensions of content knowledge (CK), school-related content knowledge (SRCK) and pedagogical content knowledge (PCK). We developed a test instrument building on this framework (see Figure 1 for sample items).

First, we conducted a curricular analysis of teacher education programs and curricula for school mathematics (both for secondary level, i.e. grades 5-13). Item development and piloting activities resulted in a total of 118 items (PCK: 31, SRCK: 34, CK: 54) that were bundled in two test booklets. One test booklet should be used with pre-service mathematics teachers for the academic track, the other for pre-service teachers for the non-academic track. However, both booklets had a considerable overlap of 81 items, in order to allow a linking of the data for analyses. The tests covered topics from arithmetics/algebra, analysis, geometry, stochastics, and numerics with a strong focus on arithmetics/algebra. With this, the test covers the characteristics of university-based teacher education as we could observe in the curricular analysis. Testing time was set to 120 minutes per booklet. The items were scored according to a scoring rubric with partly dichotomous, partly partial scores (0, 0.5, 1). For the 34 open answers, the interrater-reliability of the scoring of two independent raters was above $\kappa = 0.73$ (Cohen's Kappa), thus the objectivity of the scores was considered as sufficient.

<table>
<tr><td>PCK</td><td>Please sketch an iconic illustration to explain to a 6th grade student how to multiply a fraction by another fraction. Take $^1/_4 \cdot {}^2/_3$ as an example.</td><td>Sample Answer (scored as correct)


"Only the areas marked as  count. How many are these? How many are there in total? $^2/_{12} = {}^1/_4 \cdot {}^2/_3$ "</td></tr>
<tr><td>SRCK</td><td colspan="2">Which of the following tasks enable students to discover that rational numbers are dense in real numbers?

| | apt | wrong |
Measure the length of the diagonal of a square (length 10 cm). | ☐ | × |
Seek the smallest fraction bigger than $\sqrt{2}$. | × | ☐ |
Split 100€ into three equal portions. | ☐ | × |
Find ten fractions between $\sqrt{2}$ and $\sqrt{3}$. | ☐ | × |</td></tr>
<tr><td>CK</td><td colspan="2">We call a field extension $L{:}\,K$ finite-dimensional if $dim_K L < \infty$. Which of the following field extensions are finite-dimensional?

| | right | wrong |
$\mathbb{Q}(\pi){:}\,\mathbb{Q}$ | ☐ | × |
$\mathbb{Q}(i){:}\,\mathbb{Q}$ | × | ☐ |
$\mathbb{C}{:}\,\mathbb{R}$ | × | ☐ |
$\mathbb{R}{:}\,\mathbb{Q}$ | ☐ | × |</td></tr>
</table>

Figure 1: Sample items for constructs of pre-service mathematics teachers' pedagogical content (PCK), school-related content knowledge (SRCK) and content knowledge (CK) tests

Sample and Methods

A total of N = 505 pre-service mathematics teachers participated in the study. On average, the students were 23.3 (SD = 2.9) years old and in their 5.9 semester (SD = 2.64). About 64% of the students aimed to teach in academic track schools (German Gymnasium). In order to investigate the structure of pre-service teachers' professional knowledge the dimensionality of the data was examined. Therefore, multidimensional random coefficients multinomial logit modelling was used (MRCML; Adams, Wilson & Wang, 1997). For the final analyses, 98 items could be maintained in the sense that they fulfil the required cutoffs for item quality indicators.

Results

The analyses presented here focus on the separability of the constructs CK, SRCK and PCK. Therefore, we contrast a three-dimensional model against a one-dimensional model (g-factor model). As the SRCK construct is seen as having a cross-cutting characteristics between CK and PCK, we further contrast two alternate two-dimensional models that combine SRCK with CK and PCK respectively (see Table 1). We could not apply chi-square test of differences to compare the fit of the different models, as they were not nested. Thus, we used the Bayesian information criterion (BIC). Smaller values indicate a better model fit. Raftery (1995, p. 141) counts a BIC difference greater than ten as “very strong evidence” and greater than six “as strong evidence” for the model with the lower BIC value.

The comparison of model fit indices indicates that the three-dimensional model fits the data best, outperforming the one-dimensional, and the two different two-dimensional models (see Table 1 for details). The three scales showed further satisfying EAP/PV reliabilities (r_{CK} = .83 with scale length 41, r_{SRCK} = .80 with scale length 31, r_{PCK} = .69 with scale length 26). Hence, we succeeded in measuring CK and PCK as well as a complementing SRCK component and the scales suggest sufficient reliability.

Model	Description	n	df	BIC
3D between model	CK – SRCK – PCK	112	44023.82	44720.97
2D between model A	CK/SRCK – PCK	109	44159.14	44837.62
2D between model B	CK – SRCK/PCK	109	44069.37	44747.85
1D general factor model	CK/SRCK/PCK	107	44312.97	44979.00

n = total number of estimated parameters, df = final deviance

Table 1: Comparison of alternate models

The latent correlation between PCK and CK was estimated as r(PCK,CK) = .54 indicating a good separability of the constructs. At the same time, SRCK correlated highly with both the CK (r(SRCK,CK) = .83) and the PCK (r(SRCK,PCK) = .85) dimension on the latent level. This can be seen as an indication that SRCK has indeed cross-cutting characteristics, as conceptualized.

DISCUSSION AND OUTLOOK

The results of the KiL study provided evidence for the postulated three-dimensional structure of pre-service mathematics teachers' domain-specific knowledge. On the basis of the refined constructs of CK and PCK, we were able to separate the two constructs satisfyingly on the empirical level. A complementing dimension of school-related content knowledge (SRCK) was conceptualized as a knowledge base for applying academic mathematical knowledge in the context of school mathematics and its instruction. On the empirical level, the correlations between the measures support this intermediary role of SRCK between academic mathematics and school mathematics. Thus, we were able to model pre-service mathematics teachers' domain-specific knowledge on the basis of the KiL model. With this we laid the groundwork to empirically investigate the growth of pre-service teachers' knowledge across formal teacher education using a longitudinal study in the upcoming KeiLa project.

On the basis of our findings, we would suggest to reinvestigate the value of academic mathematics for the development of teacher professional knowledge, a key element of teacher expertise. Our investigations might have importance for the design of teacher study programs. Especially, it is a new starting point to focus on an applied mathematical knowledge for teaching that is energized by a profound understanding of mathematics and enables a teacher to solve the evolving problems of teaching mathematics.

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A STRATEGY FOR ENGAGING STUDENTS WHOSE ACHIEVEMENT HAS FALLEN BEHIND THEIR PEERS

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This paper reports results from an investigation of the impact of an intervention program, Prepare 2 Learn, *which was designed to support students who had sufficient gaps in their mathematical knowledge that made participation in mainstream classes problematic. The intervention incorporated a range of components, adapted from other successful intervention programs, with the main goal being to prepare students for their subsequent learning. The first iteration of the intervention included three year 6 students who were assessed as achieving approximately 6 months behind the expected standard in mathematics. Over the course of the iteration the students' academic results improved substantially. Even more encouraging was the improvement in the students' approaches to their learning.*

INTRODUCTION

"Governments and school communities recognise the value of identifying, early in schooling, the students who are not thriving mathematically and in providing them with more intensive instruction" (Gervasoni, Parish, & Hadden, 2012, p. 306). While this statement applies to helping students in the initial years of primary school, it is also relevant for later years of schooling as well. The assumption is that early detection and intervention has the potential to minimise long term mathematical learning difficulties. This intervention targeted students whose results were approximately 6 months behind the expected year level, with the perceived need for the study being that without assistance the most probable scenario is that the achievement gap is likely to grow (Sullivan & Gunningham, 2011). The research, aspects of which are reported here, involved designing an intervention program which took into account components of other successful intervention programs. In particular, the deliberate selection and combination of the components of other intervention programs, chosen purposefully, made this intervention unique. The program, *Prepare 2 Learn*, had two main objectives: firstly, to prepare students for their mainstream lessons, ensuring they had the necessary prior knowledge; and secondly, to increase students' awareness of the impact they can have on their own learning through particular actions and attitudes.

IDENTIFYING KEY COMPONENTS OF CURRENT INTERVENTION PROGRAMS

The *Prepare 2 Learn* program incorporates four key components: increasing mental computational fluency; building prior knowledge of mathematical language, concepts and skills to prepare students for their upcoming mainstream sessions; encouraging a growth mindset; and developing metacognitive strategies. This report focuses on the components prior knowledge, mindset and metacognitive strategies which provide the theoretical rationale for the initiative.

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The focus on building prior knowledge of mathematical language, concepts and skills addressed barriers that may inhibit students' subsequent participation in mainstream mathematics classes. This is partly based on cognitive load theory (Bransford, Brown, & Cocking, 1999) which suggests that, due to its limited capacity, difficulties in accessing prior knowledge, language and processes can cause working memory to become overloaded. Similarly, Hattie (2012) argued for the need for teachers to plan lessons taking into account what students already know and can do, with the idea that teachers can plan to bridge the gap between a student's current knowledge and understanding and the target knowledge and understanding. Sullivan and Gunningham (2011) likewise argued that many students who have fallen behind lack the necessary prior knowledge to access mainstream mathematics teaching. Sullivan and Gunningham designed the GRIN intervention to provide preliminary awareness of language, concepts and skills for low achieving students to 'get them ready' for their forthcoming mathematics lessons. Sullivan and Gunningham argued that many low achieving students lack the prior knowledge necessary to enable them to construct new understandings from the experiences in their upcoming mathematics class.

A further possible factor contributing to student reluctance to participate is that classrooms are social places. As Middleton and Jansen (2011) argued, "human beings seek relatedness with one another" (p. 9). The GRIN program sought to provide prior knowledge of upcoming lessons to encourage participating students to feel confident enough to join in classroom learning experiences, and feel part of the class group. Like GRIN, *Prepare 2 Learn* aims to provide necessary prior knowledge and awareness to enable students to participate fully in classroom learning experiences, thereby facilitating students' connection to the class group.

The second component of the *Prepare 2 Learn* program aimed to develop a 'growth mindset' in the students. Dweck (2008) described two mindsets: 'fixed' and 'growth'. People with a 'fixed' mindset believe their "…qualities are carved in stone" (p. 6), that they have a certain amount of talent and there is little they can do to alter this. In contrast, students with a 'growth' mindset believe their qualities, such as intellectual capacity, can be improved through effort. *Prepare 2 Learn* was designed to foster a growth mindset in its students, intending that this would allow students to understand the ways that their learning actions and effort can enhance their success in mathematics.

A third component of the *Prepare 2 Learn* program sought to enhance metacognitive strategies in students. Hattie (2012) argued for the need for all learners to develop metacognitive strategies: "We need to develop an awareness of what we are doing, where we are going, and how we are going there: we need to know what to do when we do not know what to do" (p.102). The focus on metacognition was chosen after reviewing Caswell and Nisbet's (2005) intervention program, which encourages students to reflect on their level of 'knowing' and the impact their actions and feelings could have on their ability to learn. As Caswell and Nisbet (2005) suggested "…the challenge exists to engage students in reflection that raises their consciousness of both cognitive and affective factors that affect their learning potential" (p. 209). Like their program, *Prepare 2 Learn* aimed to encourage students to reflect on the type of actions and attitudes that can enhance their awareness of strategies thereby helping them to manage their learning.

In short, the *Prepare 2 Learn* program intended to prepare students to take advantage of their classroom mathematical experiences by providing prior knowledge, encouraging a 'growth' mindset and teaching metacognitive strategies.

THE RESEARCH CONTEXT

The *Prepare 2 Learn* intervention program has had a number of iterations, although data from only one are presented here. This iteration supported the learning of three year 6 girls who were identified as achieving approximately 6 months below the expected level for their year. The participants were chosen in collaboration with the classroom teacher, based on their academic achievement, regular school attendance, willingness of them and their parents to be part of the program, and (non) participation in other programs.

The sessions for the *Prepare 2 Learn* intervention were planned by the intervention teacher, the author, in consultation with the classroom teacher, taking into account the knowledge that would be needed to prepare the students' for their forthcoming classroom lessons. As these sessions were aimed at supporting the mainstream classroom sessions, the tutorial sessions were in addition to them.

The iteration began with two introductory sessions. The first involved students watching two short videos about the brain. These videos allowed students to see how the brain learns new things by developing neural pathways. These pathways are established and made easily accessible, for example, through regularly practising unfamiliar concepts and skills. The next introductory session revised what students had learnt about the brain and linked this with actions of good learners. As a group, the students then designed a checklist of the actions of 'good learners'. This checklist then became a self-reflection tool in which students would note at the end of each mathematics lesson what learning actions they had done well, and what they would need to improve on in their subsequent lessons. Such meta-cognitive strategies were introduced to encourage students to become responsible for their own learning

The tutorials ran for 15 weeks with the students attending three 40 minute sessions per week. The structure of the tutorial sessions was as follows:

5 minutes: Mental computation activities based on the intended topic.

5 minutes: Teacher and students review the self-reflection checklists, and discuss how to be an effective learner using metacognitive strategies. Teacher discusses any aspects students need more help with.

25 minutes: Teacher establishes students' prior knowledge and introduces the necessary mathematical language, concepts, basic skills etc. that they will need in order to participate in the mainstream lesson/s.

5 minutes: Summary - students reflect on what they need to know to be able to engage in the follow-up mainstream lesson.

At the end of the iteration, the students discussed with the intervention teacher what they had learnt from the program and what changes they intended to implement to improve their ability to learn. These reflections are part of the data presented below.

INSTRUMENTS

The methodology informing the data presented below included elements of both design (Kelly, 2003) and action (Kuhn & Quigley, 1997) research, drawing on both quantitative and qualitative data. Prior to choosing the students all possible candidates were given the PAT (Progressive Achievement Test) Mathematics (Lindsey Stephanou, Urbach, & Sadler 2009) to establish their academic level, being approximately 6 months behind the other students. The class teacher then completed a questionnaire on each student focussing on learning behaviours.

Prior to the iteration commencing, the selected students were presented with a vignette during an interview in which the use of story encouraged a discussion on the actions of successful learners. They also completed a ladder instrument (Mornane, 2010) consisting of three statements about mathematical learning styles which the students ordered from most like, to least like, their preference. The students' data were collected via recordings, transcribed, and the statements progressively categorised to identify themes. These data, along with others not referred to in this paper, were intended to give a picture of the students' academic level as well as their actions and attitudes towards learning mathematics.

At the conclusion of the iteration, the teacher again completed a questionnaire focussing on the learning behaviours of the students. The students for a second time completed both the vignette and the ladder instrument. They were also interviewed on their thoughts about the Prepare 2 Learn program. Parents of the students were also asked to complete a questionnaire seeking indications of behavioural or attitudinal changes in their children.

As this iteration was completed over a year ago, an opportunity arose to interview the three students 12 months later when they were in year 7 in secondary school. At the same time, their parents were given a subsequent questionnaire. It was hoped that these data would give insights into the residual effects of the program.

The results below are intended to offer insights into the following research questions:

Does providing prior knowledge of mathematical language, concepts and skills, encouraging a growth mindset and developing metacognitive strategies in students result in:

- substantial improvement (of 12 months or greater) in academic achievement?
- students approaching their mathematical learning with more confidence?
- increased participation by the students in their mathematical learning experiences?

- students taking greater responsibility for their learning?

RESULTS

The first set of results relate to the students' achievement. The subsequent sections present data from one student, Rachel, which are representative of the responses of the other students. These data suggest that all three students increased their confidence in class and in mathematics, and took more responsibility for their learning. The data related to the third question, engagement in learning experiences, also confirm that all three students were more active participants as a result of the program, although those data are not presented, due to space limitations.

Improvement in academic achievement

With respect to the first question, the data from the PAT showed that all three students had improved more than the expected 12 months in achievement, and two of the students moved substantially more. These data are not presented due to space limitations.

In addition to these data, the students were interviewed 12 months after the completion of the iteration. While no assessments were conducted, all three students' academic results seemed to have improved even further against what would be expected for their year. Two of the girls, who were at the same secondary school, had received a grade of A on their half yearly mathematics report. The mother of the third girl, who attended a different secondary school, reported that she was averaging 80% and above for all her mathematics tests. These results indicate, in regards to question 1, that the *Prepare 2 Learn* intervention resulted in substantial improvement in the students' academic results, and that this improvement continued to increase rather than diminish over the succeeding 12 months.

Increased confidence in class and in mathematics

With respect to the second research question, Rachel, like other participants in the program, had improved noticeably in her confidence. This could be seen by comparing the pre-program to the post program responses.

Before the program began her teacher expressed concerns about Rachel's level of confidence when learning mathematics. The teacher made statements like, "…she's one of those children who doubts herself…she lets others sway her towards some different solutions or answers when she may in fact be correct". When the teacher was asked why she had nominated Rachel for the program, besides her results, she replied "mainly the (lack of) confidence".

Phrases such as "doubts herself", "lets other sway her", "mainly the confidence" present an image of a student who appeared to the teacher to be not sure of herself and her mathematical ability.

Rachel's mother was also concerned at Rachel's lack of confidence prior to the iteration. In the questionnaire she filled out at the end of the iteration, the mother remarked “She commenced with no confidence, particularly worded problems. She actually was unable to do (them) without assistance.” By the end of the program, however, both the teacher and Rachel’s mother spoke about an increase in the level of confidence shown by Rachel. The teacher commented:

> …works diligently and with more confidence this semester
>
> … has become more certain and competent in her thinking

Similarly Rachel’s mother stated, “Now she has complete confidence…She also feels within herself she can do (it)”. Phrases such as “more confidence”, “more certain”, “complete confidence”, “she feels she can do it” all indicate that Rachel presented as a more assured mathematics learner after the iteration.

In her answers throughout the various types of post iteration data collection, Rachel also indicated she was more confident. When asked if she believed the program had helped her with mathematics learning, she replied:

> Yes because I never used to put my hand up for anything because I was scared if I got the answer wrong, I’d be ashamed, so that’s why I kept quiet.

When questioned whether this was because she was more confident, she replied “Yes”. This shows Rachel was willing to take risks by volunteering to answer questions even though her answers may not be correct. In a subsequent question she explained further “… I used to be scared if I got it wrong or right but now I don’t care because some people may have the same answer… Or maybe I have just done the calculation wrong.” These responses illustrate Rachel has become a more confident and resilient student. Martin and Marsh (2008) referred to this as ‘academic buoyancy’.

More responsibility taken by students for their learning

The fourth research question examined whether the iteration encouraged students to take greater responsibility for their learning. The results from the pre to the post-program data indicated that Rachel and the other students could be seen to be taking more responsibility for their learning.

The following data presents examples of changes, in regards to Rachel taking responsibility for her learning, which occurred over the course of the intervention. Prior to the program the teacher described Rachel as a student who:

> …doesn’t concentrate fully…easily distracted … she needs to stay on task.

The above phrases indicated Rachel lacked responsibility for her learning as indicated by the lack of focus.

In the post-program data the teacher commented on Rachel’s heightened level of responsibility towards her learning. She described Rachel’s learning with comments like:

> …has paid more attention to teaching points……focused harder to listen…
>
> …go [sic] over calculations to check for accuracy and make changes if necessary.

> She has a go and if not successful will come to me for assistance and clarification and then be happy to go and try for herself.

The change in Rachel taking responsibility for her mathematics learning is evident in words like “paid more attention”, “focused harder”, “check for accuracy” and “if not successful will come to me for assistance and clarification”. This indicates a student who was in control of her learning and who understood the role she needed to play to make the learning happen.

Also in the data collected directly from Rachel we can see a change in Rachel’s thinking. In the preliminary interview, Rachel was presented with the following vignette

> Two students in the same class at school in year 3 had been getting the same mathematics results. However later in year 6, one of the students had started to go a lot better.

Rachel was asked why she thought this might be so. She replied “May be because she got a tutor, and she was more smarter…” This statement revealed that Rachel had a ‘fixed’ mindset. However in the post program vignette, when questioned about the improvement of the year 6 girl Rachel believed she may have improved because:

> She might do stuff at home, if she gets confused she keeps going…maybe on holidays or something else she practices her timetables or like anything like that.

In this data we can see a change in Rachel’s mindset. She now believes academic improvement comes from actions like persisting and practising. Rachel had begun to understand the impact of one’s actions on one’s learning.This increased responsibility was also evident in the data collected from the other two students in the program.

CONCLUSION

The OECD 2003 report (Artlet, Baumert, McElvany, & Peschar, 2003) begins by recognising, in the foreword, that students with stronger approaches to learning get better results at school and are much more likely to take up further study and become lifelong learners.

The data from this iteration showed the *Prepare 2 Learn* program had substantially improved students’ academic achievement as well as changed the way they approached their learning. At the conclusion of the program students showed increased confidence in mathematics, greater participation in mathematics learning experiences and had begun taking more responsibility for their learning. As such these students should now have a greater chance of taking up further mathematics study and becoming life-long learners. Such positive results indicate that this program is worthy of further investigation, especially to identify which aspects are critical in prompting this improvement in participation.

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SECONDARY MATHEMATICS STUDENTS' PERCEPTIONS OF THEIR TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE FOR TEACHING ASPECTS OF PROBABILITY

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This paper investigates senior secondary mathematics students' identification of teaching practices that they perceive are conducive to their learning. An existing framework describing aspects of pedagogical content knowledge (PCK) was used to analyse survey and interview data from two classes of students studying discrete and continuous probability distributions. Students identified that their teachers exhibited several elements of PCK as contained in the framework. The study contributes to the limited research that considers teachers' PCK from a student perspective, in the context of learning abstract mathematics.

INTRODUCTION

Effective teachers of mathematics have knowledge of students' thinking, knowledge of mathematical content, and knowledge of how to represent the content so that it makes sense to others (Hill, Ball, & Schilling, 2008). Substantial progress has been made towards identifying the constituent parts of teacher knowledge including pedagogical content knowledge (PCK) (e.g., Chick, Baker, Pham, & Cheng, 2006; Krauss et al., 2008). PCK concerns the way subject matter is transformed from the knowledge of the teacher into the content of instruction. Shulman (1986) described PCK as an intricate blend of content and pedagogy that encompasses all that is needed to teach a subject or topic in a way that makes it comprehensible to others.

It is widely accepted within the mathematics education community that PCK impacts upon teaching and learning (e.g., Ball, Lubienski, & Mewborn, 2001; Krauss et al., 2008). Research into PCK has focused mainly on pre-service and practicing teachers in the context of primary mathematics (e.g., Baker & Chick, 2006; Rowland, Huckstep, & Thwaites, 2005). Comparatively few studies have focused on secondary mathematics, as emphasised by Matthews (2013) in her review of research into PCK across grade bands. Furthermore, students' perceptions of the PCK they consider to be helpful in assisting them with their learning of abstract mathematics have been largely unexplored. The on-going concern about the level of participation and achievement in post-compulsory rigorous mathematics (e.g., Vale, 2010) underlines the importance of further research into the teaching and learning of senior secondary mathematics. This paper focuses on PCK from the student perspective by exploring the following research question: What aspects of mathematical pedagogical content knowledge are identified by students as having an impact on their learning of senior secondary (Grade 11/12) mathematics content?

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CONCEPTUAL FRAMEWORK

Several frameworks have been developed for discussing PCK as a multi-faceted category of mathematics teacher knowledge (e.g., Ball, Thames, & Phelps, 2008; Chick et al., 2006; Rowland & Turner, 2007). At present there is no widespread agreement on any one particular teacher knowledge framework (Matthews, 2013).

The framework for analysing PCK in mathematics developed by Chick and her colleagues (e.g., Chick et al., 2006; Chick & Harris, 2007) gives a detailed inventory describing evidence for identifying key components of PCK within three broad categories. These include “clearly PCK”, “content knowledge in a pedagogy context” and “pedagogical knowledge in a content context”. Space prevents the inclusion of the entire framework in this paper, but a brief description is provided. Each category comprises several elements of PCK (e.g., knowledge of examples, mathematical structure and connections). Elements are classified as “clearly PCK” when content and pedagogy are completely intertwined, such as knowledge of student thinking and, more specifically, knowledge of students’ misconceptions. “Content knowledge in a pedagogy context” includes elements that relate to the way mathematical knowledge is used by the teacher, for example “deconstructing content to key components” (Chick & Harris, 2007). The third category, “pedagogical knowledge in a content context,” is concerned with general teacher knowledge applied to a particular content area, for example “providing students with a goal for learning a particular skill.”

Chick and her associates acknowledge that the categories of the PCK framework are not necessarily exhaustive, although previous work with it in other studies has not indicated the need for additional categories (Chick et al., 2006). The framework enables researchers to investigate PCK by applying it to data such as interview transcripts, written responses to items about teaching and learning mathematics content, and actual teaching episodes (Chick et al., 2006).

UNDERSTANDING PROBABILITY DISTRIBUTIONS

Discrete and continuous probability distributions are key foci of the statistics component of senior secondary mathematics courses in Australia. Typically, discrete random variables are introduced first, followed by the use of binomial and hypergeometric distributions to model discrete random processes. Continuous random variables are then dealt with, along with the normal distribution and its applications. Although there has been limited research into students’ understanding of binomial and hypergeometric distributions per se, some studies have focused on the foundational ideas of these probability distributions such as combinatorial reasoning (Batanero, Navarro-Pelayo, & Godino, 1997). Wroughton and Cole (2013) point out that distinguishing binomial from hypergeometric distributions can be challenging for students, and the introduction of terminology such as “with replacement” and “without replacement” does not clarify the distinctions sufficiently. The findings of their small study, however, suggest that hands-on activities designed to build students’ understanding of binomial and hypergeometric distributions may assist students to recognize the differences between them (Wroughton & Cole, 2013).

Batanero and her associates (2004) highlight the complexity of teaching and learning the concept of the normal distribution as it requires the interconnection of many different statistical ideas, such as recognising when and how to apply the fact that the proportion of data within one, two, and three standard deviations of the mean is 68%, 95%, and 99% respectively. It involves interpreting and relating graphical, symbolic, and numerical information (Batanero et al., 2004). More broadly, research findings underline the role of technology in affording students the opportunity to develop their understanding of ideas such as density curve, population parameters (i.e., mean and variance), and comparing empirical and theoretical distributions. Little research, however, has been conducted into students' perceptions of the teaching strategies they find useful for their learning about probability distributions.

METHOD

This paper uses data from a larger study and explores the aspects of PCK that students identify as affecting their learning of senior secondary mathematics content. Two Grade 11/12 Mathematics Methods classes from a large metropolitan secondary college in Tasmania took part. Mathematics Methods is one of the most demanding mathematics courses offered in Tasmanian schools. It is assessed by internal unit tests and a final external examination; the major topics are function study, calculus, and statistics. The statistics component of the course focuses on the probability distributions discussed previously. Data presented in this paper were collected over a sequence of lessons on discrete and continuous probability distributions.

Participants

Participants were 16-18-year-old students enrolled in one of two Mathematics Methods classes during 2014. One class was taught by Mr Jones and the other by Mr Taylor (both pseudonyms). Of the 18 students enrolled in Mr Jones' class, 14 (five females and nine males) contributed data by participating in one or more focus group interviews and/or completing one or more short answer surveys. Similarly, seven of the ten students in Mr Taylor's class participated (three females and four males). Student names are pseudonyms in this paper.

Procedure

Data were collected over a period of seven lessons, three taught by Mr Jones and four taught by Mr Taylor. At the end of each lesson, a short-answer survey was completed by participating students and semi-structured audio-recorded focus group interviews were conducted. The survey consisted of two questions eliciting responses about the types of explanations and strategies that assisted participants with their learning of particular mathematics content. Each focus group interview involved between three and six participants and took up to 20 minutes. Participants were asked to comment on aspects of the lesson they found to be particularly helpful for their learning (e.g., How much did you know about this topic before today's lesson? What do you know now? What happened in the lesson that particularly helped this knowledge growth?).

Data analysis

Both the survey and focus-group interview transcripts were analysed to identify occurrences of the different types of PCK using an adaptation of the Chick et al. framework for analysing PCK (e.g., Chick et al., 2006). A subset of the data was first coded by all three authors and discrepancies resolved before the first author completed the remaining coding. Data were examined for students' perceptions of the most useful aspects of PCK in the teaching and learning of probability distributions. Over 200 instances of PCK were identified and coded. Survey and interview data have been combined for this paper.

RESULTS

Table 1 shows the occurrences of teachers' different types of PCK as identified by the students according to the Chick et al. framework for analysing PCK. The category "other" includes aspects of PCK that occurred infrequently (i.e., detected on no more than two occasions in each class) such as Student Thinking (and Misconceptions), Purpose of Content Knowledge and Assessment Approaches.

PCK Category (Chick & Harris, 2007)	No. of occurrences (Jones)	No. of occurrences (Taylor)	Total
Teaching strategies	9	15	24
Classroom Techniques	6	9	15
Knowledge of Examples	15	14	29
Mathematical Structure and Connections	15	13	28
Explanations	9	14	23
Methods of Solution	14	9	23
Procedural Knowledge	11	9	20
Appropriate and Detailed Representations of Concepts	8	11	19
Deconstructing Content to Key Components	6	9	15
Cognitive Demands of Task	4	5	9
Other	3	4	7

Table 1: Occurrences of teachers' different types of PCK as identified by students.

Responses in the categories of Knowledge of Examples and Mathematical Structure and Connections tended to occur more frequently than others (i.e., 29 and 28 respectively). Multiple facets of PCK were identified in the majority of responses. For example, "it was helpful going through the different types of probability distributions and explaining how they differ from each other and where to use each one" suggests evidence of both knowledge of Explanations and Mathematical Structure and Connections. Such inevitable overlap added complexity to the coding process because even though some aspects of PCK may have been clearly evident, others involved subtle interpretation or were present to a lesser degree. The tension between coding unanimously and defending a case for the incidence of a particular category of PCK was discussed in relation to inter-coding reliability. Examples of responses from students in Mr Taylor's and Mr Jones' class are given in the following sections.

Teaching Strategies and Classroom Techniques

Students highlighted the value of being asked questions by their teacher—a teaching strategy—during the instructional phase of a lesson. For example, "It helps when Mr Taylor asks me to work it out … This allows me to think properly about a question so that next time I understand and remember it" (Liam). Similarly, Christopher from Mr Jones' class said "Asking people for answers while explaining ideas/concepts on the board reinforced my understanding of binomial and normal distributions".

Representation of Concepts, Explanations, and Knowledge of Examples

Aspects of PCK including Appropriate and Detailed Representations of Concepts, Explanations, and Knowledge of Examples collectively featured strongly in the data. Many responses mentioned the use of diagrams as helpful representations of concepts. For example, the following comment from Margot relates to Mr Taylor's introduction of the standard normal curve: "diagrams helped me, specifically the example of comparing x to z and how they relate". Other students made similar comments in relation to the normal curve and its properties. Meg from Mr Taylor's class included a sketch of a normal curve with the following survey comment, "this diagram helped me to learn normal curves and how to find the values of μ and σ".

Mathematical Structure and Connections

Some students focused on the classroom incidents they perceived to be helpful in assisting them to make connections between mathematical ideas. In the following excerpt some students discussed the way Mr Taylor distinguished between the binomial and hypergeometric distributions by emphasising the role of the correction factor in the probability formula due to the fact that the hypergeometric involves sampling without replacement from a finite population.

> Researcher: Can you think of any connections specifically that you've made?
>
> Stuart: Ummmm sort of the last bit with the, the factor ummm the adjustment factor.
>
> Vincent: Umm I found it really nice how Mr Taylor showed us the similarities between the binomial and the hypergeometric and why they are different. And ah why you need the umm what's it called the, the thing on the end [refers to the correction factor often written

as the fourth factor in the probability formula for hypergeometric distribution] ummm it accounts for the difference between the two umm ways ummm distributions.

Similarly, Elizabeth from Mr Jones' class talked about making connections between the two discrete probability distributions as a result of a more general teaching strategy where the teacher had produced a flow chart of the major features of the distributions: "it [the flow chart] kind of helped draw the links between binomial and hyper geometric because when you do them all separately it is kind of hard to get your head around how they combine."

Conversely, but still related to mathematical structure and connections, other responses focused on an *absence* of connections between concepts, as indicated in the following interview excerpt based on a lesson on the normal distribution.

James: It's hard to see how it [the normal distribution] fits in with everything else we've done. I was a bit confused because we did do like basic graphs before but it's just like he has gone into it in more detail, but I didn't completely understand it and I'm not sure if it's linked.

James' comment relates to the first lesson on the normal distribution, where the class was introduced to the properties of the normal distribution including the empirical rule for the relative proportion of data within one, two, and three standard deviations of the mean. As the class had studied the binomial distribution earlier in the year, James appeared to have noticed some similarities in terms of the shape of the two distributions in some circumstances and expressed some concern about not being able to reconcile the new knowledge at that stage.

Procedural Knowledge and Methods of Solution

As indicated in Table 1 a reasonable proportion of survey and interview responses focused on the Procedural Knowledge and Methods of Solution categories of PCK. The following comments are indicative of a range of comments from students particularly in Mr Jones class.

Alan: Oh and worked solutions! That's another thing, I appreciate the working out.

Danny: Yeah and if you get some solutions where Mr Jones has written out like how he has done it and the steps he has done, or just even the process of how to do it it's a lot better.

Other cases however, indicated tension between the perceived value of worked examples and having the ability to produce solutions to problems in unseen situations. The following transcript follows on from Liam's earlier response about working it out for himself rather than copying:

Liam: It kind of puts you in a real life situation like in an exam when you have to think.

Researcher: When you are doing that are you thinking in a different way to you would normally be thinking if say you had a similar worked example on the board or something like that?

Liam: Yeah well I kind of don't understand what I'm doing; it's like I kind of rote learn it without actually understanding sometimes and I can rote learn it like that and do the same kind of question but if they change the question or, like, reword it I struggle.

Margot: Yeah, and sometimes when we are going through like an example on the board umm I'm just following through copying it down…umm it makes sense but I couldn't repeat it, like do it by myself.

Cognitive Demands of Task

The Cognitive Demands of Task category involves identifying aspects of the task that affects its complexity. The following comment from a student from Mr Jones class identified this aspect of PCK, relating to a revision worksheet the teacher had given the students based on the probability distributions the class had studied.

> With those questions that we were doing, I think they were better than the questions … in the text book because often with the text book I find that ummm they take too big a step between each of the questions … whereas with these they just had very, very slight changes between them … which is good because you're doing more questions that are quite similar to each other and also you've got … an easy progression curve. (Alan)

Other students commented on the accessibility of the questions provided by their teacher as suggested in the following survey comment: "Giving us time in class for revision helps a lot and giving us easy questions so we can better understand" (Jake).

DISCUSSION AND CONCLUSIONS

The study explored students' responses to questions seeking evidence of their teacher's PCK. The Chick et al. (2006) PCK framework provided a set of filters through which to systematically examine the responses for aspects of PCK. Students tended to highlight those aspects of PCK directly related to the explanations, examples, and representations provided by their respective teachers. Some responses highlighted specific representations such as the normal curve illustration itself as being helpful to their learning. Responses indicated that students notice and appreciate connections being made between concepts, and recognised when those connections appear to be absent such as was the case with James. Procedural Knowledge and Methods of Solutions were categories of PCK identified in several responses as being particularly important for students' learning about probability distributions. There was also evidence to suggest that although some students (e.g., Liam and Margot) valued worked examples, they also recognised limitations in terms of being able to produce solutions independently. The findings of the study suggest that student perspectives on the useful ways teachers transform mathematics knowledge for learning can provide insight into their teachers' PCK. Future study that investigate teachers' perspectives of effective PCK and compares it with students' perspectives would be useful.

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PORTUGUESE AND BRAZILIAN CHILDREN UNDERSTANDING THE INVERSE RELATION BETWEEN QUANTITIES – THE CASE OF FRACTIONS

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This study compares Portuguese and Brazilian fourth-graders (n=84) understanding of the inverse relation between quantities when fractions are presented in quotient and part-whole interpretations. It addresses three questions: 1) How do children understand these inverse relation in quotient interpretations of fractions? 2) How do children understand this inverse relation in part-whole interpretation of fractions? 3) Are there differences in performance between Brazilian and Portuguese children concerning these issues? A survey by questionnaire was applied and 16 part-whole and quotient problems were analysed. Results indicate that quotient interpretation promotes more the understanding of this inverse relation; Portuguese and Brazilian children perform differently when solving the fraction problems.

FRAMEWORK

This study focuses on a comparative analysis conducted with Portuguese and Brazilian children's understanding of the inverse relation between quantities, when fractions are involved. To understand rational numbers is one of the greatest conceptual challenges faced by children as they learn mathematics (Behr, Wachsmuth, Post, & Lesh, 1984; Hallett, Nunes, Bryant, & Thorpe, 2012; Singler, Thompson, & Schineider, 2011), since it requires a reorganization of numerical knowledge (Stafylidou & Vosniadou, 2004), as well as an understanding that the properties of integers do not define numbers in general, and thus, require other types of more complex cognitive skills (Jordan, et al., 2013).

The understanding of inverse relation between two quantities is an important skill for the conceptual knowledge of rational numbers (Hallett, Nunes, Bryant, Thorpe, 2012). Literature presents several studies focused on the students' understanding of the inverse relationship between quantities. Some are focused on the concept of fraction (see Behr, Wachsmuth, Post, & Lesh, 1984; Kornilaki & Nunes, 2005; Mamede & Cardoso, 2010; Mamede, Nunes, & Bryant, 2005; Mamede & Vasconcelos, 2014). Recent research (see Mamede, et al., 2005; Nunes, et al., 2004) consider that the conceptual knowledge of fractions comprises: (1) the invariance principle, that is, the division of a whole into equal parts, while maintaining the initial quantity; (2) the ability of representation, being written as $\frac{a}{b}$, where *a* and *b* are whole numbers (with $b \neq 0$) and the same symbols can represent different quantities (e.g., $\frac{1}{2}$ of 8 and $\frac{1}{2}$ of

12); (3) the understanding of equivalence ($\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$) and ordering ($\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$) of fractions; and (4) the diverse and complex interpretations, meanings or situations of fractions. Children's understanding of the inverse relation between numerator and denominator is crucial for the concept of fractions, and this understanding seems to be affected by the type of interpretation of fractions.

The literature presents different classifications of interpretations or meanings for fractions. Kieren (1993) distinguishes four categories known as "sub-constructs" that are relevant for the concept of rational number: (1) quotient; (2) measure; (3) operator; and (4) ratio. Berh, Lesh, Post, and Silver (1984) consider five "sub-constructs" to clarify the concept of rational number, which are: (1) part-whole; (2) quotient; (3) ratio; (4) operator; and (5) measure. More recently, Nunes, et al. (2004) presented a classification based on "situations" in which fractions are used, relying on the meaning of the magnitudes assumed in each case, distinguishing: (1) part-whole; (2) quotient; (3) operator; and (4) intensive quantities.

This study adopts Nunes et al. (2004) classification in which in quotient interpretation or situation, $\frac{a}{b}$ can represent the relationship between the number of recipients and items to be distributed (e.g., $\frac{2}{3}$ can represent 2 chocolate bars to be shared fairly by 3 children), but it also represents the quantity of an item received by each recipient (e.g., $\frac{2}{3}$ corresponds to the amount of chocolate received by each child). In the part-whole situation, $\frac{a}{b}$ represents the relationship between the number of equal parts in which the whole is divided and the number of these parts to be taken (e.g., $\frac{2}{3}$ of a chocolate bar means that this was divided into 3 equal parts and 2 of these parts were considered).

Studies focused on different interpretations of rational number have suggested that these affect differently how children understand fractions. Some authors argue that the quotient interpretation favours the understanding of the inverse relationship between numerator and denominator of fractions (see Mamede, et al., 2005).

Mamede, et al. (2005) investigated whether the quotient and part-whole interpretation of fraction influence the children's performance in problem solving tasks. Eighty children participated in the study aged between 6- and 7-year-olds, who haven't had formal instruction on fractions, but some of them were already familiar with the words "half" and "fourths" in social contexts. The authors analysed how children understand fractions in part-whole and quotient interpretations, in tasks related to equivalence, ordering, and labelling. Results indicated that children performed better in quotient interpretation than in part-whole regarding ordering and equivalence of fractions; children performed similarly when solving labelling tasks presented in quotient and in part-whole interpretations. Children's success levels in ordering and equivalence of fractions in quotient interpretation suggests that they have some informal knowledge about the logic of fractions, developed in their daily life, without school instruction.

These results emphasize the idea that different interpretations of fractions create distinct opportunities for children to understand the inverse relation between quantities.

Nunes et al. (2004) suggest that children's understanding of the inverse relation between quantities is facilitated in quotient interpretation because numerator and denominator are variables of different nature. Previous research on these issues was conducted by Mamede and Vasconcelos (2014), with Portuguese 4th graders, to understand how the inverse relation between size and number of parts in division situations is related to the concept of fraction presented in quotient and part-whole interpretations. Among other things, they found that children's performance in solving ordering and equivalent fraction problems in quotient interpretation were related to each other, as was their performance in solving ordering problems in quotient and in part-whole interpretations, but no significant correlations were found when solving the equivalence problems in quotient and part-whole interpretations.

Traditionally, in Brazil and in Portugal, fractions are introduced to children using the part-whole interpretation of fractions. If children possess an informal knowledge about fractions, and classroom practices emphasize the introduction of fractions in part-whole interpretation from the 3rd grade, how do children who already received some formal instruction on fractions understand the inverse relation between size of *n* and *n*-parts when problems are presented in quotient and part-whole interpretations? Cross-countries systematic comparisons are relevant, as both countries speak the same language, and are necessaries before making generalizations.

This study analyses Brazilian and Portuguese children's ability to establish the inverse relationship between quantities, for understanding the concept of fractions and the logical invariants of ordering and equivalence. It addresses three questions: 1) How do children understand the inverse relation between quantities when fractions are presented in quotient interpretations? 2) How do children understand this inverse relation when fractions are presented in part-whole interpretation? 3) Are there differences in performance between Brazilian and Portuguese children concerning the understanding of the inverse relation between quantities in these interpretations?

METHODS

A survey by questionnaire was conducted with 9- to 10-year-olds Portuguese (n=42; mean age = 9.69), and Brazilian (n=42; mean age = 9.88) children. The questionnaire included 22 tasks: 8 problems with fractions in part-whole interpretation (4 ordering; 4 equivalence); 8 problems with fractions in quotient interpretation (4 ordering; 4 equivalence); and 6 division problems (3 partitive division, 3 quotitive division). Due to length constrains, the analysis presented here will focuses only on problems of fractions presented in quotient and part-whole interpretations.

All fractions involved in the tasks were less than 1 and were the same for the problems proposed with quotient and part-whole interpretation. Table 1 shows an example of tasks presented for each type of fraction interpretation.

The questionnaire was solved individually and lasted for 40 minutes, and was implemented by the class teacher. Each child received a booklet with one problem per sheet to be solved. In each problem, multiple-choice questions were present, and the judgment for relative value of the quotients by using relations “more than/ less than/ same quantity as” was favoured.

Problems	Equivalence	Ordering
Part-whole	Marco and Lara have each a pizza with the same size. Marco divided his pizza into 5 equal parts and ate one part. Lara divided her pizza into 10 equal parts and ate two parts. Did Marco eat more pizza than, less pizza than, or the same quantity of pizza as Lara? Explain why.	Ana and Rita have each a chocolate with the same size. Ana ate $\frac{1}{2}$ of her chocolate and Rita ate $\frac{1}{3}$ of her chocolate. Did Ana eat more chocolate than, less chocolate than, or the same quantity of chocolate as Rita? Explain why.
Quotient	Children share two same-sized cakes. Two girls share one cake fairly; three boys share the other cake fairly. Does each girl eat more cake than, less cake than, or the same quantity of cake as each boy? Explain why.	Two girls will share a chocolate bar and each one will eat $\frac{1}{2}$ of the chocolate. Three boys will share a chocolate bar and each one will eat $\frac{1}{3}$ of the chocolate. Does each girl eat more chocolate than, less chocolate than, or the same quantity of chocolate as each boy? Explain why.

Table 1: Examples of tasks presented with fractions in quotient and part-whole interpretations.

Questions were presented to the class and read by the researcher using PowerPoint slides. Each child had to indicate the right answer on the booklet and give a written explanation. The tasks used were adapted from the studies of Mamede, et al. (2005) and Spinillo and Lautert (2011).

Results

Results of the children’s performances when solving the proposed tasks were analysed, by assigning 1 to each right answer and 0 to each wrong answer. Table 2 presents the mean of the correct answers and standard deviations, according to the type of problem, presented in part-whole and quotient interpretations.

	Portugal		Brazil	
	Equivalence	Ordering	Equivalence	Ordering

Part-whole	1.4 (1.31)	1.98 (1.47)	0.55 (0.86)	1.07 (1.14)
Quotient	2.33 (1.14)	3.0 (1.19)	1.67 (1.43)	1.57 (1.40)

Table 2: Mean and (standard deviation) of children’s correct responses according to the type of problem presented in part-whole and quotient interpretations, by country.

The results suggest that problems presented in quotient interpretation are easier for children than those presented in part-whole interpretation. Results also suggest that ordering problems are easier for children than equivalence ones. Table 1 also gives the idea that Portuguese children seem to perform better than Brazilian solving fractions problems presented in both interpretations.

Children’s performance in each type of fractions problem presented in part-whole and quotient interpretations, by country, is given by Figures 1-4.

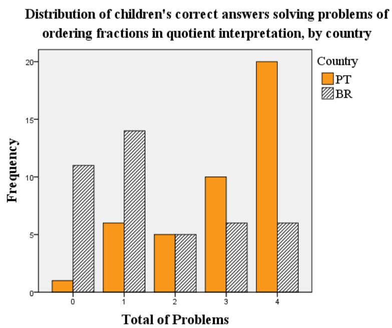


Figure 1

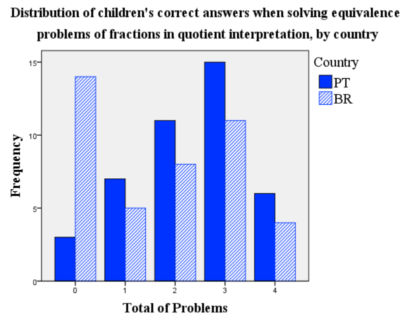
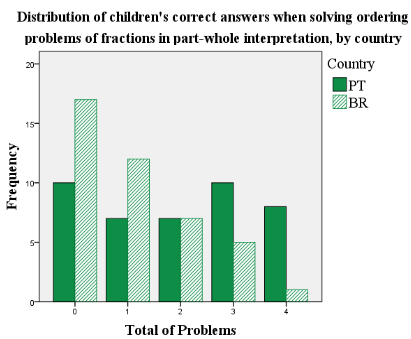


Figure 2



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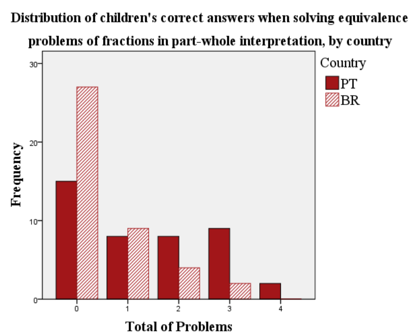

Figure 4

In quotient interpretation, when solving ordering problems, 83.3% of Portuguese and 40.5% of Brazilian children got at least 2 problems correctly solved; all of these problems were correctly solved by 47.6% and by 14.3% of Portuguese and Brazilian children, respectively. Regarding the equivalence problems, 76.2% of Portuguese and 54.7% of Brazilian children got at least half of the problems correctly solved; and 14.3% and 9.5% of Portuguese and Brazilian children, respectively, got all the problems correctly solved.

In part-whole interpretation, when solving ordering problems, 59.5% of Portuguese and 31% of Brazilian children got at least 2 problems correctly solved; all problems were correctly solved by 19% of Portuguese and by 2.4% of Brazilian children. Regarding the equivalence problems, 45.2% of Portuguese and 14.3% of Brazilian children got at least half of the problems correctly solved; and 4.8% of Portuguese children got all problems correctly solved, but none of the Brazilian children did it.

A non-parametric Wilcoxon-Mann-Whitney test was conducted to compare Portuguese and Brazilian children's performance when solving the fractions problems (equivalence and ordering in quotient and part-whole interpretations). Children's performance when solving problems in quotient interpretation is significantly better in the group of Portuguese than in Brazilian children ((U=403; W=1305; p<.001) for ordering problems and (U=647; W=1550; p<.05) for equivalence problems). Portuguese children's performance was significantly better than Brazilian also when solving problems presented in part-whole interpretation, (U=573; W=1476.5; p<.05) for ordering problems and (U=552.5; W=1455.5; p<.001) for the equivalence ones. The discrepancy of performance between Portuguese and Brazilian children when solving the tasks might be explain by the differences in the mathematics instruction of fourth graders in these countries.

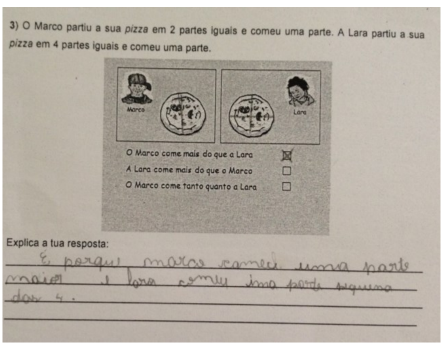

Figure 5

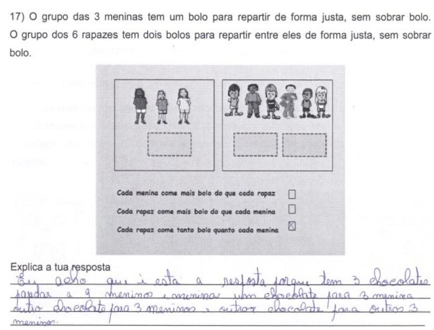

Figure 6

In Figure 5 an ordering problem was presented in part-whole interpretation to compare fractions ½ and ¼. The child justifies that Marco eats more pizza than Lara because "Marco ate a bigger part and Lara ate a smaller part from the 4.". In Figure 6, an equivalence problem was presented in quotient interpretation to compare 1/3 and 2/6. The child justifies that "I think this is the answer because there are 3 chocolate bars for 9 boys and girls, a chocolate bar for 3 girls, another chocolate for 3 boys and another for other 3 boys."

DISCUSSION AND CONCLUSIONS

This study suggests that Portuguese and Brazilian children understand the inverse relation between quantities in quotient interpretation. In spite of the some differences in performance across countries, most of the children of each country who participated in this study, succeeded in solving simple ordering and equivalence fractions problems presented to them in the quotient interpretation. In the part-whole interpretation, children of both countries found more difficult to succeed either in ordering or equivalence simple problems.

The results suggest that quotient and part-whole interpretations contribute differently to the children's understanding of the inverse relation between quantities. This idea is in agreement with previous research carried out with 6- and 7-year-olds children (see Mamede, et al., 2005), who had not received any formal instruction about fractions in school, but could succeed in solving simple fraction problems in quotient interpretation, revealing that children possess some type of informal knowledge on the logic of fractions (ordering and equivalence), developed in their daily life. If different interpretations of fractions involve distinct levels of understanding of the inverse relation between quantities for children, caution should be made when exploring these interpretations in the mathematics classes. Teachers should be aware that an absence of exploration of an interpretation of fractions may compromise children's understanding of rational numbers.

The discrepancy of performance between Portuguese and Brazilian children when solving the tasks might be explain by the differences in the mathematics instruction of fourth graders in these countries. In Portugal, children contact more informally with fractions in 3rd grade and more formally in 4th grade, according to the official curricular guidance; In Brazil, in spite of curricular guidance related to these issues, frequently teachers avoid to explore fractions in mathematics classes, compromising the development of children's understanding on the inverse relation between quantities. Possibly, this happens because teachers do not believe that their children can understand such relations or are unsure about how to explore fractions with their students. This study gives evidence that fourth-graders can understand the inverse relation between quantities, and interesting discussion moments around these subjects could take place in their classes. More research needs to be developed concerning such important topic in order to stimulate properly the children's understanding of the inverse relation between quantities, at primary school levels.

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STUDENTS' EMOTIONAL EXPERIENCES IN LINEAR ALGEBRA COURSES

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The aim of this qualitative research is to identify emotional experiences of Mexican undergraduate mathematics students in linear algebra courses. In order to obtain data, focus group interviews were carried out with 27 students. Data analysis was based on the Theory of Cognitive Structure of Emotions. The participants' emotional experiences were: A) Satisfaction and disappointment, B) Fear emotions, C) Relief and D) Self-reproach. The results showed that almost all students' emotional experiences were based on their appraisal of events in terms of active academic goals and learning achievement. Effort to achieve goals was the variable that modified most of the intensity of emotions. It was considered to be a supreme effort (compared to other courses) because linear algebra is considered a very difficult course.

INTRODUCTION

Research on emotions in mathematics education highlights the necessity to move beyond the simplistic view of distinguishing between positive and negative emotions and focus on emotions during routine mathematical experiences, rather than non-routine mathematical activities (Hannula, Pantziara, Wæge, & Schlöglmann, 2010). Our study aimed to identify emotional experiences in routine activities in mathematics classes. In order to go beyond a consideration of positive and negative emotions we used the cognitive structure of emotions theory (Ortony, Clore, & Collins, 1988). Other researchers in mathematics education (e.g Di Martino, Coppola, Mollo, Pacelli, & Sabena, 2013) have suggested that this theory is an appropriate one to use to analyse students' and teachers' emotions in mathematics.

We are aware that the analysis of narratives of emotional experiences is quite different from the direct analysis of emotions but, like Ortony et al. (1988, p. 8), we are willing "to treat people's reports of their emotions as valid, also because emotions are not themselves linguistic things, but the most readily available non-phenomenal access we have to them is through language". This is the main reason to focus on the following research question: *What are students' emotional experiences in linear algebra courses?*

THE THEORY OF THE COGNITIVE STRUCTURE OF EMOTIONS

For the cognitive structure of emotions' theory (OCC theory), emotions arise from interpretations of situations by those who experience them: emotions can be taken as

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"valence reactions to events, agents or objects, with their particular nature being determined by the way in which the eliciting situations is construed" (p. 13). Thus, a particular emotion experienced by a person on a specific occasion is determined by his interpretation of the changes in the world:

> When one focuses on events one does so because one is interested in their consequences, when one focuses on agents, one does so because of their actions, and when one focuses on objects, one is interested in certain aspects or imputed properties of them *qua* objects (Ortony et al. 1988, p. 18).

OCC theory categorises different types of situations that elicit emotions into classes according to a word or phrase corresponding to a relatively neutral example that fits the type of emotion (Ortony et al., 1988). The characterisations of emotions in OCC theory are independent of the words that refer to emotions, as it is a theory about the things that concern denotative words of emotions and not a theory of the words themselves. In terms of the distinction between reactions to events, agents and objects, there are three basic classes of emotions:

> Being *pleased* vs. *displeased* (reaction to events), *approving* vs. *disapproving* (reactions to agents) and *liking* vs. *disliking* (reactions to objects) (Ortony et al. 1988, p. 33).

OCC theory specifies classes, groups and emotion types; these are briefly laid out in Table 1.

Class	Group	Types (sample name)
Reactions to events	Fortunes-of-others	Pleased about an event desirable for someone else (*happy-for*)
		Pleased about an event undesirable for someone else (*gloating*)
		Displeased about an event desirable for someone else (*resentment*)
		Displeased about an event undesirable for someone else (*sorry-for*)
	Prospect-based	Pleased about the prospect of a desirable event (*hope*)
		Pleased about the confirmation of the prospect of a desirable event (*satisfaction*)
		Pleased about the disconfirmation of the prospect of an undesirable event (*relief*)
		Displeased about the disconfirmation of the prospect of a desirable event (*disappointment*)

		Displeased about the prospect of an undesirable event (*fear*) Displeased about the confirmation of the prospect of an undesirable event (*fears-confirmed*)
	Well-being	Pleased about a desirable event (*joy*) Displeased about an undesirable event (*distress*)
Reactions to agents	Attribution	Approving of one's own praiseworthy action (*pride*) Approving of someone else's praiseworthy action (*appreciation*) Disapproving of one's own blameworthy action (*self-reproach*) Disapproving of someone else's blameworthy action (*reproach*)
Reactions to objects	Attraction	Liking an appealing object (*liking*) Disliking an unappealing object (*disliking*)

Table 1: Emotion types according to the OCC theory.

OCC theory specifies global, central and local variables that affect the intensity of different emotions types. The global variables are: (1) *sense of reality*, which depends on how much one believes the emotion-inducing situation is real, (2) *proximity*, which depends on how close in psychological space one feels to the situation, (3) *unexpectedness*, which depends on how surprised one is by the situation, and (4) *arousal*, which depends on how much one is aroused prior to the situation.

Central variables are: 1) *desirability* of an event is appraised in terms of how it facilitates or interferes with the focal goal and the sub-goals that support it, 2) *Praiseworthiness* of an agent's actions is evaluated against a hierarchy of standards, and 3) *appealingness* of an object is evaluated with respect to a person's attitudes.

Local variables are tied to particular groups of emotions. For example, Prospect-based emotions are affected by (1) *likelihood*, which reflects the degree of belief that an anticipated event will occur, (2) *effort*, which reflects the degree to which resources were expended to achieve or avoid an anticipated event, and (3) *realisation*, which depends on the degree to which an anticipated event actually occurs.

METHODOLOGY

Context

The research was carried out in the Mathematics Faculty of the Autonomous University of Zacatecas (UAZ) in a northern state of Mexico. This university offers a four-year mathematical degree organised into eight semesters with five possible specialisations: basic mathematics, applied mathematics, mathematical education, statistics and informatics. This career offers two Linear Algebra courses. The structure of these Linear Algebra courses is based on teacher explanations. The didactic organisation of the course is highly linked to homework. There are two types of homework: 'short homework' (presented the day after it is assigned) and 'large homework' (presented a week later and consisting of six or seven exercises).

Assessment in both courses of Linear Algebra is the weighted sum of the numerical evaluation of homework and tests. There are several kinds of tests: short tests, partial tests and ordinary tests. Short tests consist of two or three questions, taken once a week. Partial tests are taken at the end of every topic. Ordinary tests, so called by the participants, are not really tests but the weighted sum of short tests, homework and partial tests (80%) and a final test (20%) that covers all topics in the course.

Participants

A group of 27 mathematics students in their second to eighth semester participated in this research. They were 12 women and 15 men, aged between 19 and 25 years old. All of them have already taken both courses of Linear Algebra; 18 of them failed both courses. Participation was voluntary.

Data gathering procedure

As the focus of the research was on the students' subjective experiences of emotions, we decided to use focus group interviews because we observed during previous research at the same university that students feel confident and comfortable in expressing their thoughts, feelings and emotions about various topics in focus group interviews.The questions asked were: 1) How do you generally feel in the Linear Algebra course (I or II)?, 2) What kind of situation stresses or distresses you in a Linear Algebra course?, 3) How do you feel when you solve a problem in a Linear Algebra course?, 4) And when you cannot?, 5) How do you feel the day of a test in a Linear Algebra course?, 6) What feelings do you relate to Linear Algebra? Why?, 7) If you failed a Linear Algebra course (I or II), how did you feel when you fail? And 8) how did you feel when you finally passed?

Data analysis

The students were identified as M*n*-G*k* or F*n*-G*k*. Where M and F indicate the participant's gender, *n* (1 to 5) indicates the participant identification number and *k* (1 to 8) indicates the focus group number. The videotaped interviews were fully transcribed. According to OCC theory, a type of emotion is identified by three specifications: 1) ***Concise phrases*** that express all the eliciting conditions of the emotional experiences. In the evidence, we highlight these phrases in bold italics, 2) *Emotion words* that express emotional experience. We highlight emotion words in italics and 3) variables that affect intensity of emotions. We underlined phrases that express intensity in the evidence.

Due to the daily use of words to express emotions, it may happen that one word refers to different types of emotions. To identify evoked emotions we took into account the eliciting conditions, just as OCC theory suggests. For example, students **F1-G6** and **F2-G6** use different emotion words ("*I feel fine*" and "*I am really happy*") to express their emotional experiences triggered by the successful solving of a problem. Both emotions are *satisfaction* emotions (pleased about the confirmation of the prospect of a desirable event) from the point of view of OCC theory.

F1-G6: I feel really fine when I solve a problem, especially if I did it alone. It is uplifting.

F2-G6: I am really happy when I solve a problem, because it is so hard.

We have included explanations in square brackets in order to clarify some of the students' expressions.

RESULTS

The participating students' emotional experiences are summarised in Table 2.

Type of emotion	Triggering situations	Variables
Satisfaction	Solving problems in class	Effort
	Solving problems at home	Desirability
Disappointment	Solving problems in a test	
Fear	Attributed difficulty of Linear Algebra course	
	Solving problems in a test	Effort
	Asking about doubts in class	
	Going to the blackboard to solve problems	
Distress	Attributed difficulty of Linear Algebra course	
	Attributed difficulty of homework	Effort
	Attributed difficulty of tests	
	Failing the course	
Self-reproach	Delay in studies	Plausibility
	Parents' disappointment	
	Repeated failure	

Table 2. Emotional experiences of students.

In the following section we show evidence of related satisfaction/disappointment emotions.

SATISFACTION/DISAPPOINTMENT EMOTIONS

Three triggering situations made students experience emotions of *satisfaction* and *disappointment.* These situations were: (1) resolution of problems in class, (2) resolution of homework problems and (3) resolution of problems in a test.

Students had different meanings for 'problem–solving'. These ranged from the narrowest sense of exploring a definition by doing concrete exercises (e.g. numerical matrixes calculus) to the widest meaning of demonstrating a theorem. In a general way, we interpreted students' meaning of 'problem-solving' as the general tasks that a teacher asked students to perform.

Resolution of problems in class (Disappointment)

Disappointment emotions are triggered by the attributed difficulty of 'solving problems'. The necessity of solving problems 'step by step' without omitting any detail to achieve the result was the most mentioned difficulty. Another frequent difficulty was the correct application of the studied theorems.

> **F2-G1**: ***Solving a problem*** *stresses me*; ***not knowing or not having an idea of how to solve a problem*** *frustrates me*. There are things I don't know. I believe that I don't know how to apply things. I understand the theorems but it is hard to apply them. It didn`t happen to me in other courses, only in Linear Algebra.

> **F1-G2**: I think I *get stress* ***with problems that are to be made with details***. Sometimes you skip the details that are the key to solve the problem but you don't realise until the teacher says so. You have to write everything even if it seems exceeding.

Effort variable (Satisfaction)

Intensity of satisfaction emotions triggered for solving a problem is increased by the effort variable. This variable reflects students' effort to achieve an anticipated event (solving a problem). Students highlighted this variable in metaphors of intense emotional state ("I feel really high", "We felt like mathematics masters"). Students did not use these adverbs to express disappointment emotions in the same activity so we considered that satisfaction emotions were more intense.

> **M1-G4**: <u>*I feel really high*</u> ***when I solve a problem*** [He changes his facial expression and smiles facing the camera directly, instead of avoiding it as in the previous questions].

> **F2-G6**: I feel really fine when I solve a problem, because <u>it takes a lot of effort and, finally, if you solve it </u>you feel really happy.

Solving homework problems (Disappointment)

The students' attributed difficulty of solving problems increases when the problem is a homework assignment. The higher level of difficulty was attributed to two circumstances: (1) solving problems in class is easier because the teacher explains the procedures, which means that homework is difficult because the teacher cannot help them, and (2) problems in class are easier because they are 'operation exercises' and homework are 'demonstration problems'.

> F2-G1: I like [Linear] Algebra, but the homework is much more complicated than examples in class.

M1-G1: You start [solving a problem] in class and say: Oh! I got it! But then you realise you don't. You ***spend a lot of time on the same problem,*** *and then you get frustrated,* ***because you could not solve*** *it.* It happens that the teacher gives a really simple example but homework is more complicated.

Effort variable (Satisfaction)

The attributed difficulty of homework problems increases the intensity (effort variable) of satisfaction emotions while solving homework.

> **M2-G6:** I heave a sigh of relief **when I solve a problem in the homework.** I feel really, really happy, it takes pressure from me.

Solving problems in a test (Satisfaction and disappointment)

Students' attributed difficulty of solving problems is even higher in a test because it is an individual activity. They expressed that their classmates, teacher and books supported them in a class, but not in a test. This greater attributed difficulty implies more effort and caused the experienced emotions of satisfaction/disappointment in a test to be more intense than during a class or homework. This showed that students perceived that solving problems in a test was the proof that they really understood the topic.

> **M1-G3:** It was a *great relief, a joy,* ***if I could solve a problem in a test.*** We could go home without a worry. I could sleep fine and said: "I can go to sleep at 10:00 yahoo!" [Joy exclamation because they could not sleep well for days in order to study]

> **F1-G4**: … It happened that I *started crying* when I finished a test ***because I believed I studied enough but I blocked out***. In a test, I read the problem and I cannot do it; I think I can't and then I don't even try.

CONCLUSIONS

The data analysis shows that the students' narratives focused on two groups of emotions: **prospect-based** group (*satisfaction, disappointment and fear)* and **well-being** group (*distress)*. The main local variables for the intensity of each group were effort and desirability respectively. Therefore, almost all the emotional experiences of the students in the class were related to reactions to events. Following OCC theory, the students' emotional experiences were based on their appraisals of desirable goals that they tried to achieve in an active way like solving problems, passing tests or passing the course.

Although OCC theory establishes that emotions in the **attribution** group (*pride* and *self-reproach*) are triggered by appraisals in terms of social rules, we identified in this study that the triggering situations for these emotions are related to academic achievement (e.g. 'finishing the degree'). Thus, students' emotions are completely triggered by the achievement of academic goals and school success. Research into mathematics education has already highlighted the central role of goals in emotional experiences. Hannula (2006) conceptualises motivation in mathematics as "goals reflected in emotions" because it is possible to direct behaviour through the mechanisms that control emotions. In this regard, some motivational research in mathematics education highlighted "fear of failure" as an important antecedent variable to direct students towards specific achievement goals (Pantziara & Philippou, 2014). According to motivational theories in school contexts, the emotions reported in this study are reactions associated to principle of *competence* of the academic motivation: "developing academic competence is both a human need and the expressed goal of schooling" (Turner, Warzon, & Christensen, 2010, p. 3).

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AN INTEGRATED TECHNOLOGY COURSE AT UNIVERSITY: ORCHESTRATION AND MEDIATION

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Investigating integration of technology into undergraduate mathematics courses is still a relatively new field. In this paper we report on intensive technology use in a large undergraduate bridging calculus class. The focus is the lecturer's modelling of technology use and orchestration of the didactical configuration to address average rate of change. We describe a possible new guide-to-investigate orchestration with evidence of local thinking and a positive effect on student usage and attitudes.

BACKGROUND

This paper addresses the issue of whether, with suitable lecturer instrumental orchestration, undergraduate students use of technology can mediate a move towards a local perspective of rate of change. Vandebrouck (2011) identifies the tendency for school mathematics to focus on pointwise and global perspectives of functions, whereas mathematics at university often requires a local perspective. He defines a pointwise property as depending only on the value of the function at a specific point x_0; a global property is defined on an interval; a local property is one that depends on the values of f in a neighbourhood of a specific point x_0. These definitions imply that differentiating between local and global properties of functions depends on the context and the type of interval considered (a neighbourhood is also an interval). The question in this research was how digital technology could assist students to construction a local perspective on function.

We subscribe to the notion that when teaching with digital technology, at any level, a shift in focus by the teacher may be required, from "seeing the technology as simply something added to the teaching of mathematics to putting the mathematics at the centre of activity, and asking how the [technology] can enable students to understand the mathematical concepts better" (Heid, Thomas & Zbiek, 2013, p. 632). In this way the technological tool can become a mediating instrument that can lead, through the development of suitable mental schemes and techniques, to pragmatic and epistemic mediation (Artigue, 2002) between the user and the mathematical constructs.

Development of these schemes and techniques by students with access to technology is not trivial and requires careful intervention of the teacher to promote instrumental genesis, described by Trouche (2004) as *instrumental orchestration*. These orchestrations exploit the *didactical configuration,* an arrangement of the tools in the learning environment, through an *exploitation mode* to achieve the teacher's goals. They may be planned and systematic or ad hoc, constituting a *didactical performance* (Drijvers, Doorman, Boon, Reed & Gravemeijer, 2010). Here we are particularly concerned with the whole-class orchestrations of a university lecturer; primarily pre-

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planned, but also comprising some ad hoc aspects. Whole-class orchestrations have been identified by Drijvers, Tacoma, Besamusca, Doorman and Boon (2013) as: technical-demo; guide-and-explain; link-screen-board; discuss-the-screen; explain-the-screen; spot-and-show; Sherpa-at-work; and board-instruction. Our analysis of the lecturer orchestrations in this study will be based on this classification.

METHOD

This research forms part of a larger design experiment study (see Oates, Sheryn & Thomas, 2014) investigating intensive technology use in a large undergraduate bridging calculus class comprising 36 one-hour lectures and 10 one-hour tutorials. By intensive use we mean that, as far as possible, students should have unrestricted use of mathematical e-environments through a multi-platform didactical configuration and that this access should extend to lectures, tutorials and all assessment (Oates et al,. 2014). Central to this study is the principle that the lecturers would model technology use, recognising the implications of teacher-privileging described by Kendal and Stacey (2001). Here this included calculators and websites such as Wolfgram Alpha, stand-alone programs such as GeoGebra and web-based programs such as Desmos. One advantage of each of these was cost-free access for students on diverse platforms such as computer, tablet or smartphone, which evidence suggests is important for student use and engagement with the technology in this project (Oates et al., 2014). A video-recording of each lecture was also provided via a local server within 24 hours.

Data reported here comes from four main sources: Observation of two volunteers as they worked in a computer lab on a rich technology-active tutorial task based on the concept of average rate of change; sample student responses to an examination question, and two questionnaires; a technology questionnaire and an attitude survey. Observation notes were made of the tutorial along with an audio recording of the students' discussion. For the questionnaires, while 62 of the 240 students consented to participate, only 12 completed the technology questionnaire and nine the attitude survey in spite of many attempts to elicit responses. Responses were anonymous so we cannot know how many were in both groups. While the number of responses seems small, we believe it still gives a reasonable indication of student reactions to the course.

Figure 1 shows examples of the questions used in the online technology questionnaire, a mix of 19 open and closed questions that investigated student use of technology in general; mathematics-focused technology use; and patterns of technology use during the course. For the attitude survey, a Likert scale was constructed with 29 randomised items, each with five possible responses (strongly agree, agree, neutral, disagree, and strongly disagree). Five subscales measured: attitude to maths ability; confidence with technology; attitude to instrumental genesis of technology (learning how to use it); attitude to learning mathematics with technology; and attitude to versatile use of technology. The versatility subscale had four questions and the others five. In addition, five questions covering possible goals in technology use, which was not a subscale. Table 1 gives examples of some items from the attitude survey.

2. Do you think the lecturers made sufficient use of these technologies to help you understand their use and value? If not, specify which you would have liked more of.

3. Which technologies do you personally own or have easy access to? [list given]

5. Which mathematics learning technologies did you personally use in the course? Please indicate your frequency of use, and whether this was the first time you had used them.

7. What activities did you use technology for? Please specify which technologies you used for each of the following activities: [Lectures, assignments, tutorials, quizzes, other]

11. Describe the kind of activities you used technology for when working on mathematics problems in the course. [Open response]

Figure 1: Examples of the open and closed questions from the questionnaire

Learning Mathematics with Technology	Instrumental Genesis
I like using technology to learn maths	Learning how to use technology is difficult for me
Using technology in maths is worth the extra effort	I work to improve my ability to use technology
Maths is more interesting when using technology	I often need to ask others how to use technology
Using technology hinders my ability to understand maths	I can understand a new technology as quickly as other people
I prefer working out maths by hand rather than using technology	Using technology wastes too much time in the learning of maths

Table 1: Examples of two attitude subscales

RESULTS

One of the key mathematical constructs targeted in the course was that of average rate of change (AROC) of a function, and its relationship to instantaneous rate of change. If we consider two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ then the average rate of change of the function on the interval $[x_0, x_1]$, $\frac{f(x_1)-f(x_0)}{x_1-x_0}$, is a global property according to Vandebroucke (2011), since it is defined on an interval. However, if we take the AROC of f over the interval $[x_0, x_0+h]$, and consider $\frac{f(x_0+h)-f(x_0)}{h}$ then is this still a global property irrespective of the size of h? Or does it become local as h becomes small. In this paper, our discussions will consider that this constitutes a local perspective. During lectures, the concept of AROC of a function was introduced using a *board-instruction* orchestration. In this mode, the lecturer wrote on paper, projected on to a large screen visible to all the students and recorded for the class lecture video (All screenshots in this paper are taken from lecture videos provided to students). The idea that a linear function and a polynomial through two points have the same AROC on the interval defined by those points was mentioned (see Figure 2a). Basic AROC calculations were carried out using function notation (see Figure 2b) and the rate of change of a function at a point x_0, the derivative, was defined as the limit as $h \to 0$ of the AROC of the function on the interval $[x_0, x_0 + h]$. Rates of change were emphasised for finding the nature of stationary points and concavity.

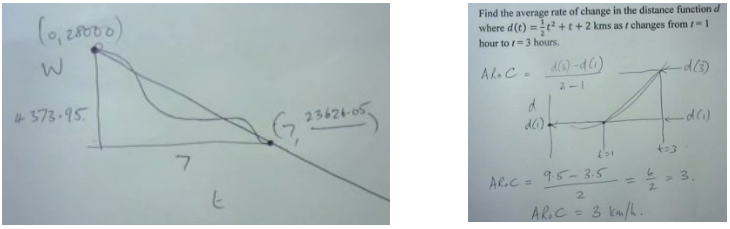
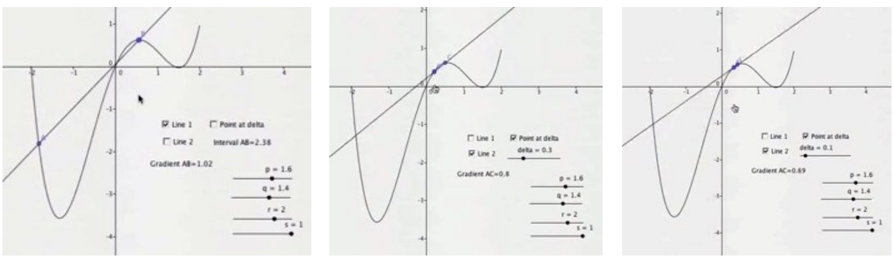


Figure 2: Two screenshots taken from lecture videos.

Whole class instrumental orchestration

During the introduction of AROC a GeoGebra program, written by the lecturer, was displayed (see Figure 3). The lecturer used dynamic dragging and an *explain-the-screen* orchestration to present examples of the AROC between two points both a variable and a fixed distance apart, linking this to mathematical constructs.



a b c

Figure 3: Screenshots showing dynamic use of GeoGebra for global and local AROC

Some of these examples (Figure 3a) could be said to illustrate global properties due to the distance between the points, while others were local properties, with a small delta, down to 0.1 (Figure 3b, c). Two further examples of *technical-demo* orchestrations using the web-based Desmos graphing program are shown in Figure 4. Here the left hand screen shows a demonstration of a technique for finding and displaying approximate solutions to equations for $2\cos x^\circ - 1 = -2$. The right hand screen shows the use of Desmos to draw functions with split domains. Students were constantly encouraged to change and extend the examples given in the lectures by investigating for themselves what the program response to various inputs would be, and 50% of the questionnaire respondents said that they used Desmos during the lectures. We see this kind of orchestration that usually followed a *technical-demo* as a new development of the Drijvers et al. classification, which we have called a *guide-to-*

investigate, with students immediately encouraged to use Desmos, or other technology in their possession, to investigate further examples.

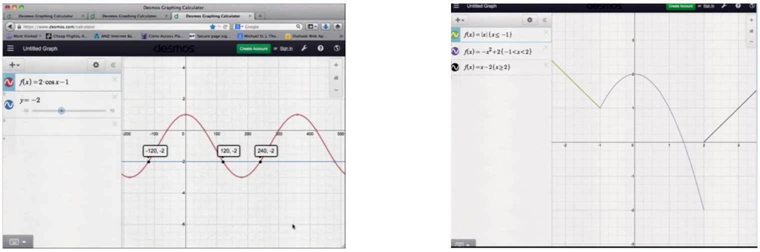


Figure 4: Screenshots showing use of explain-the screen with Desmos

The final program that featured through the course was Wolfram Alpha. In Figure 5 we provide just two examples of its use.

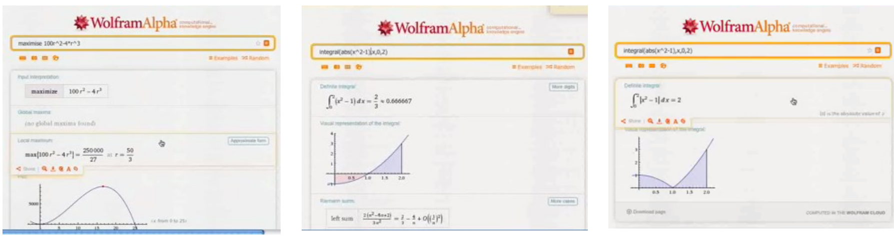


Figure 5: Screenshots showing use of explain-the screen with Wolfram Alpha

All three screens were employed in *explain-the-screen* orchestrations. The first shows how we can find the (local) maximum value of a function. Although this is a black box process, one advantage of Wolfram Alpha here is that it makes links between the algebraic and graphical representations. The other two screens enabled discussion of a valuable technique for finding the area between a function and the *x*-axis, and why a difference sometimes occurs between this area and the 'standard' definite integral.

Student technology use

Student responses to technology use were consistent with those reported earlier (Oates et al., 2014), although there was some evidence the sustained intensive technology approach might be leading to greater usage, for example 100% of respondents to the later study reported using Desmos in the course (85% in 2014). It also seems teacher-privileging may have had led to an increased use of GeoGebra with 10% in this study compared to only one student in 2014. Desmos was the most popular platform (80% very useful or useful; only 10% had never used it), with usage in lectures (50%), assignments (77.8%), tutorials (77.8%), and quizzes (77.8%). They particularly liked its ease of access and use: "very easy to use and very easy to access", "useful as it is very responsive (quick) and extremely easy to use" and "Easy to use." 91.7% said the lecturer made sufficient use of technology in the course (cf. 76.9% in 2014), with

90.9% affirming that they received sufficient help with the technology ("Lecturer always explains"). Further, 83.3% liked the extensive use of technology in the course ("The use of technology was great, seeing the graphs and how they work in Desmos was really useful"; "It provides another perspective when solving problems"; "Yes, it's nice to know that we are moving with the advancement in technology") and 91.7% thought the technology helped in their learning of mathematics, for example, helping to visualise solutions ("Graph is much easier to understand and solve problems").

In one tutorial, students were given a specially designed, technology-rich problem involving the focus concept of AROC in a financial context. Students in the tutorial were asked to respond to a description of two mock students answering a problem associated with the graph of $f(t)$, as in the condensed excerpt below:

> …Raj said this was an interesting problem but a bit difficult. He found the graph of the function $f(t) = 0.025(2\sin(t) + t\sin(2t) - 2t\sin(3t) + 65),\ 0 \leq t \leq 25$, which looked a bit like one of these graphs. He suggested that they work together to find out for this graph which *t* interval of size 2 has the greatest average rate of increase. It didn't take Sonja long to suggest a method. She said "You take the point at which the rate of change is greatest and take a *t* interval of 1 either side of it." What do you think of Sonja's method? Is she right? Investigate the greatest average rate of increase over a *t* interval of 2 for this graph. Where does it occur? If the *t* interval is 1 instead, where does greatest average rate of increase occur then? If the *t* interval is *k* instead, where $k \geq 0.5$, for what value of k does the greatest possible average rate of increase occur? If the *t* interval is k again, what happens to the average rate of increase as k gets smaller and smaller, i.e. as $k \to 0$? Describe in detail a method that would help Raj and Sonja find greatest average rates of change for graphs like this one.

All student groups immediately used Desmos to plot the given function and to zoom in on aspects of the graph or compare plots with other functions. In addition, scientific calculators were used and Wolfram Alpha employed to look up concepts they did not know or could not find in their notes, including AROC. While the technology was well used, the students in the closely observed focus group tended to perform by-hand calculations, integrated with computer use, retrieving data or ideas and moving back and forth between the two environments. There was considerable discussion between the two of them. They knew how to calculate AROC:

> A: So you work out the average rate of change between that point and that point which is going to be 3.2 take away 0.1, which is pretty much that bottom point there. Between those two. And there's only a difference of one. So you've got an average rate of change of 3.1. Are we good on that?

They demonstrated some idea of local properties, the effect on AROC of reducing the interval size, in essence thinking about limiting values.

> A: So that will give you the steepest line there. The other one is that one, which is pretty close, between the 29th and 12 o'clock on the 29th. But it's not quite as good. But as your *k* gets smaller, so as your *k* interval gets smaller and smaller and smaller, that one will become your steepest line. But then it will swap to that one.

A: …so m gets smaller and smaller…As m gets smaller, the greatest rate of change is going to effectively be steeper. Until you get to the stationary points. So the stationary points will remain the same, but as you get closer and closer…

The following question on AROC was set in the final examination:

> The London Eye (see picture) is a giant circular ferris wheel in London, UK. The height, H metres, of passenger capsule A above the centre of the wheel t hours after the wheel starts to move is given by: $H(t) = 60\sin\left(4\pi t + \frac{\pi}{4}\right)$. What is the average rate at which capsule A is rising during the period from $t = 0$ to $t = \frac{1}{16}$ hours?

It proved relatively difficult, with only 21.6% of the students fully correct and 62.5% gaining no marks out of 2. The students were comfortable using the function notation $H(0)$ and $H\left(\frac{1}{16}\right)$, although few gave the exact answer (even when close to it), resorting, not surprisingly, to calculators to work out the answer (Figure 6).

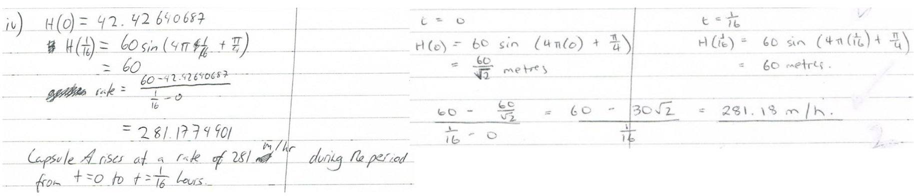


Figure 6: Sample working on the examination question on AROC

The attitude survey demonstrated, with reasonable reliability (supported by Cronbach Alpha measures), that students at the end of the course had positive attitudes towards technology to learn mathematics, to learn the techniques and construct the schemes required to do so, and a confidence to follow through on both. The subscale for *Attitude to Learning Mathematics with Technology* had a mean response of 3.24/5 (Cronbach alpha 0.71), *Confidence with Technology* a mean of 3.69/5 (Cronbach alpha 0.77), *Attitude to Instrumental Genesis* a mean of 3.62/5 (Cronbach alpha 0.64) and *Attitude to Mathematics Ability* a mean of 3.58/5 (Cronbach alpha 0.86).

DISCUSSION

The results reported here confirm the value of the intensive-technology approach suggested in Oates et al. (2014), with students consistently reporting high levels of engagement and satisfaction with the use of technology in the course and positive attitudes towards the use of technology in mathematics. It also provides further evidence of the value of teacher-privileging of technology (Kendal & Stacey, 2001), where it seems the lecturer's active modelling of diverse technologies has positively influenced student usage. The categorisation of whole-class orchestrations by Drijvers et al. (2013) proved suitable for analysing technology use by the lecturer. Evidence from the tutorial observations suggests that examples of specific orchestrations in lectures (e.g. technical-demo; explain-the-screen; board-instruction) may have mediated students' movements towards instrumental genesis when using Desmos to work on an AROC problem. We further suggest a new category for this framework, described as *guide-to-investigate,* where students immediately use technology for mathematical investigation. The conceptualisation of pointwise, local and global perspectives by Vandenbrouck (2011) enabled an analysis of students' work on the AROC problems, suggesting that the instrumental orchestration provided by the lecturer had prompted growth in students' local perspectives of the given function through pragmatic and epistemic mediation.

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YOUNG AUSTRALIAN INDIGENOUS STUDENTS GENERALISING GROWING PATTERNS: A CASE STUDY OF TEACHER/STUDENT SEMIOTIC INTERACTIONS

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Young Australian Indigenous students are underrepresented in the area of early algebra. The aim of this study is to explore the teacher and student semiotic interactions that assist these students to generalise growing patterns. Piagetian clinical interviews were conducted with six students, in which a case study is presented in this paper as a representative of the students. The results suggest that when generalising there are a series of teacher/student actions to assist students to identify pattern structures and to access mathematical language that may not be apparent. This study further extends the application and practicality of semiotic theory in the teaching/learning process of pattern generalisation with young students.

Research pertaining to Australian Indigenous students has primarily focused on pedagogical practices that support Indigenous students' learning (e.g., Jorgensen, 2009). There have been limited studies focussing on these students' acquisition of mathematical concepts. The purpose of this study is to explore how young Australian Indigenous students generalise growing patterns and to identify teaching actions that assist these students to generalise.

LITERATURE

Fundamental to the development of algebraic thinking is the ability to generalise patterns (Warren & Cooper, 2008). Research has highlighted that young students can generalise the mathematical structure of the patterns from a range of pattern contexts. For example, students can identify the structure of repeating patterns as multiplicative, and the structure of growing patterns as functions (Blanton & Kaput, 2004; Cooper & Warren, 2011). From identifying this structure, young students have demonstrated aspects of early algebraic thinking (Becker & Rivera, 2008; Radford, 2010; Warren & Cooper, 2008).

While the theory of semiotics has been long established, it is only recently that studies in the area of pattern generalisation have considered how semiotics impacts on the learning process. Those studies that have occurred have focussed on: (a) the use of semiotic resources when working on mathematical problems related to functions (Arzarello et al 2009; Radford 2009); (b) the benefits of young non-Indigenous students using hands-on materials when engaging in generalising patterns (Cooper & Warren, 2011); (c) semiotics as an analytical tool to understand how students generalise constructs in mathematics (Radford, Bardini, & Sabena, 2007; Warren &

2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 257-264. Hobart, Australia: PME.

Cooper, 2008); and, (d) semiotics in relation to cognitive models for pattern generalisation tasks (Rivera, 2010).

Semiotics has also been proposed as a means of creating a series of chaining processes to shift from culturally-embedded mathematics to Western mathematics (Presmeg, 1998). Presmeg (1998) demonstrated a series of activities that assisted secondary students to shift from a game of dominoes embedded in students' culture (culturally-embedded mathematics) to a general formula (Western mathematics) and mathematical abstractions. She developed a model to show this transfer. However, this model does not capture the complex semiotic processes between and within each step, and how students' culture impacts this learning. To date, no studies have considered what teaching actions specifically assist young Australian Indigenous students to generalise. Thus the research question for this study was: What teacher actions assist in enhancing young Indigenous students to generalise growing patterns?

THEORETICAL PERSPECTIVES

The theoretical perspectives underpinning this study were semiotics (Peirce, 1958) and Indigenous research perspectives (Denzin & Lincoln, 2008). Semiotics was utilised as a lens to interpret the interactions between teacher and students, and between students and context. From this perspective the learning of mathematics is two-fold; it involves the interpretation of signs, and the construction of mathematical meanings through communication with others (Saenz-Ludlow, 2007). In researching these cognitive interactions in young Indigenous students, it is important to acknowledge the potential for unique cultural variations with regard to how the outward displays of thought processes may be expressed. To appropriately account for these cultural sensitivities, Indigenous research perspectives were adopted. Indigenous methodologies are principally about two notions: that of relationships, and that of empowerment. Thus this study was respectful of Indigenous ways of knowing (Martin, 2003). In essence, every attempt was made to ensure that the findings of this study best reflect how Indigenous students construct knowledge and engage in the learning process.

METHODOLOGY

The data reported in this paper are taken from Piagetian clinical interviews (Opper, 1977) that were conducted at the conclusion of conjecture driven teaching experiments. The interviews provided opportunities to trial new ideas and to further discuss with students their mathematical thinking in terms of the activities being presented to them. Prior to this study students had no formal lessons on growing patterns.

PARTICIPANTS

The research occurred in one Year 2/3 classroom (7-9 year olds) of an urban Indigenous school in North Queensland, Australia. Piagetian Clinical interviews (PCI) were conducted with six purposefully selected students, three times during the year. The interviews were approximately 20 minutes in length and were video recorded where both students' gestures and the researcher's gestures were captured. Due to the limitations of this paper a case study is presented (S6). Student 6 (male, Aboriginal) was considered a low achieving student in mathematics.

DATA ANALYSIS

The data were analysed using an iterative approach. This approach is a deeply reflexive process of continuous meaning-making and progressive focusing (Srivastava & Hopwood, 2009). The researcher, due to the unique application of mathematics, semiotics and culture, constructed a data-analysis model, comprising three key stages. First, the initial video-footage was transcribed to capture students' verbal responses. These transcriptions were then analysed to consider emerging key mathematical themes from the interviews. Second, semiotics was utilised as a lens through which to reanalyse the data. The evolving data were reanalysed, focusing on semiotic bundles (signs, gestures, language) of both the student and researcher. This analysis provided an interpretation of the learning interactions between the researcher and students. Of particular importance were the students' and researcher's physical gestures, including the manipulation of hands-on materials and body language. These iconic and indexical signs were coded. Third, the data were reanalysed in line with the cultural perspective provided from the Indigenous Education Officers.

FINDINGS

As part of the full study a hypothesised semiotic learning and teaching trajectory emerged with regard to the teacher and students' interactions as they work towards generalising a linear growing pattern (see Figure 1). Figure 1 depicts the sequence of semiotic learning processes that occur as students move from the particular (immediate object) to the general (real object).

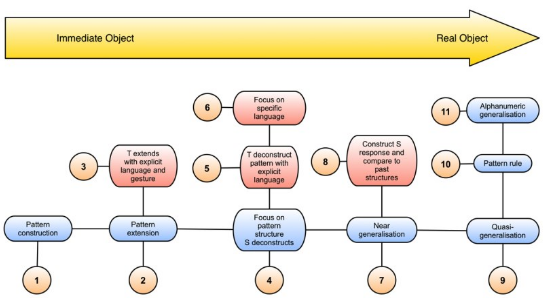

Figure 1. Semi… …m the *particular* to *the general* when engaging in pattern generalisation.

Within the trajectory, each orange numbered circle shows a point of semiotic interaction for the student. The following reports data from Interactions 4, 5 and 6, with respect to Student 6.

Interaction 4: Interaction 4 required students to focus on the pattern structure. This involved students decoding the signs, and then encoding the signs by deconstructing the pattern structure (i.e., identify the multiplicative structure by relating the pattern quantity to the pattern term). As this occurred the researcher focussed on students' gestures as these provided insights into how students perceive the structure of the pattern. For many Indigenous students, it was this juncture where they were beginning to use gesture to support their mathematical language. This interaction is particularly challenging, as students also begin to coordinate both pattern variables and their own signs (i.e., coordinating language and gesture).

S6 was asked to describe the structure of the pattern presented in Figure 2.

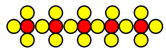

Figure 2. The daisy chain pattern used in Task 2 of Piagetian Clinical Interview 2.

Below is an excerpt from the interview of the discussion that ensued.

42 R1 Tell me what you see

43 S6 The flower joining it [*S6 gestures to a yellow counter between the two red counters*]

44 R1 And how is the flower joining?

45 S6 It's one more on each side and then there and there [*S6 points to the yellow counter on top of the red counter and then on the bottom of the red counter*]

46 R1 Can you separate it for me?

47 S6 [S6 separates the pattern into groups containing one red centre and three yellow petals around the red (see Figure 3). There are three groups and one counter left over. S6 has left this to the side of the pattern and has not attached it to a 'flower'] There is one missing from each side [*S6 points to the side where there is no yellow counter*].

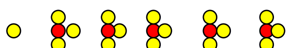

Figure 3. Deconstructed pattern created by S6.

48 R1 So, for four red centres how many yellow petals? [*R points to four red centres*]

49 S6 [*S6 using two fingers to touch the pattern while counting in his head. It appears he is counting in twos*] Thirteen

50 R1 Good boy. Can you tell me how you counted it?

51 S6 Twos. I went two, four, six, eight, ten.

While S6 was able to deconstruct the pattern so that there were groups of three yellow petals he was unable to identify the 'threeness' as he still counted in twos. Thus, it was decided to begin interaction 5.

Interaction 5: For students who had difficulties during the previous interaction, the researcher provided additional teaching actions. If the student was having difficulty deconstructing the pattern, the researcher then deconstructed the pattern incorporating a heavy use of gesture and language. In this case S6 was able to deconstruct the pattern. He was unable to link the multiplicative relationship between the number of yellow petals and the red centres. Teaching actions that assisted him included pointing to the pattern term with the student verbally highlighting the position, for example, 'Position four [researcher/student gesturing to pattern term] 4 groups of three [researcher gesturing to groups of three in the pattern structure] and one more [researcher pointing to constant]'. S6 then mimicked the gestures and language. This assisted him to make connections between the two variables. This interaction required decoding and encoding by both researcher and student. The interview began with the researcher gesturing in a semicircle around the first group of three yellow counters and asking, 'How many yellow petals in this flower?' This process was repeated three times with the researcher gesturing around the second three group of three yellow counters and finally the third group. The discussion continued as follows:

57	S6	Three and that would equal twelve.
58	R	So imagine there is a bee sitting on that flower petal. So I have four reds and how many petals do I have?
59	S6	There are 12.
60	R	Yes there are. What if I need to make the next one what do I have to do?
61	S6	Put one red and 3 yellows
62	R	Tell me how I would make a daisy chain with 7 red centres.
63	S6	Do it with 3's. 3 yellows [*S6 pointing to red centres and then gesturing the threes*]
64	R	Good boy so I would have 7 with 3 yellows and then I would have to join him on.
65	R	How do we say that in maths where we join two things together?
66	S6	The bee is sucking all the honey out of it.
67	R	Are we taking him away or adding him?
68	S6	Adding

The constant (first yellow counter of the pattern) was then highlighted with a sticker of a bee. S6 was then asked again, "Tell me how I would make a daisy chain with seven red centres. What would I do?" S6 responded, "You'd be having seven (S6 gestures to red centres) with three yellows and then I would have to join him on... the bee is sucking all the honey out of it". S6 gestured to the counter that represented the constant. This counter had been signified to the student by placing a bee sticker on the counter after he had earlier deconstructed the pattern. Discussion ensued about the bee and what it was doing in the pattern, and S6 suggested that the bee (constant) was being added to the pattern.

Interaction 6: Some students required further support with their mathematical language. Interaction 6 required the researcher to explicitly teach students the mathematical language used to assist students to express the structure. This interaction can be challenging for some young Indigenous students as they are often learning new, specific, mathematical language.

78 R	How many red centres do I have?
79 S6	Five
80 R	How many groups of 3 do I have? [*R use specific language of 'groups'*]
81 S6	3, 6, 9, 10, 11, 12, 13, 14, 15 [*S6 counting in threes gesturing to the yellow petals for 3, 6, 9 – then started counting in ones*]
82 R	Good boy. So if I have 5 red centres I have 15 petals and one more. So this is one group of 3 [*R gestures and covers one group of three*]
83 S6	2 3 4 5 [*S covers group 2 up then group 3, 4, 5*]
84 R	How many groups of 3 would I need for 7? [*R covers a group of three*]
85 S6	Seven groups
86 R	And what if I made the whole chain? Would I just make 7 groups of 3?
87 S6	Equals one ….. plus one more
88 R	What about for position 10?
89 S6	10 groups of 3 plus 1

Later in the interview S6 generalised the pattern for any position, "Any number of flowers that you want" he was pointing to the red centres as he was stating this. "Put all the yellows, three yellows around the thing [red centre]....then one more". S6 gestured a semicircle around the red centres. This gesture was identical to the gesture used by the researcher previously as the pattern was deconstructed. It appears that while the language was used in interaction 6 and the student was also using the language he did not maintain this and used gesture to support the mathematical language when generalising for unknown positions.

DISCUSSION

Indexical sign vehicles are pivotal in the teaching and learning process of engaging young Indigenous students in pattern generalisation tasks. This study adds to past research that suggests that indexical signs (such as gesture, language, and hands-on materials) contribute to making the mathematics apparent for non-Indigenous students (Radford, 2009). To highlight particular signs and structures of the pattern to the young Indigenous students, specific and purposeful gestures were used as the researcher deconstructed the pattern. These gestures were indexical sign vehicles (Saenz-Ludlow, 2007). It was essential to gesture between the variables (pattern term, pattern quantity, and constant) as the pattern was deconstructed. During this process, there was a deliberate coordination between gesture and language (Radford, 2009).

Iconic signs assist students to move quickly from recursive thinking to covariational thinking. This was achieved by using iconic signs to highlight the two variables. It has been demonstrated in past studies that young students can engage in covariational thinking (Blanton & Kaput, 2004); however, this study begins to shed light on the processes that assists students in 'noticing' the relationship between the two variables. Additionally, a recursive approach to solving growing patterns is still a major challenge for both young and older students (Rivera, 2010). The results of this present study suggest that this issue relates to the way the patterns are structured, and can be overcome by using iconic signs to highlight both variables in growing patterns, namely, the pattern number (term) and the pattern quantity.

Conclusions drawn from this study provide a positive story in relation to young Indigenous students engaging with, and learning mathematics. Additionally, it provides insights in relation to the teaching and learning of early algebra, an area where young Indigenous students are underrepresented in the literature. This present study further extends the application and practicality of semiotic theory in the teacher/learning process of pattern generalisation with young Indigenous students.

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MATHEMATICS EDUCATION AS A PRACTICE IN PURSUIT OF [INTELLECTUAL] EXCELLENCE

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In this paper we argue that mathematics is a practice as described by MacIntyre (2007) in his seminal After Virtue. We explore the link between mathematics and virtue ethics, providing an account of how the social and moral life inherent in mathematics both exhibits and promotes excellence, not just in an intellectual sense but in an ethical sense as a contribution to flourishing. Conceiving of mathematics as a practice (in MacIntyre's sense) is necessary for identifying the core virtues as they are exercised in mathematics. We then argue that the practice of mathematics can foster intellectual (and derivatively moral) excellence, and investigate whether and how mathematics education, exhibits and promotes these same virtues, and hence the extent to which mathematics education is a practice properly so called.

INTRODUCTION: CONCEIVING OF MATHEMATICS (EDUCATION) AS INTELLECTUAL AND MORAL VIRTUE

While the goals of school mathematics are typically articulated in the preliminary sections of mathematics curriculum documents throughout the world, explicit discussion of goals and purposes in school mathematics is rare. Indeed, normative questions about the purpose of education and what makes for excellence in educational practice generally are rare and often confined to the educational philosophy literature (e.g., Biesta, 2010). In the few papers discussing values in mathematics education (e.g., Bishop, 1999) these are conceived of as desirable attitudes that might be made explicit in mathematics; and hence as external, often affective, qualities that include social issues and dispositions. These are important considerations; however, the respect shown for external values does not consider the practice of mathematics as something that engages specific internal activities and hence specific virtues, and the subsequent implications for educational practice, policy and research that engages mathematics. We ask therefore: is mathematics a *practice* (MacIntyre, 2007) properly so called? What are the attributes that mark it off as such? Can philosophy's approach to virtue ethics help us to decide whether to commit to the view that mathematics is a *practice*, and to understand the implications for mathematics education that flow from this commitment? In our exploration of the proposition that mathematics is a practice properly so called, we therefore examine excellences inherent to and derivable from mathematics education.

PRACTICES AND VIRTUE

In his seminal *After Virtue* (2007), philosopher Alasdair MacIntyre defines a *practice* as:

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> Any coherent and complex form of socially established cooperative human activity through which goods internal to that form of activity are realised in the course of trying to achieve those standards of excellence which are appropriate to, and partially definitive of that form of activity, with the result that human powers to achieve excellence, and human conceptions to the ends and goods involved, are systematically extended (2007, P. 187).

It is important to note that in defining goods that are internal to a form of activity MacIntyre (2007) delineates a second type of goods, namely those that are external to the activity. External goods are characteristically objects of competition derived from social circumstance such as material goods, fame, power or status, which are possessed by individuals rather than contributing to human excellence, and which can also be obtained outside of engaging in practice (MacIntyre, 2007). We argue that this aspect of MacIntryre's definition provides a critical delineation between internal and external goods and their relationship to the virtues generally, and more specifically, to the recognition of the virtues in mathematics, and mathematics educational practice, policy and research.

It is against this definition of practice that we consider the discipline of mathematics and argue that it is a *practice*. In this section we briefly introduce key aspects of mathematics that we suggest mark it off as a *practice*. We also provide a historical overview of virtue thinking in the Western tradition in order to make clear the inextricable link between virtue and practice and provide a basis for our discussion of mathematics and virtue ethics.

Mathematics as a practice

Arguably mathematics exhibits a coherence and complexity that mark it as unique among traditional intellectual disciplines. Different philosophical traditions notwithstanding the need for rigour in mathematical proof has been characteristic of mathematics for millennia (Kleiner, 1991), and has, we suggest, led to a level of coherence evident in no other discipline. Moreover, with rare exceptions, this search for rigour and coherence has taken place within a social context marked by collaboration that transcends social and geographic boundaries (Cranshaw & Kittur, 2011). That is, mathematics as a discipline is a 'socially established cooperative human activity' (MacIntyre, 2007, p. 187).

Moreover, the activity of mathematics seeks to realise goods that are primarily internal to mathematics itself. While there can be no disputing the capacity of mathematics to model the world and hence to solve problems in the world, the overwhelming motivation for many mathematicians is to contribute to the growth of human knowledge (e.g., Hardy, 1940). Thus we regard as incontrovertible the proposition that at least one aim (if not the central aim) of engaging in the practice of mathematics in general is systematically to extend human powers to achieve excellence, and that mathematics is an exemplar *par excellence* of such an activity. As a *practice*, mathematics therefore extends human powers to achieve excellence, and hence cultivates intellectual virtues and, at least derivatively, moral virtues.

Virtue thinking in the Western tradition

In the western intellectual tradition, the role of education in the pursuit of human excellence received its earliest expression in the works of the ancient Greeks, particularly Socrates (through Plato) and Aristotle, both of whom regard the cultivation of human excellence (Greek *arête*, commonly, if not always perfectly, translated as *virtue*) to be humanity's highest aim. For Socrates, the key attributes of excellence ('the cardinal virtues') are fourfold: wisdom, justice, temperance and courage (*Republic* Book IV, 426 – 435). For Plato virtue was inseparable from mathematics, and achieving proper proportions was seen as essential for just living, wisdom and correct judgments (Kung, 1989). Aristotle presents a more complex view of human excellences in the *Nichomachean Ethics*, in which is outlined an ensemble of virtues required to be cultivated in order for humans to live maximally well (attaining a state of *eudaimonia* – the flourishing life). In Book 6 (1139a), Aristotle distinguishes between the moral virtues (as ethical dispositions conducive to living well amongst other persons in a polity, which is Socrates' chief preoccupation) and the intellectual virtues (as intellectual dispositions conducive to functioning; the cultivation of theoretical and practical knowledge and their deployment to best effect). The distinction between moral and intellectual virtues serves to characterise the practices that engage both, and so those practices arguably most worth engaging in.

Between the ancient Greeks and the modern period, the history of ideas shows a hiatus in western thinking about the role virtues play in the fostering of human excellence. During this period moral thought was dominated by theological concerns, and virtue *per se* fell into the domain of doctrinal prescriptions of conduct. While non-theological approaches to ethics emerged after the Enlightenment (e.g., Hume, 1739) and rationality came to dominate ethical thought in the eighteenth, nineteenth and early twentieth centuries (e.g., Kant, 1785 & 1788; Mill, 1861; Rawls, 1991), it was not until relatively recently that the west saw a resurgence in interest in the virtues as a plausible approach to core ethical questions ('What ought I to do?' 'How ought I to live?'). In 1958 Elizabeth Anscombe published an account of the value of thinking about the virtues in her article 'Modern Moral Philosophy', and this led to the 'aretaic turn,' influencing a revival of virtue approaches not just to questions of ethics, but in a number of areas including epistemology and education. Much contemporary debate on the virtues is centred on the theses propounded in Alisdair MacIntyre's *After Virtue* (1980; 2007), which has become a key influence of theorists considering questions of virtue and identity and how these relate to conceptions of practice. The key idea that unites virtue approaches is their emphasis on the agent's character and the cultivation of internalised qualities or attributes (virtues) that mark him or her off as a person of moral and intellectual virtue.

In particular we focus on three virtues: truthfulness, justice and courage. By truthfulness we mean a commitment to honesty and integrity as a mark of one's own internal relationship to knowledge and external relationship to others. By justice we mean a commitment to impartiality with regard to knowledge and the treatment of others on merit. By courage we mean a commitment to seek truth regardless of personal interest and a profound respect for others.

MATHEMATICS AS PRACTICE: THE TWIN PRIME CONJECTURE AND THE POLYMATH PROJECT

We propose that the following global example illustrates and provides an exemplar of mathematics as a *practice* as defined by MacIntyre (2007). The example comes from recent advances in mathematics, and highlights particularly the way in which mathematicians have collaborated in an intellectual endeavour that illustrates how engagement in mathematics both enacts and develops the aforementioned virtues.

On 14 May 2013 a relatively unknown mathematician, Yitang Zhang, who was working on the classic twin prime conjecture, announced a proof that there are infinitely many prime pairs. However, rather than differing by two as in the twin prime conjecture, Zhang's result showed that these infinitely many pairs were no more than 70 million apart. Zhang's result was the first time anyone had been able to put a finite gap on gaps between prime numbers and prompted a flurry of activity around the world. By 30 May 2013 Scott Morrison from the Australian National University announced that he had reduced the gap to 59 470 640, and on 4 June 2013 Terry Tao of the University of California launched a collaborative project as part of the Internet-based Polymath endeavour (Gowers, 2009).

The Polymath project is an open access online collaboration between mathematicians established in order to share knowledge and ideas towards solving previously intractable problems in mathematics. As a result of this online collaboration by 27 July 2013 the smallest gap between prime pairs that occurs infinitely often was reduced to 4680. Within the space of three months arguably more progress was made on the twin prime conjecture than had been made in the previous 2000 years (Nielsen, 2014).

This research provides an example par excellence of mathematics as a *practice* (MacIntyre, 2007). It shows how the establishment of a cooperative research community enabled the initiation and sustaining of intellectual engagement with a serious problem. We suggest that it was only by engaging in virtues such as truthfulness, justice and courage, and by the participants accepting the authority of the standards they brought into play, that the rules necessary to test, construct and reconstruct mathematical concepts and relationships could be employed, and hence, that the internal goods of mathematics could be realised and mathematics as a practice flourish.

VIRTUES IN THE POLYMATH PROJECT

Truthfulness

In the course of finding a solution, the participants in the Polymath project engaged in *truthfulness* through their commitment to honesty, openness and integrity in the sharing of results. The report of progress towards a resolution of the twin prime conjecture on the Polymath website is full of qualifications and reservations. Where results have not been rigorously validated, or have been conjectures that rely on some other as-yet unproven result, these caveats are carefully noted. There are assertions and leaps of faith, however these are reported as such rather than being accepted as true. Truthfulness defined the relationship between the participants and the purpose of the project. It is therefore evident in both the commitment to knowledge, in the acceptance of the judgment of the community, and in the open celebration of advances produced by others.

Justice

Justice played out in the commitment to carefulness for the facts, rigorous attention to detail, the use of uniform and impersonal facts and in treating participants on merit. It is significant that the impetus for the project was a result reported by a relatively unknown mathematician rather than by someone of established status and authority. The community of mathematicians was willing to renew their engagement in the problem, rather than overlooking it because it was reported by someone who arguably could be seen as on the periphery of the mathematics research community.

Courage

Finally we argue that the twin prime project illustrates *courage*, in that it shows a profound respect for alternative beliefs, and thus genuine care and concern for both the individual and the community. Zhang's result introduced radical techniques by looking at the problem from a completely new perspective. In doing so, Zhang took risks that were potentially endangering to his reputation and achievement. The community of mathematicians was willing to set aside conventional ways of tackling the problem and engage in what promised to be more productive lines of enquiry.

PROBLEMATIC: CAN WE CONSIDER MATHEMATICS EDUCATION, AS CURRENTLY EMBODIED IN SCHOOLS, A PRACTICE?

If virtue thinking has any value at all in the education of a person (insofar as this consists at least partly in the development of the pupil's knowledge-acquisition faculties such that it is reasonable to hold him or her responsible for the excellence of these), then conceiving of virtues as excellences brings into sharp relief the value of thinking of education as a practice according to MacIntyre's definition. Moreover, if it can be shown that mathematics education has an especially important or even unique role to play in the cultivation of excellences and so cultivates in persons the faculty to address and take responsibility for their answers to key intellectual questions ('What should I know?' 'How can I acquire this knowledge?' What should I be able to do?' 'How can I acquire this skill?' 'How, where and when is the right way to apply these?' 'What makes this the right way?' 'How do I know?'), then it follows that committing to the view that mathematics education is a practice properly so called is warranted.

However, we suggest that the dearth of literature relating to the virtues inherent in and developed through mathematics education, coupled with an increasingly managerial agenda focused on the production of external goods as measured in comparisons of achievement, make a focus on consideration of mathematics education as a practice both urgent and essential. It is debatable to what extent external goods, such as the outputs of universal standardised testing contribute to human powers to achieve excellence. Thus although external goods can be valuable within themselves (MacIntyre, 1999), when they become the primary focus of a practice, such a focus marginalises the recognition of goods internal to the practice. More crucially, such a focus silences discussion of how these skills transform, enrich and extend human powers to achieve excellence. We argue that without explicit attention to the virtues discussed in this paper, practices in school mathematics become confined to the realisation of external goods.

While we plan to make a detailed discussion of virtues in mathematics education as it is currently embodied in schools the focus of a subsequent discussion, we raise the following questions:

1. To what extent does school mathematics enact truthfulness in a commitment to honesty and integrity? Or does a focus on external goods realised through the desire to outperform rival countries or jurisdictions in standardised tests inhibit the capacity to develop truthfulness in and through school mathematics?

2. To what extent does school mathematics enact justice in its commitment to impartiality and treatment of others on merit? Or does a focus on external goods convey differential privilege on those who are willing to "fit the system"?

3. To what extent does school mathematics enact courage in its seeking of truth regardless of personal circumstances and its profound respect for others? Or does a focus on external standards of accountability marginalise those teachers or students who seek to be different?

CONCLUSION

We have argued that the notion of practice is central to mathematics. We have shown that the consideration of mathematics as a practice brings into sharp focus the virtues of truthfulness, justice and courage. Through an example from contemporary mathematics we have shown that these virtues are both characteristic of and developed through mathematical activity. Yet we have questioned whether school mathematics, as a surrogate for academic mathematics, can be considered as a practice properly so called. We argue that further consideration of the intellectual virtues, as a key characteristic and goal of school mathematics, is an urgent and important endeavour that potentially brings into question many of the currently unquestioned assumptions about what counts in mathematics education.

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WHAT FRUITFUL DISCUSSIONS DO ZAMBIAN TEACHERS HAVE IN LESSON STUDY? A CASE STUDY

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This article aims to identify what sort of mathematics related discussions Zambian teachers hold in the reflective session of lesson study, and what factors influence them. The qualitative analysis, based on the cognitive aspect of the framework of TEDS-M, revealed that primary school teachers discussed mathematics pedagogy content (MPC) and mathematics content (MC) in their reflective session, although these conversations were not developed mathematically in depth. On the other hand, two facilitators who contributed to the discussions, offered assessment of students' learning, leading to the conclusion that the existence of experienced facilitators can be the key to success of lesson study.

INTRODUCTION

Teachers' professional growth is one of the factors necessary to achieve quality mathematics education in Zambia. Japanese lesson study is known worldwide, and Zambia has adopted this approach. The Zambia Ministry of Education and Science, Vocational Training and Early Education (MOES) and Japanese International Cooperation Agency (JICA) implemented the project of lesson study in 2005. Since then, it has been extended to a large number of schools in the country.

There are a number of difficulties, however, with developing Zambian teachers' professional growth. For instance, Ishii (2011) examined the effectiveness of lesson study in the country. He concluded that the reflective session after lessons offered a chance for teachers to develop teaching techniques. However, he mentioned that teachers did not consider how students learned reflectively even in reflective sessions. Kinone (2011) found it difficult for two primary school teachers to offer problem-solving or discussions in their class in spite of their long teaching experiences. Both studies concluded that teachers' reflection on students' learning was not satisfactory. Nakawa (2015) in her action research in primary schools also concluded that teachers rarely discussed mathematical content when they reflected on their lessons. In addition, there have been only a limited number of studies on lesson study in Zambia.

It is crucial, therefore, to investigate Zambian teachers' use of lesson study and to discover what mathematically related content they are discussing in their reflective sessions. Specifically the investigation sought to answer the following research questions: What kind of discussion about mathematical content can lesson study offer? What factors influence this discussion?

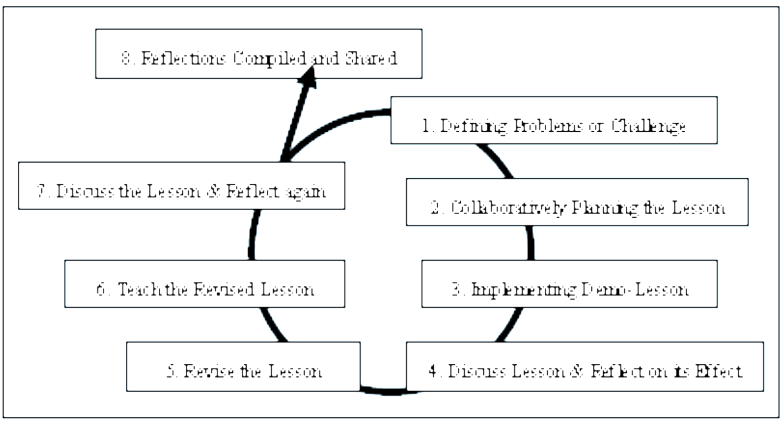
2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 273-280. Hobart, Australia: PME.

THEOREICAL BACKGROUND

Lesson study

According to Fernenandez and Yosida (2008) and Stigler and Hiebert (1999), lesson study is a collaboration-based teacher professional development. Although lesson study has a relatively long history in Japan, it remains fresh to other countries outside Japan (Murata, 2011). Some studies (e.g. Meyer & Wilkerson, 2011; Hart & Carriere, 2011; Olson, et al. 2011) discussed the implementation of lesson study in the U.S. and tackled the issues of what evidence exists that lesson study can function effectively in the country. There has also been research which documented the positive changes of teachers in *Lesson study research and practice in mathematics education: learning together* (Murata, 2011).

In the Sub-Saharan African countries, lesson study has been officially implemented by a top-down approach, unlike Japan, and is gradually spreading out to more countries in the Japanese international cooperation. Zambian lesson study is currently called SMASTE (Strengthening of Mathematics, Science and Technology Education) School Based Continuing Professional Development Project (SMASTE-SBCPD). The project, was introduced to three provinces in 2005 and expanded gradually to the whole country. Its historical changes and development are well explained in Baba & Nakai (2011). Figure 1 shows the Zambian lesson study cycle.



(Zambia Ministry of Education, 2007, p.4)

Figure 1: The cycle of lesson study in Zambia

The cycle is different from Japanese lesson study. After discussing and reflecting lesson (no. 4 in Figure 1), teachers again conduct the revised lesson, then discuss and reflect on it (no. 6 in Figure 1). JICA (2012) reported that as the result of the technical cooperation, lesson study worked effectively for teachers' professional development and improved students' academic performance. Baba and Nakai (2011) also mentioned a positive impact on teachers' professional development. They also pointed out that students participated more in lessons as the outcome of lesson study. However, its effectiveness and challenges have not been discussed academically in mathematics education.

Mathematics teachers' competencies

Teachers' competencies are intensively discussed in mathematics teacher education worldwide (e.g., Döhrmann et al., 2012). Shulman (1986; 1987) used the term Pedagogical Content Knowledge (PCK), as combining knowledge and understanding of students' learning, content and pedagogy for teaching in education. Hill et al. (2008) extended PCK into mathematics education, classifying Mathematical Knowledge for teaching (MKT). Furthermore, Döhrmann et al. (2012) explained the definition of effective mathematics teacher education in Teacher Education and Development Study in Mathematics (TEDS-M). TEDS-M regards Mathematics Content Knowledge (MCK) and Mathematics Pedagogical Content Knowledge (MPCK) and General Pedagogical Knowledge (GPK) as crucial cognitive components of mathematics teachers' professional competencies (Döhrmann et al., 2012). The conceptual model classifies teacher competencies into these cognitive abilities and affective-motivational characteristics: professional beliefs, motivation and self-regulation.

Teachers' professional development has also been a hot issue in the African context (Baba & Nakai, 2011; 2009; Walker, 1994; Wright, 1988). Unfortunately, however, there are the only few studies in Zambian mathematics in-service teacher education. Nakawa (2015) reported that two teachers showed drastic development of GPK in her action research, but found it difficult to develop their MCK and MPCK despite one- to two-month intensive meetings for improving lessons. Qualitative analysis on four teachers' reflection by Kinone (2013), revealed that reflections were mainly focused on how to make their students memorise and use formulas shown in textbooks. Thus, it seems that Zambian teachers' MCK and MPCK are not satisfactory.

METHODOLOGY OF STUDY

Lesson study was conducted in a primary school in Serenje, a semi-urban town in the Central Province of Zambia. Lesson study started in the town in 2005. There were active lesson study groups and school, and the facilitators of lesson study were well-experienced. They incorporated different challenging themes for quality lesson study, including problem-solving.

Fourteen teachers participated in the study from four different schools in total: six grade 2, six grade 3 and two senior teachers. The two facilitators from MOES also assisted with lesson study discussions. Initially a workshop was held with the group to discuss the difficulty of teaching and learning multiplication and investigate the ways to improve this. The teachers then conducted two lessons in grade 2 and 3 classes on the second day. After the lessons, the group had two reflective sessions on the same day. Due to time limitations, the group did not follow the usual way of lesson study shown in Figure 1, completing the process at no.4.

The purpose of lesson study was to overcome lower-grade students' difficulty of multiplying two one-digit numbers and to develop alternative ways of counting by using fingers. A previous study showed that Zambian students frequently used their fingers to count for 1x1 multiplication. Their frequent counting hinders answerisng more advanced questions when they reach upper grades (Nakawa, 2013). In a planning session, the group agreed to develop a lesson to use a 5x5 dot sheet (cf: Wittmann & Müller, 2012a, 2012b) for grade 2 and 3 (See Figure 2). Students were expected to understand the meaning of multiplication, to group the dots using the chart, and to add these groups to answer the question. This article focuses on the case of grade 2.

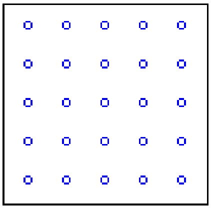


Figure 2: 5x5 dot sheet

In the introduction, teacher called students to the front while singing a song. When students randomly sat on a mat in front of the blackboard, she introduced the day's topic: multiplication. In the development of the lesson, there were four questions and activities. In the first activity, she showed the 5x5 dot sheet and made sure that students would use it. In the second question, she divided 4x5 dots into 2x5 and 2x5 and students answered how many groups of ten they had on the board. In the third activity, students were asked how many groups of a number they can form on 5x4 dots; and in the final question, she showed 2x4 dots, asking the answer of 2 times 4.

The data obtained were analysed qualitatively, because the study aimed to investigate what sort of mathematical discussions occurred in the reflective sessions. The lesson and reflective sessions were audio-recorded, and the analysis was done in the following three stages. In the first stage, the author transcribed all the data. In the second stage, the conversations were divided into semantic clusters. In the third stage, using the framework of TEDS-M's teachers' competencies except for the affective-motivational characteristics in the model, the author attentively hand-coded the transcriptions based on categories of Mathematics Contents (MC), Mathematics Pedagogical Contents (MPC), General Pedagogy (GP), comments on lesson (L) and others. A label was put on the clusters to show the summary of discussions, and more categories were labelled in one cluster, if necessary. In the final stage, the author checked the accuracy of the clusters and codes.

RESULTS AND DISCUSSION

In the reflective session, there were thirteen semantic clusters: three were discussing the introduction of the lesson and ten for development of the lesson. In the discussion of the introduction, there was only one cluster on which GP is labelled. In the reflection of the development of the lesson, one MC-related, nine MPC-related, three GP-related discussions and one L, were identified shown in Table 1.

Mentioned part of the lesson	No. of semantic clusters	No. of coding			
		About MC	About MCP	About GP	About L
About introduction	3	0	0	1	1
About development	10	1	9	3	1
About harmonisation	0	0	0	0	0
Total	13	1	9	4	2

Table 1: The result of the analysis

Table 1 shows that there were no conversations related to the harmonisation of the lesson. Most clusters were about the development of the lesson.

Only one remark regarding MC was provided by the facilitator, and it was the longest part of the reflective session. He emphasised the importance of understanding and use of multiple representations of multiplication. The part was mainly lecture-style, and participant teachers silently listened to his explanations. After his explanation, there were no questions. Secondly, regarding MPC-related discussions, the contents were arranged as shown in Table 2.

Table 2 shows that teachers had some conversations and reflections on MPC. For instance, a teacher said a positive comment about code-switching mentioned on no.5 in Table 2. Another teacher also reflected that the size of a number that a demonstrator used was appropriate on no.6. However, those remarks were not developed more by other teachers and ended up with only a single remark. Moreover, the comments such as no.7 and 8 in Table 2, mainly judge if an act of the demonstrator in the lesson was 'good' or not. They did not offer further discussions. There were no connections from one cluster to another cluster.

Apart from these discussions, there were not observations of students' learning like the tenth cluster in Table 2, except for the two facilitators' remarks. "When students think of 5 by 4, they had so many answers. They brought numbers they have felt" was an important comment by a facilitator, because the demonstrator was not able to handle the scene well in which so many incorrect answers were given by students. It was an opportunity for the teachers to consider the reason the lesson did not go well; however, the discussion did not continue after this.

No. of clusters	Label	Conversations

5	Relation between local language and teaching	In grouping, it was good that the teacher used a local language so that students could understand it well.
6	About setting of lesson	It was good that the teacher started to show ten dots at first as the number was appropriate for introduction.
7	About assessment of the material	The 5x5 dot sheet was useful because it can be extended to 10x10 dot sheet.
8	About observation of students	We should be careful that students not use fingers for counting.
9	About assessment of the lesson	It was good to say multiplication is the same as repeated addition.
10	About students' learning	When students think of 5 by 4, they had so many answers.
10	About students' learning	They were just counting, but not grouping.
11	About confirmation of lesson content and curriculum	Students must know the sign of multiplication.

Table 2: The contents of MPC-related discussions

Factors of occurring MC and MPC-related discussions

The MC and MPC-related discussions may have been subtle and at a lower level compared to Japanese and other countries' lesson studies, but it did provide evidence of some mathematics-related conversations. Teachers discussed MC and MCP-related content, which could be a positive result of the implementation of lesson study in Zambia, unlike the result of other studies mentioned. However, the challenge is that the teachers' focused on how they delivered lessons, rather than how students learned, and their discussion did not develop further mathematically.

The above results showed that the MC and MPC-related discussions were identified in the reflective discussion, and that comments on students' learning were made by the facilitators. The difference between the teachers and facilitators was that primary teachers usually do not have opportunities to develop their competencies after they graduate college, or university. Moreover, they do not have enough pre-service training. By contrast, facilitators are exposed to a number of workshops for their professional development. Thus, in the reflective session, the existence of experienced facilitators seemed to offer more insightful discussions. At the same time, however, it may be risky to rely too much on the skills of facilitators because the professional competencies of facilitators may also vary from place to place in the country. Of course, more cases of lesson study should be analysed for the development of Zambian lesson study.

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THE INTRODUCTION OF FUNCTIONS AT LOWER SECONDARY AND UPPER SECONDARY SCHOOL

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This paper is based on my longitudinal PhD study where I followed eight students in the transition from lower secondary to upper secondary school. In my study I focused on the teaching and learning of functions and in the following I will consider how the introduction of functions was carried out by the teachers at both lower and upper secondary school. Findings suggest a gap between function as a well-defined mathematical concept on one hand and explanations provided in textbooks and teaching on the other. This gap was most prominent related to the treatment of variables and the uniqueness property.

INTRODUCTION

In my PhD study, I investigated the transition from lower secondary to upper secondary school, and the teaching and learning of functions (Nilsen, 2013). The limited body of research carried out considering the lower-upper secondary transition, both nationally and internationally, to a great extent motivated the context of this study. Furthermore, the topic of functions could be conceived of as a *boundary object* (Star & Griesemer, 1989; Akkerman & Bakker, 2011) between lower secondary and upper secondary school, since functions plays a prominent role in the Norwegian national curriculum *Knowledge promotion* (LK06) at both these levels of schooling. In this paper I will focus on teaching aspects related to the introduction of functions, involving examples and explanations provided by the teachers at both lower and upper secondary, and I pose the following research question:

> *What characterizes the mediation of functions in the introduction phase, at lower secondary and upper secondary school, general studies programme?*

By "mediated" I do not just refer to the actual teaching observed in classrooms, but also to the way functions is presented in the textbooks used. In upper secondary there are two main study programmes; the general studies programme and the vocational studies programme. The vocational programme is orientated towards practical professions, while the general studies programme aims to prepare students for tertiary education. As the research question indicates, in this paper I will only focus on examples and analysis from lower secondary, and the general studies programme at upper secondary.

One of the prevailing definitions of functions stems from Dirichlet:

> y is a function of the variable x, defined on the interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way the correspondence is established. (Burton, 2003, p. 572)

2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 281-288. Hobart, Australia: PME.

This definition contains the *uniqueness property*, entailing that for each value of the independent variable (x) one should have one, and only one value of the dependent variable (y). Variants (and sometimes incomplete versions) of this definition were found in most textbook used in this study.

THEORETICAL FRAMEWORK

I am positioned within the socio-cultural perspective as elaborated and developed by Vygotsky (1978). One of the main reasons for these theoretical lenses is the role of *mediation* in this study, in terms of teaching sequences where teachers act as mediators in the mathematics classroom. Mediation of mathematical content through textbooks is also a significant aspect. The importance of language and mediation in social-cultural learning theory offers the possibility of powerful frameworks for operationalizing the process of mediation, for example by the application of semiotic models. Teaching sequences could further be analyzed through semiotic chains. In the analyses I apply Steinbring's epistemological triangle as an analytical tool (Steinbring, 2005). This can be understood as a triadic semiotic model containing the components "object/reference context", "sign/symbol" and "concept". The "object/reference context" represents what the sign/symbol may refer to. Steinbring claims that due to mathematical epistemological conditions, this mediation is not entirely subjective. This is because meaning obtained through such mediations rests on certain epistemological conditions of mathematical knowledge and the intrinsic relations between them which in turn secure some objectivity with respect to the meaning of the concept (Steinbring, 2005). This meaning, or "concept", constitutes the third corner of the epistemological triangle.

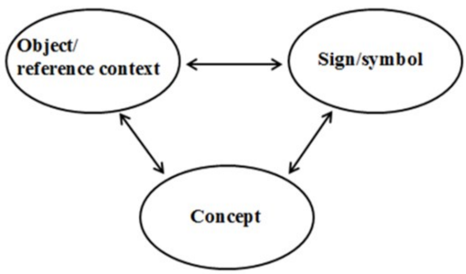

Figure 1: The epistemological triangle (Adapted from Steinbring, 2005, p. 22)

These epistemological triangles further could be used to construct semiotic chains. Building on ideas mainly from Steinbring (2005) and Farrugia (2007), I will define a semiotic chain as an *iterative movement between two signs*. The core idea of semiotic chains as these are applied here, is to identify how teachers mediate meaning of mathematical signs by linking these signs to prior (or other) mathematical signs. In my study such chains are applied to analyze the different parts of teaching and to visualize the intended progression. One example of this and how this is utilized could be found in Figure 3 in the analysis section.

Mathematical objects, including functions, are accessible to us only through representations, as “there is no other ways of gaining access to the mathematical objects but to produce some semiotic representations” (Duval, 1999, p. 4). In line with this, these representations could be understood as signs which “stand for” mathematical objects. Janvier (1978) noted four different representation forms related to functions: “Situations/verbal descriptions”, “tables”, “graphs” and “formulae”.

METHODOLOGY

This study involves four different lower secondary schools which are labelled as School A, School B, School C and School D. School A is a Waldorf School, while Schools B-D are public schools. The Waldorf School was included with the aim of obtaining some diversity in my empirical data. In total, I focused on 8 students, distributed in these schools. The selection of these students was based on criterion related to the distribution of genders, performance level and geographical locations. By a few exceptions, I observed from two to six lessons in each class, both at lower and upper secondary, when the topic of functions was introduced and further dealt with. Data was collected through the use of video camera and voice recorder. Interviews were conducted with teachers and students prior and subsequent to the lessons. At upper secondary the students attended four different schools, but most of them attended different classes. These schools are labelled School 1-4, followed be a small letter (a-c) indicating the actual class. Only one out of the eight students attended School 4, so no letter was needed in that case.

Lower secondary schools

School A (Waldorf School) **Students:** Otto, Edna **Teacher:** Kim	School B (Public school) **Students:** Lena, Olga **Teacher:** Oda	School C (Public school) **Students:** Kent, Anna, Matt **Teachers:** Tim, Tom	School D (Public school) **Student:** Thea **Teacher:** Roy

Upper secondary schools

<table>
<tr><td>School 1 (VS)
Class a
Student: Otto
Teacher: Bernt</td><td>School 2
Class a (VS)
Student: Olga
Teacher: Ronny</td><td>School 3 (GS – 1T)
Class a
Student: Kent
Teacher: Derek</td><td rowspan="3">School 4
(GS – 1T)
Student: Thea
Teacher: Kerry</td></tr>
<tr><td rowspan="2">Class b
Student: Edna
Teacher: Sonja</td><td rowspan="2">Class b
(GS – 1T)
Student: Lena
Teacher: Tommy</td><td>Class b
Students: Kent, Anna
Teacher: Greta</td></tr>
<tr><td>Class c
Student: Matt
Teacher: Henry</td></tr>
</table>

Table 1: Distribution of the students and classes involved (Kent shifted to "Class b" during the semester).

In Table 1, VS indicates "vocational studies programme" and GS indicates "general studies programme". 1P and 1T indicate respectively the 1P and 1T versions of mathematics in the general studies programme.

Pseudonyms are provided for each of the participants, and to make the analysis clearer and more readily understood I have used 3-letter names for the teachers in lower secondary, 4-letter names for the students and 5-letter names for the teachers at upper secondary school.

ANALYSIS

The following analysis mainly focuses on examples from lower secondary, even though observations from upper secondary is taken into account in the discussions. This reduces the need of additional explanations of students' background, since the topic more or less was treated by the teachers as a "new topic" to the students. From my empirical data, the introduction to functions could be analysed through four main analytical categories. These categories emerged when my empirical data was analysed and coded. *Representations and examples* refer to the treatment of functions through the use of different representations forms and practical examples. It is important to stress that this category is only applied in situations where representation forms or/and examples are the main tools for explaining the function concept and without any additional formal approach. The *function machine* emerged from teaching sequences both in lower and upper secondary school. Characteristically, function machines are different models used to illustrate the uniqueness property of functions, or the one/many-to-one principle. Common to these models is that a certain object is put into the model and a well-defined object comes out. The category *formal definition* is used when students were introduced to some kind of formal definition of functions, including the uniqueness property. This was mainly done by referring to textbooks and in most cases this was not explicitly discussed in the classroom. *Functions as co-variance* is a more imprecise version of the previous one. In this analytical category, it is emphasized in various terms that functions has to do with variables that somehow relate to each other.

The following example from School A (lower secondary) illustrates *representations and examples*. In this case movement between perpendicular walls were used as a reference context. Prior to the excerpt below, the students were asked to draw a path which always kept the same distance to each of the two walls.

Kim (teacher):	In mathematics we make use of walls like these. They're not two walls, but what do we call them?
Student:	y and x.
Kim:	Yes, we call them y and x … so when we move like this, y equals x. Always. No matter where we are along this path, the distance to y and the distance to x is the same, right?

Kim: We say that the fact that y = x, that is what we call a function, while this drawing here [points], we call a graphical presentation.

The above teaching sequence could be illustrated by Steinbring's epistemological triangle through the following semiotic chain (Figure 2):

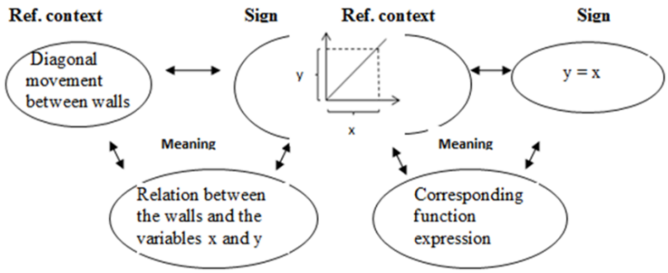

Figure 2: From movement between walls to function expression

The link between paths of movements and a more formal mathematical description of the path was made through Kim's statement "y = x". Kim's last statement illustrates how the representation form "formulae" (Janvier, 1978) is equated to the mathematical object "function". The next example is from lower secondary school, School C.

Tom (teacher): We are going to find out how to use functions in our everyday life. The simplest example is for example if you are going to the grocery store and you are going to pay for something. And we have a function here [points to the expression y = x + 1 at the blackboard].

Further, Tom links the function expression $y = x + 1$ to buying candy, where y is the total costs and x is the amount of kilograms. The gradient (in this case 1) is the prize per kilogram and the constant term (in this case 1) is the prize of the paper bag. As in the case of Kim in the previous example, the representation "formulae" and the mathematical concept of functions are being equated.

At lower secondary in particular, but also occasionally at upper secondary, different function machines were applied. Illustrated in Figure 3 are some of the examples observed:

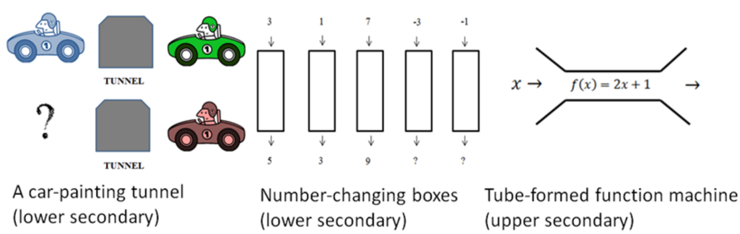

Figure 3: The different function machines observed in teaching

In the first example, the students are told that a green car drove into a tunnel and came out blue. They then guessed which colour a red car would have, as it came out of the tunnel. Since the "function" of the tunnel was "to paint cars blue", the result was always that a blue car came out of the tunnel no matter which colour it had driving into the tunnel. This example could "correspond" to any constant function, $y = k$ but this link is not explicitly made by the teacher. One could argue that the meaning of "function" in this context radically differs from mathematical definitions, as numbers are replaced by cars.

When it comes to the number-changing boxes, different numbers were dropped into each of them, and the students were asked to look for patterns. The teacher revealed that dropping the input values 3, 1 and 7 into the boxes resulted respectively in the output values 5, 3 and 9. He then asked the students what will happen to the input values -3 and -1, and they responded by suggesting the output values -1 and 1. The teacher then established the agreement that the value two was added to these numbers. It is worth noticing that it was the boxes which Roy (the teacher) denoted by y and not the output values. Hence, in the equation $y = x + 2$, y would have a double role, as it denotes both the function and the dependent variable. In this activity though, y denoted only the function.

Functions as co-variance is a category emerging from observations where links explicitly are made between the concept of functions on one hand, and a specific relation between two variables on the other. At lower secondary, School A, the teacher Kim provides the following explanation:

Kim: The function illustrates the relation between two varying magnitudes. If we vary x, then we also have to vary y related to x. We say that y is a function of x.

As *formal definition* was mainly referred to by the teachers alluding to textbooks, these became the main source of information for the students. In total six different textbooks were used (three at lower and three at upper secondary). In lower secondary, the uniqueness property and the concept of variables are only present in one textbook. At upper secondary, the uniqueness property is present in all the definitions, but "variables" are only mentioned in one of them.

SUMMARY AND CONCLUSIONS

As accounted for in the previous section, a proper treatment of the very definition of functions was absent at both lower and upper secondary. The concept of variables, like independent and dependent variables and the relation between them was minimally dealt with especially at lower secondary schools. Extensive use of representations emphasizing especially graphs, expressions and value tables was apparent in most schools. The introductory lessons at lower secondary seemed to focus on function expressions, which to a certain degree seemed familiar to the students from prior work. At some schools the uniqueness property of functions was implicitly dealt with through the use of function machines. The concept of variables and the uniqueness property were only superficially treated in textbooks. Related to the uniqueness property, Even and Bruckheimer (1998) question the traditional teaching of functions, as the importance of formal aspects like the uniqueness property is highlighted, without justifying why this criterion is important. Even and Bruckheimer's rather radical suggestion is to postpone the explicit treatment of these aspects in teaching, until the reasons for the criterion of uniqueness are more obvious. In my study the teacher interviews could suggest that teachers had similar rationales for omitting discussions about the formal definitions, but instead of emphasizing the difficulties of justifying why the uniqueness property is important, they emphasized value of practical examples for the sake of students learning outcome.

My impression was that the teaching of functions at upper secondary, general studies and lower secondary primarily differed in terms of the examples applied in the different representation forms. For example, it seemed common that functions expanded from including only linear functions in lower secondary to involve polynomial functions in upper secondary. Certain shifts in notations and mathematical symbols also took place, such as f(x) instead of y, but the f(x) notation was introduced by means of various justifications and arguments by the teachers. The complexity of this notation is pointed out by Sajka (2003) in terms on emphasizing that "f(x) can represent both the name of a function and the value of the function f" (p. 230).

As indicated in the analysis, function machines were observed in both lower and upper secondary school. One of Blomhøj's (1997) conclusions from a study involving Danish students in ninth grade, was that some students tend to "see the expression $y = x + 5$ as a recipe of a function machine, which changes the numbers put into the machine" (Blomhøj, 1997, p. 24, my translation). This suggests that an extensive and uncritical use of function machines might lead to the misconception that the independent variable is transformed or *changed into* the dependent variable.

The prior analysis belonging to the category *representations and examples* shows some examples of teachers equating a certain representation form, for example the formulae, to the function concept itself (the mathematical object). This relates to what Font, Bolite and Acevedo (2010) identify as "object metaphor". Object metaphors are "object image schema in mathematics" (p. 138) which in turn suggest that representations (e.g. graphs, formulae) are physical manifestations of the objects (functions). Utterings like "what does the function look like?" or "this is the function" are examples of such object metaphors. These could enforce an understanding that mathematical objects (like functions) are equivalent and on the same ontological level as their representations (for example graphs).

Summarized, findings suggest that the introduction of functions is done without explicitly considering mathematical aspects like the range and domain, the uniqueness property, and dependent and independent variables. The examples provided seemed rather arbitrary and in some cases, as illustrated through the use of some function machines, functions are given meanings which differs from the mathematical convention. It is my aim that the outcome of this study could increase the awareness among teachers and teacher educators, so that examples and explanations provided underpin and support the mathematical properties of functions and related sub-concepts.

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AT-RISK GRADE 1–3 STUDENTS' UNDERSTANDING OF THE NUMBER SEQUENCE AND THE NUMBER LINE

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This paper presents the outcomes of a research project aimed at developing mapping tests designed to identify students at-risk of lagging behind in mathematics. Data from the first test implementation (N1 = 2370, N2 = 2483 and N3 = 2286) are used to investigate at-risk Grade 1–3 students' understanding of the number sequence and the number line, as well as to discuss how this knowledge develops across grades. Analyses indicate at-risk students demonstrate a weak understanding of the number sequence. When attempting to identify target numbers on a structured number line, many consistently count by one from zero. Evidence suggests at-risk students do not master counting by two, five or ten. In addition, some struggle with the conventions for writing numbers. More growth is seen from Grade 1 to 2 than from Grade 2 to 3.

INTRODUCTION

Norwegian Grade 4 students scored at the international average in the 2011 Trends in Mathematics and Science Study (TIMSS 2011) (Mullis, Martin, Foy, & Arora, 2012). However, in total, 9% of students did not reach the low benchmark and only demonstrated some basic mathematical knowledge, such as adding and subtracting whole numbers. Still, the number of students reaching the low benchmark has risen substantially from 75% to 91%, from 2003 to 2011 (Mullis et al., 2012, p. 93). Previous studies have demonstrated Norwegian primary school teachers have had a tendency to 'wait and see' when they observe students with difficulties (Nordahl & Hausstätter, 2009), as well as that the majority of special needs education is given in lower secondary school (Solli, 2005). Consequently, intervention has often been delayed until students have developed real difficulties or lagged substantially behind. This pattern has worried the Norwegian Ministry of Education and Research, and in 2006, they released the white paper 'Early Intervention for Lifelong Learning', presenting a national policy for "how the education system can make a greater contribution to social equalisation" (MER, 2006, p. 1). A key aspect of the early intervention policy is to ensure every student acquires the basic skills needed to succeed in the education system, and in 2008, a mandatory national screening test for Grade 2 was introduced, followed by optional grade 3 and 1 testing in 2009 and 2011. The purpose of the implementation was to aid teachers in identifying students who had not acquired the basic mathematical concepts and computational skills necessary to provide a solid foundation for further learning. Based on the TIMSS 2011 outcomes, the strategy might be seen as successful. However, the majority of special needs education continues to be given at the lower secondary level (Norwegian Directorate for

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Education and Training [NDET], 2014), indicating teachers still either tend to wait and see or struggle to plan and carry out successful interventions for identified students.

In a What Works-report, Gersten et al. (2009) recommend screening all students to identify those at risk of potential difficulties in mathematics and to provide intervention to identified students. However, for the assessment to be a starting point for mathematical learning, teachers need to be able to use the test results to identify where learners are and where to go next (Wiliam, 2007). Consequently, assessments should be targeted to the student group of interest, and when the second generation of the mapping tests were developed, this was taken more into consideration. The assessment, launched in April 2014, is a hybrid between a screening and a mapping test. While the full student cohort takes the test, it is targeted towards the weakest 20% of students (labelled 'at-risk students') and is developed to differentiate more securely between the first and second quintile groups of students (NDET, 2011). Consequently, the tests have a ceiling effect by design and consist of many easy items. A student close to the cut-off score typically solves 70–85% of the items correctly, which provides teachers with much more information about what students can do than assessments targeted at the full student population, where struggling students are characterised by what they cannot do.

This paper draws on experiences and data from the first test implementation in the spring of 2014. The research questions for the paper include: 1) what do the mapping tests display about Grade 1–3 students' understanding of the number sequence and the number line and 2) how does this knowledge develop from Grade 1 to Grade 3.

UNDERSTANDING THE NUMBER LINE

Young children's knowledge of counting and quantity can be seen as a starting point for their understanding of numbers (Griffin, 2003). As they become more experienced with counting and numbers, their conceptual structure for whole numbers might be seen as a mental number line (Griffin, 2003). This structure functions as a mental counting line and allows students to compare the magnitude of numbers and understand place value. In school, students encounter the number line as a mathematical object representing a sequence of numbers, often starting from 0, representing only the positive numbers. However, a number line might start from a different number and have different spacing. Students need to understand how intervals with the same magnitude represent the same numerical difference. While the number line can be seen as a measurement object or a graphical representation of number, students often conceive it as a counting object (Diezmann & Lowrie, 2006). The empty number line might be used as a learning tool towards developing strategies for addition and subtraction and is often advocated in teacher education literature (for a Norwegian example, see Heiberg Solem, Alseth, & Nordberg, 2009). However, teachers might make ambiguous connections between the number line and the number track (Grey & Doritou, 2008). As a consequence, many students focus on the number line as a counting tool for handling addition and subtraction.

Poor understanding of the number sequence might hinder students' mathematical development, as this most likely signals a weak conceptual understanding of numbers. When assessing students' number concepts, number line items are frequently used as a representation of the number sequence (Diezmann & Lowrie, 2007).

TEST DEVELOPMENT AND TECHNICAL REQUIREMENTS

The Norwegian national mapping tests implemented in 2014 are designed to measure students' number concepts and calculation skills. According to the test development framework provided by the NDET (2011), tests should serve as a formative assessment tool for the first quintile group (the 20% weakest students), in that test results should function as a starting point for further classroom work. The class teacher administers tests towards the end of Grades 1–3 when students are six, seven and eight years old. While the Grade 2 test is mandatory, tests for Grades 1 and 3 are optional. The school principal or the local school administration decides on participation. Tests are scored by the teacher.

One test was developed for each grade level drawing on research literature (for instance, Geary, 2004; Griffin, 2003; Verschaffel, Greer, & De Corte, 2007), and pre-tested in cognitive labs and through small group testing before being pilot tested in spring 2013. Test reliability, measured by Cronbach's alpha coefficient, is high and larger than .93 for all tests. In addition, no gender DIF was observed. As described in the test framework (NDET, 2011), all tests discriminate around a 20% cut-off score, which is between the first (Q1) and second (Q2) quintile groups. This means that for most test items, 70–90% of students were successful. All tests have a ceiling effect by design. Tests are timed to identify students with naive or rigid counting strategies.

All tests were validated using different expert panels (primary school students, researchers, test designers, teachers, special education teachers, school leaders and international experts). Experts commented on single items, clusters of items and full tests. All experts agreed that the items described below assess some aspect of understanding of the number sequence.

Grade	No. items	Cut-off	Mean Q1	n Q1	Mean Q2	n Q2
1	50	39	36.3 (7.697)	454	42.4 (1.401)	448
2	55	41	31.7 (8.720)	500	45.6 (1.889)	541
3	72	59	48.7 (10.089)	443	63 (1.704)	488

Table 1. Test characteristics, means and number of students in Q1 and Q2.

Sample

The dataset used from this paper was collected during the first test implementation. A representative national sample was drawn, excluding schools with fewer than five students at the grade level and international schools. In all, 127 schools participated, with one student group/class at each grade level: N1 = 2370, N2 = 2483 and N3 = 2286. Only data from Q1 and Q2 students are used for this paper (see Table 1).

Items assessing students' understanding of the number line

Items assessing students' understanding of the number line were developed for each grade level, albeit with a different range of numbers: 0–20 for grade 1, 0–100 for grade 2 and 0–300 for grade 3. Four types of items will be discussed in this paper (the number of items for each grade level used in this paper is given in parenthesis).

Counting objects and indicating on a labelled number line how many (3/2/2)
Placing numbers on a number line (2/0/2)
Counting on from a given number (counting up and down) (4/5/4)
Sorting numbers by magnitude (3/3/3)

Examples will not be given, nor will the numbers used in the items, although their magnitude will be indicated. As the test items are not released, items will only be described in the results section. Response rates will be reported for single items and groups of items for the two lowest quintile groups. Due to the test design, significant differences are observed between Q1 and Q2 in all grades for all four item groups.

RESULTS

Tests designed to identify the weakest 20% of students could have been aimed at the average achievers, applying items that the weak students do not master at all. However, teachers must be able to draw conclusions about the competences of their students from student response patterns and identify weaknesses and strengths in their students' mathematical competences for the test to function as an assessment for learning for the targeted student group (Wiliam, 2007). Consequently, the tests mainly comprise items that are very easy for most students. In addition, Q2 students have a generally higher probability then Q1 of solving correctly all items. Significant differences were observed between Q1 and Q2 for all groups of items for all grade levels as measured with ANOVA, for grade 2 for instance $F(2,2479) = 386.562$, $p < .01$, $F(2,2480) = 1114.889$, $p < .01$ and $F(2,2480) = 528.767$, $p < .01$ for item groups 1, 3, and 4 respectively.

The first group of number line items asked students to count concrete objects and mark the corresponding number on the number line by drawing a line from the group of objects to the location of the number. Figure 1 displays a marked number line similar to the number lines used for Grade 1 test items. Students were asked to count small, unstructured groups of objects (maximum 15), and 56–82% of Q1 students successfully solved these items. Items discriminated satisfactorily between the two first quintile groups when students were asked to count more than 10 objects. A difference in percentage points ranging from 9 to 27 between the two groups indicated Q2 mastered the items to a much larger extent than Q1.

Q1 students took longer to finish these items. In total, 16% of students in this group did not manage to finish the last item, compared with 2% of the Q2 students. Pencil marks in the test booklets indicated many of the struggling students counted by 1 from 0 to identify the numbers on the number line. Figure 1 displays how one student counted to identify 6, 9 and 12 on a similar item, used in test development. Such marks were frequently observed in test booklets. It is hypothesised that the weakest students solve such items by double counting; first the objects followed by counting up to the target number on the number line.

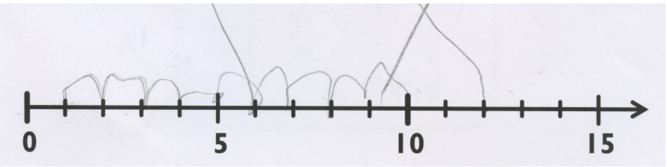

Figure 1. Student work on a structured number line, Grade 1.

To monitor growth across grade levels, a set of identical items was given to students in Grades 2 and 3. Students were asked to count structured groups of objects (grouped in tens and single units). Items included up to 39 objects (one, two or three groups of 10 and some single objects). Number lines were structured, and each whole number was marked. In addition, the structure of the number sequence was given either by labelling every 5 and 10 or by labelling every 10 only. Approximately half the Q1 students solved the items correctly in both grades, indicating these tasks are still challenging to the weakest students, even after three years of being exposed to these types of items from their textbooks. Again, students counting by 1 from 0 were observed by pencil marks in the test booklets. In addition, fewer Q2 students mastered the items in grade 3 compared to grade 2. However, Grade 3 students were allowed a shorter time to solve this item, which might explain this observation.

The second group of items asked Grades 1 and 3 students to mark on the number line the position of some given numbers. Behaviours similar to those observed for the previous group of items were observed for some Q1 students who apparently counted by 1 and ignored the marking and sequencing of the number line. This strengthens the hypothesis that many Q1 students do not understand the number line structure. As the magnitude of the numbers, the marking and the labelling of the number lines were different for the number line items for Grades 1 and 3, direct comparisons of difficulty level cannot be made. It should be noted that only number lines that marked every whole number were used for Grade 3, thus enabling students who count securely to identify the correct position of a number, provided they started counting from the correct number and position. When assigning the starting point of the Grade 3 number lines at a number other than 0, some Q1 students still started counting from 1 as if all number lines start at 0. For struggling grade 1 students, numbers larger than 10 where challenging and a marked difference was between Q1 and Q2, with 40% and 74% of students, respectively, solving these items correctly. Development is seen across grade level, and Q1 Grade 3 students confidently handled numbers smaller than 50. However, numbers larger than 100 were still challenging.

The third group of items reported in this paper were items that required students to recognise number patterns and count on from a given number. The mapping tests are paper-and-pencil tests, and students had to provide written responses to the count-on items. Consequently, these items also provided information about students' knowledge of the conventions for writing numbers (e.g., that fourteen is written 14 and not 41). Students were typically given three numbers and asked to count on from the last. Numbers ranged from 0–15 (Grade 1) and 0–70 (Grades 2 and 3). Grade 1 at-risk students confidently counted up but struggled to count down. Counting down was challenging, even to Q2 students, with approximately 42% and 70%, respectively, managing to count down from a number smaller than 15. Grade 1 students were mainly asked to count by 1. Counting by two differentiated mainly between the second and third quintile groups; only 14% and 34% of Q1 and Q2 students, respectively, solved this item correctly in Grade 1. Counting by two, five and 10 can be efficient for executing mental calculations quickly, but this rests on secure knowledge of the number sequence. This is one of the areas where the Q1 and Q2 differ substantially in grades 2 and 3. Whereas 80% and 93% of Q2 students on average solve such items, respectively, 45% and 59% of Q1 students on average answered the same items correctly, respectively, indicating weaker knowledge of the number sequence. Q1 students also took longer to solve these items, and in Grade 2, as many as 30% did not manage to finish the last item within the given time, supporting the hypothesis of a weaker understanding of the number line.

The last group of items required students to sort a given set of numbers by magnitude (up to 5 numbers). To sort and compare successfully, students need a well-developed mental number line (Griffin, 2003). Grade 1 students were asked to sort numbers in the range of 0–20, Grade 2 in the range of 0–100 and Grade 3 in the range of 0–300. While more than 80% of Q2 students at all grade levels successfully sorted the numbers, this is one area where much development in seen from grade 1 to Grade 3 among Q1 students. In Grade 1, numbers larger than 10 represent a challenge and only 55% of students in Q1 managed to sort five numbers, of which four were larger than 10, indicating almost half the students identified as being at-risk struggled to understand the magnitude and ordering of small, positive whole two-digit numbers. Q1 students in Grade 2 struggled with items where they needed to compare numbers, such as ab and ba or ab, ca and ad. A few students were likely unable to handle comparing as many numbers as five, and responses indicated these students probably compared pair-wise or a few numbers at a time when sorting. In addition, at-risk students took longer to complete these items.

The numbers given to the Grade 3 students masked some of the difficulties displayed by Grade 2 students in relation to not understanding place value or not mastering the conventions of writing numbers. However, students were also observed to struggle when comparing numbers like abc and aca. Still, many at-risk students struggled with sorting items, and of the six items given to Grades 2 and 3, between 41% and 86% of the students, respectively, sorted the numbers correctly.

DISCUSSION AND CONCLUDING REMARKS

Analyses of student response patterns on the mapping tests revealed that Norwegian at-risk primary school students displayed difficulties and a lack of conceptual understanding and procedural use of the number line, as anticipated from previous research. Q1 students also demonstrated a weak knowledge of number sequence, as might be anticipated from the number line items. Further analysis demonstrated Q1 students took longer than Q2 students to count up the number line, especially when only the end points were labelled. Items demanding 'double' counting, as in the first group of items, which asked students to count small numbers of concrete objects and tie this amount to the number line, were more challenging to Q1 than to other students. It might be assumed that more at-risk students started counting from 0 on the number line when solving these items rather than using the numbers identified by the labelling of the number line to navigate. It could be argued that these students view the number line as a counting tool rather than a measurement tool, as discussed by Diezmann and Lowrie (2006; 2007). However, their knowledge of the number sequence, displayed by the counting-on and sorting of items, is also weak, and it might also be assumed they have less support from the structuring of the number lines. For instance, some weak students struggled with the conventions of written numbers, indicating a lack of understanding of place value.

Learning the number sequence and understanding place value and the number line are crucial to learning basic arithmetic and the primary school curriculum. Consequently, items assessing these basic concepts must be included in a mapping test if the teacher is to use the test for formative purposes. Traditionally, at-risk students have been identified by their strategy use when solving simple addition and subtraction tasks (see for instance Geary, 2004; Griffin, 2003). However, number line items are well suited to identify these students and to provide valuable information on students' number concepts that can serve as a starting point for teaching interventions.

Analyses of responses to the mapping tests revealed Norwegian teachers might need further support to target teaching to students' needs, as the same use of simple counting strategies was found among students in all three grade levels; so many weak students do not progress through existing teaching activities. In addition, error patterns indicate weak students have a much slower growth of understanding from Grade 1 to Grade 3 than typical developing students, as might be expected (see, for instance, Geary, 2004). While a fairly strong development was observed from grade 1 to grade 2, little improvement was seen from grade 2 to grade 3. The weak Grade 1 students most likely do not catch up with other students during Grade 2.

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A CONTINUUM TO CHARACTERISE AND SUPPORT TEACHER INTERPRETATION OF AN INNOVATIVE CURRICULUM

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A continuum for describing the degree to which teachers interpret the various features of a curriculum is presented. The continuum has been developed based upon the observation of classroom practices and discussions with a group of teachers who are using an innovative junior secondary mathematics curriculum. It is anticipated that the ongoing use of the continuum will lead to its improvement as well as the refinement of the curriculum, more focussed support for the teachers, improved student learning, and the building of explanatory theory regarding mathematics teaching and learning.

INTRODUCTION

This paper presents a continuum that characterises teachers' interpretation of the curriculum provided in an Australian Research Council (ARC) funded project titled *Accelerating the Mathematics Learning of Low Socio-Economic Status Junior Secondary Students (XLR8)*. In this paper, the *interpretation continuum* is a scale to describe the degree to which the teachers involved in the project interpret the project's intended curriculum and transform it into the enacted curriculum of their respective classes. The paper first provides an overview of the project and then summarises the literature and approach that has led to the development of the continuum. To illustrate the continuum's application, two teachers' interpretations of the innovative curriculum are presented. Ultimately, the continuum may aid in the development of theory regarding teachers' effective interpretation of curriculum innovations, such as the one proposed in the XLR8 project.

PROJECT OVERVIEW

The XLR8 project has been designed to develop theory and practice regarding the acceleration of junior secondary students (Years 8-9) whose level of mathematical achievement is nominally at a mid-primary school level (Year 4). The project aims to improve students' potential to enter Year 10 with the requisite knowledge to successfully study mathematics and follow this with further study or employment. The project, including its underlying conceptual framework and methodology has been presented previously (Cooper, Nutchey, & Grant, 2013).

In short, to address the identified issue of underperforming students, design experiment (Cobb, Jackson, & Munoz, 2015) is used to propose and iteratively refine a curriculum for acceleration (i.e., the intervention). The XLR8 curriculum is innovative because it has been designed to carefully explore the structure of mathematical knowledge in a nested, conceptually-focussed sequence that builds students' understanding from a low-achievement level to age-appropriate level. To achieve this, the curriculum employs a pedagogy referred to as RAMR, standing for Reality-Abstraction-Mathematics-Reflection. The pedagogy is grounded in the students' reality, drawing upon suitable everyday-life examples to situate learning. It provides a clear order of abstraction activities that progress through kinaesthetic – iconic – symbolic representations while also connecting to everyday and mathematical language. Mathematical activities build students' fluency with mathematical procedures and skills as well as promoting their conceptual understanding (i.e., developing and reinforcing connections between mathematical ideas). During reflection, opportunities are made for students to reflect their learning back to their reality, thereby transferring their knowledge to new situations and further developing connections, including the formation of generalisations.

The XLR8 curriculum is presented to teachers as a series of module booklets, each nominally 5 weeks in duration. Each module is composed of several units, each of which corresponds to a single cycle of the RAMR pedagogical framework. The modules carefully explain the mathematical ideas of each unit and their structural relationships with one another. The ordering of the modules and units defines a conceptual sequence (referred to as the structured sequence) by which the structure of mathematical ideas is to be explored, which is further explained in the module booklets. Accompanying each module is a set of classroom resources, including worksheets, that serve as examples of intended classroom activities. The curriculum includes supervised test tasks which provide pre/post instruction data and which are marked in a timely manner by the research team such that they can be used by the teachers to inform their teaching. Assignment-style assessment tasks are also provided for each module. To support the teachers as they use the XLR8 curriculum, members of the research team regularly visit the teachers, both in their classes and for one-to-one meetings. During the in-class visits the researchers act as teacher-aides, assisting the teacher as needed. In the one-to-one meetings, the researchers act as a coach, discussing the curriculum with the teacher and collaborating with them to plan their teaching and to develop teaching resources. The teachers are also supported by meeting together in professional learning sessions, during which aspects of the XLR8 curriculum are presented and discussed.

Thus the curriculum is comprised of five features: 1) the structure of mathematical ideas embodied in each of the modules; 2) the conceptual sequence by which the modules and their units explore the structure; 3) the RAMR pedagogy that is described in each of the units with regard to the corresponding content (i.e., to follow the structured sequence); 4) the resources used to implement the structured sequence using the RAMR pedagogy; and 5) the assessment materials that generate diagnostic, formative and summative evidence of students' mathematical understanding.

LITERATURE REVIEW

Provided with any form of curriculum material, whether officially mandated curriculum or restructured curriculum materials, teachers are tasked with its interpretation. Via that interpretation, teachers make decisions, plan learning activities and prepare resources that will be enacted in their classroom. Teacher interpretation of curriculum and response to curriculum change is variously described in the literature. Doyle and Ponder (1977/78) identify three images of the teacher faced with curriculum change: *Stone-age Obstructionist, Pragmatic Skeptic, Rational Adopter* (Doyle & Ponder, 1977/78). The first image is of a teacher who rejects (and resists) change regardless of argument or material. The third image is of a teacher who accepts curriculum reform if good arguments are made and the materials appear to reflect these arguments. The second image is more complex and embodies the ecological consideration that teachers adapt curricula to the specific needs and environment of their students. Doyle and Ponder go on to describe the degree by which pragmatic skeptics embrace curriculum change is moderated by their perception of the innovation's practicality, in terms of instrumentality, congruence and cost. More recently, Basalam (2010) has defined a continuum of categories with which to characterise teachers' responses to curriculum change. The continuum ranges from *non-adopters* (including sub-categories of *rejecters* and *resisters*) through to *adopters* (including sub-categories of *partial-adopters, pragmatic-adopters* and *critical embracers*). In both cases, these categories seek to provide salient descriptions and insights regarding of teachers' adoption or adaptation of curriculum changes.

The interpretation of the intended curriculum to form the enacted curriculum is bound to vary in terms of its alignment to the intention of curriculum designers (Porter, 2006). This variance in teacher interpretation is influenced by a range of factors, including: their own beliefs about mathematics content and pedagogies in relation to their unique classrooms (McLaughlin & Talbert, 2001); resources provided as a part of the innovation, including the textbook (Little, 2002; Remillard, 2005); concern for immediate contingencies and consequences as a reaction to student responses rather than from evidence of long-term goals (Doyle & Ponder, 1977); and perceptions of the abilities and learning capacities of students within their classroom and their possible life trajectories and aspirations (Schoenfeld, 2008).

This literature provides a basis for identifying categories of responses to the XLR8 project's curriculum and for developing explanations regarding the varying degrees of teacher interpretation. This characterisation and explanation of individual teacher responses will in turn inform improvements to the support given to teachers such that the desirable sustained impact and long-term benefits of the project are achieved.

APPROACH

Participants in the XLR8 project in 2014 were 10 classroom teachers from four different schools, teaching approximately 180 students. Of these 10 teachers, five had also been involved in the project in 2013. The teachers had varying professional backgrounds: some teachers were relatively junior (including one first-year graduate), others were mid-career and one was an experienced teacher (who was the Head of the Maths/Science Department in one school). Most of the teachers were mathematics trained. However, some were teaching out-of-field, having been selected to participate by their respective schools based upon their experience of teaching students with behavioural and/or additional learning needs.

Data gathering in regard to these teachers' practices of curriculum interpretation has included: field notes taken during lesson observations and one-to-one coaching sessions; video recordings of discussions during professional learning sessions; and individual semi-structured interviews conducted with each of the participating teachers at the end of each year. Both the first and second authors have met with, observed and/or interviewed all of the participating teachers, and so have been able to discuss their experiences and develop a shared understanding of each teacher. In particular, they have been able to characterise typical practices of the participating classroom teachers as they interpret the XLR8 curriculum.

Data analysis leading to the formulation of the continuum for characterising teacher interpretation was conducted by the first two authors as follows. First, with regard to the five features of the curriculum intervention (structure, sequence, pedagogy, resources and assessment), the first two authors proposed, discussed and refined statements that described the observed or reported practices of the participating teachers. These statements were written on sticky notes and assembled in columns (per teacher) and rows (per curriculum feature).

Second, within each curriculum feature (row), these descriptive statements were compared and sorted into groups based upon similarity. This sorting was guided by the literature: groups that aligned to adoptive or adaptive practices were sought. The sorting was refined when it became apparent that some practices reflected non-compliance with the curriculum (similar to Basalam's (2010) non-adopter category). The imperative of the project to situate learning within the students' reality necessitates teacher modification of the curriculum to suit their students. This led to the further refinement of the adaptive category into those teachers who questioned the curriculum and those who improved it. This comparison and sorting of the descriptive statements ultimately led to the proposition of four categories along the continuum: *resister*, *follower*, *questioner* and *improver*.

Third, the collected descriptive statements for each of the four continuum categories in relation to each curriculum feature were then synthesised into general descriptive statements regarding teachers' curriculum interpretation practices. As a result of trying to synthesise the general descriptions, the sorting of the specific statements was revisited and refined until the two authors reached a consensus, both in regard to the sorting and the generalised descriptions that resulted.

RESULTS AND DISCUSSION

The final result of synthesising the general descriptions is presented in Table 1.

Feature	Resister	Follower	Questioner	Improver

Structure	Knowledge of mathematical structure not evident in discussions or teaching. Focus on each mathematical idea in isolation.	Structural knowledge evident in discussions and teaching. Learning activities develop conceptual understanding.	Critiques own knowledge of mathematics, including structure. Discusses and queries structure as presented in curriculum.	Improves own knowledge of mathematical structure. Suggests refinements of the structure presented in the curriculum.
Sequence	Planning focussed on procedural fluency with end-point ideas. Ignores, skips or in-cohesively reorders curriculum activities.	Follows sequence as a series of isolated events. Lesson-level planning, little longer-term planning to build structural understanding.	Critically discusses sequence and the structure it develops. Longer term planning to develop structural understanding.	Adjust sequence to suit students, informed by structural knowledge. Participates in discussions regarding sequence improvement.
Pedagogy	Focussed upon mathematics phase to develop procedural fluency using rote-based instruction. Limited situated learning. Abstraction sequence absent or inconsistent use.	Routinely uses RAMR sequence without adjustment (most phases). Connects mathematical activities and language. Coherent situated learning in all RAMR phases.	Actively reflects upon and discusses teaching and learning in terms of using the RAMR cycle.	Recommends refinements to RAMR-based curriculum in terms of classroom practicality and students' development of understanding.

Table 1: Feature-wise characterisation of the interpretation continuum.

Feature	Resister	Follower	Questioner	Improver
Resources	Uses own resources	Uses provided resources and	Critically reviews	Collects, creates,

	instead of those provided, which do not align to the curriculum intentions.	finds similar resources that are aligned to curriculum intentions.	resources in terms of students' needs and curriculum intentions.	improves and shares resources that are aligned to curriculum intentions.
Assessment	Formal assessment used only for reporting. Focussed upon procedural fluency not conceptual understanding or ways of working.	Uses assessment data to inform planning.	Queries content, coverage, form and language of assessment items.	Suggests improvements and makes modifications to assessment items to address perceived weaknesses.

Table 1 (cont.): Feature-wise characterisation of the interpretation continuum.

Guided by the continuum of descriptors presented in Table 1, two XLR8 teachers (Teacher A and Teacher B) were profiled. This profiling is summarised in Table 2 and then the profiles of each teacher are discussed in turn. As can be seen in Table 2, each teacher varied in the degree to which they interpreted the five curriculum features. For some features, teachers were positioned on the boundary of two categories. That is, a teacher cannot be simply categorised as Resister, Follower, Questioner or Improver.

Feature	Resister		Follower		Questioner		Improver
Structure	A				B		
Sequence		A				B	
Pedagogy	A					B	
Resources		A	B				
Assessment		A			B		

Table 2: Interpretation profiles of Teachers A and B.

Teacher A was a newly-graduated Mathematics teacher: 2014 was her first year of teaching. Overall, the degree to which she interpreted the XLR8 curriculum could be described as a resistive follower. Observations and discussions with the teacher suggested she had a weak understanding of the structure of mathematical ideas, at least with respect to the low-level content that she was teaching to her XLR8 class. She made efforts to follow the XLR8 structured sequence, but often rearranged the suggested order of activities such that the structured sequence was not adhered to. Her planning was very short-term (usually limited to the activities of the next lesson) and infrequently considered the development of big ideas across a module. She resisted using the RAMR cycle to base her teaching upon, citing that the students were unable to behave appropriately when attempting the more physical activities in the Abstraction phase. Teacher A often used her own resources, however they usually focussed upon practising procedural skills (the importance of which she emphasised during one-to-one discussions) and were sometimes misaligned to the objective of the curriculum units in which they were used. Whilst she administered the pre/post tests and assignment-style assessment tasks, she only partially drew upon the assessment data to inform her teaching, instead, relying upon anecdotal observations that were based upon her own, apparently weak, structural understanding.

Teacher B was an experienced Mathematics teacher and was the Head of Department at his school. 2014 was his second year of teaching using the XLR8 curriculum. In contrast to Teacher A, Teacher B provided evidence of a much more richly connected understanding of mathematics, was critical of his understanding and used his connected understanding to improve the curriculum sequence. This deeper structural understanding was also reflected in the way in which he refined his understanding and use of the RAMR pedagogy to better develop students' understanding and the ways in which he used assessment data to guide his teaching. Interestingly, Teacher B seemed less inclined to modify the resources that were provided, instead preferring to use what was provided in the ways that were suggested.

CONCLUSION

The XLR8 project involves teachers in trialling material developed by researchers with the outcomes of producing improved teaching and learning, innovative approaches to professional learning, classroom materials and theory with respect to teacher change and student learning. Based upon literature and data taken from the XLR8 classrooms, a continuum has been proposed to describe the degree to which the XLR8 teachers adhere to, query or improve the XLR8 curriculum with regard to its five features. It is anticipated that the best outcomes will emerge when teachers are questioning and improving the curriculum, that is, when they enhance learning in classrooms and act as co-researchers with respect to learning materials and student learning. However, as illustrated in the profile of Teacher A, some teachers tend towards resistance or following. The construction of the continuum and its use to characterise Teachers A and B has raised the question "How do the interpretation practices across the five curriculum features relate to one another?"

Moving forward, this continuum will be used as a basis to structure XLR8 classroom observations and discussions with teachers regarding their interpretation practices. Further use of the continuum will lead to the refinement of the continuum descriptors and the development of explanations regarding inter-feature relationships (e.g., the influence of teachers' interpretation of mathematical structure upon assessment) and external factors which influence the teachers' curriculum interpretation. Importantly, this more focussed data gathering and analysis will lead to the identification of opportunities for the XLR8 project to provide professional learning support that will enhance teaching practices, the curriculum and, ultimately, student learning outcomes.

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LOGICAL PROBLEMS WITH TEACHERS' BELIEFS RESEARCH

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This report explores the 'messy' field of mathematics teachers' beliefs, suggesting the problem may lie in the difficulties encountered defining what beliefs are, which have given rise to inconsistent ways of thinking about how beliefs function. This has also led to some fallacious thinking about mathematics teachers appearing to act in ways considered to be inconsistent with their beliefs. In this paper, a simple definition for beliefs is proposed, a vocabulary drawn from Fives and Buehl's (2012) review is used to discuss the role beliefs play in practice, and the logical problems associated with claims about inconsistent mathematics teachers are explored.

INTRODUCTION

There is an extensive literature spanning seven decades which explores the impact of teachers' beliefs on their pedagogical practice. It is a testament to the complexity of the field that one attempt at consolidation and clarity is partly titled "cleaning up a messy construct" (Pajares, 1992) while 20 years later Fives' and Buehl's (2012) review of 627 articles is titled "spring cleaning the 'messy' construct of teachers' beliefs". Clearly this is a difficult area, but it is hoped that some further light can be shed by focusing on claims that mathematics teachers have been observed to act in ways considered inconsistent with their beliefs. It is important to first look at the context around teachers' beliefs and why it is still an area in need of some tidying.

CONTEXT

Many have indicated that defining the construct of teachers' beliefs is difficult (Pajares, 1992) while others have pointed out that definitions are not the problem so much as getting authors to use them consistently (Fives & Buehl, 2012). Fives and Buehl (2012) identified five categories of differences across the many definitions they found in the literature. These are based on whether beliefs were defined as: a) implicit or explicit (i.e., whether teachers were conscious or unconscious of them); b) stable or dynamic (i.e., on a spectrum from resistant to change through to open to change); c) generalised or situated (i.e., consistent or variable across contexts); d) related to knowledge (i.e., whether belief are the same, different, or related to knowledge); and, e) individual or systemic (i.e., beliefs being defined as either discrete propositions or as interconnected with a broader system of beliefs).

These points of difference illustrate the diversity of roles attributed to beliefs, and hint at the explanatory power researchers believe them to have. However, such variety makes it easy to lose track of what is meant by such a common concept, a meaning often further obscured by technical definitions incorporating what appear to be functions and features of beliefs into the definitions themselves (e.g., Rokeach, 1968;

2015. In Beswick, K.., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 305-312. Hobart, Australia: PME.

Kagan, 1992; Fang, 1996). Yet underlying many of the various definitions on offer in the literature is arguably the concept that beliefs are simply what people think (or hope) are true (or probably true) – and many have defined beliefs in more-or-less this way (e.g., Ajzen & Fishbein, 1980; Pajares, 1992; Richardson, 1996).

On this view Five's and Buehl's (2012) categories of definitions are different facets of the same phenomena rather than different psychic structures per se: a) the implicit/explicit nature of beliefs come from people consciously thinking that some things are true, while they hold other implicit truth claims that they are completely unaware of. For example, a teacher might consciously think it true that open ended problems are desirable pedagogical tools (explicit), while also holding unconscious truth claims (i.e., assumptions) that these will not prepare students for standardised tests (implicit); b) Truth claims can be both stable and dynamic – a teacher may emphasise skill drills for all topics based on their own experience as a student, yet be persuaded that calculators are appropriate for some topics but not others. There is nothing that precludes a truth claim from being embraced for extended periods (stable), modified over time (dynamic), or discarded entirely; c) the generalised and situated nature of beliefs stems from whether the truth held applies in all or only some settings. For instance, a teacher might think it true that mathematics is the highest of all human endeavours, and while this truth is held across all settings (generalised), they might simultaneously think while teaching one class that geometry is their favourite topic, while with another class they prefer algebra – a truth varying by time and circumstances (situated) but which is a subset of other generalised truths; d) the question of how beliefs relate to knowledge remains an area lacking clarity (Clandinin & Connelly, 1987; Sutherland, Sinatra, & Matthews, 2001), and is too large a topic to be covered in detail here, suffice to say that although flawed, the Platonic tripartite theory of knowledge as justified true belief is preserved by the proposed definition of belief as what is thought true, since for Plato, knowledge consists of thoughts or propositions that are true and which people are justified in thinking are true; and, e) the individual or systemic aspects of beliefs are closely related to whether they are generalised or situated. In terms of beliefs as truth claims, this variation is explicable in a similar way to situated truths being viewed as subsets of more generalised truths – preferring geometry, then algebra, is an individual subset of the systemic view that mathematics is the ultimate human activity. So it seems that defining belief as what is thought to be true incorporates all of the variations identified to date whilst preserving a definition common to all.

The discrepancies in the use of the concept of teachers' beliefs, and the observation that much of the literature is focused on content and context specific descriptions or relational analysis, prompted Fives and Buehl (2012) to examine what they characterised as "perhaps the most important issue" (p. 478), namely, what do beliefs actually do? They proposed a three tier taxonomy wherein beliefs act as: interpretive *filters* – a layer of unconscious assumptions or habits that implicitly filter experiences and perception; *frames* within which problems are dealt with – this is the level at which ideological beliefs define or frame problems and situations that arise; and, action *guides* which motivate teachers to act. In other words, filtering beliefs constrain what can be experienced, frames provide a context to understand situations that arise from these filtered perceptions, and guides represent the pool of intentions and motivations that give rise to the ensuing actions aimed at addressing the problem/scenario at hand.

This approach of viewing beliefs in terms of filters, frames and guides goes some way to accounting for the different ways in which teachers' beliefs could be seen as influencing their actions. For instance, if two students are speaking with each other during a mathematics lesson, having behaviourist unconscious beliefs (implicit truth claims) might act as a filter leading the teacher to interpret them as being actively disruptive, while a constructivist filter might lead to their behaviour being interpreted as student collaboration. The teachers' filtered observation produces a situation which then becomes framed by their ideological position. If the teacher subscribes to an authoritarian model of teaching they may view active disruption harshly, while a more student centred set of beliefs might treat the situation as a minor or even positive one. Thus framed, the kinds of guiding beliefs the teacher holds will then influence how they respond to the situation. The authoritarian teacher might be guided by a narrow set of disciplinary truths involving detentions or writing out class rules, while a more laissez faire teacher might possess guiding truths involving strategies such as positioning themselves near the students, or actively ignoring them while praising other students who are obviously working on the task at hand.

However, while thinking of beliefs as filters, frames, and guides has some appeal in terms of accounting for the diverse ways in which beliefs interact to produce actions, there remains a degree of confusion. For instance, it is unclear when any given belief is acting as a filter, frame or guide. Is a belief in strict discipline actually a filter rather than a guide? Does an ideological frame filter the way one sees the world? Or are behaviourism and constructivism ideological? Or are they, in practice, pools of strategies which guide action? The confusion arises when resulting actions might be indistinguishable even though the explanations in terms of which beliefs filter, frame, or guide the actions are completely different.

So while this taxonomy retains much of the ambiguity inherent in thinking about beliefs, it none-the-less provides a helpful vocabulary around the different roles truth claims can play in shaping teachers' actions. One area where a vocabulary like this could help bring greater clarity and reveal some fundamental logical problem is in the area where teachers' actions have been deemed inconsistent with their beliefs.

INCONSISTENT TEACHERS

A significant area of research within teachers' beliefs investigates how beliefs drive teachers' practice, and how teachers' observed actions appear to contradict their stated beliefs (Thompson, 1984; Raymond, 1997). Lerman (2002) argues that an awareness of such discrepancies would motivate a teacher to attempt to change their practice. However, there are a variety of weaknesses with this approach.

Leatham (2006) identifies two potentially flawed assumptions common to a number of articles that explore mathematics teachers' beliefs: i) that teachers can easily state what their beliefs are; and, ii) that the meanings researchers take from these statements accurately reflect what the teachers actually meant. He points out that despite these two points of possible error, researchers have gone on to claim that teachers are engaging in behaviours that are inconsistent with their own beliefs, and that the teachers hold inconsistent sets of beliefs. Leatham (2006) argues that such conclusions do a disservice to both teachers and researchers, and offers instead an alternative framework for researchers to work from, a 'sensible system of belief' approach in which teachers are assumed to be "inherently sensible rather than inconsistent beings" (p. 92).

Leatham's (2006) framework proposes that beliefs influence action regardless of the actor's ability to express or even be aware of their beliefs (i.e., regardless of whether they are operating as unconscious filters or frames, or conscious guides). He suggests that researchers can only draw plausible inferences about these underlying beliefs when they have access to a number of sources with which to triangulate such inferences. Leatham (2006) essentially argues that to make plausible inferences about a teacher's underlying beliefs requires a variety of evidence, not just the teacher's statements from questionnaires or interviews. Additionally, teachers' filtering beliefs are more likely to be unconscious, rendering them completely opaque to such methods in any case.

And while it would be unusual for someone to hold a bevy of entirely independent beliefs, they need not be connected in a rigorously logical and coherent system either. Thagard (2000) provides an analogy of beliefs being like rafts floating at sea forming mutually supportive clusters, as opposed to being arranged hierarchically as they would be in a analogy based on building a house where a person starts with foundational 'truths' and builds up other beliefs using logical cement. In contrast, the raft analogy allows for the adjusting of surrounding beliefs until a coherent, mutually supporting raft of belief obtains. The beliefs are adjusted or 'tweaked' until it all makes sense to the believer, arguably the point at which they have developed clusters of functional truth claims that act as filters, frames, and guides – and one person's filter may be another person's guide. From this perspective teachers who seem to hold contradictory beliefs have already made sense of the situation and it is up to the researcher to find out how they have done so, regardless of how irrational or unjustified it may seem to the observer – for Leatham (2006), "our incredulity does not diminish another's coherence" (p.95).

And yet, while filters, frames, and guides viewed from a sensible system approach may be a better way to account for the claimed inconsistencies, there still remain other logical problems with the notion of inconsistent teachers.

LOGICAL INCONSISTENCIES WITH TEACHER INCONSISTENCY

The claim that a teacher is inconsistent entails that they were observed behaving in a manner at odds with their stated beliefs. The corollary to this is that researchers expect behaviour and stated beliefs to be in harmony with each other, which in turn is built on the assumption that behaviour is at least influenced by beliefs, if not wholly driven by them.

Unfortunately, the weaker claim – that behaviour is merely influenced by beliefs – allows for other internal or external non-belief factors to also influence actions, perhaps factors like feelings, heart rate, diet, or even planetary alignments. Without the stronger version of this assumption – that behaviour is primarily caused by beliefs – there can be no real teacher inconsistency to speak of, since any given behaviour may result from the non-belief factors that are unknown, and perhaps unknowable.

The fact that researchers have reported on inconsistencies implies an acceptance of the stronger form of the claim, which runs the risk of question begging or circular reasoning in its use. This circularity comes in the form of the 'affirming the consequent' logical fallacy:

If B exists **then** A will be observed. A is observed, **therefore** B exists.

Here B represents a teacher's belief and A represents their actions. That is, if a teacher has a certain belief (B) then they will act (A) in certain ways. Consequently if the teacher is observed to act in those ways (A), they must have the corresponding belief (B). This is identical to invalidly reasoning along the following lines: If a person is a billionaire (B), then they can afford to buy an apple (A). A person is observed to buy an apple (A), therefore they must be a billionaire (B).

Of course, a related argument could be constructed in the valid *modus ponens* form:

If A is observed **then** B exists. A is observed, **therefore** B exists.

In this case the argument runs: If a teacher acts in a certain way (A) then they must have particular beliefs (B). They are observed to act in that way (A), therefore they must have those beliefs (B). But this drastically alters the causal relationship inferred by the majority of researchers, arguing instead that actions cause beliefs, not beliefs causing actions. Some, such as Lloyd (2002) and Hart (2002), do make the case for the causal link running in this direction, that is, that changes in pedagogical practice brings about changes in beliefs, however most want to argue the reverse. The valid form of this reverse argument can be expressed in *modus ponens* form thus:

If B exists **then** A will be observed. B exists **therefore** A will be observed.

But this then runs into difficulties with the observations needing to be of beliefs, not actions. What seems to happen is that some authors assume strong B → A causation, and when the observed actions do not correspond to the inferred beliefs, contradictions are deduced. A similar situation, in terms of billionaires being able to afford apples, is that when a billionaire (B) is seen at a grocer unable to pay for an apple (A), the observer concludes she (the billionaire) must not be fully aware of her financial situation, instead of concluding that some other factors may be at play, such as the billionaire having lost her purse, or having left it at home, or only having foreign currency in it *etcetera*.

The closest valid rendering of the argument to what seems to be intended by these claims is the *modus tollens* form, wherein what is observed is the absence of an action assumed to accompany a given set of beliefs:

If B exists **then** A will be observed. A is *not* observed, **therefore** B *does not* exist.

Here the valid reasoning is that if holding certain beliefs gives rise to certain actions, and those actions are not observed, then those beliefs are not held. However, the conclusion is *not* that the teacher is inconsistent; it is that they did not hold the beliefs in question. This suggests that the problem lies in wanting to make two incompatible claims, that there exists a strong causal link between beliefs and actions, and that teachers can believe something yet act in a manner not driven by those beliefs.

If beliefs are the drivers of actions, then by definition there can never be any inconsistencies. Instead, any observed behaviour that appears inconsistent with the beliefs assumed to be driving a teacher's actions must arise from some other set of beliefs, otherwise they would not be acting in the way they were observed to act. The strong causal claim is therefore inconsistent with any claim of inconsistent actions, and the weaker causal claim is self-defeating in that beliefs are no longer the primary drivers of actions and therefore of diminished explanatory value.

DISCUSSION

The difficulties noted above arise from beliefs not being directly observable. Self-reporting of beliefs may not have a one-to-one relationship with actual beliefs, and it may be the case that other framing and filtering beliefs 'cause' teachers to undertake the act of reporting their beliefs to researchers differently to what they actually are. Connelly and Clandinin (1995) provide compelling grounds for why teachers might be unwilling to share their true beliefs, but whether they are unwilling or unable to state their true beliefs, it appears futile to ask mathematics teachers what those beliefs are.

Interactions between teachers and students are complex, but this complexity does not appear to be captured by the focus on discrete beliefs and limited observations that dominates the literature. While theories aim to simplify such interactions into more basic structures and patterns, it is inevitable that they will lose much of the richness and diversity that is present in the reality of individual classrooms. There is also a danger that overemphasising theoretical approaches will lead to simplistic interpretations and the discovery of gaps in pedagogical practice where they do not meet theoretical expectations, opening the door to conceiving of teachers in belittling ways rather than as sensible practitioners operating within a sensible system of beliefs. Such characterisations might be seen as what Schwab (1962) called the "rhetoric of conclusions" (p. 24) – stripped down knowledge claims that are neither theoretically nor practically useful to mathematics teachers, and which serve to alienate and disempower them, ironically, a direct inconsistency with what is intended.

CONCLUSION

Leatham (2006) argues that mathematics education researchers should be looking to build more comprehensive models of teachers' beliefs using teacher consistency as a guiding principle - looking not only for what teachers beliefs are, but also the ways in which they believe them, such as how strongly they are held, the links between them, and how they are grouped. He suggests mathematics teacher education should aim not just to replace or instil certain beliefs in student teachers, but also to make these desired beliefs the most sensible to adopt and follow in a coherent manner.

Fives and Buehl (2012) notably call for a shift from the predominantly cognitive approach to the study of beliefs to one that gives consideration to emotions and more affective elements of experience. This is effectively a challenge to the strong claim of causality between beliefs and behaviours, and potentially undermines the basis of much of the research to date. Arguably it may prove more fruitful to forgo an emphasis on beliefs entirely, since changes in mathematical pedagogy can culminate in changed beliefs (e.g., Kensington-Miller, Sneddon, & Stewart, 2014), and techniques which demonstrably improve students' learning and engagement with mathematics will be of great interest to mathematics teachers.

So while conceiving of beliefs more simply as truth claims that manifest as filters, frames, and guides may help remove some of the confusion that exists, it may prove more fruitful to look at grounding studies in which guides or strategies demonstrably improve student outcomes, with a view to utilising mathematics teachers as partners rather than as subjects of abstracted psychological scrutiny.

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EXPLORING HOW A MATHEMATICS LESSON CAN BECOME NARRATIVELY COHERENT BY COMPARING EXPERIENCED AND NOVICE TEACHERS' LESSONS

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We aim to clarify how a mathematics lesson can become narratively coherent by adopting a philosophy of narrative and qualitatively comparing two lessons conducted by experienced and novice teachers. We identified three characteristics of narratively coherent lessons: (1) the teacher stimulates the students to make sense of the present task by narrating their past experiences, but not by determining the future result; (2) the fundamental idea is enriched through the classroom interactions using multiple voices in a multi-layered way; and (3) the learning goal is focused on the conceptual meaning and oriented to interpret the concept based on the students' experiences. We conclude that the teachers' knowledge about a route of reconstructing the concept from students' experience leads to the construction of a narratively coherent lesson.

INTRODUCTION

As a part of a larger study clarifying the qualities of mathematics lessons referred to as "structured problem solving" (Stigler & Hiebert, 1999), we have explored how the lesson becomes narratively coherent (Okazaki et al., 2014). Several researchers have indicated that children's learning is narrative in nature, and that a quality lesson can be developed in a narrative form (Dewey, 1915; Stigler & Perry, 1988).

As Stigler and Hiebert (1999) identified, there is a pattern or script of mathematics lessons of "structured problem solving": reviewing the previous lesson, presenting the problem for the day, students working individually or in groups, discussing solution methods, and highlighting and summarising the main point. This script may play a role in making the lessons of structured problem solving. However, we should not directly equate the script with an effective mathematics lesson, because there is a range of teacher efficacy from effective to ineffective, even if the pattern is indeed adopted by most of the primary school teachers in Japan. Thus, it may be essential to clarify how such differences in teacher efficacy can be produced during conducting a lesson even when it is developed using the same script. In this paper, we aim to give insight into how a quality lesson can be produced by comparing two lessons conducted by experienced and novice teachers, as a study continued from Okazaki *et al*. (2014).

THEORETICAL BACKGROUNDS

We adopt a philosophy of narrative (or a philosophy of history) (Noe, 2005) as our theoretical background for understanding a quality lesson, where teaching may be regarded as a narrative act. A narrative includes the events, the contexts, and the time sequences, and a narrative act means "a speech act that plots the temporally distant

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events along a temporal order of beginning-middle-end" (Noe, 2005, p. 326). Bruner (1986) distinguished two modes of thought for constructing our reality: paradigmatic and narrative. He describes that the paradigmatic mode of thought "deals in general causes, and in their establishment, and makes use of procedures to assure verifiable reference and to test for empirical truth", while the narrative mode "deals in human or human-like intention and action and the vicissitudes and consequences that mark their course. It strives to put its timeless miracles into the particulars of experience, and to locate the experience in time and place" (p. 13).

An epistemological stance of a philosophy of narrative is that human acts of narrating or writing enable our reality to come into existence. In particular, humans identify facts through narrating their pasts. Conversely, the facts themselves detached from any context do not have their places in the history that we narrate. Namely, our experiences or facts do not exist without our speech acts of narrating, through which the isolated experiences are newly given their significances and become certain experiences in our memories. This theoretical stance expresses a criticism for regarding mathematics that may exist in the future as readymade entities.

Ontologically, the world then is not understood as a collection of facts, but as a network of events that are connected causally more than temporally. We mention the distinction between the materials and the plot in the narrative (Vygotsky, 1971; Karp, 2004). Materials refer to events and characters, which comprise the narrative, and plot means how these materials are composed as a narrative. We do not identify the factual and chronological sequences of the material with the plot because these often give rise to different emotions in the narrative. As the writer's work is to shape the events and to give the artistic arrangements to them, experienced teachers may compose the lesson by using and relating the students' activities and opinions with each other to allow students to "see and feel their inner significance more vividly" (Karp, 2004, p. 46).

Stigler and Perry (1988) stated that a well-formed story "consists of a protagonist, a set of goals, and a sequence of events that are causally related to each other and to the eventual realisation of the protagonist's goals. An ill-formed story, by contrast, consists of a simple list of events strung together by phrases such as 'and then…', but with no explicit reference to the relations among events" (p. 215). We emphasise that the protagonists are the students; therefore, their ideas and emotions are the central components of the narrative. It may be useful to examine the coherence of the lesson in terms of the learning goals collaboratively constructed by the teacher and students.

METHODOLOGY

We set the following three types for elementary school teachers and asked six teachers who corresponded to one of these three types to conduct a lesson: A) two experienced teachers who specialise in mathematics teaching, B) two experienced teachers who do not specialise in mathematics teaching, and C) two teachers who have a few years' experience. In this paper, we compare the results of our analysis of the lessons conducted by two of the teachers: Mr. F (type A, 35 years of experience) and Mr. S (type C, 3 years of experience).

We selected the content from a fifth-grade mathematics textbook, i.e., the ‘area of a parallelogram for which the height cannot be known from a straight line on its inside’ (Fig. 1, right). We assumed that students will have difficulty knowing the area of a parallelogram because of their difficulty in understanding the meaning of height. Moreover, we assumed that there are some differences in teachers’ behaviour related to their students’ difficulty.

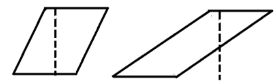


Figure 1

The lessons were recorded with video cameras and field notes, and transcripts were made. In our data analysis, we each first interpreted all meaningful events and interactions to examine what student responses a teacher’s questioning or instruction evoked, what experiences the students had, and how the teacher used such student responses and experiences in their subsequent lesson development. Next, we integrated our interpretations and identified the lesson scenes that were discriminable as units of activity/discussion, before trying to reconstruct the entire picture of the lesson structure, i.e., the ‘plot’. After that, we got together and examined each of the analyses of the events and interactions, the scenes, and the entire plot until all authors agreed. Finally, we compared the reconstructed lesson structures to clarify how the well- or ill-formed lesson narrative can be created.

RESULT 1: THE CASE OF MR F’S LESSON

We observed the eight scenes comprising Mr F’s lesson, which formed a coherent plot (Okazaki *et al*., 2014). We here present only the important scenes that constitute the plot in order to compare them with Mr. S’s lesson later.

Second scene: Setting a problem by an experience of conflict

Mr F presented a problem as follows, after reviewing the known area formula at the first scene.

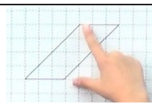


Figure 2

Mr F: I have one issue with this. I am bothered by this parallelogram. Do you understand my concern?

Student 1: The previous parallelograms had this line. This time, we can’t draw this (line) (Fig. 2).

Mr F: I tried to find the height, but there is nothing there! Oh, there is no height!

Students: But, but… (Several students raised their hands to respond.)

Mr F: But, does the parallelogram have an area?

Students: Yes, it has an area.

Mr F: Yes, it does. This is a parallelogram. However, we cannot use the area formula because we don’t know the height. Don’t you feel like crying?

The problem setting was like the beginning of a narrative in which the students were involved in an issue troubling Mr F, where the two circumstances ('there is no height' and 'the area formula can't be used') were given as the problematic aspects.

Third scene: Setting a goal by comparing the known and the unknown

Mr. F proposed setting a learning goal. He clarified the task by aligning three parallelograms and confirming that the formula could now be used only for the 6×4 and 3×1 parallelograms, which were reviewed during the first scene (Fig. 3). Students could then set a goal: to find the area of a parallelogram of unknown height using a formula. This 'aligning' implicitly prepared the students with an insight for solving the problem by seeing the parallelogram as half of a 6×4 parallelogram and as four 3×1 parallelograms.

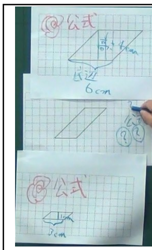


Figure 3

Fifth scene: Class discussion (1): Sharing the fundamental idea

The individual activities during the fourth scene were followed by a class discussion. We found that Mr. F employed one particular type of interaction in which he attempted to enrich the idea of using four 3×1 parallelograms using plural voices (Fig. 4). Mr. F's writing on the blackboard gradually became more detailed as he interacted with the different students. This series of interactions are multi-layered.

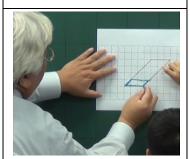
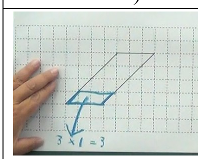
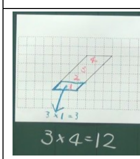
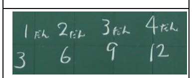
S2: Here is 1, 2, 3 and 4. Mr F: What is here? (He circled the bottom one.)		S3: The small parallelogram is 3×1=3. As there are 4, the answer is 12. Mr F: Can anybody else explain in the same way? (He wrote the formula.)		S4: The bottom one is 3. There are 4 parallelograms, 3×4=12. (He wrote the numbers.)		Mr F: What is the case of one step? Ss: Three. Mr F: What about for two steps? Ss: Six.
	⇒		⇒		⇒	

Figure 4

Seventh scene: Class discussion (3): Rethinking the goal

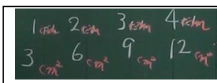


Figure 5

Mr. F proposed a rethinking of the main goal after confirming the other ideas during the sixth scene, and asked the students again to find the height of the shape. The students were not confident in their answer. Here, Mr. F told them to reflect on the idea shown in Figure 4, saying together with the students, "The height of the smallest one is 1 cm, the height of the parallelogram one step higher is 2 cm…" Moreover, he modified the table by changing the word "step" to "cm" and newly adding cm^2, indicating the area of each smaller shape (Fig. 5). Consequently, the students could reinterpret one "step" as 1 cm of height and then understand that the area formula that they already knew was actually applicable to all parallelograms.

RESULT 2: THE CASE OF MR. S'S LESSON

First scene: Reviewing the parts of the area formula and the transformation of a parallelogram into a rectangle for a known parallelogram

Figure 6

Mr. S began with reviewing the area formula for known parallelograms on which a grid of the unit squares did not appear (Fig. 6). We found that Mr. S's utterances were a series of questions that could be answered by one noun or adjective and which were strung together by phrases 'and then'.

Mr. S: Where is the height? What are the important points about base and height?

Student: Yes. It is vertical.

Mr S: Yes, it is vertical. It must be vertical. And then, what was the way to find the area of the parallelogram?

Student: It is base times height.

Here, the name of the figure, the base and height, the vertical relationship of the base with height, and the area formula were checked one after another. We note that the students' review of the meanings of the area, such as how many unit squares are there, was blocked because the parallelograms were presented without a unit square grid.

Mr. S then superposed two different parallelograms in Figure 6 and asked whether the two areas are the same. Student C responded that the areas were the same because the same formula can be applied. Then, Mr. S cut each part of the two parallelograms using scissors and moved them so that they became identical rectangles. He said, "even if the slopes are different, the areas of two parallelograms are the same given that the base and height are the same". We note that Mr. S confirmed that the two parallelograms result in identical rectangles, but did not explain why the two parallelograms were transformed into the same rectangle. As we see hereafter, Mr. S felt that "if the result is right, the process is also justified".

Second scene: Posing a problem to find the area as well as being aware of the resulting place of height

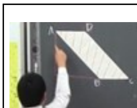


Figure 7

Mr. S posed a problem, to 'find the area of a special parallelogram' by comparing the unknown parallelogram with the known. The students noticed that the height was not inside the figure, and student C stated that the line from vertex A to the extension of BC was the height (Fig. 7). Mr. S authorised the opinion and mentioned, "We have to use the condition that the height must be vertical to the base this time". The height was conceived as "already determined".

Fourth scene: Setting a learning goal of transforming the parallelogram into the known figures

Mr. S first let the students read the summary of the previous lesson, ‘the area of a parallelogram is obtained by transforming it into the known figure’, which encouraged the students to notice that this strategy may be used this time. Then, Mr. S set a learning goal of ‘finding the area by transforming the parallelogram into the known figures’. Here we note that the learning goal is just to transform to the known figure, which contrasts with Mr. F’s lesson that aimed to reconstruct the unknown parallelogram based on the known area formulas.

In addition, Mr. S asked the students what the formula was, and one student said, “it’s 3 times 6”. In response, Mr. S stated, “That means that if we get the answer 18, it is all right to use this method (base times height)”. We again observed the misconception ‘when the result is right, the process can be justified’. The students then tried to solve the problem individually or in small group settings (**Fifth scene**).

Sixth scene: Class discussion: checking the areas of the transformed figures

Four ideas for obtaining the area were presented, where all the interactions between Mr. S and the students were very similar.

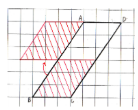


Figure 8.

Student Y: We cut the parallelogram in the middle and bring the bottom to the top. Then, it is 6 times 3. It’s 18. (Fig. 8)

Mr. S: Does everybody understand? Where is 6?

Student M: It’s at the bottom.

Mr. S: Six is here. We move it to here. It becomes double. So, 6 times 3 is 18. Please raise your hand if it is the same for you. Great! Let’s applaud Y!

We can identify two characteristics of the interaction. One is that the interaction was a simple one-response type exchange between Mr. S and the particular students: 1) some student presents his/her idea; 2) Mr. S explains the idea or asks one or two simple questions to confirm the idea; and 3) Mr. S authorises the student’s opinion by emphasising the area formula. Another is that Mr. S always ended the interaction by referring to the formula. Therefore, it seemed that giving the formula was for Mr. S a ritual or symbol for determining the idea and finishing the interaction. We also note that Mr. S checked just the formula of the ‘transformed’ figure (6 × 3), but he had never referred to the area formula of the parallelogram in question (3 × 6).

Seventh scene: Summarising the main points: Discrepancy between the learning goal and the summary

Mr. S requested the students to agree that the area of the original parallelogram was 3 × 6 without having discussed it, with the only reason that the answers of all ideas are 18. Here, we also observe in his utterance the misconception ‘if the result is correct, the process is also justified’.

When he stated, “the height is outside the figure” in writing the summary, several students questioned the words, saying "what is the meaning of 'outside'", "I am not sure that the height is not on the base". These students' doubts during the last scene suggest that the lesson was not coherent in the students' minds. This incoherence may be a result of the discrepancy between when they explored the transformed figure and when Mr. S summarised the formula of the pre-transformed figure.

DISCUSSION

We have seen two lessons that were different in quality and followed the same lesson script (Table 1).

J-script	Mr. F	Mr. S
Reviewing the previous lesson	Reviewing the meaning and application of the formula for the area of a parallelogram	Reviewing the parts of area formula and the transformation of parallelogram into rectangle
Presenting the problem for the day	Involving the students in two issues 'there is no height' and 'the area formula can't be used'	Posing a problem to find the area as well as being aware of the resulting place of height
	Setting a learning goal of grasping the unknown parallelogram based on the known figure	Setting a learning goal of changing the parallelogram into the known figure as well as being aware of the resulting formula
Students working individually or in groups	Individual activities and redefining the goal	Individual and small group activities
Discussing solution methods	Enhancing the fundamental idea in the multi-layered interaction and rethinking the goal	Checking the 'transformed' figure and its formula in the one-response type of interaction
Highlighting and summarising the main points	Applying the formula to other figures and summarising the main learning points of the day	Summarising the new area formula: discrepancy from the students' experiences

Table 1: Comparing the experienced and novice teachers' lesson scripts.

We identify several points for how to improve the narrative coherence of the lesson by taking Mr F's lesson as an exemplary case and comparing it with Mr. S's lesson.

First, we find that a quality lesson can be produced through making sense of the present task by narrating the students' past experiences, where the experiences are used as the materials for an unfolding plot. In fact, the students in Mr. F's class causally related the way to find the area, the concept of height, and the meaning of area formula among each other. This was the opposite to Mr. S's lesson, in which the future results of the area formula were exposed from the beginning, and the students confined their activities to obtain the area value independently of the future result. As a result, the students' contributions remained isolated in Mr. S's lesson.

Second, as an analogy of how the narrative is usually constructed by the involvement of multiple characters particularly in the middle of the lesson, it may be important for teachers to consider how many students as possible can actually be involved in a lesson. In Mr. F's lesson, one idea was enriched through the multi-layered students' voices. It may be appropriate to compare Mr. F's teaching method to getting as many students involved in the lesson as possible as a way of constructing a 'living theatre' (Okazaki et al., 2014), with the students as the main actors on the classroom 'stage'. Conversely, Mr. S interacted with only one or two students when one idea was presented, in a series of a one-response type of interaction, where giving the area formula was not used to enrich the meaning, but as a symbol of finishing the interaction.

Third, we find that the coherence of the lesson depends on the development of learning goals collaboratively set by the teacher and students because the narrative is followed from the viewpoints of protagonists. In Mr. F's lesson, the students' learning goal developed towards making sense of the unknown height and reconstructing the area formula by using the idea of how many parallelograms of 1 cm height would need to be stacked, whereas in Mr. S's class, the goal remained as finding the answer and transforming the parallelogram into the known figure. As a result, the students in Mr. S's class felt confused by the gap between what they experienced and Mr. S's summary of a new area formula. Also, we often observed the misconception, 'if the result is correct, the process can be also justified', which is not guaranteed in mathematics. We thus conclude that the teacher's knowledge about mathematics and the route for reconstructing the target concept based on their students' knowledge is essential for realising a narratively coherent lesson.

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PROFESSIONAL DEVELOPMENT BY EXPERIENCING THE OBJECT OF LEARNING

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In this paper, I present a model that can be used for supporting teachers' reflection on practical situations they are confronted with. The model is grounded in two concepts from variation theory: critical aspects and dimensions of variation. Analysis of the data allows for determination of what kind of reflection is used in teachers' professional development when working with algebra modules, and the teachers′ perceptions of the relevance and usefulness of the professional development concerning algebra modules. The results show that effective professional development focuses on improving instructional practice by giving teachers new knowledge and techniques for assessing learning with the ultimate goal of improving the learning of students. The results also show that the teachers practiced reflection-in and on-action.

INTRODUCTION

Sweden participates in various international knowledge measurements extending over several years and covering several thousand students. Swedish student scores in international comparative tests have declined every year since the 1990s. The latest international knowledge measurement of students' knowledge of Mathematics, TIMSS 2011, showed that the Swedish students' knowledge of mathematics was at the same level as the previous measurements, but still below the EU/OECD average. The results for PISA in mathematics had deteriorated further and the most recent measurement of the PISA 2012, showed that the results continued to fall for Swedish students. Intensive efforts are underway to reverse the negative trend. One of these is "Matematiklyftet" (Mathematics lift) that was initiated by The National Agency for Education in Sweden. Mathematics lift is a professional development program for teachers who teach mathematics (PDM), with the aim being to strengthen and develop the quality of teaching and thus increase student achievement.

BACKGROUND

Databases, including among other ERIC and the Social Science Citation Index, were searched, for the terms teacher education and teacher development from the year 2000 to the present. The content of journals were also reviewed in the areas of teacher education and professional development (PD), with a particular focus on mathematics teacher education. A study by Saxe, Gearheart and Nasir (2001) compared three groups: two PD programs on elementary school students' understandings of fractions plus a control group. The two PD programs groups offered teachers' opportunities to work with other teachers around implementing a reform curriculum unit on fractions. One program also included a focus on subject matter knowledge for the teachers, pedagogy, and student thinking. The control group used a traditional textbook and methods and received no PD support or time to work with others. The researchers found that students of teachers in the group that included a focus on subject matter knowledge for the teachers, pedagogy, and student thinking showed the greatest gains in conceptual understanding of fractions. This work indicates that integrating new knowledge for teachers around pedagogy and content, along with time to work with colleagues in meaningful, guided ways, is one way to provide effective PD that impacts students in positive ways. Effective PD focuses on improving instructional practice by giving teachers new knowledge and techniques for assessing learning with the ultimate goal of improving the learning of students (Wei, Darling-Hammond, Andree, Richardson, & Orphanos, 2009).

Fishman, Marx, Best and Tal (2003) describe a successful science PD program, which focused on student improvement, content and pedagogical knowledge, time for teachers to work together and learn, and a way to document improved outcomes. Other researchers (e.g., White, Lim & Chiew, 2006; Andrews, 2006) found that when development takes place in the classroom, teachers build practical skills both during initial teacher education and in the course of PD. Effective changes have been recorded for teachers who write role plays based on their classroom experiences.

Cordingley, Bell, Thomas and Firth (2005) carried out a rigorous systematic review of seventeen studies of collaborative PD in various contexts. They found that when teachers engage in collaborative PD, there was improvement in students' learning and behaviour, and in teacher's practices, attitudes and beliefs. Similar populations of teachers engaged in individually-oriented PD did not achieve the same outcomes: there was only weak evidence of change. They also found that collaborative PD worked best when outside expertise was brought into the teaching context, and when outside providers developed fruitful and respectful partnerships with teachers. Similar results were found by, for example, Evans et al. (2006), and Kirkwood (2001).

Yoon, Duncan, Lee, Scarloss and Shapley (2007), examined nine studies of PD efforts to determine how much time is necessary for an impact. They noted that when efforts were less than 30 hours, they showed no significant effects on student learning. Efforts that ranged between 30 and 100 hours, with an average of 49 hours, showed positive and significant effects on student achievement. They also found that PD efforts that were directly related to a teacher's practice, that were integrated with other school reform efforts, and that engaged teachers in collaborative communities, were also more effective.

Mathematics lift is a PD program for all teachers in Sweden who teach mathematics and is to be implemented between 2012 and 2016. The starting point of the program is collegial learning. It takes place locally at the schools and is closely linked to teachers' regular work. All that they are reading, discussing and planning is tried in their own teaching. Teachers work with various modules consisting of didactic material containing planning and discussions on mathematics teaching. One of these modules is Algebra. The focus in this paper is to describe the didactic idea used to design the algebra modules and some preliminary results.

METHODOLOGY USED IN CONSTRUCTION OF ALGEBRA MODULES

The ground idea in the algebra modules is located in the concept of reflection. There are different reasons to affirm that reflection is important for teacher professional development. Reflection is necessary for the teacher to recognise his own ideas on teaching concepts and to link them to the more scientific theories about teaching (Gallacher, 1997). It allows the teacher to become more aware of his own teaching by make it explicit to colleagues and students, and to develop a way of teaching that is more focused on the learning of the students. Reflection needs to include a confrontation with scientific knowledge about teaching and learning. To create opportunities for teachers to go through a reflection process, the approach used in algebra modules was to use two concepts from the theory of variation (Marton & Tsui, 2004; Marton, 2014): critical aspects, and dimensions of variation.

From a variation theoretical perspective, it is the object of learning that is the focus in a teaching situation. An object of learning has two constituent parts: the direct (is defined in terms of content) and indirect (refers to the specific capability that students are expected to develop) objects of learning. The object of learning is formed of the intended (refers to the part of the content that students should learn and which is supposed to be treated in the classroom), the enacted (is what appears in the classroom and refers to what is possible for students to experience within a learning environment), and the lived object of learning (the students' initial level of capability to the appropriate object of learning as well as the way in which students understand the object of learning). The intended and enacted objects of learning can be compared to determine whether what is being taught matches what was intended to be taught. If reflection is supposed to promote PD of teachers, then the object of learning has to be broad and deep enough. To make this possible, reflection needs to be systematic. Systematic reflection is not the same as the regular spontaneous activity of the teacher at the end of a lesson or at the end of the day, without the systematically work with the object of learning. Instead, systematic reflection in this context means to give the teachers experience with the object of learning in the PD program. An important aspect relating to systematic reflection concerns the moment of reflection. Schön (1983) makes a distinction between reflection-in-action and reflection-on-action. The process of reflection on-action, where teachers are encouraged to bring their acquired knowledge to the level of consciousness and thereby take their (teaching) actions more directly under their own control, is a component of the intended and enacted object of learning. The process of reflection in-action is seen as a component of intended and enacted object of learning.

The central idea in variation theory is that to discern certain aspects of the object of learning, a person needs to experience variation corresponding to those aspects (Marton & Tsui, 2004). Some of those aspects are critical aspects in students' learning.

A critical aspect is the capability to discern aspects related to the object of learning by experiencing them. For example, to experience an equation is to experience both its meaning, its structure and how these two mutually constitute each other. So neither structure nor meaning can be said to precede or succeed the other. If these aspects are not focused on in a teaching situation or in textbooks, they may remain critical in the students' learning (C. Olteanu & L. Olteanu, 2012, 2013). Researches results (e.g., Marton 2014) point out that, it is very important that the teacher is able to bring critical features of the object of learning into students' focal awareness.

The theory of variation serves as a useful theoretical framework to help teachers plan and structure their lessons (Olteanu, 2014). It guides them to decide what aspects to focus on, which ones to vary simultaneously, and which to keep invariant or constant. Furthermore, it guides teachers to consciously design patterns of variation to bring about the desired learning outcomes. Marton and Tsui (2004) argue that in order to discern different aspects of the object of learning, variation must be experienced in these aspects. An aspect is defined as the capability to discern the whole, the parts that form the whole, the relation between the parts, the transformation between the parts, and the relation part-whole for a mathematical concept or between different concepts (C. Olteanu & L. Olteanu, 2012, Olteanu, 2014). Previous research (Marton et. al., 2004; Olteanu, 2014) mentioned five patterns of variations which can facilitate students' discernment of critical features or aspects of the object of learning: (1) Contrast (in order to experience something, a person must experience something else to compare it with); (2) Generalization (is to see variations in the use of the object to fully comprehend it and involves recognizing that some features are not critical to the identification of that phenomenon); (3) Separation (an aspect must vary while other aspects remain invariant); (4) Fusion (several critical aspects need to be considered together); (5) Similarity (the property of two or more expressions to adapt the same meaning). The idea in algebra modules is that to improve the quality of reflection it is necessary that the teachers work to open up dimensions of variation in the aspects that are supposed as critical in students learning. Good reflection is not only an individual process, but equally a social event. This paper focuses on the following questions: What kind of reflection is used in teachers professional development when working with algebra modules?; What were the teachers´ perceptions of the relevance and usefulness of the professional development concerning algebra modules?

ALGEBRA MODULES

In this paper I briefly describe the structure of the algebra modules. There are three modules: grades 1-3, 4-6 and 7-9 and have the same structure. The difference between the mathematical content is specific for each grade and in relation to the present curricula in Sweden. The modules consist of eight parts and it is convenient to work with some parts for two weeks: Part 1- Reflection as a learning process; Part 2- Reasoning ability; Part 3- Assessment for developing the teaching of algebra; Part 4- Interaction in algebra classroom; Part 5- Algebra as a language; Part 6- Socio Mathematics norms; Part 7- Communications in Algebra classroom; Part 8- Final reflection and evaluation. Each part focuses on a specific algebraic content linked to the present curricula in Sweden, and each part consists of four sub-parts. In sub-part A (45-60 min), the teachers individually read a number of texts, look at example, films, mathematics tasks or problems. Sub-part B (90-120 min), is a collegial learning. The teachers discuss the content they read individually; then they shall jointly plan a lesson or an activity that will be implemented in their regular teaching. The idea is that teachers, together, discuss the activities planned, giving each other advice and suggestions by sharing their knowledge and experiences. In sub-part C, the teachers complete the activity/lesson planned in sub-part B in their own class. In sub-part D (45-60 min), teachers discuss and reflect together over the completed activity/lesson. The teachers also conclude what they have learned while working on the part. For large geographical distance it is possible to use technical solutions, such video conferences, for the collegial meetings.

DATA COLLECTIONS

Eight schools (two for each school year group) and 52 teachers participated in a preliminary evaluation of mathematics lifting during the autumn 2013 and spring 2014. The data was collected through interviews with teachers (total 52 teachers: 18 teachers from grade 1-3, 17 teachers from grade 4-6, and 17 teachers from grade 7-9), observations in classrooms (total 8: two for each school year group), collegial conversations (total 8: two meetings for each school year group) and documents from teachers. It was collected at two different times in each school, when teachers worked with two of the module parts. On each occasion the teachers' work with sub-parts B and D and they conducted an activity/lesson in sub-part C. Group interviews, with semi-structured questions, were the primary method of data collection. This allowed participants to ask for clarification from the researchers. During class observations and collegial conversation only the teacher's voice was recorded on account of ethical considerations. Johnson and Onwuegbuzie (2004) argued that using mixed methods allows varied sources of data to be collected and provides the opportunity for the triangulation of data, which can work to address any potential weaknesses that may be inherent in a single method approach and provides opportunities to test the consistency of research findings. After transcription the analysis of the data began with the processes of bracketing and coding in order to identify themes. During this phase of study, key statements that relate directly to the evaluation are identified. The key statements are interpreted and then examined for what they reveal about the recurring characteristics of the professional development. When bracketing was completed, the data were aggregated according to the themes that had emerged.

PRELIMINARY RESULTS

The results show that during the professional development the teachers used both a reflection in-and-on-action. The reflection in-action are expressed in terms of feedback, taking notes, setting up checkpoints in form of critical aspects, adjusting to improve the teaching of algebra by creating dimensions of variation in the supposed critical aspects. With reflection-in-action, teachers examine their experiences and responses in the collegial learning (sub-parts B and D). The teachers' descriptions of their reflective processes during their interactions with colleagues and students clearly indicated that they practiced reflection-in-action:

> While engaged in a conversation with my colleagues I may be reflecting on how I can restructure my questioning so they understand what I am saying.
>
> I always think about what I am doing and how the students respond.

Several teachers described the strategy of note taking in sub-part A as a tool for reflecting-in-action that assists them in fulfilling the goal of future activity (sub- part B and D).

> I spend some time looking at my notes. . . . I am always building my dimensions of variation for the next meeting and I reflect on critical aspects.

In the case of reflection-on-action, teachers consciously review, describe, analyse and evaluate their past teaching practice (sub-part C). The teachers described, for example, that they reflect of their teaching approaches by determining the effectiveness of their strategies or to consciously use to identify the critical aspects and vary the content for further understanding of students learning.

> When I am teaching . . . if students . . . look indifferent, it is an indication that I need discern a critical aspect.
>
> I am always thinking about ways to improve my teaching. I am always thinking about what it would take to understand the use of communication in algebra.

The teachers are looking back on the situation that has occurred in the classroom when they reflect on action. For example, the teachers described that they ask their students questions about discerned aspects of the object of learning or teaching strategies (the use of dimension of variation), for actively engage their students in their own learning, after which they collected student feedback to determine the appropriate next steps.

> Reflection helps me do self-evaluations. I have always tried to make sure that I am self-improving.
>
> What previously took 15 lessons to get students to understand now take a lesson!

The teachers´ perceptions of the relevance and usefulness of the professional development is expressed as: 1) explicit changing approaches to teaching and leave the textbook often; 2) mathematics, we can already, but have learned to vary the content; 3) understand more the thought patterns that may underlie students' wrong answers; 4) new ways of thinking; 5) the ability to argue has been used more often; 6) deeper discussions with students; 7) gained in-depth knowledge of mathematics education; 8) to provide an opportunity to learn from other teachers; 9) got suggestions on how to develop and work on lessons; 10) reasoning about critical aspects. The teachers recognize that the use of concepts from variation theory give the possibility to become more aware of his own teaching by make it explicit to colleagues and students, and to develop a way of teaching that is more focused on the learning of the students. Referring on students the teachers argued that:

> Mathematics lift has meant that students are more daring, more students have had the opportunity to demonstrate their abilities. The design also works well in larger groups and more students dare to be wrong.

Moreover, it is argued that: students will explain more and ask more; some students have "switched on" and become creative; and that the collaboration between students has increased.

SOME CONCLUSIONS

For professional development to lead to substantial teaching changes and improvements in student learning, it needs to (1) integrating new knowledge for teachers around pedagogy and content, along with time to work with colleagues in meaningful, (2) include time for teachers to reflect and collaborate during the professional development, (3) give the teachers possibility to evaluate their past teaching practice. Every school has its own unique context, and this context needs to be considered carefully in professional development.

The preliminary results show that is three aspects of teacher development that make the most difference to teachers' effectiveness, originality and enthusiasm. First, it gives teachers time to collaborate with other teachers and school colleagues. Second, it allows more sustained learning and professional development to occur since it becomes part of the work rather than an additional piece of work. Third, it allows work to be well integrated in a meaningful, concrete way that addresses specific problems teachers have in their own classroom and links to the object of learning.

The structure used to construct the algebra modules is differentiated according to the diverse needs of teachers of mathematics, and considering that the teachers have different pedagogical skills, mathematical knowledge and experience of teaching and aspirations. Algebra modules are sufficiently flexible to allow teachers to recognise and work on their different needs, and provide a structure within which teachers can identify their needs and how they might be met. Algebra modules engage teachers in reflective practice in their own classrooms and provide the basis for teachers' learning to become generative so that their knowledge and practice continue to grow and evolve.

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MATHEMATICS COMMUNICATION AND CRITICAL ASPECTS

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This paper deals with one prominent topic in the field of mathematics education: the communication in mathematics. In this article, a framework is proposed for analysing the effectiveness of communication in mathematics classrooms. The presentation is based on data collected, during a 3-year period, and consists of the students'tests, the teachers'lessons plan and reports of the lessons'instructions. In the analysis, concepts relating to variation theory have been used as analytical tools. The results show that: effective communication occurs in the classroom if it has the real critical aspects in student learning as its starting point; teachers develop new strategies to present the contents by having the focus to open up dimensions of variation.

INTRODUCTION

This paper is based on observations collected from three scholar years, 2007 (autumn semester) -2010 (spring semester). The main idea of the project has been to provide continuous professional development of mathematics teachers from pre-school to upper secondary school that really address what teachers need to develop in their teaching and identify the mechanism for an effective communication.

The three papers by Olteanu and Olteanu (2010; 2011; 2012) provide rich descriptions of the potential for analysing classroom activity through the lens of embodied communications. The three papers collectively add some analytical conceptions that are shown to clarify and help us to understand that teachers need to develop in their teaching how to analyse and reflect upon the indirect object of learning. An object of learning has two constituent parts: the direct and indirect objects of learning. The first part is defined in terms of content and the latter refers to the specific capability that students are expected to develop (e.g., Marton & Tsui, 2004). Regarding teaching quality, Krainer (2005) concludes that the teachers themselves have to work all the time for what constitutes good mathematics teaching. Olteanu and Olteanu (2011, 2012) give examples of how a teacher's actions in the classroom can be understood by applying variation theory and which decisive the concept of critical aspects, that is the capability to discern aspects presented, by experiencing them, can play in the context of professional development.

Day (1999) argues that: "teachers cannot be developed (passively) […] they develop (actively)" (p. 2). All we can do is to provide opportunities for teachers to change their teaching. In this paper, in particular, the following research questions have been pursued: What aspects of the object of learning do teachers hold towards professional development?; What previous experiences influence the communication in the classroom?; What are the mechanisms for an effective communication?

BACKGROUND

From a variation theoretical perspective, the object of learning is formed of three components (Marton & Tsui, 2004): the intended, enacted and lived object of learning. The intended object of learning refers to the part of the content that students should learn and which is supposed to be treated in the classroom. The enacted object of learning is what appears in the classroom and refers to what is possible for students to experience within the learning environment. The students' initial level of capability to appropriate the object of learning as well as the way in which students understand the object of learning is the lived object of learning. Olteanu and Olteanu (2010) identified two types of critical aspects for the object of learning:

Potential critical aspects (PCA) or intended critical aspects are what teachers suppose to be critical aspects of students' learning.

Real critical aspects (RCA) or lived critical aspects are what students' exhibit as critical aspects in their learning.

For example, the teachers suppose that $\frac{3x+5}{3}$ may not prompt students to recall the cancellation property (PCA). So then, teachers do not focus on this aspect in teaching. Despite this, the students show that they incorrectly cancel 3 (RCA) (Olteanu, 2012). As mentioned, a critical aspect is the capability to discern aspects presented by experiencing them. For example "To experience an equation or a function is to experience both its meaning, its structure (composition) and how these two mutually constitute each other." (Olteanu & Olteanu, 2012). Compositionality is the property that the meaning of any complex expression is determined by the meanings of its parts and the way they are put together (Pagin & Westerståhl, 2011).

Researchers (Marton &Tsui, 2004; Olteanu & Olteanu, 2011)) have defined the patterns of variations which can facilitate students' discernment of critical features or aspects of the object of learning: (1) contrast (C) means that to discern a quality X, a mutually exclusive quality non X needs to be experienced simultaneously; (2) the meaning of separation (S) refers to the other dimensions of variation that need to be kept invariant or varying at a different rate in order to discern a dimension of variation that can take on different values,; (3) generalisation (G) means that to discern a certain value, X_1, in one of the dimensions of variation X from other values in other dimensions of the variation, X_1 needs to remain invariant while the other dimensions vary; (4) fusion (F) is to experience the simultaneity of two dimensions of variation; 5) similarity (SI) is the property of two or more expressions to adapt the same meaning. The critical aspects can be analysed based on six general categories: the whole; the parts that form the whole; the relation between the parts; the transformation between the parts; the relation parts-whole (Olteanu & Olteanu, 2011, pp. 9-10); the relation between different wholes (Olteanu, 2012, p. 6).

The study of communication, in general, is broad, with wide-ranging contributions from a socio-cultural approach, a process-oriented approach where the focus is on the transfer of messages, coding and analysis or a semiotic approach (e.g., Nilsson &

Waldemarson, 1990; Sfard, 2008). No attempt is made to summarise each researcher's position; rather the emphasis is on briefly defining notions that will be used later in the presentation of the synthesised framework. In this paper, a framework is proposed for analysing the special role the teacher plays in communicating the mathematical contents in classroom and for describing a structure to support teachers in understanding their practice and improving it. The hope is that this framework will be valuable not only to researchers interested in studying patterns of communication, but also for teachers who want to communicating the mathematical contents in their classrooms more effectively.

A number of international studies show that a deep gap seems to exist between educational research and what is going on in the school and the classroom (e.g., Kaestle, 1993; Kennedy, 1999; Pang, 2008; Olteanu & Olteanu, 2011). To diminish this gap several researchers have proposed frameworks based on lesson or learning study concept (e.g., Stigler & Hiebert, 1999). A learning study involves a group of teachers who undertake theoretically grounded collaborative action research on their own practice and is an iterative process following a given structure: 1) choosing and defining a specific set of educational objectives; 2) finding out the extent to which the students have developed the capabilities or values targeted before the teaching begins; 3) designing a lesson (or series of lessons) aimed at developing these capabilities or values; 4) teaching the lesson (or lessons) according to the plan; 5) evaluating the lessons (or lessons) to see the extent to which the students have developed the targeted capabilities or values; 6) documenting and disseminating the aim, procedures and results obtained.

The study presented in this article does not follow the above cycle but it is an extension because the iterative process has taken place for three years and several groups of students participated in the study. Teachers' group and the direct learning object are maintained constant while the groups of students vary.

SYNTHESIS OF A FRAMEWORK TO ANALYSE COMMUNICATION

The basic idea of the mathematical theory of communication, as developed by Claude Shannon: The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message that has been selected at another point Shannon, 1949, p. 31). The success or failure of communication is a matter of the relation between thought contents of speaker and hearer (Frege, 1918). Pagin (2008) argue that a communicative event is successful just if the terminal state corresponds to the initial state. According to Olteanu and Olteanu (2011), the process of meaningful interaction among the intended, enacted and lived objects of learning is an indication of whether the communication in the classroom is successful or not. This process is illustrated in Figure 1. Olteanu and Olteanu (2010) defined effective communication as: a process by which the teacher assigns and conveys meaning in an attempt to create shared understanding, […] the process of meaningful interaction among the intended, enacted, and lived objects of learning (p. 385). This means that, if you understand a new concept, or a new theory, or a hint, you interpret it in accordance with how it was meant, and if the interpretation is not so in accordance, it has resulted in misunderstanding rather than in understanding it. In order for there to be such a difference between interpreting correctly and interpreting incorrectly, what is in or not in accordance with what was meant must be established independently of the interpretation (Olteanu, 2012).

The model in Figure 1 has as a starting point the general categories of the object learning which makes possible to analyse the meaning of any complex expression by analysing the meanings of its parts and the way they are put together. These categories are, in turn, the basis for identified the critical aspects (potential and real) and for planned and experienced dimensions of variation.

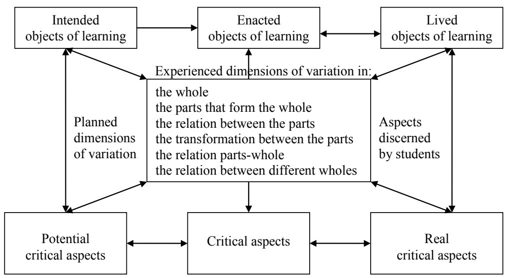

Figure 1. The interaction among the intended, enacted and lived objects of learning

RESEARCH DESIGN AND METHODS

In Sweden, the pre-school is for children up to 7 years old; the compulsory school is for all children aged 7-16 and is divided in Grades 1-3, 4-6 and 7-9. Upper secondary education provides a platform of knowledge for further studies and for a future career and is for students up to 19 years old. In upper secondary school the courses of mathematics are divided in A, B, C, D. During a 3-year period 22 teachers, from pre-school to upper secondary school, participated in a development project (6 teachers from pre-school, four teachers from Grades 1-3, three teachers from Grades 4-6, 4 teachers from Grades 7-9 and 5 teachers from upper secondary school).

The data was collected in 11 steps. The teachers examined the course module and curriculum to identify the intended object of learning, that is, what they planned to do during the lesson (Step 1). The teachers identified the object of learning, which in this article, is to simplify a rational expression (Step 2). The project continued by explaining various concepts used in the variation theory to the teachers and putting those concepts into practice (Step 3-4). Then, the teachers worked to identify potential critical aspects in students' learning (Step 5). Subsequently, tests and interviews were conducted with students to identify the real critical aspects of their learning (Step 6-7). Based on the identified real critical aspects and the difference between potential and critical aspects, the key concept of the theory of variation was explained again (Step 8). The teachers implemented six lessons (Step 9). After each lesson, the teachers wrote a detailed report using the following template: (I) General information: school, class/group, teacher, moment, object of learning, type of lesson; (II) General purpose; (III) Specific purpose: content, emotional view, psychomotor view; (IV) Prerequisites: technical aids, materials; (V) Lesson implementation according to teaching method (with focus on the opened dimensions of variation) and activities with students (Step 10). The students took different tests after the implementation of the lessons (Step 11). These steps were used in three phases of the project (Olteanu & Olteanu, 2010, 2011).

RESULTS

The qualitative data is explored by content analysis. In the first phase of the project teachers worked together to identify the potential critical aspects in students' learning and to intend the object of learning on the basis of these identified aspects. Their work was documented in written reports based on the following questions: What aspects are discerned by the students when simplifying rational expressions? What dimensions of variation can open up in aspects that are not discerned by the students?

At the beginning of the project (phase I), the teachers largely supposed that students did not discern the object of learning as a whole (A), the relation parts-whole (E) and the relation between different wholes (F). However, they do not consider that students need to better understand the constituting parts (B), the relation between those parts (C) and how to relate the parts to each other in a different way (D). Some example show that the teachers' description of potential critical aspects refers to: 1) **the whole**: calculations with natural numbers; subtraction; powers; equations; functions; derivative; 2) **the parts that form the whole**: numbers in decimal form; numbers as fractions; the exponent; constant term; variable; factors; 3) **the relation between the parts**: operations between the parts; parentheses; equal sign; argument; the operation between term in the numerator and/or in the denominator; the relation between the nominator and denominator; 4) **the transformation between the parts**: $21 - 12/3 + 8 \cdot 3 = 21 - 4 + 24$; $2(3 + 5) = 2 \cdot 3 + 2 \cdot 5$; $2(x + 3) = 2x + 2 \cdot 3$; to factorise the numerator and/or in denominator; rewritten an equation; factorise; 5) **the relation parts-whole**: $39 = 21 - 4 + 24 = 21 - 12/3 + 8 \cdot 3$; $16 = 2 \cdot 3 + 2 \cdot 5 = 2(3 + 5)$; $2x + 6 = 2(x + 3)$; value of function; coordinates and graphs; equation and solution; 6) **the relation between different wholes**: the relation between different numbers; the equivalent relation between two algebraic expressions; function and equation; derivative and function; graphs and function; graphs and derivative. The teachers assumed for example that students can discern the difference between terms and factors, and that only common numerical or algebraically factors would be cancelled in the simplification of a rational expression. Consequently, the teachers focused on the aspects in categories A, E and F and less or not at all in B, C and D. In addition, they only rarely mentioned to open up dimensions of variations in these aspects.

In the first phase of the project, none of the students discerned the aspects that teachers expected them. The students cannot work out the meaning of the whole because they have no understanding how the meaning of the whole is determined by the meanings of the parts and the mode of composition of the constituting parts.

In phase II and III the students improved their ability to discern different aspects of the object of learning from pre-school to upper secondary school. An explanation for this phenomenon is that in phase II the teachers focused on opening up dimensions of variation in the identified critical aspects. In six consecutive lessons, teachers focused on several aspects and opened up dimensions of variation by separation (S), contrast (C), generalisation (G), fusion (F) and similarity (SI). Some examples of the dimensions of variation opened up by contrast are: the difference between a fraction with unitary numerator and a non-fraction (e.g., $\frac{1}{x}$ and x); the difference between factorising a polynomial and solving an equation (e.g., $2x + 12$ and $2x + 12 = 0$); the difference between terms and factors (e.g., $2 + x$ and $2x$). Separation and generalisation was used, for example, when the teachers specified multiple times that only factors, and not terms, can be cancelled and identifying the common factor in the nominator and denominator. All dimensions of variations were used for give the students the possibility to discern that: the common factors can be simplified by any common numerical or variable factors (e.g., $\frac{2x+12}{2x} = \frac{2(x+6)}{2x}$); the use of parentheses around the nominator and denominator to highlight the whole; simplify fractions with polynomials in the numerator and denominator by factorising both and renaming them using the lowest terms (e.g., $\frac{2x+12}{2x+14} = \frac{2(x+6)}{2(x+7)}$). In addition, the teachers kept invariant the meaning of the object of learning in the classroom communication and varied the expression of the meaning. For example the teachers used the following questions when they worked with factorising rational expressions: What does factorising look like for a polynomial expression? How do we know when we are finished factorising? What is the process we use to cancel? What does cancelling look like? When do we know we are finished cancelling? All these questions have the same meaning, thus Thomas opened up a dimension of variation by similarity.

The design used in phase III was the same as in phase II and the teachers carried out the teaching in another class. Apart from the aspects focused on in phase II, the teachers focused on new aspects, such as, finding values of a variable for which an algebraic fraction is undefined as well as to understand the difference and connection between roots of a quadratic equation and factors of a quadratic expression. The enacted object of learning has enabled students to discern the aspects of the object of learning. For example the students could discern the process of factorising polynomials and to simplify algebraic expressions written as fractions. In addition, the students had the opportunity to experience: the term cancelling; that factorising is the reverse of the distributive property; both the expressions factor and cancel when working with algebraic expressions written as fractions; to use factorising, cancelling and rules of fraction operations to simplify algebraic fraction expressions. This led to a reduction of students' critical aspects in all categories.

CONCLUSIONS

In this study, a primary factors demonstrated to encourage teachers to use the research findings in the field of mathematics education in their practice was to open up dimensions of variation and identify critical aspects in students' learning. In this way, knowledge concerning the object of learning would be better connected and contribute to the specific subject matter field of research in mathematics education. The teachers who are involved in the project recognise that it is necessary to have a developmental theory of how students learn the mathematical content and understand how the theory relates to the development of knowledge and content. The identification of the real critical aspects in students' learning and opening up dimensions of variation in these aspects are key terms in this process. The teacher improve their own knowing of the meaning of all the parts of the object of learning and the semantic significance of the mode of composition of the object of learning by analysing the real critical aspects in students learning. The teachers are then able to put these pieces of knowledge together into knowledge of the meaning of the object of learning by opening up dimensions of variation.

The communication in the classroom succeed or not depending on the opportunities offered in the classroom to work out the meaning of the whole by knowing the meaning of the simple parts, the semantic significance of a finite number of syntactic modes of composition, and recognises how the whole is built up out of simple parts. In this way, it is possible for the teacher to create a meaningful interaction among the intended, enacted and lived objects of learning, which is to create a successful communication in the classroom. If the student knows the meaning of the simple parts, the semantic significance of a finite number of syntactic modes of composition, and recognises how it is built up out of simple parts, then s/he can work out the meaning of the whole. By reflecting on these general categories, the teachers constituted a complete learning object, in the sense that they were able to take up almost all critical aspects of the students' learning and open up dimensions of variation by contrast, separation, generalisation, fusion and similarity in those aspects. This resulted in an essential improvement of students' learning and successful communication in the classroom.

Implementing a lesson plan in relation to the report of the lesson with focus on creating dimensions of variation in the critical aspects of the content seems to be a powerful tool for the teachers' reflective process. This may therefore create some important mechanisms to support the cumulative nature of knowledge attained in mathematics education and its progressivity. What teachers learn about teaching is explicit and analytical than intuitive and imitative. These studies indicate the need to plan the mathematical content of the starting point in the students exhibited critical aspects. This is possible if teachers are constantly working with an iterative process in which they can discuss with each other and reflect on the implementation of lessons in relation to what students discern and what dimensions of variation are created.

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POTENTIAL FACTORS INFLUENCING SENIOR SECONDARY STUDENTS' USE OF CAS CALCULATORS IN MATHEMATICS

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The following paper reports on certain aspects of the quantitative analysis of data collected from 367 participants across six Victorian secondary schools in Australia. The data was collected using the Mathematics and Technology Attitudes Scale (MTAS) developed by Pierce, Stacey and Barkatsas (2007) which measures five affective variables examining students' learning with technology in mathematics. Using ANOVA techniques, statistically significant differences were found between the MTAS variables and gender, school, grade, year level and years of CAS experience.

INTRODUCTION

As we move deeper into the 21st century, the use of digital technologies have become such an integral part of the teaching and learning process they are now viewed more as "necessities rather than luxuries" (Bouck & Joshi, 2012, p. 115). As discussed by Hall (2010), there are a variety of technologies now accessible for teaching and learning within the classroom domain including electronic whiteboards, computers, laptops and calculators. Apart from their increased availability, research in the field of education has also "recognised the potential for mathematics learning to be transformed by the availability of digital technologies" (Goos & Bennison, 2008, p. 102). Guerrero, Walker and Dugdale (2004) noted:

> When technology is used well . . . it can have positive effects on students' attitudes towards learning, confidence in their ability to do mathematics, engagement with the subject matter, and mathematical achievement and conceptual understanding. (p. 5)

The potential benefits of technological resources in mathematics have also been acknowledged by educational organisations. In the 1996 'Statement on the use of Calculators and Computers for Mathematics in Australian Schools' by the Australian Association of Mathematics Teachers (AAMT, 1996), it was recommended that "all students have ready access to appropriate technology as a means both to support and extend their mathematics learning experiences" (p. 1). A publication by the Australian Curriculum, Assessment and Reporting Authority (ACARA, 2009), an educational body which shapes the writing of the National curriculum, also highlighted that "digital technologies allow new approaches to explaining and presenting mathematics, as well as assisting in connecting representations and . . . deepening understanding" (p. 12).

While the use of technologies have presented many advantages, Drijvers, Doorman, Boon, Reed and Gravemeijer (2010) expressed concern that the integration of technology within mathematics has fallen behind the promising expectations of the past two decades. In Australia, implementation of calculators equipped with computer

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algebra system technology (CAS) has faced various obstacles, despite becoming an important aspect of the senior secondary mathematics curriculum in the state of Victoria (VCAA, 2013). Factors such as student attitudes, teacher perceptions, time restrictions and the technical skill required to use the CAS have made integration difficult, and as such these technologies continue to play "a marginal role in mathematics classrooms" (Goos & Bennison, 2008, p. 103).

CONTEXT AND RATIONALE

In 2001, CAS calculators were introduced into Victorian secondary schools as part of a pilot study which aimed to investigate the effects that the use of 'supercalculators' would have on the senior mathematics curriculum (Stacey, McCrae, Chick, Asp & Leigh-Lancaster, 2000). Since then, the senior mathematics curriculum developed a new subject – Mathematical Methods (CAS) – which emphasised "the appropriate use of computer algebra system technology (CAS) to support and develop the teaching and learning of mathematics and in related assessments" (VCAA, 2013, p. 179). This technology is also expected to be used in the alternative subjects, Further Mathematics and Specialist Mathematics.

Geiger, Faragher and Goos (2010) highlight that CAS calculators hold many potential benefits to enhance the teaching and learning of mathematics:

> [They] not only have the capability to perform a wide range of mathematical procedures, such as function graphing, matrix manipulation and symbolic operations, but also the capacity to provide users with real time advice about errors as mathematics is done. (p. 48)

As a result, CAS calculators are not only a useful technological resource to complete mathematical work, but their time-saving capabilities also allow for a shift in the focus for learning to more conceptual understanding rather than the mastery of algebraic manipulations (Heid & Edwards, 2001). However, the advantages of CAS have been overshadowed by the polarised findings of educational research. As argued by Hall (2010), "development of a promising technology does not guarantee that it will achieve widespread use" (p. 232). While in some cases teachers and students have made use of CAS calculators successfully, others have encountered difficulties which have marginalised CAS use in the classroom. It is therefore important to examine the issue of implementation further with Hall (2010) proposing four essential questions in regards to the introduction of new technologies:

> Is it being used?
> How well is it being used?
> What factors are affecting its use/non-use?
> What are the outcomes?

While Hall (2010) refined these questions with respect to the change required to implement new digital resources, the student and teacher perspective in relation to these questions is also valuable as they are ultimately the users of these new technological innovations. Without understanding the obstacles faced by each within the mathematics classroom, the benefits of using CAS calculators are essentially lost. As summarised by Guerrero, et al. (2004):

> Technology has shown potential for positive effects on student engagement and achievement, on teaching techniques, and on the learning environment overall. [However], the extent to which this potential is realised relates to *how* [emphasis added] the technology is used within the mathematics curriculum. (p. 16)

The data analysis reported in this paper is part of a broader study which aims to explore students' use of CAS calculators in senior secondary mathematics and the possible factors which may influence their use. The purpose of the quantitative dimension of the study was to aid in the identification of potential factors (to guide subsequent interviews and classroom observations) and to determine any differences that may exist between the MTAS variables and gender, school, grade, year level and years of CAS experience.

METHODOLOGY

The questionnaire used in this study is the Mathematics and Technology Attitudes Scale (MTAS) designed by Pierce, Stacey and Barkatsas (2007). The questionnaire consists of 20 items divided into five subscales measuring the affective variables technology confidence (TC), mathematics confidence (MC), affective engagement (AE), attitude to learning mathematics with technology (MT) and behavioural engagement (BE). Four statements are allocated to each subscale and for each statement students indicate their extent of agreement on a five-point scale ranging from strongly agree to strongly disagree, or from nearly always to hardly ever (for behavioural engagement). Additional items relating to gender, school, grade, year level and years of CAS use were also added to the questionnaire.

To analyse the MTAS responses, each participant's overall score for each subscale was determined. This was achieved by adding together the scores for the four individual items in each subscale with values ranging from 5 (strongly agree/nearly always) to 1 (strongly disagree/hardly ever). Each participant can obtain a maximum score of 20 and a minimum score of 4 for each subscale. According to Pierce et al. (2007), subscale scores of 17 or above are considered to be high scores, indicating a positive response to the examined factor. Scores of 13-16 are considered to be moderately high, and scores of 12 of below are considered to be low scores indicating a neutral or negative attitude.

The 367 participants came from six secondary schools across Victoria, Australia. Three were government schools (two co-educational and one all girls'), two were independent co-educational schools and one was a catholic co-educational school. The questionnaire was administered to mathematics students in Years 11 and 12 (the final two years of secondary schooling) as these are the years in which the CAS calculator is used most extensively. A number of schools had less mathematics subjects on offer due to lack of student participation and other schools only had certain classes participate in the questionnaire based on teacher interest. Findings with respect to these schools have been made with caution as the responses provided may not be representative of the schools' senior mathematics student population.

RESULTS

Principal Component Analysis

Prior to conducting 'between groups' analyses, the 20 items of the MTAS were subjected to a Principal Components Analysis (PCA) in order to validate the scale and show that the items continue to load on the same component as seen in prior studies (Barkatsas, 2011; Barkatsas, Kasimatis & Gialamas, 2009; Pierce, et al., 2007). The PCA results revealed that the five components (all with eigenvalues greater than 1) explained 66.5% of the variance with the first component (mathematics confidence) contributing to 30.37% and the second component (attitudes to learning mathematics with technology) contributing to 13.97%.

To establish the factorability of the data, Bartlett's test of Sphericity (BTS) and Kaiser-Mayer-Olkin (KMO) measure of sampling adequacy were examined. According to Tabachnick and Fidell (2007), BTS should be significant ($p < 0.05$) and KMO values should be greater than 0.6. Analysis of the questionnaire revealed that both conditions were satisfied with BTS = 0.000 and KMO = 0.853. Additionally, each subscale was subjected to a reliability analysis. The Cronbach alpha values obtained were 0.912 (MC), 0.850 (MT), 0.784 (BE), 0.755 (TC) and 0.754 (AE) which indicated a strong to acceptable degree of internal consistency (Field, 2013).

Analysis of MTAS subscales

Analysis of variance (ANOVA) techniques were used to compare the means of each subscale against different variables (e.g. gender). In addition, post hoc analyses were also conducted to determine where the significant differences between each of the groups lie. Tukey's test was selected as it is considered one of the most commonly used post-hoc tests as it controls well for the Type I error and has reasonable statistical power (Field, 2013). Table 1 summarises the main findings from the ANOVA (p-values), highlighting where statistically significant differences were identified.

Post hoc analyses determined the following results:

- Boys achieved a higher average score on the technology confidence and mathematics confidence subscales compared to girls. (Note: data from the all girls' school was not included to remove the influence of a different learning environment).
- Significant differences between schools were evident for the mathematics confidence, affective engagement and attitude to learning mathematics with technology subscales.

	MTAS Subscales				
Variable	TC	MC	AE	MT	BE

Gender	0.001*	0.000*	0.442	0.638	0.189
School	0.122	0.000*	0.000*	0.001*	0.048*
Grade	0.923	0.000*	0.009*	0.137	0.000*
Year Level	0.027*	0.211	0.234	0.573	0.067
Years of CAS experience	0.003*	0.894	0.418	0.888	0.965

Table 1: ANOVA results for each subscale.

Students with grades in the A+/A range (80-100%) scored higher in the mathematics confidence, affective engagement and behavioural engagement subscales than students with grades in the B+/B range (70-79%) or C+/C range (60-69%).

Students in Year 12 scored higher on the technology confidence subscale than students in Year 11.

Higher technology confidence scores were evident if a student had used CAS calculators for 2 or 3 years compared to a student who had used CAS calculators for only one year.

DISCUSSION

Gender Differences

As the CAS calculator has become a more integral part of the senior mathematics curriculum in Victoria, Australia, there has been concern regarding gender equity and whether technology is accentuating gender differences in mathematics (Forgasz & Griffiths, 2006). The results obtained from the MTAS determined that there were statistically significant differences between male and female students in the technology confidence and mathematics confidence subscales. Males achieved a higher average score than females on both affective variables, which is consistent with prior large-scale studies conducted by Pierce et al. (2007) and Barkatsas (2011). Schmidt (2010) also discovered gender differences after surveying upper secondary school students in Thuringia, Germany. With respect to CAS calculators, the study found that male students experienced fewer difficulties and made more use of this technology in other lessons as opposed to female students. It is possible that the greater difficulties encountered by girls when using CAS calculators may make it more difficult to develop confidence with this technology. Alternatively, the obstacles faced by girls may be the result of lower technology confidence. It is anticipated that subsequent observations and interviews with students will provide greater insights into these differences and how they affect students' use of CAS calculators as part of their mathematics learning.

Differences between Schools

Figure 1 summarises the main findings from the ANOVA and post hoc analyses. Statistically significant differences were found for the mathematics confidence, affective engagement and attitude to learning mathematics with technology subscales and these have been shown in the box plots below. It can be seen that there were variations between schools for each of these variables.

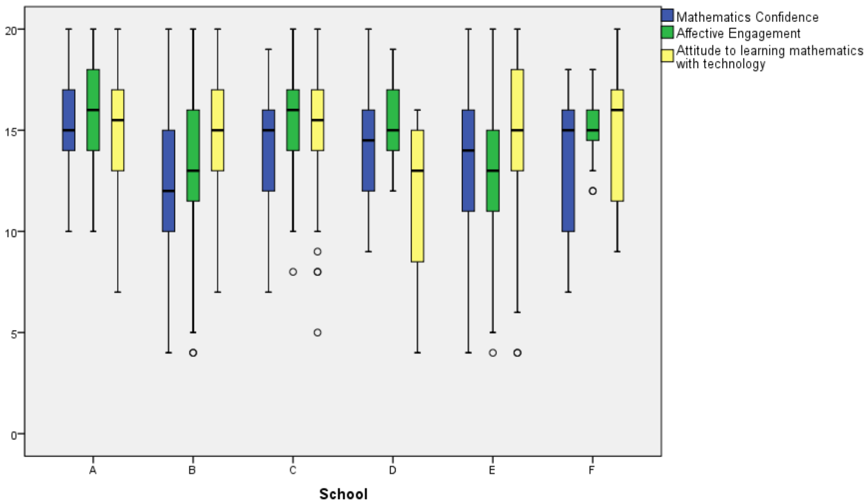

Figure 1: Mathematics confidence, affective engagement and attitude to learning mathematics with technology subscales by School

The key findings obtained from the MTAS data were that students in School B (Catholic, co-educational) obtained lower mathematics confidence scores than all other schools, students in School B and E (Government, all girls') obtained lower affective engagement scores, and students in School D (Government, co-educational) obtained a lower score on the attitude to learning mathematics with technology subscale. As the subscale on attitudes relates specifically to the CAS calculator, it is a point of interest to determine how negative attitudes may affect students' use of this technology in mathematics. It was also noted that School D had various students from low socio-economic families, which created issues of accessibility to the CAS. The difficulty in obtaining this technology, which is essential for 'technology-rich' assessments, may also have led to the development of negative attitudes in students.

Differences between Grades

Findings from the data analysis revealed statistically significant differences between student grades and the MTAS subscales mathematics confidence, affective engagement and behavioural engagement. Students who obtained grades within the A+/A range (80-100%) scored higher, on average, on the mathematics confidence, affective engagement and behavioural engagement subscales than students with other grades. These results are in agreement with the study by Barkatsas et al. (2009) who performed a cluster analysis to explore the interrelationship between student attitudes, gender, engagement and achievement. The authors concluded that "students with excellent mathematics achievement demonstrated very high levels of mathematics confidence [and] strongly positive levels of affective and behavioural engagement" (p. 569). However, Barkatsas et al. (2009) also noted that these students may be overconfident and may not consider technology to be beneficial to their mathematics learning - a point which has been explored further in the subsequent qualitative sections of this study (but are not reported here).

Year Level and Years of CAS experience

In a study by Barkatsas (2011), it was conjectured that "it may take at least two or three years for students to get accustomed to the complex functionality of CAS calculators" (p. 7). Results from the ANOVA supported these findings with statistically significant differences found in the technology confidence subscales for both Year Level and Years of CAS experience. Students in Year 12 scored higher, on average, for technology confidence than students in Year 11. Further, students who had used the CAS calculator for two or three years scored significantly higher, on average, for this subscale compared to students who had used this technology for only one year. It could be argued that the more time students have to familiarise themselves with the CAS calculator, the more confident they become with this technology. Although this subscale is not specific to CAS calculators, it still provides an avenue for investigation in the subsequent interviews and observations in this study. As different schools introduce CAS calculators at varying points in time (e.g. Year 9 or Year 11), it will be intriguing to determine how the years of experience have affected students' use of this technology in senior secondary mathematics.

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ENHANCING MATHEMATICS INSTRUCTION AND PROFESSIONAL DEVELOPMENT THROUGH LESSON STUDY

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Given the importance of improving mathematics teaching and the effectiveness of lesson study for professional development, this paper described how a specific lesson study was implemented in the Korean context. As such, it analysed in what ways a series of lessons had been changed as a group of five in-service teachers applied five practices for mathematics discussions to their teaching practices. This paper also analysed what the participant teachers learned through lesson study. It is expected to expand our understanding of lesson study and provoke international dialogue as for professional development of in-service mathematics teachers.

INTRODUCTION

Improving mathematics teaching to maximize students' meaningful learning has been one of the main issues in mathematics education. However, enhancing mathematics instruction is not an easy task even for committed teachers partly because it requires them to reflect on their teaching practices in a constructive way with deliberation. Lesson study with various forms in different educational settings has been reported as an effective tool to improve mathematics teaching and develop teachers (Hart, Alston, & Murata, 2011).

This paper describes a specific lesson study among a group of five in-service Korean teachers. Since the release of Korean students' outstanding performance in international comparative studies on mathematics achievement, there has been increased interest in teaching practices along with the competencies of Korean teachers. Some studies explored the characteristics of knowledge required for mathematics teachers in teacher preparation programs (Kim, Ham, & Paine, 2011; Kwon & Ju, 2012) or examined the quality of teacher knowledge both in mathematics and in pedagogy (Li, Ma, & Pang, 2008). Others focused on the characteristics of teaching practices in conventional lessons or reform-oriented lessons (Pang, 2009; Park, 2012).

However, there has been little known how to improve mathematical practices and develop Korean teachers' professional knowledge or beliefs in the international context. Against this background, this paper deals with a specific lesson study in which the participant teachers applied *5 practices for orchestrating productive mathematics discussions* of Smith and Stein (2011) to their mathematics instruction. As the group of teachers went through an iterative cycles of planning, implementing, and analysing a lesson of teaching mathematical problem-solving to sixth grade students, this paper focuses on the changes in lesson design and teaching practice in such a collective setting. Specifically, two research questions are examined as follows: (a) how have the lessons been changed as the teachers conducted lesson study? And (b) what do the

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teachers learn through lesson study in terms of lesson plan, implementation, and analysis? As teaching is a cultural activity (Stigler & Hiebert, 1999) and lesson study varies across different educational systems (Huang, Su, & Xu, 2014), this paper is expected to expand our understanding of lesson study and provide implications for professional development of mathematics teachers.

BACKGROUND TO THE STUDY

Professional Development for Mathematics Teachers in Korea

Due to the popularity and security of the teaching profession, only outstanding high-school graduates may enter teacher education programs in Korea. As long as they complete the four-year coursework requirements, they can gain a teaching certificate as a *second-class* teacher. They then have to pass a competitive National Teacher Employment Test to be a teacher in a public school. As the test requires prospective teachers to have professional knowledge and beliefs along with skilful teaching ability (Kwon & Ju, 2012; Pang, 2015), the quality of beginning teachers is high.

Once a teacher is employed in a public school, the job is secured until retirement. The teacher is expected to engage in two kinds of education programs throughout the teaching career: (a) qualification training and (b) duty training. First, a teacher has to take intensive training courses (more than 30 days and 180 hours) after teaching for five years. The successful completion of such courses authorizes the teacher as a *first-class* teacher. If the teacher does not plan to be a vice-principal or a principal, she or he does not necessarily seek for further qualification beyond the first-class teacher. Second, a teacher is expected to take several training courses in a yearly basis to refine her educational theory and practice. This is an optional training in practice. In fact, a survey by Park and Moon (2009) shows that about 40 % of the teachers did not take such training or at best took it less than 5 hours per year for recent 10 years. In this respect, developing expertise in teaching mathematics is rather voluntary than compulsory on the part of teachers. In addition, many training courses are criticized for being lecture-oriented and theory-oriented, which makes it difficult for teachers to apply what they learned from such professional development programs to their classrooms (Park et al., 2010).

Several approaches to enhance teacher expertise have been implemented such as (a) development of effective curriculum materials to set the basis of high-quality instruction, (b) voluntary activities of discussing lessons among groups of teachers at the same school, (c) instruction-research contests for teachers organized by educational offices in a province, and (d) appointment of chief teachers who play a leading role in upgrading the quality of instruction. These approaches are related directly to teaching practices, calling for teachers' active engagement and ongoing commitment to effective mathematics instruction.

Theoretical Framework to Enhance Mathematics Instruction

The recent curricular revisions ask mathematics teachers to enhance mathematical problem-solving, communication, reasoning, and creativity beyond emphasizing mathematical constructs which have been traditionally valued in Korea (MEST, 2011). Many teachers recognize these educational needs but do not know exactly what they have to do. It has been a challenge for a teacher how to foster students' development with regard to mathematical processes while teaching specific mathematical topics.

Five practices for orchestrating productive mathematics discussions (Smith & Stein, 2011) provide a teacher with a practical guide on what to do: (a) *anticipating* students' responses; (b) *monitoring* students' actual responses; (c) *selecting* students and their solutions to be presented during the whole class discussion; (d) *sequencing* students' presentation in a specific order; and (e) *connecting* students' responses to key mathematical topics to be taught. This paper analyses how these practices are adapted via lesson study. Lesson study includes iterative cycles of setting goals, studying curriculum and students, devising a lesson plan together, observing the implementation of the plan, and debriefing the lesson (Hart, et al., 2011).

The cycles of a lesson study are well-suited to the five practices above. Two basic components, setting goals and selecting tasks, are necessary before employing the five practices (Smith & Stein, 2011). To anticipate students' responses requires a teacher to study curricular materials and students. The five practices per se do not necessarily require joint planning or implementing of a lesson but collaboration with colleagues in implementing such practices in school context is desirable. While one teacher goes through from monitoring to connecting practices in a lesson, the other teachers in the group who observe the lesson may analyse the quality of five practices implemented.

METHODS

Participants and Setting

The participants for this study were from in-service teachers who enrolled in a graduate course, *studying elementary mathematics instruction*. A group of 5 teachers (3 female and 2 male) were analysed in this paper. Two of them had master's degree in mathematics education. The teachers read the book of five practices by Smith and Stein (2011) as the course requirement. They were asked to implement what they had learned from the book to their lessons via a specific lesson study as follows: (a) A mathematical topic related to teaching problem-solving is given to each group of 4 or 5 teachers; (b) Individual teacher prepares for a lesson plan which includes learning objectives and a series of instructional activities on the basis of his or her knowledge and research on curriculum and students; (c) The teachers meet as a group, discuss the pros and cons of each lesson plan, and develop a group lesson plan; (d) One teacher implements the lesson plan, while the other teachers observe the lesson; (e) The teacher who taught students writes a reflection report on what happened in the lesson, whereas the observers analyse the implemented lesson focusing on the five practices; (f) All the teachers discuss the pros and cons of the implemented lesson and then debrief the lesson in front of other groups of teachers to get some feedback; (g) The group of teachers revises their initial lesson plan on the basis of their own analysis as well as feedback from others; and (h) They go through the same process from (d) to (g) until all of the teachers in the same group teach a class once.

After this process, each teacher was interviewed with the following protocol: (a) what have you learned as for planning a lesson through this lesson study? (b) What have you learned as for implementing a lesson through this lesson study? (c) What have you learned as for analysing a lesson through this lesson study? (d) Is there anything interesting with regard to students' understanding of what you taught as well as teachers' instructional strategies?

Data Collection and Analysis

Five types of data were collected for this study: (a) both individual and collective lesson plans with several revisions produced during the process; (b) reports on group discussions about lesson plan and implementation; (c) videotapes of the implemented lessons; (d) individual teachers' reflection reports; and (e) interview data with individual teachers on their learning.

Lesson plans and reports with various versions were analysed to trace down in what ways the lessons have been changed using a grounded theory approach (Corbin & Strauss, 2008). Transcripts of the implemented lessons were analysed in terms of the five practices using an analytic framework developed by Pang & Kim (2013). Four levels can be identified per practice. For instance, as for the *connecting* practice, At Level 0 the teacher deals with various solution methods but does not connect them to key mathematical idea to be taught in the lesson, leaving each method isolated. At Level 1 the teacher connects solution methods one another without supporting students to do so. At Level 2 the teacher often provides students with too detailed or direct guidance for connection. At this level, meaningful connection between solution methods or between a solution method and the key mathematical idea in the lesson may occur in part. At Level 3 the teacher provides students with adequate questioning to help students draw meaningful connections. At this level, a full range of connections occur. The interview data were analysed in a way to understand the change of lessons and to identify the individual teacher's learning through lesson study.

RESULTS

Changes of Lesson Plan and Implementation

The lessons were changed in multiple ways as the group of the teachers conducted lesson study. First of all, the goals for the lesson were clarified from "students can choose an adequate solution method by comparing various methods and solve a problem using the method" to "students can (1) compare multiple solution methods while solving a problem in various ways, (2) understand the advantage of each solution method, and (3) solve a problem using an adequate method according to a given context. Note that the former goal tends to focus on a specific solution method, whereas the latter turns students' attention into comparisons among multiple solution methods as well as appropriate choice of method.

Second, the mathematical task in the lesson was changed both rigorously and meaningfully. The final version of the task used in the lesson was as follows: "The group of Jun-Gil decided to exchange a letter while shaking hands once one another during the special week of friendship. If the number of students of Jun-Gil group is 6, figure out the number of total handshakes." The teachers in this study discussed whether they would use the context of exchanging a letter and present the word 'once'. Note that the number of exchanged letters is double the number of handshakes, and this relationship is critical in understanding various solution methods used to solve the given task.

Third, the five practices were implemented in various ways. For instance, anticipating students' responses was well established at Level 3 from the beginning of the lesson study cycle because the teachers recognized the importance of such practice. The teachers developed a chart for monitoring individual students' work and effectively used the chart during the lesson study cycle. Owing to the chart the teachers were able to monitor more number of students and select whom to present and what in a specific order. However, sometimes the student selected by the teacher did not volunteer to present or changed his or her initial method while the teacher was monitoring other students. In these cases, the teachers had a difficulty to orchestrate the whole-class discussion in a planned order.

Connecting students' different solution methods remained the most difficult practice for most teachers. As for the handshakes task the students in this study used multiple problem-solving strategies such as drawing a picture, making a table, examining a simpler case, and writing an equation. During an early phase of the lesson study cycle, these strategies were presented but connected one another at best intermittently or superficially through the teacher's direct intervention (Level 1). For instance, the teacher asked students which strategy is more convenient between using addition (i.e., 5+4+3+2+1=15) and using multiplication (i.e., 6x5 divided by 2 equals 15). Only at the last phase of the lesson study cycle mathematically meaningful connections among strategies were examined on the basis of communication between the teacher and students (Level 3). For instance, the following episode happended after two solution methods (i.e., drawing a hexagon type of picture and writing an equation) were presented:

1	T:	How could you connect this picture to the multiplication expression?
2	S1:	In the picture there are 6 people and 5 lines per person, so it is 30. But drawing a line between 1 (first student) and 2 (second student) is the same as drawing a line between 2 and 1. So you divide [30] by 2 and get 15.
3	T:	S2, do you understand what S1 said? Why don't you explain?
4	S2:	If you represent the connection in the picture into an equation, as two people handshake at the same time, you need to divide by 2.

5	T:	Then where does 6x5 come from?
6	S3:	6 means the number of people and 5 means the rest of the people except oneself.
7	T:	Who can help? What is the meaning of 6x5?
8	S4:	One person can handshake with 5 people and there are 6 people. But you need to divide it by 2 in order to subtract the overlaps in handshaking

In the episode above, students were able to connect the picture to the equation (6x5 divided by 2). Note that the teacher kept probing the meaning of the equation in the picture representation. This kind of meaningful connection between students' solution methods was implemented owing to the considerate lesson planning among the teachers as a group. As the teachers recognized how difficult it was to connect a solution method to another on spot during an early phase of the lesson study, they prepared for a list of good questions during the lesson planning. They included detailed questions which were intended to explore mathematical meaning or relationships between different solution methods or representations. Thorough lesson planning made it possible for students to have the opportunity to identify the same mathematical idea underlying the different representations.

Teacher Learning through Lesson Study

First of all, all teachers recognized the importance of detailed lesson plan as a foundation of effective instruction. Specifically, they mentioned that instructional goals need to be clarified and high-level mathematical tasks to be accessible by different levels of students need to be used. The teachers also emphasized that a lesson plan should be full of the teacher's key questioning along with students' meaningful responses including their misconceptions. The following is an example from a teacher's reflection report:

> All I did in previous lesson planning was to insert the simple dialogue between the teacher and students along with the order of activities to be displayed to students. In the process of applying five practices to my teaching, I came to know that monitoring, selecting, sequencing, and connecting students' approaches result from anticipating their responses. This made my lesson plan be filled with students' mathematical interpretation, a series of solution strategies (both correct and incorrect ones), and connection between students' strategies or interpretations and mathematical ideas. My latest lesson plan was structured in a way to include my key questioning and students' meaningful responses.

Second, the teachers realized the complex nature of teaching practices, reminding that unexpected things always happened in every lesson, even though they planned it very carefully. A teacher emphasized that the degree by which a teacher deals with such unexpected things can be an important determinant of teacher expertise.

Third, the teachers understood that there are various perspectives in analysing the same lesson and that the focus of analysis should be on alternatives or better approaches. This led the teachers to recognize that the main purpose of analysing a lesson was to improve lesson plan and implementation or teacher's teaching expertise.

Finally, the teachers were able to develop a better understanding of students' responses. Specifically, they mentioned different solution strategies according to students' various mathematical ability. As noticing that most students used the strategy of drawing a picture in solving the handshake task, the teacher began with the common strategy and then connected it to more abstract strategies. The teachers also recognized that students can correct their mistakes for themselves while solving the given problem in multiple ways or extend their initial thinking by connecting their solution methods to others.

DISCUSSIONS

Lesson study urges teachers to analyse the strengths and weaknesses of teaching approaches implemented in one class and to come up with alternatives. In fact, the teachers in this study employed such alternatives through the process of lesson study and were able to teach subsequent lessons more productively. This happened easily partly because they taught the same content in different classrooms. However, from an individual teacher's perspective, he or she may not be able to re-teach the same mathematical content in the classroom. Note that an elementary school teacher in Korea charges one class and teaches main subjects such as mathematics in a daily basis. But mathematical processes such as problem-solving or communication may be re-teachable for the same students no matter what the mathematical content is covered. As mentioned above, the recent curricular revisions in Korea urge a teacher to teach both mathematical content and processes (MEST, 2011). This is a challenging task for a Korean teacher who has been focusing on teaching content. It is expected for a Korean teacher to employ lesson study in a way to enhance his or her teaching for problem-solving, communication ability, and reasoning. This may be counted as a cross-cultural variation of implementing lesson study.

Another noticeable remark is to regard the teachers as key stakeholders through lesson study. The teachers in this study learned the five practices in theory from a graduate course but played a proactive role in applying such practices to their classrooms through lesson study. Meanwhile, they were able to develop professional knowledge of lesson plan, implementation, and analysis as well as to understand the complex nature of teaching practice. These positive effects of lesson study are significant because many current professional development programs in Korea have little impact on teachers' growth or teaching practice (Park et al., 2010). As there has been lack of research on Korean teachers' professional development in the international contexts, this paper is expected to contribute to provoke discussion on the nature and characteristics of lesson study implemented in a different setting.

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