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Editors | Berinderjeet Kaur, Weng Kin Ho, Tin Lam Toh, Ban Heng Choy

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STUDYING PRESCHOOL CHILDREN'S REASONING THROUGH EPISTEMOLOGICAL MOVE ANALYSIS

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In this paper, we propose a theoretical tool for analysing mathematical reasoning using Epistemological Move Analysis (EMA) in combination with a framework focusing on arguments and the foundation of these. We also suggest the addition of evaluative arguments when talking about different types of arguments besides predictive and verifying arguments. The tool was applied on data of preschool children's mathematical reasoning. The results indicate that different types of epistemological moves are connected to the different types of or the lack of arguments, and will fill (or not fill) gaps that occurs in the reasoning.

INTRODUCTION

Research focusing on young children's mathematical thinking indicates that young children are more capable than previously has been reported when it comes to develop and demonstrate mathematical thinking including processes such as mathematical reasoning (Mulligan & Vergnaud, 2006; Säfström, 2013). Recent studies show that children can not only use different competencies in their reasoning (Sumpter & Hedefalk, 2015) but also that other skills do not have to be developed in beforehand (Säfström, 2013). However, looking at the development of mathematical thinking, there is evidence that children do not develop these competencies without someone providing the learning opportunity (Bobis et al, 2005). Mathematical reasoning is such a competence (Bergqvist & Lithner, 2012). Also, it has been indicated that if children have access to a guide, they are more likely go further in their mathematical thinking (Björklund, 2008) especially if that person is asking key questions (van Oers, 1996).

With regard to mathematical reasoning, one of the goals that Swedish preschools should aim for is that children "develop their mathematical skill in putting forward and following reasoning" (School Agency, 2011, p. 10). In order to so, following the idea of learning opportunity, teachers need to be able to pick up children's mathematical ideas (Bergqvist & Lithner, 2012; van Oers, 1996; Shimizu, 1999). This should happen independent of the activity is planned or informal since the key thing of Swedish preschool education (children age 1-5) is the emphasis of play and should not be formal schooling (School Agency, 2011). Previous research looking at education of mathematical reasoning, although on secondary level, reports that in Swedish teachers' presentations, most task solutions are based on algorithms with only rare opportunities to see aspects of creative mathematical reasoning (Bergqvist

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& Lithner, 2012). At present moment, we don't know how such results are translated to preschool level especially with informal settings as an important learning opportunity. Our future aim is to study the opportunities to develop different types of mathematical reasoning presented to children at preschool level, which would be a similar aim to Bergqvist and Lithner (2012). However, at preschool level such opportunities are most likely to occur in a play based education. Therefore, other theoretical tools are needed compared to Bergqvist and Lithner (2012). This is the aim of this paper: to propose and discuss a theoretical tool that would allow us to perform such an analysis. The tool needs to allow us to look at the conversations, interactions, between teachers and children and in particular the role of the teachers in these conversations, but at the same time focus on the mathematical reasoning and the different types of arguments in the reasoning. Here, we will test this theoretical tool on a subset of a data set to show different types of arguments in mathematical reasoning and teachers' role in these situations.

THEORETICAL BACKGROUND

We propose the parallel use of two theoretical frameworks. One framework helps us to study mathematical reasoning, in particular the different arguments in mathematical reasoning, that take takes place in conversations in play based activities. In order to study the conversations and the teacher's input, we use a method called Epistemological Move Analysis (EMA). The starting point for this study is, just as Bergqvist and Lithner (2012), an ecological perspective meaning that the teachers' choices or actions are not seen from a right/wrong dichotomy.

Mathematical reasoning

Young children's mathematical reasoning is getting more attention in research (Sumpter & Hedefalk, 2015), but a general problem in mathematical reasoning research is that mathematical reasoning is used to denote a 'higher quality' thinking without defining what this would encompass (Lithner, 2008). To avoid this, we use a framework that has a clear definition of mathematical reasoning and also allow different types of reasoning including those that are not based on deductive logic. Reasoning is defined as the line of thought adopted to generate assertions and conclusions when solving mathematical tasks (Lithner, 2008). This is a product and we see it as a sequence or several sequences that starts with the tasks and ends with an answer, where the answer could be no conclusion at all. When organizing the data, we use the following four step structure: (1) A task situation is met (TS); (2) A strategy choice is made (SC); (3) The strategy is implemented (SI); and, (4) A conclusion is obtained (C). Lithner (2008) has attached two types of arguments to two of these steps. The strategy choice can be supported with predictive arguments and the implementation with verifying arguments. The first type of arguments aims to answer the question 'Why will the strategy solve the task?'. The second type aims to answer the question 'Why did the strategy solve the task?'. While these two types of arguments focus on the strategy, no arguments focus on the conclusion and the evaluation of it: how and in what way is this an answer to the initial question? Inspired by the argumentation research in the field of artificial intelligence, we would like to add evaluative arguments to the different types of arguments. Evaluative arguments serve the purpose to persuade that something is right or wrong (Carenini & Moore, 2006). We suggest that evaluative arguments fill the void that occurs in the conclusion step answering the question 'How do the conclusion answer the TS?' We argue that evaluative arguments could function as part of control (Schoenfeld, 1985) or review (Polya, 1945) in problem solving. This is yet to be tested in this paper.

To be able to analyse the arguments, Lithner (2008) introduce the notion of anchoring. It is important to note that anchoring does not refer to the logical value of the argument since it allows us to talk about reasoning that is incorrect. This helps us to look at the foundation and how it is used (Sumpter & Hedefalk, 2015). Anchoring is seen as the fastening of the relevant mathematical properties, or what is the replacement of it, of the components that you are reasoning about. These components are objects, transformations, and concepts (Lithner, 2008). Certain mathematical properties will be surface and other intrinsic depending on the task such as when comparing fractions, the size of the numerator and denominator is a surface property whereas the quotient is the intrinsic property. In Lithner's (2008) framework, different types of reasoning can be classified. Here, we will only focus on the different types of arguments and their foundation and connect these to the teachers' input, the role of the teacher.

Epistemological Move Analysis (EMA)

EMA is an analytical method that aim to generates knowledge about the role the teacher plays in children's meaning making. The focus of the analysis is on how the teacher directs the children's meaning making in different ways (Lidar, Lundqvist & Östman, 2006; Lundqvist, Almqvist & Östman, 2012). When the children respond, verbally or non-verbally, to the teacher's direction, we call it an epistemological move. The epistemological moves from the teacher show the children both what counts as knowledge and appropriate ways of obtaining knowledge. The following moves have been identified in science and technology education in primary school and secondary school (Lidar et al., 2006): confirming, reconstructing, instructional, generative, and reorienting moves. In the confirming move, the teacher confirms that the children are recognizing the correct phenomenon, or confirms that the children are undertaking a valid process, by agreeing with what the children say or do. The reconstructing move makes the children pay attention to the "facts" they have already noticed but have not yet perceived as valid. The instructional move gives the child a direct and concrete instruction for how to act, to discover what is worth noticing. In the generative move, the teacher enables the children to generate explanations by summarizing the important facts in the context of the activity. Finally, the reorienting move indicates that other properties may be worth investigating and encourages the children to take another, alternative direction.

How the teaching affects the meaning making process is studied by analysis of practical epistemologies. Practical epistemology is used as a tool for describing the route that meaning making takes, and the meaning making processes involved. Four concepts are used in a practical epistemology analysis, namely: encounter, stand fast, gap and relations (Wickman & Östman 2002). An encounter is a specific situation in terms of what the participators interact with and here we will focus on encounters between children and teachers. What stands fast for the participator is identified in their actual use of words within the practice. When the participator uses a word without hesitation or questioning, such words are said to stand fast in the particular situation. Standing fast is a situational description of the meaning that words have in action (Wittgenstein, 1969/1992). When the participator hesitates, when what is happening cannot be taken for granted, there is a gap. When a gap is noticed it can, according to Wickman and Östman (2002), be filled through establishing relations to what stands fast in the encounter. Then it is possible for the participators to proceed in their meaning making again.

APPLYING THE TWO FRAMEWORKS

The data comes from a larger set that was used to study children's collective mathematical reasoning. For more information of how data was collected, see Hedefalk and Sumpter (2015). Here, we have chosen a part of a longer episode, divided into three parts, to apply the proposed theoretical model. As a first step, the encounter and its goal is described. This is related to TS. In this encounter, Kasper and Karolina is playing in the woods. They have found a rock that they are trying to climb. Teacher Kristina, marked with [T], sees this and interacts with the children. The main TS for this encounter is: what is rock's height in relation to other objects/ people? In the next step, we identify what epistemological moves the teacher uses with the children in the encounter and if the actions (the practical epistemology) is changed. We also analyze the arguments using the four step structure to identify the different types of arguments and the foundation of these. The last step is to connect the results from the two analysis.

Line	Person	Data	Argument	EMA
2443	Kasper:	[…] Oh, this is not so easy. Oh! Oh! Kristina, this is not so easy because it is so slippery. [trying to climb a rock. Successful.]		
2444	Kristina [T]:	I can understand that, and do you know. That one, that one is pretty big. I think it is bigger than me.	Prediction: rock taller than a specific person.	Confirming move
2445	Kasper:	Yes.	Agreeing with previous conclusion: Rock's height > teacher's height.	
2446	Kristina [T]:	Should we try?	Initiating TS: is rock's height > teacher's height?	Instructional move
2447	Kasper:	Should we measure?	SC suggested. No further arguments.	
2448	Kristina [T]:	Yes, let's measure.	Agree to SC.	Confirming move.

Table 1: Part 1 of TS.

In the first part of this episode, the teacher initiated the TS by first a confirming move and then, the actual initiation, with an instructional move. When Kasper suggests a SC with no predictive arguments, it is not challenged by the teacher but instead the SC is confirmed. This confirming move agrees that the SC is correct and/or relevant however do not encourage further arguments such as predictive arguments.

Line	Person	Data	Argument	EMA
2449	Karolina:	It is bigger than me anyway. [walks and stands next to the rock and looks up, using her own body as a measure.]	SC, SI and C: rock is taller than Karolina as a result from measuring with a Karolina as a measure unit: Karolina's height < rock's height. No further arguments are given.	
2450	Kristina [T]:	Yes, it is bigger than you anyway.	Agreeing to previous conclusion: rock is taller than Karolina.	Confirming move.
2451	Kasper	And me, too.	Another C: no arguments provided. Since Karolina and Kasper are about the same height it is plausible to think that Kasper compares his own height with Karolina and the rock. Rock taller than Kasper.	
2452	Kristina [T]:	Oh look! I think it is, maybe a bit smaller than [up] to my nose. Oh, that is big isn't?	Another SC, SI and C: teacher's height > rock's height. Going against previous conclusion.	

Table 2: Part 2 of TS.

In this part, there is a solution to a sub-task of the main task. There is one move from the teacher, a confirming move, to Karolina's conclusion. This confirming move could function as an evaluative argument: since a teacher agrees to the conclusion, this is a correct answer to the sub-TS. In line 2452, it could have been a reorienting move but since there is no change in practice, this move doesn't occur.

Line	Person	Data	Argument	EMA
2453	Kasper:	But you are as big. [meaning as tall]	Different C: Disagreeing with previous statement with a comparison: teacher as big as rock. Teacher's height = rock's height	
2454	Kristina [T]:	This stone is a bit smaller than me. Isn't?	C: Teacher's height > rocks height.	
2455	Kasper:	It is bigger, a little bit bigger.	C: Rock's height > teacher's height.	
2456	Kristina [T]:	Yes, yes…no, I am a bit bigger.	C: No argument provided. Teacher's height > rock's height.	
[…]		[The children climb the rock and are now sitting on the rock]		
2501	Kasper:	Yes, but the house is bigger than the rock.	Final C. New TS and C. Argument not provided. House's height > rock's height.	
2502	Kristina [T]:	Where?		
2503	Kasper:	The house is bigger than the rock.	House's height > rock's height.	
2504	Kristina [T]:	The house? Yes, definitely. Because the house, I can step in [the house], right?	Agreeing to C: provides argument using transitivity: Since House > Teacher, and Teacher > Rock, therefore House > Rock.	Confirming move

Table 3: Part 3 of TS.

In this part, there are two incidents where a gap occurs. When the teacher argues that the rock is up to her nose, Kasper disagrees as he says that "you are as big" (line 2453). The gap occurs as the participants in the encounter show hesitation about the size of the rock in comparison with the teacher's body. The comment from the teacher does not result in a change of epistemology, i.e. a move, as the children does not change their arguments in line with the teacher's argument. The gap is visible again in line 2454 and line 2455. In these situations, no further arguments are given. When Kasper says that the house is bigger than the rock (line 2501) the teacher confirms that it is a valid statement (line 2502) but she also gives arguments for her conclusion. Since they are related to the TS and not SC and SI, they are evaluative arguments functioning as control. In this chain of interactions, the gap is not filled.

The relations they create to what stands fast is that the rock is smaller than the house which is the final C to the TS.

DISCUSSION

The purpose of this paper was to find a theoretical tool to study mathematical reasoning in settings including both formal and informal learning. The choice was to combine EMA and Lithner's (2008) framework. EMA allowed us to identify different moves and using the four step structure, we could see when these moves occur but also when gaps occurs and if these gaps were filled. It is important to stress that gaps are not seen as needed to be filled using an ecological perspective. In this episode, an instructional move initiated the task situation which could be compared to hatsumon, the asking of a key question (Shimizu, 1999). This main TS were addressed by several sub-tasks initiated by the children. There were also confirming moves connected to evaluative arguments meaning that these arguments came from the teacher instead of the teacher initiated these types of arguments from the children. Such a situation would have been a generative move. EMA helped us to distinguish between these two different situations. Here, there were no arguments based on mathematical properties but instead a repeated statement of conclusions and the gap was not filled. If we were to use the concepts provided by Shimizu (1999), there was no 'polishing up' (neriage). Compared to Bergqvist & Lithner (2012), the proposed analysis stresses the role of the teacher but at the same time allowing a focus on reasoning. We see this as contribution to mathematical reasoning research theories besides the addition of evaluative arguments.

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INSCRIPTIONS THAT MEDIATE INTERACTIVE CONSTRUCTION OF NEW MATHEMATICAL MEANING IN A PRIMARY MATHEMATICS CLASS

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This study explores how teachers and students construct interactions to develop new mathematical meaning. By using the lens of discursive focus by Sfard, we discuss herein the progress of interaction in five consecutive fifth-grade lessons on comparing fractions with unlike denominators. In particular, we examine the written record (i.e., on the blackboard) of student thinking, which mediates the interaction. The result shows that the interaction progresses in three critical phases in which the students' early focus on the calculation moves to its meaning, which is made explicit by the creation of new words, and then refined by further clarification. The students produce drawings in parallel with this process, and these drawings are repeatedly questioned and examined and serve as a springboard for new foci.

INTRODUCTION

Research is increasing on the structure of classroom discourse to support student thinking and understanding. Researchers have now explored the conditions of high-quality discursive practices and classroom interactions that allow students to achieve desirable outcomes (see, e.g., Walshaw and Anthony, 2008).

Through a case study on mathematics lessons in Japan, we have examined how student attention is focused onto new mathematical content by the interaction between teacher and students (Funahashi and Hino, 2014). By analyzing a classroom episode, Hino and Koizumi (2014) show a progress of interaction in terms of how a vague student attention to a subject is questioned and how different study targets are presented or modified. In the present paper, to investigate the relationship between the process of focus building and the students' development of mathematical meaning, I look closer at the different foci that were questioned, presented, or modified. The modification of foci is expected to provide rich information on the features of interaction and on the roles played by the teacher.

To capture different foci developed by the students, this paper considers the written record of student thinking, as written on the blackboard. Stigler and Hiebert (1999) pointed out that Japanese teachers use visual aids to "provide a record of the problems and solution methods and principles that are discussed during the lesson" (p. 74). In the lessons analyzed in this paper, the teacher developed detailed records of student drawings and utterances on the blackboard and used them to organize the interactions throughout the lessons. Such records can be conceived as *inscriptions* in the sense that they are signs materially embodied in a medium (Roth and McGinn,

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1998). Furthermore, "because of their material embodiment, inscriptions (in contrast to mental representations) are publicly and directly available, so that they are primarily social objects" (Roth and McGinn, 1998, p. 37). As social objects, I examine the inscriptions in the lessons from the perspective of how they provide opportunities for students to propose and discuss different foci.

Thus, this paper addresses the following research question: How does student attention shift toward new mathematical meaning during interactions involving student inscriptions?

THEORETICAL FRAMEWORK

Funahashi and Hino (2014) proposed a *guided focusing pattern* to describe the interactive process in which new mathematical content is introduced to students. It comprises four phases: A proposing the problem, B eliciting student ideas, C focusing on the object of examination, and D formulating the result on the basis of the object. In this pattern, phases C and D are especially crucial because it is in these phases that students come to focus more explicit attention on the important ideas that become the foundation of new mathematical knowledge. The focus of this paper is phase C.

To capture the progression of student attention in phase C, we use Sfard's construct of discursive focus (Sfard, 2000). Sfard distinguishes three components of focus used to understand the object in question. *Pronounced focus* is "the word used by an interlocutor to identify the object of her attention" (p. 304). *Attended focus* is "what and how we are attending—looking at, listening to, and so forth—when speaking" (p. 304). Finally, *intended focus* is the "interlocutor's interpretation of the pronounced and attended foci;" this component includes "the whole cluster of experiences evoked by these other focal components as well as all the statements he or she would be able [to] make on the entity in question, even if they have not appeared in the present exchange" (p. 304). The three ingredients of focus are considered to indicate an actual, context-dependent discursive occurrence. In the commognitive approach to study learning, they are made use of articulating discursive objects and examining changes in student discourse (Sfard, 2008; Tabach and Nachlieli, 2015).

The three targets help us explore student-attention paths and the guidance given by the teacher. In the present paper, I further analyze the interaction with another experienced teacher in a prolonged discussion. By looking at the inscriptions working in the conversation, I analyze an awkward process of focus building.

RESEARCH METHOD

This study uses data from nine consecutive lessons on comparing fractions with unlike denominators. The lessons were taught by Mr. Taka (all the names herein are pseudonyms) in a university-affiliated primary school in Tokyo. The lessons were conducted as part of the Learner's Perspective Study–Primary (Fujii, 2013). The data were collected from the lessons and from interviews with the teacher and focus students (see Fujii, 2013, for a detailed description of the data-collection procedure).

The objectives of these lessons were to understand that fractions can be compared if a common unit fraction is found and to understand the methods to compare fractions by finding a common denominator. In the interview, Mr. Taka repeatedly stressed the idea of finding a common unit fraction and remeasuring the original fraction in terms of the new unit fraction. Once the new unit fraction is found, one can compare fractions and add or subtract them in the same way as whole numbers. Mr. Taka said that these concepts are important to build up an understanding of fractions as numbers.

Table 1 briefly describes the tasks and activities in the nine lessons. To direct the students toward his lesson objectives, Mr. Taka addressed repeatedly the meaning and ways of making equivalent fractions in the context of comparing fractions. The students continued to elaborate the explanation by using figural representations.

Lesson	Task and activity
1	Which is larger 2/4L, 3/4L, or 2/3L? Students explained 2/4<3/4 and 2/4<2/3.
2	For 3/4 and 2/3, a student gave a way of finding common numerator.
3	For 3/5 and 2/3, students discussed how to justify the way of finding common numerators.
4	Which is larger, 2/5 or 3/8? Students tried to further justify the approach by clarifying the meaning of “×3” or “÷3” to make 6/15 from 2/5.
5	Discussion continued. The word *unit fraction* was introduced by Mr. Taka to clarify the object of discussion.
6	Students applied the same approach to 3/8 and justified the approach to make 6/16 from 3/8.
7	Students expressed and explained 2/5 = 6/15 = 12/30 by using figural representations.
8	Which is larger, 3/4 or 2/3? Students justified the method of finding common denominators.
9	Which is larger, 1/2 or 1/3, and by how much? Students explained their ways and found that the common denominator better clarifies the difference between the two fractions.

Table 1: Brief description of tasks and activities.

The analysis was done qualitatively by catching the shift of student foci on the construction of equivalent fractions. In so doing, I identified and categorized student inscriptions on the blackboard and examined the ways in which new inscriptions are produced from older ones. Some inscriptions were referred to by several students and the teacher in their discussions. For those anchored inscriptions, I examined closely how the focus is built and what role the teacher plays.

PROGRESS OF INTERACTION WITH INSCRIPTIONS DEVELOPED BY STUDENTS

This section illustrates progress of interactions between teacher and students in lessons 1 to 5 (L1-5) because these lessons are especially rich from the perspective of building focus toward the remeasurement of a fraction. Due to space limitation, I briefly illustrate some of the inscriptions and the interactions they mediated.

Interaction with early inscription

Two early inscriptions were developed by the students to explain how to make an equivalent fraction. One was proposed by Miku in L1 when they were comparing 2/4L and 2/3L. She explained why 2/4L<2/3L by using her inscription, which was drawn by Mr. Taka on the blackboard (Figure 1).

Miku: The least common multiple between 3 and 4 is 12. So, I divided a rectangle into 12. I connected 12 blocks. This is one block [pointing to 1/12 part, Figure 1 bottom]. [Mr. Taka wrote *a block* by red.] For 2/4, I divided the blocks into 4 chunks, and 1, 2, well, I marked here [pointing to the area of 2/4 in Figure 1 top]. [She explained 2/3 in the same way.] Then we know that 2/3 is larger by the difference of 2 blocks. [Mr. Taka drew a dotted line.]

Figure 1: Tape diagram by Miku

Miku said, "The least common multiple between 3 and 4 is 12. So, I divided a rectangle into 12." She first did the calculation and then expressed its results via the tape diagram. The diagram is subordinate to the calculation in the sense that it expresses only the result of the calculation. Miku did not discuss changing the unit fraction, either. Therefore, at this point, her focus was likely on the numerical calculation. Remeasuring was not yet the object of her attention. Similar observations were made of the other early inscription.

Based on these inscriptions, Mr. Taka asked for the reasoning behind their calculation. For example, for Miku's explanation, he said, "Why must you make the denominators the same?" However, the students only repeated the calculation procedure or insisted that the tape diagram clearly showed the result.

Emergence of focus on remeasuring the fraction

In L3, the students compared the two fractions 3/5 and 2/3. After individual activity, they presented their reasoning for claiming 3/5<2/3. Ino gave the following procedural reasoning: "I found a common numerator. Six is the smallest number that divides both 2 and 3, so I used 6. Since 3 was doubled and became 6, so 5 was also doubled. [She continued her explanation.]" Mr. Taka asked if someone could show her procedure in a drawing. Ida gave an explanation based on circles (Figure 2 shows part of Ida's inscription).

Figure 2: A circle by Ida

Reflecting on Ida's drawing, Mr. Taka pointed out that the circle drawing did not explain the procedure given by Ino. Ino first doubled the numerator 3 and applied the same procedure to the denominator 5. However, Ida first divided a circle into 10 parts (denominator) and shaded 6 of them (numerator). This question invited the students to give further explanations. Among them, Naka showed an inscription (Figure 3).

Naka: First is 6, so, now, here we have three equal parts, the red part is divided into three equal parts, so we make them six equal parts. [In place of Naka, Mr. Taka divided each of the three red parts in half (see 3-1 in Figure 3)] … Well, it became 6 equal parts. But I think this state (3-1) is 6/5.

Mr. Taka: Is this OK as a fraction?

Naka: No, it isn't.

Mr. Taka: Why isn't it OK?

A student: Because it is not divided evenly.

Naka: But this state expresses exactly that case (with lauder voice). It came to be 6/5, but, we must do the same thing all over, doubling and tripling them, too (referring to 5 and 3 in 3/5 and 2/3.) We draw lines for these parts, too. [He moved his finger straight to divide each of the two unshaded parts in half.]

Mr. Taka: [By following Naka's instruction, he wrote 3-2 in Figure 3.]

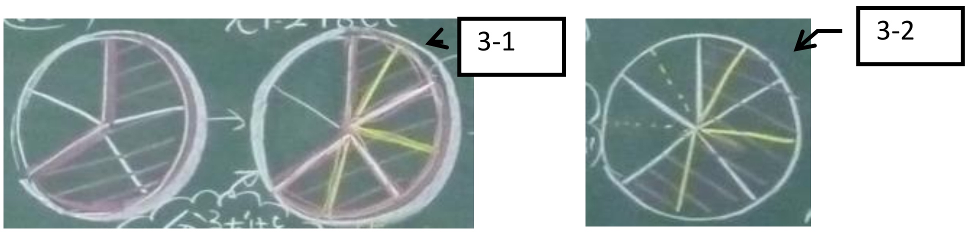

Figure 3: Three circles by Naka

The inscription 3-1 evoked different reactions from the students. Some expressed their dissatisfaction with the unevenly divided circle. Others understood Naka's intention and said, for example, "This expresses the state when only the numerator is doubled." Mr. Taka settled the argument by saying "This is not expressing 6/5 as it is …. This is the figure that was made to explain how 3/5 becomes 6/10."

Naka's inscriptions (Figure 3) were the first that expressed an attending procedure of remeasuring the fraction using a new unit fraction, i.e., *(i) by evenly dividing the part that represents the numerator of the original fraction such that the result becomes the part that represents the numerator of the equivalent fraction and (ii) using the new unit fraction to divide the remaining part*. It deliberately accompanied an inappropriate drawing of the fraction. The inscription caused some debate between the students. Yet, it contributed to attracting student attention to the procedure of remeasurement.

Creation of pronounced focus by student

In L4, the class compared the fractions 2/5 and 3/8. To clarify the object of the explanation, they concentrated on 2/5 and explained their reasoning by using drawings to show why the denominator 5 must be tripled once the numerator 2 is tripled. Initially, the students again proposed early inscriptions in which the drawings they developed only showed the result of the calculation. When a student asked about the connection between the drawing and the calculation, several students attempted to explain how to make 6/15 from 2/5 by using an inscription similar to that shown in Figure 3 (Figure 4 shows part of it). In refining the explanation, Naka participated in the conversation. The transcript below details part of the interaction between Naka and the teacher.

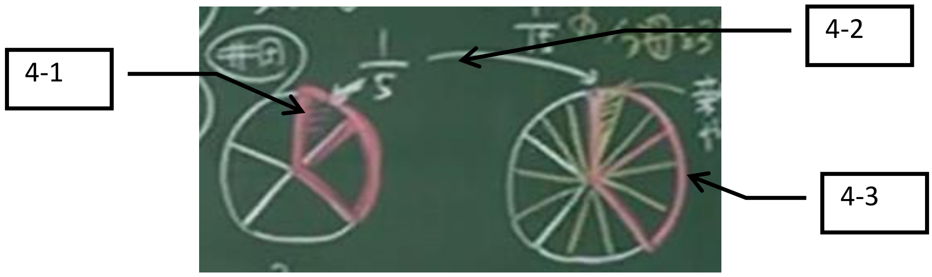

Figure 4: Two circles that mediated the interaction

Naka: The size of a whole (meaning 2/5) is not changing, but, the size of one numerator is from 1/5 to, …, well, it was, well, 1/5 was evenly divided by 3. [Mr. Taka shaded the part 1/5 by red. (4-1 in Figure 4)] … well, 1/5 became 1/15…

Mr. Taka: Yes, but you are talking about this, aren't you? [Mr. Taka drew arrows (4-2).]

Naka: 1/5, oh, one numerator … [Mr. Taka repeated 1/5] was divided evenly by 3.

Mr. Taka: Can we call this *numerator*?

Naka: One numerator.

Mr. Taka: One numerator. (For the class,) Do you understand? Now you know? Do you understand what he is talking about?

A student: It is not numerator.

Mr. Taka: It is not numerator. What he said was this, one numerator //

Naka：[Is it] *moto* (meaning *base* in Japanese)?

Mr. Taka: 1/5 became 1/15. You said *moto*?

Naka：Yeah, *moto*, …, well, the left part, …, the 1/5… [Mr. Taka repeated *moto*.]

Naka：… I mean there are two 1/5s. Well, I think 1/5 is the *moto* [of 2/5] … There are two 1/5s … And, for the new one … [Mr. Taka pointed to the 6/15 (4-3 in Figure 4)], 1/15 is the *moto*, … there are six of them (meaning *moto*).

During the conversation, Naka created a word *moto*, which he brought from everyday language, to refer to the unit fraction. From the beginning, Naka tried explaining that the size of the unit fraction changes although the size of the fraction itself does not change. The pronounced focus *moto* was created by his need to clearly convey to Mr. Taka and his peers the distinction between "fraction as a whole (or numerator)" and "one numerator." In the last utterance, he narrated 2/5 and 15/6 in terms of *moto*. Owing to the familiarity of its usage, the procedure of remeasurement became clearer.

Talking about calculation in terms of new focus of remeasurement

The explanation continued in L5. Mr. Taka led the students to connect (i) the process of remeasuring the fraction with a new unit fraction and (ii) the calculation procedure.

He asked the students the following questions: “This part was divided evenly by three. What do you call this [tracing the outline of 3/15 of the circle 4-3 in Figure 4]?” The students expressed it in different ways; for example, *quantity of numerator*, *sector’s central angle,* or *arc*. When one student said *area,* Mr. Taka repeated this answer: “Yes, it is area or how large it is.” They ensured that the area became smaller upon being divided by three. Mr. Taka then asked “as a result, what was multiplied by three?” Several students replied that the number was increased. At this point, Mr. Taka introduced the word *unit fraction,* and summarized the discussion as follows:

> Mr. Taka: The reason why the numerator is multiplied by three is that the unit fraction became smaller. The size of the whole (meaning 2/5) is unchanged. But the number of pieces increased [because] the area of one piece became smaller.

After this, Mr. Taka asked the students why the denominator was multiplied by three. After discussion, Koma finally gave the explanation: “Since a unit fraction was divided by three, I think there are five [pieces that were] divided by three.”

The students had difficulty making an explicit connection between the remeasurement of the fraction (a focus they had been building) and the procedure of multiplying both numerator and denominator by the same number. Mr. Taka focused the students’ attention on distinguishing between the “area” and the “number.” These pronounced foci clarified for the students that changing *moto* changed both the area and the number. Attending to these two aspects seems to enable the students to build such a connection.

DISCUSSION

This paper illustrates the process in which the student focus shifts toward the target that reflects the objectives of the lessons. The shifting process contains three critical phases: First, the students’ object of attention changes from a procedure for calculating a common numerator or denominator to the meaning of this procedure. This major shift is enabled by Naka’s proposal of remeasuring the fraction with a new unit fraction that accompanies an inappropriate figure of the fraction. Second, a pronounced focus *moto* was proposed, which was created by Naka’s need to more explicitly explain the concepts to his peers. Third, the new focus was connected with and used to talk about the method of making equivalent fractions. Here additional pronounced foci were produced, which seem to contribute to refining the students’ procedure.

The three phases are consistent with the result of our previous analysis (see, e.g., Hino and Koizumi, 2014). A common feature of this process is that the attended focus is repeatedly negotiated and refined by calling for new pronounced foci. By analyzing the inscriptions made by students during the discussions over five lessons, this paper reveals that the process advances nonlinearly and fluctuates between newer and older inscriptions. A salient observation is the prevalence of the students’ attention to the calculation procedure. Early inscriptions reappeared over and over.

Nevertheless, each time one appeared in a student inscription, it was questioned or problematized either by the teacher or by the students.

The student inscriptions provided fruitful opportunities for a disequilibrium to emerge between the focal ingredients that were conceived as the major drive of discursive growth (Sfard, 2000). Every student inscription reflects strongly on his or her current understanding of fractions and of equivalent fractions. His or her intended focus is reflected not only in the layout of elements but also in how they are inscribed. The act of drawing on the blackboard publicizes the intended focus and thereby invites different interpretations by other students. In this way, as social objects, inscriptions enable an iterative and dialectical process between signs and referents (Roth and McGinn, 1998) and serve to impel discursive practice.

Since this work deals with the first five lessons, the analysis will be continued by considering more lessons from the point of view of focus building. Furthermore, note that the teacher gave high priority to the student inscriptions and used them as a crucial pedagogical device. The shift of focus was made possible by his conscious lesson objectives and consistent support and guidance of the students toward these objectives (Funahashi & Hino, 2014). Articulation of this aspect is a productive future task.

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PROVING SUBGROUP'S CLOSURE UNDER INVERSES: COMMOGNITIVE ANALYSIS OF STUDENTS' RESPONSES

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Encounter with the Subgroup Test is, more often than not, the first major challenge that undergraduate mathematics students face in their first Group Theory course. This study focuses on students' responses to the proof of one of the tests' conditions, namely closure under inverses. Analysis suggests that there have emerged inaccuracies of two categories: the first is related to the involved mathematical concepts per se and the other with the actual process of proof. Additionally, analysis suggests that incomplete learning of these concepts has an unfavourable impact on the process of proving closure under inverses. For the purposes of this study there has been used the Commognitive Theoretical Framework.

INTRODUCTION

Subgroup Test is one of the first routines undergraduate students need to cope with in their first engagement with Group Theory, where they need to prove the three conditions, namely, non-emptiness, closure under operation and closure under inverses. Often, though, this apparently simple task proves to be an arduous endeavour, partly due to the abstract nature of Group Theory (Hazzan, 2001). In fact, Group Theory "is the first course in which students must go beyond 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes" (Dubinsky et al., 1994, p268). A typical first Group Theory module requires a deep understanding of the abstract concepts involved, namely group, subgroup, coset etc. In addition, the deductive way of teaching Group Theory is unfamiliar to students and, in order to achieve mastery of the subject, it is necessary to "think selectively about its entities, paying attention to those aspects consistent with the context and ignoring those that are irrelevant" (Barbeau, 1995, p140). Moreover, Gueudet (2008) suggests that many pedagogical issues emerging in undergraduate Mathematics Education are based on the transition from secondary to tertiary Mathematics, which can still occur in their second year. In fact, student difficulties in Abstract Algebra may be an indication of problematic transition, mainly due to the particular nature of this module (Ioannou, 2012). The aim of this study is to investigate the undergraduate mathematics students' responses to the concept of subgroup, and in particular in proving closure under inverses, during their first encounter with Group Theory. For the purposes of this study, there has been used the Commognitive Theoretical Framework (CTF) (Sfard, 2008), due to its great potential to investigate mathematical learning in both object level and meta-discursive level (Presmeg, 2016).

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THEORETICAL FRAMEWORK

CTF is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis. It involves a number of different notions such as metaphor, thinking, communication, and cognition (Sfard, 2008). In mathematical discourse, objects are discursive constructs and form part of the discourse. Mathematics is an **autopoietic system** of discourse, namely "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p129). Moreover, CTF defines discursive characteristics of mathematics as the **word use**, **visual mediators**, **narratives**, and **routines** with their associated metarules, namely the how and the when of the routine. In addition, it involves the various objects of mathematical discourse such as the **signifiers**, **realisation trees**, **realisations**, **primary objects** and **discursive objects**. It also involves the constructs of **object-level** and **metalevel rules**. Thinking "is an individualized version of (interpersonal) communicating" (Sfard, 2008, p81). Contrary to the acquisitionist approaches, participationists' ontological tenets propose to consider thinking as an act (not necessarily interpersonal) of communication, rather than a step primary to communication (Nardi et al., 2014).

Mathematical discourse involves certain objects of different categories and characteristics. **Primary object** (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). **Simple discursive objects** (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization. **Compound discursive objects** (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary." In the context of this study, subgroups are an example of compound d-objects. The (discursive) object signified by S in a given discourse is defined as "the realization tree of S within this discourse." (Sfard, 2008, p166)

Sfard (2008) describes two distinct categories of learning, namely the **object-level** and the **metalevel learning**. "Object-level learning […] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p253). In addition, "metalevel learning, which involves changes in metarules of the discourse […] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p254).

LITERATURE REVIEW

Research in the learning of Group Theory is relatively scarce compared to other university mathematics fields, such as Calculus, Linear Algebra or Analysis. Even more limited is the commognitive analysis of conceptual and learning issues (Nardi et al., 2014). In the context of this research strand, Ioannou (2012) has, among other issues, focused on the intertwined nature of object-level and metalevel learning in Group Theory and the commognitive conflicts that emerge.

The first reports on the learning of Group Theory appeared in the early 1990's. Several studies, following mostly a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, examined students' cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic concepts. The construction of the newly introduced d-object of group is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education mathematics to the formalism of undergraduate mathematics (Nardi, 2000). Students' difficulty in the engagement with the Group Theory concepts is partly grounded on historical and epistemological factors: "the problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today" (Robert and Schwarzenberger, 1991, p129). Nowadays, the presentation of the fundamental concepts of Group Theory, namely group, subgroup, coset, quotient group, etc. is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry. Moreover, this chasm of ontological and historical development proves to be of significant importance in the metalevel development of the group-theoretic discourse for novice students.

Research suggests that students' understanding of the concepts of group proves often primitive at the beginning, predominantly based on their conception of a set. An important step in the development of the understanding of the concept of group is when the student "singles out the binary operation and focuses on its function aspect" (Dubinsky et al., 1994, p292). Students often have the tendency to consider group as a "special set", ignoring the role of binary operation. Iannone and Nardi (2002) suggest that this conceptualisation of group has two implications: the students' occasional disregard for checking associativity and their neglect of the inner structure of a group.

An often-occurring confusion amongst novice students is related to the order of the group G and the order of its element g. This is partly based on student inexperience, their problematic perception of the symbolisation used and of the group operation. The use of semantic abbreviations and symbolisation can be particularly problematic at the beginning of their study. Nardi (2000) suggests that there are both linguistic and conceptual interpretations of students' difficulty with the notion of order of an element of the group. The role of symbolisation is particularly important in the learning of Group Theory, and problematic conception of the symbols used probably causes confusion in other instances.

A distinctive characteristic of university mathematics is the production of rigorous and consistent proofs. Proof production is far from a straightforward task to analyse and identify the difficulties students face. These difficulties have been extensively investigated for various levels of student expertise. Weber (2001) categorises student difficulties with proofs into two classes: the first is related to the students' difficulty to have an accurate and clear conception of what comprises a mathematical proof, and the second is related to students' difficulty to understand a mathematical proposition or a concept and therefore systematically misuse it.

METHODOLOGY

This study is part of a larger research project, which conducted a close examination of Year 2 mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Group Theory. The module was taught in a research-intensive mathematics department in the United Kingdom, in the spring semester of a recent academic year.

The Abstract Algebra (Group Theory and Ring Theory) module was mandatory for Year 2 mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. All members of the teaching team were pure mathematicians.

The lectures consisted largely of exposition by the lecturer, a very experienced pure mathematician, and there was not much interaction between the lecturer and the students. During the lecture, he wrote self-contained notes on the blackboard, while commenting orally at the same time. Usually, he wrote on the blackboard without looking at his handwritten notes. In the seminars, the students were supposed to work on problem sheets, which were usually distributed to the students a week before the seminars. The students had the opportunity to ask the seminar leaders and assistants about anything they had a problem with and to receive help. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

The gathered data included the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, the collected data of the 13 volunteers has been scrutinised. Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities, right to withdraw, procedures of complain, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., have been addressed accordingly.

DATA ANALYSIS

Incidences of incomplete mathematical learning appeared in eight of the thirteen (8/13) students' attempts to prove closure under inverses. Due to limited space, there will be presented only three characteristic examples of such incidences.

The most common inaccuracy, which occurred, was related to the proof of the uniqueness of the inverse. For instance, in an attempt to solve the following task: "For any $n \in \mathbb{N}$ the sets $\{g \in GL(n, \mathbb{R}): Det(g) = 1\}$ and $\{g \in GL(n, \mathbb{R}): gg^T = I_n\}$ are subgroups of $GL(n, \mathbb{R})$", Student A, successfully applies the routine for a set to be a subgroup for the first set, i.e. $X = \{g \in GL(n, R): Det(g) = 1\}$. Her solution indicates complete object-level learning of the d-objects involved, successful application of the governing metarules, as well as good connectivity across different mathematical discourses, such as Linear Algebra. For the second set, $W = \{g \in GL(n, R): gg^T = I_n\}$, she successfully applies the routine and proves non-emptiness and closure under operation, and for the closure under inverses she correctly states that the inverse in this case is the transpose. Yet she has omitted to clarify the uniqueness of inverse taken both from the right and the left as shown below. Without this clarification the algebraic manipulations would be unjustified.

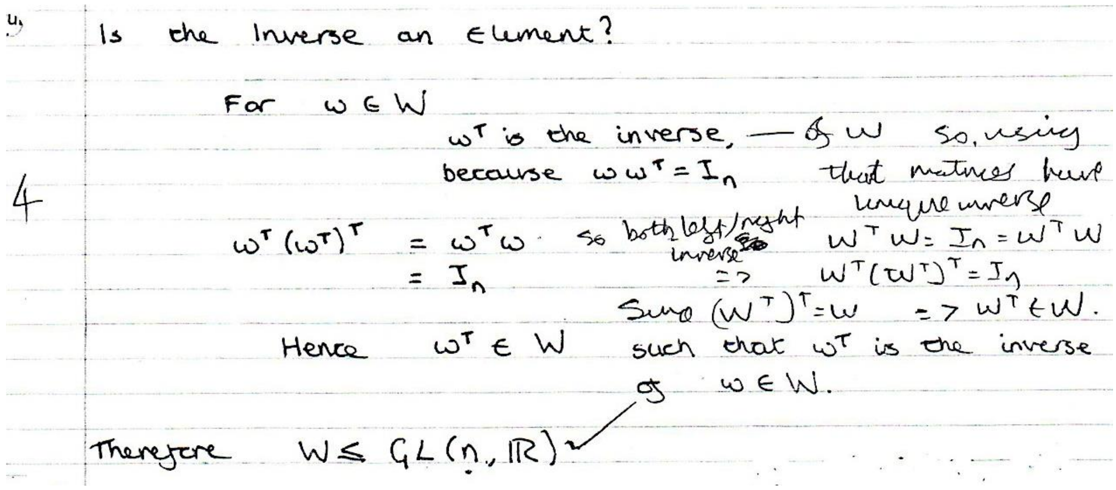

Figure 1: Part of Student A's solution

A second incidence of incomplete mathematical learning regarding closure under inverses appeared in the coursework of Student B. In her attempt to solve the following task "Suppose X is a non-empty set and $G \leq Sym(X)$. Let $a \in X$ and $H = \{g \in G: g(a) = a\}$. Prove that H is a subgroup of G.", she demonstrates a rather complete object-level learning of the involved d-objects. In addition, she successfully uses the condition that $a \in X$ *and* $H = \{g \in G: g(a) = a\}$ to prove that H is non-empty and that the closure under operation holds. Nevertheless, her attempt to prove that H is closed under inverses is problematic, due to problematic application of the governing metalevel rules. She assumed that since $g^{-1} \in G$, it is given that it belongs to H as well, instead of showing it. Instead she should apply g^{-1} in both sides of $g(a) = a$ and get that $a = g^{-1}(a)$. This suggests incomplete metalevel learning and consequently inaccurate application of metarules, particularly regarding the precision and rigor that mathematical reasoning in this advanced context requires.

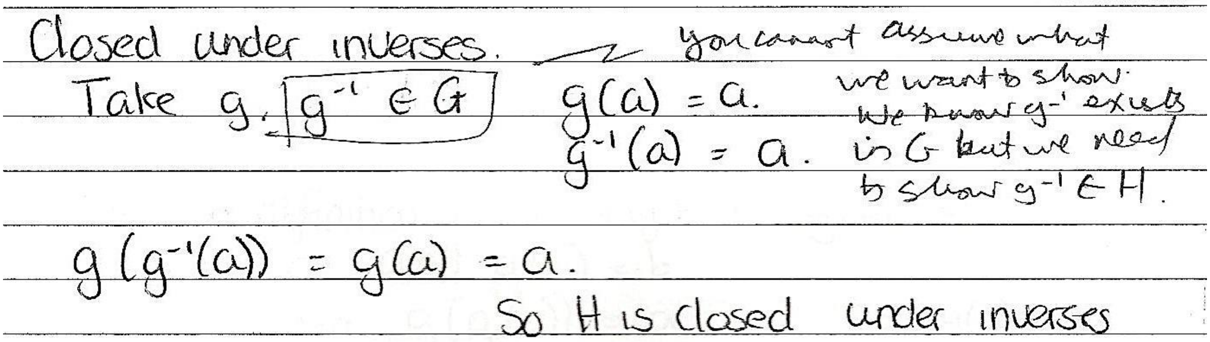
Closed under inverses. you cannot assume what we want to show. We know g^{-1} exists in G but we need to show $g^{-1} \in H$.
Take g. $\boxed{g^{-1} \in G}$ $g(a) = a$.
$g^{-1}(a) = a$.
$g(g^{-1}(a)) = g(a) = a$.
So H is closed under inverses

Figure 2: Part of Student B's solution

Apart from Student B's problematic application of the governing metarules, in this particular exercise her performance seems to be unfavourably influenced by incomplete object-level learning, particularly at the initial stages of her attempt to solve the task, as the following interview excerpt suggests.

> Um... but I did manage to sort it out eventually – I just think – I found it hard cos – I was going between X and G and H and A, there was just a lot of – different groups, that I was trying to get my head round, but um, I did manage to sort that out eventually. Student B

Student B's object-level disengagement at this initial stage is related to the identification of the difference between the various sets and the groups, which would allow her to apply with facility the routine for a set to be a subgroup.

Similarly, incomplete object-level learning seems to occasionally, but not necessarily, have negative impact in the application of the governing metarules. This is obvious in Student C's attempt to prove closure under inverses in the following mathematical task: "Suppose $(G,\cdot)$ is a group and H, K are subgroups of G. Show that $H \cap K$ is a subgroup of G."

In Figure 3, one can detect incomplete object-level learning of the involved d-objects. In particular, there are indications of problematic engagement with the d-object of subgroup as such and its elements. These indications are particularly obvious in the notation used in the narrative $(hh^{-1} \cap kk^{-1})$, since Student C possibly does not realise what hh^{-1} and kk^{-1} represent, and the circumstances under which the operation of intersection can be used.

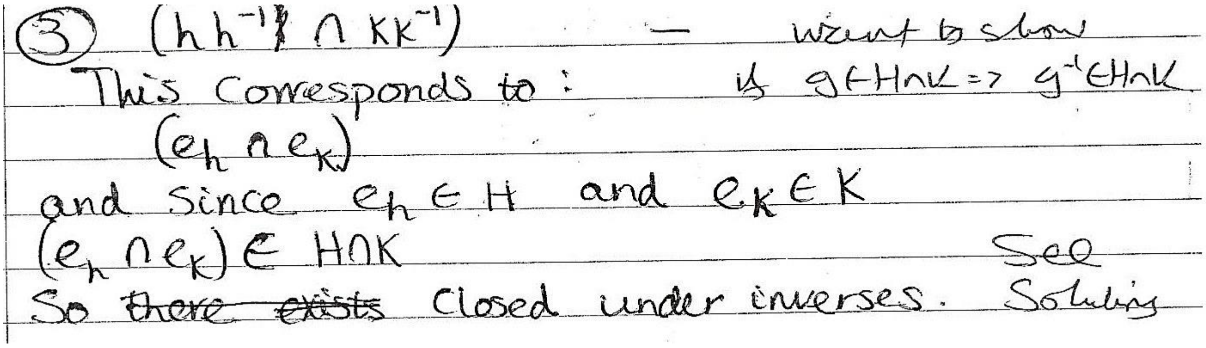
③ $(hh^{-1} \cap kk^{-1})$ — want to show
This corresponds to : if $g \in H \cap K \Rightarrow g^{-1} \in H \cap K$
$(e_h \cap e_k)$
and since $e_h \in H$ and $e_K \in K$
$(e_h \cap e_k) \in H \cap K$ See
So ~~there exists~~ closed under inverses. Solutions

Figure 3: Part of Student C’s solution

In addition to the aforementioned issues with object-level learning, there are also indications of problematic engagement with metarules. In particular, Student C seems not to have a clear idea of how and when his proof needs to be further developed, indicating some difficulty with the applicability conditions of the routine, as well as the how of the routine and the ‘course of action’. This is obvious in his attempt to prove closure under inverses, since he does not seem to be fully aware that he has to prove that $g^{-1} \in H \cap K$ if $g \in H \cap K$. Moreover, Student C expressed his concern about applying the particular routine and connected it with his ability to communicate the proof in a way that was comprehensible to others.

> But I – yeah, again, it might be – me not – it makes perfect sense, but I might not... make it – it’s just like you know – I can understand it, but it’s trying to, I mean because proof is really trying to make someone else understand it, and I say, possibly I do struggle at – giving, you know, making someone else understand it by writing it down, but, so it’s where I might lose some marks, but... Student C

More generally, Student C’s writing as seen in his scripts is personalised with signs of tentativeness on many occasions. Tentative writing occurs when his mathematical learning is incomplete. In these instances, his solutions are nonlinear and messy.

CONCLUSION

The subgroup test is the first major routine that undergraduate mathematics students are invited to be engaged with in the context of Group Theory. This study has focused on the student’s responses regarding the proof of closure under inverses, adopting a participationist perspective. In agreement with other studies in the field, there have emerged difficulties that are related to the object-level learning of the concepts of group, subgroup and set (in agreement with Iannone and Nardi, (2002); Robert and Schwarzenberger, (1991)), as well as difficulties that are related to the application of metarules and the level of rigor that the process of proof requires (in agreement with Weber, (2001); Nardi (2000)). The last characteristic example of Student C, indicates that incompleteness of object-level learning has an unfavourable impact on the application of metarules and the proof production overall.

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PRE-SERVICE TEACHERS' USES OF A LEARNING TRAJECTORY TO NOTICE STUDENTS' FRACTIONAL REASONING

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Learning Trajectories are seen as a tool that can help pre-service teachers (PTs) focus on students' mathematical thinking to make instructional decisions. The aim of this study is to examine PTs' use of a Learning Trajectory for fractions to notice students' fractional reasoning. Our results show that the use of a Learning Trajectory for fractions as a scaffold allowed PTs to interpret students' fractional reasoning. However, they interpreted students' fractional reasoning in different ways, differing in the professional discourse generated. Our results also suggest a relationship between the way in which PTs interpreted students' fractional reasoning and the instructional decisions made. PTs who used a more detailed mathematical discourse proposed more activities to help students progress in their fractional reasoning.

NOTICING AND LEARNING TRAJECTORIES

Noticing what is happening in a classroom is an important skill that teachers should acquire to actively respond to complex and challenging situations that arise in their classrooms. Although the skill of noticing has been conceptualised from different perspectives (Jacobs, Lamb, & Philipp, 2010; Mason, 2002; van Es, & Sherin, 2002), we are going to focus on the conceptualisation given by Mason (2002; 2011). For him "noticing is a movement or shift of attention" (Mason, 2011, p. 45) and he has identified different ways in which people can attend (p.47):

Holding wholes is attending by gazing at something without particularly discerning details.

Discerning details is picking out bits, discriminating this from that, decomposing or subdividing and so distinguish and, hence, creating things.

Recognizing relationships is becoming aware of sameness and difference or other relationships among the discerned details in the situation.

Perceiving properties is becoming aware of particular relationships as instances of properties that could hold in other situations.

Reasoning on the basis of agreed properties is going beyond the assembling of things you think you know, intuit, or induce must be true in order to use previously justified properties as the basis for convincing yourself and others, leading to reasoning from definitions and axioms.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 25-32. Singapore: PME.

Jacobs et al. (2010) particularised Mason's work conceptualising the skill of noticing students' mathematical thinking as a three interrelated skills: attending to students strategies (discerning details), interpreting students' mathematical thinking taking into account the details identified before (recognising relationships) and deciding how to respond on the basis of students' reasoning (perceiving properties). However, recent research has shown that the skill of deciding how to respond on the basis of students' mathematical thinking is the most difficult one to develop in teacher education programs (Choy, 2013) since "the specificity of what teachers notice while necessary, is not sufficient for improved practices" (p. 187). In other words, teachers can be very specific about what they notice without having a teaching decision according to what it has been noticed.

On the other hand, previous research has shown that the use of Learning Trajectories could focus teachers' attention on students' mathematical thinking and that when pre-service teachers attend to students learning progressions in a particular mathematical domain, they are better in making decisions about next instructional steps (Wilson, Mojica, & Confrey, 2013). In this context, students' Learning Trajectories could help pre-service teachers interpret students' mathematical reasoning and respond with appropriate instruction (Sztajn, Confrey, Wilson, & Edgington, 2012). Furthermore, the use of Learning Trajectories could provide pre-service teachers with a mathematical language to describe students' mathematical thinking (Wickstrom, Baek, Barrett, Cullen, & Tobias, 2012).

Our study is embedded in these two lines of research and analyses how pre-service teachers' learning of a Learning Trajectory for fractions helps them to notice students' fractional reasoning. Our research question is: to what extent do pre-service teachers use a Learning Trajectory for fractions to interpret students' fractional reasoning and make instructional decisions on the basis of students' reasoning?

A Learning Trajectory for fractions

A Learning Trajectory (LT) is a way of articulating the students' conceptual progress from informal thinking to a more sophisticated mathematical reasoning, and consists of three components: (i) a learning goal, (ii) a hypothetical learning process and (iii) learning activities (Simon, 1995). The learning goal of the Learning Trajectory for fractions used in this study takes into account the Spanish Primary Education's curriculum: the meaning of fraction as a part-whole relation and its different representations and, the meaning of fractions operations. The student's learning process takes into account how the student's fractional reasoning develops over time and is organised in six proficiency levels of fractional reasoning (Battista, 2012; Steffe, 2004; Steffe, & Olive, 2009) (Figure 1). Regarding to the learning activities, the Learning Trajectory for fractions used in this study includes activities to help students progress to a more sophisticated level of reasoning, particularly, activities of identifying, representing, and comparing fractions, and operations with fractions in both, discrete and continuous contexts.

In this study, we focus on pre-service primary teachers' uses of the three initial proficiency levels of the Learning Trajectory for fractions. These levels are focused on the meaning of fraction as a part-whole and its representations: the recognition that the parts of the whole must be congruent, the representation and identification of fractions in continuous and discrete contexts and the identification of equivalent fractions recognising that a part can be divided into other parts.

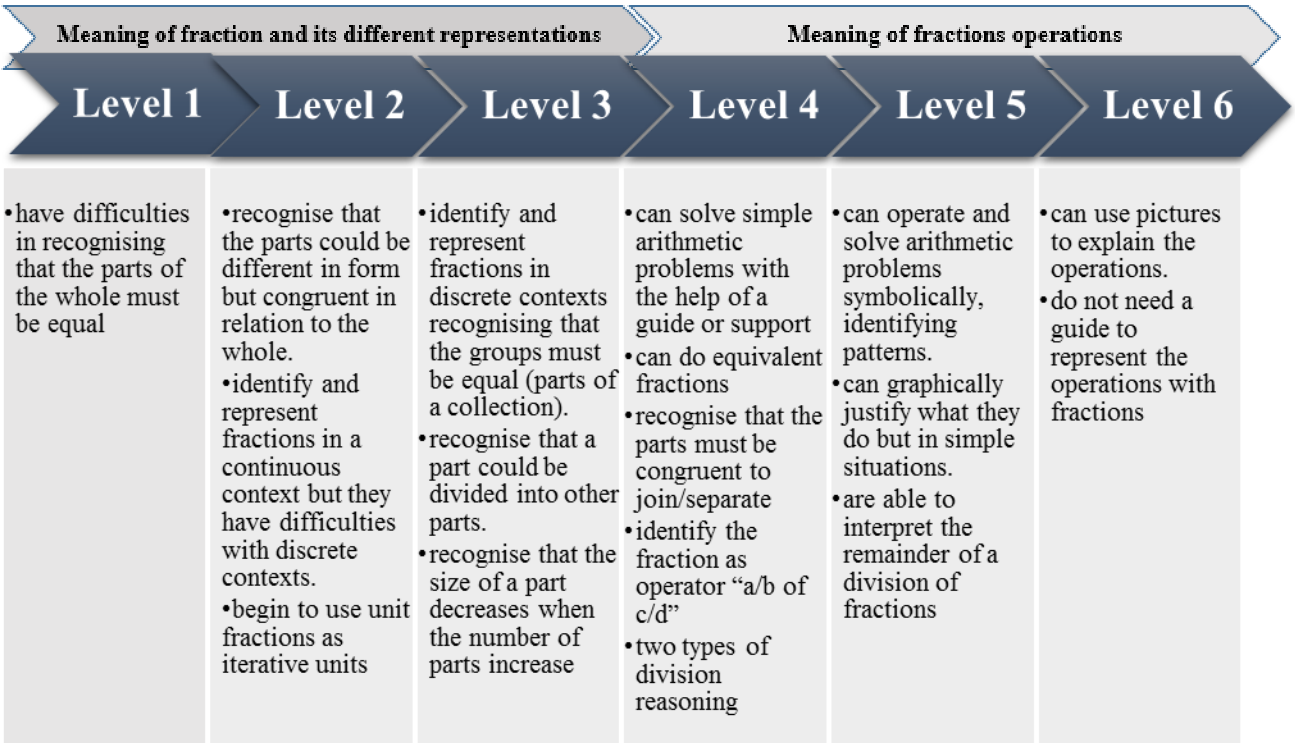

Figure 1*:* Learning Trajectory proficiency levels

METHOD

Participants and the task

Participants were 95 pre-service primary school teachers (PTs) enrolled in their third year of a degree to become primary school teachers. They were enrolled in a subject related to the teaching and learning of mathematics in primary school. In previous courses, these PTs had participated in a subject related to Numerical Sense and in a subject related to Geometrical Sense.

The task consists of the answers of three couples of primary school students that have a different fractional reasoning proficiency level in an activity of identifying fractions (Figure 2) (to see a complete version of the task, Ivars, Fernández, & Llinares, 2016). The answer of Xavi and Víctor (couple 1) shows characteristics of the level 1 since they do not take into account that the parts of a whole must be congruent. The answer of Joan and Tere (couple 2) reflects characteristics of the level 2 since they take into account that the parts must be congruent in continuous contexts but they still do not recognise that a part can be divided into other parts. This last characteristic is evidenced when they say that Figure E is not three quarters because it is divided into

24 equal parts and there are 18 shaded. Finally, Álvaro and Félix (couple 3) take into account that the parts must be congruent and that a part can be divided into other parts (they consider Figures B, D, E, and F as representations of ¾).

1. Which of the following diagrams represent ¾. Explain your answers.

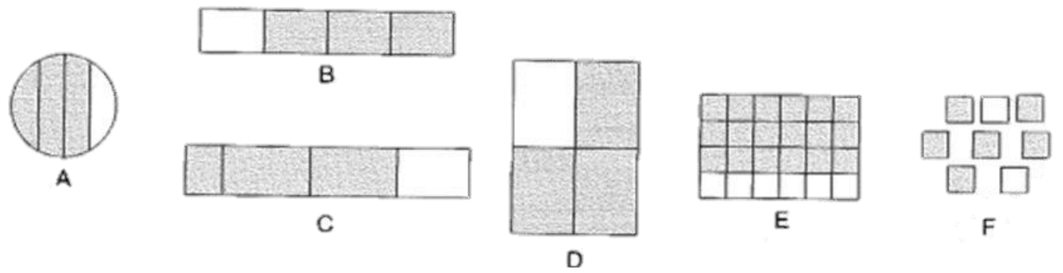

Figure 2: Activity of identifying fractions (adapted from Battista, 2012)

Considering the answers of the three couples of students, pre-service teachers had to answer the next four questions.

- Q1- Describe the activity taking into account a learning objective: what are the mathematical elements that a student needs to know to solve it?
- Q2- Describe how each couple of students has solved the problem indicating how they have used the mathematical elements involved and the difficulties they have had with them.
- Q3- What are the characteristics of students’ reasoning (Learning Trajectory) that can be inferred from their answers? Explain your answer.
- Q4- How could you respond to these students? Propose a learning objective and an activity to help students progress in their fractional reasoning.

Pre-service teachers were provided with information about the mathematical elements of the fraction concept and the Learning Trajectory for fractions (Figure 1). We hypothesise that these questions and the theoretical information could focus pre-service teachers’ attention on identifying the relevant mathematical elements of students’ answers (discerning details); on interpreting these answers (recognising relationships between the mathematical elements and students’ reasoning) and on deciding how to respond on the basis of students’ mathematical reasoning.

Analysis

We analysed PTs’ answers taking into account three aspects. Firstly, if they had identified relevant elements of the fraction concept in students’ answers (discerning details). Secondly, how pre-service teachers interpreted students’ fractional reasoning (recognising relationships between the mathematical elements of the fraction concept identified in the students’ answers and the different levels of students’ fractional reasoning). Finally, how pre-service teachers made instructional decisions (using what they have identified about students’ reasoning to propose activities that help students progress in their fractional reasoning).

To carry out the analysis, a subset of PTs' answers was analysed and coded by three researchers, independently, considering the three aspects mentioned above. Then, we put together our analyses and compared and discussed our discrepancies until we reached an agreement. Then, new data samples were added to review our allocation.

RESULTS

We can highlight two main results. On the one hand, pre-service teachers interpreted students' fractional reasoning in three different ways. On the other hand, these ways of interpreting students' fractional reasoning influenced the instructional decisions made.

Different ways of interpreting students' fractional reasoning

The analysis revealed that 90 out of the 95 PTs identified the mathematical elements of the fraction concept in the students' answers (discerning details), that is, they used the mathematical elements, *the parts must be congruent* and *a part can be divided into other parts* to describe the students' answers. Furthermore, 89 out of 90 PTs who discerned details were able to interpret students' fractional reasoning recognising the relationships between the mathematical elements of the fraction concept in the students' answers and the different proficiency levels of students' fractional reasoning (Learning Trajectory). However, these PTs interpreted students' fractional reasoning in three different ways depending on if they were able to elaborate a more detailed discourse using the Learning Trajectory:

<u>*Non-evidencers*</u>*:* These PTs interpreted students' reasoning recognising the relationship between the mathematical elements and the levels of the LT but did not provided evidence from the students' answers (23 PT). For instance, the PT 85 described Felix and Alvaro's answer indicating (emphasis is added underlying the mathematical elements recognised):

> Couple 3 (Félix and Álvaro).These students are in level 3 of the LT because, as Tere and Joan (the second couple of primary school students), they identified that <u>the parts must be congruent</u> but, they did not have difficulties in recognising that <u>a part can be divided into other parts</u>.

This PT recognised the mathematical elements in the students' answers and determined the level of students' fractional reasoning but he did not provide evidence from students' answers to support his inference.

<u>*Adders*</u>*:* These PTs interpreted students' fractional reasoning recognising the relationship between the mathematical elements and the levels of the LT and provided evidence from the students' answers but adding unnecessary information (7 PT). For instance, the PT 62 wrote in relation to couple 3 answers (emphasis is added underlying the mathematical elements not related to the activity that the PT added unnecessarily):

> Couple 3 (Félix and Álvaro). These students are in Level 3. They identified fractions in discrete contexts recognising that the groups must be congruent because they identified F

> as ¾. Furthermore, they said that E was ¾ too, so they recognised that a part can be divided into other parts. Finally, when comparing fractions they recognised that the wholes must be equal and they stablished the inverse relation between the number of parts and the size of each part.

This PT provided evidence of her interpretation from the students' answers when she wrote *"...they identified F as ¾" "...they said that E was ¾ too"*. However, she added unnecessary information about the comparison of fractions that was not related to the problem (although this information is appropriated in the level of the LT identified).

Evidencers: These PTs interpreted students' fractional reasoning recognising the relationship between the mathematical elements and the levels of the LT and provided evidence from students' answers (59 PT). For example, the PT 49 wrote for the couple 3:

> Félix and Álvaro. These students reasoned about figures A, B, C, D in the same way that Joan and Tere. However, in figure E, as the whole has 6 equal squares in each line and there are 3 lines out of 4 shaded, they said that this figure represents ¾. And, in figure F, they grouped the eight squares in groups of 2, obtaining 4 groups of 2 squares each. Then, they realised that 3 groups of 2 squares are shaded. They are at level 3 because they recognised that a part can be divided into other parts.

How the ways of interpreting influenced the instructional decisions made

The way that the PTs interpreted students' fractional reasoning influenced their instructional decisions. Taking into account that each PT had to propose an activity to each couple of students (3 activities x 89 PTs), we obtained the data of Table 1. The 23 *non-evidencers* were able to propose only a new activity in the 19% of the cases, the 7 *adders* only in the 29% of the cases, and, finally the 59 *evidencers* in the 38% of the cases. These data suggest that when PTs provided details of their interpretations from the students' answers, they were able to propose more activities to support the students' progress.

		From Level 1 to Level 2		From Level 2 to Level 3		From Level 3 to Level 4		Total
	PT's	Act.	%	Act.	%	Act.	%	%
Non-evidencers	23	3	13%	8	35%	2	9%	19%
Adders	7	3	43%	2	29%	1	14%	29%
Evidencers	59	26	44%	38	64%	4	7%	38%
Total	89		33%		43%		10%	29%

Table 1: Instructional decisions made by PTs who recognised relationships

Table 1 also indicates that the *non-evidencers* and *evidencers* groups had more difficulties in proposing activities to help students progress from level 1 to level 2 of the LT than from level 2 to level 3 (*non-evidencers* 13% vs 35% and *evidencers* 44% vs 64%). However, proposing an activity to help students progress from Level 3 to the Level 4 is the most difficult one for PTs in all the groups.

DISCUSSION AND CONCLUSIONS

Our results have shown that the use of a Learning Trajectory for fractions as a scaffold by pre-service primary teachers allowed them to interpret students' fractional reasoning since 89 out of 95 of the PTs who participated in this study were able to recognise relationships between the important mathematical elements involved in the students' answers and the different levels of the Learning Trajectory.

Nevertheless, the way in which these 89 PTs interpreted students' fractional reasoning was different. All of these 89 PTs recognised relationships between the mathematical elements in the students' answers and the proficiency levels of the Learning Trajectory, but they differed in the professional discourse generated to interpret students' fractional reasoning. In fact, *non-evidencers* generated a less detailed discourse without giving evidence from students' answers, *adders* started to use a more detailed discourse giving evidence from students' answers but adding unnecessary information and, *evidencers* generated a detailed discourse giving evidence from students' answers.

Our results also suggest a relation between the way in which PTs interpreted students' fractional reasoning and the instructional decisions made. PTs who used a more detailed mathematical discourse (using students' answers to support their interpretations) proposed more activities to help students progress in their fractional reasoning. *Noticing details* helped PTs to propose more activities taking into account the students' fractional reasoning. This data support the claim that the more sensitive pre-service teachers are to noticing details in students' answers, the more capable they are to act responsively (Mason, 2002).

In relation to noticing, the task used appears to be a powerful tool that helped PTs focus their attention on discerning the mathematical details of students' answers, on interpreting students' reasoning and on making instructional decisions on the basis of students' reasoning. Furthermore, the Learning Trajectory could act as scaffold to improve PTs' mathematical discourse since it provides to PT's with a specific language to describe students' thinking (Wickstrom et al., 2012).

Our results provide a snapshot of how pre-service teachers, through the use of a students' Learning Trajectory for fractions, begin to notice students' fractional reasoning. Further research could be focused on analysing if pre-service teachers' noticing skill is developed when they are enrolled in a learning environment that uses a students' Learning Trajectory as a referent.

Acknowledgment

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WHAT DO MATHEMATICS PRE-SERVICE TEACHERS LACK FOR MASTERING INSTRUCTIONAL DEMANDS?

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In addition to subject-specific professional knowledge, research in teacher education recently focusses on teachers' abilities to master subject-specific instructional demands. Although knowledge is seen as a prerequisite for according competences with close relation to instructional demands, the complex relationship between knowledge and according competences is not understood in detail. In order to investigate this relationship, we analysed answers to video-based instructional situations of 4 mathematics pre-service teachers. The case study illustrates that despite of sufficient teacher knowledge and perception abilities, the ability to give helpful feedback in instructional situations can be lacking. Our cases give indications, what factors might further the area of research.

RESEARCH IN TEACHER COGNITION

Subject-specific knowledge

Research in teacher education brought up a variety of models for teacher competences (e.g. Hill, Schilling, & Ball, 2004; Kunter et al., 2013). In this research, there is a tendency of narrowing teachers' cognition to declarative knowledge. Especially when it comes to subject-specific knowledge, research often focusses on teachers' content knowledge (CK) and pedagogical content knowledge (PCK). Those constructs have been successfully described, conceptualized, and operationalized in many studies so far (e.g. Shulman, 1986; Kunter et al., 2013; Hill et al., 2004). Despite some conclusive evidence that mathematics-specific knowledge is a predictor for instructional quality and student learning (e.g. Kunter et al., 2013), recent discussions pointed out limitations. In particular, it is questioned if standardized measures for teacher knowledge are sufficient to predict teachers' abilities to use this knowledge in the classroom (e.g. Blömeke, Busse, Kaiser, König, & Suhl, 2016; Knievel, Lindmeier, & Heinze, 2015). Given this issue, recent studies brought up alternative approaches of modelling and assessing teachers' subject-specific skills and abilities which are beyond declarative knowledge and complement previous research.

Subject-specific action-related competence

There are currently different approaches on expanding classical models of teacher knowledge. For the present research, we use the model of Lindmeier (2011). In this model, the understanding of subject-specific competences considers the variety of typical demands that come along with teaching a subject and, in a European tradition,

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 33-40. Singapore: PME.

defines competence as the ability to master those demands. Consequently, the model defines the ability to master "core teacher tasks" (Lindmeier, 2011, p. 108) of instructional processes as action-related competence (AC) which are characterized by spontaneous, immediate, and interactive demands (Knievel et al., 2015).

AC comes into play when teachers e.g. have to react to a conceptual misconception displayed through a student's statement during classroom discourse or have to give immediate feedback to a student's mathematical question. However, separating action-related competence from teacher knowledge from a theoretical perspective leads to the need to investigate how knowledge and competence relate to each other. It can be assumed that AC covers PCK and CK as necessary components supplemented by the ability to apply or enact this knowledge (Lindmeier, 2011). Hence, teacher knowledge is not found sufficient to master demands of actual teaching. With this study, we want to investigate which skills might be suited to disentangle the complex relation between a profound knowledge and high-quality actions.

PREREQUISITES FOR ACTION-RELATED COMPETENCE

Describing skills and abilities that are necessary for teaching on a conceptual level has relevance not only for research in teacher education but also for teacher education at university. There is evidence from the TEDS-M study indicating that characteristic differences of teacher education programmes between countries result in characteristic differences in pre-service teachers' knowledge (Wang & Tang, 2013). Evidence for the initial acquisition of action-related competence during teacher education is missing so far. Understanding the conditions that lead to teacher competence might help to improve programmes for mathematics teachers at university. In the following, we delineate individual factors that may have an effect on action-related competence.

As AC is conceptualized based on demands that typically occur during mathematics instruction, a closer look at prototypical processes of teacher action in instruction gives indications for possible influencing factors. They can be described as traits that are necessary for three steps usually modelled for such processes (e.g. Blömeke et al., 2016): 1) perceiving a situation in teaching and see what is essential e.g. paying attention to a student's production in class despite of competing attentional distractions in the complex teaching situations, 2) interpreting and making sense of the perceived, e.g. in order to identify a misconception, and 3) reacting adequately to the situation, e.g. through offering an apt hint that may turn the student's misconception into a mathematical learning opportunity. Since the demands are considered to be subject-specific, it can be assumed that teacher knowledge (CK and PCK) is needed to master these demands.

Focusing on the first steps, research on professional noticing skills may be helpful to delineate cognitive dispositions necessary at the stage of perception. For example, mathematics teachers' noticing has been described as the ability to attend (step 1) and then use existing knowledge to interpret events (step 2) that are mathematics-specific (e.g. Sherin, Russ, & Colestock, 2011). Although noticing could therefore be useful for investigating mathematics action-related competences, the need of subject-specific knowledge makes it difficult to separate perception skills from knowledge when it comes to operationalization. Another approach is followed by Miller (2011). Teachers' basic abilities to 'perceive important features in a given classroom situation' are described as teachers' situation awareness (SA) (p. 51). SA is seen as a function of general cognitive abilities which allow teachers to quickly realize simultaneous events in a situation (e.g., student 1 talks to student 2, student 3 raises his hand, student 4 is doing something under her desk). The concept of situation awareness therefore might be useful to describe teachers' perception skills for 'prototypical' instructional situations that are, to a certain extent, independent from subject-specific knowledge. However, neither the concept of situation awareness nor another fundamental perception skill has recently been investigated empirically with respect to its relation to teacher competences. Regarding the third step, skills like decision-making were suggested as influencing factors, but research on teachers is still emerging in this field.

To sum up, it is currently an open question on which traits other than subject-specific knowledge mathematics teachers' AC is based. From a theoretical perspective, situation awareness – in instructional processes – is a subject-unspecific construct which, together with subject-specific knowledge, contributes to AC.

RESEARCH QUESTIONS

Considering the need of research pointed out in the previous section, we considered the following research questions: *Is situation awareness (SA), in addition to mathematics-specific knowledge (CK, PCK), influencing action-related competences (AC) of pre-service teachers? Is there evidence for further factors contributing to action-related competence (AC)?*

METHODS

In order to investigate our research question, we administered tests for the constructs in question to a group of mathematics pre-service teachers of Kiel University (quantitative survey). On the basis of the test performance we then selected specific cases in order to investigate and identify factors influencing AC.

Instruments

This section reports on the instruments used for the quantitative survey as far as possible within the limits, as it is necessary to access the case study reported below. Mathematics AC was measured by a video-based instrument (extension of Lindmeier, 2011). Each of the 8 items contains a short video-vignette of a classroom situation typical for secondary mathematics instruction. The situations focused on problems in algebra (5 items) and calculus (3 items). Depending on the item type, the response should be e.g. an explanation that solves a students' mathematical question or an adaptive feedback that helps students with a mathematical problem without giving the solution. Since AC is characterized by its spontaneous and immediate demands, AC items had to be answered in a microphone with an oral statement under time pressure. A specialized software was used for the computer based implementation (see Lindmeier, 2011, Knievel et al., 2015 for a details on AC operationalization).The resulting audio recordings were coded and scored by three trained persons independently under usage of a detailed a-priori developed manual. Partial scores were applied (score 2: adequate; score 1: partially adequate, score 0: inadequate answer). The responses were considered adequate if they comply with the following aspects of high quality teaching (cf. Knievel et al., 2015): correctness of content, building on students' thinking, and clarity and appropriateness of explanation/stimulus without giving irrelevant information. First results for interrater reliability were acceptable with a range of $\kappa = .65–.89$ (Fleiss' Kappa).

Items for assessing mathematics PCK and (school-related) CK have been developed for pre-service teachers in previous studies (e.g. Loch, Lindmeier, & Heinze, 2015). For the present study, we used their empirical results to select items and assemble the instruments (PCK: 12; CK: 7 items). All items were in a constructed response format.

Situation awareness was conceptualized as a subject-unspecific ability of teachers to perceive critical incidents. In particular, our conceptualization of situation awareness focusses on classroom management issues and situations that typically occur in class (e.g. noticeable or inappropriate student behaviour). We developed an 8-item instrument using material of other-than mathematical instruction of a German video study (Seidel, Prenzel, & Kobarg, 2005). For each item, a short video clip is to be watched once. After that, a constructed-response question shows up offering (true and false) details of the situation in the video-clip (Figure 1).

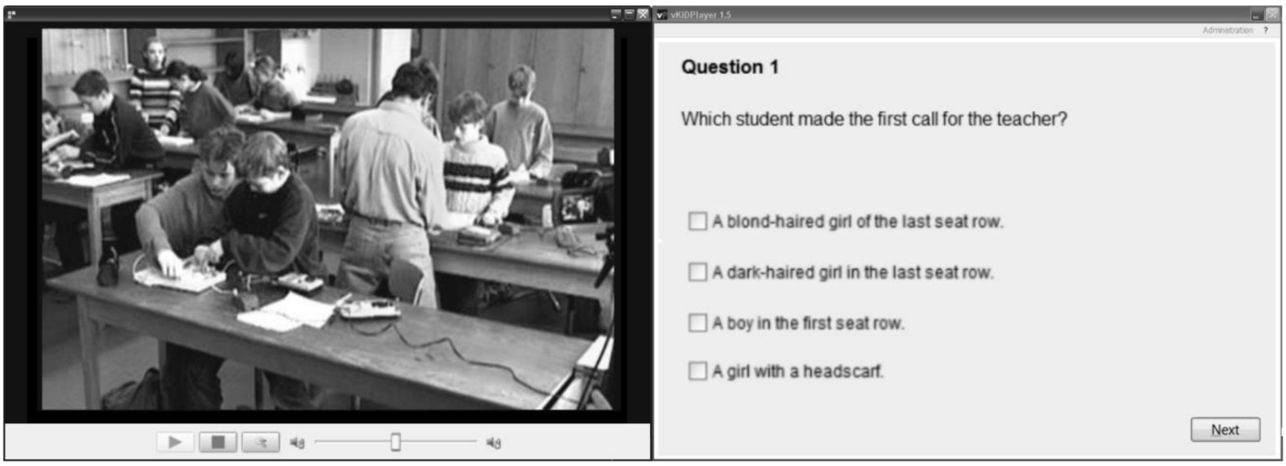

Figure 1: Item for situation awareness (left: video clip; right: single-choice question).

Case selection

The survey was conducted with pre-service teachers enrolled for the mathematics teacher programme for upper secondary level at Kiel University (Germany). For our analysis, we selected a sample of 4 cases (3 females, mean age 25.3 years) out of 41 according to the following criteria: Person shows high scores in CK, PCK and SA, but low scores in AC. Five participants reached the criterion for CK, PCK and SA, with only four of them having low AC-scores below the average. Theoretically speaking, those four participants should have good pre-requisites for achieving high scores in AC, but were not able to achieve expected scores. Overall, the AC data of this sample contains 26 responses, 4 empty responses, and 2 missing due to test abortion. Hence, analysing those responses might give evidence why knowledge and perception skills do not suffice for the mastery of teaching demands.

RESULTS

In order to get an insight why the selected participants showed a poor performance in AC, we reviewed the oral answers in the video-based test with the aim of describing why these were not adequate. For that, we analysed whether answers rated as not-adequate in the survey (score 0 and 1) can be characterized in categories that e.g. describe problems in perceiving, interpreting or reacting to the given situation.

We found that the 26 answers can be characterized using only four categories: (1) answers that are not useful at all, e.g. statements that do not contain an explanation although it was expected, (2) answers that lack correctness of content, e.g. suggestions that are mathematically wrong, (3) answers that are partially adequate but contain supplements that are irrelevant or irritating, and (4) answers that are considered not helpful for the students, although the students' problem and the problem solution is (probably) understood by the participant, e.g. explanations that are targeted at an intellectual level far beyond the skills of the student/grade level or hints that trigger a strategy that is likely to hinder the conversion of a situation into an opportunity to learn. None of the 26 answers explicitly showed a misunderstanding of the situations that might occur due to the fact that the situations had to be answered spontaneously and under time-pressure. That gives evidence that the participants did indeed not lack perception skills, what can be seen as validity evidence for the measure of SA.

The deficit that occurred – by far – most often is characterized by category 4. The participants seemed to know what the student's problem is (which indicates sufficient content/pedagogical content knowledge in line with the case selection criteria), but were not able to phrase an answer that is helpful for the student:

Situation (Item 1):	6^{th} grade, topic: total order of fractions. Three students are working on a mathematical task. The teacher asks them, if they have finished and what exactly their task was. The students reply that they had to find five fractions between 3/8 and 7/8, but that they have found only the three fractions 4/8, 5/8 and 6/8.
Participant 2:	There are more fractions than just eighth. There are also half and quarters. Maybe you can find more fractions with this hint.

In item 1, it was asked to give a helpful stimulus so that the students may find the correct solution on their own. Participant 2 correctly focussed on different representations of the fractions. That indicates that participant 2 understood the mathematical problem and the problem of students' thinking. Possibly, the participant even knew the right strategy to solve the mathematical task herself. However, the participant prompted a strategy that might infer negatively with finding more fractions as it lacks coherence with the presented situation. Therefore, this answer is considered to be not adequate and was characterized with category 4. Overall, we found 14 other responses that comparably lack instructional coherence.

Besides that, we found 5 statements that are partially adequate, but contain further information that is irrelevant and not helpful or, even worse, irritating:

Situation (Item 3): 6th grade, topic: division of fractions. Two students were asked to present their results on the board. The first student, Simon, multiplies 4 x 3/5 using (4x3)/5. The second student, Mailin, divides 2 by 2/3 using (2:2)/3.

Participant 3: Mailin, we already discussed that multiplying and dividing fractions work differently. Do you remember the reciprocal rule of division? (…) What does division mean? What does multiplication mean? Multiplication means that we get a part of something (…) and dividing means that we divide something, e.g. to people. (…) You have to turn the second fraction upside down (…). And then, you can go on just like Simon did.

For item 3, it was asked to give a solution and an explanation for Mailin's problem. Participant 3 gave the correct solution and an adequate explanation using the reciprocal rule of division. However the participant added several phrases that do not help to solve this particular problem, in contrast, might irritate as the expressed conceptions for multiplication and division are not fitting to the problems presented.

Participant	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6	Item 7	Item 8
1	4	3, 4	3	3, 4	4	1	4	4
2	4	3, 4	4	2	1	2	4	4
3	*correct*	2	3	1	*correct*	4	4	*missing*
4	1	1	4	4	*correct*	*correct*	2	*missing*

Table 1: Classification of the answers to AC items of the selected participants
(1: no useful answer, 2: lacking correctness of content, 3: irrelevant/irritating supplements, 4: feedback not helpful)

Given the characterization of all analysed answers (Table 1), some tendencies are visible regarding the participants' action under time pressure. Participant 1 showed sufficient knowledge for perceiving and interpreting the situations but deficits in providing precise and helpful teacher actions in almost every item. Participant 2 most often showed deficits in mathematical correctness, giving evidence for lacking CK. She additionally showed deficits in providing helpful feedback twice, although the required knowledge seemed to be present. Participant 3 showed multiple deficits in CK, although this case only contained a smaller number of answers that were rated non-adequate. Participant 4 skipped two items after watching the video, which could indicate problems with understanding the situation (lacking SA, CK or PCK) or missing strategies for responding to the situation (lacking PCK).

DISCUSSION AND IMPLICATIONS

The main aim of the present study was to explore factors that may contribute to pre-service mathematics teachers' action-related competence, i.e. teachers' ability to react adequately in a classroom situation under time pressure. We selected specific participants from a quantitative survey with high knowledge (CK, PCK) and situation awareness (SA). The results indicate that the low AC of the selected participants is often not simply a lack of knowledge or situation awareness. More than half of the answers did not show an adequate or helpful teacher action for the given situations, although the participants seemed to be aware of the students' problem and the problem solution. Some of the remaining answers contained adequate approaches but turned out to be only partially adequate due to irrelevant or inappropriate supplements. Again, this gives evidence that the difficulties rather resulted from difficulties in responding than understanding a problematic situation.

Based on these results we conclude that the pre-service teachers were able to apply their CK, PCK and SA to understand the challenges in the classroom situations even under time pressure. In the terms of Sherin, Russ, and Colestock (2011), the participants noticing skills were sufficient. However, they showed a weak performance when they had to use their CK and PCK for an adequate subsequent teacher action. The latter might be caused by two different reasons. First, we see (indication of category 3) that the quality of knowledge may be a factor to be considered in more detail. Instruments in teacher knowledge usually focus on declarative knowledge, therefore, measures of teacher knowledge may not reflect the usability of this knowledge for specific situations. Recent research shows how PCK can be differentiated in the types of declarative, propositional and episodic case knowledge with expected different characteristics with respect to usability (e.g. Kuhn, Alonzo, & Zlatkin-Troitschanskaia, 2016). Future approaches may hence seek to assess also different qualities of PCK with more rigour. Second, we see (indication of category 4) that despite of a good understanding of the situation, the decision-process may lead to incoherent teacher actions. The problems in phrasing helpful teacher actions may also partly be attributed to a lack of usable teacher knowledge. The findings also indicate that a closer look on teacher-specific skills for decision-making can help to explain the difficulties.

The results of this case study should be considered as only tentative as sample size, possible selection effects regarding the overall sample as well as the design of the study constitute limitations. To overcome these limitations, further studies will be conducted. Consequently, we are currently gathering data of a larger sample of pre- and in-service teachers aiming on both corroborating our findings and being able to describe in more detail the differences between more and less competent mathematics teachers. That knowledge could not only improve current models of teacher competence, but also teacher education itself as it may yield starting-points for fostering pre-service teachers' abilities to apply their knowledge already in an early stage of teacher education.

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A FRAMEWORK TO EXAMINE THE MATHEMATICS IN LESSONS OF COMPETENT MATHEMATICS TEACHERS IN SINGAPORE

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This paper outlines an analytical framework that was developed, to examine the mathematics in mathematics lessons of competent teachers in Singapore secondary schools. The framework is guided by Schoenfeld's Teaching for Robust Understanding (TRU) framework and also the field notes of the project – A study of the enacted school mathematics curriculum which is presently underway in Singapore. The framework was trialled and the indicators were suitable but may not be comprehensive. Therefore more trials and also more codes on how the teacher aided students in developing mathematical knowledge and student engagement with mathematical ideas are needed. In addition student perspectives of the lesson are also necessary to make any valid claims related to the quality of the lessons.

TEACHING OF MATHEMATICS IN SINGAPORE SCHOOLS

A few studies done so far provide us with glimpses about how teachers teach mathematics in Singapore schools (See Kaur & Yap, 1997; Chang, Kaur, Koay & Lee, 2001; Kaur, 2009; Hogan et al., 2013). In this paper we briefly elaborate the findings of two studies, the CORE 2 research led by Professor Hogan (Hogan et al., 2013) and Learner's Perspective Study (LPS) in Singapore led by Professor Kaur (Kaur & Low, 2009). As part of the CORE 2 research the quality of the enacted curriculum in Secondary 3 (grade 9) mathematics lessons was assessed using criteria and standards identified by Hattie in Visible Learning (2012). Thirty-one mathematics teachers, sampled randomly, teaching secondary three mathematics in Singapore participated in the study. Sequences of lessons of the teachers in the study were video-recorded. Two main findings from the study were that: i) teachers focused more on procedural knowledge than conceptual knowledge and engaged students in domain-specific knowledge practice in about a third of the phases of a typical lesson. Of the domain-specific knowledge practices, knowledge representation was emphasized. Also, procedural learning support was evident as teachers often helped with the "how to do" steps; ii) students were engaged in doing performative tasks (77.3%) more often than knowledge building tasks (22.7%). A performative task mainly entails the use of lower order thinking skills such as recall, comprehension and application of knowledge while a knowledge building task calls for higher order thinking skills such as synthesis, evaluation and creation of knowledge.

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Kaur (2009) in her study of grade eight competent mathematics teachers found that lessons were deemed good by students and teachers when they had the following characteristics: i) whole-class demonstration (exposition) where the teacher explained clearly the concepts and steps of procedures; made complex knowledge easily assimilated through demonstrations, use of manipulatives, real life examples and introduced new knowledge, ii) seatwork and out of class assignments where teacher gave clear instructions related to mathematical activities for in class and after class work; provided interesting activities for students to work on individually or in small groups; provided sufficient practice tasks for preparation towards examinations, and iii) review and feedback – where teacher reviewed past knowledge, and used student work or group presentations to give feedback to individuals or the whole class.

The findings of both Hogan et al. (2013) and Kaur (2009) indicate that there appears to be an apparent focus on the development of skills in mathematics lessons in Singapore schools. These findings certainly do not explain the stellar performance of Singapore students in PISA 2012 and 2015 that required students to complete tasks that were of the knowledge building type (OECD, 2014, 2016). As noted by Fan and Bokhove (2014), perhaps mathematical algorithms lead to proficiency of skills thereby stimulating thoughts about the conceptual aspects of the mathematics explored.

At present a study of the enacted school mathematics curriculum (secondary schools) is underway in Singapore. It attempts to document the practices of 30 competent mathematics teachers. The study aims to examine i) pedagogies adopted by competent teachers when enacting the curriculum, and ii) competent teachers' use of instructional materials for the enactment of the curriculum. Amongst others, one of the research questions explored is "How does the pedagogy of the teachers compare with that of mathematically powerful classrooms advocated by Schoenfeld (2011)?" To explore this question an appropriate analytical framework, comprising five parts, is being developed by the researchers of the study. This paper is based on one part of the framework which is used to examine the mathematics in the mathematics lessons of competent teachers in the study.

MATHEMATICALLY POWERFUL CLASSROOMS

The three decades of extensive research by Schoenfeld in the US on mathematical problem solving and mathematics instruction (2011) affirms that people's moment to moment decision making in teaching can be modelled as a function of their i) resources (esp. knowledge); orientations (esp. beliefs) and goals. He advocates that the five dimensions of mathematically powerful classrooms are: i)The mathematics context; ii) Cognitive demand; iii)Access to mathematical content; iv)Agency, Authority and Identity; and v)Uses of assessment. The Teaching for Robust Understanding framework proposed by Schoenfeld, Floden, and the Algebra teaching Study and Mathematics Assessment Project (2014) provides a tool for teacher learning and growth, according to the five dimensions of mathematically powerful classrooms, with regards to student learning of mathematics. Figure 1, provides a general top-level description of the Teaching for Robust Understanding (TRU) framework Schoenfeld, 2016, p. 10). In our study reported in this paper, we use the TRU framework instead to examine two dimensions, namely the mathematics and cognitive demand, in mathematics lessons of two competent teachers in Singapore.

The Five Dimensions of Mathematically Powerful Classrooms	
The Mathematics	The extent to which the mathematics discussed is focussed and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained.
Cognitive Demand	The extent to which classroom interactions create and maintain an environment of productive intellectual challenge conducive to students' mathematical development.
Access to Mathematical Content	The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed by the class.
Agency, Authority, and Identity	The extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to their development of agency and authority resulting in positive identities as doers of mathematics.
Formative Assessment	The extent to which the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings.

Figure 1. The five dimensions of mathematically powerful classrooms

METHODOLOGY

The analytical framework we created for the dimension: The Mathematics was guided by i) the respective prompts for teacher thought and discussion in the TRU guide (Schoenfeld, Floden, and the Algebra Teaching Study and Mathematics Assessment Project, 2014), and also the field notes from the project – A study of the enacted school mathematics curriculum.

Four researchers involved in the study of the enacted school mathematics curriculum, contributed towards the crafting of the indicators guided by the prompts from the TRU framework. Figure 2 shows the analytical lens that was created to examine lessons of competent mathematics teachers in Singapore for the dimension – The Mathematics. The analytical lens crafted was used to examine the lessons of two teachers. These teachers were "experienced and competent", where experience is a measure of the number of years they have taught mathematics in secondary schools and competency is a composite measure of their students' performance at examinations and their performance in class in the eyes of their students. The teachers were nominated by their respective school leaders and the research team followed up on the nominations and interviewed the teachers. A strict requirement for participation in the study was that the teacher had to teach the way she / he did all the time, i.e. no special preparation was allowed. The lessons of these two teachers were selected, as they are both lead teachers and they also taught the same topic. Teacher 1

[T1] is a male who has taught mathematics for the last 20 years and Teacher 2 [T2] is a female teacher who has also taught mathematics for the last 20 years. For both teachers sequences of their lessons were recorded according to the protocol developed for the Learner's Perspective Study in Singapore (Kaur, 2009).

Dimension 1 – The Mathematics	What we looked out for in the lessons?
Aspect	Indicators
Were the mathematical goals of the lesson apparent?	Did the teacher articulate the goal/s of the lesson? Did the teacher articulate the goal/s of the mathematics students worked on during the lesson? Did the teacher articulate the goal/s of the mathematics students were assigned to do after the lesson during out of class time?
Were important ideas in the lesson connected with those in past and future lessons?	Did the teacher connect the important idea/s in the lesson to what students already know? Did the teacher relate concepts to each other — not just in a single lesson, but also across lessons and units in past and future?
How were math procedures in the lesson justified and connected with important ideas?	How did the teacher develop mathematical knowledge in the class? (Telling and showing / developing concepts through student activities / through systematic logical steps) Did the teacher identify the important ideas behind concepts and procedures? Did the teacher highlight connections between skills and concepts?
Were students engaged with mathematical ideas during lessons?	Did the teacher get the students to participate in meaningful math learning, so that they could make sense of concepts and ideas for themselves during lessons? Did the teacher get the students to participate in meaningful math learning, so that they could make sense of concepts and ideas for themselves as part of their out of class work after lessons? Did the teacher engage the students in authentic performances of important disciplinary practices (e.g., reasoning from evidence, communicating one's thinking, clarifying doubts, etc.) Did the teacher invite the students to explain things, or just give answers?

Figure 2. Analytical lens for the dimension – The Mathematics

Altogether two, the first in the sequence of lessons of the two teachers were coded. Both teachers were teaching the same topic – Vectors and they covered the same content during their first lesson. As part of the science curriculum, students had knowledge of vectors as this topic had been taught to them by their Science teachers during Physics lessons. The mathematical ability of students in the class of T1 was slightly below average as they were from the 40^{th} percentile of their cohort and those in the class of T2 were from the 50^{th} percentile of their cohort.

The coding was done in the following manner. Two researchers viewed the video-records of the lessons. They first segmented a lesson into episodes. An episode was delineated by the beginning and end of an activity, for e.g. it may comprise the teacher beginning the lesson and telling the class about the day's lesson, or the beginning of an activity that had a specific goal such as engaging students in recall of past knowledge. Next they scanned one episode at a time for indicators of the dimension and recorded its presence. The inter-rater agreement was 83%. When a disagreement arose, the two researchers discussed their differences and arrived at consensus, either agreeing on the presence of the indicator or dismissing it.

The following show a few of the indicators with sample excerpts from the lessons.

<u>Were the mathematical goals of the lesson apparent?</u>

Did the teacher articulate the goal/s of the lesson?

> T2 – Episode 1: (2:50) we will see what are vectors, how do we represent vectors on a diagram, how do we find magnitude, add/subtract vectors, and the use of vectors.

Did the teacher articulate the goal/s of the mathematics students worked on during the lesson?

> T1 – Episode 11: 20:18) I'd like to test your understanding now….(25:16) The reason why I'm giving you this task is …

Did the teacher articulate the goal/s of the mathematics students were assigned to do after the lesson during out of class time?

> T2- Episode 11: 52:38) I want you to do some thinking on your own. You need to understand what you've just learnt.(53:33) Why do I give you part a and part b? Are they the same? (54:05) You will show me the answers tomorrow. And then tomorrow we will do addition of vectors.

<u>Were important ideas in the lesson connected with those in past and future lessons?</u>

Did the teacher connect the important idea/s in the lesson to what students already know?

> T2-Episode 2: (05:05) How do you represent your vectors when you do Science?

Did the teacher relate concepts to each other — not just in a single lesson, but also across lessons and units in past and future?

> T1-Episode 2: 42:21) Many quantities have only magnitude… you are all familiar with that in the primary school. When you come to secondary school, you started learning in physics, … These are the various quantities that you are familiar with.

<u>How were math procedures in the lesson justified and connected with important ideas?</u>

How did the teacher develop mathematical knowledge in the class?

> T1-Episode 7: (08:37) Now, what did you observe about these four vectors? How are they different and how are they the same? (09:14) What other observations did you observe? (10:08) What do you notice about OA and OC?

T2-Epiosde 4: (15:00) If your vector is not represented by a column vector, then how do you find the magnitude? … And you will use all kinds of knowledge that you have to find length. (16:49) Look at the diagram and ask yourself, what do you know? What are the concepts, what are the skills your already have? What can you use to find …

Table 1, shows the number of episodes in which the respective indicators were present in the lessons of the two teachers.

Dimension 1 – The Mathematics	Teacher 1 (18 episodes) (68 minutes)	Teacher 2 (12 episodes) (52 minutes)
Did the teacher articulate the goal/s of the lesson? Did the teacher articulate the goal/s of the mathematics students worked on during the lesson? Did the teacher articulate the goal/s of the mathematics students were assigned to do after the lesson during out of class time?	8 (2+7+1)*	6 (1+3+2)
Did the teacher connect the important idea/s in the lesson to what students already know? Did the teacher relate concepts to each other — not just in a single lesson, but also across lessons and units in past and future?	5 (3+4)	5 (3+7)
How did the teacher develop mathematical knowledge in the class? (Telling and showing / developing concepts through student activities / through systematic logical steps) Did the teacher identify the important ideas behind concepts and procedures? Did the teacher highlight connections between skills and concepts?	6 (10+7+5)	5 (8+5+4)
Did the teacher get the students to participate in meaningful math learning, so that they could make sense of concepts and ideas for themselves during lessons? Did the teacher get the students to participate in meaningful math learning, so that they could make sense of concepts and ideas for themselves as part of their out of class work after lessons? Did the teacher engage the students in authentic performances of important disciplinary practices (e.g., reasoning from evidence, communicating one's thinking, clarifying doubts, etc.)? Did the teacher invite the students to explain things, or just give answers?	5 (5+1)**	7 (7+1)

Note: in some episodes, more than one aspect was present. Also in some episodes more than one indicators of an aspect was present. * (?+?+?) shows the number of times the respective indicators in an aspect were present. ** (?+?) represents the number of episodes for the first two indicators.

Table 1: Number of episodes where the respective indicators were apparent

FINDINGS AND CHALLENGES

From Table 1, it is apparent that for all the four aspects of the dimension – the Mathematics the indicators crafted by the researchers were apparent in the episodes of the lessons of the two teachers, though with varying density. We found the indicators suitable but may not be comprehensive as they were only trialled with two lessons. Therefore they have to be trialled more extensively. In addition, we also found that specifically for the indicators:

- How did the teacher develop mathematical knowledge in the class?
- Did the teacher engage the students in authentic performances of important disciplinary practices (e.g., reasoning from evidence, communicating one's thinking, clarifying doubts, etc.)
- Did the teacher invite the students to explain things, or just give answers?

we needed sub-codes to capture the range of approaches used by the teachers. Some of these approaches may be unique to the pedagogy of mathematics learning in Singapore. Furthermore in trying to rate the lessons according to the rubric shown in Figure 3 taken from Schoenfeld (2011) we felt that the level of both lessons may be rated as high but a more fine grained rubric may be needed to differentiate between lessons at this level for our research project.

Dimension	Level	
The Mathematics How accurate, coherent, and well justified is the mathematical content?	Low	Classroom activities are unfocussed or skills-oriented, lacking opportunities for engagement in key practices such as reasoning and problem solving.
	Medium	Classroom activities are primarily skills-oriented, with cursory connections between procedures, concepts and contexts (where appropriate) and minimal attention to key practices such as reasoning and problem solving.
	High	Classroom activities support meaningful connections between procedures, concepts and contexts (where appropriate) and provide opportunities for engagement in key practices such as reasoning and problem solving.

Figure 3. Summary Rubric of Dimension 1 – The Mathematics

Also, to make any valid and rigorous claims, we feel that we have to interrogate our data from the perspective of the students and answer the following questions which were presented by Schoenfeld (2016) during his plenary lecture at PME 40 in Szeged. The questions are:

Dimension 1 – The Mathematics

- What's the big idea in this lesson?
- How does it connect to what I already know?

Dimension 2 – Cognitive Demand

- How long am I given to think, and to make sense of things?
- What happens when I am stuck?
- Am I invited to explain things, or just give answers?

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THE TEACHER IDENTITY OF MATHEMATICS TEACHERS

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This study explores how mathematics teachers' pedagogical identity develops in the social context of their classroom interactions, and what challenges teachers perceive in advancing their pedagogical identities. The study draws on a dialogical approach to identity that sees the self as something that individuals develop through interaction between their core "substantial self" and context-dependent "situational selves." Data were collected from four in-service high school teachers. The findings shed light on the variability of mathematics teachers' pedagogical identity and the processes through which they develop that pedagogical identity in the classroom context.

INTRODUCTION

Math teachers sometimes face a disjuncture between their natural inclinations as teachers and who they are expected to be in the classroom (Gainsburg, 2012). For example, many credentialing programs favor reform-oriented teaching approaches that emphasize classroom interaction. A math teacher thus educated might believe that reform-oriented teaching is the most effective approach. But if such a teacher is naturally introverted, she might struggle to implement student-teacher interactions. This study seeks to understand the nature of such struggles in terms of *teacher identity*.

Grootenboer and Ballantyne (2010) defined teacher identity as a teacher's own conception of who s/he is as a teacher, including beliefs, classroom behaviors and learning experiences. In the context of teaching math, Grootenboer, Smith and Lowrie (2006) held that teachers' identities change in response to continuing experience, continuing education and new dilemmas. Aligning with this perspective, this study seeks to learn how mathematics teachers' identities develop and interact with the ways the teachers design and implement classroom activities. The exploration of these issues will help us understand how the social setting of the mathematics classroom, which determines classroom interaction, shapes and is shaped by teacher identity.

Peressini et al. (2004) considered teacher identity (or *professional identity*) to have both *cognitive aspects* – goals, values, commitments, knowledge, beliefs – and *sociocultural aspects* – the ways in which teachers participate in the activities of their professional communities and present themselves to others in the context of professional relationships. In this study, I define the *pedagogical characteristics* of a teacher as the elements that constitute both cognitive and sociocultural aspects of teacher identity, and I refer to a math teacher's unifying and connective concept that

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brings together these elements in the classroom context as *pedagogical identity*. To understand how pedagogical identities are formed, I adopt a *dialogical approach* to identity (Akkerman & Meijer, 2011), which considers the *self* as comprising and balancing between a *substantial self* and *situational selves* (Nias, 1989). The substantial self is embodied by beliefs and values shaped in one's early years, and is relatively impervious to change. Situational selves incorporate such beliefs and values, but change over time and context. The study's main research question is: how does a math teacher's pedagogical identity develop in the social context of her/his classroom interactions? In particular, how do math teachers' pedagogical experiences prior to their teaching, such as classroom experience as a learner, inform their pedagogical identities in the early years of their teaching career? And how do teachers achieve balance between their substantial and situational selves in the classroom context?

DIALOGICAL APPROACH TO IDENTITY

The notion of *identity* has changed over time, reflecting changing value systems (Akkerman & Meijer, 2011). In the modern era, identity was perceived as singular (not varied or dynamic within the individual), continuous (keeping the core identity consistent regardless of the social context) and individual (regardless of the social environment). In contrast, in the postmodern view of identity, it is decentered into multiplicity in the sense that an individual has multiple identities, different ones of which come to the fore depending on the social setting; discontinuous in the sense that the multiple identities that emerge in different social settings are not necessarily interrelated; and social in the sense that identity is understood in a social context. These three characterizations of the postmodern view suggest that identity is *neither* an overarching and unified framework *nor* a fixed, stable entity. Rather, it is viewed as being fragmented along with the multiple social worlds that people engage in, and as shifting with time and the context of the society of which people are a part.

Akkerman and Meijer (2011) typified teacher identity as both unitary and multiple, both continuous and discontinuous, and both individual and social, with the two opposing natures (unity-continuity-individuality and multiplicity-discontinuity-sociality) taking turns in a dialogical relationship of intersubjective exchange and temporary dominance. That is, one develops identity by engaging in a dialogical relationship between the two. In this dialogical relationship, the self is understood further as both the core, substantial self (unity) and the situational selves (multiplicity) (Nias, 1989; Rodgers & Scott, 2008). Drawing upon this dialogical approach to identity formation, this study aims to understand the dialogical nature of teachers' identity development. Some studies have discussed how the personal histories of novice teachers influence their teaching in the context of the workplace (e.g., Flores & Day, 2006). But little research has taken a dialogical approach to understanding how teachers' pedagogical characteristics inform their identity as teachers.

METHODS

Using field observations, surveys and interviews, I collected data from four in-service math teachers at two high schools – School A and School B – in the United States. Table 1 summarizes the participants' pedagogical backgrounds.

Teacher	School	Experience	Classes observed	Grade levels taught
Mr. A	A	3.5 years	Pre-Calculus	9–11
Mr. B	A	5 years	Advanced Calculus 2	9–11
Ms. C	B	4 years	Algebra 1	9
Mr. D	B	18 years	(Remedial) Algebra 1	9–12 & lower levels

Table 1: Pedagogical background of teacher participants

Speer (2005) described teachers' beliefs and practices as *professed* if stated by the teachers, and *attributed* if inferred based on observations. I used field observation believing that "attributed" practices would depict the participants' behavioral patterns. During observation, I took field notes in which I focused on pedagogical habits the teachers had developed to facilitate learning, the kinds of teacher-student interactions they allowed to occur in class, what difficulties they revealed in managing classroom activities, how they dealt with unexpected student behavior, how they responded to student questions and whether they showed signs of making efforts to overcome difficulties they encountered. I conducted field observations two to nine times for each teacher over a six-week period. Observed classes were audio-recorded. Only the parts of the recordings deemed significant for the study were transcribed, including participant comments that (1) revealed the teacher's pedagogical characteristics; for example, pedagogical beliefs, and (2) characterized their teaching styles.

After I completed field observations, participants took a 17-item survey. The first eight items asked about their academic background. The other items sought the participants' perceptions of their value systems: the factors of their pedagogical knowledge they value most, their math teaching philosophy, how they compare their own learning experiences as a student with the teaching they do, and whether they experience conflicts caused by disjuncture between their personal and pedagogical identities.

The interview inquired into the participants' reasons for certain practices. The actual interview questions varied to reflect the participants' individual responses to the survey items. The interviews took about one hour and were audio-recorded for transcription and coding purposes. All interviews were transcribed completely, and coded with the *line-by-line* and *focused* coding methods commonly used in grounded theory approaches (Charmaz, 2004). Continually reading the transcripts within and across categories, I first developed preliminary hypotheses and then picked out valuable, overarching, emerging themes, using concept maps, by comparing and contrasting the participants' different pedagogical characteristics.

RESULTS

The results largely rely on the teachers' descriptions of what they thought contributed to their pedagogical characteristics. I address notable pedagogical characteristics of each teacher, and discuss them in terms of Nias's (1989) view of the self as a balance between the substantial self (*I*-position) and the situational selves (*me*-position).

***Mr. A*:** Mr. A's interview revealed an autonomous learning inclination, teaching approaches characterized by minimally interrupting his classes and imposing few rules, and an introverted social nature.

His autonomous learning inclination is evidenced in [A].

> [A] I learned it [math] by trying different things, and seeing. … [I was] trying, you know, 'how do we figure this out?' For me, it was always a puzzle. That was what math was about. Doing things you didn't know how to do given certain relationships.

As [A] implies, Mr. A's belief that students should learn by trying different things comes from his own inclination toward learning autonomously. [B] and [C] connect this teaching belief to his own autonomous learning inclination.

> [B] I try to teach it [autonomous learning] … which I think actually doesn't work very well for teaching because I think a lot of kids aren't trying different things.

> [C] …[autonomous learning] is only successful with some of the kids. For most of the kids, you still have to give them the information you wanted them to know in the end. I am not gonna say, 'Hey, you gotta figure it out. I am not gonna help you until you figure it out.'

[B] and [C] show Mr. A's frustration in implementing autonomous learning. In particular, [C] shows his compromises to meet his students' needs, and implies that he takes a *me*-position regarding his belief in autonomous learning.

Mr. A views students' working on difficult problems as a valuable learning experience. This is shown in his written ([D]) and verbal ([E]) responses to a survey item, which asked about conflicts between the math teachers' own beliefs about what/how to teach and others' expectations of what/how they should teach.

> [D] … I think that mostly a sense of inadequacy arises at times about how I conduct my classroom and how well my students perform … it is a valuable experience for kids to work with difficult math … sometimes I let them struggle rather than just hold their hand.

> [E] I would rather let kids fail than just be this constant coach. … If I have to push a kid every step of the way, I don't think they are learning. They may be learning some math, but in the end, they don't learn how to deal with difficult things in their life[1]. … If I try to help motivate kids, but if they just continually give up, then I let them give up. … If a kid puts his head down for whatever reason, day in and day out, I will try to make a real connection with them. … But I am not gonna go every class like, 'take your head up'[2].

These claims confirm that Mr. A values "learning by discovery": he will let his students struggle in the hope that they will move forward in discovery-learning (E[1]). However, he would rather give up on "strongly unmotivated students" if improving their learning requires intervening in their work frequently (E[2]). This contrast shows how he handles conflict between what he believes is ideal (having students learn by discovery) and reality (having unmotivated students). His refusal to compromise – balancing closer to his beliefs – shows his *I*-position in handling such conflicts.

***Mr. B*:** Mr. B's interview showed his passion for teaching, tendency to associate with his students, and an inclination to learn math concepts by justification. His prior learning and teaching experiences seem to have affected his pedagogical identity.

To a question asking what challenges one faces in interacting with students and how he or she manages the challenges, Mr. B made the following two claims.

> [F] I personally think these (unmotivated) kids are kids with needs that aren't being met … they are way higher up than a math education. You know, like, 'where am I going to spend the night tonight, what am I going to eat when I get home', stuff like that. I don't know if high school has an answer for those [pause], those kids.

> [G] … my job is to teach kids algebra, whatever their course is, but also to teach a lot of life skills there as well. I think they get a lot of that from me.

As [F] and [G] show, Mr. B sees the scope of his teacher role as extending beyond teaching math. This implies that he takes a *me*-position to accommodate student needs. To a question about challenges in promoting student-teacher interaction, Mr. B responded:

> [H] … it's not a typical room, in the sense that 'you'll get in there and I'll be lecturing and everyone will be writing down what I am saying.' They are freer to talk … they care about what they are doing. They are pretty directed, but <u>there are times when you have to talk to kids and say 'hey, you are not really doing what you need to be doing</u>.'

As [H] shows, Mr. B perceives student distraction as a disciplinary challenge.

Further, [I] shows Mr. B's preference for allowing students the freedom to talk over upholding discipline.

> [I] <u>The disciplinary thing is tough for me,</u> um because, you walk by some classrooms and you say, 'Wow, look at all those kids just sitting there', you know. I don't think you see that in my room. So I think it's a weakness in some regard. But, it's a sacrifice I'm kind of making. … <u>I know the disciplinary stuff might be there and I'm allowing for it to be there, but I think it's a chance that is worth taking</u>.

As [I] shows, he engages in self-reflection in his balancing, and is willing to compromise on discipline to attain student learning in his classroom context, implying that Mr. B is balancing closer to his *me*-nature than his *I*-nature.

***Ms. C*:** Ms. C had an inclination to modify her teaching strategies, and an introverted social nature. Further, her teaching experience appears to be the dominant factor informing her teaching style for the first three years of her teaching career.

Her inclination to modify teaching strategies is shown in [J] and [K].

> [J] I think throughout the college classes they weren't very good about preparing you for the real life situations. … <u>Every child had a different need, every year there were different kids, a different chemistry in the classroom, a different dynamic, so I had all these modified myself, and you know the lessons, to try to reach out to more and more students</u> … I don't think you can learn that without the experience.

> [K] I learned the hard way that those types of kids won't do homework. So, I don't assign homework anymore. … I used to give homework to everyone, and …

Ms. C's comments in [K] show a compromise in her pedagogical approach between what she had believed her students would do to learn and what they really turned out to do, providing evidence of her *me*-position. Ms. C further described the challenges that lead to her continuous modification, as in [L].

> [L] A teacher education [program] may really emphasize the set-up of lesson plans … but I learned every day isn't predictable … some days you may have to throw your lesson out the window and say 'you know what? This kid had a bad day. I'm not going to be able to teach as much as I wanted to, the kids are rowdy' and you have to keep molding it. So, my belief has changed in terms of preparation. You have to be prepared to be unprepared.

This excerpt explains how Ms. C's beliefs differ from what teacher education programs suggest and how she came to these beliefs. [L] implies that she takes a *me*-position in adapting her teaching methods to fit her class's nature.

[M] and [N] again show Ms. C taking a *me*-position, as she describes herself as an introvert who tries to act like an extrovert to bring energy to her classes.

> [M] I am shy, and self-conscious in a crowd … but in the classroom you are the center stage, so it kind of forces you to become an extrovert whether you are or not.

> [N] I may be, um, a little too friendly. … The atmosphere, I try to make it light and airy, and I try to make sure the kids are joking … I want it to be a positive experience.

[O] further shows her taking a *me*-position in designing her classes, as she carefully considers the math level of the students:

> [O] I would still bring my energy (to upper level classes). I would still bring the positive attitude. … I really want to make math fun and approachable using my personality and students' personalities, but I would do less side conversations and more material.

***Mr. D*:** Mr. D had a unique one-on-one teaching approach: he sat at his desk for most of the class, having each student come to him to teach the content individually. His pedagogical identity originated largely from his non-academic work experience.

Mr. D's understanding of the challenges struggling students face is shown in [P].

> [P] You saw me how many times in class, where some things happened. … they might start yelling … when you are a lawyer, conflict is part of the job. … you have to be able to communicate in a way where you can get your point across without creating conflict or getting all emotional. … I did represent a lot of kids in court. And I represented parents who had their children removed by the state because they were being [inaudible] by their parents. And any of these situations create, um, conflict at home, and negative situations at home. So I've been in there and I understand what, when they leave school and go back home, I know what they go to.

As Mr. D's comments in [P] imply, his conflict-handling experience as a lawyer helps him diffuse tense situations with his students. To my follow-up question, "You have this strong idea that individual relationship with the students is an important thing … where do you think this idea came from?" he responded:

> [Q] I think I know where kids are from, I mean I know what they are living … treating someone as an individual and trying to know what makes them tick, to get them to perform clearly is from coaching. And those were the people that influenced me the most.

[Q] suggests Mr. D's belief in treating students in need the way a sports coach treats players. Further, he sees the importance of treating students in ways that work for them. This shows Mr. D's taking a *me*-position in how he deals with his students.

Further, Mr. D had a theory most math conceptualization starts to happen around 5th grade. To my question, "In math education we talk a lot about the conceptual understanding versus the procedural [inaudible], do you ever think of these things when you teach?" Mr. D responded:

> [R] Absolutely … we do various types of visual type situations … <u>I did a lot of it with 5th through 8th grade, which is where your conceptualization should really be founded in the middle schools</u>. I did far more of that in the middle school (than the high school).

Mr. D's view that math conceptualization happens at a certain grade band may be attributed to either his (15 years of elementary school) teaching experience or his lack of math content knowledge (he has taken no math courses at the college level). Such experiences might have impacted how he balances between *I*- and *me*-positions. Mr. D teaches high school students with practices based on pedagogical views developed from his K–8 teaching experience. He believes that he has a good understanding of struggling students, but he does not try to understand how his high school students learn, implying that he balances closer to an *I*-position in his high school teaching.

DISCUSSION

This study shows the varying routes math teachers take to form their pedagogical identities. Each participant had found a different balance between an *I*-position and a *me*-position, and had distinct pedagogical beliefs and practices that had been shaped by different factors. For example, Mr. B expanded the scope of his teacher role beyond teaching the subject, and Ms. C, who claimed to be an introvert, acted as an extrovert to bring energy to her classes, both showing a stronger *me*-nature than *I*-nature. Mr. D, in contrast, seemed to take a *me*-position on certain issues and an *I*-position on others.

The findings of this study show how math teachers' pedagogical identities constantly change as different factors that shape identity come into play. The study contributes to the math education community by providing evidence that the formation of a math teacher's pedagogical identity impacts his or her choice of teaching approaches. By shedding light on how math teachers develop their pedagogical identities, the study provides an explanation for the disjuncture between what teachers do in the classroom and the practices that are encouraged by teacher education programs (Gainsburg, 2012). If such disjuncture are inevitable, then preservice math teachers need to be aware (1) that they will be engaged in struggles between their beliefs, or who they are, and what they face in reality in their early years of teaching and (2) that they will need to develop ways to respond to students, including incorporating student reactions in their class design. In other words, they should be prepared to take a *me*-position to reconcile their beliefs or natural inclinations to their teaching context if necessary.

This study's conclusions are largely based on the participants' professed beliefs and practices (Speer, 2005), which may lack accuracy: a teacher may not be aware or may not be frank with a researcher regarding his or her own pedagogical identity development. These limitations constrain the extent to which the findings can be generalized. Larger-scale research that depends less on self-report would provide firmer and more generalizable findings. Further research is needed to identify factors that influence how teachers' negotiation between their substantial and situational selves leads to each one's unique balance between a *me*-position and an *I*-position, and in turn affects their teaching practices.

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PRESERVICE TEACHERS' RECOGNITION OF AFFORDANCES AND LIMITATIONS OF CURRICULUM RESOURCES

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This study examined elementary preservice teachers' interaction with curriculum resources, focusing on their recognition of affordances and limitations of the resources in the context of lesson planning in a mathematics methods course in the United States. Because of the prevalence of the curriculum programs with a direct teaching model in the country, preservice teachers need to develop the capacity to use them productively to design instruction. For this reason, the preservice teachers in this study were asked to critique and modify lessons from such programs to make lesson plans in the methods course. Data were gathered in this setting and analysed to inform for better teacher preparation using curriculum resources.

INTRODUCTION

This study examined preservice teachers' (PSTs) recognition of affordances and limitations of curriculum resources in the context of lesson planning in a mathematics methods course in the United States. Recognizing affordances and limitations of resources is critical in designing instruction. For a productive enactment of curriculum, teachers need to utilize its affordances and fill the gap in it. This study focused on PSTs' critiques of written lessons with a direct teaching model in terms of the extent of student engagement with mathematical exploration and teacher support for it. The purpose of the study was to account for PSTs' reasoning about curriculum resources in order to design a better methods course that helps them develop *pedagogical design capacity* (PDC), i.e., "a teacher's skill in perceiving affordances [of resources], making decisions, and following through on plans" (Brown, 2009, p. 29). The results of the study provide implications for using curriculum resources in teacher education.

THEORETICAL BACKGROUND

Curriculum resources in this study are defined as artefacts that mediate teachers' instructional actions (Brown, 2009). They are static representations of content and pedagogy, which teachers enact and make dynamic for student learning in instruction. Prior studies examined how inservice and preservice teachers interpret reform-based materials (Atanga, 2014; Kim & Atanga, 2014; Lloyd, 2009; Nicol & Crespo, 2006; Son & Kim, 2015) and how teachers' evaluation of curriculum leads to various adaptations (Sherin & Drake, 2009). Investigating teachers' use of curriculum resources to design instruction, Atanga (2014), Kim (2015), and Kim and Atanga (2014) found that some teachers did not recognize significant affordances of the

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resources they used and failed to utilize them in critical moments during instruction. For example, whereas the written lesson includes helpful intervention suggestions for struggling learners, the teacher, not using them, mainly repeated the same procedural explanations to students in confusion (Kim, 2015).

Researchers argue that curriculum resources can be inherently educative for teachers (Ball & Cohen, 1999; Davis & Krajcik, 2005) and that teacher education should incorporate teachers' investigation of curriculum resources as a pathway to building both content and pedagogical knowledge (Drake, Land, & Tyminski, 2014; Son & Kim, 2015). In particular, Drake et al. (2014) emphasized the importance of supporting PSTs' learning about and from curriculum materials in elementary mathematics methods courses and recommended principles for using curriculum resources, such as the need for providing opportunities for PSTs to attend to educative features in the curriculum resources. Using both curriculum resources with a direct teaching model and those with a student-centered model, this study further accounts for how curriculum materials can be used in teacher preparation in order to develop PSTs' pedagogical design capacity.

METHODS

The data of the study were collected from 19 PSTs in an elementary mathematics methods course about two thirds of the way through the semester.

The Setting

Early on in the semester, the PSTs were introduced to a set of standards, including the *Mathematical Practices* of the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and *Professional Standards for Teaching Mathematics* (National Council of Teachers of Mathematics, 1991), to analyse videotapes of mathematics classrooms, such as whole group discussions and interactions between a teacher and a student. These standards were also used to examine to what extent written lessons provided opportunities for students to engage in mathematical explorations. The purpose of these examinations was to help the PSTs develop a critical thinking about curriculum resources and be prepared to use them productively to design instruction in the future. The course included an opportunity for the PSTs to look at one lesson with a direct teaching model and one with a student-centered model early on in the semester. Later in the semester a more extensive opportunity was provided to examine those two types of lessons in order to make a lesson plan with more student exploration than direct teaching. The process in which the PSTs were engaged in lesson planning includes two simultaneous steps. One was to critique and modify a lesson with a direct teaching model in class; the other was to make a lesson plan using another lesson as a project outside class in pairs. When issues arose regarding lesson planning, however, those were discussed in class as well. In these two simultaneous steps, the PSTs were also asked to examine lessons with a student-centered model as comparison that had similar mathematics contents.

The PSTs were required to use and modify lessons with a direct teaching model for lesson planning, because of the prevalence of such curriculum programs in the United States. The PSTs will be likely to teach in a school district in which one such program is being used. They need to develop the capacity to use them productively in the teacher education program.

The Procedure

The data were gathered from the two simultaneous steps (in-class and outside class) of examining lessons with a direct teaching model in order to modify them for more student inquiry. In class, the PSTs were asked to respond individually on paper to some questions about a lesson on fractions and division, including the goal of the lesson, the main task for students, useful resources included, potentially useful resources not present, and suggestions for modification. Once the PSTs finished responding to the prompts, they were asked to share what they noticed in the lesson without looking at their responses on paper. The reason for this sharing was to capture the PSTs' overall impression of the lesson along with any critical issues of the lesson that grabbed their attention. During this period of sharing and interaction among the PSTs, main ideas publicized were captured in field-notes. Individual responses on paper were gathered to examine the PSTs' initial thoughts about curriculum resources before collectively critiquing the lesson. In addition, the lesson plans that PSTs completed in pairs outside the class were gathered to examine their recognition of the critical aspects of the written lesson on fraction comparison they modified, after having opportunities to discuss various ideas about the lessons with a direct teaching model extensively. In the lesson plan, the PSTs were asked to describe the extent to which the written lesson provided opportunities for students' mathematical exploration by using some of the standards mentioned above.

The PSTs' individual responses and lesson plans, and field-notes were coded in terms of their recognition of affordances and limitations of the lessons they analysed. Then, similar codes in each data source were grouped together to find a general pattern.

RESULTS

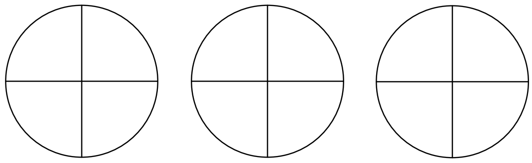
In this section, the PSTs' recognition of the critical issues is described along with each of the two written lessons mentioned above.

The Lesson on Fractions and Division

The lesson the PSTs were asked to critique and modify in class was on division involving fractional parts, such as 3÷4, in grade 5 (Charles et al., 2008). The lesson in the student text has four parts: (1) Learn, (2) Check, (3) Practice, and (4) Mixed Review and Test Prep. The first part (Learn) basically illustrates two examples of problem and solution with diagrams divided into equal parts and regrouped to show the answers. One of the examples is provided with a real-life context: "Anna, Tim, Mark, and Deb are sharing 3 quesadillas. What fractional amount does each one get?"

This example shows how to solve the problem in the section of "what you think" along with a diagram as shown in Figure 1.

Divide each quesadilla into 4 equal parts.



Each person gets 1 piece from each quesadilla. So, each person gets 3 pieces. This is the same as ¾ of one quesadilla.

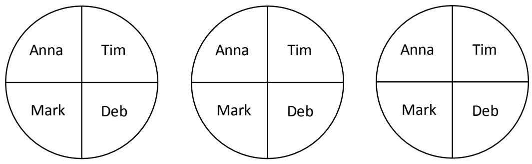


Figure 1: The solution of a quesadilla problem in the student text (Charles et al., 2008)

The other example provided in the student text only includes the problem ("Find 4÷6"), the answer (4÷6=4/6), and a diagram (four rectangles divided into six equal parts each that are shaded in six different colours). This example is also explained in the teacher's guide, suggesting teachers to "Point out that each color covers 1/6 of each bar. There are 4 bars. When the same-color squares are put together, they cover 4/6 of one bar." Basically, the two examples illustrate the same strategy. A prompt for discussion is followed by the two examples: "Explain why one piece from each quesadilla in Example A [the first example] is the same as ¾ of one quesadilla." Besides one sample response expected ("All pieces are the same size and each person gets 3 pieces"), there is no further guidance for the discussion.

Although there are some suggestions for teachers regarding how to teach the lesson, the student pages are the main resources for the lesson. The first page shows the two examples and a discussion prompt described above. Then, the next page includes 11 division problems without any context or representation, such as 9÷10 and 1÷3, in the second and third parts of the student text (i.e., Check and Practice). It is assumed that students apply what they were shown in the "Learn" part in order to find the answers of the practice problems in a mechanical way. In fact, suggestions for "ongoing assessment" and "error intervention" in the teacher's guide seem rather directive, unilateral, and procedural than supporting students' thinking, as shown below.

> If students do not see why one piece from each quesadilla is the same as ¾ of one quesadilla, then have them copy the second part of the diagram, cut out the model, cut out the pieces, and then regroup the pieces for each person.
>
> If students reverse the numbers in the numerator and denominator, then remind them that the number in the numerator represents the object being divided, or the dividend, and the number in the denominator is the number they are dividing by, or the divisor. (p. 398)

The last part of the student text (Mixed Review and Test Prep) provides three different diagrams and students are asked to write the fraction for each diagram. There is also one multiple-choice item on geometry. None of the problems in this part are related to the mathematical idea of the lesson, i.e., fraction as division.

PSTs' Recognition of Critical Issues

During the discussion of the lesson on fractions and division, the PSTs were asked to share the first thing that they wanted to talk about the lesson without looking at their responses to the prompts on paper. The main ideas that they discussed are shown below in order of sharing:

- In the lesson students find the pattern and copy it to do other problems but little understanding is promoted.
- The visuals and hands-on examples are good.
- Terms such as denominator and numerator should be explained.
- Problems are too small and too many, which are not connected much.
- Discussions on how students did the problem are needed.
- Students need to solve problems on their own without a given diagram (along with story contexts).

Besides the second and third points above, the PSTs seemed to recognize significant limitations of the lesson. They expressed concerns about providing "the" solution strategy to students and mechanical applications of the strategy to practice problems, instead of having students solve problems on their own and share different strategies.

The PSTs' individual responses on paper also indicate that they recognized critical issues from the written lesson. Table 1 presents the limitations of the lesson the PSTs identified (some PSTs mentioned multiple limitations). All but two PSTs identified significant shortcomings of the lesson.

Issue	PSTs (n=19)
Students' own exploration is needed rather than providing the strategy with the given diagram	12
Clear teacher moves are needed (e.g., assistance for struggling students, guidance for discussion, and teacher questions)	3
Different learning modes are needed (e.g., discussion and small group)	3
A clear connection among the problem, the representation and the answer is needed to support student understanding.	2
No significant recognition of limitations	2

Table 1: PSTs' recognition of limitations

It seemed that the biggest issue to the PSTs was the solution strategy given from the start. Twelve PSTs recognized that the strategy given limits students' thinking and their own ways of solving problems and this part must be modified in their lesson plan. Three PSTs noticed that the guidance to teach the lesson lacked important aspects of instruction: They looked for assistance for struggling students (rather than those in "ongoing assessment" and "error intervention"), guidance for group discussion, and specific teacher questions to probe and support students' thinking. Two PSTs did not consider that the problem, the representation and the answer were fully connected in the written lesson because the connection was rather superficial.

Besides those included in Table 1, some PSTs mentioned resources that they thought were needed but not provided in the lesson, such as resources for differentiation (eight PSTs), assessments (two), and vocabulary (two). The resources that the PSTs mentioned as helpful include guidance for struggling students (11 PSTs), visual representations (four), questions to support student thinking (two), and assessment (two). Interestingly, whereas one PST mentioned that the lesson did not have sufficient guidance for struggling students, 11 indicated that the ongoing assessment and error intervention shown previously "give teachers an idea of common points of confusion to look for and ways to mange the confusion." Also, two PSTs thought that the lesson provided questions to promote student thinking based on the discussion prompt.

The Lesson on Fraction Comparison and PSTs' Critiques

The lesson plans that the PSTs created were based on a fraction comparison lesson in Grade 3. Similar to the "fractions and division" lesson above, this lesson included moments for students' discussion on mathematical ideas and yet the written lesson treated them mechanically. For example, the most explicit part of the lesson for student exploration of size of fractions was a discussion on comparing fractions with same denominators and same numerators and unit fractions. "Is it easier for you to compare ¼ and ¾ or 2/6 and 2/8? Why?" "What happens to the size of the pieces as the denominator gets larger? ½, 1/3, ¼, 1/5, 1/6, 1/7" There was no specific guidance regarding this portion of the lesson. It seemed rather straightforward responses were expected, based on the subsequent examples for fraction comparison and practice problems on the next pages of the student text. Mainly, pictures (circles or rectangles) equally partitioned already were used to determine which fraction was larger in each pair in both examples and exercise problems. There is not much connection between the first page (the discussion portion) and the second page (examples and practice problems) of the student text of the lesson.

In order to modify the lesson, the PSTs were asked to critique it in terms of the extent to which it provided opportunities for students' engagement with two particular *Mathematical Practices* of the *Common Core State Standards for Mathematics*: (1) Making sense of problems and persevering in solving them and (2) Constructing viable arguments and critiquing the reasoning of others. They were also asked to use two particular NCTM *Professional Standards for Teaching Mathematics* to critique the lesson: (1) asking students to clarify and justify their ideas and (2) letting a student struggle with a difficulty. Some PSTs' critiques are as follows:

> The first question asked is "Is it easier for you to compare ¼ and ¾ or 2/6 and 2/8? Why?" This does prompt for a justification of why but … I feel that the student would need to prove why with clarification and may need to be promoted more explicitly. I feel a class discussion on this question could allow for this to surface, but the teacher's role would consist of more than just these two questions [the discussion prompts above].

> They are asked to simply state what happens to the size of the pieces when the denominator increases in fractions. It could be potentially extended in a discussion setting to have students explain or justify their reasoning.

> Providing the student with a strategy from the start does not allow for the process of a student working and thinking on their own to understand what the problem is asking and think of a reasonable way to come to the solution. By placing the correct way to reach the answer is skipping the step in which a student is to struggle and make sense of the problem. This can hinder the child from developing and understanding math concepts.

All PSTs (eleven lesson plans) provided similar responses in varying degree of description, concerned about the simplicity of discussion and mechanical approach to the practice problems, particularly in terms of the specific standards suggested to use.

CONCLUSION AND IMPLICATIONS

The results indicate that using the curriculum resources in this methods course supported the development of PSTs' critical thinking about curriculum resources and capacities to use curriculum resources productively to teach mathematics. Although many important ideas had been addressed earlier in the course, not all PSTs noticed some critical aspects of the written lessons before explicitly discussing them. Moreover, they identified some superficial resources as helpful. Once they shared what each thought about the lessons and discussed specific aspects of the written lessons explicitly by using some focused standards, they recognized actual limitations and affordances of the lessons more and clearly, and were able to modify them for a deeper student thinking of the mathematics of the lessons based on their examination of the written lesson.

This study highlights the importance of specific aspects to look for and detailed prompts in critiquing lessons before planning a lesson. Also examining lessons of the same mathematics content with different teaching models helped the PSTs see the difference in the nature of students' learning of the same content that lesson modification can create. This preparation (along with exploration of anticipated student thinking and struggle around the mathematics of the lesson, although not part of the data in this study) encouraged the PSTs to design and adapt tasks that support student exploration of mathematics and come up with useful instructional guidance on how to enact the tasks. In addition, the simultaneous steps of critiquing and modifying lessons in class and outside the class further prompted the PSTs to develop a critical thinking about curriculum resources and pedagogical design capacity.

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INFLUENCES OF SPECIFIED VIDEO BASED PROFESSIONAL DEVELOPMENT ON TEACHER PRACTICE

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How does video-based professional development (PD) influence teacher practice? In this paper we explore what teachers took back to their classrooms based on a specified PD experience. Survey data, focus group conversations and video from PD sessions was qualitatively analysed to triangulate data on teachers' learning and uptake from the PD. Teachers were classified into four different user categories -- Generative, Transformative, Incremental, or Non Users – based on how the teachers carried their PD experiences into their mathematics classrooms. Teachers differed in their classifications based on their mathematical understandings and contextual influences. These classifications help us to understand how and why teachers take up learning from PD programs in unique ways and to varying degrees.

INTRODUCTION

Video based professional development (PD) generally relies on selected clips for teachers to collaboratively discuss and analyse. In these cases, video serves as a tool with wide-ranging potential to guide meaningful inquiry, reflection, and learning (Brophy, 2004). The use of video clips varies greatly depending on the PD model. We posit that PD models fall on a continuum from adaptive to specified (Koellner & Jacobs, 2015). Adaptive models of PD, such as video clubs (Sherin, Linsenmeier & Van Es, 2009) and the Problem-Solving Cycle (Borko, Jacobs, Koellner & Swackhamer, 2015), generally utilize video clips from the participating teachers' own classrooms. On the other hand, in specified models of PD, video clips are typically pre-selected and come from other teachers' classrooms. Across both models of PD, video clips are expected to provoke inquiry and productive discussion relative to identified learning goals - around exploring targeted content, pedagogical strategies, and/or student thinking.

Much of the research on the uses of video clips in mathematics PD focuses on teachers' analysis of own classrooms or that of their colleagues. Less is known about using strategically selected video from other teachers' classrooms in specified PD programs to promote teacher learning and instructional change. The research questions guiding this paper are: What do teachers take up from specified video-based PD and enact in their classroom? How do teachers vary in what they take up from the same PD experience?

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THEORETICAL FRAMEWORK

Mathematics teachers come to PD workshops with varying levels of knowledge, much like the K-12 students who come to their math classrooms. Sherin (2007) argued that teachers' knowledge includes their "professional vision," which drives their ability to notice and analyse classroom interactions. VanEs and Sherin (2002) defined noticing as having three components: (1) identifying the important features of a classroom situation; (2) making connections between classroom interactions and the broader principles of teaching and learning; and (3) using what one knows about the context to reason about classroom events. Over the years, diverse conceptions of noticing have emerged, but in general most discussions of mathematics teacher noticing involve two main processes: (1) Attending to particular events in an instructional setting, and (2) Making sense of events in an instructional setting (Sherin, Jacobs, Philipp, 2011).

Participants in video-based PD do not necessarily make sense of the video clips or the classroom situations they depict in the same way; rather individuals bring differing knowledge and beliefs about teaching and learning, students, content, and curriculum to bear on what they notice (Erickson, 2011; VanEs, 2011). Furthermore, there are important individual differences in terms of what teachers bring to and take from video-based mathematics PD experiences (Kazemi & Hubbard, 2008; Kersting, Givvin, Sotelo & Stigler, 2010; Santagata & Yeh, 2014). More research is needed to categorize these differences, and connect the use of video to both noticing and uptake.

LTG MATERIALS VIDEOCASE DESIGN

The Learning and Teaching Geometry[1] (LTG) materials use videocases as the centrepiece of the PD, which is intended to improve the teaching and learning of mathematical similarity based on geometric transformations. Video alone is not viewed as a sufficient learning tool by the developers of the materials; instead the materials incorporate pre- and post-video tasks - which combined with a selected clip comprise a videocase - thereby ensuring a holistic foundation for the representation of authentic practice (Seago, Koellner, Jacobs, in press). The Learning and Teaching Geometry materials engage teachers in learning about similarity, congruence, and transformations through carefully designed and sequenced videocases that offer access to specific and increasingly complex mathematical concepts presented within the dynamics of classroom practice (Seago, Driscoll & Jacobs, 2010).

Video clips were selected for inclusion in the LTG materials based on the expectation that they would support multiple access points for teachers within the PD setting. Some clips contain challenging mathematics content, a conceptual hurdle, student misunderstanding, and/or interesting pedagogical moves. The activity that most commonly comes before watching a given video clip is working on the mathematical task that is in the clip. Solving the same task as the students in the video allows the teachers to develop an adequate understanding of the mathematical demands faced by the students, and helps them to better engage with the video clip. The assumption behind this type of pre-video activity is that teachers need a period of time to become sufficiently immersed in and familiar with the mathematics content they are about to see, so that they can readily follow the pertinent issues that arise in the video episodes.

Post-video viewing activities in the LTG materials include: careful unpacking of the ideas presented in the video clip, taking into account how those ideas apply in different mathematical contexts, discussing the pedagogical issues that were brought up by the video clip, and reflecting on how teachers can apply their emerging insights to make improvements in their own lessons (Jacobs, Seago & Koellner, in press). Facilitators of the LTG materials are encouraged to promote a culture of inquiry and reflection, supporting teachers to offer alternative and dissenting viewpoints, which at the same time focusing on specified learning goals. Furthermore, participating teachers are encouraged (but not required) to take any of the mathematical materials (such as math tasks and computer applets) from the PD to their classrooms.

THE LTG EFFICACY STUDY

The data from this paper are from an efficacy study investigating the impact of the LTG PD materials on teachers' instruction and students' knowledge. The LTG Efficacy Study aims to explore the effectiveness of the LTG PD using a randomized, experimental design. The sample is comprised of 108 mathematics teachers (serving grades 6-12) and their students from two contexts- one in the northeast United States and the other in the western mountain region. Approximately half of the teachers were randomly assigned to take part in the LTG PD in the first intervention year and half will take part in the second intervention year. The intervention consists of the entire LTG PD program, including a one-week summer institute and four days of academic year follow-up sessions beginning in Summer 2016. This paper looks at the teachers assigned to the treatment group based on data from their participation after 7 of the 9 workshop days (Note: the remaining two workshops will be held in Spring 2017).

DATA COLLECTION AND ANALYSIS

The analysis for this study entailed an examination of a small portion of the entire data corpus that included focus group interviews, a written survey by participants, and video from PD sessions. First, we studied the focus group interview and written survey from the participants. We took detailed notes on teachers reported use of the LTG materials in their classrooms as well as their perspectives on how the PD supported their learning. Lastly we used 'episodes' of video data. An episode consisted of a period of time found in the PD where teachers were discussing connections from PD content to their classroom. We then created categories of participants based on use, and identified teachers who exemplified each category.

In the next section, we detail our findings from the analysis of survey, focus group and video data. We describe four categories of teachers based on how they have taken up the PD materials in their classrooms so far, highlighting teachers who are representative of each category using illustrative quotes and other information about their experience of the PD.

FINDINGS

Based on qualitative data analyses conducted to date, we found that participants used information from the LTG PD in very different ways depending on their experiences during the PD, school context, and the mathematics courses they currently teach. We identified four categories of teachers that highlight the different ways they report using mathematics content and pedagogical strategies from the PD in their practice: Generative Users, Transformative Users, Incremental Users, and Non Users.

Generative users are teachers who went beyond the scope of the LTG workshop by using the knowledge and skills gained in the PD to generate new and innovative instructional materials for their classrooms. Generative users reported incorporating both their own newly developed instructional materials, along with materials and practices taken from the LTG PD program, in order to engage their students in the types of content and pedagogical experiences promoted by the PD. *Transformative users* intentionally took what they learned about content and pedagogy from the LTG PD into their classrooms, using many of the given materials and observed practices in a substantive way to transform their mathematics instruction. *Incremental users* took up some of the materials and/or pedagogical strategies from the PD for use in their own classroom, but not to the degree of the transformative users. Lastly, *Non Users* are participants who did not use either the LTG content-based materials or pedagogy strategies in their classrooms. In the next section, we provide examples of each type of user, highlighting what they noticed and took up from the PD program and how particular elements of the PD appeared to influence their learning.

Generative User Example

Peter was classified as a generative user because he not only applied what he learned from the LTG PD, he used that learning to create new instructional materials that expanded on critical mathematical and instructional components of the PD. Peter was heavily influenced by the emphasis on transformations in understanding geometric similarity, and he noticed that his own learning was deeply impacted by opportunities to explore technology on this topic (both through video clips and post-video activities). Peter, a high school geometry teacher with a strong math background, explained why he was driven to generate innovate classroom materials based on his PD experience:

> "I am someone who has very strong visual-spatial reasoning. I regularly manipulate shapes and objects in my mind. I know that this is not something that everyone else has. So it was very beneficial to get to see something that would allow everyone to have a common dynamic vision of similarity. Using Geogebra applets during the workshops inspired me to develop my own Geogebra Applets and also worksheets so my students

> can self-guide through some of our investigations. I even invested in a class set of tablets to make sure that I can use Geogebra applets as often as possible."

The LTG PD highlights a visual, transformations-based approach to congruence and similarity. As part of many of the post video-viewing experiences, teachers had opportunities to explore Geogebra applets that supported their visualization of the dynamic relationships among similar figures. Peter was inspired by these experiences to develop his own Geogebra applets and accompanying classroom materials that went beyond the scope of the LTG PD materials.

Transformative User Example

Whereas Peter was particularly attentive to the impact that technology could have on teaching and learning similarity, Nancy was very interested in the use of patty paper. Nancy not only found herself learning important content by watching videos of students using patty paper and then using it herself, she brought this experience to her own classroom. However, unlike Peter, Nancy did not report generating new ways to use patty paper that were different from those explored during the PD. Nevertheless, Nancy described her use of patty paper as supporting a significant shift in her students' learning:

> "I used patty paper with transformations, which was helpful because students moved them around and we haven't ever done that before. This clearly helped them learn in more conceptual ways."

Patty paper as a tool to understand transformations-based geometry is an important focus of the LTG PD materials, and is highlighted in several video clips. During those clips, students use patty paper in unique mathematically appropriate ways, which commonly influences teachers to begin exploring patty paper. Nancy, like many other teachers, was cognizant of the learning opportunities afforded by this tool and brought it into her classroom, closely following the examples of the videotaped students and the mathematics tasks used during pre- and post-video activities. Nancy is considered a transformative user because she brought critical tools from the PD into her classroom in what appears to be a substantive and appropriate manner.

Incremental User Example

Carol, who is currently teaching Algebra II and no geometry classes, is an example of a typical incremental user. Although she has not brought any of the content focused materials from the LTG PD into her classroom, she reports changes in her pedagogy that she ascribes to her PD experience. Carol explained that she has not yet had the opportunity to utilize her increased content knowledge due to the fact that she is not currently assigned to teach geometry, however she has intentionally incorporated newly learned instructional practices in her algebra classes:

> "I am trying to incorporate some of the teaching methodologies that we observed in the videos from the workshops. For instance, I am having students present and explain their work to the others and making students defend their positions by further questioning them when they are not clear in their responses."

The video clips that Carol and her colleagues viewed, discussed and analysed over the course of the LTG PD provided a mirror to reflect on her own practice, and consider aspects that she could improve on. In many of these clips, as Carol noticed, students presented their ideas to their classmates in whole and small groups, questioned each other, disagreed with each other's methods or solutions, or defended and clarified their mathematical arguments. These videocases helped Carol to recognize new pedagogical possibilities, and she is striving to incorporate them into all of her math classes regardless of the content focus.

Non User Example

Very few participants reported that they had not used any of the content materials or pedagogical tools from the LTG PD in their classrooms. However, one high school teacher, Barb, who fell into this category explained her non-use by describing the school-imposed barriers she faced:

> "I haven't used anything so far. We teach 2-hour block periods of math per semester, covering one year of material each semester. It is hard to use stuff from this PD with the rapid pace of our math blocks. The pace is harder for me as a teacher than the students."

Barb teaches in a high achieving school, and was concerned that the materials and tools used in the LTG PD program will cause her to slow down her instruction too much. Although she recognized the benefits of incorporating a transformations-based approach, she could not see a way to incorporate anything from the PD into her own classroom given her school's demands to cover a large amount of information in a short time frame.

CONCLUSIONS

The LTG PD materials, through the use of videocases, provide extensive opportunities for teachers to notice and analyse the dynamic relationships among content, pedagogy and student thinking. Videocases in specified PD provide an interesting study because while they target carefully composed content and pedagogy learning goals, individual teachers may find particular components of the videocases to be personally meaningful and relevant to their classrooms. This phenomenon is analogous to how students learn from a given mathematics curriculum. Although the curriculum is likely to have a variety of identified learning goals, students actual learning will vary widely depending upon their prior knowledge, learning styles, and classroom contexts.

Teachers who participated in the LTG PD reported many different ways the workshops impacted their practice, with video clips appearing to play a central role in their learning. For instance, Peter shared, "*The most significant thing about the video clips was the ability to analyse different "levels" of student understanding. I think understanding these different levels will help me encourage more students to share their thinking. Understanding students' levels of thinking would allow us as teachers to compare between partially correct and correct responses in class discussion. It* *actually would allow us to make rubrics that are explicitly focused on students thinking."*

Other teachers reported that seeing effective pedagogical strategies in the video clips helped them to envision how pedagogical strategies or content may play out in their classroom. At the same time, it is clear that the teachers learned not only from the video clips, but from the activities that supported viewing and discussion of the clips. The fact that teachers. As we have noted, videocases incorporate not only video clips but pre- and post-video viewing activities. As such, videocases provide teachers with multiple avenues to stimulate content learning and access pedagogical strategies in ways that are aligned with teachers' prior experiences and unique contexts.

We found that the videocases in the LTG materials anchored teachers' noticing and insights in various ways, around a multitude of topics. We conjecture that teachers' unique experiences in and learning from the PD was likely due to individual differences in their noticing skills and/or their instructional context including grade level, courses taught, and curriculum requirements. We further hypothesize that this combination of differential noticing and variation in instructional context contributed to teachers' classification as different types of users of the PD materials in their classrooms. More research should be undertaken to explore and disentangle this connection, such as by more carefully examining what individual teachers noticed and discussed during the workshops and whether those workshop experiences are correlated with their classroom use categorizations. In addition, information on teachers' observed classroom practices is essential to validating data on their self-reported uptake of information from the PD.

Note

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HOW DO UNDERGRADUATES SQUARE-ROOT IN ℝ AND IN ℂ?

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This study was concerned with undergraduates' images of the square root concept with a focus on consistencies and inconsistencies. The images were explored through square-rooting – a mental act associated with extracting roots from different radicands in the field of real and complex numbers, validating and sustaining the obtained results. The data was collected with an online questionnaire from first-year university students. A qualitative analysis of students' reasoning revealed a variety and complexity of students' webs of knowledge, in which a square root was compartmentalized into distinct domains of consistency and linked with other concepts through schemes. The phenomena of real number bias and complex number bias were indicated.

PRELUDE

Imagine Israela, a diligent student in her last year of Israeli school, who is studying towards her matriculation exam in mathematics. As part of her preparations, she decided to review various concepts and she started with roots. Israela opened Yaquel (2004), a popular Israeli textbook, and read:

"We have already met square roots many times in the past when solving quadratic equations. We recall that the domain of the expression $\sqrt{a}$ is all non-negative numbers and also $\sqrt{a}$ is a **non-negative** number, the square of which is *a*" (bold in the origin, ibid, p. 581).

The textbook's explanation made perfect sense to Israela, she thought to herself, "$\sqrt{9}$ is 3 only because $y=\sqrt{x}$ is a function and then for every *x* in the domain there should be a single *y* in the range". Then, she decided to recall how roots of complex numbers are extracted. Israela went over the conversion of a number from Cartesian ($z=a+bi$) to polar form ($z= re^{i\theta}$), De Moivre's theorem for raising a number to a natural power, and the formula for extracting roots of the *n*-th degree: $\sqrt[n]{r}e^{i\left(\frac{\theta+2\pi k}{n}\right)}$ for $k=1,2,\ldots,n$.

The formula was very confusing for Israela, "How can it be!", she exclaimed, "If according to page 581, $\sqrt{9}$ =3 and 9 is real and complex, so why does $\sqrt{9}$ =±3 on page 543? How did it grow another root in just forty pages?".

After struggling some more, Israela turned to her mathematics teacher for clarifications. The teacher flattered Israela's curiosity and said, "Don't overthink it, it is much easier than it seems. In the matriculation exam, Question 3 is the only one with complex numbers. So answer with two values there, in the rest of the exam give just one root".

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The presented anecdote is based on my conversations with secondary teachers, high-school and university students who were well-familiar with the root concept in the field of real and complex numbers. However, many of them struggled with making sense of the presented inconsistency. The study reported in this paper was concerned with undergraduates' images of the square root concept with a special focus on their consistencies and inconsistencies.

MOTIVATION FOR FOCUSING ON THE ROOT CONCEPT

This study is a part of a larger project on teaching and learning cross-curricular concepts. These are threshold ideas that appear multiple times in the students' landscape of mathematics education, when each time the ideas are reconsidered in a new domain. A domanial shift is often accompanied by a redefinition and an introduction of new properties, which entail a substantial change in the ways these concepts are approached. Kontorovich (2016) showed that a root concept appears in intermediate-school, high-school and university mathematics curricula.

An examination of various textbooks and mathematicians' approaches revealed that there is no consensus in regard to a conventional definition and notation of the concept (Kontorovich, 2016). Specifically, various answers exist to the questions whether an even root of positive numbers is single- or double-valued, whether a verbal identifier 'root' and a radical sign have the same meaning, and how the concept should be defined in the field of complex numbers. Therefore, school teachers and university lecturers could benefit from an evidence-based picture of how students can understand the concept and what inner logic can be behind their thinking.

THEORETICAL BACKGROUND

The construct of *concept image* has been introduced by Vinner (1975) to mathematics education to account for the total cognitive structure that a learner associates with the concept, which includes all mental pictures, properties, and processes. For 42 years this construct has been successfully utilized for obtaining insightful research findings (e.g., Alcock & Simpson, 2011). For instance, it has been extensively used in classification studies focused on how learners judge whether a stimulus is or is not an example of a particular concept (e.g., Alcock & Simpson, 2011; Hershkowitz, 1989; Tall & Bakar, 1992). Mathematically, the decision should be based on the *critical attributes* that appear in the formal definition (Hershkowitz, 1989). Practically, it has been found that in many cases students and teachers classify based on the critical attributes of *prototypes* – special examples that incorporate features the most highly correlated with all examples from one's concept image (e.g., Hershkowitz, 1989). Other studies showed that the reasoning for classification can be not example-based. Indeed, Tall and Bakar (1992) found that for many students, candidates for a function "should be defined for all real numbers" and their y "should equal an expression of x". I will use *prototype conceptions* to refer to a learner's pool of 'should-bes' that a stimulus needs for being accepted as a concept example.

In the last decade, research on classification and concept images has been critiqued. Alcock and Simpson (2011) argued that studies often assume *concept consistency* – students' classification of stimuli being driven by a single mechanism. In their study, the researchers demonstrated that when given a list of sequences with a request to determine whether each of them is increasing, decreasing, neither or both, the same student could rely on a definition for some sequences and on local behaviors for other sequences. Concept consistency became central for this study.

Another critique can be that research has been often concerned with students' images of a single concept (e.g., see Tall & Bakar, 1992 for a function; Alcock & Simpson, 2011 for a sequence). However, mathematics has been acknowledged for its interconnected nature, and then an individual can hold "a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (Hiebert & Lefevre, 1986, pp. 3-4). Accordingly, when working on a problem, a solver chooses the concepts that she or he is going to operate with, images, definitions and prototypes included. Harel's (2008) ways of understanding and ways of thinking were used in this study for capturing these choices.

Harel (2008) considers ways of understanding (WoU) and ways of thinking (WoT) in the context of a mental act, which is a primary notion of his framework. The researcher illustrates mathematical mental acts with proving, interpreting, connecting, modeling, generalizing and symbolizing. *WoU* are defined as products of a mental act that are observable through a person's statements and actions. *WoT* refer to the common cognitive characteristics of the WoU that emerged from repeated observations.

RESEARCH GOALS AND ANALYTICAL FRAMEWORK

Harel (2008) maintains that an exploration of one mental act cannot be isolated from other acts. This study is concerned with *square-rooting*, which consists of extracting square roots from numbers, validating and explaining the emergence of the obtained values. The goals of the study were to explore undergraduates' square-rooting WoT, and particularly students' webs of knowledge that connect an image of a square root with images of other mathematical concepts and ideas. The driving force behind these goals was to explore potential changes in students' square-rooting as a response to variations of a field (real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$) and a radicand (either from $\mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{C}\backslash\mathbb{R}$).

The constructs of *connecting schemes* and *domains of consistency* were introduced for capturing potential changes in students' square-rooting. A *connecting scheme* is a coherent cognitive mechanism that answers the questions of what and how mathematical concepts were used in one's reasoning about a mental act, square-rooting in this case (see Dubinsky & McDonald's, 2001 for a similar interpretation of a scheme). One example of a connecting scheme can be found in the Prelude where Israela relied on the concept of a function to sustain that $\sqrt{9}$ is single-valued. Obviously, one can concentrate on the prompted concept itself and reason with the prototype conceptions that she or he attributes to it. Although such reasoning does not provide data on the one's web of knowledge, it can spotlight various aspects of a person's concept image.

A *domain of consistency* characterizes the stimuli as a response to which one's cognitive products share particular attributes. The notion accounts for Alcock and Simpson's (2011) critique by answering the questions "what stimuli prompted the same WoU or WoT, and what stimuli were different?". For example, Israela's confusion in the Prelude could be explained with a struggle to make sense of $\sqrt{9}$, which was conceptualized as the same stimulus that can be classified either to the domain of real or to the domain of complex numbers. However, the connecting schemes in each domain produced different results, which contradicted the conception of sameness.

METHOD

According to Harel (2008), an exploration of one's WoT necessitates repeated observations of her or his WoU. Accordingly, the data for this study was collected with a questionnaire consisting of 12 triples of questions. The formulations of the questions in each triple differed only in the radicands belonging to the same numerical set: either $\mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{C}\backslash\mathbb{R}$. Consider an example of a triple: "In the field of complex numbers $\sqrt{-9}=?/\sqrt{-144}=?/\sqrt{-81}=?$". The assumption underlying this design was that the questions in a triple probe the same domain in one's concept image, and then she or he is expected to demonstrate the same WoU. The responses of the participants that did not align with the expected consistency were excluded from the analysis. The questions involved radicands that were convenient for manipulation to enable students to concentrate on the reasoning rather than on computational issues. Each participant was assigned with a randomized set of questions from different triples. Cronbach's alpha of the questionnaire was 0.92, which indicated a high level of internal consistency within the specific sample.

Participants were asked to complete the questionnaire via a Google form, in which the questions were accompanied by two opposite numbers, the squares of each of them equal the given radicand. The participants could respond with either one of the numbers, both of them, or provide values of their own. The participants were encouraged to provide explanations for their responses.

The questionnaire was spread in an online closed asynchronous forum in a popular social network. The forum was intended for students enrolled in first-year mathematics courses in a technological university in Israel. An average respondent was 24 years old (SD=3.54) and 96% of them were studying towards a bachelor degree in engineering. All respondents had already taken at least one course in calculus or linear algebra, and then they were exposed to roots, real functions and complex numbers in high-school and university. In line with Harel (2008), participants' WoU were associated with final numerical responses and the accompanying reasoning that they provided. WoT emerged from the content data analysis with pre-determined categories of connecting schemes and domains of consistency.

FINDINGS

The findings presented in this paper emerged from 39 students who explained their responses to the questionnaire. Overall, their square-rooting can be described with classification of a stimulus to a particular domain of consistency in their concept image and application of conceptions and schemes that were prototypical to the domain. In terms of consistency, four square-rooting patterns were identified: (1) a *single domain*, when students responses were consistent for all the questions (n=21); (2) *real* and *complex* square-rooting which were field-dependent (n=7); (3/4) square-rooting from *positive / negative* radicands, in which the students accounted for whether a radicand belonged to a particular half of a real number line regardless of the field assigned in the question (n=7/4). Some characteristics of the identified square-rooting WoT, including the patterns (1-4), are illustrated next. The characteristics intertwine, and then the analysis is focused on the underlined characteristics.

Classification according to distinct domains of consistency: Let us consider an example of a stimulus that was classified by two students with different square-rooting WoT. As a response to $\sqrt{625+0i}=?$, Alex wrote, "*it is a complex number because it is presented in its complex form*". For Ben, on the other hand, it was "*a positive number with two square roots*". The presented reasoning illustrates that the students classified the stimuli according to the domains, upon which their square-rooting depended. Indeed, Alex distinguished between real square-rooting, which was single-valued and complex square-rooting, which produced two results. She classified the stimulus to the latter domain based on the attribute of the "complex form" 625+0*i*. Note that simplifying the radicand to 625 would have complicated the classification because then the stimulus could have been considered in the field of real and in the field complex numbers, the square roots in which were different for Alex. Accordingly, preservation of the $a+bi$ form was critical. Ben, on the other hand, accounted for whether the radicand was 'positive' (and then single-valued) or not (and then double-valued). His reasoning suggested that positivity was determined by whether the radicand could be simplified to belong to $\mathbb{R}^+$. In this way, simplification was helpful for distilling the attribute that was critical for classification.

Coordination of prototype conceptions and connecting schemes: In their explanations, 17 students used some variation of "*a square root yields two numbers*", nine students wrote, "*there can be only one result for a square root*". The connecting schemes that students demonstrated aligned with these conceptions. Two popular schemes for the double-valued conception were: squaring both values to show that the result equal the assigned radicand; and making a link with the concept of inverse, which produced two values for an even function $f(x)=x^2$. The popular connecting scheme among adherents of a single-valued conception was based on an inverse function.

Five students explicitly addressed the importance of coordination between the domanial conception and the applied scheme. For example, as a response to $\sqrt{121}$ in the field of real numbers, Katie wrote, "*a root is a multivalued function. It is a worldwide convention to give a positive* [square] *root in real numbers*". When she encountered $\sqrt{16}$ in the field of complex numbers, she explained, "4·4=16, –4·–4=16 *and a root of a complex number can be negative*". Katie's reasoning indicates that for her, positivity and negativity existed in both domains, but positivity was critical for real square-rooting only; complex square-rooting, on the other hand, resulted in positive and negative values. Accordingly, her responses were tailored to align with the conceptions prototypical to the distinct domains in her concept image.

An unexpected explanation was proposed by Larry for real square-rooting being single-valued and complex square-rooting being double-valued. As a response to $\sqrt{16}$ in the field of complex numbers, he wrote: "*It is accepted in reals that square roots are positive. There is no order relation in* $\mathbb{C}$ *and then you cannot prefer 4 over –4*". Larry's reasoning showed that he was aware of the lack of order relation in the field of complex numbers, which made sign-based schemes for selecting between the root candidates invalid. Accordingly, in his comlex square-rooting he was using the same connecting scheme without filtering one of the candidates for a root. This is an interesting WoT in which an amendment of the scheme was justified with the same conception; a conception which was valid in one domain and invalid in another.

Different domains of consistency – unrelated connecting schemes: In their reasoning, five students accounted for different domains of consistency, where they square-rooted with schemes based on different concepts and ideas. For example, in the field of complex numbers, Ella accounted for three domains and demonstrated three connecting schemes. In the questions with radicands belonging to $\mathbb{R}^+$, she responded with two values and explained that, "*both equal the given number when squared*" (scheme a). When radicands from $\mathbb{R}^-$ were under consideration, her reasoning was based on the theorem stating that if a complex number *ai* is a solution of a polynomial with real coefficients ($x^2+a^2=0$ in this case) then the conjugate $-ai$ is a solution as well (scheme b). In the questions with non-real radicands, Ella converted the radicands to the polar form $re^{i\theta}$ and applied De Moivre's theorem (scheme c). As a result, Ella's complex square-rooting from real numbers was double-valued but square roots of non-real radicands resulted in a single $\sqrt{r}e^{i\frac{\theta}{2}}$.

Notably, while the connecting schemes (a) and (b) that Ella applied for positive and negative radicands were unrelated, they were compatible in the sense that each of them could be applied in both domains and result with the same values. The third scheme was effort-consuming and incompatible. It does not seem that Ella was concerned about the same concept 'behaving' differently in different cases. Possibly, this was because the incompatible scheme (c) that she used in the domain with non-real radicands computed the roots, the schemes applied in two other domains validated the provided root candidates. In terms of Harel (2008), a computation scheme could have fulfilled Ella's intellectual need for certainty by ensuring that all the roots were obtained and the need for causality by explaining the inconsistency between the domains.

Real number bias: Square-rooting of 16 students involved two connecting schemes based on the conception of positivity. For example, in the questions with positive radicands (in Ben's sense), Henry explained that an inverse function of x^2 is a bijection only for $x>0$. Accordingly, he wrote, "*a root is always positive*". When radicands from $\mathbb{R}^-$ and $\mathbb{C}\backslash\mathbb{R}$ were under consideration, he chose the responses from the upper half of the *y*-axis and the right half of the complex plane, correspondingly; he explained, "*we still need to choose the positive root*". Another student, Inga, relied on the concept of an absolute value. In her explanations, she wrote $\sqrt{x^2}=|x|$, which was interpreted in alignment with Henry's conception of positivity.

Henry and Inga's reasoning illustrate how a prototype conception can be expanded from real to non-real numbers. In $\mathbb{R}$ the connecting scheme of an inverse function required $x>0$ (the case of $x=0$ was missed by Henry). In $\mathbb{C}$ a number is often represented with $x+iy$. The students referred to $\{x+iy \mid x>0\}\cup\{iy \mid y>0\}$ as 'positive', which made $\{x+iy \mid x<0\}\cup\{iy \mid y<0\}$ 'negative'. Accordingly, the conception of positivity enabled them to apply the same connecting scheme for choosing between two complex candidates for a root.

Complex number bias: In their reasoning, ten students demonstrated awareness to distinct domains of consistency and applied variations of the same connecting scheme in them. For example, when extracting a square root of 121 in the field of real numbers, Fred wrote, "*I am solving the equation $x^2=121$ which yields two solutions according to the Fundamental Theorem of Algebra. Both of them are real and then they both fit*". As a response to $\sqrt{-36}$ in the field of real numbers he explained, "*$x^2=-36$ yields complex $\pm 6i$ and then there are no roots*". Another student, Gloria, associated the assigned radicands (real and complex) with vectors $re^{i\theta}$ on a *x*-*y* plane. She extracted two square roots $\pm re^{i\frac{\theta}{2}}$ with a variation of De Moivre's theorem and responded with the values that belonged to the assigned field.

Notably, both students square-rooted with concepts and methods that are valid in the field of complex numbers. When it was necessary, they filtered the obtained results to fit $\mathbb{R}$. Such square-rooting can be interpreted as a complex number bias since $\mathbb{C}$ is an algebraic field with an axiomatic system of its own which is not reducible to $\mathbb{R}$.

Another bias was indicated in the field of complex numbers when three students connected between $\sqrt{-1}$ and square roots of radicands belonging to $\mathbb{R}^-$. In his explanations, John wrote that $\sqrt{-1}$ equal *i* because, "*this is the definition*". While his square-rooting from positive and non-real radicands was double-valued, $\sqrt{-144}$, for instance, was approached as $\sqrt{144}\sqrt{-1}$ and resulted in a single 12*i*. I interpret this WoT as an extension of a prototype example $\sqrt{-1}$ to a new domain with a multiplicative conception $\sqrt{ab}=\sqrt{a}\sqrt{b}$; a conception which is valid in $\mathbb{R}$ but not in $\mathbb{C}$.

CONCLUDING REMARKS

I would like to conclude with remarks situating the findings of this study in university teaching and learning. It is a reality in many countries that a curriculum condensed with concepts is delivered through lecturer-centered instruction to students coming from various mathematical backgrounds. Such pedagogical setting can intensify a practice in which a lecturer interprets the errors that students make as *mis*understandings that can be ‘patched up’ with speedy reteaching (Kontorovich, 2016). This study showed that erroneous mental products can be a result of *understandings* rooted in webs of knowledge compartmentalized into multiple domains and connections with other concepts. Accordingly, an error can be not indicative of the complexity of the concept images that students hold and advanced mathematical processes that they carry out. Furthermore, it seems unlikely that as a result of a speedy exposure to the mathematically correct WoU, students will reformat their WoT and calibrate it with the one promoted by the lecturer without further assistance. Lastly, a root is just one example of many concepts that lack a consensual approach in the mathematical community. This situation invites a reconsideration of such discoursive labels as ‘erroneous’ and ‘mathematically correct’ WoU and WoT.

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SECONDARY-TERTIARY TRANSITION: HOW MESSAGES TRANSMITTED BY LECTURERS CAN INFLUENCE STUDENTS' IDENTITIES AS MATHEMATICS LEARNERS?

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This paper explores how first year mathematics students narrate their identities as mathematics learners during their transition to university. We focussed our interest on the messages that the students received from their lecturers and on the ways that these influenced their identities. The results of the study suggest that students with a strong mathematical identity maintain it and are not influenced by strongly framed messages. But for others, the degree of control exerted through the transmitted messages plays an important role in reshaping students' existing identities as they move into university. These results have important implications for teaching and learning policy and practice in HE and demonstrate the need to raise awareness of the importance of incorporating student agency into the design and delivery of mathematics education.

INTRODUCTION

The transition from school to university mathematics has been and still is a persistent and often problematic issue in the field of mathematics education. The existent literature recognises several factors as the root cause. These refer mainly to the difficulties that students face due to changes in the content of mathematics taught at university (Brandell et al., 2008), changes in the ways of thinking and communicating mathematics (Engelbrecht, 2010), changes in the way of teaching (Thomas & Klymchuk, 2012) and changes in the social and cultural context of the new institution (Holmegaard et al, 2013). In this study, we employ a socio-cultural theoretical framework to investigate how variations in the degree of power and control of messages transmitted from lecturers influence the construction of students' identities as mathematics learners in transition. For this purpose, we use Bernstein's (2000) concepts of classification and framing, and the notion of positional identity developed by Holland, Lachicotte, Skinner and Cain (1998). This is an approach that to the best of our knowledge has not been employed before in transitional studies.

THEORETICAL FRAMEWORK

Classification and framing depict the underlying structures of power and control that a message carries and here we use them to explore in what ways messages can influence how students reshape (or affirm) their identity. Bernstein (2000) developed his theory on educational transmissions to show how knowledge is reproduced by controlling what can be taught, and how, by those in power. With the concepts of

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classification and framing he intended to demonstrate how transmitted knowledge is transformed into discourse. In this study, we use these concepts to show how messages transmitted by lecturers transform the knowledge that they possess, regarding the learning of mathematics at university and the content of mathematics itself, into pedagogical discourse. In the discourse that takes place during the teaching-learning interactions the concept of classification helps us recognize the power of the message; it sets the limits of the discourse, specifies the specialty of the context and makes clear what meanings are going to be put together. For instance, consider a lecturer who strongly suggests that students use induction to prove a certain theorem. This message carries a high level of power by setting the limits of what students can do – i.e. not following the suggestion of the expert would be unwise. A message with strong classification carries a lot of power. With the concept of framing we can identify the degree of the exerted control that a message conveys; it sets the form of realisation of the discourse and shows how the meanings that the transmitter intends to address are going to be put together. When the framing is strong, the degree of control is also strong. In that case the control is with the transmitter (Bernstein, 2000). Consequently, the person who transmits the message does not leave space for the acquirer to shape her/his own thinking; the thinking of the acquirer is regulated entirely by this strong control. Drawing on the previous example, the lecturer might then remind students of the particular steps in a proof by induction; the control of this message is then strong, it shows students explicitly how they should prove the theorem and directs their thinking.

Bernstein (2000) argued that changes in the formation of identity may occur because of variation in the distribution of power and variation in the principles of control, through different ways of communication. In addition, we draw on the concept of 'positional identity' developed by Holland et al. (1998), which refers to the ways that a person identifies their position in relation to others, mediated through the ways that make them feel comfortable or constrained. We focus particularly on their concept of *space of authoring*. Here an individual can orchestrate the social discourses and practices which allow them to act in a particular way. In this sense a message transmitted by the lecturer which varies in classification and framing can influence how students participate differently in university practices (i.e. opens up or constrains their space of authoring), hence shaping their identities as mathematics learners in different ways while they participate in these practices (e.g. a brilliant mathematician, a dropout, etc.). As Lerman (2000) argues the individual trajectories in the development of a person's identity when s/he engages in social practices are influenced by the ways that this person functions in this specific practice. Through the engagement in the practice an individual is attributed a different positioning. In our context, this means that the identity that the students bring with them from school can be reshaped through their involvement with university practices. The messages presented in this paper are carried through these practices and contribute to the reshaping of students' identities by positioning students with respect to the new practices of university.

METHODOLOGY

The research took place in a UK research-intensive university among ten first year honours degree mathematics students. During semester one, we observed lectures of two compulsory modules and we employed questionnaires, focus groups and interviews with the students in order to see what messages are transmitted and how students interpret them. To explore students' identities, we gathered information about their background, the reasons behind choosing this particular degree, the ways they approach mathematics at university and the ways they deal with the changes (e.g. subject content, teaching practices, structure of the programme, etc.). We conducted two individual interviews with the lecturers of the modules (Lecturer A and Lecturer B hereafter). All the observations and interviews were audio recorded and transcribed.

We constructed narratives from students' interviews and focus groups to explore how their identities mediated the interpretation of the received messages. We investigated how these messages positioned students in relation to the practices of university, hence affecting their existing identities. These narratives emerged from the ways that students described themselves as mathematics learners during the passage between the two educational settings (Bruner, 1996), and the influence that the received messages had on them. We chose messages that were transmitted during lectures and which we considered crucial for the learning of mathematics in the new context (e.g. interaction among students, assessment methods, teaching practices). These were analysed using Bernstein's (2000) concepts of classification and framing to identify the conveyed power and control. Then we explored how these messages positioned students in the university setting, using Holland's et al. (1998) concept of positional identities, and the effect this had on their identities. We wanted to know if the same message was received by different students and if they interpreted it differently according to their particular identities, and what consequences this had for their transition.

RESULTS

We now describe two of the messages that the two lecturers transmitted about mathematical discussion and assessment methods. We present narratives of two of the students, hereafter Lesley and Jason (pseudonyms), through which we identify the influences that the messages had on the shaping and re-shaping of their identities. We chose to present these two messages here because they have the same level of classification (both strongly classified) but vary in the degree of framing (one is weakly framed, the other one strongly), and they influence the students in very different ways.

Message 1: Mathematical discussion among students

One of the strongly framed messages that both Jason and Lesley referred to concerned mathematical discussion among students and was transmitted by Lecturer A. The lecturer herself stated in the interview her intention for transmitting this message: "…being able to talk about the ideas, and being able to explain why you think what you think, even if … you are wrong, I think that's important…" In every lecture, she invited the students to engage in discussions mostly by giving them mathematical tasks, included in a set of notes that she provided, or by asking them to work on things that would come up spontaneously, such as finding a counter example for a statement discussed at that time. In the first lecture, she invited the students eight times to engage in a conversation, asking them "tell the person next to you…" whether a specific statement is true or what the explanation for a particular argument could be. She explained why it is important to do so: "At university the aim is not just for you to 'do' mathematics, but to learn to communicate clearly…". The message she intended to transmit through this practice is explicit, carries a lot of power and sets clear boundaries about the importance of being able to communicate mathematics. The power carried through this message makes the students recognise that at university they need to develop the skill and habit of communicating mathematics clearly. It also has a high degree of control; students will learn how to communicate clearly by engaging in the discussion with specific tasks chosen by the lecturer.

Message 2: Assessment methods

We notice weaker framing transmitted through Lecturer's B message regarding the assessment of the module. As a part of the assessment for this module students need to complete a two-piece coursework which contributes 50% to their overall module mark. Lecturer B admitted in the interview that by employing this coursework he wanted students to spend time working on their own (or in groups) on the material that was covered during lectures. Although the message that he intended to transmit with the use of the coursework is powerful - students have to work on the coursework in order to understand the material and achieve good marks - it was not strongly framed. During the 17 lectures we observed, Lecturer B referred to it only three times, with the most prolonged time being in the first lecture when he provided general information about the course: "Coursework information are on the other sheet [sic]… issue dates, handing dates and return dates… And you will be given three weeks to do the coursework. The first coursework is to be done individually, and the second can be done in groups of three… It can be done individually but it's advisable to work in groups… Each coursework will be worth 25% of the total module."

Jason's narrative

Jason's family and teachers were important influences on his decision to study mathematics. His parents' professional background played an important role in his interest in mathematics: "My mom is an artist and my dad is a computer programmer so he is obviously very mathematical… the combination of the two was very good..." He acknowledged that two of his school mathematics teachers inspired him with their love of mathematics: "I had two Further Maths teachers that I will always remember and they will always be an inspiration for me. They were very strict and very old fashioned and it is what I needed, someone straight, and they were very enthusiastic… I think just having two great teachers was enough for me to be inspired to want to do this for the rest of my life… I wanted to do my best for them…”. The differences in the content of mathematics taught at university did not trouble him; he approached these as opportunities to challenge his mathematical knowledge: “I didn’t mind, I wanted the challenge, I didn’t come here for a repeat of last year… I wanted new material, I wanted to find how stuff works with integration, differentiation...”.

He received the message that Lecturer A transmitted about mathematical discussion and recognised the logic behind it. Nevertheless, it did not prompt him to take further action. He explained: “I don’t speak to anyone from the course, I go there and sit on my own and make notes. It’s quite an arrogant thing to do but I find it most useful for me... I tend not to engage in the conversation about the answer, sometimes I am wrong but sometimes it’s very obvious and I don’t think it worth conversation… I hate conversations around me, they are extremely silly…”. Through his narration it is obvious that he is not influenced by this message. Jason’s identity created during previous years through schooling affects the ways through which he deals with the practices taking place at university and consequently in what ways he positions himself. He does not value any social interaction with other students, and approaches mathematics as a challenging task that needs to be done individually. This has as a consequence a disregard for this strongly classified and framed message.

On the contrary, the weakly framed message that Lecturer B transmitted had more influences on Jason. Through this message he identified a challenge to deal with. He was the only one among the students who took part in the study that interpreted the message in this way. He is independent and more prone to agree with the messages that promote such challenges. This contrasts with the rest of the students who, in most cases (as we see shortly in Lesley’s case), struggled with the different structure of assessment at university. He argued: “I like having a challenge to be set every week and to go away to try it over. Even if I get it wrong and I get a bad mark… I still enjoy it… The best example was Lecturer’s B question 4d, that was really good. It was induction of power sets and it was a proof by induction and that was a real challenge, I had to spend hours on that question try to get it right.” We notice that Jason is influenced by the transmitted message despite its weak control, he still puts work in and encounters it as a challenge. It is his identity as a mathematics learner that makes him position himself as someone who enjoys and looks for challenges. This practice makes him feel comfortable and lets him extend his mathematical horizons. His actions are not prompted by the degree of control of the message but from his intrinsic interest in the content of mathematics.

Lesley’s narrative

Lesley chose to study mathematics because of her personal interest in the subject: “I came to study maths just because I really enjoy the actual I’d say sort of methods type [of] maths, like sort of complex but not so much proofs and I thought that’s what university mathematics is.” She was encouraged by her school teachers to follow this choice: “Teachers gave me advice, like my Further Maths teachers... [They said]... basically that you can do it. Like don’t think you can’t, don’t regret it...” Eventually university mathematics was not as she expected it to be; her expectations were based on "rumours" which were supported by the view that university mathematics resembled A Level Further Mathematics: "I listened to rumours that it wasn't going to be that hard, like everyone said that the jump from school to 6th form is harder than the 6th form to uni, but they lied! ...I got told that 1st year was just Further Maths A Level, but that wasn't true either." The main difference that she recognised between the two institutions, and which made her struggle, was the necessity to study proof at university. She commented: "There is a lot more proofs and I find them quite difficult to understand, I think that's the main difference... maybe that's why I am struggling more, and missing something from the lectures..."

When Lesley talked about the strongly classified and framed message that Lecturer A transmitted regarding the discussion among peers she showed a completely different grasp of it compared to Jason. She recognised the discussion on mathematical activities as a familiar practice, like the ones taking place at school, and she acknowledged it as beneficial for students' understanding. This specific practice used during her schooling played an important role in the way that Lesley formed her identity as a mathematics learner and consequently it influenced the way that she dealt with an analogous situation at university. In the focus group she said: "School was a lot more discussions and stuff like work with your partner and things like that and I think some lecturers… try to make you discuss." A few weeks later in the interview she elaborated further: "I do find that really helpful... When you explain something to someone else it helps you as well and is nice to hear somebody else's point of view... Just to hear another explanation, it can just sort of make you remember more or in an easy way." Although she finds the differences in the new context troublesome she seems to be eager to take action from this strongly framed message which resembles school practices. The practice of talking to her peers is similar to the school practices and therefore she agrees readily with this message. In turn, this helps her deal with the problematic situation.

The same though did not happen with the strongly classified but weakly framed message from Lecturer B. Lesley did not anticipate such a difference in terms of assessment between the two educational settings: "I didn't expect it to be this much coursework... and I am not used to it, 'cause at 6th form we had exams at the end of the year and that was it for maths. So it is hard having something constantly in the back of your mind and have to hand it in... It's quite stressful but then maths at university it was always going to be stressful..." In order to cope, Lesley admitted that she focused more on the things that the lecturer says during lectures and tried to find bits that would fit as possible answers in the coursework's questions: "I feel like I am looking for certain... things that come up in the questions and then pay like extra attention and stuff like that, so I learn differently, in that way... Like this morning we had a question that used words that were really similar to a question in my coursework, so I was like, right! Extra focus on this, I do focus on quite a lot but I was like writing my notes really good for this bit, it's just sort of extra pressure kind of thing." We notice that, for Lesley, the lack of control in this message makes the gap seem bigger between the practices used at school and at university. She sees mathematics as mainly procedural and positions herself as a student that needs explicit explanations of the mathematical methods seen in lectures. The weak control in Lecturer's B message, where he gave freedom to the students to act independently, made her feel undirected. She reacted to this challenge by adopting a strategical way of approaching the coursework without grasping the intention behind the message (i.e. make students act and think independently, encourage conceptual understanding). During the lecture, she tried to link words that sounded relevant to the questions in the coursework. Through this strategy she attempted to orientate herself in specific parts of the notes in order to be efficient with the coursework. This tactic limited her space for exploring mathematics and therefore, we suggest, shaped her mathematical identity in a very different way to that of Jason.

DISCUSSION AND CONCLUSIONS

The practices and messages at university positioned the two students differently, and hence shaped their developing identities in different ways. On the one hand, we have Jason who chose to study for a degree in mathematics inspired by two enthusiastic mathematics teachers who instilled a love of mathematics in him, ready to accept the challenge in the new context, willing to work and conscious about the differences in the content. On the other hand, Lesley liked mathematics and decided to study it encouraged by her teachers that she would be able to cope with it. She held a strong belief that the content would be like A Level Further Mathematics and she struggled with the new elements, such as the need to study proof at university.

We notice that students' identities are shaped differently according to the variations in the degree of control in the transmitted messages. Jason's identity as a mathematics learner is affirmed by the weakly framed message regarding the assessment method. This happens because the degree of framing here presents individual challenges and he is more willing to take action as a consequence. Through the narration of his identity he seemed to be self-aware about his mathematical abilities and choices and well prepared for the changes that he found at university. On the other hand, the message regarding mathematical discussion among peers, made him feel constrained; and he chose not to take any action from it because, for him, mathematics is an individual task.

Contrarily, strongly framed messages appeal to Lesley, whose mathematical identity is constructed on the perception that mathematics is about methods. She tries to find similarities between the two contexts and is keen on taking actions from messages with high degree of control because they can regulate explicitly her thinking, like the message about mathematical conversations. This agrees to some extent with Hernandez-Martinez's (2016) study where students entering university were alienated because their identities did not resonate with the new practices. Similarly, Lesley finds the change of assessment methods uncomfortable due to the weakly framed message that Lecturer B transmitted about coursework. The lack of control makes her feel constrained, leaves room for independence and Lesley did not seem ready for that yet.

These two cases show how messages with different degrees of power and control position students' in different ways according to their individual identities; therefore, it would be wrong to suggest that, for example, only strongly classified and framed messages should be transmitted by lecturers. The implication of our analysis is that university actors should account for the learners' identities and agency when designing educational practices. This confirms Pampaka's et al. (2016) work on current debates on the 'what works' agenda where the authors stress the need to consider the learners' agency in policy and practice in mathematics education. As we argued elsewhere (Kouvela, Hernandez-Martinez and Croft, 2016, under review) importance should be given to the discourses taking place in the teaching-learning interactions during the transitional phase. Taking these into account we can explore in what ways messages transmitted by university actors position students in relation to the practices of the new institution and how this shapes their identities during their transition to university.

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ICONICITY IN SIGNED FRACTION TALK OF HEARING-IMPAIRED SIXTH GRADERS

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This paper reports from a project on the investigation of the influences of sign language on the development of conceptualizations of mathematical ideas. Following research in Deaf Studies, iconic aspects of an idea represented in the related sign are considered one factor impacting the understanding of the signed concept. This paper adopts this approach and proposes a categorization of the diverse types of iconic references made by the students when signing about fractions, based on interviews with deaf and hard-of-hearing students using sign language as natural everyday-language.

INTRODUCTION AND THEORETICAL BACKGROUND

How do deaf students learn mathematics? How do they think about mathematical ideas? And how can answering these questions help us gain more comprehensive insights, not only about how to better respond to the specific needs of deaf students, but also about what influences learning and conceptualisation in general? While these questions are inspired by Healy (2015), research from the field of Deaf Studies suggests approaches towards answering them by considering specific features of sign language that have been found to influence the conceptualization of the signed idea.

This contribution presents part of a larger study that aims at understanding the influence of sign language on mathematical learning and establishing sensitizing concepts to foreground the impact of sign language in mathematical discourse. Specifically, this current report focuses on examining how students sign about fractions and how this might influence their understanding of fractions. Therefore, the objective of this paper is to provide first categories to describe how students sign in fraction talk.

Assuming that knowledge is constructed by individuals through co-construction in social interaction, communication as it is carried out in the gestural-somatic mode of sign language is considered to have a non-trivial impact on this learning process from two perspectives. On the one hand, visual aspects represented in the sign might influence 'what is actually talked about' and how the signed utterance may be interpreted as a whole, similar to as it is already considered for the case of gestures accompanying speech in learning processes of hearing children (Krause, 2016). On the other hand, following the theory of embodied cognition we can assume that bodily existence and the being in and experiencing the physical world impacts how we construct meaning and what kind of meaning we construct (Núñez, Edwards, &

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 89-96. Singapore: PME.

Matos, 1999, p. 53). With respect to the role of body in cognition, Wilson and Foglia state in their *embodiment thesis* more specifically:

> Many features of cognition are embodied in that they are deeply dependent upon characteristics of the physical body of an agent, such that the agent's beyond-the-brain body plays a significant causal role, or a physically constitutive role, in that agent's cognitive processing. (Wilson & Foglia, 2016, paragraph 3)

One aspect they highlight with respect to the body's role in cognition concerns the "body as constraint", which implies that

- Some forms of cognition will be easier, and will come more naturally, because of an agent's bodily characteristics; likewise, some kinds of cognition will be difficult or even impossible because of the body that a cognitive agent has.
- Cognitive variation is sometimes explained by an appeal to bodily variation. (Wilson & Foglia, 2016, paragraph 3)

Therefore, the conditions for deaf students with respect to cognitive processing can be considered being different to those of their hearing peers due to bodily variation.

Furthermore, from a socio-cultural perspective, mathematics is mediated semiotically and the way we come in touch with mathematics – whether it is through auditive signs or mainly through visual signs – alters the structure and the flow of how we think mathematically (Healy, 2015, referring to Vygotsky, 1917). In accordance with this, it is not the question *if* deaf students can develop mathematical skills just as their hearing counterparts, but rather *how* these skills develop and how the "profound restructuration of the intellect" (Healy, 2015, p. 299) caused by the substitution of the bodily tool in semiotic mediation changes how the mathematical thinking and knowledge becomes structured.

Influence of sign language on conceptualization

Research in the field of Deaf Studies points out that certain features of sign language influence the conceptualization of the corresponding signed ideas (Grote, 2013). One of these features concerns the *iconicity* of a sign, that is, the relationship between a sign and the aspects of the idea or object that can become reflected in this sign as evoked by some kind of similarity, e.g. to an action or object. According to Grote (2013), the iconicity of the sign influences which ideas become marked as distinctively linked to the concept. While in this study, only German Sign Language (DGS) is considered, the feature of iconicity encompasses sign languages in general (see Grote, 2013).

Sign languages are naturally growing languages and as such, they have been acknowledged as languages only since the last century. While for many mathematical concepts there is no common consensus about corresponding 'mathematical signs', these signs often develop in the discourse in the mathematics classroom (Fernandes & Healy, 2014). Investigating which aspects are reflected iconically in the signs used is thus key to getting a better understanding of how this idea becomes encountered and which aspects become considered important to 'stand for' the mathematical idea.

METHODS

This study was carried out in cooperation with a German school for special educational needs that focuses on ‘Hearing and Communication”. Ten deaf or hard-of-hearing students from a grade six class, were invited to participate in the interviews. German Sign Language was the primary language of each of the students. In the mathematics lessons, the hearing mathematics teacher used sign language as well as spoken language. The topic focused on in the interviews – fraction numbers – was covered in class two months earlier.

Interview methodology

One purpose of the interview is to investigate the students’ fraction talk, that is, to find out more about *how* the students talk about fractions and ideas related to fraction numbers. Therefore, two aspects become key in the methodological approach to the interviews:

- The students have to be encouraged to talk in their natural language, that is, they need to feel free to use sign language.
- The interviewer themselves shall not provide signs to refer to mathematical ideas that stand in the focus of investigation to not influence how the students talk about these ideas.

The first issue is encountered by having the interviews carried out by a deaf assistant that already contributes in the project by subtitling video data gathered in the classroom (see also Krause, *in press*). The interviewer has neither a research nor a specific mathematical background, which required to design an interview guideline and introducing her thoroughly to the purpose and the aims of the interview. While this proceeding provides good conditions for the first of the two aspects mentioned above, it obstructed the researcher to intervene in cases where further questioning may have helped assessing the students’ ideas of the mathematical concepts.

The second methodological aspect underlying the planning of the interviews was encountered by a specifically geared interview design that made use of ‘term cards’ and ‘fraction cards’. In the course of the interview, cards have been presented to the students, each labelled with a fraction term. The fraction terms given to the students were (English translation provided in brackets): ‘Bruch’ (*fraction*), ‘Zähler’ (*numerator*), ‘Nenner’ (*denominator*), ‘kürzen’ (*simplifying/reducing*), ‘erweitern’ (*expanding*), ‘Bruchrechnung’ (*fractional arithmetic*), and ‘Brüche vergleichen’ (*comparing fractions*).

The students are asked to talk about one term after the other, initiated by the interviewer asking “*I will give you some words. How would you explain the meaning?”* (signed as “words give-to-you content meaning explain-to-me (what?)”) after a first introduction to the interview situation. Subsequently, the interviewer asks the students “what fits together what?”, lets them regroup the cards on the table and asks for an explanation for the grouping they made. This slimmed down version of a concept map is trialled to gather further insights about the aspects considered significant for the students with respect to the mathematical ideas.

Following this, two fraction cards are given to the students, one labelled with the fraction $\frac{4}{2}$, the other one with the fraction $\frac{6}{4}$. The final task consists of students comparing these two fractions and deciding which one is bigger. The students' explanations ought to provide a further perspective on how the students talk about fractions in the specific context of a concrete task.

Data preparation and analysis

The video data has been subtitled by the deaf assistant using the German words corresponding with the signs, preserving the linguistic structure of German Sign Language as best as possible. These subtitles served as basis to identify the students' use of the fraction terms to then reconstruct their iconic reference.

KINDS OF ICONICITY IN STUDENTS' SIGNS FOR FRACTION TERMS

The investigation of the iconic aspects reflected in the students' signs used in the first part of the interview showed diverse types of iconicitiy, that is, diverse kinds of iconic similarity as reflected in the sign. In the following, the different categories will be presented by means of illustrative examples.

Innerlinguistic iconicity

A large amount of 'mathematical signs' used by the students when talking about the fraction terms has been found to be based on signs used in everyday sign language. That is, the sign resembles another, possibly nonmathematical, sign in handshape and/or motion of the hand, and placing of performance of the sign. Assuming that the iconic reference fosters a stronger link to the idea referred to in the similar sign, the reference of the *innerlinguistic iconicity* and its 'fit' with the corresponding mathematical idea need to be considered for the development and appropriate use of 'mathematical signs'.

For example, the DGS-sign for 'zählen' has been used as 'mathematical sign' for the term 'Zähler' (*numerator*). As nominalization of 'zählen' (*counting*), hence 'the one that counts', the idea of 'Zähler' could be conceptually linked to 'counting' the given number of the parts the whole is divided in, embedded in an understanding of fractions as 'part of a whole' (e.g. Kieren, 1980; Lamon, 2012).

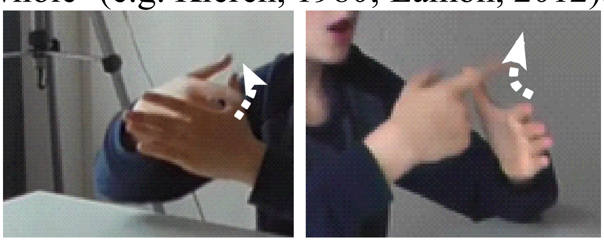

Fig.1: Sign used for "Zähler" *(numerator)* as innerlinguistically iconic to "zählen" *(counting)* in DGS (from two perspectives)

Another sign used for 'Zähler' reflected innerlinguistic iconicity to the DGS-sign for 'Zahl' (*number*). That is, the innerlinguistic iconicity to the sign for 'number' provides a link to a more general feature of the 'Zähler' – being a number – rather than providing a conceptual link to some idea of what the ‘Zähler’ could be understood as within the concept of fraction.

While potentially chosen due to the similarity of the written word ‘Zahl’ to the word ‘Zähler’ –-some kind of innerlinguistic iconicity in written language –-the sign for ‘Zahl’ furthermore also evokes innerlinguistic iconicity to the sign for “rechnen” (*calculating*). The shape of the hands matches for both signs, but the signs differ in movement insofar as the hands move down for ‘Zahl’ while they move up and down as opposed to each other for ‘rechnen’ (see Fig 2).

Fig.2: Sign used for ‘Zähler’ *(numerator*; left side*)* as innerlinguistically iconic to ‘Zahl’ *(number)* in DGS. On the right side, the sign for ‘rechnen’ (*calculating*).

That this actually seems to influence conceptualization is revealed in a student’s choice for grouping the terms in the second part of the interview. Being asked “what fits together?”, she explains her choice of grouping ‘Zähler’ and ‘Bruchrechnung’ together by pointing at the card ‘Bruchrechnung’, performing the sign for ‘rechnen’, then performing the similar sign for ‘Zahl’, placing the hand beneath the card for ‘Zähler’ and nodding before continuing with her explanation for the rest of her grouping.

The signs the students used for ‘Nenner’ (*denominator*) have been found to be similar to each other, all providing an innerlinguistic iconicity to the sign for ‘Name’ (*name*) or ‘nennen’ (*naming*). However, differences have been found in the features the sign used as ‘Nenner’ shared with the one of ‘Name’/’nennen’. The signs can coincide

- by only sharing the same shape, the DGS-sign for the letter “n” in this case. Since this is a rather general match, the link provided through innerlinguistic iconicity is a rather weak one.
- by sharing the same shape and the same motion.
- by sharing the same shape and the same motion and by furthermore being performed at the same place, the cheek in this case. The link provided here between “Nenner” and the idea of “Name”/”nennen” is a stronger one.

Iconic-symbolic and iconic-physical reference

Iconic-symbolic reference in this context concerns a signs’ reference “to a symbolic, written inscription, which in turn represents a specific mathematical entity or procedure” (Edwards, 2009, p. 138). *Iconic-physical* reference, on the other side, concerns the similarity to real objects or physical actions (Edwards, 2009). Although the students’ referred in their explanations of ‘fraction’, ‘numerator’ or ‘denominator’ often to the symbolic representation of the fraction as one of the numbers being located above the fraction bar, the other one below, none of the signs referring to the fraction terms where purely iconic-symbolic or iconic-physical. Nevertheless, all of the students used a sign for ‘kürzen’ (*simplifying*) that combined both (see Fig. 3).

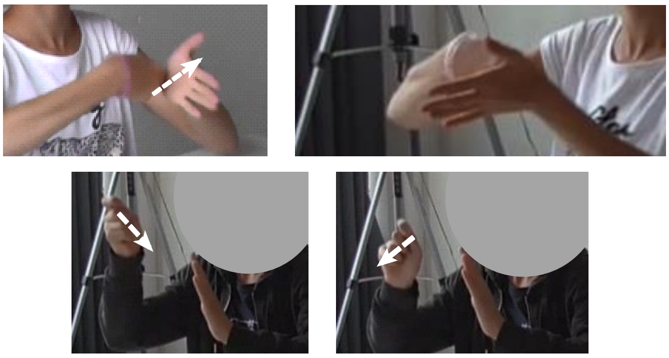

Fig. 3: Sign used for ‘kürzen’ *(simplifying)*, reflecting the action of striking off in the symbolic representation of the fraction while simplifying (above: from two perspectives; below: subsequent movements)

The signs reflect the idea of striking off when dividing the numerator and the denominator by the same number. It therefore refers iconically to a physical action that is performed within the symbolic notation of the fraction. With this, it recalls an aspects of the procedure performed when simplifying a fraction.

Iconic aspects of fraction talk in fraction comparison: an enacted iconic approach

8 out of 10 students approached the comparison of the fractions by activating area models of cake, chocolate or pizza pieces (Lamon, 1999). For this, they subsequently ‘placed’ respective imagined ‘wholes’ in the signing space in front of their body and ‘cut’ them into parts. This *enacted iconic approach* reveals an interpretation of the fraction as ‘part of a whole’, providing a *visual basis* to solve the task by means partitive division within the ‘quotient subconstruct’ (e.g. Marshall, 1993). However, all of these eight students mixed up the roles of the dividend and the divisor and identified the denominator as providing the number of wholes and the numerator as giving the parts of each whole. Since all the students visit the same class this might be explained by being prompted by some approach to fractions followed in the lessons, but not yet being fully elaborated.

CONCLUSIONS AND DISCUSSION

In this paper, I have presented diverse ways of how signed fraction talk might feature iconic aspects of mathematical ideas in the signs and gestures used and proposed how these aspects might influence the way these ideas become perceived and processed. For example, these iconic aspects might concern a certain similarity to other signs that are already used as conventionalized with another meaning and in this sense, bear an *innerlinguistic iconicity* within the specific sign language. The mathematical idea might then become linked to and interpreted against the background of some association the conventionalized meaning might evoke. Also, a sign can refer to a symbolic representation of a mathematical idea or to some sort of procedure carried out in its context. That way, it might foster a link to this representation or procedure by means of providing *iconic-symbolic* or *iconic-physical* reference to them. Furthermore, explanations carried out in sign language can provide a *visual basis* to the mathematical idea.

Grote points out that “assuming that epistemic processes are processes inherently mediated by signs, the similarity that forms the relationship between icon and referential object is constituted actively” (Grote, 2010, p. 312, translated by the author). That is, a sign does reflect iconic aspects of a referential object, or idea; it does so only for those who are aware of this iconic relationship. For signs referring to mathematical ideas, the reference has to develop hand in hand with the mathematical idea. Therefore, two intertwined processes of meaning making – of the mathematical idea and of the corresponding sign – have to be combined. In (Krause, *in press*) I describe this reconstruction of the ‘process of iconization’ to survey the gestures and signs used by a teacher while introducing the concepts of axial symmetry and point symmetry in an all-deaf classroom. The former becomes grounded in the activities of folding and mirroring, the latter in the activity of rotating around a point. The corresponding signs the teacher conventionalizes for “axial symmetry” and “point symmetry” respectively reflect these ideas iconically, showing aspects of innerlinguistic iconicity (mirroring) and iconic-physical references to folding and rotating. This raises the questions, are there general ways in which certain iconic relationships develop in processes of encountering mathematical ideas? Are these observable in the mathematics classroom?

As has also been seen in the description of the results, students do not necessarily use the same signs in their fraction talk. Still, there needs to be some degree of conventionalization if they want to communicate in the mathematics classroom. How does the use of multiple diverse signs for one mathematical idea influence the variety of conceptual links available for a student with respect to the signed idea?

The different types of iconicity presented in the examples are by no means thought of as exhaustive categories but rather as providing a first approach to describing the features of signed mathematical talk, based on a specific empirical basis. Further research needs to be done to widen the scope and uncover other categories so as to investigate the nature of mathematical signs and related visual-gestural representations as they develop and become established and used in the mathematics classroom.

Making claims about what makes a mathematical sign beneficial or hindering for learning mathematics is beyond the scope of this paper. The results presented here moreover raise awareness of how a ‘mathematical sign’ can be more than just a mere ‘name’ for a mathematical idea and how visual aspects of sign language can influence the shaping of mathematical thought. On the one hand, this provides an important baseline for attempts of developing dictionaries of ‘mathematical signs’, a current discussion in the DGS-community. On the other hand, knowledge about the influence of the shaping of mathematical signs provides a starting point for the elaboration of teaching methods in the mathematics classroom of deaf and hard-of-hearing students. In addition, research towards a more comprehensive knowledge about how those visual-gestural representations influence learning might also shed another perspective on how our body in general and gestures in particular might contribute to and shape the learning of mathematics.

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READING COMPREHENSION, ENJOYMENT, AND PERFORMANCE IN SOLVING MODELLING PROBLEMS: HOW IMPORTANT IS A DEEPER SITUATION MODEL?

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In an experimental study with ninth graders (N=165), we investigated whether presenting reading comprehension prompts would have a positive impact on students' enjoyment and performance in modelling. Contrary to our expectations, the enjoyment and modelling performance of students who received reading comprehension prompts were similar to those of the students in the control group. Further, we found that students' success in answering the reading comprehension questions was positively related to their enjoyment and modelling performance. However, after we controlled for intra-mathematical performance, the relation between reading comprehension and modelling disappeared, whereas the relation between reading comprehension and enjoyment remained significant. Implications for future research are discussed.

INTRODUCTION

Reading comprehension is an essential precondition for successful modelling, and deficits in reading comprehension can be responsible for the occurrence of errors in solving modelling problems. Hence, it is important to ask whether presenting reading comprehension prompts, which have been found to improve students' reading comprehension in different domains (Levin & Pressley, 1981; Rickards, 1976), can also lead to better modelling performance. In the present paper, we examined whether reading comprehension prompts would have a positive influence on students' performance and enjoyment in solving modelling problems. Further, we investigated whether answering the reading comprehension questions correctly would play a role. We therefore analyzed the relation between success in answering the reading comprehension questions and modelling performance and the relation between reading comprehension and task enjoyment.

THEORETICAL BACKROUND AND RESEARCH QUESTIONS

The Role of Reading Comprehension in Mathematical Modelling

The core of mathematical modelling is the translation of a real-world situation into a mathematical model. The translation process requires an adequate understanding, structuring, and simplification of the initial real-world situation. This means that students need to be able to build an adequate mental model of the situation before it can be mathematized. Even the first step of "understanding the situation" in the

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 97-104. Singapore: PME.

modelling process can be demanding for students and is often a source of errors (Blum, 2015; Leiss, Schukajlow, Blum, Messner, & Pekrun, 2010). This is not only because the reading comprehension that is necessary to solve modelling problems is cognitively demanding but also because students are used to word problems that can be solved without the effort of building an adequate mental model of the real-world situation ("Situation model") through the direct application of the given numbers in a straightforward calculation (Verschaffel, Greer, & de Corte, 2000). However, for modelling problems, which often include superfluous or missing information, such strategies are not sufficient and can lead to incorrect solutions (Krawitz, Schukajlow, & Van Dooren, 2016). Because of the fact that the real-world situation is mostly presented in written form (Verschaffel et al., 2000), it is obvious that reading comprehension is a necessary condition for deriving a situation model from the text, and it plays an important role in understanding and further structuring and simplifying the written information that is presented about the real-world situation.

A first indication of a positive relation between reading comprehension and performance in solving modelling problems comes from research on word problems. The positive relation between the two factors was found to remain significant even after technical reading skills were controlled for (Vilenius-Tuohimaa, Aunola, & Nurmi, 2008). Leiss et al. (2010) demonstrated that mathematical reading comprehension – assessed via the request to select the numerical information that was important for solving a given modeling problem – is even more important for modelling problems than for word problems. This study showed a significant relation between mathematical reading comprehension and performance in solving modelling problems (.486) and a smaller but also significant relation with performance on word problems (.183). Also, in Leiss et al.'s (2010) study, general reading comprehension was measured with a standardized reading test. In contrast to mathematical reading comprehension, general reading comprehension was not correlated with performance on the word problems or the modelling problems. This suggests that the specificity with which reading comprehension is assessed plays an important role.

Although mathematical reading comprehension was found to be important for modelling, we do not know much about how improvements in reading comprehension influence modelling processes. Because posing questions that were focused on the contents of the text was found to benefit students' understanding in research on reading comprehension (Levin & Pressley, 1981; Rickards, 1976), we applied this approach to investigate how the use of reading comprehension prompts would affect modelling performance.

Reading comprehension and enjoyment while solving modelling problems

Students' enjoyment as they solve math problems depends on whether they assign value to the activity of solving math problems and whether they perceive this activity to be sufficiently controllable (Pekrun, 2006). Because the perception of control, which is often assessed via self-efficacy, is closely related to performance, higher performance should result in greater enjoyment. Empirical evidence for this impact has been provided by the findings that students' mathematical performance in grades

3 and 6 has a positive impact on enjoyment in grades 6 and 9, respectively (Hannula, Bofah, Tuohilampi, & Metsämuuronen, 2014) and that students' grades at the beginning of the school year are positively related to their enjoyment during the school year (Ahmed, van der Werf, Kuyper, & Minnaert, 2013). As reading comprehension is an important part of modelling activities, and modelling is positively related to enjoyment (Schukajlow & Krug, 2014), higher reading comprehension should also result in greater enjoyment.

Further indications for the positive relation between reading comprehension and enjoyment in modelling has come from research in other domains. Deep reading comprehension was found to be accompanied by enjoyment (Guthrie et al., 2007). As students' enjoyment in solving modelling problems might refer to all modelling activities, deeper reading comprehension should result in greater enjoyment in modelling. Moreover, improvements in reading comprehension should also positively affect students' enjoyment in modelling.

However, to the best of our knowledge, we could not find research that had investigated the relation between students' reading comprehension and their enjoyment while solving modelling problems.

Research Questions

These considerations led us to pose the following research questions:

1. Does the presentation of reading comprehension prompts have a positive effect on modelling performance? Is higher reading comprehension positively related to modelling performance?
2. Does the presentation of reading comprehension prompts lead to greater enjoyment in solving modelling problems? Is higher reading comprehension positively related to enjoyment?

We expected that presenting reading comprehension prompts would lead to better modelling performance because answering the reading questions might lead to a deeper comprehension of the real-world situation presented in the text (Levin & Pressley, 1981) and thus to better solutions on the modelling tasks. Further, taking into account previous research (Leiss et al., 2010), we expected a positive relation between reading comprehension and modelling performance. Regarding the extent to which students enjoyed solving the modelling tasks, we expected benefits of presenting reading comprehension prompts because previous research showed that deep text comprehension was accompanied by enjoyment (Guthrie et al., 2007). Moreover, because of the positive impact of prior performance on enjoyment (Ahmed et al., 2013; Hannula et al., 2014; Pekrun, 2006), we expected that students who answered the reading comprehension prompts correctly would show greater enjoyment when solving the modelling tasks.

METHOD

Sample and design

Data were collected within the Taiwanese-German research program (TaiGer) on the influence of cultural-societal factors on mathematics education. The sample involved 65 ninth graders (46 % female, mean age = 15.12 years) in seven middle-track classes (German Realschule) at three different schools. Students in each classroom were randomly assigned to an experimental (EG) or a control condition (CG). All students had to take a 60-minute test that included three descriptions of real-world contexts (here, called situation statements) and corresponding modelling problems. The test version for the experimental condition also included reading comprehension prompts corresponding to the situation statements.

Measures

Two of three situation statements and the related modelling tasks were adapted from previous studies, and we developed other tasks on our own. In the following, one of the three situation statements from the test is presented as an example (see Figure 1).

Fire Brigade

In 2004, the Munich fire brigade got a new fire engine with a turn-ladder. Using the cage at the end of the ladder, the fire brigade can rescue people from great heights. According to the official rules, while rescuing people, the truck has to keep a distance of at least 12 meters from the burning house. Technical data of fire engine are shown in the table below.

Table 2

Technical data of fire engine	
Engine model:	Daimler Chrysler AG Econic 18/28 LL - Diesel
Construction year:	2004
Power:	205 kw (179 HP)
Cubic capacity:	6374 cm^3
Dimensions of fire engine:	Length 10 m width 2.5 m height 3.19 m
Dimensions of ladder:	Length up to 30 m
Weight of unloaded truck:	15540 kg
Total weight:	18000 kg

Figure 1: Situation statement of the real-world context "Fire brigade"

The test version for the experimental condition included six reading-comprehension prompts (two prompts for each situation statement). For the "fire brigade" context, one of the two reading comprehension prompts was:

"What is the longest possible length of the ladder?"

The answers to the reading comprehension prompts were scored dichotomously as right or wrong. The mean of the six answers to the reading comprehension prompts was used to assess reading comprehension performance. Thereby, the internal consistency was low, as expected, since the reading comprehension prompts were developed to address different aspects of the problem (Cronbach’s $\alpha = .518$).

Students in both groups were given six modelling problems (two for each situation statement). All of the modelling problems referred to the Pythagorean Theorem. For the “fire brigade” context, one of the modelling problems is presented here as an example:

> “What is the maximal height from which the Munich fire brigade can rescue people with this fire engine? Find one possible solution and briefly explain your solution.”

The solutions to the modelling problems were coded by applying a three-step coding scheme (from wrong = 0 to right = 2). The reliability for the six modelling problems was acceptable (Cronbach’s $\alpha = .719$).

In line with Schukajlow and Krug's (2014) study, students’ enjoyment was operationalized in a prospective and task-specific manner. Therefore, the students were first asked only to read the modelling problems and to use a 5-point Likert scale ranging from 1 (not at all true) to 5 (completely true) to rate whether they would enjoy working on the tasks (“I would enjoy solving these problems”). After answering this question, the students solved the respective modelling problems. The reliability of the scale was satisfactory (Cronbach’s $\alpha = .738$).

Moreover, an intra-mathematical performance test on the Pythagorean Theorem was administered (10 minutes, Cronbach’s $\alpha = .635$). The intra-mathematical performance test was used to control for students’ intra-mathematical abilities in assessing the relation between reading comprehension and modelling performance or enjoyment, respectively, and also to verify the comparability of the groups. A sample task is presented in Figure 2.

1. Calculate the length of the diagonal *d* of a rectangle with the length of 3 cm and the width of 4 cm.
   How long is the diagonal *d*?

   ☐ 5 cm ☐ 6 cm ☐ 6.5 cm ☐ 7 cm ☐ 8 cm

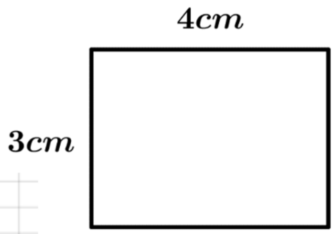

Figure 2: Sample task from the intra-mathematical performance test

We removed three of the 165 students from our analysis because they did not answer the enjoyment questions. We included the remaining 162 students ($N_{EG} = 81$; $N_{CG} = 81$) in our analysis. Missing values on the reading comprehension prompts and the modelling problems were coded zero, whereas for enjoyment the mean of the remaining items was calculated.

RESULTS

As a preliminary result, we found that the two groups had comparable intra-mathematical performances (EG: M = .29 (.20); CG: M = .29 (.21); t(160) = -0.048, p = .962). This result confirmed the comparability of the groups.

To investigate whether the presentation of reading comprehension prompts had an effect on modelling performance (research question 1), we used an independent t-test to compare the modelling performance of the EG ($M = .27$, $SD = .35$) with that of the CG ($M = .24$, $SD = .33$). The results showed that the groups did not differ significantly in their modelling performance ($t(160) = .609$, $p = .543$). Thus, the reading comprehension prompts did not have a significant effect on modelling performance. A correlational analysis (Pearson correlation) was used to examine the relation between students' modelling performance and the correctness of their answers to the reading comprehension prompts (research question 1). A low correlation between reading comprehension and modelling performance was found ($r(79) = .198$, $p < .05$, one-tailed). However, a much greater proportion of the variance in modelling performance was explained by intra-mathematical performance ($r(160) = .519$, $p < .01$, one-tailed), and the relation between reading comprehension and modelling performance disappeared after intra-mathematical performance was controlled for (partial correlation: $r(79) = .077$, $p = .248$, one-tailed).

Regarding students' enjoyment (research question 2), the EG ($M = 2.58$, $SD = .97$) reported nearly the same enjoyment as the CG ($M = 2.59$, $SD = .94$, $t(160) = -.123$ $p = .902$). Thus, the reading comprehension prompts did not have a significant effect on students' enjoyment. However, similar to the relation found in research question 1, reading comprehension was positively related to students' enjoyment ($r(79) = .220$, $p < .05$, one-tailed). Moreover, this relation remained significant even after intra-mathematical performance was controlled for ($r(79) = .202$, $p < .05$, one-tailed).

SUMMARY AND DISCUSSION

In the present study, we investigated the effects of reading comprehension prompts on students' modelling performance and enjoyment. Further, we examined the relations between students' success in answering the reading comprehension prompts and their modelling performance and enjoyment. Contrary to our expectations, the results showed that presenting reading comprehension prompts did not lead to an improvement in students' modelling performance or enjoyment. This indicates that the positive impact that was previously found from asking questions about text comprehension (Levin & Pressley, 1981; Rickards, 1976) could not be directly transferred to modelling performance in the current study. Thus, simply providing reading comprehension prompts does not seem to be sufficient for improving modelling. It is possible that students answered the prompts superficially so that the intended engagement with the text and the expected deeper understanding was not fulfilled. This explanation was supported by the results of our correlational analysis, which showed that success in answering the reading comprehension questions was positively related to students' modelling performance and enjoyment, respectively. Students who answered the reading comprehension questions successfully showed better modelling performance and greater enjoyment of the modelling tasks. Hence, it is not the presentation of the reading comprehension prompts on its own but rather students' actual engagement with the questions that seems to be the determining factor. The positive relation between reading comprehension and modelling performance confirms findings from previous studies, although the correlation we found was lower (.198 compared with .486, which was reported by Leiss et al. (2010)). This may have occurred because the reading comprehension test in our study focused on the construction of a situation model, whereas in Leiss et al.'s (2010) study, the students were asked to select the information that was important for solving the problem. Thus, in addition to general reading comprehension activities, the students in the previous study had to idealize and structure their situation model, and therefore were strongly engaged in modelling activities.

The positive relation between answering the reading comprehension questions correctly and students' modelling performance disappeared after we controlled for intra-mathematical performance. Hence, students' intra-mathematical performance seems to be crucial for students' modelling performance. However, the positive relation between students' reading comprehension and their enjoyment of the tasks remained even when we controlled for intra-mathematical performance. Students with deeper reading comprehension enjoyed solving the modelling problems more than students with surface reading comprehension, even when the two groups of students had comparable intra-mathematical abilities. This confirms the previous finding that a deeper understanding is accompanied by greater enjoyment (Guthrie et al., 2007) and moreover indicates that a deeper comprehension of the real-world situation results in a greater enjoyment of modelling.

Finally, we want to acknowledge the following limitations of our study. The benefits of prompting students to answer reading comprehension questions were hypothesized because of the findings of prior studies. In the present study, we thus used the reading comprehension prompts to enhance students' understanding of the text as well as to measure their reading comprehension. With this design, however, it is not possible to examine whether the prompts led to better reading comprehension in the experimental group compared with the control group. In addition, the modelling problems we used were found to be very demanding for the students in terms of constructing a mathematical model, so it is possible that this interfered with the examination of the interplay between reading comprehension and modelling because even students with a good understanding of the situation potentially had trouble solving the modelling problems.

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STUDENTS' EPISTEMOLOGICAL FRAMES AND THEIR INTERPRETATIONS OF LECTURES IN ADVANCED MATHEMATICS

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In this paper, we present a comparative case study of two students with different epistemological frames watching the same real analysis lectures. We show that students with different epistemological frames can interpret the same lecture in radically different ways. These results illustrate how different students' interpretations of a lecture are not inherently tied to the lecture, but rather depend on the student and that student's perspective on mathematics. Thus, improving student learning may depend on more than improving the quality of the lectures, but also changing student's beliefs and orientations about mathematics and learning.

INTRODUCTION

In recent years, several researchers have explored the relationship between students' epistemological beliefs and their learning of advanced mathematics. In particular, some scholars have claimed that some students struggle to learn mathematics because they lack the epistemological beliefs to support this learning (e.g., Alcock & Simpson, 2004, 2005; Dawkins & Weber, in press; Lew et al., 2016; Solomon, 2006). The primary aim of this paper is to extend this research. In particular, we introduce the notion of epistemological frames, a construct from physics education (e.g., Redish, 2004), and illustrate how students who hold different epistemological frames can interpret the same advanced mathematical lecture in different ways. In particular, using Mason's (2002) account of/account for distinction, we give an account of two students' epistemological frames in an advanced mathematical setting and then use these frames to give a fine-grained account for their different interpretations of the same utterances by a lecturer.

THEORETICAL PERSPECTIVE AND RELATED LITERATURE

Goffman (1997) introduced the notion of *frame* to describe how individuals develop expectations to help them make sense of the complex social spaces that they inhabit. For instance, most adults in the Western world have a "restaurant frame" consisting of expectations that are activated when they enter a restaurant. In a restaurant, an individual likely would expect that the restaurant employees will prepare food for the individual, the individual will be obligated to pay for this food, the menu consists of the food items that the individual may order, and so on (Schank, 1990).

Physics educators have introduced the notion of an individual's *epistemic frame* (or e-frame) as consisting of their epistemological expectations about a pedagogical

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situation. These consist of an individual's responses to questions such as "what do I expect to learn?", "how do I build new knowledge?", and "what counts as knowledge or an intellectual contribution in this environment?" (Redish, 2004). If a teacher and her students approach the same pedagogical activity with different e-frames, the students likely will not learn what their teacher intends, which Redish illustrated with a student focused on computation while the professor's aim was conceptual thinking.

We are not aware of any mathematics education research that has specifically used the specific construct of e-frames to account for students' behaviors, but many scholars have explored the general relationship between students' epistemological beliefs and their concomitant cognition. For instance, students' e-frames can be viewed as a subset of the sociological norms of a classroom (Yackel & Cobb, 1996) and conventional e-frames can be viewed as part of the didactical contract (Brousseau et al., 2014). Thompson (2013) illustrated how students who have a procedural orientation toward mathematics are likely to ignore or misinterpret the conceptual explanations that their mathematics teachers provide.

Logical versus psychological understandings in advanced mathematics

In this paper, we distinguish between two ways in which a mathematical proposition can be known. We say that an individual knows a proposition *psychologically* if the individual has grounds for believing that proposition is true. The individual knows a proposition *logically* if the individual perceives how the statement is a deductive consequence of other propositions that are known or assumed to be true. We also focus on a type of activity that deVilliers (1990) has coined *systematization*. In this activity, mathematicians transform an existing theory—i.e., a constellation of concepts and related statements that are accepted as true—into a unified whole. Mathematicians do so by creating a system of axioms and definitions and then demonstrating that commonly accepted statements are deductive consequences from this system of axioms and definitions. As deVilliers (1990) noted, with systematization, "the main objective is clearly not *to check whether statements are really true*" (p. 21, emphasis was the author's). In our interpretation, the purpose of systematization is not to *psychologically* know, rather, the purpose is to *logically* know statements (or to logically know these statements in a novel way).

METHODS

Rationale

In this paper, we report a comparative case study (Yin, 2013) in which we attempt to illustrate how a particular phenomenon unfolds within a given context. Specifically, we aim to illustrate how students' e-frames influence their interpretation of a mathematical lecture To accomplish this we interviewed two students as they watched real analysis lectures that had previously been posted on youtube. Here, the students can act as if they were attending an actual lecture yet the interviewer or student could pause the video to discuss their in the moment impressions of what was being discussed.

Participants

Two participants, Alice and Brittany (pseudonyms), agreed to participate in this study. Both participants were mathematics majors at a large state university in the northeast United States. At the time of the study, both students had completed a transition-to-proof course, were not currently enrolled in a real analysis course, but their university considered them appropriately prepared for one.

Procedure

The lectures consisted of Professor Su, a mathematics professor at Harvey Mudd College, beginning a real analysis course by constructing the rational numbers and then the real numbers from the integers. Prior to conducting the study, the research team studied the lecture and parsed the lecture into five to ten minute segments in which coherent mathematical content was being presented.

Each participant met individually with the first author of the paper once a week for four video-recorded clinical interviews. Interview 1 was a one-hour interview in which the participant discussed their experience in their transition-to-proof course to provide the interviewer with a sense of the participants' understanding of the course content (e.g., proofs, rationals) as well as their learning strategies and dispositions.

Interviews 2, 3, and 4 were two-hour interviews in which the research team attempted to explore the e-frames, ways of knowing, and any associated mental schemes that each participant used to interpret the mathematical lectures. During each interview, the participant watched Professor Su's lecture and was instructed to stop the video whenever something noteworthy occurred. The interviewer would also stop the tape to probe the participant's thinking when the professor had stated something that the research team had identified as important or at the end of a segment. The next interview began with the interviewer asking that the research team had generated after watching the previous video. The next interview then had the student watch more of Professor Su's lecture.

After all four interviewers were conducted, we transcribed all four interviews and clarifying our initial hypotheses of participants' e-frames from the prior concurrent analysis. We then engaged in cyclic retrospective analysis, using Mason's (2002) account of/account for distinction, and had two main purposes: (1) we first aimed to analyze broad characteristics of Alice and Brittany's behavior in our interview to give an account of the e-frames that they are using; (2) we then analyzed specific interpretations that they gave to Professor Su's lectures and used their e-frames to give an account for these interpretations. To identify e-frames, for each segment of the lecture, we summarized Alice and Brittany's comments. We focused on what mathematical contribution Alice and Brittany perceived was being made and, when possible, inferred the criteria against which they were evaluating that contribution. For each aspect of a participant's hypothesized epistemological frame, each member of the research team individually read the transcripts, identifying all instances that either supported or disconfirmed that the participant held this frame. The research team then met to determine how well the proposed epistemological frame was supported by the data and either kept the frame, or, as needed revised it and re-coded or removed the aspect from the e-frames attributed to the student. The result of this retrospective analysis was epistemological frames for Alice and Brittany that were grounded in our data. With these hypothesized e-frames, we revisited episodes where Alice and Brittany had different interpretations of the meaning of segments of Professor Su's lectures. We chose differences that we felt were representative of the data set and engaged in interpretive analysis in which we accounted for Alice and Brittany's different interpretations via our posited e-frames.

DATA AND RESULTS

We first note two aspects that both Alice and Brittany appreciated, understood, and enjoyed mathematical proof and that both exhibited an internal locus of control.

Alice's e-frames

One needed to define a concept to be able to reason about it

At the end of Interview 1 and before watching Professor Su's lectures, Alice was asked what the real numbers were. Alice's response was revealing: "That's an excellent question [long pause]. I don't know the formal definition of a real number". This was representative of Alice's tendency to express an epistemic need to see concepts defined, something which she displayed throughout her four interviews. For instance, in Interview 1, Alice was asked if the fractions 9/15 and 12/20 were "the same thing". She responded, "You need to assign a definition. Same thing does not tell me anything. […] So based on how we want to define in the same thing, they may or may not be". The importance that Alice assigned to concepts being defined led her to continually seek out definitions when she was watching the lectures.

When constructing a system, you need to distinguish between what you know through experience and what you are allowed to know within the system

As Professor Su constructed the rational numbers, Alice continually distinguished between what she knew based on her psychological understanding of the rationals and what was permissible to assume as the rationals were being constructed, her logical understanding. At 14 different points, Alice stressed the need to differentiate between the two, reminding herself and the interviewer that "we only assumed that we have knowledge about the integers" and "we don't know anything about what [rational numbers] do or look like if they are not an integer".

What were the lectures all about?

In the last interview after the conclusion of the lecture, the interviewer asked Alice "how do you understand the rationals?". Her response was as follows:

> [I understand the rationals] on a very simplified level. [The rationals] are just fractions of an integer, numerator and denominator, and I've been working with those types of fractions all my life. […] But on a construction level, we

> are trying to build them. It's like I want to already know this but the attitude that it is newly explored material which is a little ironic. It's the attitude that you kind of have to have.

In the first excerpt, Alice distinguishes between what she knows on a "simplified level" (what we call knowing in a psychological sense) and at the "construction level" (knowing in a logical sense), noting that you are trying to construct what you already know simplistically (justifying logically what you know psychologically). In the second excerpt, Alice expressed a similar sentiment. You are trying to discover logically what you already knew psychologically. To Alice, the point of these lectures was to ignore everything we knew about the rational numbers and construct a logical foundation for the content from definitions.

Brittany's e-frames

Brittany believed definitions were used to enhance understanding

Brittany viewed the role of definitions as to help her understand a concept. This belief was exhibited in multiple ways. First, at six points, Brittany said that Professor Su was presenting definitions to ensure that the class had a shared understanding of what was being discussed. At six other points, Brittany recognized that Professor Su's characterization of the rational numbers as equivalence classes of ordered pairs differed from her understanding. In each case, Brittany figured that Professor Su was trying to enhance her understanding of the rationals by providing an alternative perspective on the topic, saying "you're seeing it [the rationals] in a new way".

Brittany would use what she knew about the rationals to answer the questions that Professor Su discussed in class

Brittany rarely expressed a distinction between what she knew from prior experience, her psychological understanding, and what she knew from deduction from definitions, her logical understanding. Only twice during our four interviews did Brittany question what she was allowed to assume. At 18 other points, she invoked facts about the rational numbers that had not been stated in the lectures to answer questions about the rational numbers, meaning she consistently relied on her psychological understanding of the rational numbers.

What were these lectures all about?

Brittany primarily viewed the purpose of these lectures as an extended review of the rational numbers. When asked about the main purpose of the lectures, Brittany said, "yeah, that [referring to the construction of the rationals], I guess, was important to take away." When asked what it meant to construct the rationals, Brittany responded, "I think he was just going over properties of it—order, addition, multiplication, what it means putting them all on the number line". Brittany was generally frustrated because she wanted to learn new material and did not find value in what she perceived as an extended review, as can be illustrated in the following exchange:

> Interviewer: So what I'm hearing you say is it’s more interesting to talk about things you don't know than things you do know, to answer some questions that you might not really know that are interesting?
>
> Brittany: Yeah, I think that’s like true for everything.

In general, Brittany wanted to apply her robust understanding of the rational numbers to the content that Professor Su was discussing and was annoyed that he did not.

Differing interpretations of the lecture

At multiple points, the two students interpreted Professor Su’s lecture in different ways. Professor Su defined addition by $<(a,b)> + <(c,d)> = <(ad+bc, bd)>$. When Alice was asked what Professor Su was trying to convey, Alice responded that providing this definition was necessary.

> Alice: [Without the definition], we wouldn’t know what addition is. We want to keep that mentality that the whole thing that we are doing is we are defining that construction, so we need to make these rules these definitions.

Alice proposed one such criterion for evaluating a definition for addition was verifying that the definition implied that $<(a, 1)> + <(b, 1)> = <(a+b, 1)>$. Recall that Alice’s e-frames specified that new concepts required precise definitions and one could not use their previous knowledge about the rationals in justifying claims about the rationals. Our interpretation is that these e-frames led Alice to see the necessity of defining addition precisely. But, while she recognized the importance of justifying the adequacy of the definition, she also knew it would “work” based on her prior experience with rational numbers:

> Alice: The other half of me, well I know how to get to this. Do I really want to seem like lay it all out or do we just accept this definition? Like I know why cause it works and that’s just what I’m told […] I feel like a lot of this would be considered valuable but I wouldn’t say its significant and new.

Our interpretation of this excerpt is that although Alice appreciates the need to justify that Professor Su’s definition of addition is an adequate one (it “would be considered valuable”), a part of her does not want to see this justified because, based on her experience, she knows it is going to work.

When Brittany was asked about the definition, she thought the definition that Professor Su provided was adequate, saying, “I liked the definition because it’s true. I can totally see how he got it. I thought it was going to be that. It proves I know what’s going on”. However, later in the interview, Brittany also complained that she saw little value in the lecture in its entirety, saying, “it’s not that useful because I already know what addition is, know what rational numbers are, and what fractions are”. We had previously discussed that in Brittany’s e-frames, definitions were used to enhance understanding and good definitions were comprehensible. Our interpretation is that Brittany liked this definition, as it was consistent with her previous experience and she was able to comprehend it. However, because she already understood what the rational numbers were, the definition was superfluous.

After defining rational addition, Dr. Su then argued that the operation was well-defined. To illustrate what he meant by well-defined, Professor Su presented two other candidates for addition, one of which was well-defined but useless (an operation whose output is always <(0, 1)>) and another operation that not well-defined (<(a, b)> + <(c, d)> = <(a+c, b+d)>). Alice claimed she understood Professor Su meant by the term well-defined, saying "we can put in different elements of the same equivalence class, and we should still expect the same result". Nonetheless she objected, "when he says this definition is well-defined, the specific definition requirements for something being well-defined was not gone over. The term well-defined was actually not well-defined". We suggest that Alice understood the concept of well-defined psychologically, but without a formal definition she could not understand the concept logically. Due to her e-frame that concepts need to be defined precisely, she found Professor Su's presentation inadequate.

Brittany viewed the definition favourably.

> Brittany: I like the definition of well-defined. It was really clear and understandable because well-defined is a word we use a lot. When he did the example with the bad definition of arithmetic and then he used two equivalent fractions and got a different answer. He was like 'so we use two things that are supposed to be the same'. So it worked.

We claim Brittany found Professor Su's examples adequate to get a psychological understanding of what the concept of well-defined meant and so she was satisfied.

DISCUSSION

We use the general finding that students' epistemological frames can enable or prevent students from interpreting mathematical lectures in a productive manner to make two points. First, previous research on lectures in advanced mathematics has generally focused on what the professor says but did not consider student's interpretation of what was said. Our results illustrate how students' interpretations of a lecture are not inherent in the lecture itself but also depend on the student doing the interpreting. Consequently, we argue that ignoring students' interpretations of lectures is a significant shortcoming of most studies on lectures in advanced mathematics. Second, our results suggest that the key to improving students' learning from lectures does not consist only of improving the quality of the lectures. Rather, it is important to attend to their epistemological frames as well, a point that Solomon (2006) and Dawkins and Weber (in press) argue has received limited attention in the mathematics education literature. While we showed how a student's distinction between logical and psychological understandings led students to interpret and evaluate the mathematical contributions of a systematization lecture differently, we believe this is representative of the more general phenomena that e-frames influence how lectures are interpreted and what is learned (e.g., Lew et al., 2016).

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CONTEXTUAL PRINCIPLES FOR THE DEVELOPMENT OF PROBLEM SOLVING MATERIAL

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Plethora of research on problem solving undergoing since the 1970s identified pivotal practices for problem solving instruction. Despite almost five decades of this accumulated knowledge teachers lack practical teaching materials to systematically foster students' development of problem solving competence. In the context of this reform agenda, problem solving material was developed, implemented, and evaluated with respect to problem solving theoretical foundation. As a result contextual principles for the development of problem solving material were derived.

INTRODUCTION

The (inter-)national educational standards (e.g., KMK, 2003; NCTM, 2000) and mathematics educators (e.g., Bruder & Collet, 2011; Liljedahl et al., 2016; Schoenfeld, 1985) have strongly endorsed the inclusion of problem solving in school mathematics. Empirical studies and large-scale studies, however, portray a different picture: Students are often unable to solve problem tasks and many teachers lack teaching materials to systematically foster students' problem solving competence (Kuzle, 2016). This is not surprising as problem solving is an extremely complex human endeavor involving much more than the simple recall of facts and concepts, or the application of well-learned procedures (Schoenfeld, 1985).

Problem solving competence relates to *cognitive* (here heuristic), *motivational* and *volitional* knowledge, skills and actions of an individual required for independent and effective dealing with mathematical problems (Bruder & Collet, 2011). Accordingly, students should *learn approaches (heuristics)* for solving mathematical problems and *how to apply* them in a given situation, develop *reflectivity on own actions*, and develop *willingness to work hard*. Consequently, problem solving instruction should include teaching practices aligned with those goals. Concretely, theoretical foundation guiding the problem solving instruction should base on problem solving teaching approaches, theories of self-regulated learning and self-regulation in problem solving, and theories of motivation (Bruder & Collet, 2011).

In this paper I report on a small part of the SymPa – implementation research project (Systematical and material based development of problem solving competence) focusing on research-based development of problem solving material for grade 6 students. The guiding question was: *How can research-based material supporting findings from mathematics education research on problem solving be developed in practice?* In this paper I outline the material design basis and its design before outlining its implementation and evaluation. As a result of evaluation, I discuss contextual principles for the development of problem solving material.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 113-120. Singapore: PME.

PROBLEM SOLVING THEORETICAL FOUNDATION

There are at least seven practices for problem solving curriculum that researchers (e.g., Kilpatrick, 1985; Liljedahl et al., 2016) have claimed to be important for helping students grow in their problem solving ability. These are, however, isolated practices. In the recent years Bruder and Collet (2001) developed a problem solving teaching concept in combination with self-regulation focusing around Lompscher's (1975) idea of "flexibility of thought", which has shown to improve students' problem solving competence in higher grades of middle school (Bruder & Collet, 2011). This long-term teaching and learning concept encompasses five phases:

Intuitive familiarization: In this phase teacher serves as a role model when introducing a problem to students. This is achieved through moderation of behaviors typical for the problem by engaging in self-questioning pertaining to different phases of the problem solving process. In that way the teacher guides the students.

Explicit strategy acquisition: During this phase the students get explicitly introduced to new heuristics on the basis of a reflection from the first phase. Here the particularities of the heuristics get discussed and are given names.

Productive practice phase: During this phase the students practice solving the problems using new heuristics. Differentiation should be a guiding principle during this phase, so that the students can choose at what cognitive level they want to work and adapt the observed problem solving behaviors.

Context expansion: In this fourth "delayed" phase the students practice the use of heuristics independent of a mathematical context aiming at more flexibility use.

Awareness of own problem solving model: Awareness of own problem solving model can be induced by having students reflect and document their problem solving model.

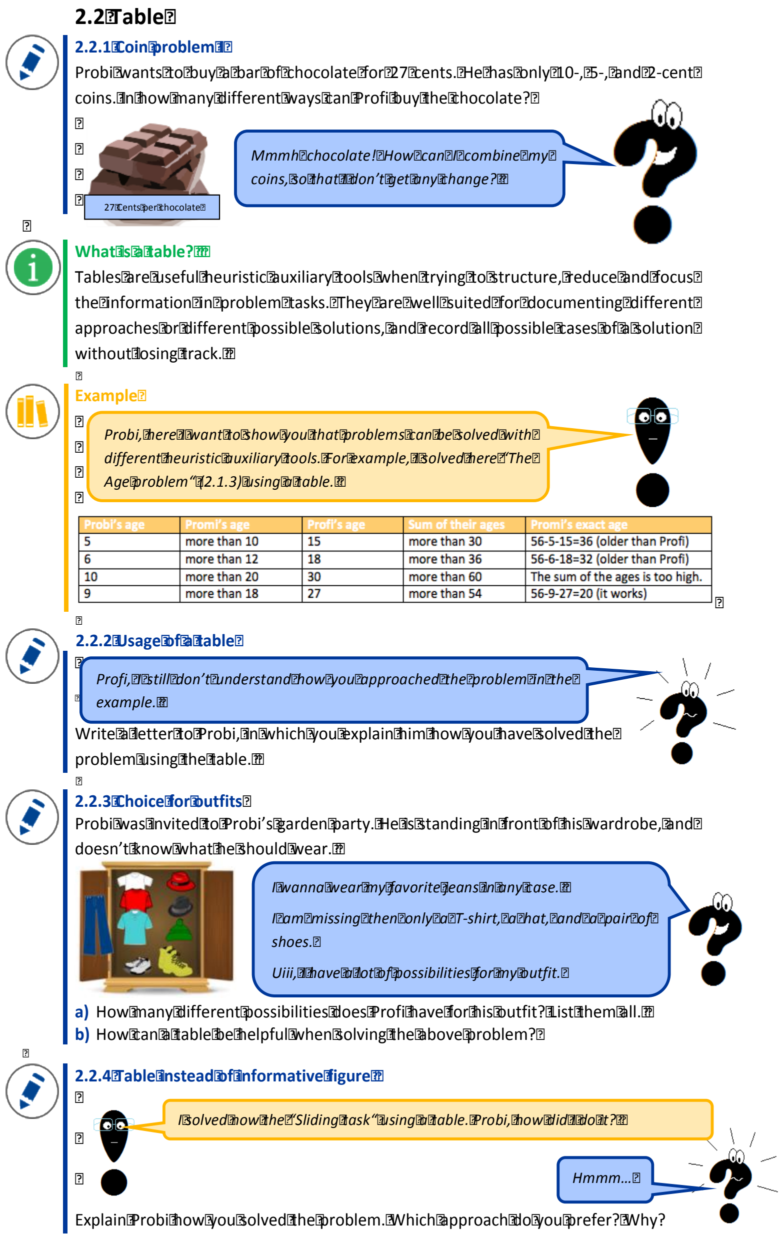
This teaching model focuses on the two subcomponents of the problem solving competence. However, problem solving competence encompasses also the ability to work hard (Bruder & Collet, 2011; KMK, 2003), which is related to motivation. Student motivation is a major factor for the successful problem solving. Without an effort from the learners, there will be no successful learning. Bruder and Collet (2011) summarize criteria for motivating tasks as follows: understandable and clear problem, age-appropriate choice of context, and visible competence growth.

DESIGN: IMPLEMENTATION OF THEORY INTO PRACTICE

The implementation of theory into problem solving material can be seen in Figure 1, which I elaborate on in connection to the research base used in the project.

Heuristic training: Learning approaches (heuristics) for solving mathematical problems

In the phase of *intuitive familiarization*, students solve a representative problem for the heuristic in focus (e.g., 2.2.1 Coin problem) together with the teacher, who serves as a role model. Here the imitation of teachers' behavior takes place through self-questioning. The problem represents the students' first encounter with the new heuristic. In the phase of *explicit strategy acquisition*, the new heuristic gets explicitly introduced through a short student-centered information text and a sample problem. In the phase of *productive practice*, usually three problems (2.2.2-2.2.4) of different cognitive level are presented. This allows for differentiation, where each student can solve as many problems as he or she can. In addition, problems from different mathematical content areas are covered, to allow for transfer (*context expansion phase*), which pertains to the fourth phase of the teaching concept. In addition, the heuristics are interrelated, so it is important that the students comprehend this. The last task (2.2.4) allows students to make this connection by comparing the two heuristic techniques and reflect on the solving process. In that manner students get to build on their problem solving model. This concept was used throughout.


2.2 Table

2.2.1 Coin problem

Probi wants to buy a bar of chocolate for 27 cents. He has only 10-, 5-, and 2-cent coins. In how many different ways can Profi buy the chocolate?

What is a table?

Tables are useful heuristic auxiliary tools when trying to structure, reduce and focus the information in problem tasks. They are well suited for documenting different approaches or different possible solutions, and record all possible cases of a solution without losing track.

Example

Probi's age	Promi's age	Profi's age	Sum of their ages	Promi's exact age
5	more than 10	15	more than 30	56-5-15=36 (older than Profi)
6	more than 12	18	more than 36	56-6-18=32 (older than Profi)
10	more than 20	30	more than 60	The sum of the ages is too high.
9	more than 18	27	more than 54	56-9-27=20 (it works)

2.2.2 Usage of a table

Write a letter to Probi, in which you explain him how you have solved the problem using the table.

2.2.3 Choice for outfits

Probi was invited to Probi's garden party. He is standing in front of his wardrobe, and doesn't know what he should wear.

a) How many different possibilities does Profi have for his outfit? List them all.
b) How can a table be helpful when solving the above problem?

2.2.4 Table instead of informative figure

Explain Probi how you solved the problem. Which approach do you prefer? Why?

Figure 1: A sample page from the material on the heuristic auxiliary tool of table.

With respect to heuristics, focus laid on those heuristic techniques prescribed for grade 6 students (KMK, 2003), namely heuristic auxiliary tools (informative figure, table), heuristic strategies (working systematically, working forwards, working backwards), and heuristic principles (composing and decomposing). With respect to mathematical content, problems covered the content areas of 5th and 6th grade mathematics (operations with natural numbers and fractions, combinatorics, measurement pertaining to 2- and 3-dimensional figures).

A particularly high quality of self-regulated problem solving can be achieved, when its different aspects with short occasional successive phases (*before* - objective, motivation, *while* - introspection, willingness to work hard, *after* - reflection (self-evaluation, self-reaction)) are promoted in the problem solving process (Landmann & Schmitz, 2007; Pólya, 1945/1973). For that reason, a problem solving question catalogue was given in the form of a table at the end of the material (see Figure 2). The students formulate questions (e.g., “What technique did I use to solve a similar problem in the past?” “What is the problem asking for?”) independently, as the teacher moderates the problem solving process in the phase of intuitive familiarization. The questionnaire created in this way was intended to serve as a reference in order for student to be able to progress independently in further phases of the problem solving concept by means of self-regulatory questions.

before problem solving	while problem solving	after problem solving

Figure 2: A problem solving question catalogue.

Reflectivity on own actions

Self-regulation plays a special role during a problem solving process. Structured reflection on the problem solving process (self-monitoring, reflection) has presented a key variable on four different levels in the problem solving material: on the one hand, reflection was called for in the individual tasks, for example by comparing the heuristics with one another or by expressing preferences (see task 2.2.3b, 2.2.4). On the other hand, at the end of each problem solving chapter, there were separate reflection questions connected to that chapter (see Figure 3). The aim here was that the students develop an overview and thereby reflect their own preferences by reviewing the chapter and identifying connections between the heuristics.

Through these different levels of reflection, students are prompted to reflect on their problem solving behavior (for example, concrete planning, application, goals, strategies), but also reflect on their learning behaviors, identify conducive and non-beneficial conditions, and then use them constructively for the further development of their problem solving behavior. Through these activities and documentation of these, the students are in the last phase of the teaching concept on problem solving (awareness of own problem solving model).

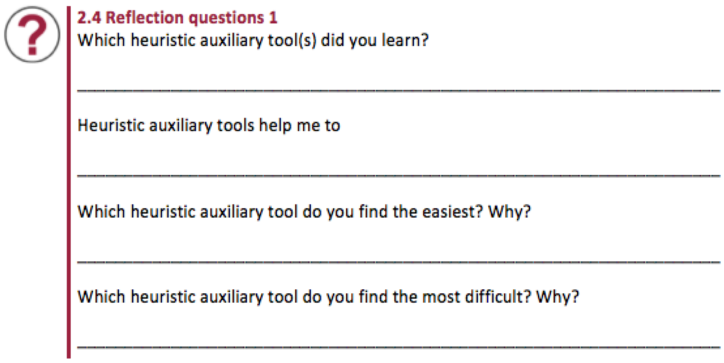

2.4 Reflection questions 1
Which heuristic auxiliary tool(s) did you learn?
____
Heuristic auxiliary tools help me to
____
Which heuristic auxiliary tool do you find the easiest? Why?
____
Which heuristic auxiliary tool do you find the most difficult? Why?
____

Figure 3: Reflection questions at the end of first problem solving chapter.

Willingness to work hard

The problems focused on contexts that are motivating and appropriate for middle school students. Transparent organization of the problem solving material was important, as it was intended for students to learn to work independently with it. For that reason, an icon/color-concept was used (see Figure 4), which reflected different components of the problem solving teaching concept. Through continuous connection between the problems and reflection, individual competence growth was aimed at.

Figure 4: Icon/color-concept used in the problem solving material.

Lastly, one other design element was used to support students willingness to work hard, namely two figures who came in a "direct contact" with the students (see Figure 1). Probi was a figure in a shape of a question mark and offered a problem context. Students were supposed to help Probi solve the problem, as he himself was not able to do so. Profi represented a supporting figure in a form of an exclamation mark with glasses. He gave hints of different nature or prompted different thinking, with the goal of supporting and guiding the students' problem solving process. In other words, Profi illustrated a professional problem solver. Only two figures were used in the problem solving material in order not to distract the students too much.

METHODOLOGY

The project was implemented in one urban school in Germany. Students of 6th grade were chosen to participate in the project lasting one school quarter, which is about 16 lessons (1 lesson = 45 min). Each heuristic was covered within 2 lessons. During the lessons students worked with the material and solved the problems according to the teaching concept as was outlined earlier. The material got implemented 7 times and in total 107 students participated in the project, who were randomly chosen. Data collection took place after the implementation phase. For that purpose a student questionnaire was developed, which entailed 5-level Likert items (1 = strongly disagree, 5 = strongly agree) pertaining to material's design elements (see Table 1). The first scale entailed 3 categories of the problem solving teaching concept (intuitive familiarization, explicit strategy acquisition, productive practice / context expansion) with accompanying subscales. The second scale entailed 3 subscales related to different levels of reflection (in-task reflection, chapter reflection, problem solving catalogue). The third scale entailed 3 subscales pertaining to elements related to students' willingness to work hard (problem context, material transparency, figures). Descriptive statistics was calculated for all quantitative data from the questionnaire.

Scale	Sample item	Cronbachs-α
Heuristic training (20 items)	Information text was helpful to understand the new heuristic.	0,81
Reflectivity on own actions (12 items)	Problem solving question catalogue helped me solve the problems independently.	0,73
Willingness to work hard (8 items)	Profi motivated me to work harder.	0,71

Table 1: Scales and reliability of problem solving material's design elements.

RESULTS

Here I address the project's research question and discuss the extent to which design elements supported the development of problem solving competence (see Table 2).

Heuristic training

Constant prompts: Material offering constant prompts guarantee gradual familiarization. Through this process students develop a need to question their actions, which is necessary for long-term problem solving.
Supporting strategy acquisition: Material offering several accesses to learning new heuristics supports learners of different styles. Access through sample problem or text information or its combination is important for reaching diverse learners.
Wide spectrum of problems with differentiated difficulty: In order for students to work on different cognitive levels, differentiated problems must be offered. This can be achieved by means of choice problems, open problems or problems with an increasing cognitive demand level.
Independent work: Using problem solving catalogue that was created through students' own language selection allowed them a closer relationship to it. In addition, it supported students to work independently during different phases of problem solving concept, especially in the phase of productive practice. In the phase of intuitive familiarization this was limited despite well-chosen representative problem.

Scale	Subscale	Mean (SD)	Median
Heuristic training I. phase	representative problem	3.52 (0.63)	3
	problem solving catalogue	4.40 (0.67)	5
Heuristic training II. phase	text information	3.91 (1.01)	4
	sample problem	4.31 (0.48)	4
Heuristic training III./IV. phase	number of problems	4.04 (0.45)	4
	increasing cognitive level of problems	4.20 (0.52)	4
	differentiated problems	4.51 (0.50)	5
	problem solving catalogue	4.68 (0.54)	5
Reflectivity on own actions	in-task reflection	4.23 (0.57)	4
	chapter reflection	3.98 (0.27)	4
	problem solving catalogue	4.35 (0.70)	4
Willingness to work hard	problem context	4.50 (0.50)	5
	material transparency	4.53 (0.80)	5
	figures	4.73 (0.47)	5

Table 2: Mean ranking, standard deviation, and median ranking of students' evaluation of problem solving material's design elements.

Reflectivity on own actions

Implicit reflection: The familiarity with the figures can be used to train reflectivity. The students help their "friends" to understand something by explaining their approaches. Such behavior is more comprehensible than explaining one's own action.

Explicit reflection: Constant prompts for reflection after each task or problem solving unit guarantee gradual self-questioning. Through this process students develop a need to question their actions, which is necessary for long-term problem solving. Problem solving catalogue contributed to reaching this goal.

Willingness to work hard

Transparent material structure: A material structure, that reflects teaching concepts and approaches for heuristic training, unburdens the teachers on the one hand, and ensures the compliance with the teaching concept on the other hand. A transparent structure focused around the entrance task, information text and sample problem, and differentiating problems supported students' independent work. Suitable icon/color-concepts serve as helpful visualization.

Use of figures in motivating contexts: For the transport of problems as well as for the provision of support, figures for the sixth graders represent a motivational moment. A separate world with everyday problems is opened to the students. They want to help their "friends" and recognize in the solution of a problem a benefit. The use of figures throughout the material allows students to get familiar with the figures and by doing so the support of the figures may be given a special value.

CONCLUSION

In this paper I focused on the question of how research-based material supporting findings from the research on problem solving can be developed in practice. The material was developed through collaboration between the researcher and the school. As a result context-related design principles for the development of problem solving material for grade 6 students were developed. The results show that students need an emotional incentive (here by the figures) in order to want to solve problems and to prompt their reflective behaviors. Transparency of the material structure and problem solving catalogue support students' independent work. Material design (differentiation, transparent material structure with explicit reflections) is an important factor in the development of self-regulatory processes when problem solving. Lastly, various design elements (text information, sample problem) allow for explicit strategy acquisition and 3 to 4 problems seem optimal for flexibility use. To what extent these context-related design principles apply to other contexts, can only implementation followed by an evaluation in other schools or other grades show.

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TWO LANGUAGES – SEPARATE CONCEPTUALIZATIONS? MULTILINGUAL STUDENTS' PROCESSES OF COMBINING CONCEPTUALIZATIONS OF THE PART-WHOLE CONCEPT

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Different languages are said to provide slightly different conceptualizations for mathematical concepts, e.g. for the part-whole concept of fractions. But how do multilingual learners make use of these different conceptualizations in their individual conceptual pathways? This case study investigates how fourteen German-Turkish seventh graders develop their part-whole concepts in a bilingual design experiment. The qualitative analysis shows that they use the conceptualizations across both languages and merge them into a multi-facetted part-whole concept. These findings provide a topic-specific empirical elucidation for the general idea of dynamic and interconnected multilingual repertoires.

Starting points and Theoretical backgrounds

Language-related conceptualizations and research needs for multilingual learners

The observation that languages sometimes provide different structures and conceptualizations has fueled controversial academic discourses on the so-called linguistic relativity hypothesis since von Humboldt and Sapir and Whorf (Lucy 1992; Gumperz & Levinson 1996). In mathematics education, the discourse led to empirical comparative studies that investigate if speakers of one language have advantages or disadvantages for their (habitual, not potential) mathematical thinking compared to speakers of other languages (e.g., Miura et al. 1988; Leung 2016). These comparative studies tend to adopt a monolingual perspective, assuming each student to be shaped predominantly by one language. However, for *multilingual* students, the question is not whether they are acquainted to one *or* the other conceptualization, but how the *interplay of different languages and conceptualizations* shapes their learning processes on the micro-level. This shift of research focus corresponds to the idea of dynamic and interconnected multilingual repertoires in multilingual communication rather than separate language proficiencies (Cummins 2000; House & Rehbein 2004; Lüdi 2006).

In this paper, we contribute to this research need by studying a case involving German-Turkish seventh graders' bilingual teaching and learning processes of the part--whole concept of fractions. The research question is: *How do learners in a bilingual teaching intervention adopt and combine (possibly language-related) conceptualizations of the part-whole concept across both languages?* After presenting the theoretical backgrounds and the methods, the qualitative analysis of 14 students shows that they relate several conceptualizations to each other in mostly fruitful ways.

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Language-related nuances in conceptualizing the part-whole concept of fractions

In most countries, the part-whole concept of fractions counts as one major meaning of fractions, besides rates and ratios (Cramer et al. 1997). However, different languages seem to provide different nuances in the conceptualization of the part-whole concept, connected to the reading and writing order: In Western languages, fractions are read and written top down (*3 fifths* in English or 3 *Fünftel* in German), whereas in most Asian languages fractions are read and written bottom up ("five parts, take two" in Mandarin (Bartolini-Bussi et al. 2014), or *beşte üç*, "five-therein three" in Turkish).

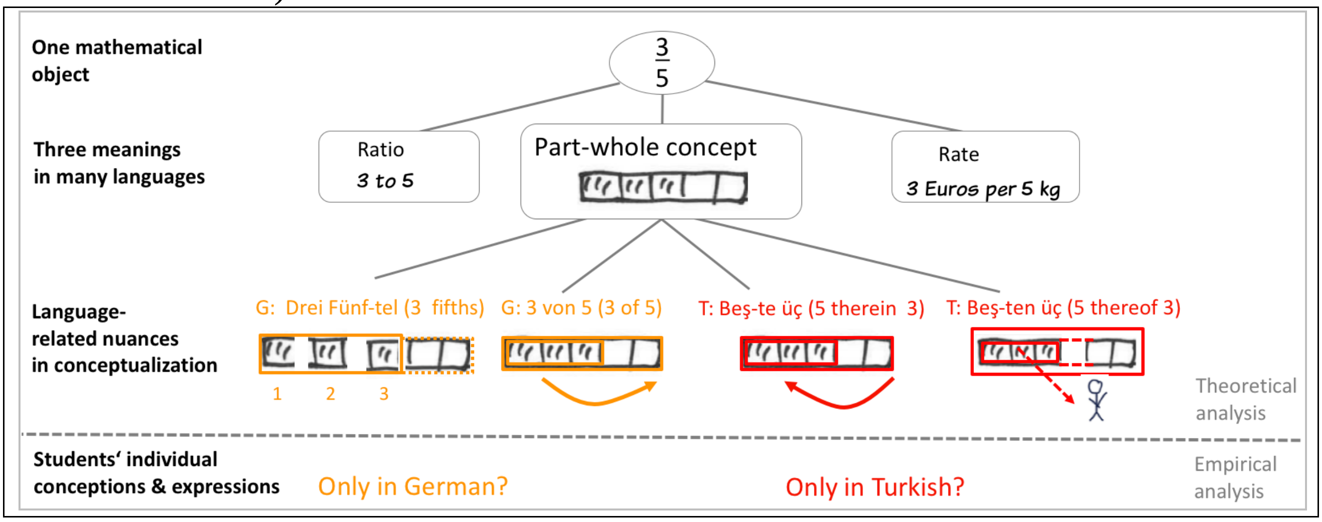


Fig. 1: Different conceptualizations in German (G, orange) and Turkish (T, red)

These language-related differences seem to not be restricted to single words but to more general habitual modes of thinking (Lucy 1996) and ways of linguistic action. This was confirmed in the case of geometric constellations by Leung's (2016) analysis of how order differences can shape mathematical thinking: Leung distinguishes the typical Asian *analytic approach* (first consider the whole, then the pieces) from the Western *synthetic approach* (first consider the pieces, then the whole) and shows far-reaching consequences for students' problem solving. In the case of fractions, the distinction of synthetic and analytic directly corresponds to the different conceptualizations in Turkish (with its Asian influences) and German (a Western language).

The German and English expressions "three fifths" also reveal a second nuance called quasi-cardinal conceptualization: considering a fifth as a unit and then counting the units (Cramer et al. 1997). The Turkish everyday language of German-Turkish immigrant students also provides a second expression, with another grammatical case: the ablative suffix *–ten* (used for movement away conditions) instead of the locative suffix of *–te* (used for static position conditions). In sum, Figure 1 shows four different nuances of conceptualizations for the part-whole concept.

So far, little is known how multilingual students with their access to both languages adopt and combine these different conceptualizations.

Alternative hypotheses for students' use of language-related conceptualizations

The idea of functional distinctions of languages for different purposes that underlies much research on code-switching (summarized in Barwell 2009) leads to *Hypothesis H1:* Multilingual students will learn all four conceptualizations and use each in the language in which it can be expressed best. The alternative hypothesis is shaped by the idea of dynamic and interconnected "multilingual repertoires" rather than separate languages (Lüdi 2006). On this basis, *Hypothesis H2* is that students adopt the conceptualizations across different languages.

Methods of the learning-process study

Research context. The research question was pursued in a learning-process study that was part of the larger mixed-methods project MuM-Multi. The larger project combined a randomized control trial with German-Turkish seventh graders (n = 139) in a teaching intervention on fractions in groups of 2-5 students (Schüler-Meyer et al. 2017) with several in-depth case studies analyzing videos and transcripts with respect to the integration of verbal and nonverbal communication (e.g., Wagner et al. 2016).

Data corpus of the learning-process study. From the large video data corpus, we selected about 230 minutes of video material for the learning-process study presented here. We concentrated the analysis on seven tasks (treating the part-whole concept in contexts of comparing fractions) done by n = 14 focus students (who were sampled according to contrasting backgrounds in their German and Turkish language proficiency and pre-test results on conceptual understanding of fractions). All students spoke at least German and Turkish, all were educated in Germany without prior formal experience of learning mathematics in Turkish.

Methods for qualitative data analysis. The transcripts were analyzed with respect to students' conceptual development across languages. For this purpose, an analytic tool for fractions was adopted based on Vergnaud's (1996) theorems- and concepts-inaction. After sequencing the transcript, the individual theorems- and concepts-in-actions were extrapolated for each sequence and then related to the language-related socially shared conceptualizations from Figure 1, e.g., the individual theorem-in-action <For comparing fractions, I compare the size of the pieces> (which is only adequate for unit fractions) is shaped by the individual concept-in-action <fraction as size of the pieces> rather than <fraction as part-of-whole>. In the analytic schemes (e.g., in Figures 3 and 4), the individual conceptualizations are presented graphically to show proximity/variation to the socially shared language-related conceptualizations; the colors of the utterances signify the language used by students to express them. The color codes allow the relation between language used and language-related conceptualizations to be conveyed. Two episodes (chosen to show the phenomena in a nutshell) are analyzed in some detail. The result of the complete analysis is summarized in a table that reveals more global pattern with respect to Hypotheses H1 and H2.

EMPIRICAL RESULTS

The following two episodes show that both hypotheses can apply in the multilingual learning processes.

Episode 1: Ilknur and the complementary use of languages

In Session 2, Task 3, the students are asked to draw 1/2, 2/3, 3/4, 4/5, and 5/6 in fraction bars (see Figure 2). One of the students, Ilknur, explains the task to her partner Akasya.

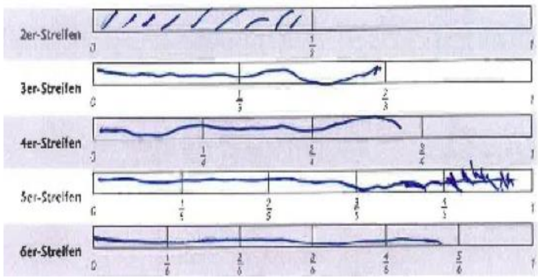


Fig. 2: Ilknur's fraction bars for Task 3

Turn Person	**Original** (Turkish in black, German in grey)	**English Translation** (from Turkish in red, from German in orange)
90 Akasya	Ich versteh das nicht!	I do not understand that!
91 Ilknur	Guck, ich zeig dir! Şimdik bak! Şöyle bakıyorsun. Burda ne yazıyor? Ein Halb! [...] O zaman bunu boyuyorsun.	Look, I show you! Now look! [*points at the 2-element bar in the fraction bar table*] You look this way. What does it say here? One half! [...] Then you have to color this one here [*colors the first piece in the 2-element bar*]
92 Akasya	Niye ama?	But why?
93 Ilknur	Weil es ein halb ist und das auch ein Halb	Because it is one half [*points at the fraction on the worksheet*] and this also one half [*points at the first piece in the 2-element bar*]
94 Akasya	Dann kann ich doch zwei Drittel hier kann ich doch auch das hier anmalen, wo ist zwei Drittel hier?	Then I can two thirds, here, [*points at the share on the worksheet*], I can yet color it also here, where is two thirds here? [*looks irritatedly at the fraction bar table*]
95 Ilknur	Bis zwei Drittel'e bakacaksın	Up to two thirds you have to look [*points at end point of the second piece of the 3-element bar*]
96 Akasya	Ah, zwei Drittel burda! Dann muss ich ja das hier alles anmalen oder was?	Ah, two thirds here! Then I must color all of this here or what?
97 Ilknur	Ja!	Yes!
98 Akasya	Und hier drei Viertel	And here three fourths [*points at the end point of the third piece in the 4-element bar*]

Ilknur explains how to draw the fractions in suitable bars of the bar table (see Figure 2). She starts using an analytic, localizing perspective where the whole is only implicit (*Buni boyuyorsun*, "this one here") in #91. When her partner Akasya signals misunderstanding, Ilknur changes languages and, with it, perspectives (a strategy often found for teachers and students; see Wagner et al. 2016): Within a Turkish phrase, Ilknur switches to German, *bis zwei Drittel* (until two thirds), in #95 to express a quasi-cardinal view, counting the pieces of the bar from the left zero until the two- thirds point in the bar.

From the German preposition *bis* ("until") we infer that using the German expression really corresponds to the conceptualization in her thinking at that moment. She repeats this conceptualization in #95 (*bis dahin*, "until here") and later in #99 and #101. In #96, Akasya connects both conceptualizations by combining the German expression *zwei Drittel* ("two thirds") with the Turkish *burda* ("here") and *das hier alles anmalen* ("color all of this here").

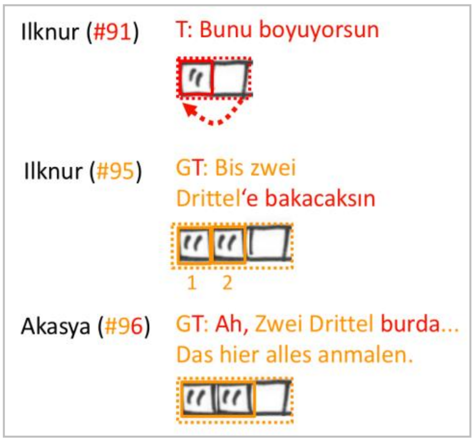


Fig. 3: Analysis of Episode 1

In sum, the analysis of Episode 1 (in Figure 3) shows a typical example of code-switching in Ilknur's complementary use of languages that supports Hypothesis H1. Akasya's reaction shows how combining two nuances of conceptualizations in two languages can enhance conceptual understanding, a phenomenon that could be found by means of deep linguistic analysis in various cases (Wagner et al. 2016).

Episode 2: Emir and the travel of conceptualizations from Turkish to German

In another group, the students work on the same Task 3, which asks them to draw 1/2, 2/3, 3/4, 4/5, and 5/6 in fraction bars (see Figure 2). Emir and Osman immediately draw one half, and the teacher asks them in Turkish where they see the half.

Turn Person	Original (Turkish in black, German in grey)	English Translation (from Turkish in red, from German in orange)
27 Emir	Eh iki komplett	Err, two, complete [*gestures with his pen to refer to the complete fraction bar*]
28 Teacher	Mhm.	Mm-hmm [yes]
29 Emir	Bi tanesinde alıyoz.	And one in it, we take.
….	[*Students proceed with other fractions*]	
47 Osman	Zwei Stück anmalen. Zwei Drittel, drei Viertel.	Coloring two pieces [*marks two thirds in the 3-element bar*] two thirds, three fourths
48 Osman	Drei Stück anmalen	Coloring three pieces
49 Ismael	Das gehört zu…?	That belongs to…?
50 Osman	Drei Stück an—warte—ja	Three pieces at—wait—yes [*marks three fourths on the 4-element bar*]
51 Emir	Und dann vier anmalen.	And then, color four.
…		
53 Osman	Vier Stück? Bei beş'li çubuk?	Four pieces? In 5-element bar?
54 Emir	Mhm, ja. Und bei 6er musst du fünf	Mm-hmm, yes. And in six, you must five
….	[*Students finalize drawing and discuss what to write down for the question "What can you discover? Write down your observation."*]	
67 Emir	Also, ich schreibe jetzt, dass unten […] Das zwei- das von ein Zweitel ist eh- Einhalb ist zwei als Ganze und du nimmst eins.	Well, I will write that below […] That two, that of one second, is Err, one half is two as the whole and you take one.

When asked to explain how to draw one half, Emir speaks in a mixed utterance, and refers to the Turkish –*therein* expression for the analytical conceptualization in Turkish (in #27 and #29). Again, we see an initial moment of code- switching. But then, Osman adopts the typical German synthetical conceptualization: He first names only the parts ("two pieces" in #47, "three pieces" in #48) and completes the whole only implicitly. In #53, he addresses the five-element bar as the whole explicitly (in Turkish). Emir, in contrast, keeps his Turkish reading order even when speaking German in #54. When asked to write down what they saw, Emir prefers to explain in #67 the Turkish –*thereof* conceptualization in German words.

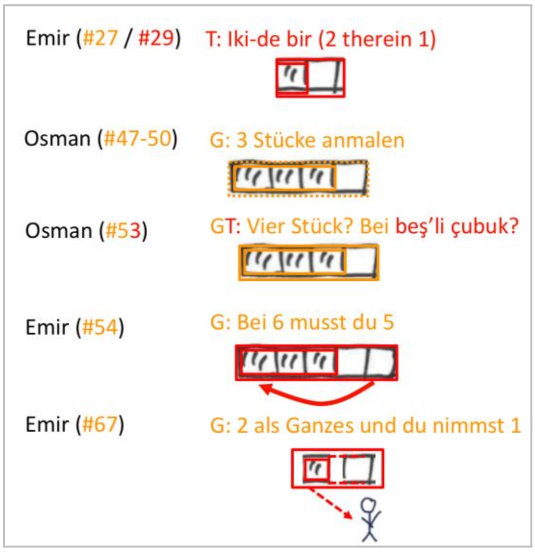


Fig. 4: Analysis of Episode 2

Hence, Episode 2 gives evidence for both Hypothesis H1 in the beginning and then later for Hypothesis H2. Figure 4 summarizes the analysis and shows the travel of conceptualizations from Turkish to German.

Overview on more cases and tasks: Travel of conceptualizations through languages

Both episodes show phenomena that could also be found for the other cases of students and tasks. Table 1 shows a summary of the analysis of five of the 14 focus students. For each student, the sequence of uttered nuances of conceptualizations for the part-whole concept is ordered from up to down and is in arranged in columns according to the socially shared conceptualizations to which the utterance refers. The transcript lines are shown in different colors for German (G) and Turkish (T) so that shifts of nuance become visible by placement along the horizontal axis and switches in language by changing colors along the vertical axis.

This summary illustrates that all students, not only Ilknur and Emir, start activating a nuance of conceptualization in one language, but when they use other languages, then also refer to the same nuance of conceptualization in the other language and in a mixed code. This might be different in other groups. What is made visible here for a sequence of four tasks seems to apply even more when considering several sessions of the intervention and more nuances of conceptualization than those presented in Figure 1.

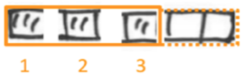
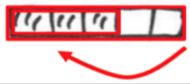
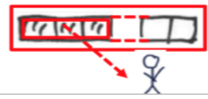
	T: Drei Fünf-tel (3 fifth) 	T: 3 von 5 (3 of 5) 	T: Beş-te üç (5 therein 3) 	T: Beş-ten üç (5 thereof 3) 
Ilknur	Task 3, #84 (G) Task 3, #91 (GT) Task 3, #95 (T) Task 3, #99 (G)	Task 3, #177 (G) Task 4, #114 (GT)	Sess1, Task 7, #3 (GT)	
Akasya		Task 2, #81 (T) Task 2, #107 (G) Task 3, #96 (GT) Task 3, #100 (G) Task 3, #159 (G) Task 3, #173 (T) Task 4, #67 (G)	Sess1, Task 7, #10 (T)	
Emir		Task 1, #20 (G) Task 1, #43 (T) Task 1, #76 (G) Task 2, #15 (G) Task 3, #90 (T) Task 4, #26 (T)	Task 3, #29 (T) Task 3, #54 (G)	Task 3, #67 (G)
Ismail		Task 1, #81 (G) Task 2, #2-7 (G) Task 2, #31 (T) Task 3 # 39 (G) Task 3, #90 (T)	Task 4, #28 (T)	Task 4, #64 (T)
Osman		Task 2, #25 (G)	Task 3, #50ff. (GT)	

Table 1: Travel of conceptualizations through the languages for five focus students

DISCUSSION

Two languages, two separate conceptualizations? By investigating the learning processes of 14 seventh graders on their pathways towards the part-whole concept of fractions, we found moments, such as those in Episode 1, that conform with Hypothesis H1 in that language is used complementarily for different conceptualizations. These complementary uses seem to enrich the multi-faceted conceptual understanding. However, across all of the video material and transcripts of the 14 students, we find more moments that conform to Hypothesis H2: In their learning processes, most students activate different conceptualizations, and, in the long run, address them across both languages. The case studies presented and the larger analysis of the data provide evidence that the **travel of conceptualizations across languages** can enrich the conceptual understanding by merging the different conceptualizations into a multi-faceted part-whole concept.

Although these findings are still shaped by methodological limitations such as the limited number of focus students and the specific tasks, it is already an interesting contribution to the idea of dynamic and intertwined multilingual repertoires which resonates with Cummins's (2000) and Lüdi's (2006) arguments against considering multilingual learners as having separate language proficiencies that may work only complementarily. The findings correspond to previous findings from the same project that it is not the complementarity, but the *connection* of languages (like the connection of different representations) that can substantially enhance students' conceptual development (Wagner et al. 2016, Schüler-Meyer et al. 2017). The findings motivate further studies of the connections of languages as specific resources of multilingual.

Acknowledgment.

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EXPLORING GENERATIVE MOMENTS OF INTERACTION BETWEEN MATHEMATICS TEACHERS ON SOCIAL MEDIA

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Stimulating sustainable mathematics teacher collaboration can be challenging in many commonly found professional development contexts. Despite this, an unprompted, unfunded, unmandated, and largely unstudied mathematics teacher community has emerged where mathematics teachers use social media to communicate about the teaching and learning of mathematics. This paper presents an analysis of one episode where teachers engage in a prolonged exchange about responding to a common mathematical error. Analytical tools drawn from complexity theory are used to explain moments of productivity. Results indicate that enough redundancy and diversity among members is necessary to make conversations productive. Identified sources of redundancy indicate the 'taken-as-shared' values of this group.

INTRODUCTION

Teacher professional development is essential for enhancing the quality of teaching and learning in schools (Borko, 2004). As such, various approaches to professional development, such as lesson study (Stigler & Hiebert, 1999) and communities of practice (Wenger, 1998), have been explored. What is known from this research, is that the robustness of a professional development initiative is dependent on ensuring both teachers and facilitators adopt a stance of inquiry, activities reflect and are driven by teacher needs and interests, and community building and networking are at the core (Lerman & Zehetmeier, 2008). This means that ongoing teacher collaboration is indispensable. However, due to constraints around time, funding, and facilitation, teacher professional development initiatives are commonly limited to sparse one-time professional development workshops held in face-to-face synchronous settings. Such workshops, due to their temporal nature, are generally unconducive to building communities that engender ongoing professional growth.

In contrast to these centrally organized, and sometimes compulsory, professional development initiatives, teachers from across North America are participating in decentralized, virtual, and autonomous professional communities. One such community involves mathematics teachers who regularly use Twitter and blog pages to asynchronously communicate their musings and practices, and have come to be identified as the Math Twitter Blogosphere (MTBoS) (Larsen, 2016). This unprompted, unfunded, and unevaluated teacher community is a rich phenomenon of interest that is largely unstudied and deserving of attention. As such, the study presented in this paper is driven by the overarching question – what can participation in the MTBoS occasion for mathematics teachers?

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THEORETICAL FRAMEWORK

With an aim to understand the autonomous organism of the MTBoS, this study is guided by complexity theory (Davis & Simmt, 2003; Davis & Sumara, 2006). Complexity theory provides the tools to describe a system of individual agents who seem to generate emergent macro-behaviours. Complex systems don't merely exist, they also learn. In complexity theory, learning is expanding the space of the possible and is primarily concerned with "ensuring conditions for the emergence of the as-yet unimagined" (Davis & Sumara, 2006, p. 135). The goal of complexity theory is not to identify interpersonal collectivity, as do other social theories of learning, but rather to understand 'collective-knowing', where knowledge is not attributed to any one member, but sits atop of the social network.

To this end, Davis and Simmt (2003) identify five interdependent conditions necessary for complex emergence, that is, for a complex system to learn. These conditions include *internal diversity*, *redundancy*, *neighbour interactions*, *decentralized control,* and *organized randomness*. Davis and Sumara (2006) further theorize these conditions into complementary pairs: specialization (tension between *diversity* and *redundancy*), trans-level learning (*neighbour interactions*[1] enabled through *decentralized control*), and enabling constraints (balancing *randomness* and *coherence*). These conditions form the basis of the theoretical framework that informs the overall study. For purposes of brevity, only the first pair of conditions, *diversity* and *redundancy,* will be used in the analysis presented in this paper.

The interplay between *diversity* and *redundancy*, also referred to as the 'zone of creative adaptability', is a key contributor to the ability of a system to adapt to changing conditions. *Diversity* allows for novel actions and possibilities because it refers to the diversity among the agents, while *redundancy* allows for stability and coherence because it refers to the common ground among agents. Without *redundancy*, agents may not be able to communicate, but without *diversity*, agents may never have anything to communicate about. Therefore, both are necessary for a system to be productive. Further, because of *decentralized control,* no agent is ever in a position of final authority, and knowledge is always tentative. Holding authority within a complex system means to have the capacity to use a prevailing discourse, or to act within the consensual domain of the system, with the overall aim of occasioning 'collective-knowing' (Davis & Sumara, 2006).

As such, this study takes interest in the possibility of 'collective-knowing' in the MTBoS, and pursues the question of how *diversity* and *redundancy* can contribute to the complex emergence of 'collective-knowing' in the MTBoS.

METHODS

Given that the MTBoS began developing as early as 2007 when mathematics teacher bloggers began to incorporate the use of Twitter into their blogging practice, and that there are over 500 self-identified MTBoS members, many of whom post multiple times a day, the sheer mass of data that has accumulated over the past few years makes the phenomenon too large to study within the confines of this paper. As such, a very specific subset is chosen as the data set for this paper. This subset contains all responses to a given Twitter post made by one particularly well-followed member. This conversation reflects the breadth and depth of MTBoS because it includes both very brief responses that do not continue conversation, and responses that initiate further conversation, both of which are generally encountered within the MTBoS.

Since Twitter is an ultra-personalized environment where users only see posts made by members they subscribe to as ‘followers’, we have taken an ethnographic approach as participant observers by immersing ourselves in the MTBoS community and subscribing to over 500 mathematics teachers who engage in the MTBoS. Without such an immersion, noticing and identifying the data set would be near to impossible. In addition, Twitter offers a feature which gives updates on the most relevant and most replied-to tweets one has missed. This feature enabled us to identify one particular post that generated a significant number of replies from mathematics teachers around the world. This post was made by Michael Fenton, who has over 4000 followers, and asked about how users would respond to a student’s mathematical error (see fig. 1).

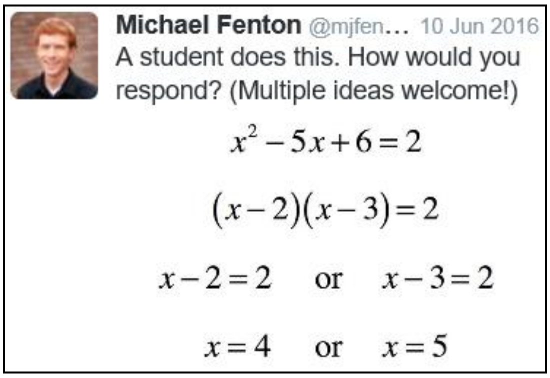

Figure 1: Fenton’s initial math mistake query

Fenton’s post elicited 254 replies from a total of 87 users, 52 of whom identify themselves as mathematics teachers. Replies included explaining the error, explaining why the error could have been made, describing a teaching approach to help the student come to a deeper understanding about the nature of the error, and generating activities to use with students to help mitigate this error. With an effort to maintain the reply structure as well as the chronological order of the posts, the data was organized into threads. Some of these threads were considered as *non-continuing* replies because they were made by one user and spawned little to no discussion. Other threads were considered as *continuing* because they included conversation between at least two users and elicited more than four subsequent replies. Out of the total 254 tweets, 84 were identified as non-continuing, 155 were identified as continuing, and 15 were irrelevant. The 155 continuing tweets were reconstructed into ten threads based on both chronology and logical conversation order.
With an aim of understanding the substantial variance in engagement in this conversation and to illuminate the complex emergence and ‘collective-knowing’ of

the MTBoS, we analysed the data using the five conditions for complex emergence, as outlined by Davis and Simmt (2003) and further elaborated by Davis and Sumara (2006). As mentioned, in this paper, we discuss only the aspects of *diversity* and *redundancy* within the continuing threads in pursuit of the more specific question – what are possible sources of internal *diversity* and *redundancy* within a series of self-organized neighbour interactions in the MTBoS, and what complex emergence do they contribute to?

RESULTS AND ANALYSIS

In what follows, we exemplify continuing interchanges through the presentation of three interchanges along with an analysis of each with respect to *diversity* and *redundancy*, and draw out key conclusions.

Example 1: Check your answers

Kathy Howe (@kdhowe1) responds to Michael Fenton's (@mjfenton) math mistake query by explaining that she would get students to check their answers.

> That's a popular error. I focus on "lots of things multiply to 2, so there are lots of answers to that factored equation" … also, "Great! Do those answers check in the original equation? Oh, they don't? Why not?" (@kdhowe1, June 10, 2016, 7:15 AM)

Fenton provokes her by responding with a sample student response to her approach.

> "But Mr. Fenton, I checked the first one, and it worked. I figured the second one would work too." (@mjfenton, June 10, 2016, 7:23AM)

Howe then notes that she explains to her students what counts as a valid response.

> I explain to them early on that "right for the wrong reason" is still not a correct solution. (@kdhowe1, June 10, 2016, 7:54AM)

Howe's last comment is ignored, and the conversation does not continue further.

In example 1, some *redundancy* is evident in that Howe and Fenton seem to both have familiarity with the student error and with the mathematics. They have a 'taken-as-shared' understanding of a general classroom context where a teacher explains to students what to do. They both can envision a prototypical student. This enables them to communicate. However, there is *diversity* in approaches. Howe focuses on explaining to students that they need to check their answers and that they should know what's 'right' and what's 'wrong'. Fenton offers a potential student response to Howe's request for checking answers. Fenton is not only challenging the request for 'checking answers', but is also illuminating that he chose to design the mistake so that one factor works and the other doesn't. There is an opportunity to continue discussing the design of the task here that is not recognized by Howe. Fenton's responses elsewhere in the data indicate that he is interested in more than a typical response. The *diversity* in intentions seems to halt the conversation, and Howe's last comment is ignored. This *diversity* can also be attributed to the different levels of membership in the MTBoS between Howe and Fenton. Howe is a newer member, with less than 200 followers, while Fenton joined early on and has over 4000 followers. In this example, there seems to be too much *diversity* between Fenton and Howe in terms of how they approach interpreting each other's posts, their pedagogical approaches, and their membership in the MTBoS to continue conversation.

Example 2: Looking for patterns

Avery Pickford (@woutgeo) responds to Michael Fenton's (@mjfenton) query by expressing she loves the mistake and offers a string of equations from which she'd have students notice patterns.

> <3 this mistake. I'd probably try to subtly slip them (x-2)(x-3)=0, (x-2)(x-3)=9, & (x-2)(x-3) = 13 & ask them 2 look for patterns. (@woutgeo, June 10, 2016, 4:00PM)

Pickford further notes that she thinks discussion around this mistake can lead to new approaches to finding roots.

> what i love about this mistake is that i can see it naturally leading to a new method for finding roots involving factor pairs. (@woutgeo, June 10, 2016, 4:04PM)

A few hours later, Max Ray-Riek (@maxmathforum) asks her to predict patterns that could be noticed. He also asks if these patterns could "get kids thinking about the new method of factoring [she] mentioned" (@maxmathforum, June 10, 2016, 6:27PM). Pickford responds with a few options.

> idk. maybe 1) not the same answers (hmm) 2) 1st is easy, 2nd is medium (should have made it =20, not 9), 3rd is hard (@woutgeo, June 10, 2016, 6:17PM)

Ray-Riek agrees with Pickford that this is a useful mistake to entertain and claims that "it stretched [his] math brain" (@maxmathforum, June 10, 2016, 6:14PM). The conversation does not continue further.

In example 2, Pickford and Ray-Riek seem to have a fair amount of *redundancy* in their pedagogical approaches, which both involve asking learners to notice patterns among several examples chosen specifically to illuminate properties *without telling*. In fact, Pickford invokes a 'problem string' structure, known as a practice where "students answer related questions, the teacher models student thinking, [and] students construct relationships and connections" (Harris, n.d., para 3). This structure shows up elsewhere in the data, and is used by members who are relatively active in the MTBoS. It is referred to as an *instructional routine*, and acts as a source of *shared language*. Pickford and Ray-Riek are both familiar with this approach, and both agree that using a 'string' helps students notice mathematical properties without direct instruction. They also both entertain the idea of finding some sort of new mathematical approach given this student scenario. Since Ray-Riek offers similar examples as Pickford elsewhere in the data, the only source of *diversity* is in the specific examples they provide, and the choices they make in ordering and selecting numerical values with aims of illuminating various features. Pickford emphasizes the increasing difficulty in the examples, while Ray-Riek focuses on merely changing the product in different ways. In this example, there seems to be too much *redundancy* between Pickford and Ray-Riek to generate any further conversation because they both agree on their approach to interpreting each other's posts, and their pedagogical approaches. They are both also relatively well-connected with the MTBoS and its overarching values.

Example 3: Generating strings

Ray-Riek responded to Fenton's post earlier that day, responding to himself several times in a journal-like fashion.

(x-3)(x-2) = 2 still only has 2 answers ... there is only one set of factors of 2 that make this true. Why those? Hmm ... (@maxmathforum, June 10, 2016, 7:50AM)
I think the direction I'd go is to look at solving a bunch of quadratics that = 2. They all have different factors. Compare to = 0 (@maxmathforum, June 10, 2016, 7:56AM)
I think I'd look at (x+8)(x+4) = 12, (x-1)(x-2) = 12, and (x-6)(x-10) = 12. Analytically we could come up w/ different sol'ns … (@maxmathforum, June 10, 2016, 8:22AM)

About five days later, Michael Pershan (@mpershan) replies to Ray Riek's musings with examples of 'equation strings'.

How does the approach this equation string aims at compare to what you'd be aiming for? $(x-2)(x-4)=15$ $(A-3)(A-5)=15$ $(A-3)(A-5)=35$ $(Y-3)(Y-10)=0$ (@mpershan, June 15, 2016, 5:19PM)

Ray Riek responds by saying that he's "not thinking of it as eqn string . . . [but that] each has solns at different factors of 2" (@maxmathforum, June 15, 2016, 7:30PM). However, he then notes that he can see the 'string' Pershan is referring to.

oh now I see the string you are talking about. Is the idea here that 1) is easy and 2) is not b/c hard to get 7*5? (@maxmathforum, June 15, 2016, 7:30PM)
oh! Now I see the whole string. X=7, A=0, A=10, Y=3 or 10 ... No, I don't think your string gets at the same idea I had. (@maxmathforum, June 15, 2016, 7:33PM)

Pershan replies that he thought it was referring to the same idea because he's emphasizing multiplication in his example. However, Ray-Riek notes that although it's related, he wants "three problems that all = 12 but in different ways" (@maxmathforum, June 15, 2016, 3:35AM). They continue discussing their intentions, and both offer additional examples. Ray-Riek explains he expects that students ignore negative and non-obvious solutions, and wants to emphasize this through different ways of factoring. Ray-Riek then offers an alternative option that may further elicit the type of student noticing they both expect.

> @mpershan @mjfenton I wonder about a #wodb with
> $A: (x-2)(x-1) = 12,$
> $B: (x-2)(x-1) = 0,$
> $C: (x-5)(x+2) = 0$ What might kids notice? (@maxmathforum, June 15, 2016, 5:55AM)

Ray-Riek's 'which one doesn't belong' example attracts another member to engage in thinking through the options and entertaining what students may notice. This conversation includes a total of 29 tweets, and prompts Pershan to post further about it in other threads.

In example 3, similarly as in example 2, both Ray-Riek and Pershan are active members of the MTBoS, and exhibit *redundancy* around the way they interpret each other's posts through inquiry and their general pedagogical approach of *teaching without telling* by asking students to observe patterns within a series of examples, guiding them towards *mathematical generalization*. They are both familiar with the *instructional routines* of 'problem strings' and 'which one doesn't belong', both common approaches to teaching discussed throughout the MTBoS, and they are able to communicate their intentions through examples of these. However, there is a slight amount of *diversity* in their approaches to and representations of the mathematics and to the *instructional routines*. They seem to use the *redundancy* to explore sources of *diversity* in a productive manner that leads them to generating several examples for use in teaching mathematics.

Overall, members who are connected to the MTBoS exhibit patterns of interaction such as *thinking like a learner*, *generating examples*, *invoking shared language*, and *using instructional routines*. They also indicate 'taken-as-shared' pedagogical approaches of *teaching without telling* that involve a teacher helping students arrive at a generalization through carefully chosen examples that will be discussed, which follows the 'notice and wonder' approach commonly exhibited in MTBoS discussions. These are all sources of *redundancy* in the MTBoS that allow users to communicate meaningfully. When this *redundancy* is not available, as in example 1, the conversation cannot become generative. When this *redundancy* is not paired with enough *diversity*, as in example 2, the conversation ends with agreeance. However, when this *redundancy* is paired with enough *diversity*, which is exposed through communication, there is possibility for the system to generate new as-yet unimagined tasks and approaches.

CONCLUSIONS

Engaging in the MTBoS with authority means to act within the consensual domain, which is to share sources of *redundancy* unique to the MTBoS. This investigation shows that the consensual domain of the MTBoS includes patterns of interaction such as *thinking like a learner*, *generating examples*, *invoking shared language*, and *using instructional routines*, as well as being guided by pedagogical values related to *teaching without telling* and guiding students towards *mathematical generalization*. Without these sources of *redundancy*, it is difficult to communicate productively.

However, it is also essential for there to be *diversity* around approaches and representations of mathematical ideas to allow for emergence of novel ideas for teaching and learning mathematics. Those with authority over the consensual domain of the MTBoS have greater capacity to push new meanings, and in turn, contribute to the complex emergence of the MTBoS.

This study indicates the potential for complex emergence in the MTBoS, and points to sources of *redundancy* and *diversity* that can contribute to the MTBoS as an autonomous asynchronous complex system that occasions space for generating an ideational network of mathematical tasks, pedagogy, and beliefs about mathematical teaching and learning. Further study should explore other cases where productivity occurs within the MTBoS to identify conditions that contribute to this productivity. The products of the MTBoS have great potential implications for teaching that need to be explored given that they are quickly unfolding and are developing at every moment.

Note

[1]Neighbour interactions refer to ideational interaction rather than social interaction. However, a physical component such as oral or written expression through various representations is often used for ideas to interact.

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INVESTIGATING PRESERVICE TEACHERS' WRITTEN FEEDBACK ON PROCEDURE-BASED MATHEMATICS ASSESSMENT ITEMS

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This study investigates patterns exhibited by preservice teachers (PSTs) during feedback practice in responding to student work in a procedure-based mathematics assessment. First, we developed an analytical framework for understanding PSTs' written feedback. Second, we looked into how a learning module on a multimedia platform influenced PSTs' feedback, and identified the ways PSTs improved their written feedback through revision. Along with an operational list of emergent patterns in PSTs' written feedback, our findings suggest that about two-thirds of the PSTs showed improvement in providing written feedback after they completed the feedback module. The implications for developing PSTs' written feedback skills through an emerging teacher education curriculum are discussed.

INTRODUCTION

Assessment is the process of gathering and interpreting evidence related to student learning with the goal of improving instruction. Therefore, a teacher's assessment practice serves as a crucial link among learning outcomes, teaching strategies, learning activities, and ultimately promoting a productive cycle of teaching and learning in the classroom (Black & Wiliam, 1998; Hattie & Timperley, 2007). In mathematics education, research has shown that effective formative assessment strategies enable teachers to shift their aims from merely grading and fixing students' work to increasing the understanding of student thinking and moving students forward in their learning (Allsopp et al., 2008; Collins, 2012). In particular, descriptive and detailed feedback from the teacher can guide students to take active steps for improving their work (Bee & Kaur, 2014).

Well-designed assessments, among other key factors, include non-evaluative, specific, timely, and personalized feedback related to the learning goals (Gearhart & Saxe, 2004; Jenkins, 2010). To that end, mathematics teacher educators are beginning to refocus their attention on training PSTs to develop skills in providing personal, relevant, and informative *written* feedback for students rather than relying only on the numbers, marks, or letter grades of summative assessments. Given the evidence in prior research that mathematics teachers have several weaknesses with respect to written feedback (Bee & Kaur, 2014), further research is necessary to investigate how teacher education programs can help improve this skill. In this study, we investigated patterns exhibited by PSTs for crafting written (i.e., detailed and descriptive) feedback on students' mathematical solutions in procedure-based assessment items.

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First, we developed an analytical framework for understanding PSTs' written feedback comments. Second, we looked into how an online module designed to foster emerging feedback skills influenced PSTs' feedback; the module utilized the Lesson*Sketch* platform (available at www.lessonsketch.org). Third, we identified the ways in which PSTs improved their written feedback through revision.

LITERATURE REVIEW

Effective Feedback Practice

Studies regarding the impact of feedback indicate that feedback has the potential to significantly impact students' learning achievement (Callingham, 2008; Volante, 2010). As such, the teacher should provide feedback in a strategic way so as to create opportunities for students to use this feedback. That is, the feedback students receive should tell them what they are doing well, where they need to improve, and what they should do next. Research has shown that students' learning improves when they get informative and constructive feedback on their work; feedback also must clearly relate to the learning goals (Crisp, 2007; Gregory & Kuzmich, 2004). In addition, feedback is more effective when it presents achievable goals with a high degree of sensitivity to self-esteem (McFarlin & Blascovich, 1981). By contrast, the impact of feedback on learning achievement is low when feedback is focused on praise, rewards, or punishment (Hattie & Timperley, 2007).

Written Descriptive Feedback

Written descriptive feedback has been the primary method of teacher feedback for writing tasks in language arts education (Goldstein, 2006), especially in the form of evaluative comments along with a letter grade. The main factors influencing the effect of such feedback on students' writing activities are the nature of feedback, which should be clear and specific with opportunity for revision based on the feedback (Hyland, 1998).

Because students' mathematical work largely exist as written work, the same principle of the effectiveness of informative written feedback to improve student learning applies in mathematics education. To date, mathematics education research has focused on teachers' approaches to student errors, including the value of addressing student errors with written feedback on test papers and homework. That is, researchers have weighed in on the efficacy of written feedback specific to student errors, rather than on the cognitive and affective issue of how different types of written feedback impact student learning. Although the efficacy of written feedback is still under scrutiny (Sadler, 2010), the consensus is that student learning can improve through feedback based on analysis of student work as well as clarity of learning goals, and explication of criteria for success (Shepard, 2006). In keeping with this trend, more research on the ways preservice teachers conceptualize written feedback and develop feedback skills through the teacher education curriculum is warranted.

METHODS

Participants in this study were 42 elementary PSTs and 40 secondary PSTs at two university-based teacher education sites in the U.S. Participants were in their junior or senior year of teacher preparation programs. Each PST was enrolled in an elementary or a secondary mathematics methods course.

A learning module with a series of five tasks was implemented in two sections each of the elementary and secondary mathematics methods courses; the module was administered towards the end of the fall semester of 2015 and again in the spring semester of 2016. In the first task of the module PSTs read graphic frames in a comic format which scripted a conversation between a methods professor and a preservice teacher about feedback (see Figure 1). Then the PSTs described important attributes of constructive written feedback on students' mathematical work. In the second task the PSTs reviewed the strengths and weaknesses evident in a sample student solution to the question: *What is the slope of the line defined by the equation $8x + 2y = 5$?* The PSTs then composed written feedback for the student. In the third task, the PSTs compared their feedback with the feedback of other PSTs on the same work (see Figure 2). In the fourth task, the PSTs reviewed another scripted dialogue (see Figure 3) designed to help them reflect on meaningful feedback comments. In the last task, the PSTs revised their initial feedback.

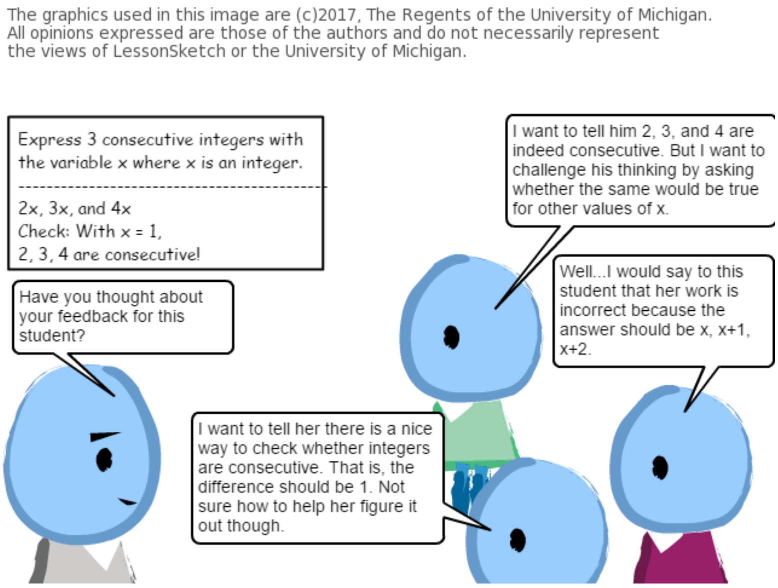

Figure 1: Scripted conversation about feedback

To analyze the changes between PSTs' initial and revised feedback, we used the inductive content analysis approach (Grbich, 2007). Initially we organized raw data into a spreadsheet, read the responses, and created codes. Drawing on the literature about effective feedback practices as well as levels of feedback skills, we developed an analytical framework. Due to the technology-based context of the feedback tasks in our study, we revised the initial analytical framework to reflect these two conditions: (a) the setting is technology-based (i.e., PSTs should provide written feedback on the Lesson*Sketch* platform) and (b) PSTs provide a sample of feedback in response to student solutions for a procedure-based mathematics assessment item. Then we analyzed PSTs' feedback with the revised framework (see Table 1). Finally we interpreted the data using both quantitative and qualitative methods.

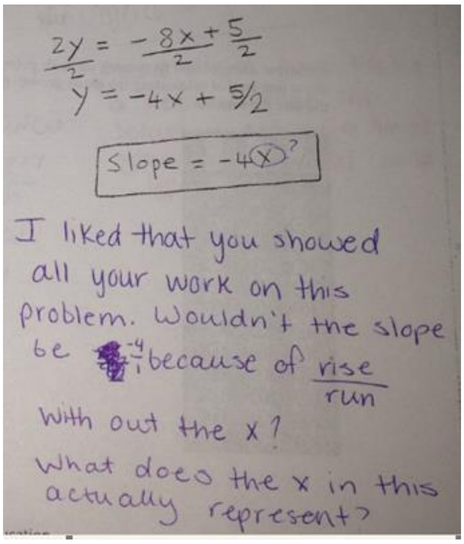

Figure 2: Sample feedback

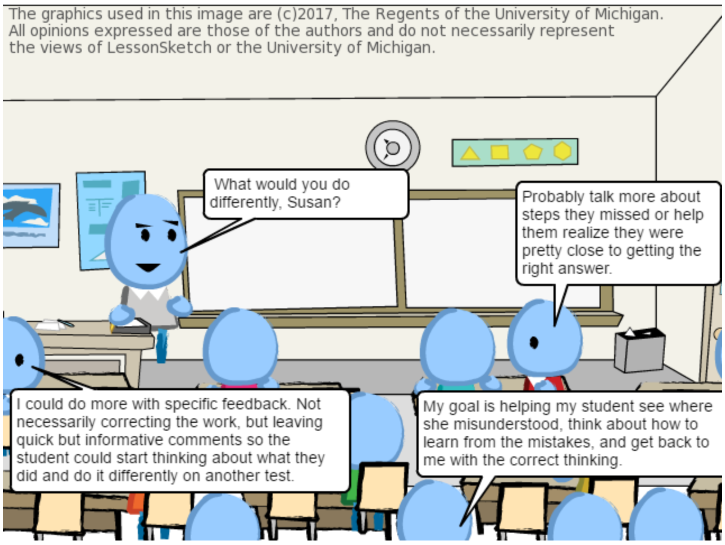

Figure 3: Debriefing feedback task with PSTs

Level	Descriptions
1A	Feedback is praise (e.g., *nice work*; *great job*) or vague comments (e.g., *better than last time*) unrelated to mathematics content.
1B	Feedback comprises general comments on test-taking skills (e.g., *double check your work*; *read instructions carefully*).
2	Feedback provides correct answers or identifies student errors (e.g., *you added the denominator incorrectly*; *you did not simplify*).
3	Feedback is given to directly remediate student errors (e.g., *be sure to use this formula*; *add the exponents and see what happens*).
4	Feedback provides an analysis of student strengths and areas of improvement.
5	In addition to Level 4 feedback, feedback provides opportunities (e.g., challenges) for new learning and encourages students to reflect on their thinking.
N/A	Feedback is not provided.

Table 1: Revised descriptions of PSTs' written feedback

RESULTS

For elementary PSTs, before going through the feedback module on Lesson*Sketch*, the most common level was level 2 (36%), followed by level 3 (30%) and level 4 (17%). However, after completing the module, level 5 had the highest frequency (33%), again followed by level 3 (26%) and 4 (17%) (Table 2). For secondary PSTs, before the module, the majority of feedback was at level 3 (47%) followed by level 2 (24%). However, after the module, most of these PSTs demonstrated either level 3 or level 4 (see Table 2).

Level	Elementary PSTs		Secondary PSTs	
	Before	After	Before	After
1A	3 (8%)	0	2 (6%)	0
1B	2 (5%)	2 (5%)	6 (18%)	3 (9%)
2	15 (36%)	7 (17%)	8 (24%)	6 (18%)
3	13 (30%)	11 (26%)	16 (47%)	16 (47%)
4	7 (17%)	7 (17%)	2 (6%)	8 (24%)
5	1 (2%)	14 (33%)	0	1 (3%)
No response	1 (2%)	1 (2%)	0	0

Table 2: Distribution of PST feedback levels before and after use of the module

Overall, more than half of the PSTs (58% of elementary PSTs and 59% of secondary PSTs) demonstrated improvement in providing feedback after going through the module, while 40% of PSTs remained at their pre-module levels (see Table 3). These results showed that the module was helpful for PSTs to develop feedback skills -- most PSTs learned to go beyond praising and fixing student errors in order to investigate student thinking.

Levels	Number of elementary PSTs	Number of secondary PSTs
Increased	24 (58%)	20 (59%)
Did not change	16 (38%)	14 (41%)
Decreased	1 (2%)	0
N/A	1 (2%)	0

Table 3: Change in PST feedback skills

DISCUSSION

The module implemented in this study offers examples of scenario-based feedback tasks in methods courses. The design of the tasks is such that PSTs are situated in the classroom interacting with students on their mathematical work. Our findings suggest that both elementary and secondary PSTs respond positively to feedback tasks in which they are asked to craft teacher comments and review comments of peers and have the opportunity to revise their comments. We also found several distinct patterns of feedback between elementary PSTs and secondary PSTs. Initially most PSTs had level 2 or 3 feedback skills, but a number of elementary PSTs were able to demonstrate level 5 after they completed the module -- whereas the secondary PSTs' progress was more incremental. This suggests that there may exist some barrier for secondary PSTs to frame their written feedback as a way to promote metacognition, or new learning, beyond the scope of the mathematical concept confined within the mathematics item at hand. Our hunch is that secondary PSTs may perceive feedback as an opportunity to engage in immediate content learning (i.e., levels 3 and 4) while overlooking the role of feedback as a way to motivate students to revisit the work and think on their own (i.e., level 5 and beyond).

While this study generally supports a case for a curriculum in teacher education that nurtures PSTs' emerging feedback skills, what is particularly important is the development of curricular materials with the following ideas in mind. First, the curriculum should provide curricular materials that encourage PSTs to develop an interest in the teaching skills necessary to provide feedback. Second, the curriculum should focus on opportunities for teacher educators and PSTs to co-construct various feedback comments with clear reference points to students' solutions; this practice should also be accompanied by a discussion of how these comments communicate awareness of students' strengths and weaknesses and how this awareness prompts students' further thinking and reasoning. Third, the focus of the curriculum should be on how high quality feedback helps students to identify the next steps in their own learning.

Traditionally, mathematics teacher educators have relied on a lecture-seminar format, which typically involves assigning research articles followed by either round-table discussion or having students write reflection papers. This may have been the dominant approach to teaching feedback strategies or any instructional practice in methods courses. Given the current disconnect between learners' need for effective feedback and the base knowledge of how to provide it, as well as the level of interest in developing effective feedback skills among PSTs, we believe they need meaningful and multiple experiences to practice crafting teacher comments on their own, which can then serve as the basis for discussion in the methods course. Such discussion can support a model for incorporating theoretical knowledge into the re-construction of comments, thereby establishing a pattern of improving feedback skills through revision. Thus we suggest a shift in the structure of teacher education toward instructional patterns that provide PSTs with opportunities to compose teacher comments, engage in the analysis of various teacher comments, and reflect on their own comments through revisions. As for the specific module in our study, we caution that some PSTs may not be receptive to the comic format of the graphic frames. However, Lesson*Sketch* as an online multimedia platform can be useful for creating optimal learning conditions. For example, we found such negative reactions were ameliorated when students observed the graphics privately rather than in a whole group setting, which was where objections to viewing comics like graphic frames arose.

IMPLICATIONS

This research contributes to the current literature on written feedback practices in mathematics education. In particular, this study has implications for designers of mathematics education courses for PSTs, as well as for researchers pursuing deeper understanding of PSTs' written feedback skills. For example, we found that a majority of PSTs easily advanced past levels 1 and 2 when they realized their feedback was too general and consisted mostly of praise. This rapid advancement suggests that the teacher education curriculum should focus on helping PSTs achieve higher-level feedback skills. It should also focus on helping PSTs compose appropriate feedback comments for various contexts (i.e., descriptive vs. evaluative vs. affective) and revise their strategies depending on how students are likely to respond.

Our future studies will investigate whether and how procedure-based items could restrict opportunities to provide quality feedback, and whether and how open-ended items can better provide a meaningful space for PSTs to develop written feedback skills. Related, we plan to refine our research to identify the type of learning in the math methods course that directly contributes to PSTs' development of feedback skills by enabling them to (1) create teacher messages that motivate students to think more deeply about mathematics and (2) to go beyond the correctness of a solution. Ultimately, we are interested in research on creating the type of learning opportunities in teacher education through which PSTs can develop the skills necessary to examine student work and plan for the next steps in meeting their needs. These learning opportunities will enable teachers to develop their written feedback skills as an integral part of effective and *affective* teacher language that motivates students to refine and extend their thinking and reasoning.

Acknowledgment

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TOOL-BASED MATHEMATICS LESSON: A CASE STUDY IN TRANSITIONS OF ACTIVITIES IN DIDACTICAL CYCLE

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Tools are made by human aiming at not only solving technical problem but also developing high-order thinking. Teaching and learning with tools create student-centred learning environment fostering rich interactions between students and teachers. This paper presents a case study investigating the implementation of a tool-based lesson in secondary level. The study explored the interactions among students, teacher and tools used, which mainly focused on the transitions of activities conducted in the lesson. Didactical Cycle is the main frame for analyzing the data collected via document review, interviews, classroom observation and worksheets completed by students. Reversible transitions in the didactical cycle frame between various tool-based activities were found in the case.

INTRODUCTION

In traditional mathematics classrooms, teachers implement mathematics lessons in deductive and authoritative teaching approaches while mathematics knowledge is developed in de-contextualized approach that usually standardizing as introduction followed by application. These approaches, including teachers' talk, often offer students few opportunities to construct mathematics knowledge on their own.

One role of tools played in mathematics classroom may pedagogically flip from teacher-centred to student-centred learning approach. Mathematics task designed by teachers which involves the use of tool is known to be tool-based task. Leung and Bolite-Frant (2015) defined tool-based task as doing or acting on thing in order for students to activate an interactive tool-based environment where teacher, students, and resources mutually enrich each other in producing mathematical experiences. Tool-based mathematics lesson is therefore viewed as a lesson including tool-based task that conceives these interactive activities. Furthermore, different from traditional teacher-centred approach, social interaction in the acquisition of knowledge is encountered in the interactive activities involved in tool-based task. To conceive interactive and collaborative mathematics experiences, the use of tools mediates the connection between construction of mathematics knowledge of students and actions taken by them. Thus, tools can be viewed as mediation between the generated production of signs while the tool-based task is being accomplished and the sign related to mathematics knowledge (Bartolini Bussi & Mariotti, 2008). The idea of signs mentioned by Bartolini Bussi and Mariotti (2008) was inspired by Vygotsky that involving words, drawings, gestures and the like accompanying actions produced in learning and teaching environment.

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In tool-based lesson design and implementation, Bartolini Bussi and Mariotti (2008 and 2012) developed an iteration called *didactical cycle* based on semiotic mediation, which consists of three phases of activities conducing to the construction of mathematics in the lesson. The ultimate goal of the didactical cycle is to foster the development of shared meanings recognizable and acceptable by teachers and students through the active activities performed by the students in the mathematics classroom (Mariotti, 2012). Research studies on the didactical cycle contextualized in mathematics and science lessons focusing on the individual phases of activities were conducted in the past (e.g. Bartolini Bussi & Mariotti, 2008; Corni, Giliberti & Mariani, 2011; Mariotti, 2012). However, previous studies mainly focused on individual phase only. This research study aims at exploring and analysing the transitions of the activity phases captioned in the didactical cycle in Hong Kong classroom context in order to compensate the analysis of the didactical cycle and contribute to a research gap where the transitions of the activity phases are essentially considered.

This paper presents a study aims at exploring pedagogical interaction in a school mathematics classroom with a teacher designing mathematical task making use of tools in Hong Kong. We explicitly investigate the tool-based mathematics lesson with the didactical cycle, specifically analyze the transitions of activity phases which aspires to develop a contextualized model of tool-based pedagogical interaction.

THEORETICAL FRAMEWORKS

In tool-based task design and implementation, mathematical knowledge can be constructed via a semiotic process where two types of sign production are generated: 1) personal written or verbal signs and 2) mathematics signs (i.e. formal mathematics knowledge) (Bartolini Bussi & Mariotti, 2008). Bartolini Bussi and Mariotti introduced semiotic mediation to describe the interrelations among tool, mathematics, teacher and students through sign productions. Tool as a mediator cognitively simulates students with the features of it through interactions between students and the tool guided by the teacher. The sign generated from tool manipulation creates twofold cognitive functions. One function is for the students to accomplish the mathematics task. Another function is the sign production related to the process of interpretation of the exchanged information and the subsequent socio-semiotic process for the communication between collaborating parties like groupmates and teacher. However, this developmental process is not automatically activated without teacher's intervention (Bartolini Bussi & Mariotti, 2008). Therefore, from this viewpoint, the role of tool is a social-semiotic mediation to produce signs that stimulate teaching and learning in the mathematics classroom. Using tools in the classroom enhances students' cognitive ability to engage in the experience of generating personal mathematics signs from the manipulations of tools on the one hand, while on the other hand, the generated of signs helps students to conceive mathematical ideas from it with teachers' guidance. This dual tool-based sign functions intertwine in the mathematical knowledge acquisition process. The didactical cycle consists of an iterative cycle of three phases (see Figure 1):

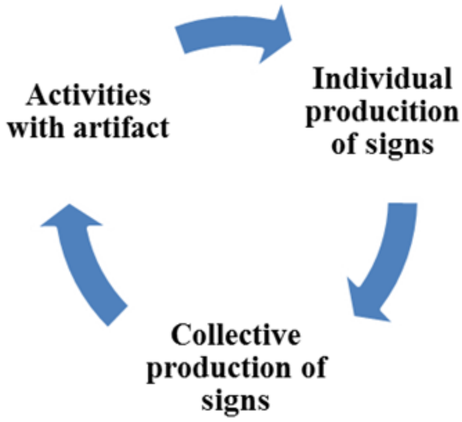

Figure 1: The Didactical Cycle. Adapted from (Mariotti, 2012)

Phase 1: Activities with tools/artifacts: It is generally a starting point of the didactical cycle where tools become indispensable elements for students to act on the task. These actions or activities are usually formed in small groups in social settings that promote social exchange accompanied by words, sketches and gestures done by students.

Phase 2: Individual production of signs: It engages students to undertake different semiotic activities individually concerning mainly the written signs for the next step. The production of signs does not require students to produce formal mathematical language (mathematics signs), but the signs rooted in the tool and the given task, which creates different artifact signs (e.g. tool-related production) for collective activities and discussions followed.

Phase 3: Collective production of signs: It includes *Mathematical Discussion* (Bartonlini Bussi, 1998) which is the core of semiotic process orchestrated by teachers. The individual signs collectively produced by the students in the previous step are shared and discussed for analyzing, commenting and elaborating. The discussion explicitly directs students to transform the personal signs to mathematical signs in cognitive dialectics process with the guidance of teacher. Therefore, the main purpose of the teacher is to collect personal signs and convert them to mathematical signs. Semiotic mediation advocates another view of constructing mathematics knowledge by producing polysemy signs representing the tool and mathematics.

"This cycle was not rigidly fixed and was open to changes, according to the particular conditions of activity" (Bartolini Bussi & Mariotti, 2008, p.763). According to Bartolini Bussi and Mariotti, the didactical cycle is a theoretical framework to guide a tool-based mathematics lesson; in addition, it allows diverse pedagogies to occur. This study will examine the issue in a situational context in a case that originally proposed to investigate the transitions of the phases in the didactical cycle.

RESEARCH METHOD

This research study is a naturalistic inquiry research on the formation of pedagogical practices. It is also a part of a tool-based research project with over twenty participating mathematics teachers from primary and secondary schools in Hong Kong. The participants have conducted tool-based lessons which were video-recorded. Each teacher implemented the designed tool-based lesson(s) with pre-lesson clinical interviews and post-lesson interviews which aimed at probing teachers' perceptions of tool-based task design and the performance in the lessons respectively. For the mathematics content, diverse types of topics and grades from primary to secondary levels were performed.

As a part of the research project, we present here a single case of a secondary mathematics teacher teaching an elite class of secondary three students (Grade 9) and the topic in estimation of the volume of a concrete object (a frustum-liked container). Primary data sources were taken before, during and after the lesson. Audio record of the pre-lesson and post-lesson interviews, video record of the implemented lesson along with the photocopies of the worksheets of the students was gathered as the data of the case. In addition, the interviews and implemented lesson were transcribed for analysis.

Research Lesson

The tool-based lesson was designed and conducted by a experienced secondary mathematics teacher. The teacher discussed the design of the task with researchers in the project and modified it for implementation. The mathematics knowledge in the lesson emphasized by the teacher in the interviews was the estimated process with tools and the accuracy of estimation. In the lesson, he allowed the students to select and bring any tools themselves. The flow of the lesson was mainly designed and implemented in four parts, 1) introduction; 2) carrying out the estimation process; 3) group presentations; and 4) checking the volume.

Data Analysis

In order to focus on student-tool, student-student and student-teacher interactions, a coding scheme was devised to characterize the phases of the didactical cycle in the implemented tool-based lesson. Three phases mentioned in the didactical cycle are coded and named as A1, A2 and A3, according to the iterative activities. Three complementary sub-codes (rt - Related to tool; nt - Not related to tool; o - Other) under these three levels were used to document the classroom interactions in a more complete manner in order to identify potential subtle forms of interactions. The conversations and actions from the transcription were analyzed and coded to precisely discern the phases for deeper understanding of addressing role of tool in the activity (e.g. ‘rt’ coding refers to the conversation related to the tool involved). Additionally, pattern analysis was focused on the transitions between the three phases in the didactical cycle.

RESULTS

The iteration of activity phases in the cycle and notable transitions of phases are presented for addressing critical factors that determined the phase changes. The excerpts below are three examples extracted from the analytical transitions.

Iteration of the didactical cycle

The didactical cycle presented and theorized as a unidirectional flow of phase changes. However, in the implemented research lesson, we have found a non-cyclic sequential transitive flow of phases of the didactical cycle where the phases were not strictly on a track of A1→A2→A3 pattern. The phase transitions within the didactical cycle were interchanged among phases, therefore, reversible flows of phases were observed. The result showed each phase may direct to the other two adjacent phases or stayed on its own stage. Therefore, the phases in the didactical 'cycle' may not be implemented as a unidirectional cyclic pattern in the observed tool-based lesson.

Transition of A1 → A3

The students were engaged in the manipulations of tools (i.e. A1) to find slant height of the container. During the manipulation, the teacher intervened (i.e. A3) and questioned the group.

Excerpt 1 (10:08)

1 Ss: *(A group of students is starting to plan how to measure and is trying to use rulers to measure the slant of the container.)* (A1)

2 T: Slant will make it longer. Then the next question will be how much does it slant? You should think about whether this measurement is accurate or not. (A3, rt)

3 S1: Oh, I see. *(A student from the group is jotting down the steps they have discussed)* (A3, rt)

Excerpt 1 showed the transition of pattern from A1 to A3. Verbatim 1 revealed the students worked in groups as a social setting on the task to generalize artifact signs (Mariotti, 2012). It is gradually emerged at the beginning of the lesson that allows students to manipulate the tools to produce personal meaning signs. In general, the didactical cycle proposed that A2 will follow to produce signs (either in verbal or written forms) by the manipulators. However, in verbatims 2 and 3, the teacher had an intervention to the group by probing question about the features of the container linked with the measurement. The guidance of teacher would be viewed as collection and clarification of production from previous activities. Therefore, the transition of phases from A1 to A3 emerged.

In the didactical cycle, activities in A1 mainly conceive possibility for producing sign from individual or group, i.e. A2; while A3 follow for the collection of the production in A2. However, the translation from in excerpt 1 blurred the phase of A2. The teacher tried to provide guideline, even though some signs were not produced by students, to the students when the students were planning and manipulating the tools with struggles and confusions. When the teacher discovered the students focused on the experiment itself but did not produce any sign, he reminded them to mathematically focus the construction of slant of the container.

Transition of A2 → A1

The students manipulated the tools individually and collaboratively in order to produce verbal and written personal signs.

Excerpt 2 (13:32)

4 Ss: *(A group of students is drawing tangents to find the centre of a circle on a paper)* (A2, rt)

5 S2: It's ok. Let's calculate. *(A student from the group is marking down measured values on a paper)* (A2, nt)

6 S3: *(Another student from the same group is counting the radius on a grid paper)* (A1)

This episode showed an interchangeable transition between A1 and A2. The group started the experiment in a way of tracing the concrete location of centre of a circle (the base of the container) in the paper in order to find its radius. The group produced mathematical terms, e.g. circle, radius, touch at a point (i.e. tangent), and the students consented that the measurement was the radius. Verbatims 4 and 5 also revealed that the measured values were substituted in formula written by the group. After all, instead of analyzing the calculation, verbatim 6 showed the group was using another ways of measurement to find the radius. The students triangulated the answers obtained from various methods.

In the didactical cycle, activities for producing signs in A2, are followed by A3 which is the collection of the signs. On the other hand, excerpt 2 showed the group was moving back and forth between the phases A1 and A2. As said in previous paragraph, manipulating tools in A1 helps students to produce their own signs, i.e. A2 activity. However, students may 'return' to repeat the same experiment or conduct another one, i.e. A1 again, instead of analyzing the production, in order to triangulate the measurement and check the consistency of the calculation. Therefore, collection of production of sign was not readily emerged. In addition, the first two phases (A1 and A2) were inseparable in this situation.

Transition of A3 → A2

The activity for the collection of students' production simulated the students to justify their productions and to modify their ways of experiments.

Excerpt 3 (36:12)

7 S4: *(A student is presenting her idea to the whole class)* We used two rulers, one vertical and one horizontal. And the slant height. (A3, rt)

8 T: Can you draw a figure? What is the goal of your calculation with this figure? (A3, nt)

9 Ss: *(The presentation was stopped and the students were thinking)*

10 S5: *(Another student from the group is going to say something)*

11 S4: Oh, I know, I know. We add two rods here to form a pyramid…(A2, rt)

In verbatim 7, a student was presenting the manipulation and measurement of what their group had performed. The terms, i.e. artifact and/or mathematics signs, were used by the student and shared with the whole class in order to consolidating formal mathematical thinking. After the teacher probing a question to the group, they tried to generate and refine the ideas in the presentation by doing an experiment shown in verbatim 11.

Theoretically, the collective activity of signs, i.e. A3, gradually completes a 'cycle' and re-starts again at A1. However, the phase A3 did not end the cycle in this case. Students were trying to produce signs/terms that convince others what experiment had done. Therefore, the collective phase includes students' production of signs with the help of using tools as a representation.

REMARKS

The theoretical framework of the didactical cycle guided us to analyze the interactive activities in the empirical tool-based lesson. Three types of activities in the cycle interacted with each other and transited from phase to phase. The first remark is that the transitions of phases were not practically restricted in unidirectional pattern of A1→A2→A3. Particularly, reversible transitions of phases were found in the research lesson. In addition, we enact a modified didactical cycle and name it as *Didactical Interaction* (see Figure 2). The word 'interaction' inclusively considers not only the cyclic flow of phases captioned by the didactical cycle, but also accommodates reversible directional transitions of phases hence multidirectional transitions of the phases. The second remark is that various kinds of transitions of phases were emerged in the lessons. In fact, teacher intervention was critical in these different kinds of transition. The students in the lesson were inspired by the teacher's question and modified their plan of experiment when they had received the feedback from the teacher. These kinds of action taken by the teacher initiated the phase change that justified the model of the didactical cycle which some phases were being skipped, blurred and reverted in the iteration direction. In addition, iteration analysis told us that the some phases in the cycle were difficult to be discerned in terms of its features. For example, students' manipulation of tools included <u>individual</u> instrumentation and instrumentalization (Rabardel, 2003) which naturally produced some terms and calculation. These kinds of productions were closely related to the activities with the tools, e.g. A1 and A2, which are inseparable. The third remark is notably placed on the role of tool in associated with the transitions of the activities. The transitions were changed according to the actions taken by the students and the teacher that the relevance of the tools was considered in the analysis. For example, the teacher questioned the students about the features of the tool that semiotic potential of the tool was critically encountered in the conversation (See verbatim 2). Another example showed that the students discussed on calculation which was extracted from the tool (See verbatim 5). In short, the cyclic feature of the didactical cycle should be critically reframed according to the implemented lesson. For example, Leung has discussed that a nested epistemic process in tool-base activities are conducive to learning (Leung, 2011). Therefore, further analysis will be conducted to study deeply the interrelationships among these activity phases.

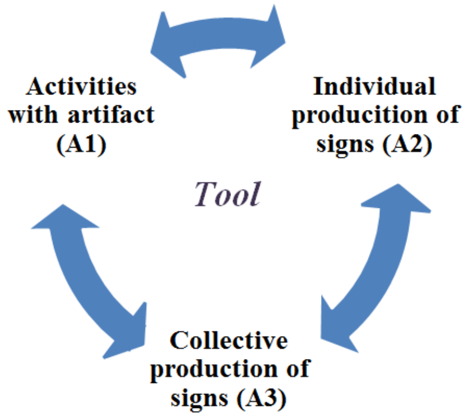

Figure 2: The Modified Didactical Cycle/The Didactical Interaction

As a part of the research project, several in-depth investigations will be continued. For examples, analysis of questioning from teachers, manipulations of tools by the students and correctness of calculation will be further studied. Moreover, similar analysis on the relationship of transition will be conducted in other research lessons conducted by other participants in the project to enrich and modified the didactical interactions.

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MATHEMATICIANS’ EVALUATIONS OF THE LANGUAGE OF MATHEMATICAL PROOF WRITING IN THREE DIFFERENT UNDERGRADAUTE PEDAGOGICAL CONTEXTS

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This proposal discusses the extent to which mathematicians agree on some of the linguistic conventions of mathematical proof writing. Mathematicians (n=128) responded to an online survey indicating whether proof excerpts were unconventional in each of three undergraduate contexts: how proofs appear in mathematics textbooks, what instructors write on the blackboard in mathematics courses, and how students write proofs in these courses. These data point to a lack of agreement among mathematicians on the linguistic expectations of the proofs written by their students.

INTRODUCTION

Research has shown that undergraduate mathematics students have difficulties when constructing (Weber, 2001), reading (Conradie & Frith, 2000), and validating (Selden & Selden, 2003) mathematical proofs. Among several reasons for why undergraduates struggle with constructing mathematical proofs, Moore (1994) included students’ unfamiliarity with the language of mathematical proof writing. However, there is a dearth of empirical and systematic research in the field of mathematics education on the language of mathematical proof writing at the advanced undergraduate level.

In particular, how advanced undergraduate mathematics students and mathematicians understand and use the technical language of mathematical proof writing is largely unknown. Lew & Mejía-Ramos (Under review) showed that the mathematicians and undergraduate students who participated in their study did not agree on the extent to which one should attend to English grammar, the introduction of new objects in a proof, and the context in which a proof was constructed when considering the exposition of said proof. While the interviews provided a clearer picture of how some mathematicians and students perceived the language of mathematical proof writing, the present study investigated how a larger sample of mathematicians evaluated parts of the same proofs via an online survey. Lending a quantitative perspective on how mathematicians understand technical mathematical language, this work further informs researchers’ and instructors’ understanding of mathematicians’ expectations regarding the presentation of mathematical proofs at the undergraduate level.

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RELATED LITERATURE AND THEORETICAL PERSPECTIVE

There is little systematic, empirical work on the language of mathematical proof writing. Konior (1993) studied over 700 mathematical proofs written in academic textbooks and monographs investigating the construction of mathematical proofs. He identified a common structure that framed the arguments of a proof by highlighting the plan of procedure and using cues to direct the reader through the proof. Burton and Morgan (2000) found that the norms suggested in professional mathematical writing guides (e.g. Gillman, 1987; Krantz, 1998) are sometimes broken by, especially by those highly regarded in the field. Selden and Selden (2014) also described seven features of the style in which mathematicians write proofs (e.g. not including the statements of entire definitions within written proofs). While these studies begin to further the understanding of professional mathematical proof writing, research on the language of proof writing at the undergraduate level is lacking.

As referenced above, a number of mathematicians (e.g. Gillman, 1987; Krantz, 1997; Higham, 1998) have written texts describing proper and effective use the language of mathematics for professional purposes such as journal articles, dissertations, and books. Meanwhile, since these guides were written based on the authors' assumptions and personal experiences, further work is necessary to investigate the extent to which these expectations of advanced mathematical proof writing are shared by the general population of mathematicians and how these conventions apply to different contexts.

Linguistic Conventions of Proof Writing in Different Contexts

As a particular type of mathematical writing, we see mathematical proof as a particular genre of the language of mathematics. Mathematician Armand Borel (1983) equated mathematical proofs to the genre of poetry in natural language, emphasizing not only that the language of mathematics is distinct from the vernacular, but also that one must be knowledgeable in the language of mathematics in order to understand mathematical proofs. In this work, we assume that the genre of proof is a way of using mathematical language defined by both the formal properties and structures of language, as well as the communicative purposes of texts in particular contexts. This view of genre is consistent with the genre theory literature (Hyland, 2002). Our consideration of proofs in this light is in the pursuit of helping students to understand and follow the linguistic conventions of the genre, as work has done in other discourses (Hyon, 1996).

To study the genre of mathematical proof writing, we sought to identify and validate the existence of linguistic conventions of proofs. We assume conventions are rationally justifiable customs of practice to which members of that practice are expected to conform in the manner of Jackman (1998). Thus, we take linguistic conventions to be rationally justifiable customs of linguistic communication. Existing literature (e.g. Gillman, 1987; Higham, 1998) has suggested some conventions of writing proofs for professional contexts, such as correctly situating notation within a sentence according to proper grammar, and structuring the proof to guide a reader through the argument.

Meanwhile, it is important to consider how the context of the proof might affect how conventions are followed as suggested by mathematicians in Lew & Mejía-Ramos's (2016) study. In particular, we investigate how mathematicians believe conventions of mathematical proof writing apply in the contexts of undergraduate textbooks, and in two classroom contexts: the way proofs are written on the board in class, and the ways in which proofs are written by undergraduate students. The consideration of this variation of context allows this work to highlight important similarities and differences in the contexts created by mathematical discourse, as Bondi (1999) had in her study of research papers, textbooks, and newspaper articles in economic discourse.

Researchers in higher education (Becher, 1987), linguistics (Hyland, 2004), and composition (Bizzell, 1982; Batholomae, 1985) have highlighted that different disciplines have characteristic discourse practices and that without knowledge of such practices students will struggle to successfully enter the discipline. We extend this necessity to acquire specialized literacy to undergraduate students of advanced mathematics, who—we argue—must understand the genres and conventions of mathematical discourse which includes the genre of mathematical proof in the different contexts that pervade their undergraduate study. Given the fundamental role of proof in mathematical practice (e.g. Thurston, 1994; Rav, 1999), understanding the language of mathematics in which proofs are written is of utmost importance for undergraduate students studying advanced mathematics.

In the present study, we investigate the conventions of mathematical proof writing from the perspective of mathematicians – the most prevalent instructors and examiners of undergraduate students' proof writing. As such, the present study investigates the following question: To what extent do mathematicians agree among themselves on what the linguistic conventions of mathematical proof writing are in the three contexts of textbook proofs, blackboard proofs, and student-produced proofs? Do conventions exist for the language of undergraduate mathematical proofs? Does the context of said proofs affect what conventions are upheld in mathematical proof writing?

METHODS

In order to evaluate how mathematicians perceive linguistic conventions in mathematical proofs, the survey adopted the methodology of breaching experiments in the style of Herbst and Chazan (2003). The survey asked participants to make evaluations regarding the language used in several partial proofs, which were based on student work, but truncated to discourage participants from focusing on the logical validity of the purported proof being evaluated. Four of the seven partial proofs used in Lew & Mejía-Ramos's (Under review) study were included in the survey.

These breaches were identified by Lew & Mejía-Ramos (2015) as common, potentially unconventional uses of mathematical language found in student-produced proofs from 149 exams at the introduction to proof level. The breaches were categorized based on suggestions from mathematical writing guides and the authors' personal experiences with proof writing at the undergraduate level. One of the partial proofs and potential breaches included in the survey is illustrated below. Figure 1 shows the marked partial proof exhibiting the use of the unspecified variable, z, and the explanation for why someone might think it's unconventional, as presented in the survey. The explanations used in the survey are based on the mathematicians' discussions of the same potential breaches and proofs in Lew & Mejía-Ramos (Under review).

Each potential breach was presented on a separate page of the survey. Participants were provided a marked partial proof and an explanation of why a colleague might believe the corresponding proof excerpt had been written in an unconventional manner, as shown in Figure 1. For each of the three contexts (a textbook proof, a blackboard proof, and a student-produced proof), participants indicated if they agreed the proof excerpt was indeed unconventional for the stated reason and to what extent it affected the quality of the proof.

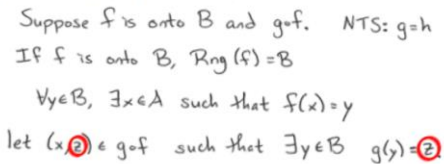
Marked partial proof exhibiting the potential breach: Uses an unspecified variable	Explanation of the potential breach
Suppose $f: A \to B$, $g: B \to C$, $h: B \to C$, for sets A, B, and C. Prove: If f is onto B and $g \circ f = h \circ f$, then $g = h$.  Suppose f is onto B and g∘f. NTS: g=h If f is onto B, Rng(f)=B ∀y∈B, ∃x∈A such that f(x)=y let (x,z) ∈ g∘f such that ∃y∈B g(y)=z	A mathematician suggested that this is unconventional mathematical writing because the variable *z* should be introduced prior to its use in the proof.

Figure 1: Example potential breach and explanation presented in the survey.

Participants (128 mathematicians) were recruited from 25 of the top mathematics departments in the United States through email solicitation through their department secretaries, which included a link directing those choosing to participate to the survey.

Analysis

The analysis for this study included investigating if the mathematicians answered the various aspects of the survey differently – in particular, whether they agreed or disagreed on which the potential breaches were unconventional in the three contexts. Table 1 presents some of the findings from this study, indicating the proportion of the sample that agreed that the proof excerpt was unconventional for the reason provided, for each of the three contexts. To evaluate if the proportions of agreement indicated a high agreement, the thresholds of a high agreement that a potential breach was unconventional and was not unconventional were set to 75% and 25% respectively. Chi-squared tests for equality of proportions were conducted to check for proportions p=0.75 and p=0.25 with a level of significance of α=0.05/42 (fourteen potential breaches in each of three categories). The results of these Chi-squared tests are indicated with ++ and --, respectively. Proportions of agreement were categorized in the following ways: high agreement that the use is unconventional (significantly different and greater than 75%), high agreement that the use is not unconventional (significantly different and less than 25%), or not shown to have high agreement.

RESULTS

Survey results suggest a lack of agreement amongst mathematicians whether the potential breaches are unconventional of mathematical proof writing for the reasons provided. Table 1 shows that fewer than half of all judgments made yielded agreement percentages significantly different and above 75% or significantly different and below 25%. However, in textbook proofs, responses also showed more internal agreement.

Do mathematicians agree the potential breaches are unconventional uses of mathematical language for the reason provided? *(% Agree)*				
Proof	Potential Breach of Mathematical Language	Textbook Context	Blackboard Context	Student Context
1	Uses non-statement	100% ++	100% ++	98% ++
	Uses an unspecified variable	59%	34%	37%
	Includes statements of definitions	41%	18%	12% --
	Lacks punctuation and capitalization	95% ++	34%	50%
2	Uses formal propositional language	88% ++	74%	66%
	Uses unclear referent	93% ++	67%	70%
	Overuses variable names	98% ++	95% ++	93% ++
	Mixes mathematical notation and text	88% ++	28%	45%
3	Fails to make the proof structure explicit	70%	29%	28%
	Uses mathematical symbols or notation as an incorrect part of speech	72%	19%	24%
	Uses informal language	77%	45%	48%
4	Fails to state assumptions of hypotheses	64%	34%	40%
	Uses an unspecified variable with an existential quantifier	85% ++	55%	54%
	Lacks verbal connectives	97% ++	52%	72%

++ Significantly different and greater than 75% of the sample, -- Significantly different and less than 25% of the sample (α=0.05/42)

Table 1: Mathematicians' responses indicating if they agree that the proof excerpt was unconventional for the reason provided in each context.

Figure 2 shows the percentage of participants who agreed the potential breaches were unconventional in each of the three contexts. Lines connect the agreement percentages for evaluations in the same context and the shaded sections indicate the percentages significantly different and greater than 75% or significantly different and less than 25%. This section of the proposal discusses the types of potential breaches for which participants' responses showed high agreement and provides a post hoc analysis of the potential breaches for which the samples' responses did not show high agreement.

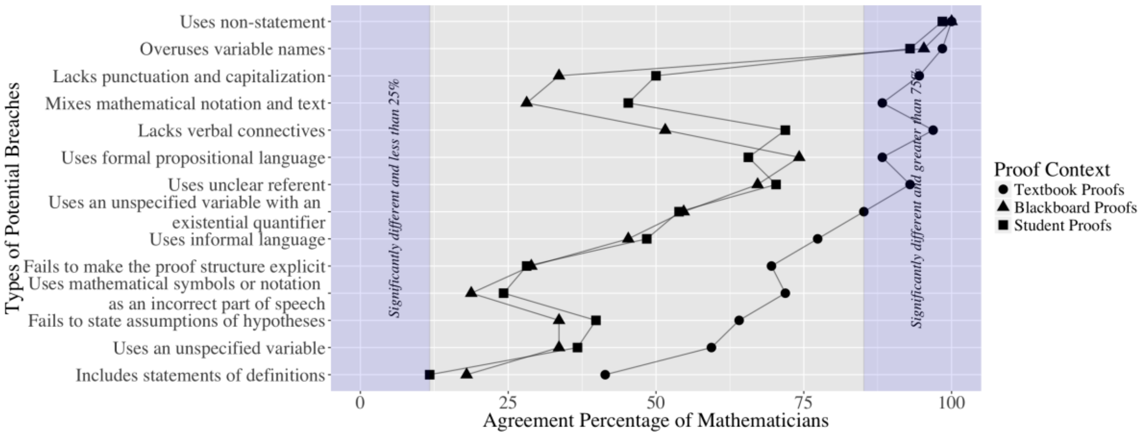


Figure 2: Mathematicians' agreement percentage for each potential breach by context

Potential breaches for which the participants' responses showed high agreement

Based on Figure 2, the mathematicians' responses showed high agreement that eight of the fourteen potential breaches were unconventional in the context of a textbook proof for the reasons presented. Moreover, for the proof excerpts exhibiting the use of non-statements and the overuse of variable names there is high agreement among mathematicians that these potential breaches are unconventional in all three contexts. These findings provide further evidence that these eight potential breaches of the conventions of mathematical language are indeed unconventional in the context of textbook proofs for the reasons provided. The proof excerpts that overused variable names or used non-statements were also indicated to be unconventional in the two classroom contexts by the mathematicians.

Finally, Figure 2 also shows the percent of mathematicians that agreed the inclusion of statements of definitions in a student-produced proof was unconventional is significantly different from and less than 25%. That is, there is a high agreement among the mathematicians that a proof excerpt including the statement of definitions is not unconventional in a student-produced proof. Moreover, fewer than 42% of mathematicians agreed the inclusion of statements of definitions was unconventional in any of the three contexts considered. We note this is in contrast to claims that mathematicians do not include statements of entire definitions within written proofs (Selden & Selden, 2014). While the scope of the present study focuses on proofs at the undergraduate level, we note two of the contexts considered are written by mathematicians (textbook proofs and blackboard proofs). Thus, it may not be that the features of proof writing described by Selden and Selden (2014) extend to different contexts of proofs written by mathematicians or proofs written by students.

When the samples' responses did not show high agreement

For 29 of the 42 judgments made by the mathematicians, agreement percentages did not cross the thresholds for high agreement. Figure 2 further shows that for five of the potential breaches, mathematicians' responses did not show high agreement in any of the contexts. When we restrict analysis to only classroom contexts (blackboard proofs and student-produced proofs), eleven of the fourteen types of potential breaches the results did not show high agreement. Finally, Figure 2 highlights that a number of these agreement percentages are close to 50%. In particular, eight of the 42 agreement percentages were between 40% and 60%, including two judgments in the textbook context. These findings suggest that beyond failing to give confirmation that many of these potential breaches are indeed breaches of linguistic conventions in proof writing, that the disagreement among mathematicians may be higher in classroom contexts, and that for some types of potential breaches the disagreement amongst mathematicians may be particularly extreme, even in the context of textbook proofs.

Moreover, it is clear that a larger percent of the mathematicians agreed that a potential breach was unconventional in the textbook context than when the same potential breach was assessed in either of the other contexts. In fact, Figure 2 suggests for some of the potential breaches, the fewer mathematicians that agreed a proof excerpt was unconventional proof writing in the textbook context, the fewer that perceived the same excerpt was unconventional in the classroom contexts.

CONCLUSION

The findings of this report highlight the existence of some potential breaches of mathematical language that mathematicians widely agree are unconventional in the context of textbook proofs. Specifically, mathematicians in our study widely agreed that including incomplete statements, overusing variable names for different mathematical objects, lacking proper punctuation and capitalization, carelessly mixing mathematical notation and text, failing to use connectives to bridge steps, using formal propositional language, using pronouns with unclear referents, and using an unspecified variable are all unconventional usage of mathematical language in textbook proofs. Moreover, mathematicians widely agreed on the specific rational justifications for why the proof excerpts breached linguistic conventions or mathematical proof writing on that context. On the other hand, mathematicians also widely agreed that one of the potential breaches studied (including full statements of definitions within proofs) was not unconventional in the context of student-produced proofs for the reasons provided, which suggests that Selden and Selden's (2014) claim that mathematicians do not include definitions in their proofs may not extend to other contexts and to mathematicians' expectation of how students write proofs.

Meanwhile this report gives insight on how these mathematicians' evaluations differed of the language of mathematical proof writing at the introduction to proof level in the classroom contexts. In particular, the results suggest that mathematicians' linguistic expectations of student-produced proof are unclear. In the student context, the mathematicians' responses did not indicate high agreement for twelve of the fourteen types of potential breaches, which may indicate the possibility that mathematicians do not have a shared understanding or expectation of how students should write proofs.

If it is indeed the case that there is not a consensus among mathematicians of how their students in introduction to proof courses should write their proofs, then how are instructors of these courses presenting proof writing to their students? Discussions amongst mathematicians, especially those who teach introduction to proof courses, concerning their expectations for language use in the writing of proofs by their students would be a useful step towards a shared understanding of linguistic conventions of proof writing in the context of student-produced proofs. Further research is necessary to understand these varied expectations amongst mathematicians and how to address students' confusion when it comes to their professors' expectations of their proof writing. In turn, better understanding of mathematicians' expectations of their students' writing could enable the creation of interventions and curriculum to help undergraduate students in the transition to abstract and advanced mathematics courses.

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CREATIVE PROCESS VS. CREATIVE PRODUCT: CHALLENGES WITH MEASURING CREATIVITY

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In this article, we look closely at the relationship between the creative process and creative products. Using a combination of theoretical and empirical analysis we call into question the validity of measuring creativity by examining products – using products as a proxy for processes.

INTRODUCTION

At PME 40 there were 21 papers presented (2 plenary activities, 10 research reports, 6 oral communications, and 3 poster presentations) on the topic of creativity. Seven of those articles (1 PL, 3 RR, and 3 OC) describe instruments that, in one way or another, for one purpose or another, measure creativity. In the case of Leu, Luo, and Lo (2016), for example, the intent of the RR was to measure creativity for the purpose of comparing mean scores of groups from different countries. Gilat and Amit (2016), on the other hand, measured creativity before and after an intervention to compare learning gains from a control and an experimental group.

Regardless of the intent or the means by which creativity was measured, the aforementioned research is predicated on an assumption that creative products are indicators for an *a priori* creative process. In this paper we look more closely at this assumption.

CREATIVITY

Torrance (1966) defined creativity as

> a *process* of becoming sensitive to problems, deficiencies, gaps in knowledge, missing elements, disharmonies, and so on; identifying the difficulty; searching for solutions, making guesses, or formulating hypotheses about the deficiencies: testing and retesting these hypotheses and possibly modifying and retesting them; and finally communicating the results. (Torrance, 1966, p. 6)

In his pursuit to understand this *process*, and building on Guilford's theory (1967), Torrance (1974, 1966) designed a test to measure a person's creative thinking through the proxies of fluency, flexibility, originality, and elaboration. The Torrance Test of Creative Thinking (TTCT) is based on Guilford's *Alternative Uses Task* (e.g., name all uses for a brick) and adds several test formats such as the *Ask and Guess Test* in which participants are requested to ask questions to given drawings. Other parts of the TTCT include non-verbal assignments such as the *Picture Completion Test* which

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consists of the completion of incomplete figures. The results on each of these items is scored independently and compiled to produce a measure of creativity. But there is a question as to whether these metrics, these measures of products, "capture the essence of creativity" (Leikin & Pitta-Pantazi, 2013, p. 160). It is exactly this question we are interested in pursuing in this paper. More specifically, we are interested in the relationship between the originality of a solution and the creativity of the process that spawned it.

To get at the answer to this question, however, we need to first understand more clearly the relationship between creative process and creative products.

Creative Process

In 1902, long before Torrance came up with his test for creative thinking, the first half of what eventually came to be a 30 question survey was published in the pages of L'Enseignement Mathématique, the journal of the French Mathematical Society. Édouard Claparède and Théodore Flournoy, two Swiss psychologists, who were deeply interested in the creative process, authored the survey. During this same period Henri Poincaré (1854–1912), one of the most noteworthy mathematicians of the time, had already laid much of the groundwork for his own pursuit of this same topic and in 1908 gave a presentation to the French Psychological Society in Paris entitled *L'Invention mathématique*—often mistranslated to Mathematical Creativity (c.f. Poincaré, 1952). Inspired by this work, Jacques Hadamard (1865–1963), a contemporary and a friend of Poincaré's, began his own empirical investigation into the creative process. Hadamard retooled the survey and gave it to friends of his for consideration—mathematicians and scientists such as Henri Poincaré and Albert Einstein, whose prominence were beyond reproach. In 1943, Hadamard gave a series of lectures on mathematical invention at the École Libre des Hautes Études in New York City. These talks were subsequently published as *The Psychology of Invention*[1] *in the Mathematical Field* (Hadamard, 1945).

Hadamard's treatment of the subject of invention at the crossroads of mathematics and psychology is an extensive exploration and extended argument for the existence of unconscious mental processes. To summarize, Hadamard took the ideas that Poincaré had posed and, borrowing a conceptual framework for the characterization of the creative process from the Gestaltists of the time (Wallas, 1926), turned them into a stage theory consisting of four separate stages stretched out over time. These stages are initiation, incubation, illumination, and verification (Hadamard, 1945). The first of these stages, the initiation phase, consists of deliberate and conscious work. This constitutes a person's voluntary, and seemingly fruitless, engagement with a problem. Following the initiation stage the solver, unable to come to a solution stops working on the problem at a conscious level and begins to work on it at an unconscious level (Hadamard, 1945; Poincaré, 1952). This phase is referred to as the incubation stage of the inventive process, can last for any period of time from minutes to weeks, and is inextricably linked to the conscious and intentional effort that precedes it. After the period of incubation, a rapid coming to mind of a solution, referred to as illumination, may occur. After illumination the correctness of the emergent idea is evaluated during the fourth and final stage, verification. In the end, the verification step may show that the solution revealed in the moment of illumination is, in fact, incorrect. For Hadamard (1945) such failures were as much a part of the creative process as the successes and that the creative process is not judged based on the correctness of the solution.

The creative process, extended over time and being punctuated by the sudden appearance of a solution, has traditionally been researched through the *a posteriori* self-reports of this private and subjective experience (Hadamard, 1945; Liljedahl, 2013; Poincaré, 1952). More recently, however, Liljedahl (2013) has argued, and used the fact, that illumination is largely an affective experience which results in an observable emotive response.

Creative Product

Especially in its beginnings, research on creativity focused on self-reports of exceptionally talented individuals as well as analyses of their works (e.g., literary, musical compositions, or scientific discoveries) (cf. Silver, 1997). This lead to the so-called *genius* view of creativity which is often associated with exceptional knowledge or products that change our perception of the world (Sriraman et al., 2014).

Since then, research has turned away from the assumption that only geniuses can be creative and researchers have focused their attention to ordinary or everyday creativity (*ibid.*; Pehkonen, 1997).

> For a professional artist, some new, ground-breaking technique, product, or process that changes his or her field in some significant way would be creative, but for a mathematics student in lower secondary school, an unusual solution to a problem could be creative (Sriraman et al., 2014, 110).

And, for the sake of objectivity, researchers shifted their attention away from the self-reporting of the creative process and towards the evaluation of products. Within this new paradigm, a solution to a mathematics problem is determined to be creative if it is deemed to be original with respect to the rest of the solutions produced within a cohort of participants. As such, quantitative studies looking at creativity through the lens of solutions will, along with other metrics, calculate frequencies of occurrences for each solution found within a cohort of students.

CREATIVE PROCESS VS. CREATIVE PRODUCT

The question of whether creative products can stand as proxies for the creative process, then, can only be answered if it can be shown that there is a one-to-one correspondence between the creative process and the originality of a solution. More specifically, we would need to show that, in every problem solving situation

1. a student goes through a creative problem solving process and produces a creative solution, or

2. a student goes through a routine problem solving process and produces a routine solution.

Further, we would need to show that there exist no problem solving situations in which

3. a student goes through a creative problem solving process but either does not solve the problem, or produces a routine solution, or
4. a student goes through a routine problem solving process, yet produces a solution that is deemed to be creative.

These four scenarios can be summarized into a 2x2 grid (see fig. 1) where creative vs. routine process is on one axis and creative vs. routine product is on the other axis.

		PROCESS	
		creative	**routine**
PRODUCT	**creative**	1	4
	routine	3	2

Figure 1: The four scenarios represented on 2x2 grid.

Data reflecting scenario 1 and 2 would show that creative products are indicators of a creative process while scenarios 3 and 4 would show the opposite. In what follows we look at student problem solving data through the lens of the aforementioned four scenarios.

METHODOLOGY

Data for this study comes from student work on the *triangle* problem (see fig. 2), one of three *multiple solution tasks* (MST's) from a German project on mathematical giftedness in upper secondary school (MBF_2). This project looked at gifted and non-gifted students with a focus on traits that help to identify mathematical giftedness in upper secondary students.

Consider triangle ABC. Points P and Q, and S and R, divides side AB and AC into three equal parts.

What is the area of the quadrilateral PQRS with respect to the area of triangle ABC?

Find as many different ways to solve this problem as possible.

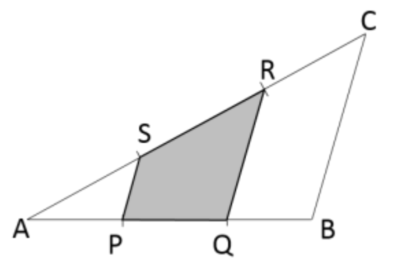

Figure 2: The *triangle* problem

Twenty students from grades 11 and 12 (age 16 – 18) participated voluntarily in this project, coming to the university every second week to work on non-routine problems. These students chose to participate in this project mostly because of their great interest in mathematics with many having shown great achievement in their regular mathematics classes.

The students had 30 minutes to work on each of three MST's. After each task was completed, the students participated in a group discussion in which the students presented their findings and reflected on their processes. The data for this project, then, are the solutions to the three MST's as well as video recordings of the students working on the MST's and the group discussions.

These data were analysed from two different perspectives – process and product. That is, the video recordings of the students solving the MST's were analysed for evidence of creative process, mostly through attention to affective responses that could signal an illumination having occurred. The group discussions were also analysed for self-reported utterances of a creative experience.

From the other perspective, the creativity of student solutions to the MST's were analysed collaboratively by a team of researchers. More specifically, the creativity of a solution was determined by its *originality* compared to all solutions to the same MST produced by all participants. The creativity of a specific participant was determined through the aggregation of all of their individual solution creativity scores. For the purposes of this paper the relevant results have been translated from German to English.

RESULTS

In what follows we present four cases from the data. Each of these cases corresponds with one of the four aforementioned scenarios.

Scenario 1: Creative process & creative product

Kirsten (fig. 3, upper left) was the only student in the group that solved the triangle problem by clearly stating the similarity of the three triangles in one of her solutions. Because of the uniqueness of this argumentation, this solution was deemed to be *original* and, thus creative.

The video of her problem solving process shows clearly that she initially did not know how to address this problem. Her work was anything but routine as she alternated the use of a variety of heuristics with getting stuck. After a time, however, she suddenly had the solution. Taken as a whole, her process was also deemed to be creative.

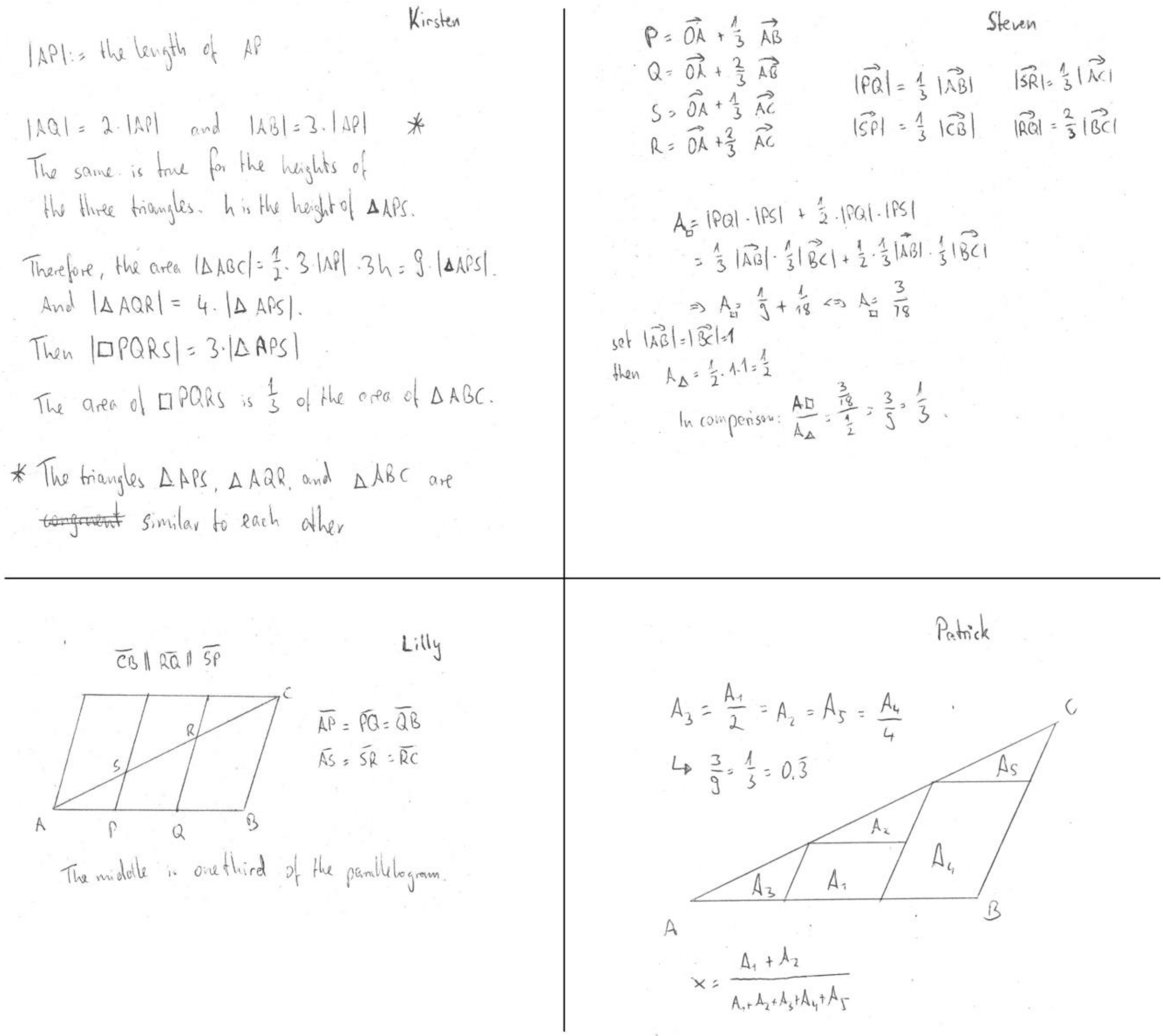

Figure 3: Four solutions to the triangle problem sorted by the grid from fig. 1

Scenario 2: Routine process & routine product

Patrick (fig. 3, lower right) solved the problem by dividing the triangle *ABC* into smaller triangles and parallelograms. This allowed him to compare the area of *PQRS* to that of *ABC*. This solution was very common within the presented cohort and, therefore, not rated as *original*.

Patrick's process related to this solution was identified as not being creative because he came to the solution very quickly and because, in the discussion phase, he admitted that this kind of geometrical problem was very familiar to him.

Scenario 3: Creative process & routine product

Lilly (fig. 3, lower left) solved the triangle problem by rotating the triangle *ABC* around the midpoint of *AC*, creating a parallelogram in the process. The middle segment of this parallelogram is one third of the whole parallelogram. Therefore, Lilly reasoned that the area of *PQRS* must be one third of the area of *ABC*. Not only is this solution incorrect, but it is not even original in that a lot of students produced the exact same solution.

Her videotaped process, however, shows that she struggled for a long time with this problem. She drew two completely different sketches, then stopped writing for several minutes. Suddenly she started writing again, producing the solution involving the point of reflection. Therefore, the process that led to her incorrect solution was judged to have been creative.

Scenario 4: Routine process & creative product

Steven (fig. 3, upper right) used linear algebra (defining vectors, calculating areas) in his approach to the triangle problem. No other student used linear algebra while working on this problem so his solution was deemed to be *original* and, as such, creative.

His process, however, reveals that for Steven, this approach was not creative. At the time the problem was posed to him, he was taught linear algebra at school and in the group discussion, he stated that solving this kind of problem was a routine task for him.

DISCUSSION

There is no doubt that there some problem solving processes are creative and that some are routine. Likewise, there is no doubt that there are original solutions and that there are routine solutions. The question we asked in this paper is whether creative processes can always be attributed to original solutions. That is, can an original solution "capture the essence of creativity" (Leikin and Pitta-Pantazi, 2013, p. 160)?

The case of Kirsten clearly shows that this can be the case. Kristen's original solution was the product of a creative process. Further, Patrick's routine solution was the product of a routine process. However, the cases of Lilly and Steven show the opposite relationship. The case of Lilly shows that the creative process does not necessarily produce correct and unique solutions. And the case of Steven shows that unique solutions can be the product of routine processes.

Taken together, it is clear that the originality of a solution is not a reliable indicator of the creativity of a solution that produced it. Where then does this leave the research has relied on, at least in part, the scoring of originality as an indicator of student creativity?

The results of the research presented here only argues that originality is not a good indicator of creative process. We say nothing about the inherent value in scoring, measuring, or ranking student solutions to MST's. Within this context, the methodologies used are, no doubt, effective instruments for categorizing, ranking, and quantifying the relative creativity of these solutions. Our research simply shows that these metrics cannot be used to reliably say something about the *a priori* creative process.

Note

[1]Within the context of creativity research the terms creativity, discovery, and invention are often used interchangeably.

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PATTERNS OF LANGUAGE USE IN TWO MODES OF WRITING MATHEMATICAL SOLUTIONS

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The study examined prospective middle school mathematics teachers' use of written symbolic and narrative language in presenting mathematical solutions. Results indicate that there exist different patterns in the use of language between these two modes of writing. The narrative solutions provided more logical connectors than the solutions featuring only symbols. The two types of solutions were different in regard to the sequence of procedures used in problem solving. Furthermore, those who demonstrated more formal language, such as rich mathematical vocabulary, complex sentences, or high adherence to standard syntax in mathematics, were not necessarily more successful in problem solving than those who used less formal language.

INTRODUCTION

Academic language (AL) generally refers to the language of schooling to acquire new or deeper understanding of the content and to facilitate communication within the practice of formal disciplines (Schleppegrell, 2004; Sfard, 2000). For example, the learning of mathematics is considered to be the appropriate context for the development of mathematical AL. Considering the critical role of teachers in students' acquisition of AL, it is essential to assess prospective teachers' (PTs') current use of AL as well as to create activities that promote PTs' use of AL. To do so, teacher educators must understand how PTs process and use AL in mathematics and the degree to which it facilitates learning. However, research regarding the ways in which PTs use academic language, especially in writing to solve, reflect upon, describe, or explain their own mathematical reasoning, is limited.

The study provides analysis of the baseline data regarding the ways PTs process and use AL to express their mathematical thinking. Specifically, it examined the following research question: When given an identical set of mathematical problems to solve and write mathematical solutions, to what extent do PTs demonstrate the use of AL in two different modes of writing (i.e., symbolic and narrative writing)?

THEORETICAL PERSPECTIVES

Academic language

The notion of AL is fraught with ambiguities in terms of its elements, structure, role, and value in the learning process (Bailey et al., 2007). However, communicative approaches in mathematics education (Morgan, Craig, Schuette, & Wagner, 2014) might characterize AL sufficiently for teacher educators to approach it pedagogically, without the limitations imposed by register and similar specialized uses of language. This study proposes a working definition of academic language and its qualifier:

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 169-176. Singapore: PME.

Language is academic when there is optimal semantic correspondence between the message sent by the teacher and the message received by the learner. Negotiation of meaning between teacher and learner is fundamental to the process and goal of effective communication within an instructional setting. Pedagogically, therefore, the goal of AL is *effective communication* regardless of the requirements and nuances of a specific register or whatever set of linguistic criteria one wishes to designate as necessary for students to learn.

Symbolization and narrative writing in school mathematics

Students study mathematical texts and participate in dialogic interactions with peers about mathematics (Chapman, 1993). In mathematical texts, symbolization—including numbers, operations, various syntax or conventional grammar of relationships—is proposed and strictly enforced; mathematicians learn to use this symbolic language to represent their ideas (Morgan, 1996). The AL of mathematics may not be part of students' colloquial language as it includes specialized ways of using words and symbols (i.e., the mathematics register) with certain syntactic preferences; this is especially true for written mathematics. Mathematical texts can be characterized as rhetorical, narrative, or argumentative (Dietiker, 2013), and the arguments in these texts are made in words and sentences; therefore, the use of language is evident. The language demands in mathematical texts include various rhetorical structure patterns more formal than the structures in colloquial language. The school mathematics register is most readily identifiable through highly specialized vocabulary. This vocabulary includes terms with definitions that are specific to mathematics, such as: parallelogram, polygon, trigonometry, or quadratic equation. Some of these terms, like line or factor, are used extensively in general language but have precise, math-specific meaning (Thompson & Rubenstein, 2000).

The grapheme system

Narrative skills relate to an ability to tell stories with literary precision and clarity with regard to organization and semantics (Conle, 2000). In order to differentiate writing systems in our study, we used the graphemic system to refer to the total writing system, inclusive of mathematics symbolization. A variable is a symbol representing an object in mathematics; so *variable* is the differentiating feature for writing consisting of symbols. In narrative form, a *word* (lexical) is the differentiating feature. Thus, for our theoretical and coding purposes, we used Variable Graphemic Symbol Set for mathematics (VGS) to refer to the mode of writing solutions in which the participants used symbols only. We used Lexical Graphemic Symbol Set for narrative (LGS) to refer to the mode of writing solutions in which the participants used words and sentences in addition to symbols and diagrams. We acknowledge that all language systems are symbolic and that the use of *symbol* to refer specifically to mathematics is a convention.

METHODS

Ten middle school PTs from a large state university in the southern United States participated in the study in the spring of 2014. The notion of academic language was previously addressed to varying degrees in the teacher education programs from which participants were drawn. Our data included the participants' written responses to five tasks in VGS and LGS respectively.

Each task had two parts in response to the same math item. Part I (VGS) asked to solve a problem using only symbols. Part II (LGS) asked to provide a solution using various forms of language. Each part took up to 25 minutes, and about an hour of break was given between Parts I and II. Considering that participants are middle school PTs, five intermediate algebra problems were posed (see Table 1).

1. Find the equation of the line in slope-intercept form, which passes through the points (-2, 7) and (3, -8).

2. The sum of three consecutive integers is 108. What are the three integers?

3. Solve the inequality and then graph its solution: $x^2 - 6 > 3$

4. Find the distance between two points, A(1, 3) and B(5, 9). Be sure to use the Pythagorean theorem in your solution.

5. Find the value of $\sqrt{50} + \sqrt{17}$ to the nearest tenth. (Please do not use a calculator.)

Table 1: List of algebra problems used in the study

The analysis focused on the patterns of language use and processes evident in the data (see Table 2). Use of VGS forms indicated the frequency of formulas to guide calculations and LGS forms to represent concepts. Use of logical connectors included symbols (e.g., arrows or numerals to indicate sequence). Linking words were analyzed to determine whether the narrative provided a logical argument with appropriate points of connection. Calculations included frequency of mathematical work exclusively for operational computations. Syntax errors included frequency of forms that were not preferred or acceptable. Correctness indicated whether the particular participant's solution was correct or incorrect. Syntactic complexity was rated by counting syntactic compound sentences versus complex sentences and by determining whether the overall writing style was basic or complex. Vocabulary use was analyzed by counting specialized terms and technical terms.

Modes	Elements of Analysis
VGS	Use of forms and diagrams; Use of logical connector; Calculations; Syntax errors; Correctness of solution
LGS	Use of specialized and technical mathematical terms; Use of forms and diagrams; Calculations; Syntax errors; Correctness of solution; Use of logical connector; Key characteristics of the solution; Audience and Subject; Complexity level

Table 2: Elements of AL in the analysis of written mathematical solutions

RESULTS

Patterns of using mathematics symbols

First, an obvious finding was that most solutions in LGS did not include as many math symbols as those in VGS. The symbols used by PTs were limited to those that referenced algebraic variables as part of calculations rather than as forms or notations relating to concepts and reasoning. Figure 1 illustrates the use of symbols in the computational process of solving for b with the equation $y = -3x + b$ where $(x, y) = (-2, 7)$ and the process for validating the solution $b = 1$ by using a point (3, -8) on the line, $y = -3x + b$ where $b = 1$. An example of symbols that can represent a translation of a geometric figure from a right triangle in the coordinate plane to a right triangle as a rigid object with the measures of each leg was indicated. Other idiosyncratic symbols include ↕, ⤻, □, @, ▶, #, *, ♪, !, ✓, and ×.

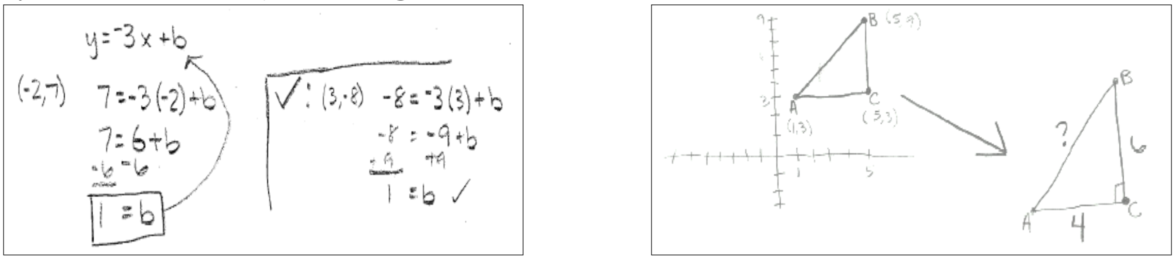

Figure 1: Use of symbols in the computational process vs. the illustration of objects

Less obvious, but evident, was the pattern of symbolic forms in solutions. Table 3 shows the frequency of symbols used in each solution of LGS. The variance indicates within-participant variability. For example, the participant, Pan (*Var* = 5.3) was less consistent in the use of symbols than the participant Ladybug (*Var* = 1.0). It is worthwhile to note that Ladybug, with low variance (i.e., consistent), also used the largest number of symbols in the group. So it is convincing that Ladybug can use multiple symbols consistently. To the contrary, Pan has the highest variance and a low number of symbol uses. While both Ladybug and Pan have a similar number of symbols for items 1, 2 and 4, Pan provided no symbols for items 3 and 5, which contributed to the high variance and the low mean value. The mean score is the average value of the frequency across all items. Most participants used about 2 – 3 symbols in each solution. However, it is obvious that overall the participants used very few symbols for items 3 (M= 1.9) and 5 (M= 0.8). Item 1 generated the highest use of symbols.

ID	Item 1	Item 2	Item 3	Item 4	Item 5	Variance	Mean
Debby	5	3	1	2	0	3.7	2.2
Ladybug	5	3	3	5	4	1.0	4
Bear	5	3	1	5	0	5.2	2.8
CREW	5	3	4	2	0	3.7	2.8
Carmen	5	3	1	4	0	4.3	2.6
Spaghetti	5	3	3	0	3	3.2	2.8
Gatorade	2	3	1	4	0	2.5	2
Turtle	5	3	4	0	1	4.3	2.6
Pan	5	3	0	4	0	5.3	2.4
Jersey	5	3	1	2	0	3.7	2.2
Mean	4.7	3.0	1.9	2.8	0.8		

Table 3: Frequency of symbols used in each solution of LGS

Second, a clear pattern of symbol uses emerged: one use for concept and the other use for procedure-based computations. In our analysis we asked two university-based mathematicians to review the math items and decide, by consensus, the number of key symbols that should be present in VGS solutions in order to represent concept and calculations respectively. We used this predetermined level to indicate excessive use of symbols by (+) and deficit use of symbols by (–) (see Table 4).

	Item 1		Item 2		Item 3		Item 4		Item 5	
ID	Con	Cal	Con	Cal	Con	Cal	Con	Cal	Con	Cal
Debby	-2	+4	-2	+4	-5	+4	-5	+4	-3	-2
Ladybug	-2	+4	-2	+4	-4	+4	-4	+4	-1	+2
Bear	-1	+4	-2	+4	-5	+4	-3	+4	-3	0
CREW	-2	+4	-1	+4	-2	-2	-5	+4	-3	0
Carmen	-2	+4	-2	+4	-5	+4	-5	+4	-3	0
Spaghetti	-3	+2	-3	+4	-3	+4	na	na	-3	-2
Gatorade	-3	+4	-2	+4	-5	+4	-5	+4	-3	+4
Turtle	-2	+4	-2	+4	-3	+4	-6	-2	-2	0
Pan	-2	+4	-3	+4	-4	+4	-4	+4	-3	-2
Jersey	-1	+4	-2	+4	-5	+4	-6	+4	-3	+2
Mean	-2.0	+3.8	-2.1	+4	-4.1	+3.4	-4.8	+3.3	-2.7	+0.2

Table 4: The indication of economical use of symbols in VGS in two ways

(Con = Concept; Cal = Calculation)

For example, the participant Debby needed 2 more symbols to represent key concepts related to item 1 but used 4 excessive symbols to indicate calculations associated with the solution to item 1. Table 4 shows that participants overall did not use a sufficient number of symbols to represent the concepts but instead used too many symbols and variables for trivial calculations and procedures. Taking the two broad patterns together it was determined that the participants used symbols to show calculations more than to represent mathematical ideas. It is also found that the LGS solutions showed a large reduction in the number of symbols.

Patterns of narrative language

There were participants whose narrative language helped in clarifying or correcting both solutions and errors in VGS. In these cases narrative solutions helped identify misunderstandings, and the logical connectors in LGS helped the writer to discover missing steps in VGS. For example, the following narrative (see Figure 2) was written in response to item 2 in which the participant was asked to write a general form of consecutive integers, n, $(n+1)$, $(n+2)$ by first defining n, and then writing the forms for the next two consecutive numbers. This process was not evident in the participant's VGS response, which resulted in the incorrect solution.

The pattern related to the use of academic language, chiefly rich vocabulary (i.e., specialized and technical math terms) and syntax in LGS, did not necessarily result in more successful solutions. We counted the frequency of vocabulary in each solution of LGS and found that higher variance (inconsistency) occurred in the participants who demonstrated higher average vocabulary counts than among those who averaged 3.6 to 8.4 words. For example consider the cases of Ladybug, who used the largest number of vocabulary (8.4 words), and Debby who used only 3.6 words per item. Although Ladybug (*Var* = 10.8) is not as consistent in using vocabulary as Debby (*Var* = 6.3), it is clear that Ladybug always used more vocabulary than Debby regardless of the items.

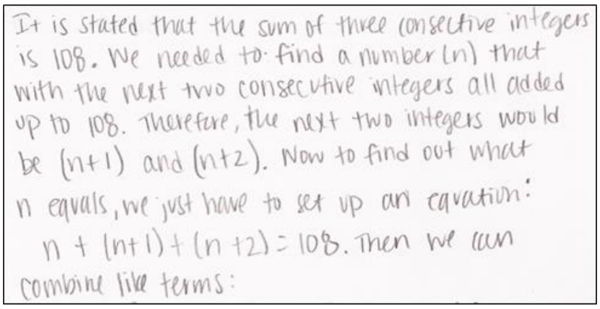
It is stated that the sum of three consective integers is 108. We needed to find a number (n) that with the next two consecutive integers all added up to 108. Therefore, the next two integers would be (n+1) and (n+2). Now to find out what n equals, we just have to set up an equation:
n + (n+1) + (n+2) = 108. Then we can combine like terms:

Figure 2: A sample narrative solution which helped identify misunderstandings

The syntax errors in VGS were little related to the correctness of the solutions. There were many correct solutions that contained syntax errors. We also examined the level of syntactic complexity in narratives. There existed a very low variance across the participants (the level ranges from 1.5 to 4.2), indicating that most participants remained consistent towards the use of narratives with syntactic sophistication. For example, Ladybug and Jersey both demonstrated high syntactic complexity, which corresponds to their use of vocabulary as well. However, Ladybug was a top performer in symbols, but Jersey lagged behind because he struggled with items 3, 4, and 5. As illustrated by these cases, there existed little evidence to indicate any correlation between the use of symbols and the use of narratives.

Patterns of interrelation between VGS and LGS

The solutions in VGS were not necessarily sequenced the same way as those in LGS. The narrative solutions displayed more logical connectors with linking words than did symbolic solutions. Evidence of logical connectors in the VGS included arrows and numerals. Five solutions out of 50 in VGS used 'arrows' or 'numbers' to indicate sequence, but 32 solutions in LGS used linking words; one solution used numbers to indicate the sequence of thinking. Although procedural description was a primary pattern of LGS, the solutions in LGS were conceptual or procedural depending on the question. It was not rare that the same participant wrote a procedural narrative for a problem, then wrote a conceptual narrative for another one.

There were six cases where the solutions in VGS were not congruous with the solutions in LGS. Among these cases, the symbolic solution helped identify notational mistakes and the narrative solution helped identify conceptual misunderstanding. For example, the first image below indicates a mistake of using "and" in writing the solution set. More formally, the solution should be $\{x|\ x < -3 \cup x > 3\}$; the "and" should have been the disjunction, "or". This suggests that either the participant was not attentive to the language of sets or did not process the solutions as elements of the set for all x where $x^2 - 6 > 3$. In the second figure, the participant proposed a quadratic function, $y = x^2 - 6$ with (0, -6) as the y-intercept, and stated that the parabola across the x-axis at x = -3 and x = 3. This implies that the participant may not understand that the critical numbers (i.e., boundary values) for the inequality $x^2 - a > b$, where a and b are real numbers, are equivalent to the zeros of the quadratic function $y = x^2 - a - b$.

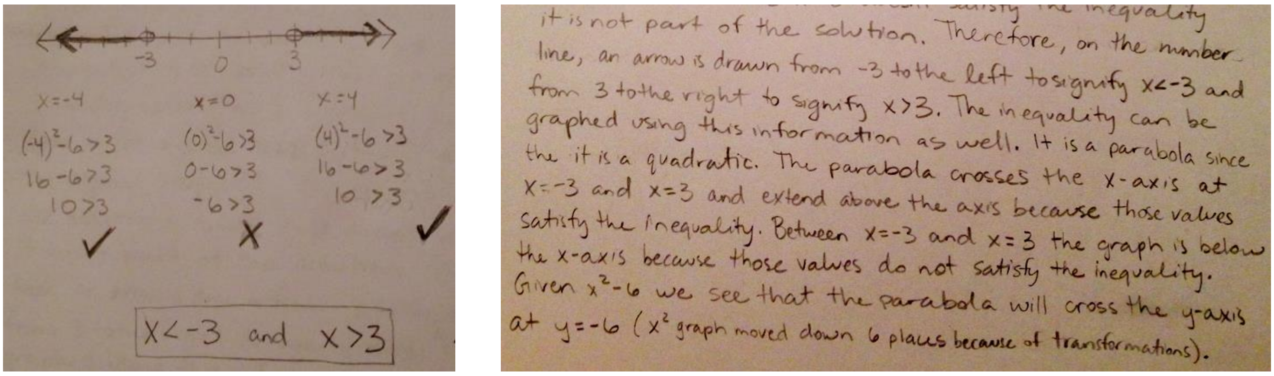

Figure 3: The symbolic solution with notational mistake and the narrative solution with conceptual misunderstanding

IMPLICATIONS

Research seeking to identify the process and nature of mathematical learning with regard to the importance of math symbols, narrative language, and how the two are perceived in future teaching of mathematics is necessary. For example, the case in which narrative solutions helped identify misunderstandings, and the logical connectors in LGS helped discover missing steps in VGS indicates the importance of narrative language and its potential to guide the learner by using the language to facilitate logical thinking and argument. Some participants recognized the role of language in the teaching of mathematics by using narratives to "teach students correct procedures," and "language use allows teachers to see student's conceptual understanding."

This supports the view that writing narratives in mathematics can ameliorate the tendency of relying solely, or even predominantly, on calculations or procedures and can create opportunities for productive discourses. Future research should seek to determine, to a greater degree of precision, how language facilitates the understanding, expression, and transmission of mathematical content understanding. If we understand how VGS and LGS relate to each other, particularly in terms of capitalizing upon their respective instructional and learning strengths, then we will be able to enhance, and perhaps refine, how PTs learn mathematics in order to better support their students' learning needs and improve their instructional skills.

These teachers need their students to have meaningful engagement with activities that complement subject-matter instruction while also developing English language proficiency. Ideally AL aspects of language proficiency would be sought; however, the goal should be substantive movement towards academic achievement. In this way the probability of successfully completing the schooling cycle is improved, as is preparation for college and beyond.

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WHICH MATHEMATICS CLASSROOM DO YOU LIKE BEST? COMPARING THE CONCEPTIONS OF MATHEMATICS CLASSROOM TEACHING HELD BY FIFTH-GRADERS, PRE-SERVICE TEACHERS, AND IN-SERVICE TEACHERS

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A follow-up to Lin and Ho (2015, 2016), this study compares the conceptions of mathematics classroom teaching held by students and two groups of teachers. The participants comprised 53 fifth graders, 59 pre-service teachers and 38 in-service teachers, all of whom were asked to rank six selected drawings of mathematics classroom teaching based on their preferences. In addition, the student group was asked to describe their feelings about mathematics teaching in the drawings while the two teacher groups were asked to predict the students' rankings. Overall, students' and teachers' preferences were fairly similar, with both preferring reform-based teaching over traditional methods. However, all groups of participants expressed a high affinity for traditional teaching if it was characterized by extensive student-teacher interactions. Pre-service teachers' preferences were more closely correlated to those of students than in-service teachers' preferences were.

INTRODUCTION

This study extends our previous work that used a drawing-based method to investigate teachers' conceptions about mathematics classroom teaching, along with students' reactions to a selection of six drawings based on sketches made by teachers (Lin & Ho, 2015, 2016). For the present research, we recruited pre-service and in-service teachers and asked them to rank the same teacher drawings that were used in our previous studies, and to predict how fifth-grade students would rank them. Then, the teachers' ranking results from this study and the students' results from our earlier study were compared.

The drawing method has been used for more than two decades, and is especially popular for research in the social science (e.g., psychology, education) (Lee & Zeppelin, 2014; Mitchell, Theron, Stuart, Smith, & Campbell, 2011). It has generally been recognized as a way of accurately capturing teacher conceptions or beliefs about mathematics classroom teaching, the implicit or unconscious nature of which can be difficult to access via traditional methods such as Likert scales or interviews; this is because a drawing usually shows a person's whole picture of something, including aspects s/he is not fully conscious of (Mitchell et al., 2011). Moreover, researchers have confirmed that using a drawing method can help obtain information such as attitude, emotion or identity that may strongly influence teachers' conceptions or beliefs about mathematics classroom teaching. An argument can even be made that

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teachers' conceptions or beliefs will not be well understood if the drawing method is *not* utilized (Lee & Zeppelin, 2014).

This study extends the drawing method through the use of ranking and the prediction of ranking by others. A single drawing of classroom teaching may reflect one's whole picture about mathematics teaching in the classroom. As such, ranking classroom teaching drawings could help us understand how students and teachers think about these drawings (different types of teaching) and their preference to these drawings may reveal their conceptions of good mathematics teaching in the classroom. Drawings used in this study could help us to investigate the relationships between the differing conceptions held by students and teachers both more efficiently and more broadly (e.g., collect feeling data by asking participants' emotion about a drawing). To the extent that we accept conceptions and beliefs as the core of teacher change (Philipp, 2007), this will eventually benefit teacher educators responsible for mathematics pre-service teacher training and in-service teacher professional development.

METHOD

We adopted a survey method that used questionnaires to collect the participants' opinions about teacher-generated drawings of mathematics teaching.

Participants

Student group

This part of the sample consisted of 53 fifth-grade students (28 male, 25 female) from two classes at a school in Taiwan (also see in Lin & Ho, 2016).

Pre-service teacher group

We recruited 59 pre-service elementary teachers (12 male, 47 female) from a public university in the north of Taiwan that specializes in teacher training. Three of them were master's students majoring in Taiwanese language teaching, and the other 56 were undergraduates. Among these 56, only four were majoring in scientific subjects, as compared to 43 in education and nine in other social-science subjects.

In-service teacher group

This group consisted of 38 in-service elementary teachers (4 male, 34 female), of whom 23 had bachelor's degrees; of these 23, 13 were pursuing master's degrees in mathematics education. The other 15 in-service teachers already held master's degrees. The group's mathematics teaching experience ranged from four to 30 years, with the average being 9.4 years. More than two-thirds of the in-service group (n=26) reported using a mixture of teacher-centered and student-centered approaches when teaching elementary mathematics. Eight others favored a more teacher-centered approach, and two implemented a more student-centered one. The majority (n=21; 55%) of the in-service teachers taught at urban elementary schools in the north of Taiwan, while the remainder were employed in suburban schools (n=8; 21%) or rural ones (n=9; 24%) were employed at suburban schools and rural schools, respectively.

Student Questionnaire

In order to create our student questionnaire (SQ), six distinctive drawings of mathematics classrooms were selected from among 32 such drawings that had been created by a separate group of pre-service teachers in our earlier study (Lin & Ho, 2015). Each drawing represents a specific type of mathematics teaching, ranging from very traditional (*Math Test*; hereafter, MT) to reform-based teaching (*Play a Math Game*, MG); all are shown in Figure 1.

The MT drawing shows a teacher watching a mathematics test in the classroom, which may imply the conception that teaching mathematics is centered on testing. The drawing *Walk Around* (WA) also depicts a teacher-centered approach, but with some teacher-student interactions, and notably, the teacher seems very happy in her role. The drawing *Mini-whiteboard* (MW) shows a mixed teaching approach: a teacher is lecturing, but students have chance to write down their own thinking and to discuss it with their peers. *Interactive Whiteboard* (IW) relates to the use of the titular technology in a lecture-based instructional setting, and *Group Work* (GW) represents a typical reform-based teaching style in which the teacher and students work together on a hands-on activity. Lastly, MG shows an ideal form of teaching, with the teacher and students playing a mathematics game together.

In order to reduce bias, these selected drawings were re-drawn by a skillful cartoonist in a uniform style. For example, the teachers and students look similar across all six drawings. In the SQ, all students were asked about (1) how much they liked each drawing (five-point Likert scale), (2) their reasons for these preferences, (3) their feelings about each of the drawings, and (4) their reasons for these feelings. Lastly, (5) they were asked to rank all six drawings according to how much they liked them. The SQ is more fully described in Lin and Ho (2016).

Teacher Questionnaire

Our teacher questionnaire (TQ) was revised from the SQ used in Lin and Ho (2016). In the TQ, the respondents were firstly told that they would see the six selected drawings of mathematics classroom teaching, based on sketches made by pre-service teachers in our earlier study (Lin & Ho, 2015). Then, each drawing was shown to each teacher participant on a separate page, along with annotations that described what the teacher and students were doing. It is noted that these annotations were written by the pre-service teachers in Lin and Ho (2015) who made the drawings. Next, the respondents were asked to rank all six drawings from “Like most” to “Like least” based on their personal preferences, and then rank them again based on their prediction of fifth-grade students’ preferences. They were also asked to write down their reasons for both their own preferences and their predictions of student preferences.

Data Analysis

Respondents’ rankings – including student preferences, teacher preferences and teacher predictions – were scored from 1 (“Like least”) to 6 (“Like most”), and the reasons given for ranking choices were carefully and repeatedly reviewed by the research team for their underlying logic/arguments.

The mean scores and also the total number of positive and negative feelings were calculated using SPSS 22.0. Additionally, Kendall's tau (τ) was employed to analyze the relationships between the differing conceptions of mathematics classroom teaching held by students, pre-service teachers and in-service teachers, also within SPSS 22.0.

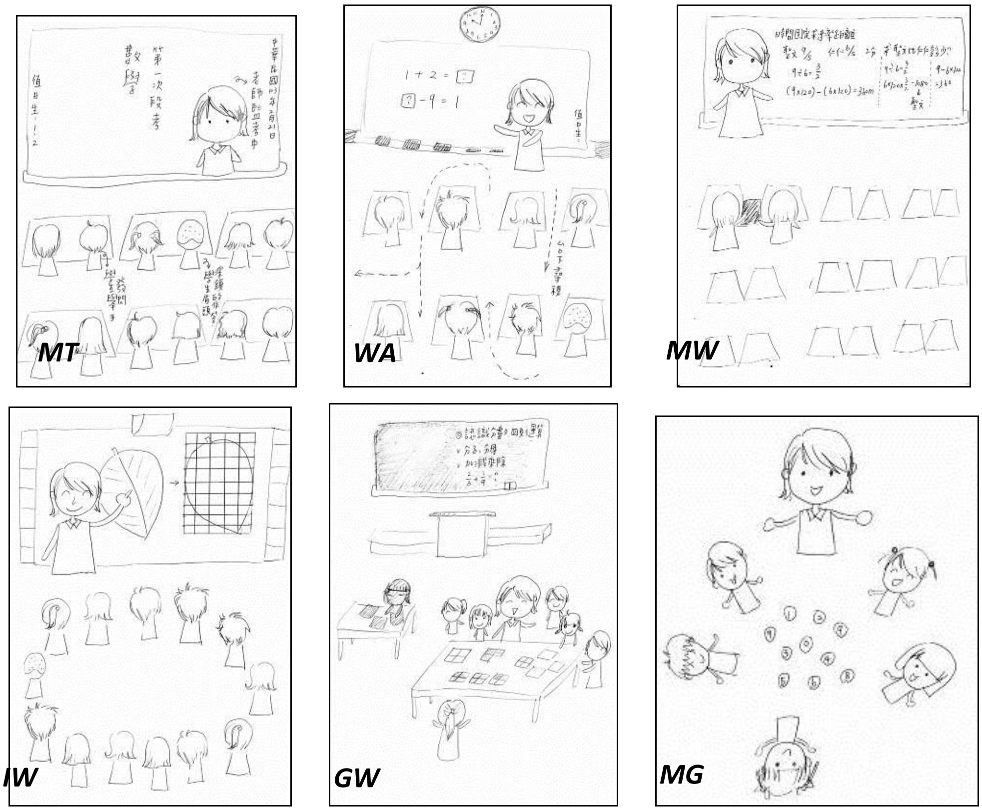

Figure 1: The six selected drawings, ranging from very traditional: watch a math test (MT) to very reform-based: play a math game (MG)

RESULTS

Table 1 shows ranking orders for the six drawings by (1) the fifth-graders' mean preference scores, (2) their mean ranking scores, (3) their numbers of positive vs. negative feelings, (4) the in- and pre-service teachers' mean self-ranking scores, and (5) the mean teacher-prediction scores for student rankings.

Overall, the results across all scales and groups were somewhat similar. As we might expect, MT was the least liked drawing among all three groups and across all scales, consistently receiving the lowest scores (all rank 6 in Table 1). Moreover, all the in-service teachers identically predicted that drawing MT would be least liked by students (Mean rank scores=1, SD=0). GW and MG were the two drawings liked the most by all three groups, but the students favored MG, whereas both teacher groups preferred GW. Nevertheless, both in- and pre-service teachers successfully predicted the student rankings that would be assigned to drawings MG and GW (MG: Student Rank=In-service Pred=pre-service Pred=1; GW: Student Rank=In-service Pred=Pre-service Pred=2).

ID	Students (n=53)				In-service (n=38)		Pre-service (n=59)	
	Pref.	Rank	N+	N-	Self.	Pred.	Self.	Pred.
MT	*6*	*6*	*6*	*6*	*6*	*6*	*6*	*6*
WA	*2*	*3*	*4.5*	*4*	*3*	*5*	*3*	*5*
MW	*5*	*5*	*4.5*	*5*	*4*	*4*	*5*	*4*
IW	*4*	*4*	*3*	*3*	*5*	*3*	*4*	*3*
GW	*3*	*2*	*2*	*2*	*1*	*2*	*1*	*2*
MG	*1*	*1*	*1*	*1*	*2*	*1*	*2*	*1*

Note. Pref=student preference (5-point Likert scale); Rank=student ranking; N+= number of positive feelings (interested + happy + innovative + others); N-= number of negative feelings (bored + scared + worried + others); Self.=teacher self-ranking; Pred.=prediction of student ranking

Table 1: Rank orders of the six drawings by different scales and groups

For the remaining three drawings (WA, MW, and IW), the results across different scales and groups exhibited more variation. WA garnered an overall rank-3 position across different groups (Student Rank=In-service Self.=Pre-service Self.=3), but the mean ranking scores varied across other scales (Student Pref.=2; N+, N-= 4; In-service Pred.=Pre-service Pred.=5). Drawings IW and MV likewise attained overall rank-4 and rank-5 positions, but again based on varying results across different scales and groups (e.g., IW: Students Rank=4, In-service Self.=5; MW: In-service self.=4, Pre-service Self.=5). As such, the results pertaining to drawings WA, IW, and MT were of greater interest than those of the other three drawings, and are therefore further discussed below.

Drawing WA. Among both in- and pre-service teachers, there was a marked difference between the teachers' own preference for drawing WA (both Rank 3) and their prediction of students' preference for it (both Rank 5). However, the students' actual ranking of this drawing (Rank=3) was much closer to the teachers' ranking of it, while the students' mean preference score for it were even higher (Rank=2). The in- and pre-service teachers reported liking drawing WA because it was a typical mathematics classroom setting (the way they usually do every day) and the activity it depicted made it easy to convey concepts. However, due to drawing WA depicting a teacher-centered approach, teachers tended to believe that students would not like it as much they did. As one in-service teacher (In-Teacher 01) explained:

"This is a common teaching strategy used in the classroom. Teachers are able to demonstrate concepts and check the correctness of students' thinking at the same time. Maintaining control of their classrooms all the time. [But regarding the prediction]: Would it be better if [teachers] could provide a slightly different teaching approach for students' learning?" (Self=1st, Pred=3rd)

This was echoed by a pre-service teacher (Pre-Teacher 03), who said:

"This teaching approach could better help students build foundations of concepts. [But regarding the prediction]: Students prefer more activities, instead of listening all the time." (Self=3rd, Pred=5th)

However, many students said they liked WA because it reflected the caring relationships they expected to have with their teachers:

Student 03: *"[Walking around] can let the teacher and students interact more closely, and the teacher can actually supervise student learning."* (Pref=4 of 5 point, Rank=1st)

Student 35: *"[I]f some students are afraid of asking questions, they have a chance to ask the teacher when she walks nearby."* (Pref=4 of 5 point, Rank=4th)

Drawing IW. Technology is not frequently used in Taiwanese mathematics classrooms. Therefore, we had no strong sense of how students or teachers would respond to this depiction of an innovative classroom technology. As can be seen from Table 1, none of our three respondent groups seemed to like it very much (Student Rank=4, In-service Self=5, Pre-service Self=4). Nevertheless, both teacher groups predicted that students would like it better than teachers did (Pred=3>4). In their written responses, some students said they did not like IW because they were uncomfortable about being seated in a circle, or that they did not think using an interactive whiteboard would make any difference: *"If we are seated in a circle, students who sit at the sides or at the back cannot see the board. It reduces our learning efficiency"*, as Student 7-03 put it.

Drawing MT. Even though Drawing MT received the lowest overall rank (Table 1), some students (n=9) still expressed a preference for taking examinations. For example, Student 1-10 reported, *"it can help us know how much we have learned."* However, more than half of the students who said they liked MT (n=5) associated it with a negative feeling – "worried" – and only one student gave MT both a high preference score and a positive feeling (the remaining three expressing neutral feelings). The student who responded the most positively (Student 1-10) wrote: *"Sometimes, you can pretend it is playing a game."*

The great majority of the teacher participants (In-service: n=35/38, Pre-service: n=54/59) gave drawing MT low ranks (Rank 5 or 6). However, two in-service and two pre-service teachers have it high ranks (Rank 1 or 2). One in-service teacher (Teacher 07) said she preferred it because *"teachers can take a rest when watching an exam"*. Both teacher groups were able to successfully predict the ranking that students would assign to drawing MT (Pred=Student Rank=6).

Table 2 shows the Kendall's tau correlation coefficients across our different scales and groups. Overall, the correlation coefficients among the student-, in-service and pre-service groups were somewhat positive. Notably, the in-service and pre-service teachers' predictions of student rankings were exactly the same (τ=1, p<.01), and both sets of teachers' predictions significantly correlated to the student group's ranking and feelings either strongly or very strongly (τ_{SRank}=.733, p<.05; τ_{SN+}=.966, p<.01; τ_{SN-}=.867, p<.05). However, the teachers' predictions did not have a significant correlation with student preferences ($\tau_{SPref.}$=.600, p>.05).

If we compare the in-service and pre-service teachers' self-rankings shown in Table 2, it is noteworthy that the pre-service group's self-rankings were more correlated to students' rankings than the in-service teachers' self-rankings were. The correlation coefficient between pre-service teachers' and students' rankings was τ=.867 (p<.01), as compared to τ=.733 (p<.05) for in-service teachers. In addition, pre-service teachers' self-rankings were significantly correlated to the rank of students' mean preference scores (τ=.733, p<.01) whereas pre-service teachers' self-rankings were not (τ=.600, p>.05). More importantly, the pre-service teachers' self-rankings seemed to work better as predictors of student rankings than their predictions of student rankings did ($\tau_{PreSelf-SRank}$=.867 > $\tau_{PrePred-SRank}$=.733, $\tau_{PreSelf-SPref}$=.733 (p<.05) > $\tau_{PrePred-SRank}$=.600 (p>.05)).

	SPref	*SRank*	*SN+*	*SN-*	*InSelf*	*InPred*	*PreSelf*	*PrePred*
SPref	--							
SRank	*.867**	--						
SN+	*.690*	*.828**	--					
SN-	*.733**	*.867**	*.966***	--				
InSelf	*.600*	*.733**	*.552*	*.600*	--			
InPred	*.600*	*.733**	*.966***	*.867**	*.467*	--		
PreSelf	*.733**	*.867**	*.690*	*.733**	*.867**	*.600*	--	
PrePred	*.600*	*.733**	*.966***	*.867**	*.467*	*1.00***	*.600*	--

Note. "S"=students; "In"= in-service teachers; "Pre"=pre-service teachers

Table 2: Kendall's tau correlation coefficients across different scales and groups.

CONCLUSION

The results of this study indicate that the conceptions of mathematics teaching held by Taiwanese fifth graders, pre-service teachers and in-service teachers are reasonably consistent. In particular, both groups of teachers' predictions of students' ranking results were exactly the same. This result is somewhat inconsistent with Murphy, Delli, and Edwards (2004) which shows beliefs about good teaching between elementary students, pre-service teachers and in-service teachers were somewhat different. In addition, to our surprise, pre-service teachers' self-rankings were even more closely correlated to students' preferences, rankings, and feelings. Such results contradict previous studies' findings that experienced teachers know students better than inexperienced teachers do (Herbst & Kosko, 2014). Conceivably, though pre-service teachers have not developed solid knowledge of students in teaching mathematics, they may nevertheless have a level of rapport with students due to their age and their own current roles as students.

Our results also indicated that, although both teacher groups and the student group tended to prefer reform-based teaching (e.g., MG), drawing WA appealed to all three of them. This may imply that traditional teaching is not totally disliked by students. Provided that it contains appropriate student-teacher interactions, students will still like it. More importantly, this drawing helped us identify a misconception among teachers that students did not like this kind of traditional teaching as much as they themselves did.

In short, the results across the three scales that we used – preference, rank and feeling – were mostly highly correlated, but not exactly the same; and we also found that the student positive-feeling was more correlated to teachers' predictions than the other two scales were. However, it remains unclear which scale best reveals students' own conceptions, and we recommend that future studies further explore this issue.

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PRE-SERVICE TEACHERS' REFLECTIONS OF THE SUMMARISE PHASE OF A LESSON STUDY

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There is growing consensus that we need to improve the quality of pre-service teacher education. This paper reports on pre-service teachers' observations and reflections of the 'summarise phase' of a lesson study in which Year 5 and 6 students engaged with a challenging mathematical task. The findings suggest that the lesson study assisted pre-service teachers to challenge their preconceived notions of the teachers' role within a lesson to support learners, and to identify critical aspects within the 'summarise phase' of a lesson that demonstrated effective pedagogy. Such an experience may equip them to better understand student learning and current pedagogical practices for teaching primary mathematics.

INTRODUCTION

Recently within Australia there has been an increased focused on how best to improve teacher education. The Teacher Education Ministerial Advisory Group (2014) recommended that teacher education should be integrated with practice in schools as part of pre-service teachers' professional learning. This view concurs with those of earlier scholars who argued that teachers learn best when activities are conducted in a school context (Darling-Hammond, 1998). Given that, recent scholars (Saito & Yeo, 2017), suggest that purposeful learning experiences for primary pre-service teachers might include lesson study. Such a study provides participants with opportunities to develop understanding of how students learning and current pedagogical practices including how to guide productive discussion.

The primary pre-service teachers, reported here, had an opportunity to experience a lesson study as part of their course. The purpose of the study was to link theory and practice in relation to the lesson sequence and use of a challenging mathematical task. The emphasise within the lesson structure used in this study focused on students' learning from each other during the discussion or the 'summarise phase' (Sullivan et al., 2015) of the lesson. It is the pre-service teachers' observations and reflections of the 'summarise phase' of the lesson study that is reported in this paper. The following research question guided our research.

What are the critical aspects of the 'summarise phase' that pre-service teachers report on during a lesson study?

REVIEW OF LITERATURE

Challenging student thinking

Students should be provided with rigorous experiences when learning mathematics and encouraged to rely on their knowledge in different contexts including unfamiliar

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situations (Sullivan et al., 2013). The teacher plays an important role when planning these experiences which require students to take responsibility for their own learning. For instance, the teacher might minimise the lesson introduction to enable students to first attempt the task by themselves and engage in productive struggle (Livy, Holmes, Ingram, Linsell, & Sullivan, 2016).

Whole class discussion is integral to effective teaching and learning as it provides an opportunity for students to clarify their understanding, justify their thinking, consider different solution strategies, and provides teachers with insights into student thinking (Anthony & Walshaw, 2009). A teacher's role in discussion is to listen, hold back from telling, invite different students to contribute, and encourage students to construct and evaluate their own and others' mathematical ideas (McDonough & Clarke, 2003). To orchestrate such discussions requires skill, and Smith and Stein (2011) argue that novices require a set of practices to use to facilitate productive mathematics discussion. They advocate the following five practices: anticipating, monitoring, selecting, sequencing and connecting. Sequencing and connecting practices are included within a 'summarise phase' (Sullivan et al., 2015) where the lesson is paused, and selected students discuss and model their response, and share their thinking and strategies. Experiences such as this extend the understanding for other class members (Smith & Stein, 2011)

Perceptions of learning and teaching

Lesson study is an approach of professional development and generally speaking the participants – (1) collaboratively plan the study, (2) implement the study lesson, (3) discuss the lesson, (4) revise the lesson plan (optional) and (6) share thoughts about the lesson (Fernández & Yoshida, 2004). Pre-service teachers participating in lesson study have an opportunity to revisit their judgements about students' abilities and teacher expectations, strengthen their subject knowledge, and extend their understanding of the complexities of teaching by reflecting together in a mutually supportive climate (Cajkeler & Wood, 2016). Cluphf, Lux, and Scott (2012) agree that by collaborating with professional teachers, pre-service teachers can reduce their initial anxiety by applying what they learnt in their teacher education programme to the real classroom contexts. A recent study highlighted that pre-service teachers and their mentors learnt together whilst focusing on the improvement of teaching (Cajkler & Wood, 2016). We adopted a lesson study approach for our study as it provided a structure for the pre-service teachers' observations, and post lesson reflection and discussion.

METHODOLOGY

The data reported in this paper were collected during 20 pre-service teachers' course experience that included a lesson study and observation of a Year 5 and 6 (N=23) lesson related to geometric reasoning. The first author taught a lesson asking the students to solve a mathematical task related to geometric reasoning and in particular the size of angles within pattern blocks (Figure 1). The pre-service teachers and the second author observed the lesson, taking field notes.

In a circle there are 360 degrees. Work out the exact size of as many of the angles in this shape as you can. Explain how you worked them out.

Figure 1: Working out the size of angles.

The approach chosen for the study was a lesson study including pre- and post-testing (of students), implementing and observing the research lesson, evaluating the research lesson, and reporting (Saito & Yeo, 2017). The process of lesson study in this case, was designed to support and extend pre-service teachers' pedagogical knowledge for teaching geometry. The lesson included three phases, *launch, explore, summarise.* In the 'launch phase' students are expected to attempt the task without help from the teacher or their peers. During the 'explore phase' the teacher monitors and selects students to present in the 'summarise phase', and provides prompts for students requiring help or extension. In the 'summarise phase' the teacher stops the lesson and selected students share their responses to the task. This phase can occur more than once during the lesson (Sullivan et al., 2015). After the lesson the pre-service teachers met with the classroom teacher and first two authors to discuss and reflect on their experience.

Data collection and analysis

Qualitative data collected from the pre-service teachers included their reflections of the research lesson and written assignments reporting on their experiences and observation of one student. This paper reports on the results from the analysis of the pre-service teachers' assignments relating to the 'summarise phase' of the lesson. The data from the assignments were collated to identify descriptions categorised according to themes that emerged from the analysis (Miles, Huberman & Saldana, 2014). The first two authors checked each other's coding for consistency and identified a total of 12 categories. These categories and their descriptors are presented in Table 1. While it could be argued that all of these categories relate to pedagogical practices, the purpose of this paper is to identify the particular aspects pre-service teachers noticed or attended to when observing an experienced teacher in a mathematics lesson.

Category	Descriptor
Timing	Spacing of student sharing during each 'summarise phase'
Nature of the task	Describing aspects of the task such as challenge, open-ended
Choice of teacher questioning	Strategic questions used to probe or challenge student thinking or orchestrate student led discussion
Selection of students	Teacher strategically chose students to share their thinking
Scaffolding student learning	Experiences that enhance student learning and understanding (e.g., use of enabling and extending prompts, questioning, wait time)
Awareness of how children learn	Pre-service teachers' discussion of how children learn mathematics
Challenging student thinking	Students present different strategies to their peers, and peers challenging strategies used
Learning how to teach	Pre-service teachers' discussion of the pedagogies the teacher used

	to orchestrate the 'summarise phase' during the lesson
Minimal teacher instruction or assistance	Teacher held back from assisting students to allow them to experience 'productive struggle' and refrained from summarising student learning and thinking
Students learning from others	Student recording and sharing their thinking provided a springboard for other students to explore alternative pathways
Strategies students use	Noticing the range of strategies students used
Positive disposition	The teacher valuing all students' contributions including incorrect answers

Table 1: Categorisation of critical aspects of the 'summarise phase'

RESULTS AND DISCUSSION

From the analysis of the 20 pre-service teachers' assignments it was evident that they all reported on at least one category of the critical aspects of the 'summarise phase' of the lesson. The frequency of use of each category is presented in Table 2. The most common category was *students learning from each other* (15) followed by *how children learn, and student disposition* (13), *scaffolding student learning* (12), then *timing, learning how to teach*, and *strategies students use* (11). The fact that so many pre-service teachers reported on these aspects highlights the benefit of engaging them in a lesson study. The fact that so many comments related to *how children learn* mathematics is reflected of their learning experience provided during tutorials and their academic reading, hence important when developing their knowledge for primary mathematics teaching. A key feature of the lesson was that student misconceptions or errors were discussed as part of the learning and students had an opportunity to assist with identifying the correct solution or error (Alice, Rose and Erin). The three least common categories were selection of students, challenging student thinking, and minimal teacher instruction or assistance (4). A possible reason for the pre-service teachers' lack of attention to these categories was due to the fact that the lesson approach that they observed was not typical of their education program (practicum experiences) or their own schooling.

Category	Frequency of use
Timing	11
Nature of the task	8
Choice of teacher questioning	7
Selection of students	4
Scaffolding student learning	12
Awareness of how children learn	13
Challenging student thinking	4
Learning how to teach	11
Minimal teacher instruction or assistance	4
Students learning from others	15
Strategies students use	11
Positive disposition	13

Table 2: Frequency of pre-service teachers' (n=20) reporting on aspects of the 'summarise phase' of the lesson

Having seen the distribution of pre-service teachers' insights, providing examples of these in Table 3 indicates the range of perspective on each category and the nature of their noticing. For instance, some pre-service teachers commented on how particular aspects impacted on the students' learning (scaffolding learning, and learning from others) while others related to their own learning (Kylie, Dora, Lily). Some overlap between the categories is evident, for example, Carla's comment on timing related to learning how to teach in this way. Dora's reflections were insightful as she was particularly sceptical before the lesson about the structure and how students would be supported in their learning. Limited space within this paper precludes us from reporting on all the pre-service teachers' insights.

Category	Evidence extracted from pre-service teachers' assignments
Timing	Every ten minutes there was sharing after the independent learning time... providing a chance to hear other solutions and work out whether their way of thinking was getting them to the right answer. (Casey)
	The working time between each student presenter gave students a chance to think about their answers and apply their knowledge or strategy they learnt from their peers. (Naomi)
Nature of the task	Challenging tasks encourage students to connect ideas and apply to another context. (Sally)
	Providing a challenging task allowed students to learn outside their comfort zone and approach tasks with new strategies. (Libby)
	Throughout this experience, I learnt about the benefits of offering students challenging tasks and a lot about myself as a mathematical learner. (Kylie)
Choice of teacher questioning	Each prompt is targeted and should be well planned and thought out ... I was able to see that this approach could work despite my reservations. (Dora)
	The teacher probed Imogen's thinking to merely try to help her to justify her answer and understand her way of thinking. (Alice)
Selection of students	These summaries were ordered in a particular way to provide the most benefit to the students. Starting with the easier shapes... then moving onto the harder ones was an effective strategy to help students who may be struggling with the task and to help them gain a greater understanding. (Rose)
	When selecting students to share it is important that teachers select some who have incorporated common strategies, and some that lead to incorrect answers. (Alice)
Scaffolding student learning	The summary phases promoted the sharing of worthwhile ideas, which then facilitated the learning of those in the class who may have been struggling or needed a prompt. (Maria)
	Following the [second] summary Hazel realized she had the wrong angles for her triangles and corrected her mistake, then following on from this she applied this knowledge to the rest of the triangles in the hexagon. (Rose).
	The summary stage was especially helpful in reassuring Penny was on the right track when she saw that other students had found the same angles as her in the red hexagon. (Erin)
	The emphasis on peer discussion allowed students who were both struggling and who were confident in their own working out to see other students' way of thinking and ways of understanding different procedures of discovering an answer. (Megan)
Awareness of how children learn	Leaning by exploring different strategies. (James)
	By highlighting common misconceptions, the teacher positions the students to be corrected and learn from other students rather than merely being told they have answered incorrectly. (Alice)
Challenging student	The lesson highlighted to me that students require a specific level of challenge, and that students' benefit immensely from this type of challenge... (Eva)

thinking	It also helped to make learning more concrete for those selected to share as they were challenged to express their knowledge in words that their peers would understand. (Maria)
Learning how to teach	Having the opportunity to see the summary phase in action allowed me to see the benefits to student learning. (Dora)
	It was interesting to watch the class have a go on their own and then share different solutions roughly every ten minutes. Teaching the class in this way gave students independent learning time while at the same time, gave them a chance to hear others' solutions and whether their way of thinking was getting them to the answer. (Carla)
Minimal teacher instruction or assistance	The students were able to work on their own through the challenging task with minimal instruction and minimal assistance from teachers and pre-service teachers. (Cassie)
	Noticing how little talking the teacher does during the lesson was powerful. (Dora)
	Providing a problem-solving question allows students to delve into their own knowledge without listening to a teacher tell them how it should be completed. (Libby)
Students learning from others	Highlights the influence of peer-oriented learning has on student potential. (Erin)
	It was an interesting way to see students use their prior knowledge and ideas from their fellow classmates to progress through the learning tasks as students had little or no help from their teacher. (Anne)
	From the class discussion, Maisie learnt how to work on some of the shapes she was struggling with and she also applied the new learning she learnt from others. (Carla)
	The emphasis on peer discussion allowed students who were both struggling and who were confident in their own working out to see other students' way of thinking and ways of understanding different procedures of discovering an answer. (Megan)
	Even if a student already knew how to solve the problem their knowledge was expanded as they were exposed to and considered other approaches. (Maria)
Strategies students use	When students had to explain their strategies, convince others of their answers, students collaborated together to try and help each other to explain their working out. (Naomi)
	Within the lesson I became familiar with various methods students used when answering a particular learning task. The summary phase highlighted the student thinking which was an aspect of the lesson I found most stimulating, these students managed to use existing knowledge on division, the angles of an equilateral triangle equaling 180 degrees as well as a straight line. (Lily)
Positive disposition	The teacher did not let on that a solution was wrong but asked the class if anyone could help out… by doing so this showed the students that all mathematical thinking is valued. (Dora)
	Creating an engaging classroom can allow students to feel engaged in their work and allow them to achieve more. (Libby)
	The findings from the lesson demonstrated that a challenging task and effective peer learning can influence a student's ability to be persistent and successful when exposed to an unfamiliar task. (Rhonda)

Table 3: Categorisation of critical aspects identified by pre-service teachers

The pre-service teachers' reflection within each category demonstrates the breadth and depth of the value of this experience. Two key aspects that many pre-service teachers noticed were the conscious spacing of timing of students' independent working time on task and sharing of student discussion, and how the teacher sequenced the student learning by pausing the lesson at regular intervals as part of the 'summarise phase'. Not only did the lesson study link theory to practice, it enabled them to see the specific skills required by the teacher to facilitate a productive mathematics discussion and how to enact the five practices (Smith & Stein, 2011). Their reflections also indicated the power of the 'summarise phase' to scaffold student learning and for students to learn from their peers in a supportive learning environment (Anthony & Walshaw, 2009; McDonough & Clarke, 2003), and the impact this had on their pedagogical content knowledge (Cajkler & Wood, 2016).

An overall reflection by Alice highlighted the nature of the task and pre-service teachers learning how to teach.

> Throughout the process of observing this lesson and later analysing the student's achievement and responses, I learnt that the discussion during the 'summarise phase' of a lesson has more instructional benefit than any other stage of the lesson. For this to occur, teachers must select worthwhile tasks in order to promote worthwhile discussion.

CONCLUDING REMARKS

Critical aspects of the pre-services teachers' responses highlighted that engaging in this experience helped to dispel some of their preconceived notions of how student learn mathematics and the teacher's role within a lesson to support learning. Learning experiences during tutorials can assist pre-service teachers to develop their knowledge of how to teach mathematics, how children learn, and suggestions for differentiating learning. However, the opportunity to observe a lesson, interact with students enabled the pre-service teachers to identify critical aspects of the 'summaries phase' and realise the importance of minimal teacher talk or assistance and consider how they might implement these strategies within their future teaching. These results highlight the many opportunities for pre-service teachers to extend their knowledge for teaching by observing a lesson study and provided an example of one way to possibly improve teacher education.

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DECODING MAP ITEMS THROUGH SPATIAL ORIENTATION: PERFORMANCE DIFFERENCES ACROSS GRADE AND GENDER

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This investigation examines the spatial orientation performance of a nationally-representative cohort of secondary school students drawn from national mathematics assessment data. This investigation analyses the changes in performance on spatial orientation tasks (maps) of Grade 7 (ages 12-13) and Grade 9 (ages 14-15) cohorts across two time periods. Although spatial orientation performance increased significantly from Grade 7 to Grade 9, effect sizes were relatively small. Males outperformed females on both items, with performance differences increasing from Grade 7 to Grade 9. Both males and females had difficulties re-orientating themselves within the map, especially when the question had multiple changes in orientation.

INTRODUCTION

There is a strong nexus between mathematics outcomes and assessment, with similar links between assessment, learning and instruction. In Australia the national assessment platform is the National Assessment Program –Literacy and Numeracy (NAPLAN). This assessment covers mathematics content and numeracy elements from the national curriculum and is conducted annually across Grades 3, 5, 7, and 9. One feature of the NAPLAN is the use of spatial items, that is, items that require spatial thinking and require minimal, if any, number computation. These items are directly associated with the *using spatial reasoning* element of Numeracy embedded throughout the Australian curriculum. Spatial reasoning has been identified as an important aspect of mathematical understanding (Lowrie, Logan & Ramful, 2016; Sinclair & Bruce, 2014) and studies suggest strong associations between spatial reasoning and success in Science, Technology, Engineering and Mathematics (STEM) disciplines (Wai, Lubinski, & Benbow, 2009; Uttal & Cohen, 2012). Hence, items that assess spatial reasoning provide insight into a somewhat silent aspect of many curricula, but one that may have important implications as students' progress through the schooling years.

Another feature of the NAPLAN is the repetition of items across grade levels within any given yearly test. For example, the same item may appear in the Grade 3 and Grade 5 assessment. This is undertaken to ensure consistency and validity across grade levels. This paper reports on two such spatial items that appeared across Grade 7 and Grade 9 assessments. These spatial items were map questions, where the students were required to navigate and orientate themselves within the question

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space. This study investigated the difference in performance between the two grade levels and considers gender differences with respect to performance and multiple-choice responses on the two map items.

UNDERSTANDING AND INTERPRETING MAPS

Maps provide a relatively authentic context for learning mathematics and assessing spatial knowledge, with the ability to interpret or decode maps requiring students to analyse locations (through position and placement) and attributes (what is actually represented); and understand that the map representation is presented within some form of scale (Wiegand, 2006). However, students do not always find their interpretation straightforward. For example, Diezmann and Lowrie (2008) reported that 10- to 13-year-olds experienced difficulty with some of the vocabulary presented in maps; students were distracted by different foci on the map; and information critical to understanding was often overlooked. Other difficulties identified in Liben's (2008) research relate to children misinterpreting the representation of symbols and confusion over perspectives and angles used to represent different maps (for example, elevation view and birds-eye view). As a consequence, decoding and understanding a map require knowledge of map attributes. Wiegand (2006) identified a framework with three levels of sophistication involved in map interpretation. This framework was used to interpret the data. The first stage involves *extracting information from a map* and generally reading symbols, texts and attributes. The second *analysis* stage, involves ordering and sequencing information, recognising perspective and wayfinding. Finally, *interpretation* requires higher levels of problem solving and decision making involving the application of information, such as multiple navigational cues and scale. In addition, proficiency with map tasks requires perceptual and cognitive processing associated with visualization and spatial orientation ability respectively.

Spatial orientation and gender differences

Spatial orientation relates to the self-to-object representational system (Kozhevnikov & Hegarty, 2001) which is seen as establishing "spatial relations in body-centered coordinates, using the body axes of front-back, right-left, and up-down" (McNamara, 2003, p. 181). Generally, spatial orientation is associated with navigation, wayfinding and perspective taking. Previous research has examined the differences between males and females on spatial orientation tasks, with males generally outperforming females (Bosco, Longoni, & Vecchi, 2004). However, Wolbers and Hegarty (2010) highlighted that much of the research manifests from the different strategies and approaches used to navigate and wayfind by males and females. Lawton and Kallai (2002) found that females preferred route-based information strategies, for example, receiving directions that explained the number of streets to pass before turning. By contrast, males preferred orientation-based information strategies such as keeping in mind the direction from which they came and keeping track of the relationship between where they were and the next place to change direction. Other studies have indicated that females prefer using landmarks and known routes, focusing on more environmental signs in order to stay orientated. However, males prefer to use cardinal directions and Euclidean information and are more likely to utilise geometry-based thinking to remain oriented within an environment (e.g., Bosco, Longoni, & Vecchi, 2004; Coluccia & Louse, 2004; Lin et al., 2012; Saucier et al., 2002).

This body of research highlights the need to consider performance differences (or otherwise) with respect to gender, but also the strategies employed to solve these spatially-demanding tasks.

DESIGN AND METHODS

This study utilised a secondary data analysis design, from a large nationally-represented data set. The data are drawn from the Numeracy assessment of the NAPLAN provided by the Australian Curriculum, Assessment and Reporting Authority (ACARA). Within this paper three research questions are explored:

- *Are there performance differences between students from Grade 7 and Grade 9 on the same spatial orientation items?*
- *Are there performance differences between males and females on the same Map items across grade levels?*
- *What difficulties do males and females experience on Map items across grade levels?*

The Spatial Orientation Map Items

The two items chosen for this paper were typical map items found across the NAPLAN and required students to contend with multiple changes in orientation (see Appendix for items). The Plum Road item was selected from the 2010 NAPLAN and the Park Map was selected from the 2013 NAPLAN. Both items were repeated in the Grade 7 and Grade 9 assessment of the respective years, and were of varying difficulty based on reported means (see Queensland Studies Authority, n.d.).

Participants

The number of participants for each item across the two grade levels was as follows: Plum Road Grade 7 = 20,441 (Female = 9,954) and Grade 9 = 29,369 (Female = 14,073); and Park Map Grade 7 = 18,947 (Female = 9,772) and Grade 9 = 29,552 (Female = 15,128). The average ages of students taking the NAPLAN were: 2010 Grade 7 = 12 years, 6 months and Grade 9 = 14 years, 5 months; and 2013 Grade 7 = 12 years, 6 months and Grade 9 = 14 years, 6 months.

RESULTS

The first two research questions were investigated through two, 2-way analysis of variance (ANOVAs) to determine whether there were statistically significant differences between Grade and Gender on the two items, namely; Plum Road item and Park Map item. The first ANOVA revealed statistically significant differences for the Plum Road item across Grade [F(1,49806=595.9, p ≤.000] and Gender [F(1,49806=151.2, p≤.000]. The second ANOVA revealed statistically significant differences for the Park Map item across Grade [F(1,48495=472.9, p≤.000] and Gender [F(1,48495=155.6, p≤.000]. For the Park Map item, there was also a significant interaction between Grade and Gender [F(1,48495=10.8, p≤.001]. Table 1 presents the mean and standard deviation for grade and gender across the two items.

	Year 7			Year 9		
Item	Male	Female	Total	Male	Female	Total
Plum Road	.60 (*.49*)	.55 (*.50*)	.57 (*.49*)	.71 (*.50*)	.65 (*.48*)	.68 (*.47*)
Park Map	.40 (*.49*)	.36 (*.48*)	.38 (*.49*)	.52 (*.50*)	.45 (*.50*)	.48 (*.50*)

Table 1: Mean (*and Standard Deviation*) for the Two Map Items by Grade and Gender.

Post-hoc analysis revealed that the Grade 9 students outperformed the Grade 7 students on both items. While this may be expected, effect sizes of d = 0.23 for Plum Road and d = 0.20 for Park Map highlight relatively low performance differences across two years of schooling.

With respect to Gender, males outperformed females on both items, at both Grades. For the Park Map item, the significant interaction highlights that the males' improvement was greater than the females in Grade 9 (see Figure 1). In order to better understand such effects, the multiple choice responses of students were collated to establish the difficulties encountered during the assessment.

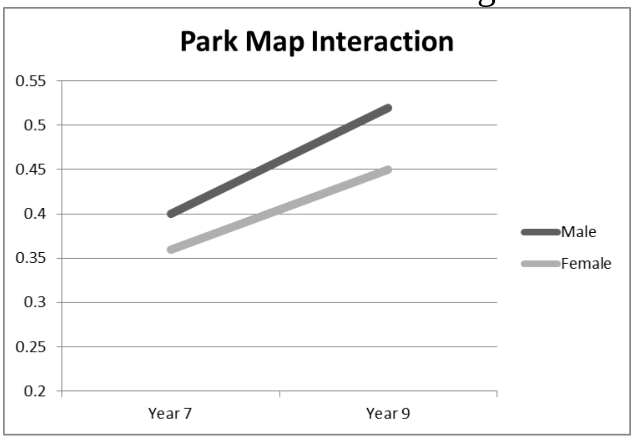

Figure 1: The Park Map interaction effect for Grade and Gender.

The third research question was investigated through descriptive analysis of the multiple choice responses. Often, the multiple choice responses of students reveal insights into their thinking about certain tasks and through analysing these data, it is possible to highlight some of the difficulties encountered by students as they solved the questions. Table 2 presents the percentages of the multiple choice responses for the two map tasks, organised by grade and gender.

For the Plum Road item (correct response "D"), the main incorrect response was "A – north-east". It appears that students who chose "north-east" were unable to apply the second aspect of directional information, where they needed to provide the direction **after** the car turned right. Noteworthy, in both grades, more females chose this option. The fact that the item does not show a compass or have North orientated toward the top of the page may have caused difficulties for students as the prototype direction for North on maps is toward the top of the page.

	Plum Road				Park Map			
	Year 7		Year 9		Year 7		Year 9	
Option	Male %	Female %	Male %	Female %	Male %	Female %	Male %	Female %
A	20	25	13	18	15	12	13	10
B	9	9	7	7	26	29	18	23
C	10	12	9	10	18	22	17	22
D	60	54	70	65	40	36	52	44

Note: due to a small percentage of missing data and rounding, totals may not equal 100%

Table 2: Percentage of Multiple Choice Responses to the Two Map Items by Grade and Gender.

For the Park Map item, in Grade 7, “B” was the main incorrect response, however, by Grade 9 the distribution of incorrect responses spread across “B” and “C”. Those students who chose location “B” on the map seemed to have started at either the top or bottom gate on the map and have not applied the first navigational cue that he walked south-east along the path. Whereas, those who chose “C” seemed to have entered the park through the correct gate, but turned left instead of right. This highlights that they weren’t able to re-orientate themselves in space or to visualise movement from a different perspective. An interesting finding for response “C” is that the relatively high proportion of females choosing this option did not differ from Grade 7 to 9, suggesting that the females tended to interpret the first required movement correctly but they struggled with the subsequent directional change.

DISCUSSION AND CONCLUSIONS

Our study examined the performance differences of Grade 7 and Grade 9 cohorts on two spatially-demanding mathematics items. Although the Grade 9 cohorts were more successful in solving both map items, performance differences were not large given the additional two years of schooling the older students had. In fact, the effect sizes between Grade on the two tasks were less than one quarter of a standard deviation. Given the fact that the administration of a test is likely to produce an effect size of d = 0.3 and Hattie (2008) suggested that one year of educational improvement equates to an effect size of approximately $d = 0.4$, it was anticipated that the Grade 9 performance would have been much greater. Each cohort (by grade level) found the Park Map item to be much more difficult to solve than the Plum Road item. For both tasks, the students were required to make orientations decisions 135^{o} from North orientation. Thus, the initial spatial orientation processing demands were similar. In fact, three aspects of Wiegand's (2006) decoding requirements were similar—that is the map's perspective, the need to wayfind within similar context (i.e., a road), and the fact that the map represented space. We maintain that the increased cognitive demands associated with the use of scale, and the more challenging use of symbols and texts, raised the item complexity. The relatively low performance increase across Grade 7 and Grade 9 cohorts suggest that insufficient attention may be afforded to these important perceptual elements. As Diezmann and Lowrie (2008) indicated, explicit teaching of the various information graphics embedded in mathematics tasks is required, since there are low performance associations across maps and graphs.

There were significant performance differences in relation to gender, across both tasks and grade levels. These results support sustained research findings that identify performance differences in favour of males on spatial orientation items (Wolbers & Hegarty, 2010). In fact, the Grade 7 male cohort's performances were similar to that of Grade 9 female cohort's, on both the Park Map and Plum Road items.

The difficulties faced by students as they were asked to orientate and re-orientate themselves within the map where relatively similar across grade and gender. The uncertainty of the starting point on the Park Map was evidenced by the higher percentages across the three incorrect options. For both items, the requirement for one or more re-orientations within the map proved a barrier to many students, especially the females. Such spatial reasoning relates to the self-to-object representational system (Kozhevnikov & Hegarty, 2001) in which movement or orientation is considered relative to the position of oneself.

IMPLICATIONS

The two implications drawn from the study are associated with the relationship between assessment, learning and instruction, namely: (1) the need to provide explicit pedagogical attention to spatial orientation in the school mathematics curriculum; and (2) increased support for females' spatial development. Evidence from this investigation indicates that instruction in mathematics needs to provide opportunities for students to become proficient in interpreting (and creating) map questions that require spatial orientations, especially multiple orientation processing. Since the performance differences between males and females increased over time (especially with the more difficult Park Map item), more instructional attention needs to be given to analysing and interpreting maps. These two higher levels of Weigand's (2006) framework go beyond the less sophisticated reading and extracting aspects of map content presented in school curricula. These distinct differences need to be addressed, especially in an age where spatial reasoning is becoming increasingly important to life aspirations. Given spatial reasoning is closely associated with success in STEM professions (Uttal & Cohen, 2012), and women are much less likely to transition into these profession, school instructional practices need to attribute more attention to these spatial dimensions of intelligence.

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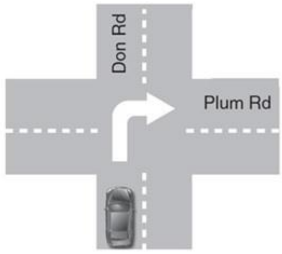
APPENDIX

A car is travelling **north-east** along Don Road.
The car is about to turn right into Plum Road.

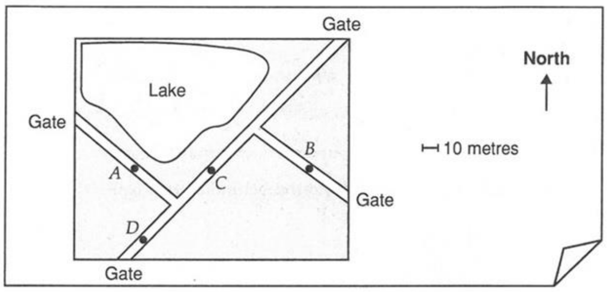
In which direction will the car be travelling **after** it turns right?



- ◯ north-east
- ◯ south-west
- ◯ north-west
- ◯ south-east

The Plum Road item

This is a map of a park.



Josh entered the park through on of the gates.
He then walked south-east along a path.
After 90 metres he turned right.
He then walked another 30 metres and stopped.

Which point on the map shows where Josh stopped?

A ◯ *B* ◯ *C* ◯ *D* ◯

The Park Map item

THE INFLUENCE OF STUDENTS' SPATIAL REASONING ON MATHEMATICS PERFORMANCE ACROSS DIFFERENT TEST MODE FORMATS

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This study compared the performance of students who solved mathematics tasks in either a traditional pencil-and-paper test (PPT) format or computer-based test (CBT) format. Specifically, this study examined the effect students' spatial reasoning had on their performance across the respective test formats. The results of the study revealed (1) no performance differences across the two test formats; however, there were (2) significant performance differences in the favour of students with higher spatial reasoning; and (3) there was an interaction effect between test format and students' level of spatial reasoning. The students with lower levels of spatial reasoning performed better in the CBT format, while the high spatial ability cohort performed best in the PPT format.

INTRODUCTION

The utility and function of spatial skills seem increasingly important in our technology-rich societies—from navigation via global positing systems to health diagnosis from medical imaging. Unsurprisingly, the notion of "learning to think spatially" has been embedded in most school curricula for the past ten years (Downs, & DeSouza, 2006). During the same time period, there has been a dramatic shift in how mathematics is assessed, both in terms of task representation and the medium in which the tasks are presented. Mathematics tasks used to assess students' performance are much more likely to contain graphical information. For example, in countries such as Singapore and Australia, graphics-based tasks have replaced more traditional word problems in national assessments. Australia's National Assessment Plan: Literacy and Numeracy (NAPLAN) consists of approximately 70% graphics tasks while Singapore's Primary School Leaving Examination (PSLE) constitutes 41% of items that are graphics based (Lowrie & Logan, 2015). To some degree, the focus on graphics-rich tasks has evolved from advances in technology, with mathematics assessment reflecting applications of mathematics concepts. These graphics-rich tasks contain more spatial features and require decoding of information associated with rotations, translations, location and arrangement. As testing agencies move toward digitally- and adaptive-based testing, how students encode and decode mathematics information is likely to change. For example, the Programme for International Student Assessment (PISA) has been gradually introducing digital-based testing since the first optional electronic module in 2006 and has included innovative item formats since 2015.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 201-208. Singapore: PME.

BACKGROUND

Decoding and encoding mathematics tasks

According to Brizuela and Gravel (2013), representations refer to products and processes that we create or interpret in order to “capture, understand, and translate an idea, an event, or a phenomenon” (p. 1). Representations can either be decoded or encoded. Decoding takes place when the problem solver is required to interpret the symbol systems, graphics and text embedded within a task—the process involves interpreting information they have not constructed. The information includes specific conventions that need to be interpreted (Roth, 2002). In the current investigation, participants are required to decode information from both traditional word-based tasks and more graphics-rich geometry tasks. By contrast, encoding involves constructing “one’s own” representations, which are usually developed from specific heuristics or personal (and sometimes idiosyncratic) constructions. Encoded representations can be produced “in the mind’s eye” (Smith & Kosslyn, 2013), or externally through some physical or concrete approach. Elsewhere, (Lowrie, Logan & Ramful, 2016) we have found that students are more likely to externally encode mathematics tasks when the mode of delivery is PPT, including drawing pictures and diagrams to process information. By contrast, students are more likely to solve problems by internally encoding (using visualisation) or using mental computations in a CBT mode.

Spatial reasoning and mathematics performance

A number of studies have demonstrated the strong association between mathematics performance and spatial reasoning. Students who perform well on spatial tasks typically perform better on mathematics tasks (Holmes, Adams, & Hamilton, 2008; Rasmussen & Bisanz, 2005). Mathematics concepts are spatial in nature, since students need to be able to imagine and visualize information (Battista, Wheatley, & Talsma, 1982). In fact, spatial visualization ability predicts talent in mathematics (Wei, Yuan, Chen, & Zhou, 2012).

The relationship between spatial and mathematics ability is evident from the early years of school (Kurdek & Sinclair, 2001) and are still prevalent with college graduates (Wai, Lubinski, & Benbow, 2009). These relationships seem most plausible when students encounter geometric mathematics tasks (Battista, 1990) since the transformation of 2-D and 3-D objects require spatial reasoning. Nevertheless, moderate relationships between spatial ability and traditional word problems exist (Hegarty & Kozhevnikov, 1999), even though the mental or physical manipulation of objects is not required. More recently, Mix et al. (2016) found that mental rotation best predicted mathematics performance in younger students, while spatial visualization was the best predictor of performance by Grade 6 (especially place value, word problems and algebra concepts). These associations, across both geometry and traditional word-based mathematics tasks, have been confirmed even when expertise is accounted for (Sella, Sader, Lolliot & Kadosh, 2016).

DESIGN AND METHODS

The research questions of the study were:

- Does Mode of Delivery (PPT or CBT) effect student performance on mathematics assessment tasks? and
- Is spatial ability an influential factor in student performance on mathematics tasks and Mode of Delivery?

Participants

The participants (N = 162; 81 Male, 81 Female) comprised Grade 6 students (Mean Age = 11 years, 4 months) from four Australian primary schools. One hundred and six students completed the mathematics test on iPads (67% male) and 56 completed the test on paper (33% male). Males were represented in the low, mid- and high spatial groups at rates of 27%, 35% and 38% respectively.

The instrument and administration

The participants in the study completed a 45-item Spatial Reasoning Instrument (SRI), which comprised an equal number of mental rotation, spatial orientation and visualisation items (see Ramful, Lowrie & Logan, 2016). The students also completed a Mathematics Test (MT)—a 12-item instrument used to determine students' performance across mathematics tasks. It consisted of five number and seven geometry and measurement items. The tasks were drawn from the Australian standard test for grade 5 students (NAPLAN) and reflected the format of the assessment with 75% of items containing a graphic relevant to the task.

Two members of the research team attended the participating schools during their morning classes. The two Instruments were administered to whole (intact) classes to minimise disruption to both the school and the students' daily classroom routine. The classroom teachers and the research staff administered the activity.

Data coding

The participants were scored according to the number of tasks they answered correctly. Hence, the highest possible score for the SRI was 45 and MT was 12.
Students were classified as low, middle or high spatial ability based on their scores on the Spatial Reasoning Instrument. Range and sample size for each group are presented in Table 1.

Spatial Reasoning Category	Range	*N* (%)
Low-spatial Reasoning	5-20	56 (30%)
Mid-spatial Reasoning	21-27	68 (37%)
High-spatial Reasoning	28-42	61 (33%)

Table 1: Distribution of students in low, mid and high spatial groups

RESULTS AND DISCUSSION

A factorial Analysis of Variance (ANOVA) was conducted to examine the two research questions; with scores on the mathematics test as the dependent variable and spatial reasoning ability (low, mid, high) and mode of delivery (PPT or CBT) as factors. There was no main effect for test presentation (Mode of Delivery) on mathematics test scores, $F(1, 161) = .92$, $p = .34$. Although these results are consistent with a comprehensive meta-analysis conducted by Wang et al., (2008), they are in contrast to studies conducted with students of this age group (ie., primary school students). In a study with over 800 Singaporean students, Lowrie and Logan (2015) found that student performance was significantly higher in a CBT (iPad) mode than a PPT mode. In the current study, it is worth noting that this cohort of Australian students was not as mathematically able as the Singaporean cohort.

A main effect for spatial ability level on mathematics test performance was found, $F(2, 161) = 52.67$, $p < .001$, $d = 1.55$, with high spatial students performing better on the mathematics test than medium and low spatial ability students, and mid-spatial students performing better than low spatial students. Means are presented in Figure 1. To some degree, these results are unsurprising since there is a substantial body of literature that that shows strong positive correlations between spatial reasoning ability and mathematics performance (Mix et al., 2016). Noteworthy is the magnitude of the differences (effect size = 1.55), indicating the large differences in student performance across the low, mid and high categorisations of spatial reasoning.

There was also significant interaction effect between mode of delivery and spatial reasoning rank, $F(2, 161) = 3.52$, $p = .03$, $d = .32$.

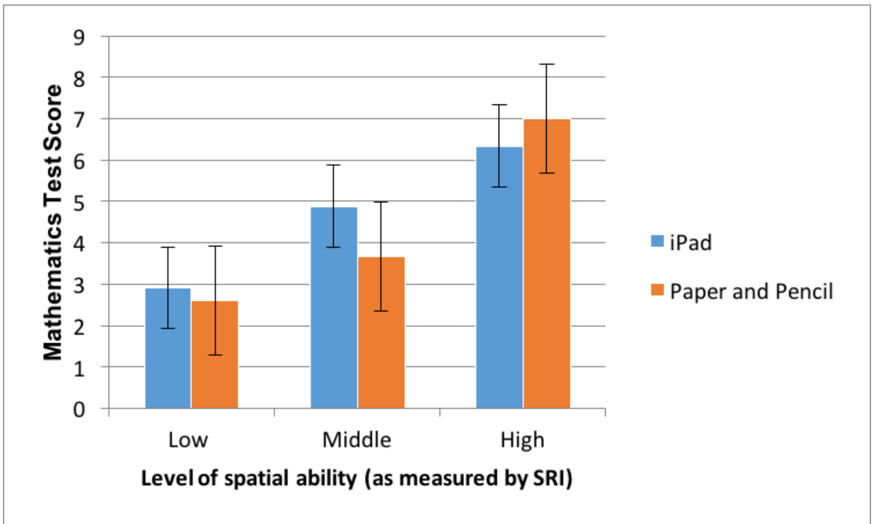

Figure 3: Means on MT by testing mode and spatial ability (bars denote standard error)

Students within the low- and middle-ranked spatial reasoning group scored higher on the mathematics test when it was delivered in a CBT (iPad) format, whereas the high spatial reasoning cohort performed better in the PPT format (as represented in Figure 1). This result was unexpected—since we envisaged that the increased demands of processing information in the digital form would impact on performance. Moreover, students are less likely to use "working-out paper" on mobile devices, which result in higher visualisation demands (Lowrie & Logan, 2015).

Further analyses addressed the interaction effect of test mode and spatial ability on the two mathematics streams incorporated into the test, namely geometry and measurement items. An interaction effect was found for the number items, $F(2, 161) = 5.13$, $p = .007$, $d = .44$, but not for the geometry and measurement items, $F(2, 161) = .94$, $p = .39$. Means and standard errors for both sets of items are presented in Table 2.

Graphic representations of the five number items are displayed in Appendix A. Three of the five number items (namely, Q.1; Q.4; and Q.5) highlight interactions between spatial reason rank and mode of delivery. In each instance, the high-ranked spatial students' performance on the PPT mode increased, and the mid-ranked spatial student's performance decreased.

Mode of Delivery	Low spatial	Mid-spatial	High spatial
	M (*S.E.)*	M (*S.E.*)	M (*S.E.*)
Geometry (7 items)			
iPad	1.77 (*.22*)	2.94 (*.23*)	4.03 (*.21*)
Paper and pencil	1.40 (*.34*)	2.26 (*.25*)	4.07 (*.35*)
Number (5 items)			
iPad	1.15 (*.15*)	1.94 (*.16*)	2.31 (*.14*)
Paper and pencil	1.20 (*.23*)	1.41 (*.17*)	2.93 (*.24*)

Table 2: Descriptive statistics for mathematics assessment items

CONCLUSION

The two major findings of the study are associated with (1) the influence of spatial reasoning on students' mathematics performance and (2) differences in students' performance in relation to mode of delivery, especially for number-concept items.

There were no performance differences in students' mathematics scores across mode of delivery. Although these results are inconclusive across the literature base, such findings support the large meta-analysis undertaken be Wang et al., (2008). There were substantial differences between student performances when the cohort was categorised according to spatial reasoning performance. There were significant differences between high-performing and mid-performing students, and between mid-performing and low-performing students. These results are consistent with a burgeoning literature base (including recent studies of Mix et al., 2016; Sella et al., 2016; Wei et al., 2012).

The second finding of the study highlighted an interaction effect between spatial reasoning and mode of delivery. To our knowledge, this is the first time such results have been reported in the literature. Students are more likely to use encoding strategies and heuristics in a PPT form than a CBT mode of delivery. By contrast, a CBT mode tends to encourage students to utilise visualisation strategies and mental imagery processes (Threlfall, Pool, Homer & Swinnerton, 2007). In addition, students are less likely to encode representations from a CBT mode, since the transition to another format presents different challenges in terms of re-representing information that needs to be decoded (Lowrie & Logan, 2015; Yahya & Hershkowitz, 2013). The multiple representations provide additional cognitive demands. Consequently, we hypothesised that students with lower levels of spatial reasoning would tend to be more successful in the PPT form, since they could draw on diagrams, encode information on the test booklet itself and generally monitor their thinking from one point of reference (Logan, 2015). Research suggests that interactions with technology in problem solving can take different forms (Jacinto & Carreira, 2013). It is important for future work to examine the different strategies employed when using CBT to ensure low spatial students are not disadvantaged by technology-based assessment.

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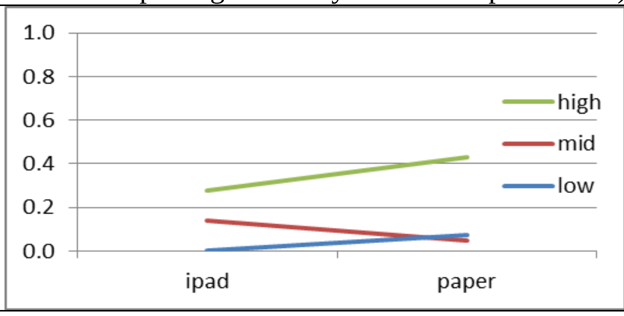
Appendix A

[Q.1] A baker made a total of 175 rolls on the weekend.

She made 15 **more** rolls on Saturday than on Sunday.

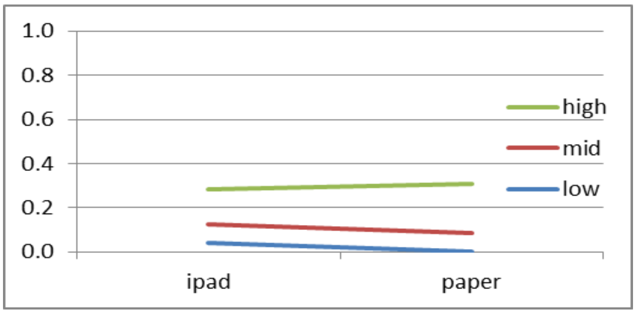
How many rolls were made on Sunday?



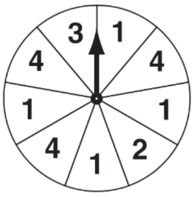
[Q.2] Ben has 2 identical pizzas.

He cuts one pizza equally into 4 large slices.
He then cuts the other pizza equally into 8 small slices.
A large slice weighs 32 grams more than a small slice.

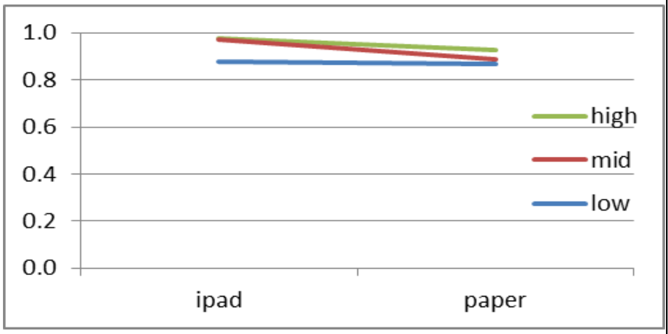
What is the mass of one whole pizza?



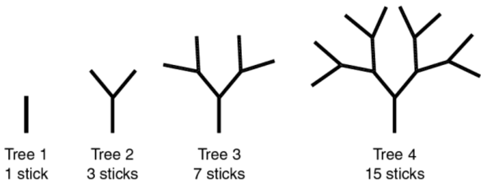
[Q.3] The spinner is used in a board game.



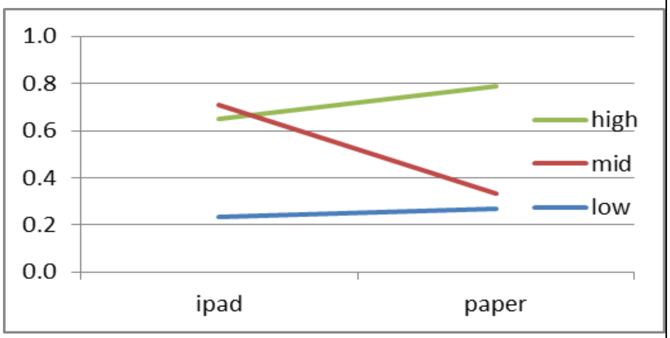
Sanjay spins the arrow.
On which number is the arrow most likely to stop?



[Q.4] Lucy made 4 tree designs using sticks.
There is a pattern in the way the trees grow.



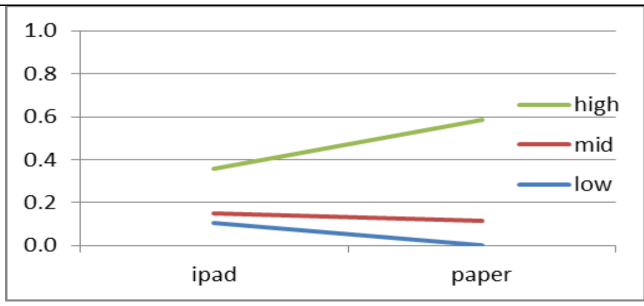
Lucy continued the pattern in the same way.
How many sticks will Tree 5 have?



[Q.5] The sum of the opposite faces of a standard six-sided dice is always 7.

Hannah rolls three dice.

The sum of the top faces is 11.
What is the sum of the three opposite faces?



MATHEMATICAL KNOWLEDGE AS MEMORIES OF MATHEMATICS

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I propose that an understanding of a mathematical concept is comprised of both a conceptual understanding of, and recollections of working with that concept. That is, a mathematical concept may not be immediately distilled in its abstract form from lived experience, didactical or otherwise, and this milleu is brought along in subsequent recollections of the concept. In an effort to balance pedagogical recommendations for increased conceptual teaching/understanding, I propose that memories of encountering a mathematical concept improve its utility in novel problem situations. I support this claim by drawing on the literature on episodic future thinking and on our developing understanding of how users of mathematics perform in authentic mathematical situations.

INTRODUCTION

Students receive a constant stream of experiences when learning mathematics — both external and mental, and taking the forms of sensory, cognitive, social, and otherwise — this stream of experience is interpreted and coloured by the student's current knowledge of and dispositions towards mathematics. From this they distil off mathematical knowledge. How exactly this process of mathematical knowledge formation takes place has long been a central subject of the cognitive psychology of mathematics learning. This article argues that mathematics education literature written from a cognitive psychology perspective has maintained too narrow a focus on the mathematical content of mathematics learning and ought to consider a student's broader remembered experience.

A discussion of the role of memory in mathematics learning is noticeably lacking in contemporary research despite advancements in memory research in psychology over the last two decades. This is not a new observation, having been identified in the mathematics education literature three decades ago (Byers & Erlwanger, 1985). This may be due in part to a conflation of memory and *memorizing*. I take the stand that, from a cognitive perspective, knowledge is a part of memory. Therefore, any serious discussion of the cognitive psychology of mathematical knowledge acquisition and development must consider the role of memory. My focus here is on the mathematical knowledge constucted in the mind of an individual. What led to this constructed knowledge, whether social interaction, bodily movements, individual reflection, or other situations, is not explicitly considered. I emphasise, however, that these modes of knowledge construction are not entirely disregarded; as will become clear, they form the substance of episodic memories.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 209-216. Singapore: PME.

The main impetus of this work is to gain insight into how a mathematics learner forms their idiosyncratic, personal understanding of mathematical concepts. That is, I am concerned with understanding the development of a student's *concept image* of a mathematical concept. Tall and Vinner define concept image as “the total cognitive structure associated with a mathematical concept” (Tall & Vinner, 1981). This is a powerful defintion, but has not realised its potential in the literature. I argue, and substantiate with examples in the Results section, that personal memories – so called *episodic memories* – of learning or working with the concept comprise a portion of this cognitive structure, especially during the initial stages of concept image formation. As such, these memories ought to receive greater attention in the mathematics education research literature. In the conclusion of this paper I conjecture that episodic memories associated with a student’s concept image are what may facilitate the utilization of the concept in novel situations. This conjecture meshes well with the current understanding of the role of episodic memories in planning for general, to-be-experienced events found in the psychology literature.

Episodic and Semantic Memory

The human memory system has been described by a number of qualitative models, the one used here – attibuted to Tulving (1983), having built on earlier work – has come to be widely accepted in the psychology community. In this model there is first a distinction between short- and long-term memory. Short-term memory — also often called *working memory* — lasts only less than a minute and is a key resource when interpreting a current experience. Long-term memory is partitioned into *implicit*, or *procedural*, and *explicit*, or *declarative*, memory systems. Implicit memory is the memory of rote tasks, those that are completed without conscious thought, such as walking, riding a bicycle, or teaching calculus. Declarative memories are those that can be explicitly recalled and stated. A further refinement of declarative memory into two qualitatively, and perhaps neurologically, different memory systems was proposed by Tulving (1983). *Semantic* memories are those that are not fixed to a particular individual's experience and can be known by anyone. That is, semantic memories are memories of shared, socially-available knowledge. *Episodic* memories are held by an individual and pertain to an event experienced by that individual. They are highly idiosyncratic, contain perceptual and temporal information, and can only be known by the individual. A memory of learning to ride a bicycle, for example, is episodic. Even though riding a bicycle is a fairly universal activity, each individual forms their own unique episodic memories of learning to do so. The analogy with constructs in the mathematics education literature is clear: episodic memories of a mathematical concept are a part of a student’s concept image of the underlying concept.

The purpose of this paper is to present evidence that students experience personal memories of mathematics when recalling mathematical concepts. These memories were often voiced freely, without prompting, by the student participants in this study, suggesting that the memories form a strong component of the students’ concept images. These personal memories, I argue, are valuable; in the wider field of the psychology of memory, episdoic memories are known to improve problem-solving ability (Taylor, et al., 1998; Schacter, 2012). Further, episodic memories may prove valuable to education theoreticians, insofar as they often reveal discrete moments in time in which a student's knowledge evolves.

METHODS

Student volunteers were recruited from two first-year mathematics courses, covering linear algebra and calculus, at a major, research-intensive New Zealand university. In total, 11 students volunteered; 9 from the general stream of the course, intended for science and business students, and 2 from the advanced stream for mathematics and science honours degree students.

Students were interviewed individually in two sessions. The results from the second set of interviews are reported in companion articles (Maciejewski & Barton, 2016; Maciejewski, Roberts, & Addis, 2016). In the first set of interviews, which forms the set of data used in the current study, each student was presented with a list of topics from their mathematics course and asked to rank them according to their own, personal familiarity with each; 1 for least familiar and 10 for most. The intention with the personal ranking of the topics was to have an increased diversity of episodic recollections. The researcher proceeded to ask the following set of questions for the topics ranked 1, 5, and 10.

For each topic, general questions about thoughts experienced by the participant when thinking about the given topic were asked first: "Describe the contents of your thoughts when thinking about [topic X] in as much detail as possible. Importantly, we are not looking for mathematical accuracy at this stage, we'd just like you to describe everything that comes to mind when you think about that topic. This may or may not include 1) mental imagery, 2) conceptual knowledge, and 3) personal memories."

More specific questions followed: "Can you describe how you came to understand (topic x) as you do now? Do you recall when you first encountered this topic? When was that? Can you describe that in detail? Do you understand this topic differently now than when you first encountered it? What led to this change? (If specific events are mentioned: Can you describe this event in detail?) Did these experiences come to mind in the first part of this study (even if you didn't talk about it)?"

Each of the interviews were recorded and subsequently transcribed. The transcriptions were analyzed from a *phenomenographical* perspective: a qualitative, interpretive methodology that seeks to understand individuals' idiosyncratic experiences of a common reality (Marton, 1986). The intention is to describe and categorize the range participants' experiences with, critically, equal weight given to each experience; no effort is made to identify which are the most prominent. In this way phenomenography is a powerful method for empirically uncovering possible lived experience.

The particular phenomenographical analysis is as follows. Student utterances were first categorized roughly as *episodic*, *semantic*, and *other*. Semantic utterances were those that contained "factual" information from an experience and no specific reference to the personal nature of the lived experience; semantic memories are abstract in the sense of not being tied to a particular experienced event. Semantic memories, and their associated utterances, are not the focus of this study and will fenceforth not be considered.

Episodic utterances were those that contained particular details specific to a lived experience. Explicit mention is made of some context of the experience. For example:

> Interviewer: When you think about these methods, do you have any images that come to mind?
>
> Participant [104]: Well, the image that comes to mind from that is the page in the course book and just the way the lecturer ... just explains it ... I can see the page and the way she laid it out and the method, how she goes step-by-step to solve it.

The participant recalls a specific instance of learning a mathematical method while sitting in class. They recall the lecturer talking and refering to the page in the textbook which, though not mentioned here, is displayed on a projector for the class.

Utterances classed as “other” were those that did not fit into either of the other two categories and often included clarifying statements, or comments unrelated to the questions.

The next stage of the analysis involved creating a categorization of the episodic utterances following phenomenographical methods (Marton, 1986). First, the collection of episodic utterances were read through and broad categories were formed. These are *temporal* – referencing the, at least relative, time an event occurred; *physical* – concerning the interaction of the participant and their environment; and *emotional* – the participant's recalled emotions during the experience. These three cateogories, though originating organically from the data, agree well with Tulving's description of episodic memories (Tulving, 1983).

Having formed these categories, each utterance was grouped into the category it best fit. Often an utterance contained elements of more than one category and was therefore duplicated in the corresponding categories. The categorisation part of the analysis was halted at this stage; further analysis appeared to lead to too fine categories which resisted succinct descriptors.

The subcategories were summarized by the researcher and combed for representative utterances. These are presented in the next section.

RESULTS

I present my analysis of the interview data according to the three identified categories – *temporal, physical,* and *emotional* – and present the corresponding subcategories. In each of the following subsections, I weigh the category topic against the students' self-reported familiarity of the mathematical topics. This allows for a richer analysis and informs conjectures about the role of memory in conceptual development we make in the subsequent section.

First, I comment on the prominence of these memories. All participants recalled episodic memories associated with at least one of the concepts. These recollections, notably the emotional recollections, were often offered by the participants without prompting. I take this as an indicator of how pervasive episodic memories are in students' mathematical conceptions.

Temporal Aspects

There were two types of responses in this category: 1) first encounter with the concept, 2) subsequent use of the concept. Participants generally recalled their first encounters with each concept. However, the vividness of these recollections varied with the participants' stated level of understanding. Less understood concepts tended to be associated with more vivid memories of a first encounter.

> I: Do you recall when you first encountered [least well understood concept]?
>
> S106: Yep. Because [the instructor's] accent just made it sound so cool, so, ya I do remember doing it and using matrices to solve something for it. But I don't remember what to do with it or anything.

More well-understood topics were associated with less vivid first-encounter memories.

> I: Do you remember when you first encountered [most well understood concept]?
>
> S106: I don't think so, actually ... I have a vague recollection of when I was supposed to have learned it.

Perhaps not surprisingly, more memories of subsequent work with the concept arose when the participants discussed more well-understood concepts. Of course, this may be because more well-understood concepts have been used more and so the participants have had greater opportunity to form memories of these concepts. This does not seem to be the case, however. Participants often mentioned using less understood concepts while solving problems, but these recollections were of "going through the motions" with the concept.

> S105: ...I came across a question in the assignment concerning Taylor Series, I had to answer it, so I kinda looked in the course book, I looked on the internet, asked my friend how to do it. He said it's kinda complicated so I looked on the internet and compared the answer, tried using one method to see if I got the answer...tried using another...just repeat that until I got the correct answer...I don't really like it, so after I answered that question, I sort of avoided Taylor Series.

Physical Aspects

Participants mentioned only a small number of physical aspects. Therefore, I present them here without grouping them further into larger categories. These are:

1) sitting in class:

> I: But what comes to mind when you think about [the concept]?
>
> S110: Our lecturer talking about it and me kinda not listening … it was a Friday morning, which isn't so conducive to learning. Um, and she was kinda talking about it and I wasn't listening 'cause they kept relating it to real life and I can't be bothered about real life … It's the last part of linear algebra as well and the test was coming up and I ignored it.

2) reading a book:

> S104: I kinda associate these concepts more with the page in the textbook...I remember pictures. And like, certain bits of the course book that I thought were more important. So, I can visually remember how the things look on the page.

3) attending a tutorial:

> S102: I remember … I was in the tutor room and one of the maths tutors actually thought I was stupid for asking such silly questions. And he came over and he actually explained step by step what he was doing for a couple of things, and he showed me a couple of little tricks … it clicked and then I could do those at least, and then it was just a case of applying that to everything else.

4) working with friends:

> I: Can you think of what led to that change of understanding for you?
>
> S100: When I was doing the Taylor Series with my friend … she was helping me … she had written a different number for the denominator, and I didn't understand why because from the example it seemed like what I was doing was correct, but then she explained to me that she was using factorial and I was 'timesing' the number, so that obviously made a difference.

and, 5) revising/reviewing/studying for an exam:

> I: Do you recall specific events that led to this change in understanding?
>
> S105: I think the assignment that we had and the test. 'cause I remember cramming for the test, 'cause I didn't understand, like, Taylor Series, really. I wasn't quite sure of them, so, studying for the test with friends and going over practice questions and searching examples online again.

Emotional Aspects

Though participants were not specificially asked about emotions they experienced when learning mathematics, all mentioned emotions in connection with at least one mathematical concept. These covered a wide range. Less understood concepts were associated with less favourable emotions, such as anxiety, trepidation, confusion.

> S101: It was just, like, pretty overwhelming...I really was not looking forward to learning something new.

Perhaps not surprisingly, more well-understood concepts had more favourable associated emotions: familiarity, enjoyment, happiness, and confidence, for example.

> S101: [I learned this concept on my own] because I was just bored...and I just was reading it and I kind of got it...and it felt...because, like, once you know how to do it, it becomes really easy, and it comes with practice, so yeah...I have an image of just sitting in class feeling pretty smug because I had already know how to do it...so yeah, it was a lot more pleasant than [the least understood concept]

DISCUSSION

The development of a student's mathematical knowledge may proceed episodically or semantically, or both. I propose that an exclusive accumulation of either one is necessarily undesirable. This is certainly the case for episodic memories; indeed, much of the research in mathematics education in the last half-century cautions against the accumulation of context-bound knowledge. Given that the same literature encourages the growth of context-transcending knowledge, it may seem an odd suggestion that students should not focus on acquiring semantic memories exclusively. I highlight a contemporary result from the psychology literature that may substantiate this claim, while being mindful of the need for further investigation in mathematics learning.

Contemporary reseach on the psychology of planning for to-be-experienced events indicates the humans often mentally simulate how an event might unfold and, in so doing, create episodic memories of the event before it takes place. This *episodic future thinking* (Atance & O'Neill, 2001) can facilitate planning and improve outcomes in general problem solving domains (Taylor, et al., 1998; Schacter, 2012). The key observation is that episodic future thinking relies on the same neurological regions and processes as are utilised in recalling episodic memories. Therefore, effective planning for to-be-experienced events is closely related, and influenced by, recollecting past events. Given the emphasis placed by some authors (Pólya, 1945; Schoenfeld, 1985) on *planning* in mathematical situations, it appears a worthwhile endeavour to investigate episodic future thinking in mathematical situations, its effect on planning, and how such future thinking is affected by episodic memories of mathematics.

Not much is known about episodic future thinking in specialized, context-specific domains, such as mathematics. There is emerging evidence that both mathematicians (Maciejewski & Barton, 2016) and mathematics students (Maciejewski, Roberts, & Addis, 2016) engage in episodic future thinking when solving mathematics problems. This is, of course, not the exclusive way of solving mathematics problems; some problems may invoke automaticity or an existing problem schema, or nothing at all (Maciejewski & Barton, 2016). What is needed is a better understanding of how a problem might relate to its solver and of which types of these relationships are likely to promote episodic future thinking. I conjecture that it is for those problem situations that are not too familiar to invoke a schema yet are familiar enough that progress can be made. This conjecture fits well with the literature on general problem solving behaviour, and further research in the context of mathematics is highly desirable.

One further point to be made is that episodic memories could act as signposts for the educational researcher. They signal discrete moments, locating the genesis of an idea or the punctuated evolution of understanding. Treating these memories as such may aid in deepening theoretical models of knowledge development.

CONCLUSION

This paper presents observations that personal experiences of mathematics pervade students' thoughts when recalling mathematical concepts. These episodic memories are a part of a student's understanding of the concept and present challenges and opportunities to educators. On the one hand, an exclusive reliance on episodic memories of using a concept could result in too-rigid knowledge without wide applicability. On the other, a diversity of rich episodic memories of mathematics may facilitate more effective planning in mathematical situations for a student. It is not clear to what extent educators ought to promote the formation of students' episodic memories of mathematics nor is it clear how best to do this. What is clear, however, is that students will continue to form episodic memories of mathematics whether or not they are attended to by educators. It is up to educational researchers to further investigate students' episodic memories of mathematics and ways in which they may be harnessed to aid students in reaching their potential.

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CONFIDENCE AND COMPREHENSION BUILDING PROCESSES REGARDING MATHEMATICAL CONTENT

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The authors contend, following from Damasio and Grounded Theory, that the convincement and comprehension experienced by a student (through distance interaction with her tutor) occur around a process that results from the confluence of sub-processes, where the student's states are a consequence of her actions -in which she mobilizes content (of linear equations) based on certain forms of sustentation-, actions that are in turn explained on the basis of previous confidence and comprehension, added to other tutor conditions.

BACKGROUND AND RESEARCH QUESTION

Several lines of research have highlighted the weight of convincement and security regarding mathematical facts that classroom agents experience during learning processes. Krummehuer (1995) for instance, highlighted convincement associated with argument backings, which he analyzed on the basis of the Toulmin Model. The omission of Modal Qualifiers Q is salient in his application, as Inglis, Mejia-Ramos & Simpson (2007) pointed out. The latter authors hold that one of the goals of instruction should be to develop the ability of students to "adequately" equate types of warrants with modal qualifiers Q (p.3). From that vantage point, in a detailed analysis of the confidence states of students, Foster (2016) suggests that students 'well calibrated' in a topic trust their correct answers and doubt their incorrect answers. In contrast, this paper, which follows some ideas of Damasio and takes direction from Grounded Theory, is interested not only in describing, but in explaining how the states of confidence, presumption or doubt regarding the mathematical content that a student experiences with her tutor are built.

THEORETICAL FRAMEWORK

In this study a functional analysis of the arguments proposed by a student is performed, using the guidance of the Toulmin Model (1984). In that model, an argument is composed of a Claim (C), data (D) supporting the claim, warrants (W) that bridge the logical gap between the data and the conclusion, a backing (B) that includes a general framework on which the argument is based, and the modal qualifiers (Q). Two components of B are identified in this work. First, one consisting of the resources upon which the argument is grounded; those having the characteristics of "invariant organizations of behavior" are referred to as "epistemic states" of sustentation by Rigo (2013). According to the latter author, while some grounds are rooted around mathematical reasons, such as instantiations of general rules, others are articulated based on extra-mathematical reasons, such as operational schemes that are activated when a rule is introduced without justification. Another component of B refers to mathematical content that is mobilized in the argument and

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in said epistemic states. The fragments chosen for this paper deal with solving linear equations, so in this paper one can distinguish content related to the transposition of terms and the properties of equality. The 3UV Model (Ursini, Escareño, Montes & Trigueros, 2005) is taken as the mathematically accepted version for solving equations in this paper. According to said model, two of the aspects that must be fulfilled when solving linear equations are: interpreting the symbolic variable that appears in an equation as a representation of specific values (I1) and determining the unknown amount that appears in equations or problems, by performing algebraic or arithmetic operations or both (I4). Comprehension is evaluated according to the above (Fig. 1) taking the mathematical standards of content, epistemic schemes and logical connections as reference.

C1	*Content (B)*	The content corresponds to the accepted mathematical meaning (specifically, with aspects I1 and I4 and the properties of transposition and equality).
C2	*Epistemic Schemes (B)*	Mathematical-type schemes are mobilized (e.g. generalized induction from specific instances).
C3	*Logical Connections (W)*	Warrants are conclusive (the allow for the steps to be made from the data to the conclusion).

Figure 1: Theoretical-methodological Instrument for Distinguishing Comprehension

Rigo & Martinez (in press) suggest that, associated with the epistemic schemes, as well as with the mathematical content mobilized within them, students experience states of confidence, presumption or doubt, which Rigo (2013) calls *"epistemic states of convincement" (esc)*. Toulmin et al (1984) introduce Modal Qualifiers Q in the functional analysis, and said qualifiers correspond precisely here to the *esc*. Following the view of Damasio (2010), Rigo & Martinez suggest that the *esc* are certain types of emotions and feelings. Damasio holds that emotions are a complex set of chemical and neural responses forming a distinctive pattern. These responses are produced by the brain when it detects an emotionally competent stimulus, that is, an object or event which presence, real or as a mental remembrance, triggers the emotion. Rigo and Martinez suggest that beliefs and epistemic schemes act as stimuli that activate the *esc*. For Damasio, emotions and feelings are changing phenomena that act as links triggering chain reactions, where one thing leads to another. Just as with *esc*, which are temporary and lead to new beliefs and epistemic schemes that operate as stimuli that could in turn modify the initial states, leading to a process of continuous transformation. The *esc*, in terms of emotion and feelings, are expressed more or less firmly through patterns of behavior and bodily expression, some of which are used here as criteria for identifying *esc* (see Figure 2).

Elements of Speech	The person uses emphasizer language that may reveal a higher degree of commitment to the truth of what he says. For instance, when the person uses the indicative form of verbs (e.g. "is").
Action	The subject's actions are consistent with his speech.
Determination	The person is determined and spontaneously manifests adherence to the truth of a mathematical sentence.
Interest	The participations of a person intervening with interest regarding a specific mathematical fact in a virtual forum are: *systematic* (i.e., the subject answers all questions addressed to him with the greatest detail possible), *informative* (his assertions, procedures and/or results are sufficiently informative), *clear and precise*.
Consistency	In his various interventions, the person shows consistency.

Figure 2: Theoretical-methodological Instrument for Distinguishing confidence

METHOD

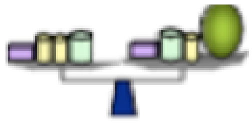
The research was undertaken on a distance program which objective was to strengthen the training of tutors instructing adults in algebra. The data used in the study was recorded in the Moodle platform for subsequent analysis, and is part of the interactions between a tutor (Author 1) and his students (Laura, specifically). The description and, especially, the explanation of the process of building epistemic states turns to the methodology of Grounded Theory (Corbin & Strauss, 2015), particularly in terms of the axial analysis and the process-based analysis. *Axial analysis* allows for a complex action to be deployed in categories related to *conditions* (reasons upon which the subject bases her actions-interactions, and that may be differentiated as those inherent to the subject, i.e. micro conditions, and those beyond the subject i.e. macro conditions); *actions-interactions* (people's responses to events or problematic situations) *and consequences* (anticipated or real results of the action-interaction). The *process* analysis is understood as the changes that take place in the action-interaction taken as the response to changes in the conditions. The set of conditions, actions-interactions and consequences are considered *sub-processes* (of a general iterated process), which consequences may act as conditions of a new sub-process, which combined with others lead to new actions, and so on and so forth.

ANALYSIS OF RESULTS PART ONE: SUB-PROCESSES

Following Damasio and the Grounded Theory, the process of building the *esc* is thought to be composed of a series of sub-processes in which we may identify: A set of *actions-interactions* performed by the student, described in the steps Pn (v. 1ª. Column Fig. below); the interest here is to contrast the additive-related steps with the division-related steps. Said actions-interactions are examined (v. 2ª. Column Figures) under the scope of Toulmin's functional analysis; the epistemic schemes (S) activated by the student in her participation stand out in B, as do the contents (C), while the W refer to logical connections (L) on which Laura bases her solutions. In keeping with the Theoretical Framework, Laura`s comprehension and *esc* are *consequences* generated during and that stem from her actions.

First Sub-process: First Participation of Laura and the Tutor

The interaction began with the tutor stating the following:



> We have certainly solved an equation, but have we reflected on its meaning and usefulness? … Let's do the following. If a can yellow container weighs 2 kg; a can green container weighs 4kg; a purple box weighs 1 kg. a) Is the scale balanced? b) What would make the scale imbalanced? c) If it were imbalanced, how could we restore the balance?
> d) Using the above, how would you explain the process for finding out the weight of the green sphere to a learner?

Here, the tutor highlighted the properties of equality. However, in the first participation (v. Fig. 3) Laura discarded the suggestion and followed her own strategy:

Laura's Participation	Functional Analysis	E.S. and Comprehension
P1: 9= 7+x; a) we perform the famous "solving", b) if 7 is being added, it moves over subtracting, and we get: c) x=9-7; **P2:** x=2.	**P1:** 9=7+x → x=9-7 **W:** **L:** If 7 is being added we move it over by subtracting **B:** **C:** Isolating (transposing) **S:** Operational and Explicitation Scheme	Operational comprehension and Confidence

Figure 3: Analysis of Laura`s first participation.

In P1 (Fig. 3), Laura *conclusively* (C3) converted an initial equation to an equivalent one by transposing terms (*as per* I1 and I4, C1), which she introduced without justification, thus activating operational-type epistemic *schemes* (C2). Moreover, the presence of an action scheme is salient (as we shall see, it acts as an epistemic scheme of sustentation); we will refer to it as an explicitation scheme, and it is of an extra-mathematical type; it entails making general properties involved in solving a task explicit by using natural language (v. P1a), using natural language to explain how these general properties apply to specific cases (v. P1b) and making the mathematical properties involved explicit by using mathematical language (v. P1c). It follows from the above that Laura reached an operational-type level of comprehension here (one in which the content and logical structure are consistent, but are based on operational schemes). Additionally, Laura showed confidence in the content, given the following: she used her conclusion in the following step of the resolution; she avoided use of mitigators when stating her plan; she showed determination in using a different strategy from the one suggested by her tutor; and she demonstrated interest in explaining it. The 2nd sub-process will show how this confidence and comprehension serve, together with the intervention of the tutor, as conditions for the subsequent actions.

Second Sub-process: Second Participation of Laura and the Tutor

Faced with Laura`s response, the tutor stated the following in an effort to activate within Laura mathematical epistemic schemes in her solution of equations:

> Once the equation is stated we often "transpose terms", but why does it work? To find out ... Click the link and assemble the equation on the scale. Describe each step of your solution. For example: -2x-4=4x-4; To isolate x I do the following: *T1*. I add 4 on both sides. We now have -

2x=4x; *T2*. I add 2x to both sides. Then we have: 0=6x; *T3*.- I divide both sides by 6. We get: 0=x. the solution is 0!!!

Figure 4 shows Laura's response to the Tutor's participation.

Laura's Participation	Functional Analysis	E.S. and Comprehension
P1: a) **Equality Property:** adding some number to both sides of an equality. -3x=x+8 b) when we add +6 c) -3x+6= x+8+6; **P2:** 3x = 14x **P3:** when we subtract -3x 3x-3x= 14x-3x; **P4:** x = 11x	**P1:** -3x=x+8 → -3x+6 = x+8+6 **W:** **L:** When we add +6 **B:** **C:** Property of equality not directed towards solving equations **S:** Explicitation and Mathematical Scheme.	Gestating confidence and comprehension in addition, and doubt and lack of comprehension in division.
P5:x/11=11x/11; **P6:**0=x So x= 0!!! **P7:** I found that addition and subtraction in the equality property always apply, however, there are cases in which multiplication and division don't apply.	**P5:** x=11x → x/11=11x/11 **W:** **L:** When we divide by 11 **B:** **C:** Property of equality directed towards solving equations **S:** Explicitation and Mathematical Scheme.	

Figure 4: Analysis of Laura`s second participation.

The presence of the explicitation epistemic scheme stands out anew (v. P1a-P1c), which Laura using it again to suggest an induction (mathematical scheme) or generalization of the additive property (V.P1a), that the tutor only mentioned in a specific instance (v. T1), and that further served for her as a guide to understand the new content the tutor introduced, which by the way acted as a macro condition. In this step P1, the student obtained equivalent equations (*content* consistent with I4, C1) *conclusively* (C3), but did not reduce them (inconsistent with I1, C1); this shows some degree of comprehension of the additive case of the property of equality (derived from the activation of mathematical and extra-mathematical epistemic *schemes*), but a lack of comprehension regarding its use in solving equations. In step P5 -related to the *content* associated with the property of dividing both sides by some number- the student once again turned to the explicitation *scheme*, although she only used it to state the property at stake in mathematical language, for which the tutor incidentally only described one way of using. Unlike in step P1, she justified step P5 with a property of equality that in addition to allowing her to obtain equivalent equations (*content* consistent with I4), it also enabled her to reduce the equation (consistent with I1). Here Laura shows comprehension of the division case of the equality property and comprehension of its application in solving equations. In terms of her epistemic states, the student showed greater confidence regarding the properties of equality in terms of addition and subtraction in her comment of step 7, which in her words "apply to all equations", but less confidence in the multiplicative properties of equality (in addition to a lack of understanding of the topic), which -the student says- "don't always apply". Future participations will show how her confidence regarding the additive properties of equality increases throughout the episode alongside increased comprehension, while doubts regarding the properties associated with division gain depth in parallel to her lack of comprehension.

Third Sub-process: Third Participation of Laura and the Tutor

In their third participation, the tutor asked the following questions:

> Hello Laura. Very well! You are talking about the properties of equality.; 1.- What are the properties of equality and what do they do for us?; 2.- Could you share with us what each of the properties of equality that you mention in your participation refer to?

Figure 5 shows Laura's responses.

Laura's Participation	Functional Analysis	E.S. and Comprehension
P1: a) The properties of equality are addition, subtraction, multiplication and/or division by a number on both sides of the equation, and they help us solve for a variable or an unknown quantity on one or both sides of the equation.	Does not apply.	Consolidation of confidence and comprehension in the addition property, and maintains doubt and lack of comprehension of the division property.
P2: a) I insist that the property of equality is to add and/or subtract a number to both sides of the equation, here is an example from the Advanced Operations book, page 120 7 + 8 = 10 + 5 b) when we add 12 to both sides we have: c) 12 + 7+ 8 = 10 + 5 + 12; 27 = 27 **P3:** b) and if we subtract we have: c) 7 + 8 - 12 = 10 + 5 – 12; 3 = 3 that's how they are applied.	**P2:** 7 + 8 = 10 + 5 → 12 + 7+ 8 = 10 + 5 + 12 **W:** **L:** when we add 12 **B:** **C:** Property of equality not directed towards solving equations **S:** Explicitation and Authority	

Figure 5: Analysis of Laura`s third participation.

In P1, Laura included division-related contents, to which she had associated doubt and lack of comprehension in her previous participation. In this third participation, she maintained those states, which can be seen by her brief description and lack of action regarding the content. In contrast, for the properties of equality referring to addition and subtraction (v, P2), with which she associated confidence in her previous participation, the student activated her explicitation *scheme*. For the addition property, she attached a scheme by authority to that explicitation scheme, by copying an example from a book. She then transferred that knowledge to the subtraction case, where she made her own example (with *content* C1 and *structure* C3 consistent). With these actions, Laura consolidated her confidence (revealed by her use of the emphasizer "I insist") and her comprehension of the properties of equality for addition and subtraction.

Fourth Sub-process: Fourth Participation of Laura and the Tutor

In this new intervention, the tutor asked the student to use the properties of equality to solve: -9x+4=12x-15. Fig. 6 analyzes Laura's response.

Laura's Participation	Functional Analysis	Epistemic State
P1: 9x+4=12x-15 9x+4-4=12x-15-4	**P1:** 9x+4= 12x-15 → 9x+4-4=12x-15-4 **W:** **L:** When we subtract 4 **B:** **C:** Property of equality directed towards solving equations **S:** Explicitation and Repetition.	Increase in confidence and comprehension in the addition property, and doubt and lack of comprehension of the division property.
P2:9x=12x-19		
P3:9x+3x=12x+3x-19		
P4: 6x=15x-19		
P5: 15x-6x=-19		
P6: 21x=-19		
P7: If I want them to become positive I must multiply by -1.		
P8: Wow! I did this one several times and always got the same answer, so I'm sure of my work.		

Figure 6: Analysis of Laura`s fourth participation.

Two matters are salient in this participation. For the additive case of the equality properties, about which Laura showed confidence in previous interventions, activation of the explicitation *scheme* is noteworthy, in mathematical language to obtain equivalent equations (*as per* I4, C1) and the value of the variable (*as per* I1, C1, which she only does for the independent term). In this sub-process, Laura showed confidence and increased comprehension, transferring additive properties to the context of solving equations. Possibly in order to increase her confidence, the student turned to the epistemic scheme of repetition; she recognizes that this scheme acted as a source of confidence in the result obtained when she states "I did it several times ... so I'm sure of my work" (P8). In contrast, of note is the fact that Laura performs increasingly fewer actions related to the properties of equality that refer to division, thus showing (as in previous cases) her doubt and lack of comprehension regarding that property.

Laura's Fifth Participation: Explicitation of Epistemic States

The comprehension and confidence in the additive properties of equality, which Laura consolidated in her fourth intervention, and her doubt and lack of comprehension regarding division also in said participation and of which she became aware to a certain extent, both served as conditions for her to finally make those states explicit in her fifth participation:

> [In the fourth participation] ... I was ... doubtful, as despite using several properties of equality (addition, subtraction, multiplication) I always got the same result, and that's why I'm sure of my work. I also kept thinking about whether I failed to apply another equality, do you think I did? Classmates, can you help us?

ANALYSIS OF RESULTS PART TWO: GENERAL PROCESS

Fig. 7 illustrates the configuration process for epistemic states that, imbricated in the evolution of comprehension, are broken down into iterated sub-processes. The consequences that stem from them (epistemic states and the student's comprehension) together with the conditions of the tutor act as new conditions that lead to actions-interactions of the student, in turn deriving into certain states of convincement and comprehension, reinforcing the former, maintaining or mitigating them. In this process, confidence in and comprehension of an addition topic coexist with doubt and lack of comprehension in division.

FINAL CONSIDERATIONS A NOTE ON DIDACTICS

Epistemic states make up a sophisticated caneva, which components constantly speak to each other so as to configure themselves one step at a time. This process is relevant to learning. According to Damasio and the research presented, stimuli that produce confidence or doubt are never definitive. That is to say, the connections between beliefs and grounds, on the one hand, and epistemic states, on the other, are not fixed. The connections are re-trainable and can be re-defined under certain conditions. One possible re-training strategy for epistemic states could consist of subjecting a student to continuous experiences, accompanied by the teacher and opportune guidance, so the student becomes aware of the confidence gained by mathematical assertions and the epistemic states that support them, and of the confidence that rules of inference activate in him, by reinforcing or re-directing his connections. Indeed, awareness of her epistemic states and the comprehension gained in her fifth participation were possibly the result of Laura's constant activation of her explicitation scheme, fostered deliberately by her tutor, which enabled her at that point to ask for help suited to her learning needs.

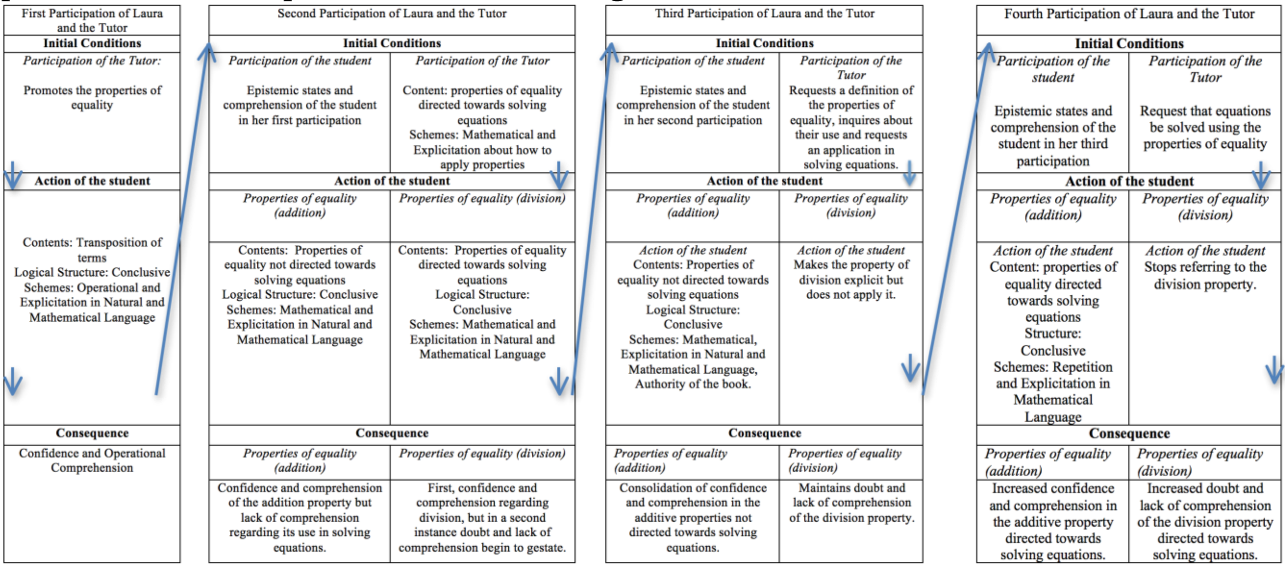


Figure 7: Process of Epistemic States and Comprehension

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STUDENTS' RELATING OF INTEGRALS OF FUNCTIONS OF TWO VARIABLES AND RIEMANN SUMS

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Action-Process-Object-Schema Theory (APOS) is used to study students' geometric understanding of partition of a rectangular domain and corresponding Riemann sum of an integral of a function of two variables. In this paper we mainly consider the most basic case of a partition, that consisting of a single rectangle (the domain itself). Semi-structured interviews were conducted with ten students who had just finished taking a traditional course in multivariable calculus. Results show that these students had many difficulties with even the most basic mental constructions needed to relate Riemann sum and double integral. This is an important observation since some of these mental constructions are commonly assumed to be obvious to students.

INTRODUCTION

There is not much research in the mathematics education literature dealing with the integral multivariable calculus. In one of the few papers we know about, McGee and Martínez-Planell (2014) report on the positive effect on student learning of consistently using a specific semiotic chain to guide instruction of integration of functions of two and three variables. This semiotic chain stressed the importance of conversion processes relating geometric and numeric representations, and also stressed the role of treatments relating different symbolic representations: finite extended sum, sum in sigma notation, the limit of that sum, and the double integral in standard notation. While the case of the multivariable integral calculus has been much neglected in the literature, the teaching and learning of integrals of functions of one variable has received considerably more attention. In particular, we base some ideas of our work on that of Sealy (2014) who proposed a framework for student understanding of Riemann sums and definite integrals consisting of an Orientation Pre-layer, and four other layers: Product, Summation, Limit, and Function. She found that the Orientation Pre-layer, in which students attend to the individual meaning of $f(x)$ and Δx, and the Product Layer, in which the product $f(x)\Delta x$ is given meaning, played a key role in allowing student understanding. This is an important observation given the apparent simplicity of the operations involved in forming such a product. In this article we build on the work of McGee et al. (2014) by stressing the importance of relating geometric, numeric, and contextual representations, as well as by initiating a detailed theoretical study that attempts to explore from the perspective of APOS

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Theory the reasons, in terms of mental constructions, for students' improved performance in the semiotic chain mediated instructional approach used by McGee et al (2014). We also base part of our work on Sealy's (2014) observations, which are now interpreted in terms of integrals of functions of two variables. Consistent with her underscoring the importance of the Orientation Pre-later and Product Layer, we restrict our attention on this article to what would be the equivalent ideas of attending to the individual meaning of $f(x,y)$, Δx, Δy and giving meaning to the product $f(x,y)\Delta x\Delta y$ relative to the corresponding double integral of *f*.

THEORETICAL FRAMEWORK

We use APOS Theory (Arnon, et al, 2014). In APOS, an Action is a transformation of a mathematical object that is perceived as external. It may be the rigid application of an explicit algorithm or of a memorized fact or procedure. The Action is external in the sense that it is relatively unconnected from other mathematical knowledge so that the individual will not be able to justify it. Repetition and reflection may allow an Action to be interiorized into a Process, where the Action is now perceived as internal, so that it may be imagined and reflected upon without having to explicitly perform the Action. The Process is perceived as internal in the sense that it has meaningful connections to other mathematical knowledge so that the individual will be able to justify it. When an individual feels the need to apply Actions on a Process and is able to apply or imagine applying such Actions, then one says the Process has been encapsulated into an Object. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed schema related to a specific mathematical notion. In this article we focus on the interiorization of Actions into Processes and do not directly consider the mental constructions involved in constructing an Object conception or Schema development.

When applying APOS it is necessary to use a genetic decomposition (GD). This is a conjecture of mental constructions students may do in order to understand a specific mathematical topic. The GD is based on the mathematics itself, teaching experience, and any previous data or research study. A GD is not meant to be unique. It is used to analyse students' mental constructions when solving mathematical problems on the specific topic of interest. This potentially results in refinements to the GD to improve its capacity to predict student behaviour and guide instruction. In the following paragraphs we include the portion of the GD which was tested in this study with student interviews.

Recognition of rectangle and function

Actions are performed on a given function in any representation with domain restricted to a rectangle, to produce the geometric representation of the restricted domain either as a subset of the Cartesian plane or as a subset of 3D space (identifying (x,y) with $(x,y,0)$). Actions are performed on the same function to obtain values of the function on the given domain and to represent them in the 3D space. These actions are interiorized into a Process to represent the graph of the function over the given rectangle together with the rectangle so that the student can imagine the relation between function and rectangle as a graph in space over a rectangle in the *xy* plane.

Forming one term of a Riemann sum

Actions of evaluating the given function of two variables at a specific point of a given sub-rectangle of its domain, multiplying it by the length and width of the rectangle to obtain a product $f(a,b)\Delta x\Delta y$ are done. These actions are interiorized into a Process which can be coordinated with conversion Processes between different representations of function, rectangle, and given point. The resulting Process allow representing the product in space as a rectangular prism and also recognizes the units of this product when necessary.

Recognition of underestimate, overestimate, and exact value

Given a continuous function in different representations defined on a rectangle, with the function simple enough so that its maximum and minimum values on the rectangle may be quickly recognized without doing any explicit computation, the Actions of obtaining an overestimate and an underestimate of the product $f(a,b)\Delta x\Delta y$ are done. These Actions are interiorized into a Process when these estimations are calculated for the same function on different rectangles or for different functions in different rectangles. Actions are performed to change the chosen point in order to construct a rectangular prism that better approximates a given exact value of the volume between surface and rectangle. These Actions are interiorized into the Process that makes it possible to recognize that for a continuous function defined on a rectangle, there is a point somewhere on the rectangle that will produce the exact value of volume between surface and rectangle or of the quantity being computed.

Forming a partition and computing a value in each sub-rectangle of the partition

Given two small specific positive integers (not in symbolic form, but actual numbers), *n* and *m*, the Action of subdividing given intervals [a,b] and [c,d] into subintervals of equal length $\Delta x = (b-a)/n$ and $\Delta y = (d-c)/m$ both numerically and geometrically in order to obtain a subdivision of the rectangle $[a,b]\times[c,d]$. These Actions are interiorized into a Process of subdivision of rectangles so that the student can imagine how for any given positive integers, *n* and *m*, the respective subdivisions of [*a*,*b*] and [*c*,*d*] give rise to a subdivision of the rectangle $[a,b]\times[c,d]$ without having to explicitly do so for any other specific values of *n* and *m*. The Action of choosing a prescribed point (x_i, y_j) on each sub-rectangle of the given partition and producing the products $f(x_i, y_j)\Delta x\Delta y$ is repeated for different points and the result is interpreted numerically (as a collection of numbers) and geometrically (as a set of rectangular figures in space), and verbally (interpreting the products in terms of its units). These Actions are interiorized into a Process that enables imagining forming such products for the collection of sub-rectangles of any given partitioned rectangle. At this point the student might not think of adding the products over all rectangles in the partition.

METHOD

The above portion of the GD for integral of functions of two variables was used to prepare an interview instrument to test it. The instrument was used in semi-structured interviews with 10 students taking a multivariable calculus course in a public university in Puerto Rico. The interviews took place in the last week of a semester course. The students were chosen by their professor so that 4 were over average, 3 average, and 3 under average, as defined by the course average grade they had before presenting the final exam. The professor had more than 25 years of experience, and had taught the course repeatedly over the years. The course was "traditional" in the sense that most of the classroom time was dedicated to lecturing, and the textbook (Stewart, 2012) and syllabus were followed very closely. In particular, since the professor was not one of the researchers of this article, no classroom or homework activity was explicitly guided by the GD. Thus APOS is used in this paper to describe the mental constructions demonstrated by a group of students who completed a traditional lecture/recitation course, as discussed in Arnon et al. (2014, p. 106).

The interviews lasted an average of 46 minutes. They were recorded, transcribed, individually analysed by the researchers, and differences were negotiated. Each interview problem was also graded on a 0 to 2 scale.

The interview problems are summarized below.

(1a) The following is the complete graph of function $z = f(x,y)$. Represent the domain of *f* in the figure [See Figure 1; the graph appeared in all parts of problem 1, except 1b]. (1b) Let $g(x,y) = x^2 + y$ be a function with domain restricted to $0 \le x \le 2$ and $1 \le y \le 2$. Represent the domain of the function in three-dimensional space. (1c) The above functions *f* and *g* are the same. If $\Delta x = 2$ and $\Delta y = 1$, what is the numerical value of $f(0,1)\Delta x\Delta y$ and what does it represent geometrically? (1d) Let $\Delta x = 2$ and $\Delta y = 1$. How does $f(0,1)\Delta x\Delta y$ compare with $\iint_D f(x,y)dA$? [No numerical computations are needed in parts d, e, f, and g] (1e) How does $f(2,2)\Delta x\Delta y$ compare with $\iint_D f(x,y)dA$? (1f) Is there any point (*a*,*b*) in the domain *D* of *f* such that $f(a,b)\Delta x\Delta y$ is equal to $\iint_D f(x,y)dA$? (1g) Let $\Delta x = 2$ and $\Delta y = 1/2$. Consider the Riemann sum *f* (0,1)ΔxΔy + *f* (0,1.5)ΔxΔy + *f* (1,1)ΔxΔy + *f* (1,1.5)ΔxΔy of the integral $\iint_D f(x,y)dA$. What does the Riemann sum represent geometrically and how does its value compare to that of $\iint_D f(x,y)dA$?

(2) Let *p* be a function defined on a region *D* of the plane. Suppose that *D* models a thin plate whose surface has a contaminant. If *x* and *y* are measured in centimetres and $p(x,y)$ is the density of the contaminant in units of *mg/cm*^{2}, what does a term of the form $p(0,1)\Delta x\Delta y$ in a Riemann sum represent and what does $\iint_D p(x,y)dA$ represent?

RESULTS

Students showing an action conception

APOS states that the overall tendency of students when dealing with different problem situations involving a specific concept, will be different, depending on whether the student thinks of the concept as an Action, a Process, or an Object. Nine of the ten interviewed students showed an Action conception of Riemann sum and double integral over a rectangle. We include below excerpts showing one representative student's responses and how his difficulties can be explained in terms of the lack of interiorization of some of the Processes conjectured in the preliminary GD. In this short article we are constrained to showing a few sample transcripts of one specific student, without examining the other students' overall tendency and specific difficulties. The reader should keep in mind that all students showing behaviour consistent with an Action conception did so in multiple instances, and that each of the difficulties exemplified below were shared by several students.

As observed in Martínez-Planell and Trigueros (2012), students tend to have difficulty understanding the notion of domain of functions of two variables, and particularly, restricted domains. This was observed with Luis and 8 of the 10 interviewed students.

> Interviewer: (in 1a) so you used set notation [correctly] to tell me what is the domain. Could you represent it as part of the figure? [See Figure 1]
>
> Luis: I can tell you what the domain is but if I don't have a function I don't think I can tell you the exact point where each of the points in the graph is. I can tell you that in *x2* and *y1* it will be this point here [darkening (2,1, *f*(2,1)) in Figure 1], and in *x0* and *y1* it will be this point here [darkening (0,1, *f*(0,1))].
>
> Interviewer: So, is the graph part of the domain?
>
> Luis: No, the domain is obtained from the graph. I can obtain the domain having the function but to do so I have to define the function.
>
> Interviewer: Is there any way to represent that set of ordered pairs you gave me as part of that figure?
>
> Luis: Well, having these two limits the only thing I can tell you is that the function is enclosed by these two limits [referring to $0 \leq x \leq 2$ and $1 \leq y \leq 2$] but I can't tell you which is the function because if I'm going to graph these two domains I'd be left with a rectangle and the function is inside this rectangle, so it would be... [See Figure 2]. The only thing this tells me is that it is enclosed by all this and I'm only talking about the *xy* plane, I'd be missing *z*, that is, with *x*,*y* I can get *z*.
>
> Interviewer: So the domain, is it only *x* and *y* or may it also include *z*?
>
> Luis: The domain may include the *z*.

In the above excerpt, Luis gives evidence of not having interiorized a Process to recognize the relationship between domain rectangle and function. He seems to be aware that the domain is formally a set of ordered pairs but also seems to believe that each ordered pair (*x*,*y*) in the domain is represented in three-dimensional space by its corresponding point on the graph rather than by (x,y,0). As could be expected, he went on to show difficulty representing terms of a Riemann sum and partitions graphically.

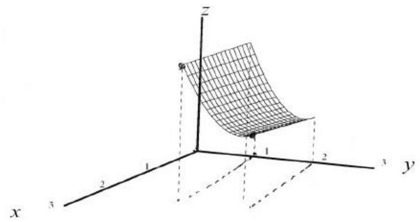
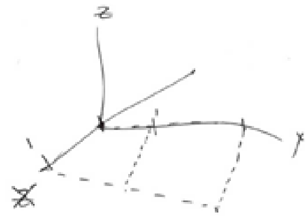
Figure for Problem 1a and Luis' drawing	Luis' drawing for problem 1a
	

Figure 1: Luis' work and drawings for Problem 1a

Interviewer: does the product $f(0,1)\Delta x \Delta y$ represent some geometric entity?

Luis: I think of $\Delta x \Delta y$ as an area, as the area at that point.

Interviewer: And $f(0,1)$, how do you think of it geometrically?

Luis: I see it as a *z*... as see it as the height in that function... at that same point, what is the height of that function...

Interviewer: So you just referred to $\Delta x \Delta y$ as an area and to $f(0,1)$ as a height, then when one multiplies area times height...

Luis: That is what I don't know

The above excerpt exemplifies that a student might know the meaning of the individual components of $f(0,1)\Delta x \Delta y$ but might not be able to do the Action of putting them together to form one term of a Riemann sum and to interpret it as the volume of a rectangular prism, as conjectured in the genetic decomposition. So in problems 1d, 1e, 1f he could not do the necessary Process to relate the given term to an underestimate, overestimate or exact value of the integral, another of the Processes contemplated in the GD. Luis went on to state that the double integral represented the surface area, and then latter on to argue that it represented the volume of the surface itself. He was also unable to make sense of the four-term Riemann sum in problem 1g nor relate it to the integral. In problem 2, where the function was given as a rate, Luis also showed not to have interiorized the Process involved in forming a term of the Riemann sum. When analysing the units of $p(0,1)\Delta x \Delta y$ he stated:

Luis: ... it will end up in milligrams since I have cm^2... which is Δx times Δy ... I have the function... density of the contaminant in units milligrams per cm^2... this will give me a constant in milligrams.

Interviewer: ... What does the double integral of $p(x, y)dA$ represent?

Luis: It would be the total density of the figure.

Interviewer: What units would you get from computing the double integral?

Luis: What it measures is volume.

It can be observed that Luis responds according to memorized scripts, consistent with an Action conception. He did not relate Riemann sum to the integral.

Some of the nine students who were able only to perform Actions considered *f*(0,1) as "a point" but would not relate it to a length of a line segment. Some thought of it as the thickness of the surface and argued that the double integral would compute the volume of the surface itself. Some of these students did not show flexibility in the use of variables and still thought of Δx and Δy as related to a rate of change, derivative or slope. Even when shown the geometric meaning of $f(0,1)\Delta x\Delta y$ with a drawing of a rectangular prism in space most still had difficulty making sense of the partition and different terms of the Riemann sum in problem 1g. These results evidence the importance of constructing the Processes conjectured in the GD so that students can overcome these difficulties.

Students showing a process conception

Only one of the ten interviewed students, Fermin, showed behaviour consistent with a Process conception of Riemann sum and its relation to the integral of a function of two variables over a rectangle. In problems 1a and 1b he gave evidence of having constructed a Process to imagine function, rectangle and their relation (See Figure 2). In problem 1c he also gave evidence of doing a Process to put together the terms *f*(0,1), Δx, Δy and interpret the resulting product as the volume of a box (See Figure 2).

Drawing for Problem 1a	Drawing for problem 1c	Drawing for problem 1g

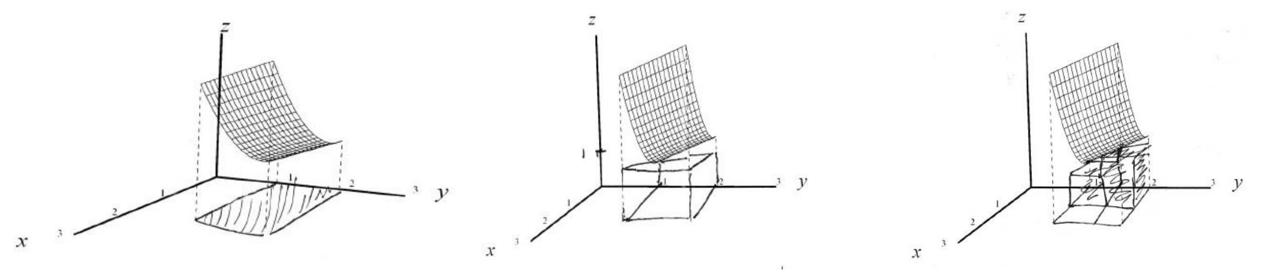


Figure 2: Fermin's work and drawings for Problems 1a, 1c, and 1g

In problems 1d, 1e, and 1f he was able to argue the cases when he obtained an underestimate, an overestimate, and, with some help, the exact value. Also, in problem 1g he gave evidence of understanding the corresponding partition and the relation between the Riemann sum and the double integral.

> Fermin: ... the change in *y* is 1/2 the change in *x* is 1. Let me draw region *D* on the *xy* plane, and one took as sample points, (0,1) which would be here, (0,1.5)... [He draws two of the boxes; See Figure 2] one would then take the four cubes under the function. How does this value compare? It would be a more approximate value... to the value of the double integral of *f*(*x*,*y*)... it would be a smaller value.

DISCUSSION AND SUMMARY

Although one would think that recognizing the relationship between a rectangle, the sample point, and the function in a partition corresponding to a Riemann sum of a double integral of a function of two variables, putting these quantities together in a product, and representing that product as the volume of a corresponding box in three-dimensional space, is a simple idea, readily understood by most students, that seems not to be the case. This stresses the fact that these mental constructions must appear in the GD of integrals of functions of two variables and should be considered in instruction. Indeed, the interviews were scored and of the possible 20 points that could be obtained, the best performing student, Fermin, obtained a score of 17; the next best performing students obtained scores of 7.5, 7, and 6.5 respectively. The average for the remaining six students was 1.6 out of 20 possible points. Only one student showed behaviour consistent with a Process conception and the nine other students gave evidence of being limited to an Action conception, applying memorized facts, not being able to imagine Actions without explicit computation, rigidly applying algorithms, and not relating symbolic and geometric representations. For the most part, these students did not recognize the geometric interpretation of a Riemann sum, the units and the meaning of a Riemann sum when the function is given as a rate, or its relation with the corresponding double integral. Students also showed not to have constructed Processes for recognizing underestimates, overestimates, and the possibility of choosing a sample point to obtain the exact value of a double integral over a rectangle. Also, most students did not manage to discuss the partition in problem 1g. Hence this demonstrated that all portions of the GD are needed in the construction of the integral of two variable functions. The case of Fermin suggests that it is possible to construct a Process conception of Riemann sums and their relation to a double integral in a traditional classroom. However, this study's results also suggest that the traditional mode of instruction may play an important role in limiting the possibility of students' understanding. An investigation of this aspect is left for future studies.

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"EXPLANATORY" TALK IN MATHEMATICS RESEARCH PAPERS

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In this paper we explore the ways in which mathematicians talk about explanation in their research papers. We analyze the use of the words explain/explanation (and various related words) in a large corpus of text containing research papers in both mathematics and physical sciences. We found that mathematicians do not frequently use this family of words and that their use is considerably more prevalent in physics papers than in mathematics papers. In particular, we found that physicists talk about explaining why disproportionately more often than mathematicians. We discuss some possible accounts for these differences.

INTRODUCTION

The notion of explanation in mathematics has received a lot of attention in both mathematics education and the philosophy of mathematics. In mathematics education, scholars have been particularly interested in proofs that explain mathematical theorems (i.e. proofs that provide an insight into why a mathematical claim is true) and their role in the mathematics classroom (e.g. Hanna, 1990). Philosophers of mathematics have discussed at length possible equivalents for mathematics of existing philosophical theories of scientific explanation (e.g. Steiner, 1978). Some of these discussions bring to bear the extent to which explanation is relevant to the actual practice of mathematicians and often cite individual mathematicians' views on mathematical explanation (more often than not that mathematician seems to be Henri Poincaré, Paul Halmos, or William Thurston). In this report we explore the extent to which mathematicians talk about explanation in their research papers, and the ways in which they do so.

LITERATURE REVIEW

In an influential paper in mathematics education, de Villiers (1990) argued that proof serves several different roles in mathematics, that proof is not only used in mathematics as a way to verify results, to provide conviction of the truth of those results (see also Bell, 1976). One of those other functions of proof was to *explain* mathematical results, to provide an insight or understanding into why these results were true, as opposed to just evidence in support of that result. Hanna (1990) made a similar distinction in the context of the teaching and learning of mathematics, discussing the idea that certain proofs fulfilled this explanatory function better than others, to the point that among the set of all proofs one could identify proofs that explain why a theorem is true, while others simply demonstrate that a theorem is true.

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Mathematics educators have generally suggested that in the mathematics classroom, mathematical explanation should be an important, if not the primary role of proof (de Villiers, 1990; Hanna, 1990; Hersh, 1993).

This distinction between proofs that explain and proofs that demonstrate has a longer history in the philosophy of mathematics. Steiner (1978) put forward a model of mathematical explanation, arguing that a mathematical proof could be better defined in terms of what he called *a characterizing property* of a concept in the theorem, as opposed to other alternative defining characteristics such as the abstractness or the generality of the proof. Steiner's top-down approach to modeling mathematical explanation by providing a general definition of explanatory proof (and thus creating an absolute distinction between explanatory and non-explanatory proofs) has been criticized by other philosophers of mathematics. In particular, Hafner and Mancosu (2005) argued that ascribing explanatoriness to specific proofs should be done based on practicing mathematicians' evaluations, not philosophers' own intuitions (such as Steiner's). The extent to which practicing mathematicians not only agree with philosophers' characterization of mathematical explanation, but simply talk about explanation in their practice plays an important role in the general argument for the *existence* of explanation in mathematics (which not all philosophers believe). As such, it is not uncommon for a discussion of mathematical explanation to mention how much mathematicians talk about it. For example, Steiner claimed that "mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (p.135), and Hafner and Mancosu (2005) supported their claim that mathematicians seek and value explanation in mathematics by presenting several examples of what they called *"explanatory" talk* in mathematical practice: passages of research mathematics papers in which the authors explicitly discuss the role of explanation in their own work. However, we do not currently have empirical evidence, other than these small selections of introspective accounts, about the extent to which talk about mathematical explanation is part of mathematical discourse. We believe one of the reasons this has not been studied at a larger scale may be methodological: a researcher would have to be able to process and analyze a large number of mathematical research papers or conversations among mathematicians.

One method of studying mathematical discourse at such a scale is to use the techniques of corpus linguistics, a branch of linguistics that statistically investigates large collections of naturally occurring text, known as corpora. Methods developed by corpus linguists can be used to investigate many different types of linguistic questions. Here, we report a study that employs some of these techniques to address the following questions: to what extent do mathematicians discuss explanation in their research papers, how does it compare to the extent to which they discuss other important related notions (such as *showing* or *proving* given mathematical results), and how does it compare to discussions about explanation in other types of scientific discourse?

THEORETICAL PERSPECTIVE

Discussions about mathematical explanation tend to differentiate between explanations of other mathematics (i.e. mathematics X explains mathematics Y, or X is an explanatory proof of theorem Y), and explanations of physical phenomena (i.e. mathematics X explains physical phenomenon Y). Colyvan (2011) refers to these two types of explanation as *intra-mathematical* and *extra-mathematical*, respectively. Here we focus on intra-mathematical explanations.

Hafner and Mancosu (2005) further differentiated between two uses of intra-mathematical explanations: those that are "instructions" on how to master the tools of the trade (as in explaining how to employ a certain mathematical technique), and those that "call for an account of the mathematical facts themselves, the reason why" (p. 217). While Hafner and Mancosu considered the latter to be a "deeper" use of mathematical explanation, which is also the focus of the larger philosophical discussion around explanatory proofs, others have emphasized the importance of the former type of explanation in mathematical practice. For instance, Rav (1999) insisted that one of the main reasons mathematicians read proofs is because of all the mathematical know-how embedded in them, emphasizing the mathematical methodologies and problem solving strategies/techniques contained in proofs. According to Rav, "proofs are for the mathematician what experimental procedures are for the experimental scientist: in studying them one learns of new ideas, new concepts, new strategies—devices which can be assimilated for one's own research and be further developed." (p. 20) Indeed, there is empirical evidence (from both small scale interview studies and large scale surveys) that mathematicians maintain that one of the main reasons they read proofs is to gain insights into how they can solve problems that they are working on (Weber & Mejía-Ramos, 2011, Mejía-Ramos & Weber, 2014).

An interesting question related to the specific ways in which mathematicians talk about explanation in their papers, relates to these two types of *"explanatory" talk*: to what extent do mathematicians discuss explanations of why a certain mathematical statement is true, compared to their talk about explanations of how to do something in mathematics?

METHODS

One of the main ways in which mathematicians around the world communicate about mathematics is through research papers stored in the ArXiv. The ArXiv is an online repository of electronic preprints of scientific papers in the fields of mathematics, physics, astronomy, computer science, quantitative biology, quantitative finance, and statistics. These papers constitute a large corpus of scientific text that can be used to analyze mathematical discourse.

We downloaded the bulk source files (mostly TeX/LaTeX) and converted the source code to plain text, which we could then analyze using standard software packages for corpus analysis. We then sorted these articles based on their subject classification

(Alcock et al., 2017, discussed the details about the processing of these source files). All analyses reported here are based on a proper subset of this corpus, containing all mathematics and physics articles (based on their primary subject classification) uploaded in the first four months of 2009. This left us with 5087 mathematics papers (30,892,695 words) and 11787 physics papers (58,859,660 words).

RESULTS

Frequency of explicit *"explanatory" talk* in mathematics papers

Table 1 shows the frequencies of all words linguistically related to the word *explain* (henceforth *explain-words*) in our corpus of 5087 mathematics and 11787 physics papers. Explain-words showed up 4871 times in the set of mathematics papers, or approximately once every 1.04 papers. While this certainly provides an existence proof of explicit *"explanatory" talk* in this corpus, it is not very surprising (it would very rare if no word based on the word explain showed up in these many mathematics papers). In comparison, explain-words showed up 21305 times in the set of physics papers, approximately once every 0.55 papers, or about twice as often as they showed up in the mathematics papers. In order to get a sense of the extent to which these frequencies were high or low in this type of mathematical discourse, we compared them against the frequencies of words related to other important mathematical activities.

Explain-word	Mathematics	Physics
explain	1827	7768
explained	1690	6513
explanation	498	3564
explains	484	1601
explaining	175	914
explanations	119	675
explanatory	51	62
unexplained	22	177
unexplainable	4	8
explainable	1	23
Total	4871	21305

Table 1: Frequency of words related to *explaining* appearing in the mathematics and physics papers

Table 2 presents the frequencies of words linguistically related to the notions of showing, solving, and proving, which were chosen based on their relevance in mathematical explanation. Measured against these other frequencies, mathematicians used explain-words rather infrequently. Indeed, mathematicians used explain-words in their papers approximately 11 times less frequently than show-words or solve-words and nearly 23 times less often than prove-words.

Show-word	Frequency
show	31691
shows	12890
shown	10235
showed	2414
showing	2129
Total	59359

Solve-word	Frequency
solution	25845
solutions	15956
solve	2204
solving	1717
solvable	1618
solved	1342
solves	1071
solvability	429
solver	145
unsolved	95
solvers	56
nonsolvable	39
unsolvable	32
cosolvable	29
equisolvable	18
unsolvability	12
Total	50608

Prove-word	Frequency
proof	56452
prove	29481
proved	12842
proves	4160
proofs	3892
proving	2661
proven	1902
provable	159
reprove	58
disprove	43
provability	29
reproved	29
disproved	17
unprovable	13
unproven	12
reproving	11
disproving	10
reproves	10
prover	7
unproved	7
subproof	5
disproof	4
Total	111804

Table 2: Frequencies of words related to *showing*, *solving*, and *proving* appearing in the mathematics papers

Finally, the search for explain-words may be thought of as requiring an extremely explicit discussion of explanation, one that would leave unnoticed a significant amount of the *"explanatory" talk* in these papers. Hafner and Mancosu (2005) offered a list of eight expressions that they had found to be commonly used in the mathematics and philosophy of mathematics literature to describe the search for explanations. Table 3 lists these expressions along with the specific concordance search we made to investigate their prevalence in both the mathematics and physics papers. We note that the total number of occurrences of these expressions is only about 10% of the total amount of explain-words in each set of papers (with disproportionately more occurrences of these expressions in the physics papers than the mathematics ones) and thus this analysis does not affect the finding made by only investigating the appearance of explain-words.

Alternative expression	Concordance search	Mathematics	Physics
"the deep reasons"	deep* reason*	5	16
"an understanding of the essence"	understand* the essence	0	5
"a better understanding"	better understand*	161	767
"a satisfying reason"	satisfy* reason	0	0
"the reason why"	reason* why	312	924
"the true reason"	true reason	3	1
"an account of the fact"	an account of the fact	0	0
"the causes of"	cause* of	16	609
	Total	497	2322

Table 3: Frequencies of alternative expressions of related to *"explanatory" talk*

Explaining why vs. explaining how

In order to investigate mathematicians' discussion of *explanations of why* a certain mathematical statement is true (Hafner and Mancosu's "deep" explanation), in comparison to their talk about *explanations of how* to do something in mathematics (related to Rav's notion of mathematical know-how), we created a concordance of the corpus of papers and identified every instance an explain-word had been immediately followed by the words why or how (e.g. unexplained why, explanation how). We did this by searching the concordance for *expla* why and *expla* how, and checking that all results were indeed uses of explain-words. We then repeated the process with the corpus of physics papers. Table 4 shows there is a clear difference in the ways that explain-words are used in the mathematics and the physics papers.

We note that when taken together the total of *expla*-why and *expla*-how expressions were roughly as common in math papers as they were in physics papers, with approximately one of these expressions showing up every 7-9 papers in the corresponding set, and also a relatively small subset of the wider use of explain-words (roughly 14% and 6% of explain-word usage in mathematics and physics, respectively). However, the distribution of these two different types of expressions in the two sets of papers was significantly different (Fisher's exact test, $p < .001$), with mathematicians using nearly twice as many *expla*-how expressions than *expla*-why expressions, and physicists on the other hand using a little under three times as many *expla*-why expressions than *expla*-how expressions.

	Mathematics	Physics
expla why	247	952
expla how	458	353
Total	705	1305

Table 4: Frequencies of explain-words immediately followed by the words *why* or *how* in the mathematics and physics research papers

DISCUSSION

Our analysis of *"explanatory" talk* in a large sample of mathematics papers does not offer support for a claim often made in the philosophy of mathematics: that this type of talk is prevalent in mathematical discourse. When compared to explicit discussion of other related mathematical practices (showing results, solving problems, and proving theorems), mathematicians do not seem to discuss explanation nearly as much. Furthermore, when compared to another scientific discourse, we found that mathematical discourse contains only a fraction of *"explanatory" talk* as research papers in physics. Indeed, we believe these findings suggest that the prevalence of *"explanatory" talk* in mathematical discourse has been widely exaggerated.

Furthermore, by analyzing the frequency with which variations of the expressions *explain why* and *explain how* occur in mathematics and physics research papers, we found that, to the extent to which they engage in *"explanatory" talk*, mathematicians seem to be much more interested in discussing explanations of how to do something in mathematics, than in explanations of why things are the way they are in mathematics. In physics we found the situation to be the opposite. This is particularly interesting given mathematics educators' and philosophers' of mathematics preoccupation with the type of intra-mathematical explanations of the form X explains why Y (where X and Y are mathematical assertions), and particularly with the notion of explanatory proofs (in which proof X explains why theorem Y is true). This focus may have been inherited from the more traditional study of the notion of scientific explanation, which is not only naturally concerned with this type of explanations (the desire to explain the real world is full of why-questions), but according to our findings may also be more commonly discussed in scientific discourse in terms of answers to why-questions. However, our findings suggest that this focus may also be misguided for those interested in studying the notion of mathematical explanation as it more commonly occurs in the discourse of professional mathematicians. Indeed, as suggested by Rav (1999), it seems that when it comes to proofs and explanations, mathematicians are primarily interested in learning how to solve other problems, possibly over learning the reasons why some mathematical results hold true.

Now, one must be careful about several inferential jumps made in this kind of analysis. First, while the ArXiv may well be the largest, most widely used repository of this type of preprints and postprints in the world, we have analyzed a very specific type of mathematical discourse, leaving open the possibility that studies of mathematical discourse in others settings (conversational or other digital communications) could lead to contrasting findings. Second, we have analyzed these research papers for a limited type of *"explanatory" talk*, one required to contain explain-words or a limited number of alternative, related expressions. While this was an obvious place to start to investigate *"explanatory" talk* in mathematical discourse, it is certainly possible that the analysis of other expressions related to mathematical explanation may skew our results. These limitations of the present study indicate clear avenues for future empirical research on mathematical explanation.

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"WHAT DO YOU SEE THAT YOU CAN NAME?" DOCUMENTING THE LANGUAGE TEACHERS USE TO DESCRIBE PHENOMENA IN MIDDLE SCHOOL MATHEMATICS CLASSROOMS IN AUSTRALIA AND THE USA

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Different communities, speaking different languages, employ different naming systems to describe the phenomena of the mathematics classroom. The International Lexicon Project has documented the lexicons of middle school teachers of mathematics in nine countries. This paper reports on aspects of the naming systems in use by Australian and U.S. middle school mathematics teachers. Members of the research community and groups of practitioners in both countries participated in the process of negotiating their own lexicons, whilst the education community at large assisted in the validation of that lexicon. Despite cultural similarities, the professional language available to mathematics teachers in Australia and America is different in content and in structure, with implications for comparative research.

INTRODUCTION

The research outlined in this paper is being undertaken as part of a larger international project. The International Lexicon Project has initiated cross-cultural dialogue to identify pedagogical terms from selected educational communities and use these as analytical tools to categorise, interrogate and enrich classroom practice, classroom research, and educational theorising. This project seeks to identify and compare the naming systems employed in teaching communities in Australia, Chile, China, the Czech Republic, Finland, France, Germany, Japan and the USA. Documenting these lexicons, "the vocabulary of a person, language, or branch of knowledge" (Stevenson, 2015), will allow for the expansion of the variety of constructs available for the purpose of theorising about classroom practice, and for identifying the characteristics of accomplished practice. In this paper, we focus on the lexicons of the only two English-speaking teaching communities in the project.

Lortie (1975), in his social portrait of the 'Schoolteacher' reported a lack of 'technical language' in teaching. Lampert (2000) agreed that "no professional language for describing and analysing practice has developed in the United States" (p. 90). More recently, Grossman and her colleagues (2009) also concurred that the teaching profession's 'grammar of practice' was under-developed. Certainly, Connell (2009) has noted that a lively occupational culture in teaching which includes "the informal processes by which practical know-how is passed to new teachers in on-the-job learning" is not always present. We suggest that the promotion of such a culture would be dependent on a suitable professional language by which the teaching

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community might discuss its practice. Lampert (2000) concluded that the lack of opportunities for American teachers to work collaboratively with peers on the problems of practice result in “a language of practice [that] remains flat or nonexistent” (p. 90). This is in contrast with a well-articulated structure in China and strong traditions in Japan of educators and teachers discussing research lessons (Lampert, 2000).

Of interest also is the recent focus on noticing as a key component of teacher expertise (e.g., Sherin, Jacobs & Philipp, 2011). The idea here is that because of the complexity of instruction, teachers cannot notice everything with equal weight, and instead must choose from among this complexity where to focus their attention. What we notice is, of course, constrained by our knowledge and experience and, we argue, by what we can name. The Sapir-Whorf hypothesis suggests that our lived experience is mediated significantly by our capacity to name and categorise our world.

> We see and hear . . . very largely as we do because the language habits of our community predispose certain choices of interpretation (Sapir, 1949, p. 162).

Our interactions with classroom settings, whether as learners, teachers, or researchers, are mediated by our capacity to name what we see and experience. If the Australian (or American) teacher’s conception of the mathematics classroom is constructed around activities that they can name, then it may follow that they are unlikely to engage in activities that they cannot name. Marton and Tsui (2004) suggest that categories not only express the social structure but also create the need for people to conform to the behaviour associated with these categories (p. 28). Thus teachers’ activity in the classroom is channelled by those practices they are able to name, obliging their behaviour to correspond to this normative construction of practice. Comparison of the lexicons of Australian and American middle school mathematics teachers indicates the variation possible within two teaching communities speaking the same language but with different pedagogical traditions.

THE RESEARCH DESIGN

Protocols and approaches common to both Australian and American settings

Compiling the national lexicons from each country involved the assembly of local terms used to identify classroom practices reflecting the well-established pedagogical traditions by which each of the participating communities describe the activities of the mathematical classroom. These terms were supplemented with the clearest possible operational definitions (a description with examples and non-examples) describing both the form and function of each named term.

The composition of the local research team in each country was stipulated to include the team leader (senior researcher), junior researchers and at least two experienced teacher practitioners, with strong preference given to mathematics teachers of grades seven to nine who were currently teaching.

Each of the nine country teams contributed video material, time-stamped transcripts and classroom supporting material for one lesson of mathematics at year eight. These nine lessons were re-packaged as “three-ups” (see Figure 1) and each local research team was given access to the entire stimulus package of nine lessons.

Figure 1. The video “three-up” (three camera angles with time-code and subtitles)

The lesson videos presented in their combination a variety of instructional approaches in classroom settings both familiar and unfamiliar to the research team members in each country. Each team used a standardised recording template to record anything in the lesson for which they had a name. The initial prompt used for stimulating thought about the video was, “<u>What do you see that you can name?</u>” This very general prompt and approach was crafted so that restrictions were not placed on what could be named. The use of the video material was to stimulate thinking about possible lexical terms. Importantly, it was not necessary that every term refer to something occurring in one of the videos. Terms that came to mind during the viewing that were not present in the video material were also recorded. The fundamental criterion for the inclusion of a term in the lexicon was that it was familiar to at least two-thirds of middle school mathematics teachers in that country.

Detailing local protocols and approaches: Australia

The Australian national team consisted of four university researchers and three practising teachers. They all viewed the video of the Australian lesson, however, the remaining eight video-recorded lessons were assigned to team members using a matrix structure ensuring at least one experienced teacher viewed each lesson and each lesson was viewed by a minimum of four team members.

The Australian team met regularly to share terms or phrases that were felt to be possible candidates for inclusion in the Australian Lexicon. Team consensus was required for the inclusion of a term in the lexicon and, in problematic cases, authority was accorded primarily to classroom experience and the team member’s capacity to argue that the term was in current use by teachers. The essential point was to record single words or short phrases that are consistently and widely used within the mathematics teaching community.

An important matter for the Australian team was distinguishing the language of the discipline (mathematics) from the language of practice (mathematics teaching/learning). On occasion, a purely mathematical term would be considered for inclusion; these opportunities allowed us to reaffirm that the aim of the Lexicon project was to identify terms and short phrases that relate specifically to classroom phenomena, not those that describe solely mathematical activity (without implied reference to the mathematics classroom). An example of this is the familiar activity of *Graphing a Linear Equation*. We can identify and name this activity in the videos, it is clearly recognisable, however, the activity is not being named as a specifically classroom practice such as 'Worked Example' or 'Explaining' and, therefore, would not be included in the Australian Lexicon.

While identifying terms for inclusion in the lexicon thought was given to the possible structure or format that would best communicate the content of the lexicon. A university class of practicing teachers was invited to assist in the grouping of lexical items into categories of their own choosing. As a result the items in the Australian lexicon have been organized in five categories. The five categories include those that were identical across different working groups (Administration, Assessment, Classroom Management) and two additional ones that captured the spirit of the teachers' suggestions (Learning Strategy, Teaching Strategy).

Detailing local protocols and approaches: USA

The U.S. national team initially consisted of two university researchers and two practising teachers, with an additional researcher and teacher joining later. The first four members all viewed and discussed the U.S. lesson, and then teams of one researcher and one experienced teacher watched each of the remaining eight lessons.

As with the Australian team, the U.S. team met regularly to discuss terms and phrases that arose through viewing the videos and that might be included in the U.S. lexicon. The first draft of the lexicon included all terms that the four initial team members agreed were in current use by middle school mathematics teachers, as well as terms that the participating teachers highlighted as very familiar within their teaching communities. Additional teachers and researchers were also consulted to propose new terms, in case the list generated from the video viewing and associate discussion was incomplete. A total of 157 terms were identified as candidates for inclusion.

Subsequently, through focus groups with teachers at three different schools in a large midwestern city and a survey taken by more than 250 teachers around the United States, we solicited feedback on the lexicon as it then existed. Drawing on ratings of familiarity from teachers around the U.S., we developed a final national lexicon that includes 100 terms. Some details of the lexicon will be discussed in the section below.

Two key challenges arose in the process of developing a U.S. lexicon. First, teaching contexts differ widely across different states, districts, and schools in the U.S. Thus, it was especially important for us to seek input on our lexicon from teachers in a range of geographic locations and school types (public/private, rural/urban/suburban, etc.). Although we did receive survey responses from teachers in a variety of different contexts, we acknowledge that many contexts are underrepresented in our sample, and new terms would likely emerge with feedback from an even broader set of teachers. A second challenge concerned the difference in familiarity of terms among even experienced teachers in the greater Chicago area. For example, some terms were very familiar to teachers in one district, while teachers in another district were unfamiliar with those same terms. Although levels of familiarity sometimes differed considerably, we strived for consensus whenever possible, and our final lexicon represents terms that were familiar to at least 75% of participating teachers.

THE LEXICONS

For the purposes of the project, specification of the lexical terms required the combination of the following elements: i) a description, ii) examples, and iii) non-examples (see Table 1).

Assessment (from Australian Lexicon)	Any activity undertaken by the teacher or a student(s) with the primary purpose of generating information about student learning or achievement.	For example: • The teacher administers a test. • The teacher observes students while they work, making notes on each student's progress. *Non-example:* • *Assigning homework, unless the teacher explicitly indicates that the purpose is assessment.*
Practising (from Australian Lexicon)	The activity of repeating a procedure for the purpose of improving efficiency or accuracy in its use.	For example: • A student solves ten consecutive tasks all involving the addition of fractions. • A student works through the problems on past exam papers. *Non-example:* • *A student attempts to make use of the property of similar triangles in a real-world context for the first time.*
Warm-Up (from U.S. Lexicon)	Brief activity used at the beginning of class, often for review or entry into a new topic.	For example: • Two short problems are written on the board for students to begin working on when class begins. • At the start of class, students are asked to identify whether they agree or disagree with three mathematical statements. *Non-example:* • *At the start of class, students review the homework.*
Worked Example (from U.S. Lexicon)	Step-by-step demonstration of how to solve a problem. Often provided by teacher to students as a model.	For example: • The teacher solves a problem out loud at the board while students follow along. The teacher explains each step as she completes the problem. • The teacher shares a completed solution to a problem with the class, discussing each step in the solution with the class. *Non-example:* • *Students use an answer key to check a solution to a problem.*

Table 1: A selection of lexical terms developed for the Australian and U.S. Lexicons

The Australian National Lexicon consists of 63 terms that are familiar and in widespread use (e.g., *Assigning Homework, Rephrasing, Worked Example*). The lexical items have been organized in five categories as follows: Administration (8 terms); Assessment (11 terms); Classroom Management (6 terms), Learning

Strategies (27 terms) and Teaching Strategies (50 terms). A lexical item appeared in more than one category if the Australian team decided on the basis of teacher advice that there was a strong association with each category.

The U.S. Lexicon consists of 100 terms that are widely familiar in the United States (e.g., *Going Over Homework, Classroom Environment, Worked Example*). The lexical items have preliminarily been organized into eight categories as follows: Administrative Practices (9 terms); Classroom Climate (14 terms); Forms of Participation (7 terms), General Classroom Practices (18 terms); Math Practices (21 terms); Tasks and Activities (11 terms); Teacher Assessment (10 terms); and Teacher Tools/Approaches (12 terms). Currently, a lexical item appears in more than one category if it has multiple definitions. Moving forward, we will seek further input from teachers to validate this categorisation scheme.

One feature of the Australian Lexicon is that none of the 63 terms identifies a practice unique to the mathematics classroom. The terms all refer to general pedagogical practices. Also worthy of note is the prevalence of 'gerunds' (a verb form that also functions as a noun; "teaching" and "learning" are relevant examples) in the Australian National Lexicon. The generic character of the Australian Lexicon content suggests that the lexicon might also be applicable to other school settings besides the mathematics classroom.

In a preliminary validation exercise over two-thirds of 83 respondents described the terms in the Australian lexicon as "familiar" or "very familiar." By this criterion, all of the 63 terms were validated for inclusion in the national Australian Lexicon. When questioned about the use of these terms in conversations with colleagues, however, responses spanned the full five-point scale from 'Used extremely often' to 'Not at all used'.

An interesting feature of the U.S. Lexicon is that despite the widespread familiarity of terms, teachers reported using terms to varying degrees. While all terms were either very or extremely familiar to more than 75% of the responding teachers, some terms were used by most teachers only monthly or yearly (e.g., *Assign Seats, Student Presentation, Extra Credit*). In contrast, many other terms were used either daily or weekly by most of the teachers (e.g. *Asking Questions, Listening, Struggling*). Variations in usage may be attributable to several different factors that merit further systematic investigation.

A comparison of the lexical term names and organisational structure of the lexicon of these two teaching communities reflect interesting similarities and differences. There are 37% fewer lexical items in the Australian lexicon and 23 terms (approximately a third of the entire terms in the Australian lexicon) are present in the U.S lexicon (see Figure 2). A further four terms were noted as highly similar (see Figure 3). There are currently only two categories in the respective organisational frames that match (Administration and Assessment). A selection of distinctive terms in each lexicon is shown in Figure 3; whether these differences are superficial or represent profound differences in pedagogical orientation remains to be investigated.

Terms present in both the Australian and U.S. Lexicons								
	Australia	**USA**		**Australia**	**USA**		**Australia**	**USA**
1	Answering Questions	Answer Questions	8	Formative Assessment	Formative Assessment	16	Reasoning	Reasoning
2	Assessment	Assessing	9	Group Work	Group Work	17	Reflecting	Reflection
3	Assigning Homework	Assign Homework	10	Justifying	Justifying	18	Reviewing	Review
4	Clarifying	Clarifying	11	Modelling	Modeling	19	Scaffolding	Scaffolding
5	Differentiating (Differentiation)	Differentiation	12	Note-taking	Note-taking	20	Test/Testing	Test, Testing*
6	Explaining (Explanation)	Explaining	13	Pair Work	Partner work	21	Wait Time	Wait Time
7	Feedback	Offer Feedback	14	Practising	Practise	22	Whole Class Discussion	Whole-Class Discussion
			15	Questioning	Questioning	23	Worked Example	Worked Example

* two entries in US Lexicon

Figure 2: Terms that appear in both the Australian and U.S. Lexicons

Similar Terms in the Australian and U.S. Lexicons		
	Australia	**USA**
1	Checking	Check Answers
2	Correcting	Correcting mistakes
3	Guiding	Guided Practice
4	Handing Out Equipment	Distribute Materials

Australian Terms (not in the U.S. Lexicon)*	
Board Work	Defining
Elaborating	Elicit Understanding
Encouraging	Group Discussion
(use of a) Hook	Motivating
Peer Support	Posing Problems
Prompting	Rephrasing

U.S.Terms (not in the Australian Lexicon)*	
"Aha" Moment	Brainstorming
Building Rapport	Challenge
Compare Multiple Strategies	Creative Thinking
Exploring	Investigation
Making Connections	Mastery
Struggling	Think-Pair-Share

* not an exhaustive list

Figure 3: Similar and Distinctive Terms from the Australian and U.S. Lexicons

Additional details of each lexicon will be published elsewhere, following a process of national validation, currently underway in each participating country. The key features reported here serve to indicate just how much variation in professional language exists, even between teaching communities that might be thought to have much in common.

CONCLUSION

Our purpose in this paper has been to contrast not only some features of the Australian and American lexicons as these have emerged, but also to share the processes whereby each lexicon was developed. In both process and product, the Australian and American lexicons necessarily reflect differences in context, community, educational culture, and pedagogical history. These differences are particularly interesting, arising from the comparison of two teaching communities that are both English-speaking, new world affluent societies. As is evident from other studies (Lamb & Fullarton, 2002) middle school mathematics classrooms in these two countries share many common features yet the professional language available to mathematics teachers in Australia and America is both different in content and different in structure. Further analyses will identify other similarities and differences in both the lexicons and in the communities and cultures that they reflect. It is expected that comparisons undertaken as part of The International Lexicon Project, particularly where the teaching communities differ even more profoundly in culture and in national language, will provide powerful insights into the way in which each community has constructed and named the practices of its mathematics classrooms.

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YOUNG CULTURALLY DIVERSE STUDENTS' INITIAL UNDERSTANDINGS OF GROWING PATTERNS

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There have been limited studies which have investigated young Pāsifika and Maori students' understanding of growing patterns. The aim of this study was to explore young culturally diverse students' initial understandings of growing patterns before formal introduction within the classroom. Data are drawn from a class of Year 2 students (6-year-old), including analysis of pretest interview questions. Results indicate that young culturally diverse students had more success continuing growing patterns that were drawn from their own culture. A small group of students were able to identify recursive generalisations from the growing patterns.

INTRODUCTION

Important changes have been proposed for mathematics classrooms of the 21st century in order to meet the needs of a "knowledge society". A key aspect of proposed changes is greater emphasis on the teaching and learning of early algebra in primary classrooms (Blanton & et al., 2015). In part, this emphasis has arisen in response to the growing recognition of the inadequate algebraic understandings many students develop during their schooling and the role this has in denying them access to prospective educational and employment opportunities (Knuth, Stephens, McNeil, & Alibabi, 2006). In response, some curricula including New Zealand (Ministry of Education (MoE), 2007) advocate teaching arithmetic and algebra as a unified strand across the mathematics curriculum. This approach focuses on using students' informal knowledge and numerical reasoning to build early algebraic thinking. Tasks involving functions and numeric patterning activities offer an opportunity to integrate early algebraic reasoning into the existing mathematics curriculum. The focus in this paper is on an exploration of young culturally diverse students' initial understandings of growing patterns before formal introduction within the classroom.

Mathematical achievement of culturally diverse students is a challenge in many countries. Similar to other countries, New Zealand has a changing student population that is increasingly culturally diverse. This includes the largest group of Pāsifika students in the Western world as well as indigenous Maori students. Students of a Pāsifika background are not from a single ethnicity, nationality, language or culture but are a diverse group including those born in New Zealand, those who have migrated from the Pacific Islands, or those who identify themselves with the Pacific Islands and culture (Coxon, Anae, Mara, Wendt-Samu, & Finau, 2002). In regards to mathematical achievement, both Pāsifika and Maori students are characterised by unenviable statistics in which a large percentage are under-achieving compared to

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their peers. Educators frequently attribute this under-achievement to the learners themselves and position Pāsifika and Maori cultures as being mathematically deficient (Hunter, et al., 2016). However, both Pāsifika and Maori cultures have a rich background of mathematics including a strong emphasis on patterns used within craft design (Finau & Stillman, 1995). This includes geometrical designs which are used in repeating and growing patterns. There have been limited studies which have investigated young Pāsifika and Maori students' understanding of growing patterns. In this paper, we investigate student responses to growing pattern tasks. This includes an analysis of the responses related to a pattern typically used in Pāsifika art-work and one which would be more typical of a Western mathematics classroom task.

RESEARCH LITERATURE

Studies with non-Indigenous primary students demonstrate that engaging with early algebra assists students to develop a deeper understanding of mathematical structures that can lead to mathematical generalisations (Radford, 2010). One particular path for developing this thinking is through students working with growing patterns (Warren, 2005). Research studies have focused on fostering early algebraic thinking through the use of patterning activities, in particular geometric patterns (e.g., Radford, 2010; Rivera & Beckner, 2011). Growing patterns are characterised by the relationship between elements which increase or decrease by a constant difference. As a means to develop early algebraic thinking, students in the elementary school engage in activities that provide students with the opportunity to copy, continue, and extend growing patterns. Eventually, there is a need for the student to see the relationship between the pattern and their position (stage). This relationship can be termed a generalisation.

There are two ways students generalise growing pattern structures; recursive generalisations and functional generalisations (Blanton et al., 2015). Recursive strategies are commonly used by young students as a means to generalise a functional relationship (Radford, 2010; Rivera & Beckner, 2011). While this recursive strategy assists students in predicting the next element in a pattern, students are not identifying the underlying structural relationship between the pattern and the position (two data sets) in order to identify the underlying general rule (Moss & Beatty, 2010).

This study aims to explore the following research questions:

1. What are young culturally diverse students' initial understandings of growing patterns prior to formal introduction at school?
2. How do young culturally diverse students describe growing pattern tasks with a cultural connection or those typically used in Western mathematics classrooms?

THEORETICAL FRAMEWORK

Mathematics as a subject was long considered by many to be value and culture free (Presmeg, 2007). Despite this belief, in the past few decades researchers (e.g., Bishop, 1991; D'Ambrosio, 1985) have shown that mathematics is a cultural product. We take the perspective that the teaching and learning of mathematics cannot be decontextualised from the learner. In this view, the teaching and learning of mathematics is wholly cultural and is closely tied to the cultural identity of the learner. The underachievement of specific groups of students (such as Pāsifika and Maori within New Zealand) is related to a mismatch between the practices within the classroom and the cultural background of the students (Bills & Hunter, 2015). One key aspect of developing a culturally responsive classroom is ensuring that mathematical tasks are set within the known and lived, social and cultural reality of the students. An example of this within the context of early algebra is drawing upon the patterns within Pāsifika and Maori culture for exploration in the mathematics classroom.

RESEARCH DESIGN

This research reports on one aspect of a larger study focusing on young culturally diverse students' developing understanding of functional patterns. It was conducted with one classroom of Year Two students in a low socio-economic, high poverty, urban school in New Zealand. Twenty-nine students (aged 6 years old) participated in the study including 17 male and 12 female students. The students were predominantly of Pāsifika descent (n = 24), with three students from an indigenous New Zealand Maori background, and two students from South East Asia.

To explore the students' initial understandings of growing patterns, each student participated in an individual task-based interview. The interview tasks were designed by the researchers and focused on growing patterns. In the New Zealand curriculum, students by the end of Year Two are expected to create and continue repeating patterns (MoE, 2007). Students in this classroom had engaged with tasks involving repeating patterns but growing patterns were unfamiliar as this is not a curriculum expectation until Year Four (MoE, 2007). The interview consisted of four tasks and took between 15 to 20 minutes. It included two patterns drawn from Pāsifika and Maori culture and two patterns used typically in New Zealand or Western mathematics lessons. This study will focus on two of the four tasks which will be described below.

One of the tasks used a design which was based on a border pattern of a tapa cloth (Figure 1).

Figure 1: Tapa cloth

Tapa cloth is a decorated bark cloth of social importance which is often given as a gift. This was chosen as it is common across many of the Pacific Island nations and also displayed within schools, therefore it would be a familiar authentic pattern to these students. The task was introduced within a cultural context. The second task was de-contextualised and involved a growing pattern of rows of squares. Figure 2 dispays both tasks.

Figure 2: Growing pattern tasks.

For each task, the students were asked to continue the pattern. Follow-up questions asked the students to draw and describe the pattern for the ninth position. Students were also asked to provide a growing pattern rule. All the interviews were video-recorded and the interviewer also recorded the student responses in written form.

The systematic approach of constant comparative method was used to analyse the interview data. The video footage of the interviews was wholly transcribed and analysed to identify themes. To manage these documents a coding system was utilised to determine how to examine, cluster, and integrate the emerging themes (Creswell, 2008). Researchers coded the data at each phase with respect to early algebraic thinking and tasks design and met to discuss their themes and recode any data. Insights gained from the students' initial interview are presented in the following sections.

RESULTS

Each section in the results will provide an overview of the student responses to the tasks shown in Figure 2.

Continuing a growing pattern

The first interview question asked the students to continue the pattern. It appears that students had more success continuing the tapa cloth pattern (see Table 1), than the square pattern (see Table 2). The following tables show the types of student responses, examples, and frequencies of the responses.

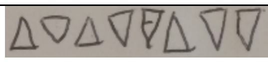
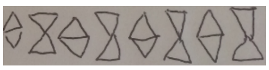
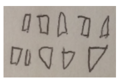
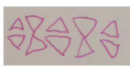
Type of Response	Example	Frequency
Did not accurately replicate the pattern		8
Continued the pattern but drew a different pattern position (e.g., 8^{th} position)		7
Continued the pattern but did not draw the alternating design (up and down)		4
Successfully continued the pattern		9

Table 1: Types of student responses, example and frequency of continuing the tapa cloth pattern

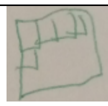
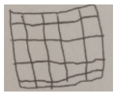
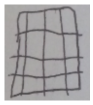
Type of Response	Example	Frequency
Did not accurately replicate the pattern		9
Drew a multiplicative structure		10
Drew a square number (four by four)		8
Successfully continued the pattern		2

Table 2: Types of student responses, example and frequency of continuing the square pattern

Pattern for the 9th position (near generalisation)

Students were asked to draw the pattern to the 9th position of the tapa cloth pattern. Two of the students drew the triangle pattern to the ninth position. One student drew nine sets of pairs of triangles but did not follow the alternating pattern of the original design and another drew nine sets of pairs of circles. Other students (n = 8) drew varying sets of pairs of triangles or wrote numbers which did not correspond with the pattern (n = 6). The remaining students (n = 11) gave non-relevant responses (e.g., *16 months; or it will be down because the sun goes down)* or did not respond. When asked to describe how the pattern was growing, the most common response (n = 18) was to identify that the pattern was getting bigger. Six of the students did not respond to the question. A small group (n = 4) identified that the pattern was growing by two triangles each time. One student focused attention on the direction of the triangles as a repeating pattern and described the way the pattern was growing as: *it is going up and down, and then up and down.*

One student successfully drew the square pattern to the ninth position and another indicated that it would be a nine by nine grid. Some of the students (n = 4) drew nine individual squares and others (n = 8) drew a multiplicative array of varying amounts. The remaining students (n = 15) either gave irrelevant responses or did not respond. When asked to describe how the pattern was growing, two of the students identified that the pattern was growing by three: *it's growing in the three times-tables.* One student began to skip count in threes but then made an error: *Three, six, nine, 13, 18.* Five students identified that the pattern was growing bigger: *another layer of squares.* The remaining students (n = 21) did not answer or gave irrelevant answers. It appeared that the students found this question more challenging than the tapa cloth pattern.

What is my growing pattern rule?

Students were asked to identify the growing rule for each pattern. Seventeen students provided a response for the tapa cloth pattern. The most common response (n = 6) was that it was growing: *by twos or counting up in twos*. Three students identified that it was getting bigger, and three students stated that it was counting up or adding more on. The remaining students (n = 5) gave irrelevant responses.

Twenty-five students provided a response for the square pattern. Four students identified that the square pattern was: *counting on by threes*. The most common response from students was that: *it was getting bigger* (n = 9). One student attempted to count in threes, while another described that it was getting an extra layer each time. The remaining students (n = 10) gave irrelevant responses.

DISCUSSION

While past research has indicated that having a culturally responsive pedagogical approach to teaching and learning mathematics benefits students from diverse cultural backgrounds, there are few studies (e.g., Miller, 2014) focused on the teaching of growing patterns to young culturally diverse students. Acknowledging that students bring their own cultural knowledge to the classroom provides an opportunity for culturally diverse students to make more meaningful connections. While the young students in this study had difficulties with both tasks, it appears that they were better able to continue the tapa cloth growing pattern rather than the square pattern. It is conjectured that this is because the task was using the students' known world (Matthews et al., 2005). This shows the importance of drawing on authentic tasks in a non-tokenistic way to provide opportunities for young culturally diverse students to make connections to their own contexts when learning new mathematical concepts.

As the tapa cloth pattern was a familiar pattern structure, we conjecture that this assisted students to 'see' the structure of the pattern. Past research has examined students engaging with growing patterns from a mathematical context (e.g., tiling patterns), where students have to continue, predict, find missing elements, determine the additive rule, and generalise geometric growing patterns (Moss & Beatty, 2006; Warren, 2005). It is argued that these types of patterns are initially challenging for young students (as evidenced in the square pattern). The results of this study indicate that it is important to consider how the context of the pattern impacts on students' ability to access the structure and relationship between the variables. In other words, how the type of context beyond the visual display used for the pattern impacts on their ability to see and potentially generalise the pattern structure. Past studies have indicated that the types of geometric growing patterns presented to students did not impact on their ability to extend the pattern (Leung, Krauthausen, & Rivera, 2012). However, in contrast in the initial stages of the present study, the context of the pattern did impact on students' ability to extend the pattern. For example, students were more competent at extending the familiar triangle tapa cloth pattern rather than the growing pattern represented by decontextualised geometric shapes (e.g., squares).

The ways in which these young culturally diverse students generalise growing pattern rules mirrors aspects of how non-Indigenous students generalise growing patterns.

For example, we observed recursive thinking, where students focused on the additive component of the growing pattern (Blanton, et al. 2015). When predicting near generalisations, (e.g., 9th position), some students were able to provided general rules that were defined by a recursive element of the pattern (e.g., adding 2 each time). In the case of the square pattern, similar to young Australian Indigenous students (Miller, 2014), these culturally diverse students were more concerned with giving the pattern quantity (e.g., total number of squares required) than focusing on the general structure of the pattern.

CONCLUSION AND IMPLICATIONS

This study begins to shed light as to how young Pāsifika and Maori students engage in tasks involving functions and numeric growing patterning activities. It is apparent that these young students can engage in early algebraic concepts, such as continuing growing patterns, before formal introduction within the classroom. It appears that they have greater success when these patterns are come from a context that is familiar to the students, as in the case of the tapa cloth task. An important implication is the need for researchers and teachers to consider drawing upon familiar pattern structures when introducing culturally diverse students to growing patterns.

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DEVELOPING A SCALE TO MEASURE AWARENESS OF MATHEMATICAL PATTERN AND STRUCTURE (AMPS)

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Awareness of Mathematical Pattern and Structure (AMPS) has been described as a general construct that underpins early mathematical development. To measure and validate AMPS, a 14-16 item interview-based instrument (the Pattern and Structure Assessment, PASA) was re-constructed, administered to a reference sample of 618, 5 to 6-year olds, and subjected to a Rasch analysis. PASA provided a reliable and valid measure of AMPS and was found highly correlated with a test of mathematical achievement (PATMaths). PASA advances early numeracy assessment and learning by integrating structural aspects: sequences, shape and alignment, equal spacing, structured counting and partitioning, linked to a pedagogical program (PASMAP).

INTRODUCTION

It has long been regarded that "mathematics [is] the science of patterns" and that the main occupation of mathematics is with abstract structures (Resnick, 1999). In parallel to this belief, many mathematics educators believe that the recognition of mathematical patterns and the abstraction of their underlying structures lies at the heart of mathematics learning (Mason, Stephens, & Watson, 2009). Over the past decade research in early childhood mathematics education has turned attention to the importance of a range of mathematical domains including patterning and early algebra, spatial skills, measurement, data exploration, and mathematical reasoning (Carraher, Schliemann, Brizuela, & Earnest, 2006; Clements & Sarama, 2007; English, 2012). An emerging line of research, focused on *mathematical patterns and structures*, aims to provide a more coherent picture of the common underlying bases of mathematical development (Mulligan & Mitchelmore, 2009). These studies have focused on how children can develop connected mathematical knowledge leading to generalisation—through the development of patterns and structural relationships. The Australian *Pattern and Structure Mathematical Awareness* project has investigated, in a series of related studies, the development of patterning and structural awareness among 4 to 8 year olds across a range of mathematical concepts (Mulligan, Mitchelmore, English, & Crevensten, 2013). In these studies, students' responses to a wide variety of individually administered pattern-eliciting task have repeatedly confirmed two findings:

1. Responses can be reliably classified into the five ordered structural categories defined in Table 1.
2. Although the same student's responses to different tasks invariably vary from task to task, students who respond at a high structural level on one task tend to

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respond highly on other tasks. The similar pattern is found for students who respond at a low structural level.

Examples of these studies have previously been reported at PME28, PME29 and PME33 and a summary of these is available in Mulligan et al. (2013). Mulligan reported a 2-year longitudinal evaluation study of 316 Kindergartners assessed by a PASA interview and a standardised measure of mathematical achievement. An intervention program was trialled with an experimental group over the entire first year of schooling. They found highly significant differences on the PASA between intervention students and the 'regular' group at the retention point ($p < 0.002$) and increased levels of structural development for intervention students. The study validated the instrument (PASA) and constructed a Rasch scale indicating item fit.

As a result of these studies, the authors identified and described the construct of *Awareness of Mathematical Pattern and Structure* (AMPS) comprising two interdependent components: one cognitive — knowledge of structure, and one meta-cognitive — a tendency to seek and analyse patterns (Mulligan & Mitchelmore, 2009). It is our hypothesis that the degree of a student's AMPS determines how readily they develop relationships in mathematics and form simple structural generalizations. The research question to be addressed in this paper is:

- Can AMPS be measured validly and reliably?

We shall answer this question by describing the construction of a Rasch scale for the measurement of AMPS.

METHOD

A *Pattern and Structure Assessment* (PASA) interview, developed for prior studies of Kindergarten and Grade 1 students was redeveloped for assessing AMPS in students in the first three years of formal schooling (students aged approximately 4.5 years to 8 years). The study employed a new sample of 618 students to validate the instrument, and to also assess the same students for general mathematical achievement using PATMaths (Stephanou & Lindsey, 2013). Because of the observed variation across these three years, three separate forms were constructed: one for the beginning of the first year (PASA-F), one for the end of the first year or the beginning of the second year (PASA-1), and one for the end of the second year or the beginning of the third year (PASA-2). Each PASA focused on similar core concepts ranging from 14 to 16 items from Foundation to Grade 2. These included repeating and growing patterns, partitioning 2-D and 3-D shape and space, skip counting and base ten structure, arrays and grids, distance and scale, and units of length, area, volume/capacity, mass and time.

Procedures: Three forms of PASA were administered by the researchers and a group of trained research assistants to a sample of 618 students in the first two years of schooling in two schools in Sydney. PASA-F was administered to a total of 213 students, PASA-1 to 189 students, and PASA-2 to 216 students. The two schools were chosen to be typical of Sydney schools in general but cannot be regarded as a representative sample. The researchers trained six interviewers and piloted protocols for conducting the interviews and coding responses, obtaining inter-rater reliability of 0.82. The PASA interviews were conducted consistently following protocols from previous studies (see Mulligan et al., 2013). The interviewers coded student responses to each item as one of five structural levels according to Table 1 using detailed guidelines aligned with each task: prestructural (L1); emergent (L2); partial (L3); structural (L4); and, advanced structural (L5). Data from this administration were then used in a Rasch analysis.

Response category	Characteristics of response	Example (drawing a clock face)
Advanced structural	An accurate, efficient and generalised use of the underlying structure	Places 12, 3, 6 and 12 accurately and then fills in the remaining numbers
Structural	A correct but limited use of the underlying structure	Places the numbers approximately equally spaced by eye
Partial structural	Shows most of the relevant features of the pattern but inaccurately organised	Makes an unsuccessful attempt to space the numbers equally
Emergent	Shows some relevant features of the pattern but incorrectly organised	Writes the numbers 12, 1, … around the circumference, leaving a large gap between 11 and 12
Prestructural	Shows at most limited and disconnected features of the pattern.	Declares the task is too difficult

Table 1: The five response categories used in scoring PASA items.

The wide range of tasks developed in previous studies (See Mulligan et al., 2013) were first analysed both in terms of their content and their item discrimination. It was found that all the items represented one or more of the five structures shown in Table 2. Some new items were devised and others discarded to achieve a balance between the five structures. For example, a task assessing the Equal Spacing structure required students to draw a representation of a ruler; this task was judged only suitable for PASA-2 because younger students may well not have used a ruler.

Items that had previously shown to a narrow range of categories at a particular age (i.e., were either too easy or too hard) were discarded or limited to a particular form. Otherwise, preference was given to tasks that were suitable for all three forms. For example, a task inviting students to fold a strip of paper into thirds was found fairly difficult by the youngest students but gave a good spread otherwise. This task was retained for all three forms, but a similar task inviting students to fold the paper into halves was included in PASA-F. All new tasks were pretested before being included in the resulting PASA.

Structure	Definition	Example
Sequences	A series of objects arranged in a definite order	Red, white, blue, red, white, blue, …
Shape and alignment	Spatial relationships	Parallel, horizontal, square, aligned
Equal spacing	A series of marks arranged at equal spacing	The scale on a ruler
Structured counting	Counting in groups	Counting 10 objects as 2, 4, 6, 8, 10
Partitioning	Dividing an object or set of objects into equal parts	Partitioning an object into halves

Table 2: The five structures measured in PASA.

RASCH ANALYSIS

The Rasch model (Andrich, 2005) assumes that each student has a certain amount of AMPS that can be captured by a single score locating the student on an AMPS scale. For each task, the five response categories should represent increasing and non-overlapping amounts of AMPS. The data were analysed using the Quest-Interactive program (Adams & Khoo, 1996) to determine how well these conditions were met. Initial analysis showed that several items did not meet the second condition. In some cases, the threshold between the categories could not be determined with sufficient accuracy; in others, the mean AMPS score of the students giving responses in one category was not greater than for students in a lower category. To solve this problem, categories were combined to create "measurement categories" for each task. For example, for the task of folding a strip into thirds in PASA-1, the emergent and partial structural categories were combined and the structural and advanced structural categories were combined—thus giving three measurement categories for this task. The process of combining response categories was carried out in an iterative fashion for each PASA form, until the analysis gave adequate results for goodness of fit to the Rasch model for that form.

Combining the results from the three forms to yield a single scale presented further problems: The measurement categories for the common tasks were often different in the different forms, so that they could no longer be considered the same task. However, there remained sufficient genuinely common tasks for a single scale to be constructed. This scale was adjusted to give a mean of 100 and a standard deviation of 15. Table 3 shows some characteristics of this scale for the various forms.

Form	Range	Mean	Standard deviation	Reliability
PASA-F	40-138	83	9	0.72
PASA-1	50-148	92	12	0.76
PASA-2	48-163	110	11	0.84

Table 3: Characteristics of the AMPS scale for the three PASA forms.

It will be noticed that each of the three forms gives a reliable estimate of a student's AMPS. To assess content validity, a detailed analysis was made of the response characteristics of each of the measurement categories for each task in each form; these characteristics were then plotted against the corresponding segment of the AMPS scale and commonalities sought. It was found that the AMPS scale could be divided into four levels, as indicated in Table 4. (Due to measurement error, there is some overlap between the four levels). These four levels showed the expected development in AMPS and support the argument that PASA really does measure AMPS.

Level	AMPS range	Student characteristic	Examples
4	From 118	Aware of the generality of some fundamental structures	• Quickly draws an accurate grid • Explains structure of 2-digit numbers
3	97-123	Aware of some fundamental structures	• Draws an accurate measurement scale • Uses “counting on” for addition
2	77-102	Recognises some simple patterns	• Extends alternating block • Divides a length into halves by eye
1	Up to 83	Only recognises some very simple patterns	• Copies block patterns by matching • Counts up to 3 groups of two

Table 4: Descriptions and examples of the four AMPS levels.

The fact that mean AMPS scores increased as students got older (Table 3) is a further indication of content validity. We do however, recognise that the conceptual coherence of the AMPS construct is limited to the scope and content of the items that were designed explicitly to measure it.

A similar analysis was completed for each of the five underlying structures (see Table 1) and again the expected relations to a student's AMPS and age were found. For further details, see Mulligan, Mitchelmore and Stephanou (2015). To assess concurrent validity, PATMaths (Stephanou & Lindsey, 2013) data were collected from the students who had been administered PASA-1 and PASA-2 data (N=371). Although the two assessments provide different information about the child's mathematical competence they were found to be highly correlated; Foundation (0.72), Year 1 (0.76) and Year 2 0.84). The AMPS scale makes it possible to compare children's level of amps across year (grade) levels, regardless of which PASA assessment form they are given. This is considered very high given the reliability of PASA and the fact that PATMaths is a group test that is not scored for awareness of pattern and structure. Further analysis (Stephanou & Lindsey, 2015) confirms that the two instruments are essentially measuring different aspects of mathematical ability.

IMPLICATIONS AND FUTURE RESEARCH DIRECTIONS

Classroom teachers can use PASA as a simple but effective tool for assessing the AMPS of their students. After entering students' response categories into an Excel sheet provided by the Australian Council for Educational Research, teachers obtain AMPS scores for each student as well as their scores on each of the five structures listed in Table 2. These data enable the teacher to identify students with particularly high or low AMPS, as well as identifying particular aspects of core mathematical structures that may be above or below average. Similarly, the data would show which aspects of AMPS are particularly high or low in a significant number of students in the class. Such information can be invaluable in designing planning their learning experiences that address deep conceptual understandings and the students' ability to seek and use mathematical relationships.

In the course of the *Pattern and Structure Mathematical Awareness* project, the authors developed and evaluated a wide range of tasks and materials that could be used for teaching the awareness of pattern and structure. A formal evaluation (Mulligan et al., 2013) showed that the learning tasks and the accompanying pedagogical approach were indeed effective. Because pattern and structure underlines the mathematics school curriculum, an early mathematics program has been developed based on promoting children's AMPS. This program, the *Pattern and Structure Mathematics Awareness Program* (PASMAP), takes the form of two volumes of Learning Pathways organised around the structural groupings measured by PASA (Mulligan & Mitchelmore, 2016). A pegagogical approach focused on promoting and connecting concepts and relationships, and ultimately generating simple mathematical generalisation directs learning sequences to particular AMPS levels in particular structures, giving the teacher explicit descriptors and examples to inform their pedagogical choices.

An important outcome of the development of PASA and PASMAP has been an understanding of the fundamental role of spatial reasoning in developing AMPS. Two of the five basic structures that have been identified (Table 2) are explicitly spatial: Shape and alignment and Equal spacing. The other structures (Sequences, Structured counting and Partitioning) appear to focus on be numerical processes, but these integrate core aspects of spatial structure. For example, repeating sequences are basically visual, grids and arrays are closely linked to structured counting, and partitioning requires spatial skills.

Further study, utilising network analysis (Woolcott et al., 2015) provides visual links between the five AMPS structures as network maps of connectivity. This form of analysis complements Rasch analysis because it highlights the connections, or lack thereof, that children make between structural groupings and specific items.

A new Australian government funded project (2017-2020), *Connecting Mathematics Learning through Spatial Reasoning* (Mulligan, Woolcott, Mitchelmore & Davis) utilises network mapping to investigate the relationships between the structural groupings and spatial reasoning tasks. The project will develop a new PASA interview and a Spatial Reasoning Mathematics Program (SRMP) for students in Grades 3 through 5 (students aged approximately 7 to 11 years). The PASMAP will be expanded in both scope and depth to include a larger component on Spatial Reasoning, such as spatial transformations, spatial structuring of 2-D and 3-D shapes, spatial measurement, angles, axis differentiation, collinearity and direction perspective taking and dynamic spatial representations (Bruce et al., 2015).

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VERBALISING SPATIAL KNOWLEDGE: AN EMPIRICAL INVESTIGATION OF STUDENTS' STRATEGIES FOR SOLVING SPATIAL-VERBAL TASKS

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This study analyses how students solve spatial-verbal tasks by focussing on the strategies used to describe spatial-geometrical configurations. German Fifth-grade students were required to solve spatial tasks with language as a mode of representation, i.e. verbalizing the (re)construction of a spatial object and its spatial-geometrical characteristics. An interpretative-based analysis of students' spatial language was used for identifying strategies employed by students in their spatial discourse. Findings shows that students use a wide spectrum of strategies for solving the spatial-verbal tasks, which will be categorized by using Barrat's (1953) spatial ability strategy groups.

BACKGROUND OF STUDY

The domain of spatial ability has been intensively researched both in psychology and in mathematics education. Most studies about spatial ability have either focussed on describing a suitable model consisting of abilities which are relevant for spatial ability (e.g. Thurstone, 1950; Pinkernell, 2003) or tend to emphasize the assessment of student's spatial ability using pencil-and-paper tests (e.g. Linn & Petersen 1985; Büchter, 2011). However, few researchers have addressed the issue of understanding spatial thinking by considering students spatial discourse and how students solve spatial tasks verbally. The aim of this study is to investigate how students verbalize their spatial thinking, which should be done by focussing on the development of strategies in their spatial discourse. An analysis of spatial language in solving spatial-verbal tasks should induce results regarding the following questions:

1. Which strategies do students use to describe spatial objects in spatial-verbal tasks?
2. To what extent can the identified strategies be categorized using Barrat's (1953) established strategy groups in spatial ability research?

THEORETICAL CONSIDERATIONS

Metaphors in mathematics educational research

Mathematics and mathematical discourse is based on the use of metaphors (cf. Pimm, 1981), which can be defined as words or phrases which serve as a support for understanding abstract ideas by referring to more concrete objects or experiences (cf. Lakoff & Núñez, 2000). From a mathematics education perspective, metaphors are

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 265-272. Singapore: PME.

projections from source domains to target domains, in which the source's properties and characteristics are assigned to the target (cf. Lakoff & Núñez, 2000; Font et al., 2010). Lakoff & Núñez (2000) differentiate between two types of conceptual metaphors: grounding and linking metaphors. Grounding metaphors are metaphors which project a source outside the fields of mathematics to a target within mathematics, e.g. 'classes are containers'. In contrast, both source and target in linking metaphors originate within mathematics (cf. Lakoff & Núñez, 2000; Font et al., 2010). Hence, the varieties of languages present in mathematics classroom – primarily everyday or mathematics language – play an important role in differentiating between grounding or linking metaphors in mathematics education.

Spatial ability and spatial language

The notion of spatial ability has been a research object in psychology and in mathematics education for the last decades. As one of the eight different intelligences, spatial ability concerns the ability to solve spatial tasks in navigation, to visualize objects in different angles and to recognize space and other spatial characteristics (cf. Gardner, 2006). Mathematics educator Pinkernell (2003) describes spatial knowledge as the ability to act on spatial objects in space both mentally and in real terms, to recognize, understand, and describe spatial objects by referring to their geometrical properties, and to interpret and construct different forms of representation of spatial-visual objects (verbal, pictorial and action-based).

Solving spatial tasks requires students to develop and use strategies, which are approaches used to accomplish the tasks' goals and are flexible by allowing several ways to reach the underlying goals (cf. Fülöp, 2015). Two well-known strategy groups for solving spatial tasks – holistic and analytic strategies – have been introduced by Barratt (1953). Holistic strategies denote students' mental transformation and manipulation of spatial objects for solving spatial tasks. In contrast, learners using analytic strategies focus more on details of spatial objects in spatial tasks (cf. Barratt, 1953).

The notion of spatial language is important when considering spatial-verbal tasks. Spatial language refers to a variety of language used to speak about spatial objects, their spatial position and spatial relations between two or more objects. Levinson (1996) indicates the importance of analysing spatial language to understand the underlying spatial concepts. In particular, Coventry, Tenbrink & Bateman (2009) emphasize the importance of developing spatial language in a dialogue, because the interaction in a dialogue gives learners the opportunity to participate more actively and creates a less artificial setting.

METHODOLOGY

Research method

Based on the interplay between language and spatial ability introduced in the theoretical background, an adequate research method is needed to create and develop effective spatial discourse between learners. Spatial discourse should be developed in a dialogue between learners and hence enable an analysis of students' spoken spatial language about spatial objects. The reconstruction method was chosen as a research method since it fulfils the above mentioned criteria. The reconstruction method is a data collection method in which two learners seated in a back-to-back position communicate with each other to solve a particular task together. Tasks implemented in the reconstruction method are characterized by their dismantling in a series of steps and two different roles – describer and builder roles – assigned to the participating learners by the researcher or teacher. The describer instructs the builder what to do and the builder performs and interprets the instructions, mostly by using the provided hands-on manipulatives. The name for this research method derives not only from the real reconstruction by using the hands-on manipulatives, but also from the opportunity for the researchers to reconstruct the (re-)constructed knowledge (in the case of this present study, spatial and heuristic knowledge) among the learners participating in the reconstruction methods. Due to the embedding of the task in the research method itself, the design and instructions of the spatial task will be introduced in the next section for a deeper understanding of the research method.

Design of spatial task

The task implemented in the reconstruction method involved two students – describer and builder – working together to describe and reconstruct a spatial object. During the task, the researcher gave the following instruction to the learners:

"*In this experiment you [the describer] will be given an object made up of these building blocks, which can be put together. You must give him/her [the builder] instructions on how to build this object, so that he/she [the builder] can reconstruct the same object. The colour of the building blocks is not important and whilst you [the describer] are describing you can also touch and move the object as you like, but the object structure has to remain unchanged. At the end, the objects' structure must be identical.*"

Such a spatial task requires an adequate spatial object which can be described by the describer so that the builder can rebuild the spatial objects using the provided building cubes according to the giver's instructions. The structure of the spatial object in the spatial task was intended to activate student's spatial and geometrical knowledge and allow different use of strategies during its description. The following criteria were considered for the spatial objects illustrated in Figure 1 and Figure 2: three-dimensionality (students are required to describe along the three dimensions), break-down (students can break down the object in different ways), and spatial relations (orthogonal spatial relation between different parts of the object).

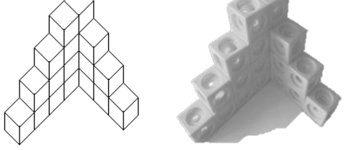

Figure 1: The structure of the spatial object A

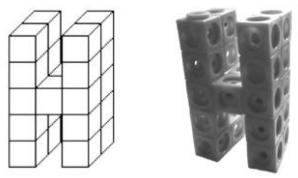

Figure 2: The structure of the spatial object B

Implementation and data analysis

By using a theoretical sampling considering different influencing factors when solving spatial-verbal tasks (i.e. students' language proficiency, spatial ability and gender), sixteen fifth graders from German secondary schools were chosen to describe the spatial objects (see Figure 1, 2) to another receiver in the reconstruction method consecutively. Both describer and builder were seated in a back-to-back position and were given the above task instructions to solve the spatial task in a dialogue during the whole task solving process. The participants were filmed and their discourse was transcribed. Considering an interpretative-heurmeneutic qualitative approach (cf. Jungwirth, 2003), the describers' utterances in the collected data were analysed for strategy use regarding the first research question and the underlying theoretical assumptions in this present study.

Results

Spatial metaphors strategy

An important strategy for describing spatial objects in the spatial-verbal tasks is spatial metaphors. Spatial metaphors are metaphors which transfer properties from concrete objects or other experiences to spatial objects in spatial construction (cf. Mizzi, 2017). An example of two spatial metaphors, 'staircase' and 'walking up/down', for describing spatial object A's structure and position in space respectively, is illustrated in Table 1.

Source	*Target*	*Transcript*
Staircase / Ability of walking up and down on the spatial object	Spatial object A	*"And now do three steps at the other staircase which you have done. Place it in a way in front of you as if you would walk up (...) and now take the other stairs which you have done now, and set it in the most front (...) in a way as if you would walk up at the front and then go down again the other staircase."*

Table 1: Examples of spatial metaphors "staircase" and "walking up/down" in a student's spatial discourse

Further examples of spatial metaphors and a model about the different dimensions of spatial metaphors – the linguistic, the spatial and the conception dimensions – can be found in Mizzi (2017).

Object break-down and assembling strategies

The strategy of object break-down denotes the students' mental break-down of spatial objects in internal parts (generally consisting of more than one cube) in order to reduce the complexity of its structure and facilitate the spatial description. The way of how the object has been broken down can be investigated by an analysis of spatial language, whereby learners assign particular names to internal parts based on the particular spatial characteristics of the underlying part. Consider for instance, the following transcript in which Student A uses a letter-based metaphor 'H' for the break-down of spatial object B in two internal parts, which are visualized in Figure 3:

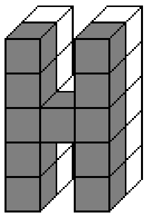

Figure 3: Student A's ntended break-down of spatial object B

Student A: "*It has five cubes at the right and left. And it has two times cube, well two cubes are next to each other. (...). It is quite a fat H. Well, there are two Hs*".
(…)
Student A: "*So two Hs, they look exactly the same, and then just do them together, so that they are exactly on each other, not next to each other, but on each other (...)*".

A further strategy, assembling strategy, is a strategy which denotes the assembling of internal parts in which the spatial object has been broken into in the underlying spatial discourse. An example of assembling strategy is illustrated in the second turn in the above transcript excerpt from Student A's spatial discourse. Whereas assembling strategy requires the prior use of the break-down strategy, assembling strategy is not always used by students after the mental breakdown of the object in spatial discourse, and hence it should be regarded as a strategy on its own.

Rotation strategy

Rotation strategy is a strategy which denotes the rotation of an object or its internal part around one of the three axes (vertical, horizontal, and frontal) embedded in the describers' spatial discourse. The intention of this strategy use is the change of spatial position of a spatial object, as the following transcript excerpt from Student B's description of spatial object A illustrates:

Student B: "*Then at the top uhm start left there too, then three stones. (...) Then uh at the top again and then two stones. Then turn it around so that you, so that it looks away from you, the staircase. Then you build three rows (...) at the bottom left.*"

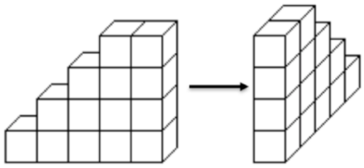

Figure 4: Visualization of Student B's intended actions upon use of rotation strategies

In the above transcript excerpt, Student B describes how an internal part of the object must be built (illustrated on the left part of Figure 4). Then he instructs the builder to turn it around, the degree of rotation is emphasized by the utterance, "so that it looks away from you [builder]".

Controlling strategies

Two types of controlling strategies have been identified in students' spatial construction – cubes controlling and structure controlling strategies – which were used in order to self-regulate and control the description process in spatial discourse. Both controlling strategies were characterized by the requirement for an elaborate feedback about the reconstructed object from the builder in the reconstruction method. The cubes controlling strategy was used to control whether the reconstructed part or object has been built as intended by reducing the spatial object to its number of cubes. An example of cubes controlling strategy use is illustrated in the following transcript excerpt of Student B (continuation of the excerpt in the previous section on rotation strategy) and Student C (the builder student):

Student C: "*When I turn it away, there is only the fifth row, well the row…*".
Student B: "*How many stones have you built already?*" (Student B counts silently)
Student C: "*Uh fourteen*".
Student B: "*That's correct*".

Another controlling strategy for monitoring the descriptive process in the spatial task is structure controlling strategy. This strategy denotes the describer's demand to the builder to describe the structure of the reconstructed object. An example of controlling strategy is illustrated in the following transcript excerpt, wherein Student D is describing spatial object A to the builder, Student E.

Student D: "*Okay. How does it look like?*"
Student E: "*(...) It looks almost like, no idea. Wait! (...) Well now I have a ladder, where there is a huge staircase*".
Student D: "*Okay*".

In comparison to the cube controlling strategy, the use of structure controlling strategy demands a more detailed description, which can consist of a spatial metaphor (as in the case of Student D and Student E's discourse excerpt above), rather than merely the number of cubes.

Discussion

In the above section, a range of strategies used by students describing spatial objects in the reconstruction methods has been described. Although Barrat's (1953) strategy groups for spatial tasks are not sufficient to represent the diversity of the identified strategies, they can be adequate for categorizing the identified strategies into analytic or holistic. In the strategy of spatial metaphors, describers emphasize particular spatial characteristics or properties of spatial objects, which would rather allocate it to the analytic strategy group. The strategies of object-breakdown and assembling can be assigned to the group of holistic strategies, because describers are required to mentally break-down and assemble the object or its internal parts in their spatial discourse. Similarly, rotating strategy is also of a holistic nature, because internal parts of the objects or the whole objects themselves are being transformed and manipulated in discourse. In contrast, both controlling strategies – cubes controlling and structure controlling – can be regarded as analytic strategies because they focus more on feature or property comparison between two spatial objects – the original and the reconstructed object – by referring to the number of cubes used or the structure of the spatial object respectively.

CONCLUDING REMARKS

This empirical study has described how students solve spatial-verbal tasks in which students are required to describe how to (re-)construct spatial objects using hands-on manipulatives. The identified strategies include spatial metaphors, object-breakdown, assembling, rotating strategy, cube controlling, and structure controlling strategies. As it was pointed out in the discussion of the results, previous established strategy groups for spatial tasks can be useful for classifying the identified strategies, but do not necessarily replace the categories identified in this present study. Hence, this study suggests that students employ diverse strategies in their spatial discourse for solving spatial tasks, which were identified by an analysis of student language. Thus this study highlights the importance of analysis of students' spoken spatial language for understanding how students solve spatial tasks and their underlying spatial thinking. Further questions which these findings raise include whether the use of such strategies differ among students with different mathematics performance and to which extent an analysis of students' spatial language can play a role in assessing students' spatial ability.

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LEARNING AS A PROCESS OF STRUCTURING ATTENTION IN THE FIELD OF MEANING

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The purpose of this study is to capture and explain the roles that signs and attention play in the fraction learning process, through a previous study that employs Deleuze's perspective on sign and the role of attention. From this case study of elementary school students, we found that signs are a prerequisite for learning and that learning takes place as different forms of attention shifts. The various types of semiotic resources used by teachers and students have been found to play an important role in coordinating collective attention between teachers and students.

INTRODUCTION AND THEORETICAL BACKGROUND

Over the past 30 years, the importance of semiotic approaches in mathematics education has been emphasised (Presmeg et al., 2016). Studies attempting to elucidate the relationship between the subject and meaning through discussions of signs point out that a significant part of the meaning-making process using signs is implicit, embodied, and passive (de Freitas & Sinclair, 2012; Seeger, 2011; Thom & Roth, 2011).

In this context, de Freitas & Sinclair (2012, 2013) argue that agency in mathematical cognitive processes should be distributed to the materials, analysing the process in which mathematical meaning appeared in front of the learning subject from a new point of view. The fact that the process by which subjects encounter mathematical meanings is largely embodied, implicit, and passive leads to the assertion that mathematical meaning presented in front of the subject is highly ambiguous. As Merleau-Ponty (2002) noted, meaning is ambiguous, and there is always a dimension of meaning that is not explicitly revealed with regard to the subject.

As noted above, many researchers have presented insight through a semiotic approach. However, it is difficult to find a study analysing the process of learning by paying attention to the phenomenon of a sign itself unfolding a field of ambiguous meaning in front of a learning subject.

According to Deleuze, the subject of learning is located in a field of ambiguous meanings which signs forcibly unfold when encountering signs (Deleuze, 1994). The field of meaning is a field of ambiguous meaning mixed with all types of meanings. The process of manifesting a field of ambiguous problems in front of a subject is a passive and involuntary process (Deleuze, 1994); within the space of the virtual that signs permits and lead to, human beings can begin to act on their attention toward meaning. Deleuze (1994) defines learning as a process of encountering signs. Human

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 273-280. Singapore: PME.

beings encounter various signs, forming new assemblages with signs constantly (de Freitas & Sinclair, 2014).

If learning is regarded as a process by which the subject of learning becomes the subject of mathematical meaning, Deleuze's perspective on signs can be regarded as a prerequisite for teaching-learning. However, in order for learning to be successful, it is necessary to clarify ambiguous meanings. Thus, how is it possible to clarify meaning clearly in a field of ambiguous meanings? In this paper, it is argued that a shift of attention is required to clarify ambiguous meanings.

Several prior studies have emphasised the role of attention in mathematics learning. Radford (2009) referred to the semiotic node as a space in which complex adjustments of various signs and sensory modalities take place. In the semiotic node, the attention of the learning subject continues to change, and through the dialectical process of such a change of state, the students come up with a mathematical concept (Presmeg et al., 2016).

Seeger (2011) points out that the ability of the mind has been overemphasised in mathematics education, emphasising the implicit dimension of knowledge and the importance of pre-logical sense. In this context, he emphasised the need to study the role of attention in mathematics education research and the possibility of its education.

Acoording to Watson & Mason (2005), mathematics learning is a process of recognising the mathematical structure as the attention of learners is structured. In particular, Mason (2004) distinguished five types of attention, insisting that mathematical learning is achieved as the five types of attention constantly shift.

The five types of attention and the relationship between attention and learning presented by Mason have strong power to explain mathematical learning. However, the attention to which they refer is limited to personal attention. As noted by Radford & Roth (2011) and Towers & Martin (2015), human cognition is a collective emergence during interaction with the other. Ultimately, they do not posit collective attention.

They also failed to capture the function of ambiguous attention directed at the ambiguous field of meaning unfolded by signs. For a full understanding of the teaching-learning phenomenon, it is necessary to discuss the function of attention directed toward the field where the meanings are not clearly revealed and the meanings are mixed. Here, we can refer to Zagorianakos & Shvarts (2015) in a study which emphasises the role of operative intentionality in the learning process. Operative intentionality refers to intentionality that always operates as passive on a fundamental level before active intentionality works (Merleau-Ponty, 2002). It should be noted that operative intentionality does not belong to the realm of explicit attention but can be seen as an act of attention that is always pre-logically operating on an implicit level.

Through the above discussion, we can categorise attention as follows.

	Personal attention	Collective attention
Explicit attention	EP	EC
Operative attention	OP	OC

Table 1: Categorization of attention

Mason (2004) states that learning is a process by which the five types of attention are structured, but actual learning must be viewed as a complex process that takes place as the EP, EC, OP, and OC types of attention are continually transformed.

Of course, the process of shifting attention cannot occur unless it is on a field of meaning forced by signs. Watson & Mason (2005) note that learners' attention can move naturally, but as Radford (2010) points out, students' attention is not easily transformed into a socio-cultural mathematical structure. A student's attention will not move smoothly unless the subject of the learning is uncomfortable because a field of vague meaning is unfolded in front of the student.

Based on the discussion thus far, this article defines the teaching-learning process as follows.

First, by signs, the field of ambiguous meaning is forcibly unfolded in front of the teacher and student.

Second, through the joint action of the teacher and the student, the attention of the learner is structured. As the student's EP, EC, OP, and OC types of attention are transformed during this process, ambiguous mathematical meaning gradually becomes clearer.

METHODOLOGY

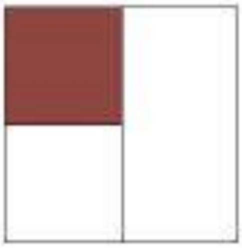
In this paper, a case study is attempted using data from a classroom situation collected in 2016. Participating students were their third year in elementary school in Korea, and they were already familiar with fractions in their regular classes.

This class was conducted for 40 minutes and aimed at confirming the students' understanding of fractions and clarifying the meaning of fractions. In this study, the students were asked to perform an activity that required them to perceive a fraction as a "whole-part" and "equal part" through a specially designed task. This task is a modified version of the task designed by Deborah Ball, Laurie Sleep and Meghan Shaughnessy and used as part of the University of Michigan Elementary Mathematics Laboratory project.

The following tasks are presented to students.

1. Observe the following figure. What is the fraction of the painted part in the large rectangle? Find as many answers as possible and explain why your arguments are correct.

2. Observe the following figure. What is the fraction of the painted part in the large rectangle? Find as many answers as possible and explain why your arguments are correct.



As noted above, in order to capture the student's learning process fully, it is necessary to capture how operative and collective attention occurs, as well as explicit and personal attention. In this study, we focus not only on personal attention, but also on the collective attention which appears when the act of utilising the semiotic resources of the subject creates an irreducible relationship with the use of the semiotic resources of others. We also want fully to understand the learning process, paying attention to the effects of operative attention on students' judgment processes.

To do this, a data analysis is conducted on the one hand by identifying how the attention types of learners are collectively coordinated through the use of semiotic resources such as gazes, gestures, and utterances and on the other hand by identifying how operative attention affects subject's judgment.

Three camcorders and two recorders recorded the responses of the teachers and the students. We will analyse the various types of data obtained through triangulation.

RESULTS

When encountering problem 1, ST5 answered, "One-third and no other answers." This means that problem 1 failed function as a sign. In order for students to pay attention to the meaning of equal parts and the whole-part, the ambiguous meaning of the fraction must be unfolded in front of the learning subject.

The following depicts the situation of the class after the students encountered problem 2.

ST3: Is it not a typographical error? Is this fraction possible?

ST2: We are not learning this fraction in the third grade.

ST4: Typographical error. The problem is strange. Error.

ST3: Is this one third?

ST1: I wrote it. Hey. I wrote it. It's one fourth or it's one third. I do not know why.

This means that problem 2 breaks the stable state of the students and succeeds in putting the field of ambiguous meanings in front of them. One-third is the answer given by the students because they pay attention to the number of rectangles, and one-fourth is the answer because the students pay attention to equal parts. At this point in the process of the formation of the students' judgment, the EP for the number and the OP for the equal parts are operating at the same time.

Figure 1: ST2's gesture

Figure 2: The researcher emphasises the rectangle

Figure 3: ST2 pointing to the picture on the blackboard and ST3 focusing on it

As shown in Figures 1, 2 and 3, collective attention is taking place between the teachers and the students, and the student's attention is shifted by the teacher's actions. In this process, the various types of semiotic resources that teachers and students use are successful in transforming the students' attention.

ST5 (Focusing on the teacher's gesture): But what is on top is one third of the equally divided, and that below is not equally divided.

Acoording to the actions of the researcher and the teacher, the attention of ST5 could move to equal parts.

After students' collective attention to equal parts had formed, ST1 suddenly referred to a mixed fraction. The second picture on the chalkboard was highlighted by the researcher as a rectangle with different colours. It is likely that ST1 would pay attention to the highlighted rectangle, judging the rectangle as 1 and the rest as a half. In order to view a given picture as a one and a half, students must be able to interpret the large rectangle mentioned in the problem as a rectangle rather than an entire square. Here, students direct collective and operative attention to the whole.

In the teaching-learning situation thus far, the students gradually became able to pay attention to numbers, equal parts, and the whole. Considering that we need to pay attention to the meaning of the whole and equal parts in order to understand the meaning of the fraction, we can judge that the process of teaching-learning is succeeding. In subsequent discussions, the students demonstrated that they were structuring attention and recognizing the meaning of fractions through semiotic resources.

Figure 3: ST1's attention on the picture on the blackboard

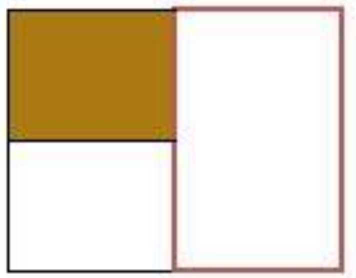


Figure 4: The shape of the figure to which ST1 paid attention

Figure 5: The teacher showing the whole and the half through a gesture

ST3: Why do you think this, this, this is not one and a half?

ST5: The fraction refers to part (making a gesture that divides the square into two rectangles) of the whole (which draws a large square), but this (a gesture that draws two rectangles) is not the whole thing. This (a gesture representing two rectangles) is one half.

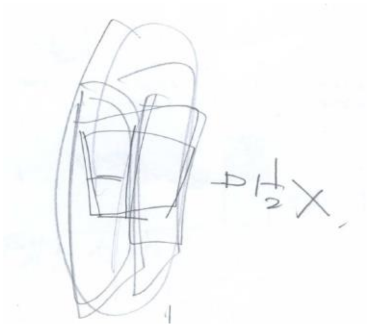


Figure 6: Figure drawn by ST5 denying a mixed fractions

At this point, ST5 pays explicit attention to the equal parts, the whole, and to the relationship between the whole and the parts. ST5 explains his claim to ST3 and succeeds in intentionally structuring his attention using words and gestures. Learning about fractions has been accomplished.

DISCUSSION AND CONCLUSION

In the semiotic approach, the mechanism of the teaching-learning process surrounding the subject, the teacher, and the meaning depends on how the concept is conceptualised. In this paper, I introduced Delueze's concept of signs and the notion of collective attention. The semiotic approach using these proved to be a great help in understanding complex teaching-learning phenomenon.

From the results of this study, the following suggestions can be put forth.

First, for successful learning, a sign must be presented that will disrupt the student's stable state and lead the student to a field of ambiguous meaning. That is, signs must be considered as a prerequisite for learning. In particular, it is more effective if an ambiguous question is added that prevents the student's attention from being fixed.

Second, as can be seen from the situation in which meaning of the mixed fraction is actualised, in the class situation, meaning comes in front of the teacher and the students accidentally through contact with the materials. That is, learning is characterised by contingency. In this sense, learning can be regarded as essentially a creative process (de Freitas & Sinclair, 2014).

Third, the teacher must constantly identify where the student's attention is headed. In particular, students can use multimodal semiotic resources such as gestures, diagrams, and tone, termed by Arzarello (2006) semiotic bundles, to indicate where their attention is heading. The teacher must continue to identify the students' attention while remaining sensitive to this.

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THE ENACTMENT OF A FLIPPED CLASSROOM APPROACH IN A SENIOR SECONDARY MATHEMATICS CLASS AND ITS IMPACT ON STUDENT ENGAGEMENT

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Once considered the domain of higher education, the flipped classroom is increasingly being implemented in secondary school settings. Although enactments of the approach vary, it typically involves the use of digital technologies to shift direct instruction to the home environment, providing for more targeted in-class teaching. This paper describes how a flipped classroom approach was enacted in a senior secondary mathematics classroom and reports on the students' and teacher's perceptions of the impact of the approach in terms of students' engagement with mathematics. It adds to the limited research in this area through providing an account of how flipping the classroom works in practice and its potential for engaging students in mathematics.

BACKGROUND

According to Bergmann and Sams (2012), there is no such thing as *the* flipped classroom, but the basic concept is "what is traditionally done in class is now done at home, and that which is traditionally done as homework is now completed in class" (p. 36). Bergmann and Sams (2012), who have been credited with pioneering the approach, advocate that there is no single way to flip a classroom, no specific methodology to be replicated or checklist to follow; hence the importance of providing descriptions of cases such as the one documented in this paper. Proponents of the approach note benefits such as differentiated teaching for a range of student abilities, increased student-teacher interaction, self-pacing and greater student engagement. The potential for the approach to engage students is an important consideration as student disengagement in mathematics is of ongoing concern (e.g., Skilling, Bobis, & Martin, 2015). The autonomous nature of the flipped classroom approach suggests that it may influence students' motivation to engage in mathematics, as autonomy, along with competence, and relatedness, has been linked with increasing extrinsic and intrinsic motivation (Abeysekera & Dawson, 2015). The study discussed in this paper investigates those links through specifically addressing the following research questions: What is the nature of a flipped classroom approach enacted within a senior mathematics classroom? What are the students' and teacher's perceptions of the impact of this approach on their engagement with mathematics?

REVIEW OF THE LITERATURE

The terms 'flipped classroom', 'inverted classroom' and 'flipped learning' appear to be used interchangeably in the literature, but a flipped classroom does not necessarily mean flipped learning, which is defined as:

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 281-288. Singapore: PME.

> … a pedagogical approach in which direct instruction moves from the group learning space to the individual learning space, and the resulting group space is transformed into a dynamic, interactive learning environment where the educator guides students as they apply concepts and engage creatively in the subject matter.
>
> (Flipped Learning Network (FLN), 2014, para. 1)

There are different interpretations of the approach and associated variations in implementation strategies, with flipped learning and mastery of topics being the ultimate goal. According to Bergmann, Overmyer, and Wilie (2013), flipped learning is characterized as a: means to increase interaction and personalized contact time between students and teachers; space where students take responsibility for their own learning; classroom where students who are absent do not get left behind; class content is permanently archived for review or remediation; class where all students are engaged in their learning; place where students can receive a personalized education. Straw, Quinlan, Harland and Walker (2015), who explored how flipped learning could be delivered in UK classrooms, identified the following features that distinguished flipped learning from more traditional approaches: Homework time is typically used to deliver new content to prepare students for lessons, as opposed to being used for consolidation and revision; greater use is made of online learning such as videos, presentations and exercises than offline learning such as textbooks and worksheets; teachers spend more time in lessons coaching and facilitating learning and less time providing whole class instruction and demonstration. While the ultimate aim of flipped learning may be for students to achieve mastery of topics that are "individually based and student paced" (Guskey & Gates, 1986, p. 74), Bergmann and Sams (2013) acknowledge that teachers may adopt flipped learning principles to varying degrees, without necessarily achieving full mastery.

Enacting the flipped classroom

A variety of enactments of the flipped classroom are represented in empirical studies. The most prevalent appears to be the model typically used in tertiary settings whereby lectures and instructional videos are prepared and recorded by teachers for their students to access at home, and class time is spent on more practical tasks with the students essentially working at the same pace (e.g., Clarke, 2015; Strayer, 2012). Other studies document enactments whereby teachers access online resources, such as Khan Academy, and make those available for students to view prior to attending class (Straw, et al., 2015). A limited number of studies have documented enactments of varying degrees of mastery (e.g., Muir, 2016). Common findings from the studies indicate positive impacts upon teaching and learning practices and students' engagement, learning and skills. Straw et al. (2015), for example, found that the approach provided more time for practicing and applying knowledge and skills, questioning and higher level discussions, individualised support and increased understanding of students' learning styles. They also reported that students showed increases in engagement in learning, knowledge and understanding, Fulton (2012) reported that students in a secondary school context enjoyed working at their own pace, appreciated being able to review material by replaying videos, and completing more challenging problems in class rather than at home. In a study conducted in senior secondary mathematics classes, Muir (2016) found that in contrast with traditional teaching practices experienced in the past, students found the video tutorials prepared by their teachers to be relevant, engaged their attention, provided for greater autonomy over their learning and enabled them to attain their goal of mastery over their learning.

THEORETICAL FRAMEWORKS

The study discussed in this paper uses the Four Pillars of FLIP framework (Flipped Learning Network (FLN), 2014), to interpret the enactment of the flipped classroom approach as it occurred in a selected senior mathematics class. The framework identifies and describes key features necessary for learning to occur in a flipped classroom: flexible environment; a shift in the learning culture; intentional content; and professional educators. These features are enacted for example, when teachers create flexible learning environments, prioritise concepts used in direct instruction, differentiate content to make it accessible and relevant, and make themselves available to students as required (FLN, 2014). In order to investigate the impact of the flipped classroom on the teaching and learning of mathematics with the selected class, self-determination theory (SDT) (Deci & Ryan, 2008) was used to investigate whether or not students' needs of competence, autonomy and relatedness were being met. According to Deci and Ryan (2008), intrinsic motivation (a natural inclination toward assimilation, mastery, spontaneous interest, and exploration), is catalyzed when conditions such as competence, autonomy and relatedness are present. Motivation is considered to underpin engagement, with both playing a large part in influencing students' drive to participate and learn at school (Martin, 2007).

METHODOLOGY

The study employed a mixed-methods approach (Creswell, 2003) whereby sequential methods were used to inform the collection of qualitative data. An exploratory case study methodology was selected in order to bring new understandings to the fore (O'Leary, 2010) with data sources including an online survey, interviews and classroom observations. The survey contained 24 questions consisting of responses to Likert-scale items about the use of online resources and seven open-ended questions. The items were adapted from an existing instrument designed to investigate students' self-initiated use of video tutorials including logistical and attitudinal aspects (see Muir & Chick, 2014). Semi-structured interviews were designed to allow the researcher to probe more deeply into students' experiences of the flipped classroom approach as reported through the survey and were conducted with focus groups. The teacher interview schedule was designed to elicit information about the enactment of the approach and its impact upon the teaching and learning of mathematics. Classroom observations were used to triangulate the data collected from the surveys and interviews and to answer the first research question.

The participants were 15 Grade 12 students (aged approximately 17 years) who completed the survey (5 male and 10 female) and their Grade 12 mathematics teacher, Ms Brown[1]. Eight students participated in focus group interviews which were audio-taped and conducted after survey completion and classroom observations. The study was conducted in a large independent metropolitan school with a 'Mathematics Methods' class. Mathematics Methods is a senior secondary pre-tertiary course which covers topics such as functions, calculus and statistics and is externally examined. This was the fifth year that Ms Brown had taught the course and her first year with producing her own videos (she had used Khan Academy videos in the past). Quantitative data from the survey were analysed using descriptive statistics expressed in percentages for the Likert scale items. Qualitative data from the surveys and interviews were transcribed and analysed using reflexive iteration (Srivastava, 2009) whereby each sentence in the transcripts was coded, initially through emerging themes. The transcripts were then re-analysed and instances of the components related to the Four Pillars Framework and SDT were identified. This process limited researcher bias in that the researcher was open to the possibility of other themes emerging and not restricted to narrowing the data to pre-determined themes.

RESULTS

The flipped classroom in practice

Lesson observations showed that Ms Brown's enactment of the flipped classroom involved an expectation that her students had watched her pre-prepared video tutorial prior coming to class, with the majority of class time spent on completing exercises in the prescribed textbook. The two lessons observed for the purpose of this paper involved the students solving simultaneous equations using matrices. Figure 1 shows an example from the textbook that students were expected to solve.

Find the determinant of the matrices:

a $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ **b** $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ **c** $C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$

Figure 1: Finding the determinant of a matrix (Jones, Evans & Lipson, 2012, p. 726)

Each lesson began with an eight minute 'warm-up' where students worked individually from their textbooks. Ms Brown then facilitated students' oral responses to the problems, then briefly revised some of the content from the video tutorial that most students indicated they had watched. The remainder of the lesson (approximately 40 minutes) was spent on working through allocated questions in the textbook, with Ms Brown individually assisting students who required assistance. Students indicated in the interviews following the lessons that that was typically what happened in their mathematics classes.

At the time of the study Ms Brown had recorded approximately 20 video tutorials, all based upon topics in the textbook and all lasting for about an hour in duration. In her interview, Ms Brown said that she preferred to create a video for each topic and then direct students to watch different parts of it, rather than break it up into shorter videos. She used Powerpoint with an OfficeMix add on to record her videos, which students accessed through an emailed link. This was provided to students at least three days prior to class. Student survey data showed that 100% of students agreed that the tutorials helped them to understand a concept and that the tutorials were helpful. Just over half (54%) of students indicated that the tutorials were of the right length, with only 38% indicating that they watched all of the tutorials from beginning to end. Interestingly, 77% indicated that they found the tutorials boring, yet 85% of students indicated that they accessed all or most of the video tutorials that were made available. Student interview data provided an insight into how the students accessed the video tutorials at home:

> I watch it all, but then if she says something I don't understand, I might go back and watch it all again. [Anna]

> She sends [emails] us sheets and questions – like summary pages of the chapters and that's what she goes through on the video so I'll have them with me and be writing down what she said … and if you don't get it, you can ask her questions [later]. [Helen]

Students were also asked in the interviews how their classes were different this year as compared to previous classes which were not flipped. The following is illustrative of the comments received:

> We didn't do questions like this, not all the time, like we used to sit and listen, but now she's doing more questions in class so that gives you more time with her one on one if you have questions, whereas I can remember some other topics, we would just sit and listen, and … we wouldn't do as many questions like we were doing today. [Hayley]

> It's better having the video and watching it at home and being able to come and ask the teacher if I am still unclear about how to do something or a particular concept … I think it's better than last year where we would go through the book and rather than have lengthy explanation in class, it's better to have an idea before you get to class. [Anna]

Students also identified that the prepared video tutorials were a good source of information, but viewed them as complementary, rather than a replacement, for either the teacher or the textbook, as the following responses indicate:

> You can access it easier than a teacher in a class with other students, as well as it being specific to the question you need. [open-ended response, survey]

> If you don't understand the book, watching another person explain the concept can help you gain a better grasp of the ideas and skills. [open-ended response, survey]

> For me, if the teacher said, watch this video as compared to doing 20 questions in the text book, I would do the video – it's more appealing, [Anna, interview]

Motivating factors

Data collected from surveys and interviews provided evidence that students' motivational needs for *competence, autonomy* and *relatedness* were addressed through this approach. In addition, another strong theme, *relevance*, emerged as being influential in students' motivation to access the video tutorials.

In terms of meeting students' need for *competence*, 92% of students agreed that they performed well in class tests because they watched the tutorials, while there was 100% agreement that the tutorials helped them understand the work undertaken in class. Qualitative comments which referred to this aspect included the following:

> Sometimes they [videos] explain it differently so that I may understand better. [open-ended response, survey]

> If I don't understand the concepts I can go back afterwards and revisit so it's like Ms Brown teaching me again. [Anna, interview]

Students also recognized that competence varied between individuals, alluding to the individualized nature of the approach, which was also noted by Straw et al. (2015). Brittany, for example, in her interview noted that:

> Sometimes with a whole class of say 20, if everyone has a question, then the whole class time is taken up with question time … but in the video, lots of questions are answered there and even if people do have questions, instead of about 18, there's maybe 2 or 3, so you've got much more time and much more availability to talk to Ms Brown separately.

Comments related to *autonomy* often included reference to self-pacing, accessibility and convenience. Open-ended responses in the survey, for example, included "You can access the videos and information from anywhere", "You can work at your own pace without being pushed ahead or slowed down …" and "They are a great way to get a head start on the next lesson". In her interview, Helen stated that "You can always go back and view them, not like last year when you had to continuously ask for help".

Students varied in their perceptions as to whether or not it was important that the videos were prepared by Ms Brown. Helen, for example, stated that:

> I think the way Ms Brown does it, she does it the easiest way possible so that we can remember it and do it by ourselves … I feel like Ms Brown does it better than a lot of other teachers would …

Abigail and Hayley agreed, with Abigail stating that:

> You understand it better when it's someone you know … and they can explain it again in a similar way in class if they have to.

As was found in other studies (e.g., Muir, 2016), having a sense of *relatedness* with the teacher was a strong motivator in its own right. The following comment from Anna was illustrative of students' appreciation of the work involved with the creation of the videos:

> You can tell she's really put in the effort so you know it's an easy task in return to her … so it's no work just to watch it.

Anna, however, also felt that:

> I don't think it's really important who does it – whether one teacher does the video or the entire maths faculty … but what's good about a teacher from school doing it as opposed to Khan Academy is that they know what the curriculum is and know what's important to focus on … The few times I did that [looked up on Google] it was extremely lengthy and only a few relevant points so it is easier having Ms Brown give us the videos – it's a lot more concise and relevant to what we want.

The above comment indicates that *relevance* is a key motivating factor; this was also endorsed by Brittany who stated that accessing other online resources were "not very helpful … the language used was a bit formal and not very easy to understand".

DISCUSSION AND CONCLUSION

The results indicate that students in Ms Brown's class were engaging with the flipped classroom approach and that her enactment included some of the features associated with flipped learning as identified by the FLN (2014). While classroom observations showed that the lessons were largely dominated by students' completion of textbook problems, there was little whole class demonstration by the teacher which was experienced by these students in the past. While a flexible environment, in terms of spaces and time frames, was not a key feature in this context, Ms Brown did demonstrate that she was intentional about the content she would present to students via the video tutorials and how that would be supplemented with the textbook exercises. Consistent with flipped learning features, Ms Brown determined what needed to be taught and which materials students should explore on their own (FLN, 2014). She also made herself available to all students for individual, small group, and class feedback in real time as required (FLN, 2014). Students perceived their experiences of this approach positively, with the results indicating that it met their needs for competence, autonomy, relatedness (Deci & Ryan, 2008), and relevance. The study has implications for mathematics teachers who may find this approach beneficial in terms of providing students with increased autonomy over their learning, leading to achieving competence with a subject that can be challenging and inaccessible for many.

Note

[1]pseudonyms used for teacher, school and students throughout

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UNDERSTANDING THE NATURE OF SELF-EFFICACY IN LEARNING MATHEMATICS

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This paper investigates the importance of self-efficacy in learning tertiary mathematics using quantitative measures. In line with Bandura's (1997) theoretical framework of self-efficacy, multiple regression data show that metacognitive self-efficacy (Self-belief in using cognitive, motivational, selection processes and Self-belief for self-regulated learning) are key predictors of success in learning mathematics. Further results reveal a positive correlation between self-efficacy in problem-solving and mathematics results. Therefore, an important point for tertiary practitioners to consider is to introduce these ways of developing self-efficacy in mathematics curriculum and student support in accord with the theory of self-efficacy.

OVERVIEW

This paper is influenced by an extensive study in New Zealand led by Mike Thomas, which reported that several practitioners perceived that first-year tertiary students lacked confidence in learning (Thomas et al., 2010). A pertinent result was that the high-achieving tertiary students felt that their level of confidence in mathematics was lower at tertiary level than in their secondary education, which indirectly lowered their level of preparedness in learning mathematics. Following their research, this paper will investigate the self-efficacy levels of first-year mathematics students in a New Zealand (NZ) tertiary institution. Our research questions are 1) What is the nature of self-efficacy? 2) In what way does self-efficacy predict success in learning mathematics?

THEORETICAL BACKGROUND

Self-efficacy is concerned with human enablement rather than personal judgement of one's ability (Bandura, 1997). People with high self-efficacy tend to make an effort and overcome difficulties because they are driven by personal affirmation which draws on one's self-knowledge (based on prior mastery experiences) and adapt their knowledge and skills to successfully accomplish future tasks. This sense of efficacy increases one's determination to succeed as well as promotes the use of self-regulation strategies for planning and organizing instructional activities, utilising resources, adjusting one's own motivation. It has been observed that having a strong belief in using self-regulation strategies determines academic success. Students are agents of their own learning so when they develop self-belief in using these strategies, they become more self-regulated learners. Mulat and Arcavi (2009) have

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 289-296. Singapore: PME.

reported that university mathematics students attributed their success to using self-regulation strategies such as, studying without distraction, completing homework, seeking peer and teacher support, paying attention in class, preparing well for examinations, persistence in solving challenging tasks, and making concerted effort on school tasks. Their results suggest that using self-regulated strategies reflect the students' metacognitive belief in learning which in turn, translates their will to achieve into learning processes and effort to produce positive outcomes.

In mathematics education research, Cretchley (2008) stated that to advance affect research, it is important to clarify the terms based on Bandura's theoretical framework since past researchers tend to generalise its concepts rather than assess~~ing~~ it within specific contexts of learning, which tends to result in misconceptions in self-efficacy research. Therefore, this study conceptualises the nature of self-efficacy in mathematics education. In his theory, Bandura (1997) states that self-efficacy beliefs produce learning outcomes through cognitive, motivational and selection processes. First, cognitive processes are described as thinking processes which involve the acquisition, organization and use of information. These processes underpin purposive learning behaviour, which is a function of self-appraisal of capabilities, resides in forethought and in the self-regulation mechanisms by which forethought is translated into incentives and guides for purposive actions. The stronger the self-efficacy, the higher the goals individuals set themselves to attain performances. People with high self-efficacy mediate through cognitive processes by visualising success, which in turn provide cognitive support and guides for attainment. Secondly, self-efficacy plays a key role in the self-regulation of motivation via motivational processes. These include causal attributions, outcome expectancies, and cognized goals. In causal attributions, Bandura (1997) states that people with high self-efficacy attribute poor outcomes to lack of effort whereas those with low self-efficacy attribute failure to low ability. Next, in outcome expectancies, people expect their behaviour and actions to bring about valued outcomes so people with high self-efficacy are more likely to persevere and attain successful outcomes because their goal setting is governed by the cognitive processes of motivation. Thirdly, in selection processes, individuals are partly the product of their environment because they choose the social and physical environment and types of activities that they judge themselves to be capable of handling. In a nurturing learning environment, people are predisposed to achieving their goals and make deliberate choices to manage challenging activities in these situations. Therefore, based on the abovementioned processes in self-efficacy, this study aims to conceptualise these metacognitive forms of self-efficacy and examine their relationships with outcomes of learning mathematics.

Empirical studies have revealed a positive relationship between strong self-efficacy in solving mathematics problems and high mathematics performance but some researchers suggested that there was a need to examine their bi-directional relationships and factors of learning. In an international study, Williams and Williams (2010) argued that causal relationships between self-efficacy and mathematics performances have been difficult to prove as researchers were forced to assume one position or other when they used recursive statistical models to estimate the model. To illustrate this point, they have modelled the concept of reciprocal determinism, which refers to the psychological functioning involving behavioural, cognitive and environmental elements (Bandura, 1986) using structural equation modelling to report bi-reciprocal relationships between cognitive form of self-efficacy and achievement of secondary mathematics students in twenty-four out of thirty-three nations.

Other researchers have shown that self-efficacy predicts success in mathematics performance (Hailikari, Nevgi, & Komulainen, 2007; Hall & Ponton, 2005; Marcou & Philippou, 2005; Pajares & Kranzler, 1995; Pajares & Miller, 1994; Skaalvik & Skaavik, 2011; Stevens, Olivarez, Lan, & Tallent-Runnels, 2010). Marcou and Philippou (2005) reported that motivational beliefs as a function of self-efficacy correlated with problem-solving performances of fifth and sixth graders. In line with the social cognitive theory of self-efficacy (Bandura, 1997), Marat (2005) investigated determinants of self-efficacy with secondary mathematics students. Their discriminant analysis showed a positive correlation between high achievers in mathematics and high scores in self-efficacy in solving algebra problems, belief for self-regulated learning, selection and motivation strategies. A study of middle and high school mathematics students have found that self-efficacy was a better predictor of mathematics achievement than prior achievement (Skaalvik & Skaavik, 2011). This result was also evident for tertiary students of calculus in study by Hall and Ponton (2005) wherein it was found that university calculus students who reported high self-efficacy gained better results than other remedial students who also had low prior experience. To take another case in point, the path model data showed that there was a positive relationship between mathematical achievement and self-efficacy in problem-solving of ninth-grade and tenth-grade mathematics Caucasian students (Stevens et al., 2010). By comparison, Hispanic students scored poorly in mathematics and their confidence level, which suggests that some students succeeded in mathematics due to their high abilities and confidence. Pajares and Kranzler (1995) have concluded that students had high self-efficacy because they exhibited more effort and perseverance in challenging problem-solving situations. The abovementioned studies suggest that investigations of the way self-efficacy affects mathematical performance (at tertiary level) have been limited. Hence, more research is warranted to understand the psychological functions of self-efficacy in learning mathematics, particularly in tertiary education.

Literature suggests that positive self-efficacy breeds success whereas negative self-efficacy spawns failure in learning mathematics. Conversely, past successes increase self-efficacy levels and past failure diminishes it. In reality, this phenomenon might reflect a misconception of tertiary mathematics students. On one hand, lecturers might perceive first-year mathematics students to be confident. On the other hand, for many under-prepared students, the reverse is true. While lecturers focus on teaching mathematical concepts in class, such students become disenfranchised with the lack of opportunities to increase self-efficacy and possibly experience failure in learning mathematics. Nevertheless, some university studies have investigated the development of self-efficacy. Parsons, Croft, and Harrison (2011) interviewed seven engineering mathematics students at the Harper Adams University College, who reported that the provision of student support has somewhat helped students to develop their cognitive processes. Hence, the confident students set high goals of mastering all the topics whereas the less confident students avoided doing the difficult mathematics. They also developed a low self-belief in using motivational processes as they were less motivated to work hard and tried to avoid difficult mathematics questions, which lowered their self-confidence and made them choose alternative questions in the examinations. Further results showed that selection processes were reflected by their deliberate choices to study mathematics. More positive results were reported by Falco, Summers, and Bauman (2010). Their study skills programme was effective because their students developed greater self-efficacy, self-regulated learning, interest and engagement in learning mathematics and achieved better achievement scores. Therefore, although these studies were carried out in specific educational settings, these findings are likely to have important consequences for the broader domain of affect in mathematics education because understanding the role of self-efficacy sheds new light on its applications in learning mathematics.

METHOD

For this study, the participants were 166 tertiary students enrolled in the Business and Engineering programmes in a NZ tertiary institution. With ethics approval and participants' consent, their final assessment results were collected and linked to their survey responses. Originally designed by Marat (2005), the Refined Self-efficacy Scale was appropriate because it accords with the Motivated Strategies and Learning Questionnaire (Pintrich, Smith, Garcia, & McKeachie, 1991) and the social cognitive theory (Bandura, 1997). This survey consists of five-point Likert type scales which has Cronbach's alpha ranging between 0.76 and 0.91. The sub-scales included cognitive self-efficacy: Self-efficacy in solving numerical and measurement problems (SEI), geometry (SEII), algebra (SEIII), statistics (SEIV), Self-efficacy in using mathematical processes (SEV) and metacognitive self-efficacy: Self-belief in motivational, cognitive, selection strategies, Self-belief for self-regulated learning (SEVI). At the end of the scale, students had to assess how well they were doing of the course using a 9 -point numeric scale (1 as 'Very Badly'; 5 as 'about average'; 9 as 'Very well').

FINDINGS

Of the 166 students, 67 students (40%) completed the Refined Self-efficacy Scale (Marat, 2005). The majority of the participants were young (17-25 years old) and male (55.3%). Of those who had passed mathematics examination (79%), the same proportion of participants scored either A or C grades (31%). Considering each subscale SEI-VIII, we found that the participants had the highest scores for SEI (3.87), Following the aforementioned sub-scales, the overall average of cognitive self-efficacy level (SE I –V) was 3.47 and metacognitive self-efficacy level (SE VI) was 3.55.

Self-efficacy in mathematics and grades

The statistical software IBM SPSS Statistics 21 was used to analyse the quantitative results. Correlational analyses showed a direct correlation between self-efficacy in mathematics and grades at 0.01 and 0.05 significance levels. Based on Dancey and Reidy (2004)'s categorisation of the strength of correlation, strong correlations range from R = 0.7 to 0.9, moderate to be 0.4 to 0.6, weak as ranging from 0.1 to 0.3. In this study, mathematics examination results are a proxy of mathematical performance since the summative work constitutes 50% course weighting and is a uniform yardstick for assessing students' performance. Table 1 shows that the main findings were SEVI correlated more strongly with the expected grades (R=0.64, p=0.000) than the actual grades (R=0.30, p=0.018). In terms of mathematics self-efficacy, there were moderate but significant correlations between SEI (R=0.44, p=0.001) and SEII (R=0.35, p=0.035) with the expected grades whereas there were weaker correlations between SEIII (R=0.28, p=0.028) and SEIV (R=0.29, p=0.018) with the actual performances. Their expected results were correlated strongly with the actual grades (R=0.55, p=0.000).

	Exam marks	Expected grades	SEI	SEII	SEIII	SEIV	SEV	SEVI
Exam marks	1							
Expected grades	$.55^{**}$	1						
SEI	.10	$.44^{**}$	1					
SEII	$.35^{**}$	$.35^{*}$	.19	1				
SEIII	$.28^{*}$	.22	.14	$.58^{**}$	1			
SEIV	$.29^{*}$	.23	.000	$.52^{**}$	$.52^{**}$	1		
SEV	.079	.030	-.079	$.28^{*}$	$.45^{**}$	$.57^{**}$	1	
SEVI	$.30^{*}$	$.64^{**}$	.30	$.53^{**}$	$.46^{**}$	$.41^{**}$	$.38^{**}$	1

p<**0.05, p<**0.01

Table 1 Pearson Correlations (*n*=55)

Predictors of student performance

There were six predictors of success, we used to understand the concept of self-efficacy and how this affects the results of students. According to Nardi (2006), "regression analysis does not tell [us] about one particular respondent, since the statistics are based on aggregated data. ….Mostly what [we] do with regression is to construct a profile of characteristics related to the dependent variable from past data and use that to explain what already exists or to predict subsequent outcomes" (p.208). In order to establish a profile of successful students, we set up the independent variables self-efficacy (SE I to SEVI) and examination scores as a dependent variable and chose the linear regression model which assumes that the error term has a normal distribution with a mean of 0, the variance of the error term is constant across cases and independent of the variables in the model. When conducting the regression analyses, we tested if the linearity, normality and data independence assumptions of the dependent variables were satisfied. This method of analyses produced a model summary, which shows 32.7% of the variation in results is a result of the factors. We found that the low p value ($p=0.040$) in the analysis of variance table ($F=2.715$), suggests that the model is a better fit than using the mean of the sub-scales and that self-efficacy in using cognitive, motivation and selection strategies, self-regulated learning are significant predictors of the model (Beta=0.482, $t=2.335$, $p=0.027$).

DISCUSSION AND IMPLICATION

In response to the initial research question, some noteworthy results were positive correlations between the cognitive (Self-efficacy in solving mathematical problems), metacognitive self-efficacy (Self-belief in using cognitive, motivation and resource management strategies, Self-efficacy for self-regulated learning) and performance as measured by grades. Associated with this predictor was the finding that there was a positive correlation between expected marks and examination marks. This aligns with the theory that greater self-efficacy raises one's expectation to achieve high marks, which in turn, projects the actual performances. The correlation data somewhat matched past literature (Hailikari et al., 2007; Hall & Ponton, 2005; Marcou & Philippou, 2005; Pajares & Kranzler, 1995; Pajares & Miller, 1994; Parsons et al., 2011; Skaalvik & Skaavik, 2011; Stevens et al., 2010), suggesting that self-efficacy is not only about having a strong belief in problem-solving but a disposition to develop cognitive and metacognitive processes in learning. According to Bandura (1997), high performance in a particular task promotes self-efficacy, which in turn, emboldens individuals to work harder and develop further skills necessary for attainment in future tasks. Bandura further explains that in skill development "efficacy beliefs contribute to the acquisition of knowledge and development of sub-skills, as well as drawing upon them in the construction of new behaviour patterns" (Bandura, 1997, p. 61). The point is that having an expectation of positive outcomes and cognitive self-efficacy, alone are not sufficient, other functions of self-efficacy need to work in concert with it so that successful students gain mastery of mathematics skills.

To investigate the next research question, a significant linear regression finding was that the most appropriate predictor of successful performance in mathematics was self-efficacy in using cognitive, motivation, selection strategies and belief for self-regulated learning. Consistent with past research (Marcou & Philippou, 2005; Mulat

& Arcavi, 2009), this result suggests that success in learning mathematics is determined by metacognitive dimensions of self-efficacy. In line with Bandura's notion of self-efficacy, by forming selection, cognitive and motivational processes, students could manage their learning by taking ownership of their own learning through self-regulated learning behaviour, goal setting, expenditure of effort and intrinsic motivation. Evidence of under-prepared tertiary mathematics students in Thomas et al's (2010) study further confirms the role of self-efficacy in the teaching and learning environment. Therefore, enhancing student learning is about overcoming low self-belief in learning as well as forming motivational, cognitive, selection and self-regulatory strategies in learning.

These findings raise another question: How could educators increase their chance of achieving success in learning? Given greater political impetus to improve student achievement, practitioners should seriously consider the influence of self-efficacy on their learning. Ultimately, the value of such programmes could outweigh the high cost of failure borne by students and staff. Teaching faculties could incorporate both metacognitive and cognitive forms of self-efficacy in their curriculum, which were shown to be somewhat effective in previous studies (Falco et al., 2010; Parsons et al., 2011). In order to produce desired outcomes in affect development, tertiary institutions could offer more incentives for developing self-efficacy in mathematics programmes in order to raise mathematical achievement.

CONCLUSION

Our study findings show that self-efficacy enhances mathematics results. This tends to shift the onus of learning onto tertiary students who may receive appropriate support for learning mathematics. With improved self-efficacy, these students tend to succeed in learning mathematics, which can serve as a gatekeeper in engineering and business programmes. Although the correlation data did not show causal relationships, we found that cognitive and metacognitive self-efficacy were positively correlated with performance and the most appropriate predictor of success was metacognitive self-efficacy. In this respect, our results suggest that as a result of lower self-efficacy, first-year students may be at-risk of failing mathematics. However, if students develop their metacognitive and cognitive forms of self-efficacy, their chances of achievement will increase. Therefore, an important point for practitioners to consider is to introduce new ways of developing self-efficacy in mathematics curriculum and student support in accord with the theory of self-efficacy.

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TEACHING MATHEMATICS FOR JUSTICE: PEDAGOGIES OF DISCOMFORT, CONTRADICTIONS AND DIALOGUE

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This paper explores learning to teach mathematics for social justice (MfSJ). As practicing teachers, graduate students and teacher educators the five authors with three colleagues from five countries came together to dialogue on possibilities and challenges of teaching MfSJ. Social justice projects were designed and some implemented in mathematics classrooms. Transcripts of each dialogue meeting inspired questions for the next. Results indicate contexts for teaching MfSJ were inspired by perceived curiosities of students and personal interests of instructors on local and global issues. Challenges were analyzed in terms of contradictions and expressed discomfort. The study highlights complexities of learning to teach MfSJ and offers pedagogies of dialogue and discomfort toward continued learning.

INTRODUCTION

There is no shortage of global issues requiring attention: more than 65 million people world-wide are displaced (UNHCR, 2016); 9 of the 10 warmest years on record have occurred since 2000 (NASA, 2017); global average sea level has risen 178 mm over the past 100 years (NASA, 2017); 1 person in 10 worldwide lacks access to safe drinking water (WHO, 2015). Addressing such complex global issues requires analysis and discussion with various stakeholders (e.g. students, teachers, communities, experts) across multiple perspectives (e.g., political, social, economic, and cultural) and fields (e.g., sciences, humanities, and arts). Preparing students and teachers to engage mathematically in understanding and responding to such complex issues, including concerns of diversity and equity, is a goal of teaching mathematics for social justice (Gutstein, 2012; Kumashiro, 2015; Wager & Stinson, 2012) and critical mathematics education (Skovsmose, 1994, 2016). There is limited but increasing research on conceptualizing a critical approach to mathematics education at elementary and secondary school levels (Gutstein, 2012; Skovsmose, 1994) and teacher education (McLeman & Piert, 2013). Our research adds to this growing field by analyzing the issues and challenges of a critical approach to mathematics education for justice through our own dialogue circle (Freire, 1970/2000) of mathematics teacher educators, graduate students and practicing teachers. Our inquiry draws upon our own experiences as relatively new social justice mathematics educators asking: What justice issues are identified as possible in mathematics education across international contexts? What are the challenges across different perspectives?

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RELATED LITERATURE

Decades ago Frankenstein (1987) in the United States and Skovsmose (1994) in Denmark independently developed the idea of critical mathematics education to challenge the neutral, acultural view of mathematics and examine the relationships of mathematics to social, political and cultural contexts in which mathematics is used. Frankenstein (1987) drew upon Freire's (1970/2000) ideas of liberating education to develop pedagogical approaches for teaching adults mathematics by using statistical data to help "students to reconsider their previously 'taken-for-granted' beliefs [and] deepen and increase the range of questions they ask about the world" (1987, p. 197) thereby engaging students in critical questioning of mathematics and their social reality.

Gutstein (2006, 2012), extended Freire's ideas of literacy development to mathematics education by helping students learn to read mathematics (to do mathematics), learn to read the world mathematically (to understand how mathematics is used for social and political decision making) and learn to write the world with mathematics (to change the world with mathematics). An example Gutstein (2012) offers is the Boundaries Project developed from an immediate and tense community issue that directly affected his students: redrawing the enrolment boundaries between two community schools. Students were asked to discuss and research "a fair solution for both communities" by (Gutstein, 2012, p. 304). They examined school attendance data, calculated acceptance probabilities, and engaged in discussions about immigration issues and movement. Gutstein (2012) found students engaged and motivated, with opportunities to "examine their lived experiences, deepen their sociopolitical awareness, and learn mathematics" (p. 306). Similarly, Skovsmose (1994, 2016) proposes a critical mathematics education–one that allows any group of students with any type of mathematics "to address critically any form of reading and writing with mathematics" (2016, p. 6). Skovesmose offers the idea of mathemacy as a tool allowing students to participate in understanding and transforming society. Characterizing mathematics as a language Skovsmose (1994) points to how a "grammar influences the possibilities available for what we can express and the purposes for which we can use our language" (p. 3). In this way Skovsmose draws attention to what he calls the formatting power of mathematics.

There are few studies documenting learning to teach mathematics for social justice by teacher candidates, practicing teachers, and teacher educators. An exception is Bartell's (2013) case study of eight secondary mathematics teachers as members of a graduate level course on learning to engage with ideas of teaching MfSJ. Bartell reported teachers experiencing tension in negotiating the dual goals of mathematics and social justice, sometimes losing the complexity of the issue or separating the goals leading students to "express existing socially produced misconceptions rather than interpreting, resisting, or rewriting them" (Bartell, 2013, p. 158).

This review of related literature highlights a body of work that has helped conceptualize what might be involved in teaching mathematics for social justice. Yet we know little about what is involved in learning to teach for social justice. Our study addresses this gap exploring our own developing understandings of social justice, the possible contextual issues for exploring math and social justice, and the perceived challenges involved in teaching math for social justice.

THEORETICAL FRAMEWORKS

We are inspired by the work of Freire, Gutstein, Frankenstein and Skovsmose and their views of moving students to engage in a critical mathematics education. This involves, as Freire (1970/2000) argues, a problem-posing education–a humanist liberating education that has students "develop their power to perceive critically *the way they exist* in the world *with which* and *in which* they find themselves" (p. 83).

Gutstein (2006) offers a framework distinguishing three types of knowledges used to design and make sense of teaching mathematics for social justice. Community knowledge refers "to what people already know and bring to school with them" (p. 300). Critical knowledge "is knowledge about the socio-political conditions of one's immediate and broader existence" (p. 301) while classical knowledge "refers to formal, in-school abstract knowledge" (p. 302) making up what is often referred to as school mathematics. Although the relationships among these types of knowledges requires further problematizing it provides ways of imagining and critiquing contexts for mathematics problems connected to students' lives, of societal relevance, and with opportunities to investigate social realities.

Vithal (2012) explores connections between mathematics education, democracy and development and suggests a pedagogy of conflict and dialogue that has contradictory themes of freedom and structure; democracy and authority; mathematics and context; equity and differentiation; and potentiality and actuality. Vithal suggests "both conflict and dialogue are needed in a pedagogy that attends to issues of democracy and development from a critical perspective" (p. 8). We use Vithal's (2012) framework to examine the challenges of learning to teach social justice mathematics education.

Finally, we draw upon Zembylas' (2015) "pedagogy of discomfort" to examine the perceived risk and challenges of engaging in social justice mathematics education. Zembylas considers the tensions of a pedagogy of discomfort and ways of countering possible ethical violence that could arise. A pedagogy of discomfort can be described as "the feeling of uneasiness as a result of the process of teaching and learning from/with others" (Zembylas, 2015, p. 170). We use pedagogical discomfort as a way to theorize the multitude of emotions involved in a critical pedagogy of mathematics education.

With these frameworks we examine our own experiences as beginning social justice educators with a goal to contribute to Gutstein's (2012) call that "there is much work to do in theorizing and practicing social justice mathematics" (p. 300).

RESEARCH METHODS

We form a group of participatory researchers interested in social justice mathematics education. We are a team of eight with five of us coming together to write this paper. We perform overlapping identities as masters graduate students – practicing teachers (NC, DG and TH), doctoral students – researchers – instructors (KY, SB, and VR), and teacher educators – academic researchers (LB and CN). Our cultural/country backgrounds are diverse and include: Australia, Canada, China, Ghana, and the United States. As a group our interest in a critical mathematics education began in 2015-2016 with many of us enrolled in a course (taught by CN) focused on bringing math, community and culture together.

Following the course we met as a group to dialogue about our developing ideas of teaching mathematics for social justice. We structured these meetings in similar fashion to Freire's (1970/2000) dialogue circles where we engaged in open communication, critiquing our ideas and assumptions. Over a period of 15 weeks we met 11 times with each meeting being between 1 hour and 3 hours in length. We created a secure password-protected website where we collected and offered texts to extend our thinking about social justice. We audio recorded all dialogue circle meetings. For some meetings we met via a web-based instant messaging program (WhatsApp) to accommodate our diverse geographic locations of Australia, Canada and Ghana. We transcribed all meetings and exported text versions of our WhatsApp conversations. Transcripts from one meeting inspired questions and issues for the next meeting. We explored ideas such as what teaching social justice mathematics might look like, how to prepare school and university level mathematics students as well as ourselves as teacher educators to engage in a critical mathematics education, and given our diverse cultural backgrounds and life experiences what might count as contexts for generative themes of interest to our students and their communities as well as our own curiosities.

For this paper we analyzed transcripts in response to our inquiry questions: 1) possible contexts for engaging in issues of social justice with each other and our students across our international backgrounds; and 2) our perceived challenges of offering a critical mathematics education. Our dialogues provided opportunities for both data collection and analysis with the movement of our ideas, assumptions and responses/re-actions to each other's thoughts as more recursive than linear.

RESULTS

Resources for social justice problems

Analyzing transcripts of our dialogues we found questions around possible issues or contexts for teaching social justice mathematics permeated our conversations. Some of us had opportunities to design, implement, and reflect on problems created for teaching social justice mathematics, while others were in positions of imagining and exploring pedagogical possibilities. One group member, for example, developed and implemented four lessons connecting high school social studies and mathematics education. One of these four lessons involved Grade 9 students examining graphs and statistics to make connections between the events leading to the French Revolution and the 2011 Occupy Movement. What is interesting about this example is the teacher's intent of making the mathematics in a social studies lesson more explicit and worthy of examination. In this case, we could say that the community knowledge the teacher drew upon was the students' experiences in social studies as well as their interest in mass protests. Although we generally found it challenging to consider problems that brought mathematics and social justice issues together, each of us drew upon our individual strengths, curiosities, local experiences and cultural backgrounds to consider problems we thought would engage our students.

Group member, VR, developed a context for her calculus university students that focused on housing affordability. Vancouver, Canada has one of the most unaffordable housing markets in the world making it difficult for many, including university students, to find reasonable accommodation. The problem involves students using mathematics to understand the issue and using related rates to examine what continues to make it an issue. Another problem developed by TH began with an investigation of air quality in China. We considered how this issue could be made real for students both locally and globally. LB suggested beginning with an image she found of school children in China sitting outside at desks, in thick smog, wearing protective facemasks. Such an image could provoke questions around the role of mathematics in understanding air quality along with who 'owns' the problem of poor air quality, locally and globally. Other contexts for problems created included: democracy, under-representation and electoral voting in Ghana and Canada; air quality, pollution and environmental sustainability in China and Canada; and, the manipulation of mathematics in framing news in the Australian media.

We recognized our resources were limited by our own expertise and confidence in unpacking justice issues with mathematics. Nonetheless, although each of these problems was developed from experience and issues found in local contexts we were able to make connections across contexts, moving between local and global spaces.

Contradictions, discomfort and risk

Contradiction, discomfort and risk were threaded throughout our dialogues across three spectrums: mathematics, social justice, and pedagogy. The issue of mathematical expertise was raised and offered as a possible source of discomfort for some teachers as seen in the remarks made by this group member:

> I think a lot of the people teaching social justice may not have high levels of knowledge or comfort in math, and therefore they kind of choose not to approach it from that lens. (Group member NC)

Others voiced concern in risking professional status. We were a group of experienced teachers; three of us received recognition for their mathematics teaching through being nominated or receiving awards. Yet, as one member stated, "I feel like I'm a new teacher again reliving all the emotion that goes with the uncertainty" (LB). Two of us have graduate degrees in mathematics and noticed contradictions in expertise that could impact teaching. For example, VR claimed, “I feel I don’t know enough about social justice to lead my students in a meaningful direction in the same way I know I can do with mathematics.” This comment speaks to a concern of the group: it was more comfortable engaging in a problem-posing curriculum based on school mathematics (classical knowledge) than one that also included social justice issues (critical knowledge). The mathematical and pedagogical terrains were more familiar and therefore comfortable to some of us than the critical social terrain. Yet, there was also encouragement and hope in our dialogue on the positive aspects arising in risking such discomfort, freeing oneself from traditional classroom structures, and being “open to the curiosity, to the not knowing” (CN). Nonetheless we recognized some students and their parents might be uncomfortable with a mathematics curriculum inclusive of social justice issues. In addition engagement with some contexts/problems/issues can stir hopelessness when problems are large and ways of responding seem out of reach as seen in our air quality problem. Discomfort could be and was experienced at multiple levels.

As a group we also wondered about the place of social justice in/for mathematics at all levels. This is seen as one group member, a university-level mathematics instructor, questioned the limits of teaching all mathematics with social justice:

> Perhaps Real Analysis II is not going to be the place for social justice in math but I also don't want to totally discount any location [for teaching MfSJ]. (Group member SB)

This comment points to the uncertainty but also willingness of staying open to possibilities of bringing mathematics and social justice together. Our dialogue raised the importance of recognizing risk, uncertainty and emotions related to experienced teachers teaching in unfamiliar terrain.

Mathematics education as a social justice pedagogy

The nature of teaching MfSJ was a re-visiting topic for discussion during our dialogue sessions. A question we grappled with was, “How might MfSJ be a pedagogy?” One idea offered by CN suggests a pedagogy of MfSJ could also involve attending to respectful relationships within the class, such as students’ relationships with each other where “creating a just(ice) classroom environment makes it possible to engage in difficult mathematics with others, to learn together and make mistakes together, to fail together.” Our dialoguing around MfSJ as pedagogy deeply moved some members’ thinking about the nature and opportunities of MfSJ. For example, LB is aware of her changing understanding of MfSJ with her comment: “I feel earlier I was still considering it as a topic to be taught rather than a way in which we teach.”

Considering MfSJ as pedagogy arose in response to a concern raised by TH that for some educators and cultures bringing political/social justice issues into the classroom is discouraged. We challenged each to consider, as VR notes, how MfSJ could be a place “to develop a democratic classroom, where we work toward opportunities for all students to engage in mathematics.” Teaching MfSJ thus became both issue-based and pedagogically-based for the group. Our emergent understanding of MfSJ as a pedagogy is one way in which our experiences in working toward building democratic classrooms could be enacted.

DISCUSSION

Our study focused on developing ideas of experienced mathematics teachers and teacher educators dialoguing on the nature and complexities of teaching MfSJ. We drew upon varied resources for creating contexts to explore mathematics and social justice. None of us began with Gutstein's definition of classical mathematical knowledge to develop a problem. Instead, each of us began with current political issues generated from our own interests, curiosities and local contexts as well as our understandings of our students' interests and community knowledge. It was recognized for some that limited mathematical knowledge prevented engaging in teaching MfSJ, for others it was limited critical or social justice knowledge.

Results indicate a willingness and ability to create contexts for exploring mathematics and social justice yet also point to conflicts. Using Vithal (2012), we noticed conflicting themes of: 1) mathematics and context in the ways in which one may be privileged over the other; 2) democracy and authority in our perceived and actual cultural contexts for pursing teaching MfSJ; 3) potentiality and actuality in our understandings of our expertise in being critical mathematics educators; and 4) freedom and structure in recognition of the uncertainty and messiness of teaching MfSJ.

Our results also speak to the emotional aspect of teaching for social justice. Each of us expressed contradictions, discomfort, and risk in teaching MfSJ. This suggests the need to attend to such discomforting feelings and we are drawn to Zembylas' (2012) 'pedagogy of discomfort' as a way to acknowledge the discomfort for students, parents, teachers and communities in teaching and learning MfSJ. We argue, as does Zemblyas, that if a goal of social justice education is to become more critical, to challenge and stir-up unsettled beliefs, then "some discomfort is not only unavoidable but may also be necessary" (p. 164). How do we prepare ourselves and our students for a pedagogy of discomfort when teaching MfSJ?

We found dialogue as method offered us a space to engage with this discomfort and conflict. It offered a space to listen to new ideas, challenge our own assumptions, think out loud together, and imagine new possibilities. Yet dialoguing is difficult work. We were challenged to be sure that all voices were heard conceptually across our different teaching contexts and perspectives, as well as culturally and geographically across international contexts. We were not always successful but recognize that dialoguing itself involves a pedagogy of discomfort, requiring on-going efforts to support an open and critical stance in relation to each other.

Our study contributes to the call for further studies exploring learning to teach MfSJ. Conflict and contradiction, lead to pedagogies of discomfort, but addressed through dialogue provide opportunities for possibilities of hope for change.

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ARE WE PLAYING CHINESE WHISPERS? ISSUES IN QUESTIONNAIRE DEVELOPMENT

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In this paper, using gender as the topic in focus, we argue that transferring items and surveys from one cultural context to another might be highly problematic. For instance, in the Nordic context, gender issues are addressed in teacher education that reflects how equity is viewed on a societal and political level. Consequently, research on teacher students' gendered beliefs should acknowledge and take into consideration their knowledge of gender equity during item development. Rather than being translated and adapted, items should be re-constructed and embedded in the context in which they are to be used in order to achieve a valid, reliable instrument. In addition, how gender equity is expressed develops over time, which differs in different cultural contexts. Consequently, time is also a factor to consider.

INTRODUCTION

Both Norway and Sweden have interesting gender patterns regarding mathematics education. In both countries, girls either outperform or perform equally to boys in mathematics, this including upper secondary school. However, in higher education, more men than women opt for mathematically intense programmes, which is a sign of more traditional gender roles (OECD, 2015). Mathematics teacher education, as part of the educational system, also has strong gender patterns in recruitment, with an overrepresentation of female students in kindergarten and primary teacher education programmes (UKÄ, 2016). The result is a segregated workforce (Johnson & Muse, 2016; UKÄ, 2016) and this in two countries with strong traditions regarding equity and equality in both society and education (Imsen, Blossing, & Moos, 2016). Previous research suggests that teachers are one of the main contributors to students' views on mathematics and other STEM subjects and to their active choice to pursue these subjects in further education (Regan & DeWitt, 2015). Hence, the gender imbalance in the teaching profession might contribute to the gender imbalance in higher education in Norway and Sweden.

Hence, the gender imbalance in the teaching profession might contribute to the gender imbalance in higher education in Norway and Sweden. Researchers have identified a need to understand gender perspectives in mathematics education from a Nordic perspective (Imsen et al., 2016). Therefore, exploring the views of mathematics teacher students' about gender and mathematics could be key to understanding some of the driving factors of the observed gender patterns. Still, when attempting to reproduce a study by Leder and Forgasz (2010) that has been tested in other countries and on teacher students (Gómez-Chacón, Leder, & Forgasz, 2014), we failed.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 305-312. Singapore: PME.

After the instrument was adapted and carefully translated into Norwegian and Swedish, the questionnaire was piloted in two Norwegian universities. During piloting, it became apparent that Norwegian student teachers were reacting negatively to some survey items. Although, judging by their response patterns, they found the topic of the survey relevant to some extent, a typical written comment was ‘We are living in a modern world, why do you ask these questions in this way?’. In the words of Andrews and Diego-Mantecón (2015), we realised that we were playing a variation of the game ‘Chinese whispers’, despite adhering to existing good practices for questionnaire translation and adaptation (e.g. Osborn, 2004; Schraw & Olafson, 2015). Beliefs differ considerably from one cultural context to another, making comparative research challenging (Andrews & Diego-Mantecón, 2015; Tuohilampi et al., 2015). However, to better understand how educational beliefs differ culturally valid and reliable research instruments must be developed that legitimise both the act of comparison and the categories used (Clarke, 2013).

The purpose of this research report is to discuss criteria and best practices in developing culturally valid questionnaires that enable investigation of student teachers’ gendered beliefs about mathematics and mathematics education in more than one educational context.

CULTURE, GENDER AND THE NORDIC MODEL

Previous research has identified several issues regarding cross-cultural studies in mathematics education (Andrews & Diego-Mantecón, 2015; Clarke, 2013). A major issue is how to define culture. One solution, although somewhat narrow, is to view cultures as national contexts (e.g. Andrews and Diego-Mantecón, 2015). A more general approach can be found in Clarke (2013), who discusses the cultural validity of cross-cultural comparisons in mathematics education. Although not distinguishing the various levels of how a culture can be operationalised, his definition enables a view of culture as functioning as a lens that is not restrained to national borders. Here, we follow Clarke’s (2013) definition. Furthermore, we argue that within a national context, several particular cultural, that is, sub-cultural, practices and identities exist that must be taken into account when conducting cross-cultural research.

The Nordic education model can serve as an example of a specific cross-cultural context, spanning several countries, including Norway and Sweden. It has been identified with attributes shared by the various countries regarding equal access to education, including no segregation by abilities, social class or gender (Imsen et al., 2016). Here, we focus on the latter aspect, in which gender refers to ‘feminine and masculine, characteristic and culture dependent traits attributed by society to men and women’ (Wedege, 2007, p. 252). A main trait of the Nordic education model is the over-arching perspective regarding gender equity and equality: both Norway and Sweden have implemented national laws and policies securing equal rights in terms of education (Hedlin, 2013; Imsen et al., 2016). This Nordic perspective has traditionally been reflected in the national curricula: for instance, in Sweden it was clearly stated in 1962 that education in all subjects should be given to every child as a legal right, based on the ideas of equity and equality (Imsen et al., 2016). While equal rights are explicit in policy documents, students and teachers at various levels express gendered views about mathematics and mathematics education (OECD, 2013; Sumpter, 2012; Sumpter, 2016). At the same time, many teachers (in Sweden) believe that gender issues are not relevant to their teaching, indicating that the ‘gender problem’ is solved (Gannerud, 2009). This illustrates a tension among various driving factors: on one hand the observed structural gender and symbolism at the individual and group levels, and on the other hand the constituted societal and political views.

WHERE WE WENT WRONG – THE PILOT STUDY

This study started out with the intention to replicate the study by Leder and Forgasz (2010) utilizing a tool previously applied in other studies involving various kinds of samples and cultures (e.g. Forgasz et al., 2014; Gómez-Chacón et al., 2014). As a first step, the survey was translated from English into Norwegian and Swedish individually by the two authors of this paper. To validate the translation, both translations were compared and back-translated, following the procedures presented in Andrews and Diego-Mantecón (2015). In addition, between-language comparisons were conducted. Prior to translation, some adaptions were made. For instance, as the target population was prospective teachers, background questions aimed at identifying teacher-education programme and year of study were added. We also added a question about whom the respondents believed to be best suited to teach mathematics, a female or a male teacher. The question, ‘Should students study mathematics when it is no longer compulsory’ was removed, as this is not relevant in a Norwegian and Swedish contexts, in which mathematics is compulsory and all students must take mathematics courses in upper secondary school. In addition, some items were carefully adapted to better conform to Norwegian and Swedish ways of expression. For example, the question ‘Who are more suited to being scientists, boys or girls?’ was changed to ‘Who are more suited to work in professions in which you apply mathematics, e.g. engineer and chemist?’. A third person with knowledge of the three languages was responsible for digitizing the questionnaire, comparing the three versions and performing external validation.

Next, students in two Norwegian universities were invited to participate in the pilot study, in accordance with good practice for questionnaire development (Andrews & Diego-Mantecón, 2015; Schraw & Olafson, 2015). The responses indicated that some questionnaire items were culturally skewed, including the item ‘Who is better at mathematics, boys or girls?’. The majority of students (56%) stated that boys and girls are equally good, and 28% stated that boys are better. However, comments indicated that some students experienced a dilemma when replying to this item. One male student commented:

> When we speak about 'good at mathematics' I am thinking that you think of this related to school outcomes (not the ability to assimilate knowledge and problem solving skills). And then, I believe that previously boys have been more interested in mathematics and science programs in higher education, but that now more and more girls study mathematics. And I believe that if someone works with something (for instance mathematics) they can be really good at it.

Based on this and other comments, including expressions that the survey questions were inappropriate because we are 'living in a modern world', we suspected that the students in the pilot study were experiencing the tensions between a local culture and the constituted national context of gender. Simply put, we were asking the wrong questions if we wanted to study prospective teachers' views about gender and mathematics education. Furthermore, the survey did not allow the participants to demonstrate knowledge of how gender can be viewed in various cultural contexts, both within and between different groups. Hence, to continue the work of adapting the survey, we needed to address various cross-cultural research criteria.

WHAT IS MISSING? CULTURALLY AWARE CRITERIA

The steps for adapting questionnaires presented by Andrews and Diego-Mantecón (2015) involve the following four principles taken from Osborn (2004): conceptual equivalence, measurement equivalence, linguistic equivalence and sampling. Measurement equivalence and sampling apply only to the measurement phase and therefore are not instrumental to addressing the cultural aspects of questionnaire development. Consequently, these principles will not be discussed further. Linguistic equivalence can be achieved through good translation practices (Andrews & Diego-Mantecón, 2015), hence leaving us with the challenge of constructing conceptual equivalence. The goal of conceptual equivalence is to 'provide conceptual definitions that have equivalent, though not necessarily identical meaning in various cultures' (Osborn, 2004, p. 269). However, Osborn (2004) does not discuss what it means to be equivalent while not identical, or how researchers can achieve conceptual equivalence other than by using an inside/outside perspective as part of the development. A potential inference is that what is missing is a more nuanced understanding of what cultural awareness could constitute. Therefore, we turned to Clarke (2013), who uses seven dilemmas when discussing how international comparative research using questionnaires might be undertaken: (1) cultural-specificity of cross-cultural codes; (2) inclusive vs distinctive; (3) evaluative criteria; (4) form vs function; (5) linguistic preclusion; (6) omission; and, (7) disconnection. These dilemmas might be used to develop the needed culturally aware criteria. In this discussion, we will focus on six of these dilemmas. Form vs function mainly concerns classroom or teaching activities that could be interpreted as having the same form but different functions. As our study does not focus on activities, this dilemma is less relevant than the others are.

When creating questionnaires, something inevitably will be omitted. This is the sixth of Clarke's (2013) dilemmas: when research is unable to capture everything that is or needs to be observed. When we asked prospective teachers about their gendered views, their written comments suggested that we had indeed omitted changes in the construct that have taken place in society over time. Item 6 from the original study illustrates this dilemma. The item addressed the question of whether girls or boys are better at mathematics. To elicit more information about the students' beliefs, this item was accompanied by a follow-up question: 'Has this changed over time?'. Previous research has concluded that gender is indeed a construct that has been changed over time (Hedlin, 2013; Imsen et al., 2016). Student responses to this and other follow-up question indicated that some students acknowledged that changes have taken place and some found the very question(s) outdated. From the students' comments, we concluded that not only did we need to formulate items differently, but also that follow-up questions needed to be added to several more questions in future versions of the questionnaire. This, for instance, could enable us to distinguish between groups who believe that gender differences have never existed and those who think that there has been development (c.f. Sumpter, 2016). Besides allowing us to distinguish between these subgroups, we might also be able to generate nuances in views with time as a factor.

Clarke's (2013) fifth dilemma, linguistic preclusion, is addressed somewhat by Osborn's (2004) linguistic equivalence and also is closely linked to Clarke's first two dilemmas, cultural-specificity of cross-cultural codes and inclusive vs distinctive. In Norwegian, 'kjønn' is used to illustrate gender, while in Sweden, separate concepts exist to distinguish gender and sex (genus and kön). In the questionnaire, all three terms are omitted, and 'boys' and 'girls' and 'male' and 'female' are used consistently to address Clarke's evaluative criteria and the linguistic preclusion criteria. In Norwegian and Swedish, the chosen terms, 'boys' and 'girls', can hold all necessary meanings/interpretations. As stated above, this is related to the first of Clarke's (2013) dilemmas, cultural-specificity of cross-cultural codes. When applying concepts across cultures, awareness of potentially different connotations is crucial. Our experiences with the first version of Item 6, asking whether girls or boys are better at mathematics, illustrates this: the question was relevant in previous cross-cultural studies (Forgasz, Leder, & Tan, 2014; Gómez-Chacón, Leder, & Forgasz, 2014), and possibly to some of the participants in our pilot, judging by the response patterns observed. Still, to others, this question represented an ambiguity that positioned them between the culture of the Nordic model and the observed gender patterns regarding school outcomes and recruitment into higher education, as theorised in the Introduction. This ambiguity applies to the second dilemma, inclusiveness vs distinctiveness, as well. Sacrificing distinctive (cultural) details results in questionnaire items that respondents regard as too open and/or too general. Hence, we suggest that providing a contextual frame could resolve this issue for our survey, since this frame could address the tension between the Nordic model and observed practices. Therefore, our solution is that instead of a series of independent, multiple-choice questions as used in Leder and Forgasz (2010), a series of items linked to the same contextual frame could allow respondents a culturally acceptable interpretation span (see Figure 1).

The following questions are all about beliefs about mathematics and mathematics teaching and learning. Such beliefs might change over time or they might remain stable in some parts of society. We want to ask you about different groups of people, what you think their attitudes are about girls and boys and mathematics. We also want to ask you about your own beliefs.

Question 1

Different groups within society might have differing thoughts about who are better at mathematics. Who do you think each of the groups below believe to be better at mathematics, girls or boys?

	Girls	Boys	Equally good	Uncertain
Male teacher students	○	○	○	○
Female teacher students	○	○	○	○
Fathers	○	○	○	○
Mothers	○	○	○	○
Male teachers	○	○	○	○
Female teachers	○	○	○	○
Girls	○	○	○	○
Boys	○	○	○	○
People in general	○	○	○	○
You	○	○	○	○

Figure 1. Item 1 of the revised questionnaire.

In addition, different cultures value different concepts differently. This most likely also applies to various groups within each culture and, hence, in the present study, it could be addressed by reformulating the original item into the form in Figure 1. This reformulation enables us to address various interpretations of the topic addressed and at the same time allow students participating in the study to show by their responses their interpretations of the views of the groups listed. In addition, students could be given a follow-up question that might ask them to reflect on changes over time for one or more of the listed groups.

Regarding the seventh and last dilemma, which is misrepresentation through disconnection, any survey targeting individuals' beliefs or views will be disconnected from the context being asked about (c.f. Tuohilampi et al., 2015). Although not a perfect remedy, providing a short introduction that aims to provide a cultural context might address this dilemma to some extent.

CONCLUDING REMARKS, INCLUDING IMPLICATIONS

Our failure to reproduce research undertaken in other cultures brought the issue of cultural validity to the surface. Through our discussions, we have sought to elaborate on this issue from various perspectives. Many of the established good practices cannot alone provide the awareness necessary when conducting cross-cultural research, at least when we speak about phenomena that are socially constructed and culturally/context conditional, as is gender. To avoid accidentally playing 'Chinese whispers', besides focusing on content and construct validity, we also need to consider a cultural validity in which culture encompasses more than just national borders (c.f. Andrews & Diego-Mantecón, 2015). We argue that within a national culture, several 'cultures' exist, and it is necessary to shape questionnaires in such a way that participants can interpret the construct in similar ways, as much as that is possible. Our solution is (1) to provide framing that explains the cultural setting in which the questions are posed to enable interaction with the items and (2) to offer multi-dimensional choices that allow flexibility in the responses. Another possible implication is that our solution for constructing questionnaires in cross-cultural research is applicable not only to beliefs about gender but also to other beliefs or other affect research focusing on individuals' conceptions. We make this inference in reference to the conclusions of Tuohilampi et al. (2015), who argue that you must understand the construct in focus from the specific contexts and 'that theoretical affective models should be considered with respect to a given culture' (Tuohilampi et al., 2015, p. 1644). Based on our failed attempt, we could not agree less.

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AN EXPLORATION INTO TEACHERS' TAKE UP OF PROFESSIONAL DEVELOPMENT TEACHING RESOURCES

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There are indications of a complex relationship between opportunities to learn in PD contexts, teachers' experiences of these, and what they ultimately do in their classrooms. Adler (2002) argues that take up from PD is personal and contextual and thus inevitably uneven across teachers who participate in the same programme. This paper contributes to our understanding of the uneven nature of teachers' take up of PD teaching. In particular, our study of teachers' take-up of PD teaching resources offered in a programme in South Africa, suggests that a key element of uneven take up is "control", where control is evidently both personal and contextual, and important.

INTRODUCTION

Teacher learning, according to Adler (2002), involves a process of "take up" elaborating that "teachers take up aspects of the programme, and different teachers do this in different ways" (p.9). The implication of such conceptualization of teacher learning is that while all teachers who participate in PD are likely to learn something, they are likely to learn differently or enact what is learned differently depending on their personal preferences and contextual enablements and constraints (Adler, 2002; Gresalfi & Cobb, 2011). Furthermore, such a conceptualization allows for individual agency despite contextual issues that might hinder the implementation of what teachers learn in PD contexts. As such, instead of organizers of PD seeking to 'fix' participating teachers, the notion of teacher learning as take up enables us to explore what opportunities to learn are offered (Horn & Kane, 2015), how they interpret their learning, and why there are variations in their enactment of PD teaching resources. This move enables us to highlight teachers' agency in relation to what they have learned (or not) "rather than trying to determine whether or not teachers have changed in directions intended" (Adler, 2002; p.9). Rather than viewing the lack of fidelity in teachers' enactment of what they learn in PD contexts as evidence of failure, Adler (2002) argues that teachers' learning and implementation of what they learn in PD contexts is a long-term journey with different things being learned at different points, a phenomenon confirmed in our current research. In a longitudinal video study (2012-2013) of a cohort of teachers who participated in a mathematics PD programme known as the Wits Mathematics Connect Secondary (WMCS) project in South Africa (SA), Adler & Ronda (in preparation) show that although there were general improvements in what they refer to as teachers' mathematical discourse in instruction (MDI – elaborated further below) , the nature of improvement was uneven

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 313-320. Singapore: PME.

despite the PD programme building on mathematics instructional practices that teachers were already engaged in on regular basis (McDonald, Kazemi, & Kavanagh, 2013). This leads to a simple question, why?

This paper contributes to our understanding of the uneven nature of teachers' take up of PD resources from their participate in particular PD programmes, and as such responds to Sztajn, Borko, and Smith's (2017) claim that "… a gap exists between what is known about PD and what teachers actually experience in PD programs" (p.1). We sought to investigate the question: what accounts for the uneven nature of participating teachers' take up of WMCS teaching resources? Specifically, why are some opportunities to learn easily taken up while other aspects appear to be challenging? In doing so, we will build the argument that while motivation (Gresalfi & Cobb, 2011) and context (Adler, 2002) are crucial in understanding the variability in teachers' take up of PD ideas and practices, another important element which mediates these two factors is the issue of *control*.

BACKGROUND OF WMCS PROJECT

Concern about quality of instruction and professional competencies of teachers teaching in schools serving the poor or marginalized students groups is not unique to SA (cf. Meaney, Trinick, & Fairhall, 2013). There is concern worldwide to address this situation, as it is now well known that for such students, access to the specialized mathematics knowledge is through what and how they are taught, and so through their teachers (Adler & Venkat, 2014). Redressing this issue however, requires that teachers who teach in these schools are equipped to teach effectively. It is for this reason that our concern was to explore in-depth how a group of secondary school teachers who participated in the WMCS PD in 2016 took up the teaching resources offered them, and in ways that would enable us to explain the inevitable unevenness of their take up, and so inform PD practice.

WMCS focuses on teachers teaching in schools in low-income communities where students' performances lag behind their colleagues in more affluent schools (Heyd-Metzuyanim & Graven, 2016). The WMCS project, in response to concerns about the insufficiency of content knowledge and quality of instruction of SA teachers, particularly in Grades 8 and 9 in the secondary school, offers a PD programme which is focused on deepening and /or revisiting participating teachers' mathematical knowledge for teaching. The content of the PD is structured in such a way that 75% of contact hours is focused on content in the secondary school curriculum (mostly grades 9-11) with the remaining 25% devoted to the how of teaching. The rationale for this is that the opportunities for teachers to revisit and/or learn new content, depending on the grade individuals teach, will enable them to build on, deepen, and extend their "existing knowledge of the mathematics at hand" (Pournara, Hodgen, Adler, & Pillay, 2015, p.3).

Another unique component of the WMCS project is the development of a mathematics teaching framework (MTF) which serves as a mediational and cultural tool. The MTF is an adaptation of the MDI framework which is research informed and offers opportunities for teachers to reflect on "…the complexity of teaching mathematics…" in ways that are respectful of their everyday practices and also responsive to their contextual challenges (Adler & Ronda, 2015, p.238). The MTF has four interrelated components, namely; the object of learning, exemplification, explanatory communication and learner participation (see Adler & Ronda, ibid for extended discussion on the framework) that work together to create a potentially coherent and connected lesson. The MTF, therefore, mediates the how of teaching mathematics in WMCS, functioning as a tool to guide the work of teachers so that their students have access to specialized mathematics knowledge.

PARTICIPANTS AND DATA COLLECTION

Participants in WMCS were predominantly Grade 8 and 9 mathematics teachers teaching in one province in SA in 2016. All teachers (48) responded to an initial survey that elicited (amongst other questions) their motivation for joining WMCS. Two distinct projected identities emerged from the authors' interpretations of individual teachers' responses to the pre-survey: Type 1: A learner identity where teachers' motivations centred on how to support their students' learning. For Type 2, teachers' motivations for joining WMCS connoted a perceived lack of content knowledge and expertise to teach effectively. Following the analyses of the pre-survey five teachers teaching in grade 9 were selected using maximum variation sampling technique. This technique allowed for a selection of teachers representing a broad spectrum of characteristics such as their content knowledge and motivation for joining WMCS, and so from Types 1 and 2.

Each of the five sampled teachers was then interviewed individually using a semi-structured interview guide. The purpose of the interviews was to elicit from teachers their interpretations of WMCS ideas and practices. Additionally, in response to Sztajn et al.'s (2017) call to explore the personal experiences of teachers who take part in PDs, the sampled teachers responded to questions addressing WMCS resources (MTF) that they have found easiest or challenging to enact in their classrooms. While two consecutive lessons on grade 9 functions taught by each of the teachers was videotaped and analysed as part of the larger study, for this paper, we focus on the self-reported data (interviews). In doing so, and following our conceptualisation of take up outlined above, our intention is not to evaluate how well teachers interpreted (and enacted) the teaching resources offered but to gain insight into their take up - how they interpreted the resources offered and what they said was easier or challenging to enact in their classrooms.

DATA ANALYSIS PROCEDURE

The interview data was transcribed and then analysed both inductively (open coding) and deductively (as suggested by the interview guide), (Hatch, 2002). This involved a focus on the five teachers' take up of WMCS ideas about mathematics (specifically opportunities to learn more about functions), and their interpretations of what it meant to teach (functions) effectively and aspects of the MTF that they found easiest or challenging to enact in their classrooms. Table 1 below illustrates our coding procedure.

Code	Excerpt from interview	# sources
Opportunity to learn about teaching. Being a reflective teacher	(before WMCS) I was just giving them – not mixing the questions. Maybe when I'm giving them the questions, if there are questions in the text book, I (will) say "Do number one to five". (Ms. V).	4
Opportunity to learn how to teach_being a reflective teacher	I wanted to them to know that we have four different types of tools to represent just one thing. So I needed them to get that, that you can actually move from a flow chart or a flow diagram to finding a rule of that, to having a table, an equation. (Ms. B)	4
Aspects of MTF easiest to implement_ Exemplification	To implement. I would say the exemplification because I can do that on my own, there's no one else but me - Like I'm in control of that. (Ms. P)	4
Aspects of MTF challenging to implement_ Explanatory communication	I would say maybe again…explanatory communication…So usually what we say and how we explain it, sometimes I think the issue of language is also a bit of a problem. They are not all conversant with English. (Ms. E)	3

Table 1. Excerpts of codes and coding procedure of individual interviews

FINDINGS

All teachers, irrespective of whether their motivation for joining WMCS was content-related or pedagogical, appreciated the opportunities to revisit or deepen their content knowledge (Pournara et al., 2015). Space limitations lead us, however, to focus our discussion here on findings related to: 1) Teachers' interpretations of the opportunities to learn the how of teaching and 2) aspects of the MTF that teachers found either easiest or challenging to enact in the classroom. While all teachers found opportunities to learn the how of teaching (as structured and mediated by the MTF) valuable, we found that what was possible to enact easily or appeared challenging to them was a function of how much "work" the sampled teachers had to do.

TEACHERS' INTERPRETATIONS OF HOW TO TEACH

Considering the widespread underperformance of students in their schools, the need for a change in practice serves as common thread in responses by teachers. This is exemplified in the following comment by Ms. E:

> … mainly being the level of performance that we are having at the school, it tells us maybe that we need some change anyway. We need to do some introspection and change maybe our way of teaching….

Ms. M commented on her experiences of learning about how to teach - that “it changes one's life”. When asked to explain what she meant by this comment, she said:

> In the classroom you find like every day you want to give learners something. You find like you don't get frustrated to go to class. There's something new that you want to share with the learners because at least now things are easy.

She expressed excitement about participating in the WMCS sessions indicating that she looked “forward [for] every day” wondering “what is it that [I'm] going to learn today…It's like wow; it's another day for something new.”

The sense of liberation from what appeared to be drudgery was echoed by Ms. E:

> I think the programme helped me to think more about what I'm doing and saying in class; unlike just going there and saying: now I've done my job. What is it about the job that we have fulfilled, we have done? It makes me to think deeper about what you are doing, what questions you are asking, how are the learners doing? I'm now paying attention as to how they will be reacting to whatever that I've been doing.

In summary, through participation in WMCS, teaching became something more than a chore to be performed by these teachers. All five sampled teachers were excited about the opportunities to learn made available to them (Horn & Kane, 2015) in WMCS.

Yet, with regard MTF, and as we go on to discuss below, take-up was uneven across the elements of the framework. Of specific interest is what emerged from those that tried to be more inclusive of learners. Their expressed difficulties appeared to a function of the amount of “control” they had over a learned instructional practice.

ASPECTS OF MTF THAT TEACHERS FOUND EASIEST OR CHALLENGING TO ENACT IN THEIR CLASSROOMS

Through our analyses of their interview responses, the issue of unevenness reported by Adler & Ronda (forthcoming) is a matter of 1) how routine an instructional practice is to a particular teachers; e.g. Ms. V, talks about choosing examples (a component of the MTF related to exemplification) as “easier” to implement, “because you are always giving them examples” and 2) the amount of “control” that the teachers felt they had related to a particular practice.

All sampled teachers were asked to indicate aspects of the MTF that they found easiest to implement. In choosing the lesson goal (in addition to two other components) as the easiest to implement, Ms. M commented:

> …normally they get so easy because ever since when I go to teach, I tell them that today we are going to learn about this. At the end of the lesson, you must do [this]". This one I can't go without because I'll not know what I will be teaching them, you know.

Ms. B therefore appeared to find this component easier to implement due to familiarity or a routine. This idea is further exemplified in a response by Ms. V who also stated that exemplification was "easier" for her: "because always you are always giving them examples". The comments by the two participants highlight how relating PD to what teachers do on a daily basis (McDonald et al., 2013) and strengthening them rather than a wholesale repudiation of everything they do has promise in crossing PD boundaries into the classroom context. However, all classroom interactions require some form of interaction and learner involvement and, enacting the components of the MTF related to learner participation and explanatory communication appeared challenging to most teachers.

Ms. P, Ms. B, and Ms. E talked either implicitly or explicitly about issues of control relative to enactment of particular components of the MTF. The following comment by Ms. P typifies how the issue of control can lead to an uneven enactment of WMCS teaching resources, especially as teachers sought to teach in ways that were more participatory which was implicitly promoted in the PD. After commenting that exemplification is the easiest for her to enact "because [I] can do that on my own, there's no one else but me, like I'm in control of that", she identified explanatory communication as quite challenging for her. She explained:

> Like sometimes what they say I may not understand or even like a language barrier. Sometimes if I want to say something, they don't understand the language I'm using but in my head it sounds beautiful. Like I know I'm explaining it beautifully but they don't get that so that for me is very difficult, to be honest.

This excerpt highlights how classroom context leads to challenges in Ms. P's enactment of what she learned because of cultural disconnect (teaching in a multicultural classroom). She contrasts her lack of control when it comes to explanatory communication with exemplification which is easiest for her to enact: "... exemplification because I can do that on my own, there's no one else but me - Like I'm in control of that". Ms. B also addresses the issue of her lack of "control" during the instructional process: "…What they say is important because that informs you on [sic] what they understand. So if they can't say anything, then it's a problem….."

Her comment indicates that not all instructional practices are wholly dependent on the teacher. This is more profound in the SA contexts where there is a pervasive classroom culture of students verbalizing little beyond chorused responses (Heyd-Metzuyanim & Graven, 2016).

In contrast, Ms. B introduces another element of control indicating that:

> Before we start with the lesson we've got games or whatever, the quizzes, you know, just to get our mind going…If we can have that in schools where you start with a little game, it would be nice but it's not because they (subject advisors) are looking at when you are finishing the ATP.

She raises issues of accountability (contractual obligations), (Gresalfi & Cobb, 2011) and introducing new practices. As a teacher who is being monitored in terms of coverage, Ms. B did not perceive herself to have control over what to teach which influenced her take up of WMCS teaching resources.

In summary, the more control teachers reported having (e.g. Ms. P and Ms. B) the easier it was for them to enact the MTF component. In contrast, the less control teachers perceived to have the more challenging they found that component to be (e.g. learner participation as challenging because of unpredictability of students and a SA culture of silence on the part of students and explanatory communication – multilingual classrooms).

CONCLUSION

In this paper we have attempted to show that what was possible to take up (enact easily or not) was dependent on what the sampled teachers felt they had control over, both personally and contextually. While they all expressed having agency to act in new ways, the amount of control they perceived themselves to have served as a mediator between what they learned and what they were actually able to do in their classrooms. We thus suggest that take up of PD teaching resources is related to teachers' feeling in control of what to do, how to do it, in short, being able to manage the teaching-learning process. Where teachers have control (e.g. of selection and sequencing of examples) they perceived this as easier to enact. In contrast, they were challenged when they perceived themselves having little control (e.g. over communication cultures or accountability processes). We offer this perspective on teachers' "work", and deliberately adopt the word "control" as explicitly used by one of the teachers. Learning to interrupt dominant cultures, and challenging accountability processes is not trivial. We suggest that recognition of teachers' managing (being in control of) their practice is important for all PD. We do this in full awareness of the negative associations of 'control' positing that more work is needed to understand how concerns with control mediate teachers' practice.

Acknowledgement

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DIMINISHING EPISTEMIC AUTHORITY: A LEVER FOR MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT

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Professional development is a process that contributes to the improvement of both the individual teacher, and the educational system as a whole. This paper describes professional development of 17 primary school mathematics teachers that collaborated with their students in an inquiry of an unfamiliar mathematical topic. The results indicated that a teacher, who is carrying out a teacher-students collaborative inquiry on an unfamiliar mathematical topic, might diminish his epistemic authority and subsequently leverage his professional development and change his relationships with the students.

INTRODUCTION

Teachers' beliefs, attitudes and philosophy are closely linked to their practices and teaching approaches. For that reason, an acquaintance with teachers' worldview is necessary for designing frameworks aimed at supporting their professional development [abbr. PD], which in turn might enhance educational processes (OECD, 2009). Current approaches to teaching, suggest that teachers should no longer be the exclusive source of knowledge of their students, but rather be initiators and facilitators of learning situations in which they themselves take an active part (Swann, 2012).

In contradiction to this approach mathematics teachers perceive themselves as a primer source of knowledge for their students more than teachers of other subjects such as biology, history or literature do (Raviv, Bar-Tal, Raviv, Biran, & Sela, 2003). A previous study that served as basis for this research has shown that among other things, mathematics' teachers recognized epistemic authority as central to their role perception (Nutov & Shriki, 2016). Following these results, the current research aims to examine in-depth the perceived role of mathematics' teachers of their epistemic authority and its place in PD process.

LITERATURE REVIEW

The search for appropriate ways of preparing students for life in the 21st century has led many educators to position the inquiry approach to teaching and learning as a top priority to ensure understanding while gaining knowledge. This teaching and learning approach is acquiring more and more followers in the mathematical educational community. The inquiry learning approach encourages learners to be involved in a learning environment that allows them to gather information, explain it, and answer open-ended questions while making use of collected evidence (Bell, Urhahne,

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 321-328. Singapore: PME.

Schanze & Ploetzner, 2010). Usually when teachers or students are engaged in mathematical inquiry, they are well aware of the fact that the phenomenon or the features, which they found, are not real "findings", and the world of mathematics already knows them well (Shriki, 2010). Nevertheless, the teachers' role in the inquiry process is challenging, and includes choosing the subject of the inquiry, teaching the students the skills needed to carry it out and guiding them through the process. The teachers' role becomes even more complex when the mathematical topic of the inquiry is unfamiliar to the teachers. This might undermine teachers' epistemic authority that is a source of determinative influence on the formation of individuals' knowledge (Raviv, Bar-Tal, Raviv, Biran, & Sela, 2003).

Raviv et al. (2003) examined perceptions of teachers and 7th and 10th graders, regarding the teachers' epistemic authority. It turned out that teachers were inclined to perceive themselves as being more of an epistemic authority than their students considered them and that teachers believed that students perceived them as being more of an epistemic authority than the students actually did. These gaps were prominent mainly in the case of mathematics teachers who participated in the study (compared to history, biology and literature teachers). Among others, the researchers attributed this gaps to the higher self-efficacy and self-perception expressed by the mathematics teachers and to their perception of mathematics:

> *Teachers of mathematics perceived themselves to be more of an epistemic authority to their students in the disciplinary knowledge domain than teachers of other subject matters. This indicates that mathematics teachers perceive their discipline differently than do other teachers and, because of the status of their discipline as an exact science, they perceive themselves to be knowledgeable experts (p. 37).*

Based on the belief that teachers' personal experience in authentic mathematical inquiry can gain deep personal insights, reduce apprehension and promote similar experiences in the classroom, we designed a PD process in which teachers executed a collaborative inquiry with their students on an unfamiliar mathematical topic – fractal geometry. The research results, among others, indicated that teachers are stressed if their epistemic authority diminishes (Nutov & Shriki, 2016). Thus, the current research aimed to examine in-depth the role that mathematics' teachers attribute to their epistemic authority and its place in PD process.

RESEARCH METHOD

Since we were unable to find a theory that deals with processes related to collaboration of teachers and students in the inquiry of fractal geometry, the study was exploratory and used grounded theory qualitative research methodology. This methodology is suitable in order to examine phenomena and processes as reflected in the eyes of the participants, with the aim to construct an initial grounded theory (Strauss & Corbin, 1998).

The study was carried out in the framework of a two M.Ed. courses that was intended for mathematics teachers. Since action research is considered a major tool for empowering teachers, the goal of the course was to support mathematics teachers' professional growth through this method (Mcniff & Whitehead, 2016). During their experience, the teachers were instructed to reflect on their beliefs and roles, as changes in these beleifs may serve as an indicator of professional growth (Korthagen, 2004). As part of their action research, the participants explorerd the collaborative inquiery of fractal geometry that was conducted together with their students. Moreover, the teachers exprienced an unfamiliar pedagogical approach and a mathematical topic they were unacquainted with (Nutov & Shriki, 2016).

Both courses were composed of primary school mathematics teachers (12 and 5 participants, respectively). On average, the teachers had 11.5 years of teaching experience (SD=4.7).

To strengthen the credibility of the findings we used a triangulation of data (Lincoln & Guba, 1985), and employed four research tools: (1) A preliminary questionnaire aimed at identifying the participants' initial perceptions regarding their role as mathematics teachers and their mathematical knowledge of school curriculum topics; (2) Transcripts of recorded class discussions that were held during the course sessions; (3) Teachers' journals entries, in which they described reflectively and criticaly their various experiences; (4) Teachers' seminar paper that described the action research they carried out in their classes.

To identify the main categories and sub-categories, the findings were analysed through a process of open and axial coding (Corbin & Strauss &, 2008).

RESULTS

The grounded theory, which was constructed in the research, suggested that, teachers who are carrying out a teacher-students collaborative inquiry on an unfamiliar mathematical topic, may reduce their epistemic authority and consequently leverage their PD and change their relationships with the students.

Due to space limitations, only partial results are presented here in order to portray a general change that the participants had undergone and summarize it as a 3-phase model (see Figure 1). The excerpts below are taken from the teachers' reflective journals and the seminar papers.

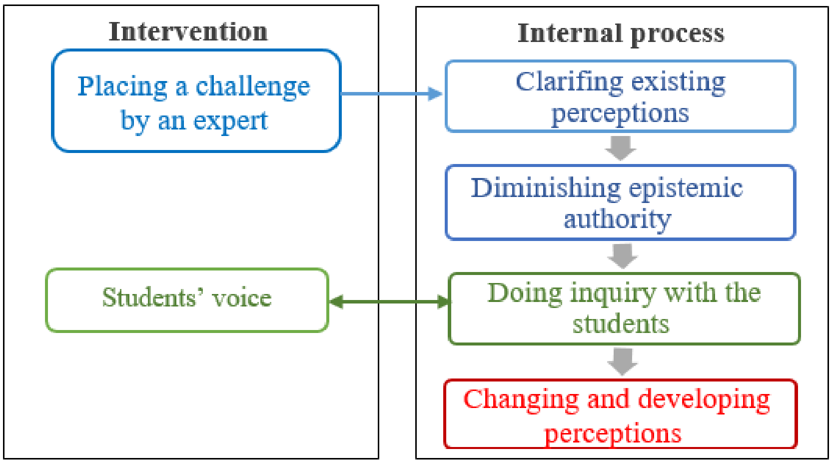

Figure 1: 3-phase model

Clarifying existing perceptions and diminishing epistemic authority: The importance that teachers attributed to their epistemic authority was encapsulated in statements such as: "Know it all" or "to have 100% knowledge of the subject". Namely, the teachers believed that they must thoroughly know the subject that they teach and that they must be able to answer immediately any questions the students ask: *"I used to believe that I posses all the knowledge I need and I'm not used to discovering that there are things that I don't know and that there are questions I can't answer".* This perception was rooted so deeply in the teachers' hearts, that the proposal to do inquiry with students on an unfamiliar mathematical topic provoked feelings of fear, frustration, discomfort and resistance:

> *The most difficult stage was the proposition "to take a problem that you do not know what the answer is, and find it together with the students". At this point, I felt that maybe I was in the wrong place. How can I teach the students, when I don't have a 100% of knowledge of the subject?*

The teachers feared that diminishing their epistemic authority will prevent them from supporting the needs of their students and as a result, their students will be intimidated by mathematics:

> *I was worried that my lack of knowledge will lead my students to lose confidence because they will feel that they lack the professional guidance for dealing with the issue... and because of this frustration, students will feel more threatened by mathematics.*

An additional threat to teachers' epistemic authority came from an unexpected place – the teachers' mathematical knowledge. The questions that addressed mathematical concepts in the pre-questionnaire revealed a gap between the knowledge the teachers thought that they possess and their real knowledge:

> *It all started from the questionnaire the professor gave us. The questions dealt with geometry and made me think about the answers for quite some*

> *time. It made me realize that there are some definitions that I do not know perfectly. I felt stressed.*

The stress that teachers reported had multiple sources: The discovered gap between the knowledge that they thought they have and their actual knowledge, the collaborative inquiry with the students on an unfamiliar to them topic, the fear of harming the students, and damaging their professional image in their own eyes and in the eyes of the students. This stress also demonstrates the importance that teachers attribute to their epistemic authority and the fear of diminishing it. However, the threats on teachers' epistemic authority promoted their PD at least in two ways: They became more aware of their students learning experiences and they learned mathematical topic, fractal geometry, in a new way – inquiry:

> *It made me think about how students who are exposed to new concepts every day and even few times a day deal with this explosion of new ideas. What is the difference between the strongest students and the students who struggle with mathematics: How do they cope with new and completely different topics? Even though I consider myself as a sensitive person, this experience refined my sensitivity of the learning experiences of my students.*

> *It was the first time I realized I can explore mathematical phenomena by myself… it strengthened my self-efficacy and my confidence as a teacher of mathematics.*

Doing the inquiry: During the collaborative inquiry on fractal geometry, the teachers' epistemic authority was diminished and it set a stage for a creation of a new balance of power in the classroom. On the one hand, when the teachers were more "vulnerable" it allowed them to be more attentive to their students. On the other hand, the students were empowered and this new status encouraged them to express their opinions on various topics and discover abilities that they did not present in the regular learning setting. In this new setting, the students supported their teachers to go through the collaborative inquiry and experienced independent learning:

> *When I told the students that we will learn together a new topic, they immediately said that they have no problem with that, even if I don't have all the answers to all the questions. It was very reassuring for me, especially when Noam said: I have a smart teacher, even if she doesn't know all the answers to all the questions.*

> *During the inquiry, students discovered things about themselves, learned about their abilities, and competences as learners. At the end of the process, I heard them saying: "I can do much more than I thought I can"; "I can reach the sky"; "I am clever ". The focus was on the students, they came up with ideas, tried them and suggested generalization. I just guided them. When the spotlight was on them, not me, they could perceive themselves as autonomous learners.*

Actually, students' feedback has become a tool for teachers' PD. Teachers' professional image improved in their own eyes and in the eyes of their students, they have adapted professional flexibility and strengthened their faith in the abilities of students:

> *Immediately, at the first lesson, I was not at the centre and I did not knew all the answers to all the questions. However, I felt that my professional image in the eyes of the students has improved. They cooperated with me, trusted my leadership, and I felt that they were satisfied and more engaged in learning.*

Changing and developing perceptions: The reflections written by the teachers about the collaborative inquiry with their students indicated that action research deepened insights about the processes associated with it, and in particular focused their attention on the change of the perception about their role as mathematics teachers: From teaching to guiding the learning process. This change is reflected in at least four respects: (1) From the teacher who "knows it all" to "the teacher does not have to know the answer for every question"; (2) from "teacher-centred learning" to "students as equal partners in the learning process"; (3) from "the students learn from the teacher" to "students' feedback as professional development tool for teachers"; (4) students have more capabilities than initially believed by the teachers.

All components of the action research created the conditions for these changes, made it most possible for teachers to examine them and to recognize the importance of carrying out an action research:

> *The teacher and students learn a new subject together. The teacher does not know all the answers to all the questions, so he is looks for answers to questions together with the students. The teacher changes his role and his place in the classroom, from speaker to listener, from information presenter to a partner in knowledge gaining, from the centre of the attention to a member of a group.*

The action research enabled the teachers to examine thoroughly their didactical approaches, and to find a gap between their beliefs and their actions:

> *I have to learn to release control, lecturing less, and more student guidance. I have to teach them how to construct new knowledge, based on their existing knowledge, to help them to take responsibility for their own learning, to encourage them share ideas, to have discussions, and to understand the meaning of a learning community.*

This approach, creating knowledge rather than distributing it, may on the one hand diminish teachers' epistemic authority (Amit & Fried, 2005; Raviv et al., 2003). However, on the other hand, it sets stage for students to express their mathematical ideas more freely and for the teachers to realize their students' capabilities. Both the teachers and the students become engaged in a significant teaching-learning process.

> *I realized that when I teach topics I know well, I don't really listen to what students have to say, and I often push them towards the answer I want them to give me.*
>
> *Only from this place of knowing almost nothing about the topic I was teaching, I was able to listen to my students, learn about the way they think, and admire them for being so smart.*

DISCUSSION AND CONCLUSIONS

Teaching mathematics is a complex task that requires expertise from the teacher in content knowledge and in various pedagogical approaches, as well as to be in a perpetual search for the optimal conditions for student learning. The analysis of the research data indicated that the mathematics teachers have real difficulty to diminish their epistemic authority, but if they are willing to do so, there may be a change in the balance of power in the classroom, which brings to teachers and students proximity. Due to this change, it seems that teachers developed professionally in terms of their role perception (from a teacher to a guide of the learning process), personal characteristics (giving up the need to be at the centre of attention, being part of a group), and their relationships with students.

As mentioned above, due to changes in the balance of power in the classroom, the students had more space to express their mathematical ideas, and as a result, the students developed as mathematics learners. They could propose their help to the teachers when they did not understand and they even gave their teachers meaningful feedback for their PD.

The change in teachers' perceptions which is based on the learning experience they had (Brown & McNamara, 2011), adheres to the requirements from students and teachers in the 21st century. In other words, the teachers and the students experience can be summarized as learning ***with*** *the experts, rather learning* ***from*** *experts*, as Stefan Heppell, one of the leading scholars in education predicted. Therefore, if we want to change the teaching of mathematics in schools, and adapt it to the needs of learners in the 21st century, we need teachers to have more learning experiences similar to those they would design for their students.

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THE SPREAD OF MATHEMATICAL IDEAS IN ARGUMENTATIVE CLASSROOM DISCUSSIONS: OVERT AND COVERT PARTICIPATION

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Argumentative discussion in the classroom is beneficial for learning mathematics. The goal of the current study is to characterize the spread of mathematical knowledge about the subject of quadratic functions in a whole class discussion among ninth graders. To accomplish this goal, we used the Documenting Collective Activities (DCA) methodology. The data collection method was unique. We recorded the activities of pairs of students while at the same time recording the whole class discussion. We found that knowledge was spread in two parallel layers: Students participated in an overt layer, spreading ideas in the public discussion. At the same time, they also spread ideas privately with their peers in a covert layer. Moreover, incorrect mathematical ideas were spread via the same mechanism.

INTRODUCTION

Mathematical ideas are learned and understood in an argumentative environment as a result of collaboration between teacher and students. Hence, the following question arises: How do mathematical ideas spread in the classroom as part of teacher-student collaboration (Saxe et al., 2009)? This paper is part of a broader study that monitored students in a ninth grade mathematics classroom as they studied quadratic functions during 20 lessons in a variety of learning environments: in a whole class discussion (WCD), in pairs, and in pairs in a computerized environment. The self-explanatory term WCD refers to everything the teacher and the students openly express during the plenary discussion. The goal of the study is to characterize the spread of mathematical knowledge about quadratic functions during a whole class discussion.

THEORETICAL FRAMEWORK

Mathematical Knowledge Spread in the Classroom

The process of knowledge construction has been studied based upon different theories (Cobb et al., 2001; Rasmussen & Stephan, 2008; Saxe et al., 2009; Tabach, Hershkowitz, Rasmussen, & Dreyfus, 2014). Researchers have raised questions regarding how knowledge is constructed in the classroom community, how it is shifted between individual students and the collective, and vice versa.

Cobb et al. (2001) describe a methodology for analyzing the collective learning of a classroom community based on the development of sociomathematical norms relative to two perspectives. The first, the social perspective, emphasizes the normative ways

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 329-336. Singapore: PME.

of reasoning that are *taken-as-shared* in the classroom community. The second, the psychological perspective, emphasizes the diverse ways in which students participate in the classroom community during these activities. Saxe et al. (2009) proposed another theoretical and methodological approach that involves analyzing shifts of constructed knowledge in the classroom through the interaction between a student's individual activity and the collective activities. Saxe referred to this as the dynamic *travel of ideas* between the student and the whole class community.

Documenting Collective Activity framework

Rasmussen and Stephan (2008) developed a methodological approach to document collective activity (DCA) that focuses on how mathematical ideas or ways of reasoning *become normative* in the whole class discussion. The term *become normative* indicates the existence of empirical evidence that an idea or a way of reasoning *functions-as-if-shared*. The term *as-if-shared* is used rather than *shared* to denote variation in the ways by which different individuals perceive mathematical ideas. DCA methodology uses Toulmin's model of argumentation (1969) as an empirical tool.

According to DCA methodology, the following three criteria are used to determine when an idea or a way of reasoning, such as an algorithm or a mathematical argument, *functions-as-if-shared*: (1) when the backing and/or warrants for a particular claim are initially present but then drop off; (2) when any part of an argument (data, warrant, claim, or backing) shifts position within subsequent arguments; and (3) when a particular idea is repeatedly used either as data or as warrant for different claims across multiple days (Cole et al., 2012). The DCA methodology can be implemented only for an argumentative class discussion in which students make claims, provide data to support their claims, and if needed provide warrants to support the linkage between data and claim. In case of rebuttals, students can add backing to support their warrants or to add qualifiers to a claim.

Tabach et al. (2014) examined knowledge shifts in the classroom and suggested several definitions: (a) *A knowledge agent* is the first member of the classroom community who, according to the researchers' observations, expresses a new idea that was not previously expressed and that at least one additional member adopts. (b) *A follower* is a member of the classroom community who appropriates the knowledge agent's idea by repeating it, elaborating on it or objecting to it (Hershkowitz, Tabach & Dreyfus, 2016). (c) *Knowledge shift* is the way by which ideas spread in the mathematical classroom. A shift of mathematical ideas can take place between the knowledge agent and his followers, within the whole class, within a group, between groups or from a group to the whole class (*uploading*), or from the whole class to one or more groups (*downloading*). In this paper, we seek to answer the following question: What characterizes the spread of mathematical ideas within an argumentative discussion that takes place in a whole class setting?

METHODOLOGY

Participants and tools

The study monitored one class of 30 ninth-grade students studying in a high-level mathematics track. The researchers observed an entire 20-lesson learning unit on quadratic functions and documented it in various ways: through video recordings of the WCDs; by recording the work of two pairs during group work; through simultaneous recordings of the voices and handwriting of two pairs of students (who used *Livescribe* pens) while they worked as a group and participated in the WCDs.

Data analysis

In this study, we analyzed the first nine lessons in the sequence. As we had several simultaneously recorded data sources, we first transcribed the WCD and numbered all the utterances in a table beginning with one. Next, we transcribed each of the pairs and numbered the utterances in separate lists, each starting from one. Finally, we synchronized the three lists by merging the utterances of each of the students in the two pairs into additional columns in the WCD utterances table while keeping track of their timing. We were able to synchronize the pairs' utterances with those of the WCD because their microphones also picked up the WCD.

We analyzed the argumentative discussions in the WCDs based on the three criteria of the DCA methodology. The analysis yielded a list of the arguments raised in the WCDs (argumentation log). Each claim was analyzed at the micro level and the arguments were then mapped to show the macro level relationships among them. We analyzed the students' roles in the spread of mathematical knowledge in the WCDs both as knowledge agents and as followers (Hershkowitz, Tabach & Dreyfus, 2016). We then used the data from the pairs to expand our DCA analysis, paying particular attention to the students' participation in the WCDs as knowledge agents and as followers.

FINDINGS

In this section, we report two main findings from three successive arguments from the second lesson. The first example in argument 2.4 describes the spread of mathematical ideas between private participation and the WCD. The second example, arguments 2.5 and 2.6, illustrates the spread of mathematical ideas in the WCD.

Spread of mathematical ideas between private participation and the WCD

While analyzing the spread of mathematical ideas in the WCD, we uncovered "private" processes that take place in parallel to the WCDs and learned the roles of the participants in these parallel private channels. We found that the "private" activities were actually part of the argumentative discussion of the students who participated in them, although they were covert for most of the community. The excerpt from the WCD in Figure 1 documents the utterances made as part of the WCD during argument 2.4 (gray background, numbered 3xx) and those made in private by the two pairs (white background, numbered 1xx). The two pairs sat close to each other during the WCD.

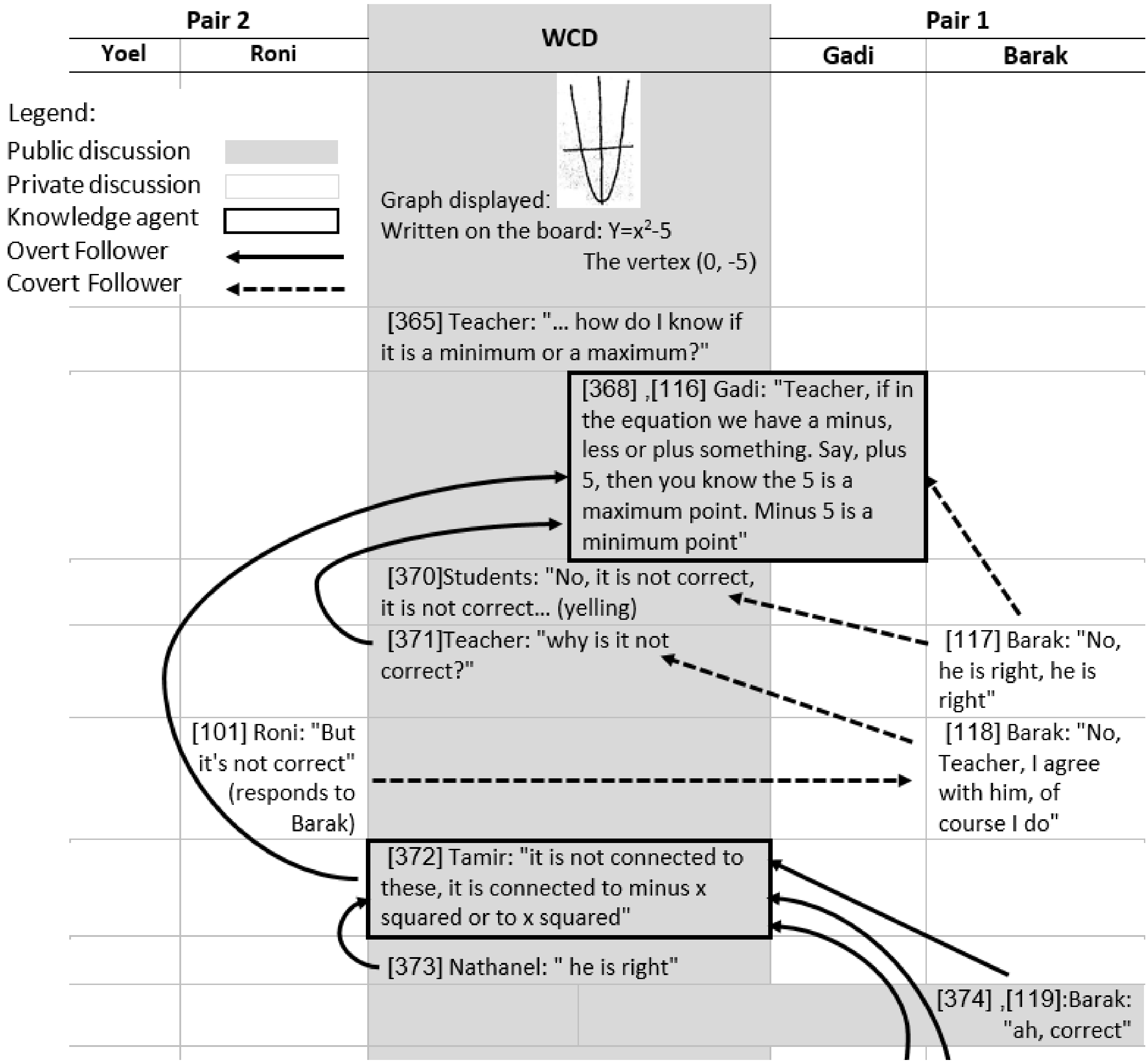

Figure 1: Argument 2.4, students characterizing an extremum point of $y=x^2-5$

As Figure 1 shows, Barak [117] privately agrees with Gadi's claim in the WCD ([368], [116]) and disagrees with the students' objections [370] claiming that Gadi is wrong. The teacher [371] encourages the students to explain and support their claim. Barak is still convinced that the students are wrong, but expresses this privately [118]. In utterance [101], Roni overhears Barak and responds to him in private that Gadi's claim is wrong. Then Tamir [372] openly responds to Gadi in the WCD, offering another claim and a backing. Nathanel [373] supports Tamir's claim and backing, after which Barak [374] changes his mind and openly adds his affirmation in the WCD. Barak, who initially agreed with Gadi, has "downloaded" ideas from the WCD into a private discussion, and after reaching a conclusion, "uploaded" it to the WCD.

Spread of mathematical ideas in WCD

The analysis of the WCD regarding the next two arguments, uncovered additional characteristics about how mathematical knowledge spreads in the classroom.

We first present part of the discussion in argument 2.5 regarding the domain in which the function is positive. In the following excerpt from the relevant DCA analysis, we shade parts of students' remarks and mark them according to Toulmin's model (1969) as data [D], claim [C], warrant [W], backing [B], rebuttal [R] or qualifier [Q].

The initial data for this argument is the same function ($y=x^2-5$) and its graphic representation, as displayed in argument 2.4.

414	Teacher:	… a function is positive, means that the Y is positive. Now I ask. Look at this graph. For which values of X is Y positive?
415	Yoram:	X is smaller than -2 and x is bigger than 2 [C2.5]

The teacher asks Yoram to repeat his answer. He does. Then she asks the students to refer to it.

421	Yoel:	Yes, because it is not -2 [R2.5], [C]
422	Teacher:	What then?
423	Yoel:	Because it has to go from the moment it is equal, this is x. The function will be positive once x is smaller than minus root 5 or bigger than root 5 [R2.5], [D]

The discussion continues. The teacher marked the point (2,-1) to demonstrate that it is not at the limit of the requested domain. Then she marked the X values ($-\sqrt{5}$, $\sqrt{5}$) at the points where the graph intersected with the X-axis (Figure 2). To conclude this part of the discussion, the teacher wrote the correct definition of the requested domain on the whiteboard: $x<-\sqrt{5}$ or $x>+\sqrt{5}$.

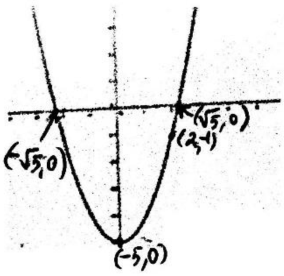

Figure 2: Graph for end of argument 2.5

Note that Yoram made an erroneous claim [415], which was rebutted by Yoel [421], who gave the correct answer [423], data. This section of the WCD ends when the teacher writes the correct domain for X on the whiteboard.

The discussion then continued to argument 2.6 regarding the domain for which the function is negative. The initial data for this argument is the same function ($y=x^2-5$) and its graphic representation as displayed for argument 2.5.

530	Teacher:	… when is the function negative? Let's first look at the graph. Where is the function negative? For what domain of X is Y negative?

531 Shiran: When x is smaller than 2 and bigger than -2 [C2.6]

The teacher asks Shiran to mark the part of the graph for which Y<0. Shiran marks this correctly on the whiteboard by coloring the graph line where the function is negative in green. Then the teacher discusses it with the students.

566 Teacher: So now you tell me, what can you say about the X values in the green area here (pointing with her hand)

567 Mati: They are bigger than -2 and smaller than 2 [C2.6]

568 Teacher: Does anyone want to comment on what Mati said? He said that they are bigger than -2 and smaller than 2.

569 Gadi: … he is right (referring to Mati) [B2.6]

570 Teacher: Is he right? Why is he right?

571 Eran: Are you sure? (Mocks Gadi's answer)

572 Roni: It's not correct [R2.6]

573 Yoel: He is correct, but it's only one specific part [Q2.6], [C]

574 Teacher: Yoram whispered something to me. Yoram, what did you say?

575 Yoram: – wrong" (he was not heard well) [R2.6]

576 Teacher: He said he repeated the mistake (referring to Yoram [575])

577 Yoel: Teacher, can we be more precise? Teacher!

578 Teacher: Who said he wanted to be more precise? Precision of what?

579 Yoel: The domain he gave is correct but it is not the entire range [Q2.6], [C]

580 Teacher: OK. What do you mean?

581 Yoel: It is true that if it is smaller than 2 it will be negative, but there are more numbers between 2 and Root 5 [Q2.6], [D]

In argument 2.6, we can see that Shiran [531] and Mati [567], with backing from Gadi [569], used the incorrect claim made by Yoram [415] in argument 2.5 as data, in spite of the correct data presented at the end of the discussion on argument 2.5. Here we see a position shift, from an incorrect claim made in argument 2.5 to data used in argument 2.6. According to the second DCA criterion, the incorrect claim *functioned as-if-shared*. This also represents a shift in knowledge regarding the X values (-2, 2), from Yoram as agent to Shiran, Mati and Gadi as his followers. In this argument, Yoram [575] rebutted by reverting to the X values, contrary to his claim in argument 2.5. Yoel [579] qualified this and then specified it [581] with the data.

DISCUSSION

At the outset of this paper, we asked: What characterizes the spread of mathematical ideas within an argumentative discussion that takes place in a whole class setting?

Our findings reveal two possible characteristics regarding the spread of mathematical ideas, adding to the existing literature in mathematics education.

The first characteristic is that in an argumentative classroom discussion, mathematical ideas spread in two layers simultaneously. Analysis of the students' utterances and their timing revealed that the learning processes occurred in two layers: an overt layer constituting the public discussion led by the teacher, and a covert layer in which the students' private discussions occurred in parallel. Figure 3 illustrates this idea.

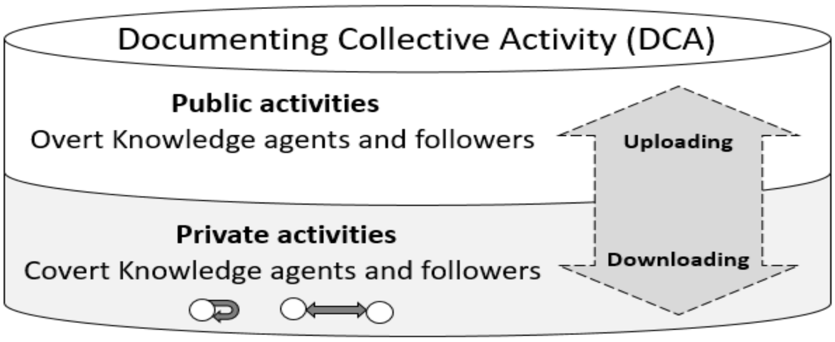

Figure 3: Overt and covert layers in the WCD and their interrelations

Our data collection method is unique in that it included recording of private activities in parallel to the public activities taking place during the WCD. We identified knowledge agents and followers that are overt or covert. Both helped generate arguments and thus contributed to the learning progression. Therefore, we can say that the WCDs provide a different learning experience for each student that includes the overt WCDs as well as the covert discussions and activities in the student's immediate surroundings.

Many studies have analyzed the teacher-student interaction in WCDs as a single setting in the classroom and have focused on its public aspect. A few studies (Koole, 2007; Rob & Jenefer, 2013), especially in the field of language teaching, have focused on students' private discussions and activities that take place parallel to the WCD and in which the teacher does not take part. These studies mainly investigated the behavioral, social and emotional aspects of these parallel activities and not so much the cognitive aspects. The importance of the current study is that it shows that mathematical arguments do not develop exclusively in the public layer of the WCDs, but rather in both the public and the private layers.

The private discussions are beneficial to the class and sometimes even critical for learning. It is important to note that the class does not have only one teacher-student interaction. Rather, several private interactions occur in parallel to the main interaction led by the teacher. One example of a parallel activity is that of a student who does not feel comfortable speaking in the WCD but rather prefers to clarify a matter related to the WCD with a friend (Koole, 2007). Understanding the processes taking place in parallel to the WCDs can help improve the practices of the classroom teacher.

We also found evidence that mathematical knowledge that *functions-as-if-shared* in the WCD can be incorrect. This means that an erroneous argument can spread in class via the same mechanism as a correct one.

On a theoretical level, this study expands the concepts involved in the spread of mathematical knowledge and offers a more detailed description of the participants in the WCD. On a practical level, the study contributes to developing teachers' awareness of the role of the private processes taking place in parallel to the public WCD, for learning.

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VARIATION AND ALGEBRA

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The aim of this article is to contribute to understanding the relationship between teaching and learning algebra at school in order to identify how schools can be supported to improve students' learning outcomes. The students' tests and examinations of their mathematical work and the teachers' lessons plan and reports on the lessons' instructions were the base data for this article. The analysis indicated that, if the teachers base their instructions on the critical aspects identified in students' learning and open up patterns of variation in these aspects, they seem to facilitate students' learning. The findings suggest that helping teachers develop an understanding of the students' critical aspects can be a productive basis for helping them to make fundamental changes to their instructions and to improve their mathematical communication in the classroom.

INTRODUCTION

In recent decades, there has been a surge of interest in how students understand In recent decades, there has been a surge of interest in how students understand algebra and, in particular, how they reason about algebra (e.g. Carraher, Martinez & Schliemann, 2008). This renewed interest stems from an increasing awareness that algebra has not only become the general language of science, but also the epitome of symbolic thinking, a language of effective communication, and the language of generalisation (Radford, 2008; Olteanu, 2014). Despite this, the results for Swedish students in TIMSS (Trends in International Mathematics and Science Study) 2007 and in TIMSS 2011 indicated that, for example, only 5% (TIMSS, 2011) of the students had developed procedures to solve the task 'Simplify the expression $\frac{3x}{8}+\frac{x}{4}+\frac{x}{2}$' and only 16% of the students understood the structure in the formula $P=\frac{3kl}{5}$ and could thereafter calculate the value of P when $k = 7$ and $l = 10$.

Because of the importance and the power of algebra, all students should have opportunities to learn it and to express themselves through it. Despite the extensive research done on algebra (e.g. Demby, 1997), there are many reasons to gain further understanding of how teachers and students experience the mathematical content related to the intended curriculum. This article is based on current curriculum in Sweden, and its focus is on examining students' *simplification of rational expressions* with an emphasis on the relationship between the content treated in the classroom and students' learning of this subject matter. The focus is also on understanding the aspects the teacher focuses on in the classroom and the aspects that the students

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discern. A close examination of this relationship may help provide a better understanding of ways to improve teachers' teaching practice in order to advance student learning. The research question in this article is: What actually happens inside the classroom when variation theory approach is used?

REVIEWING THE LITERATURE

Two trends can be distinguished in previous research on student learning of algebra. One trend, which is found in current mathematics education research, suggests that technologies have the potential to help learners make connections with and between mathematical concepts and to enrich learners' mathematical thinking. The other trend, which is found in early mathematics education research (more than three decades of research), recognised that students have difficulty when they try to manipulate (simplify) rational expressions (e.g. Davis, Jockusch & McKnight, 1978; Demby, 1997). These difficulties are still evident today for Swedish students (e.g. TIMSS 2007, 2011). That early research made some important findings on what contributed to these difficulties. These included the visual cues present in algebraic structures; cancellation error; seeing an algebraic expression as a process or a sequence of instructions; the proceptual divide; the difference between procedural and static interpretations; and the process-product dilemma (Demby, 1997; Freudenthal, 1983; Gray & Tall, 1992; Kieran, 1992; Sfard & Linchevski, 1994).

To overcome the difficulties presented above, the researchers suggested solutions that included the following: making explicit algebraic thinking inherent in arithmetic in children's earlier learning (e.g. Warren & Cooper, 2006); explicit teaching of nuances and processes of algebra in an algebraic and symbolic setting (e.g. Stacey & Chick, 2004), especially in transformational activities (Kieran & Yerushalmy, 2004; Stacey & Chick, 2004); using multiple representations, including the use of technology (Kieran & Yerushalmy, 2004); and recognising the importance of embedding algebra into contextual themes (e.g. Stacey & Chick, 2004).

The research reported in this article took a different approach, which was grounded in teaching and learning mathematics through using variation (e.g. Marton, 2015).

THEORETICAL FRAMEWORK

What the teachers focus on and what the students discern in mathematical communication are aspects of the object of learning. The central idea in variation theory is that in order to discern certain aspects of the object of learning, a person needs to experience variation corresponding to them (e.g. Marton, 2015). The aspects of the object of learning can be the whole, the parts that form the whole, the relation between the parts, the transformation between the parts, and the part–whole relation for a mathematical concept or between different concepts (Olteanu, 2016; Olteanu & Olteanu, 2013). For example, to simplify the rational expression $\frac{x^2}{2x}$, it is necessary to perceive x^2 as $x \cdot x$ and $2x$ as $2 \cdot x$. In addition, it is necessary for the students to experience that anything divided by itself is just '1'. Olteanu (2014) states that *communication* is a collectively performed patterned activity in which an aspect that is critical for one or more students (A) is focused on by action of the teacher or other students (B) so that A discerns the aspects focused on by B.

A verbatim presentation of Olteanu (2014, 2016) works is reproduced in this paragraph. When communication succeeds, thought content (critical aspect) is shared between speakers and addressee(s) in [a] joint activity and there is shared understanding. *Critical aspects* are those [features] necessary to understand the content worked out in the classroom in order to develop the ability to communicate the content in algebra and mathematics more generally (Olteanu, 2016). 'Critical' here refers to a critical difference in the learners' ways of grasping and becoming acquainted with the object of learning (Olteanu, 2014). Critical also refers to subjective positioning of the participant vis-à-vis the object of knowledge. The critical aspects are divided into the *intended critical aspects* (ICAs) – the aspects of the content that teachers intend to present in the classroom; the *enacted critical aspects* (ECAs) – the aspects of the content that teachers focus on, and the *lived critical aspects* (LCAs) – the aspects that the students distinguish (Olteanu, 2014). Meaningful interaction among the ICA, ECA, and LCA shows whether the communication has been effective (Olteanu, 2016; Olteanu & Olteanu, 2013).

Marton (2015) argued that in order to discern different aspects of the object of learning, students must experience variation in these aspects. When these aspects are not discerned, they become LCAs and are what students' experience as critical aspects in their learning (Olteanu, 2016). For example, if teachers assume that the expression $\frac{3x+5}{3}$ would not prompt students to recall the cancellation property (ICAs), they would not focus on this aspect in teaching. If students incorrectly cancel 3, that is an LCA for them.

The patterns of variations which can facilitate students' discernment of critical features or aspects of the object of learning are the following: (1) contrast (C), which means that to discern a quality X, a mutually exclusive quality non-X needs to be experienced simultaneously; (2) separation (S), which means that in order to discern a dimension of variation that can take on different values, the other dimensions of variation need to be kept invariant or varied at a different rate; (3) generalisation (G), which means that to discern a certain value, X_1, in one dimension of variation X from other values in other dimensions of the variation, X_1 needs to remain invariant while the other dimensions vary; (4) fusion (F), which means experiencing the two dimensions of variation simultaneously; and similarity (SI), which is the property of two or more expressions with the same meaning (e.g. Marton, 2015; Olteanu, 2016).

METHODOLOGY

In this paper, we discuss the results of students' learning and the teaching of rational expressions in an upper secondary school. The research approach used was educational design, which is characterised as being pragmatic, rooted in praxis, interventionistic, iterative, collaborative, flexible in design, and theory oriented.

Educational design research involves understanding the link between teaching and learning through cycles of intervention, with a view to improving the next intervention (e.g. McKenney & Reeves, 2012; Plomp & Nieveen, 2009). The aim is to produce useful and sustainable results for regular use in a school.

Over a three-year period, two teachers (here called Thomas and Patrik) and 65 students (23 in Phase I, 18 in Phase II, and 24 in Phase III) in the Swedish Natural Science Programme participated in the study. The analysis was grounded in 30 exercises and 12 written reports. The data was collected in 11 steps. The teachers examined the course module and curriculum to identify the intended object of learning (Step 1). The teachers identified the enacted object of learning (Step 2). The researchers explained various concepts used in the variation theory to the teachers and the teachers put those concepts into practice (Steps 3 and 4). The teachers worked to identify ICAs (Step 5). The teachers conducted tests and interviews with the students to identify LCAs (Steps 6 and 7). The researchers explained the key concept of the theory of variation again (Step 8). The teachers implemented six lessons (Step 9). After each lesson, the teachers wrote a detailed report on the following: (a) general information (school, class/group, teacher, moment, object of learning, type of lesson); (b) general purpose; (c) specific purpose (content, emotional view, psychomotor view); (d) prerequisites (technical aids, materials); and (e) lesson implementation according to teaching method (with a focus on the open dimensions of variation) and activities with students (Step 10). The teachers conducted different tests with the students (Step 11).

The initial analysis entailed coding students' responses for the types of aspects discerned and the teachers' reports for the types of aspects focused on. In the Phase I, the teachers had worked together to identify the ICAs and the intended object of learning, based on these ICAs. Their work was analysed based on the following questions: What aspects did the students discern when simplifying rational expressions? What patterns of variation could be opened up in the aspects that the students did not discern?

RESULTS

The initial analysis identified six categories of the object of learning (Table 1).

At the beginning of the project (Phase I), the teachers, to a great extent, had assumed that students did not discern rational expressions as a whole (W), the relation parts-whole (PW) and the relation between different wholes (RDW). In addition, they did not consider that students needed to have a better understanding of the constituting parts (P), of the relation between those parts (RP), and of how to relate the parts to each other in a different way (RPD). The teachers assumed, for example, that students could discern the difference between terms and factors and so the students would only cancel common numerical or algebraically factors in the simplification of a rational expression. Consequently, the teachers intended to focus on the aspects in categories W, PW and RDW and focus less or not at all on categories P, RP and

RPD. In addition, in their notes, they occasionally or rarely mentioned using patterns of variation in these aspects.

Table 1. *Categorisation of aspects and examples of non-discerned aspects*

Categories	Rational expression	What has to be done to discern	Examples of non-discerned aspects
whole (W)	$\frac{4x^2-12x+9}{4x^2-9}$	to discern/focus on the relation between the numerator and denominator	$\frac{4x^2-12x+9}{4x^2-9} \Rightarrow -12x-1$
parts (P)	$4x^2 - 12x + 9$ and $4x^2 - 9$	to discern/focus on the composition of nominator and denominator	$\frac{a^2-b^2}{5(a-b)} \Rightarrow \frac{a-b}{5}$
relations between the parts (RP)	$\frac{4x^2-12x+9}{4x^2-9}$	to discern/focus on the operation between term in the numerator and/or in the denominator and the relation between the nominator and denominator	$\frac{a^2-b^2}{5(a-b)} \Rightarrow$ $\Rightarrow \frac{a\cdot a-b\cdot b}{a\cdot a\cdot a\cdot a\cdot a-b\cdot b\cdot b\cdot b\cdot b} \Rightarrow$ $\Rightarrow \frac{1-1}{a\cdot a\cdot a-b\cdot b\cdot b}$
transformations between parts (RPD)	$\frac{(2x-3)^2}{(2x+3)(2x-3)}$	to discern/focus on to factorise the numerator and/or in denominator	$\frac{4x^2-12x+9}{4x^2-9} \Rightarrow$ $\Rightarrow \frac{(2x+3)(2x+3)-12x}{(2x+3)(2x-3)} \Rightarrow$ $\Rightarrow \frac{(2x+3)-12x}{2x-3}$
relation between parts and whole (PW)	$= \frac{(2x-3)^{2}}{(2x+3)(\cancel{2x-3})}$	to discern/focus on cancelling	$\frac{7x+x}{7x+3x} \Rightarrow \frac{1}{3}$
relation between different wholes (RDW)	$\frac{4x^2-12x+9}{4x^2-9} = \frac{2x-3}{2x+3}$	to discern/focus on the equivalent relation between two algebraic expressions	$\frac{30x+x^2}{x} \Rightarrow$ $\Rightarrow \frac{30x^3}{x} \Rightarrow 30x^2$

In Phase I, only a few students discerned the aspects that teachers expected them to. In this phase, one or more students had aspects that were critical (P, RP and RPD) for them, but they did not have an opportunity to discern them because the teacher and other students did not focus on them. As a consequence, those students could not work out the meaning of the whole because they had no understanding of how the meaning of the whole is determined by the meanings of the parts and by the mode of composition of the constituting parts.

In Phases II and III, the students improved their ability to discern different aspects of a rational expression. An explanation for this phenomenon is that in Phase II, the teachers focused on LCAs by using different patterns of variation. In six consecutive lessons, Thomas focused on several aspects and opened up dimensions of variation by separation (S), contrast (C), generalisation (G) and fusion (F). Thomas used several tasks in which the nominator varied and the denominator was kept invariant or vice versa. Some examples of what was taught and how critical aspects were taught using patterns of variation were:

- the difference between a fraction with a unitary numerator and a non-fraction (e.g. $\frac{1}{x}$ and x) (C);
- the difference between factorising a polynomial and solving an equation (e.g. $2x + 12$ and $2x + 12 = 0$) (C);
- the difference between terms and factors (e.g. $2 + x$ and $2x$) (C);
- specifying multiple times that only factors and not terms can be cancelled (S, G);
- identifying the common factor in the nominator and denominator (S, G, SI);
- specifying that common factors can be simplified by any common numerical or variable factors (e.g. $\frac{2x+12}{2x} = \frac{2(x+6)}{2x}$) (C, S, G, F, SI);
- using parentheses around the nominator and denominator to highlight the whole (C, S, G, F, SI);
- simplifying fractions with polynomials in the numerator and denominator by factorising both and renaming them using the lowest terms (e.g. $\frac{2x+12}{2x+14} = \frac{2(x+6)}{2(x+7)}$) (C, S, G, F, SI);
- identifying and factorising the difference of two perfect squares (S, G, F).

In addition, Thomas kept asking the following questions and kept them invariant in the communication in the classroom: What does factorising look like for a polynomial expression? How do we know when we are finished factorising? What is the process we use to cancel? What does cancelling look like? When do we know we are finished cancelling? All these questions have the same meaning, thus Thomas opened up a dimension of variation by similarity.

The design used in Phase III was the same as that in Phase II. The differences were that Patrik taught in another class that had gone the same process as in Patrick's class. In addition to focusing on the aspects that Thomas had in his class (Phase II), Patrik focused on finding the values of a variable for which an algebraic fraction was undefined as well as on the difference and connection between roots of a quadratic equation and factors of a quadratic expression. In addition, the beginning of each new lesson, Patrik repeated the content discussed in previous lessons at.

The enacted object of learning in Thomas and Patrik's classes enabled students to discern the process of factorising polynomials and to simplify algebraic expressions written as fractions. In addition, the students had the opportunity to experience the following: the term 'cancelling'; that factorising is the reverse of the distributive property; the expressions 'factor' and 'cancel', when working with algebraic expressions written as fractions; the use of factorising and cancelling and the rules of fraction operations to simplify algebraic fraction expressions. In addition, students in Patrik's class could discern that 'undefined' means that the denominator of a fraction is zero. This led to a reduction in students' critical aspects in all categories. An explanation for this is that an aspect that was critical for one or more students was focused on by an action of the teacher or by other students, by using several patterns of variation. The shared understanding increased during the project (from Phase I to Phase III) and the result was reflected in a reduction in critical aspects in students' learning in all categories, especially in categories P, RP, RPD and PW (more than a 50% reduction). It can be concluded that in Phase III, there was a further reduction in the students' critical aspects, and an explanation for this may be the systematic repetition done in Patrik's class.

DISCUSSION

In this study, one of the primary factors demonstrated in order to encourage teachers to use the research findings in the field of mathematics education was the identification of critical aspects in students' learning and the use patterns of variation. The learning theory of variation serves as a useful theoretical framework for teachers in planning and structuring their lessons. It guides teachers to decide what aspects to focus on, what aspects to vary simultaneously, and what aspects to let remain invariant. Furthermore, it guides teachers in how to consciously design patterns of variation to bring about the desired learning outcomes.

One of the LCAs identified in this study was discerning the difference between terms and factors. The students found it especially hard to discern the aspects that appear in the following: the relations between the parts (RP), the transformations between the parts (RPD), and the relation between the parts and the whole (PW). By reflecting on these general categories, the teachers formed a complete learning object, in the sense that they were able to take up almost all of the critical aspects of the students' learning and open up dimensions of variation by contrast, separation, generalisation, fusion, and similarity in those aspects. This resulted in an essential improvement in students' learning and successful communication in the classroom. Classroom instructions that are based mainly on a theoretic variation approach have a positive impact on students' learning of factorising rational expressions. What teachers learned about teaching was explicit and analytical rather than intuitive and imitative.

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THE INFLUENCE OF TEACHER PERCEPTIONS AND TEACHING APPROACHES ON SENIOR SECONDARY MATHEMATICS STUDENTS' USE OF CAS CALCULATORS

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The focus of this paper is to examine the extent to which teacher perceptions and approaches to CAS calculator use influence whether students utilise this technology in senior secondary mathematics. Using an embedded multiple case study approach, data were collected from two Year 11 and two Year 12 mathematics classes and their respective teachers. While the results presented highlight the key role of the teacher in fostering a CAS environment, other contextual factors also need to be taken into consideration.

INTRODUCTION

Within the past few decades, researchers in the field of mathematics education have recognised the potential for information and communications technology (ICT) to transform the teaching and learning of mathematics (Goos, Galbraith, Renshaw, & Geiger, 2000). The effective use of ICT has also been seen as a key capability within the Australian curriculum and is incorporated across all learning areas, including mathematics (Australian Curriculum, Assessment and Reporting Authority, 2015). However, while the use of technology has presented many advantages, "the vision of using information and communication technology (ICT) to transform the teaching and learning process ... [has been] far from becoming a reality" (Rodríguez, Nussbaum, & Dombrovskaia, 2012, p. 81), with concern that ICT integration in school mathematics has fallen behind the promising expectations of previous decades (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010).

Computer Algebra System (CAS) devices are an example of one such technology that have faced various obstacles in its integration within the school mathematics domain. "In spite of the long history of work with CAS in educational settings, the impact of technology on school mathematics has to date been marginal, and the incorporation of CAS in classrooms has been even slower" (Heid, Thomas, & Zbiek, 2012, p. 599). Student attitudes, time restrictions, and the technical skill required to operate CAS technology are just some of the factors that have made CAS integration within school mathematics difficult to achieve successfully (Barkatsas, Kasimatis, & Gialamas, 2009; Drijvers, 2002; Schmidt, 2010).

Goos et al. (2000) also noted that the role of the teacher was crucial in developing a technology-rich learning environment. However, as highlighted in a study by Teo (2011) involving Singaporean teachers, "perceived usefulness, attitudes towards use,

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and facilitating conditions [all] have direct influences on behavioural intention to use technology" (p. 2437). With CAS calculators forming an integral part of the Victorian senior secondary mathematics curriculum in Australia, and the significant role a teacher may play in their successful integration, the main focus of this paper will be to report on how teacher perceptions and approaches regarding CAS calculator use may influence students' utilisation of CAS technology in mathematics.

THEORETICAL FRAMEWORKS

Within this study, two theoretical frameworks were used to investigate and describe teachers' use of CAS-based systems within the mathematics education domain. These frameworks are summarised briefly below.

Technological Pedagogical Content Knowledge (TPCK)

Mishra and Koehler (2006) introduced the theoretical framework Technological Pedagogical Content Knowledge (TPCK) to describe the relationship between teaching and technology. In order to successfully integrate technology as part of teaching practice, teachers need to have three essential forms of knowledge: Technology Knowledge (knowledge of how to use technology), Pedagogical Knowledge (knowledge of teaching practice) and Content Knowledge (knowledge of the subject matter). The interaction between each of these three forms of knowledge is crucial to the development of a teachers' TPCK. As summarised by Koehler, Mishra and Yahya (2007): "at the heart of TPCK is the dynamic transactional relationship between content, pedagogy and technology. Good teaching with technology requires understanding the mutually reinforcing relationships between all three elements" (p. 741).

MSPE Framework

In their three-year longitudinal study, Goos et al. (2000) identified four metaphors to describe the interaction between teachers and technology (the MSPE Framework). Technology plays the role of *Master* if a teachers' implementation of technology is limited by their technological knowledge and skills. Technology plays the role of *Servant* when its only purpose is to support preferred teaching practices (e.g., using a calculator solely for its speed and efficiency to replace pen and paper techniques). Technology is used as a *Partner* when it aids in the implementation of a teaching practice that gives students more control over their learning, such as sharing or mediating mathematical discussions. Finally, technology is used as an *Extension of self* when its "powerful and creative use ... forms as natural a part of a teacher's repertoire as do fundamental pedagogical and mathematical skills" (Goos et al., 2000, p. 308).

METHODOLOGY

The findings presented within this paper form part of a larger mixed methods study which utilised a quantitative phase followed by a qualitative phase. The population investigated for this study were Year 11 and Year 12 mathematics students and their teachers. Local schools in the region of Melbourne, Victoria, Australia were invited to participate via a letter explaining the nature and intention of the research. Within the time frame allocated, six schools agreed to participate in the study and the findings of the quantitative phase involving a questionnaire can be found in a prior paper (Orellana & Barkatsas, 2015).

The participants for the qualitative phase, presented in this paper, came from two of the initial six schools who participated in the quantitative phase of the study. Findings from the quantitative phase aided in the selection of participants through examination of key differences between schools on variables such as technology confidence, years of CAS experience and attitudes towards using CAS calculators in mathematics. Within each participating school, one Year 11 and one Year 12 mathematics class (taught by the same teacher) were selected with the aid of the school's mathematics coordinator.

An embedded multiple case study approach was used for data collection incorporating classroom observations and interviews with students and teachers. Observations were non-participant and overt while also adopting a semi-structural approach to allow for greater flexibility and responsiveness to naturally occurring events (Flick, 2006). Interviews with participants also allowed for greater flexibility by being semi-structured and one-on-one, without diverging too far from the research aims (Berg, 1995). For the interview process, students were selected with the aid of the classroom teacher and restricted to those with parental consent. In total, 20 students were interviewed along with the two respective teachers from each school (henceforth labelled Teacher A and Teacher B).

The collected data were initially analysed using a preliminary exploratory analysis in order to obtain a general sense of the data (Creswell, 2005). Once complete, the data were analysed using a thematic analysis procedure as outlined in Braun and Clarke (2006) in order to identify key patterns or themes within the observational fieldnotes and interview transcripts. While the larger study, of which this qualitative phase forms a part, focuses on both teachers' and students' use of CAS technology, this paper will report on the findings with respect to Teacher A and Teacher B.

RESULTS

Teacher A

Teacher A was a female, secondary mathematics teacher who had been teaching in schools for over 30 years. Teaching students from Years 9 to 12, she was introduced to the CAS calculator between the years 2006-2007, when the CAS became a compulsory element of the Victorian Certificate of Education (VCE – the final two years of secondary schooling) study design. Having had no prior experience with this technology, initially her use of CAS was limited due to the fact that she was not involved in teaching VCE mathematics: "Back then [I] didn't use it as much as what I do now because I wasn't involved in [years] 11 and 12". However, with professional development and support from her colleagues, along with self-teaching, Teacher A became proficient with CAS and used it more frequently in her teaching practice.

With respect to her views on CAS calculators in mathematics, Teacher A is fairly positive about using this technology and believes it has aided her students' achievement as well as providing a means to explore mathematical concepts. However, she has also encountered various obstacles along the way, such as trouble resolving complex technological errors, over-reliance on CAS by students, and changing students' mindsets "to get them to realise that they don't have to do everything by hand". The approach adopted by Teacher A when teaching with the CAS calculator involved a more student-centred approach by using open-ended questions to get students thinking about the mathematics being taught. The calculator was used as a means to 'explore' mathematics and to aid in the development of conceptual understanding.

The Year 11 and Year 12 classes taught by Teacher A were Mathematical Methods (CAS) classes, a subject focused on calculus, algebra, functions, and probability. Students were first introduced to the CAS calculator in Year 9 and were expected to bring this technology to every class. From the observational data, and from student interviews, it was found that students in both classes encountered difficulties with the CAS calculator including syntax errors, problems with settings, and interpreting output. Students in the Year 11 classroom also struggled to understand when CAS calculator use would be more efficient than by hand methods, which may also be why the latter techniques were preferred by students regardless of whether the task was considered technology-rich or technology-free. In contrast, the Year 12 classroom used the CAS calculator more frequently. However, at times, students were too reliant on the technology, using it for questions that would have been faster by hand.

Teacher B

Teacher B was also a female, secondary mathematics teacher with over 30 years of teaching experience. Teacher B focused exclusively on VCE mathematics, teaching only Year 11 and Year 12. With no experience using the CAS calculator prior to its implementation, she learned to use the CAS through guest speakers and practising in her own time with the instruction manual. She uses CAS everyday as part of her teaching practice and incorporates it into every mathematics class.

Teacher B displayed a very positive attitude towards using CAS calculators in mathematics and was enthusiastic about learning as much as she could about this technology. She did not encounter many difficulties with the CAS and many students turned to her when they experienced any problems with hardware or software: "I fix the calculators for them". Teacher B believed that her students benefited greatly as a result of having CAS technology available, particularly lower achieving students, with improved student results, attitudes, and confidence. However, while she noted that the CAS calculator could help lower achieving students "pass exams", she also acknowledged that it may not necessarily improve their mathematical understanding:

> For those students who do not really know the value of the numbers, actually it's just useless for them ... I would say the calculator actually is spoiling them ... they don't understand how ... they get the answer because the calculator don't show them step by step.

With respect to her teaching style, Teacher B's approach focused on the development of conceptual knowledge before introducing CAS as an efficient way of obtaining solutions. However, depending on the level of achievement of her students, she has also introduced the CAS calculator as soon as possible for students who appeared to be struggling with mathematics. Unlike Teacher A, CAS was not used as a means to investigate mathematical constructs, but rather as a faster, alternative means of solving mathematical problems.

The Year 11 and Year 12 classes taught by Teacher B were General Mathematics and Further Mathematics classes respectively. These subjects place more emphasis on topics such as geometry, data analysis, and business related mathematics rather than calculus or algebra. Students were first introduced to the CAS calculator in Year 11 and were expected to bring this technology to every class. From the observational data, and from student interviews, it was found that students in the Year 11 class displayed moderate CAS calculator use. Some students used the CAS frequently, others preferred by hand methods, and others did not know how to use it. However, students were more inclined to use the CAS when they were shown how efficient it was. Additionally, students in the Year 11 class experienced difficulties knowing how and when to use the CAS calculator, requiring a lot of teacher support and guidance. In contrast, the students in the Year 12 classroom used CAS frequently to complete mathematical work, making seamless transitions between by hand and CAS techniques. CAS use came naturally to these Year 12 students as they understood both how and when to use this technology.

DISCUSSION AND CONCLUSIONS

From the results presented above, it was evident that both teachers had various characteristics in common, such as their years of mathematics teaching experience and their lack of familiarity with the CAS when it was first introduced. However, one difference between the two teachers was their experience with VCE level mathematics. Whereas Teacher B had more familiarity with Year 11 and 12 (teaching only these year levels), Teacher A had only recently become involved with VCE mathematics and thus had not used the CAS calculator as frequently when it was initially introduced. Thus, it could be argued that Teacher B had a greater understanding of how to use the CAS calculator as part of VCE mathematics teaching (TPCK) with more years of experience in this regard. Her high level of CAS knowledge (the 'T' of TPCK) also allowed her to solve a majority of her students' problems when they arose in class, whether they were technical in nature or due to incorrect notation.

Although less experienced with VCE mathematics, Teacher A was confident in her mathematics teaching knowledge (PCK) to take a more constructive and open-ended approach with technology, using the CAS calculator to explore mathematical concepts in whole class discussions. This type of instruction also reflected her belief that one of the many advantages of CAS lie in its capacity to allow students to see patterns and to think about mathematical constructs. Using the metaphors developed by Goos et al. (2000), Teacher A could be described as working with the CAS as a *partner* using this technology "creatively in an endeavour to increase the power students collectively exercise over their learning" (p. 307). For example, when introducing students to the inverse function, Teacher A asked students to take out their CAS calculator and input three functions: $f_1(x) = f(x)$, $f_2(x) = g(x)$ (the inverse), and $f_3(x) = x$. She then prompted a discussion with students, asking questions such as "What do you see?" to allow them to discover what the inverse represents. These types of discussions took up a large portion of Teacher A's classes.

In contrast, Teacher B used a more traditional and structured mode of instruction, explaining concepts on the board before moving onto CAS-based examples and questions. CAS, in this context, was used as an efficient and alternative way to solve mathematical problems (as a *servant*), encouraged as a means to save time in examinations rather than changing the nature of classroom activities (Goos et al., 2000). For example, when introducing the topic of graphs and relations, Teacher B briefly went through the key points and definitions before moving into an example where CAS could be used, outlining step-by-step instructions on the board: "Menu → Stat → Stat calc→ 2 → x list x, y list y → select $y = a + bx$ or $y = mx + c$ ". Teacher B announced to students that if they had forgotten how to find the equation of a straight line given two points, then the CAS was a "useful" way to obtain a solution. Explanations and discussions did not take up as much of the lesson as those of Teacher A, and the main focus was to complete set questions from the textbook or from provided worksheets. Thus, while there was a preference for students to develop conceptual understanding prior to learning CAS procedures, there was also an emphasis on performance and procedural knowledge. This may be why lower achieving students in her classes were, at times, introduced to CAS-based procedures earlier than their peers, despite Teacher B preferring the development of conceptual understanding prior to this.

Analysis of the classroom results revealed various differences between Teacher A's and Teacher B's participating classes. While in Year 11 the results were quite similar for both classes, students in Teacher B's classes appeared to use the CAS more frequently and efficiently than students in Teacher A's classes. In particular, Teacher B's Year 12 class encountered fewer difficulties when working with the CAS calculator and had higher levels of CAS knowledge, knowing both when and how to use this technology. Having endeavoured to learn as much as she could about the calculator, Teacher B's familiarity and confidence with the CAS appeared to have positively influenced her students. Even though students had fewer years of CAS experience compared to students in Teacher A's classes, students in Teacher B's classes were able to quickly overcome any CAS-related issues with the aid of their teacher, and appeared to be developing greater confidence with this technology, consequentially influencing their use of CAS in the classroom.

In comparison, students in Teacher A's classroom displayed a lack of CAS knowledge, encountering more difficulties when using the CAS calculator. Common errors included issues with syntax and settings, which were also evident in Year 12, although to a lesser extent. Knowing when to use the CAS was also an area students struggled to understand. For example, not using CAS for technology-rich questions, or using CAS to solve $V(t) = 0$ for $V(t) = 10^3 (90 - t)^3$. Teacher A's approach to addressing these difficulties reflected her teaching style. If students encountered errors, Teacher A preferred to use prompts to help students discover where they went wrong rather than provide them an answer. She would also use discussions to find out the different solutions that students had to solving a mathematical problem with the CAS calculator rather than giving them step-by-step instructions: "How can I use my calculator to get the rule for the inverse function?" Thus, the results, in a sense, have been counterintuitive as this approach has been suggested in prior research by Drijvers (2002), who proposed using CAS obstacles as opportunities for learning:

> Instead of trying to ignore the obstacles encountered, I suggest to make them the subject of classroom discussion ... such an approach turns the obstacles of computer algebra use into opportunities for learning, and enriches mathematical discourse in the classroom. (p. 228)

In summary, the results presented in this paper showed that the participating teachers in this study found the CAS calculator to be useful for different reasons and have incorporated this technology into their teaching based on these beliefs. However, it is difficult to determine the extent to which the teachers' perceptions and teaching approaches in this study influenced students' use of CAS calculators considering the potential contextual factors involved (e.g., mathematics subject) and the limited sample size. Further research is needed to compare and contrast the views of teachers and students with respect to CAS technologies to determine how they may potentially influence each other.

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PRINCIPLES IN DESIGNING TECHNOLOGY-INTEGRATED GEOMETRY TASKS FOR TEACHING TEACHERS

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When technology is used as a teaching tool without adapting it to human cognition, the learning becomes focused on the tool rather than on the learner. Hence, the current research seeks to examine the advantages of technological tools in designing research tasks in geometry for teaching teachers, taking into consideration that even teachers have different learning attributes. Adapting geometry tasks to variations between teachers is an art that can be learned from research literature on task design and on designing multimedia tasks using technological tools in the context of human cognition. The current study adapted design principles from the research literature to design geometry research tasks for teaching teachers.

THEORETICAL BACKGROUND

This section outlines the theoretical concepts for constructing research tasks so as to create a distinct framework for designing geometry tasks to be solved using technological tools. Such a framework can be implemented for teaching teachers. The concepts are presented in the following contexts: the advantages of dynamic computerized tools for solving geometry problems; theories of constructing mathematical tasks; theories of constructing tasks using technological tools while taking the learner's cognitive structure into consideration.

Integrating Dynamic Geometry Software (DGS)

Research in education constantly seeks methods to improve the quality of teaching. One issue often focused upon is the integration of technology. Dynamic geometry software (DGS) is a technological tool that allows mathematical equations to be represented and mathematical objects to be constructed so as to provide constant feedback to the user (Alakoc, 2003; Martinovic & Manizade, 2013). DGS allows users to create and then manipulate constructions and properties. DGS provides students with different opportunities to engage with geometric objects and their measures and has the potential to help them develop a deeper understanding of properties and theorems (Leung, 2008). Dragging in DGS is one of the pedagogical values of a dynamic geometry environment. Such an environment enables users to express their geometrical thoughts in visual-dynamic ways that can help them construct abstract knowledge. While dragging different points and objects, users can identify properties and relations between them, while generating a multitude of specific examples and hypotheses (NCTM, 2010).

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Constructing Mathematical Tasks for Teaching

The current study describes an experiment in which geometric tasks that integrated a technological tool were designed for mathematics teachers. Construction of the tasks was based on the theories of three researchers regarding the design of mathematical tasks. Zaslavsky (2008) proposed several ideas for intelligent construction of mathematical tasks for teachers. Chen, Kalyuga, & Sweller (2016) referred to the architectural structure of human cognition and Mayer (2014) advocated integrating multimedia into learning. We discuss each of these theories in brief and use them to construct the conceptual language for this research study.

The first theory (Zaslavsky, 2008) outlines themes that characterize the criteria for designing mathematical tasks to promote teachers' learning processes, as follows. **Developing adaptability:** Teachers must consider making changes in questions and tasks, looking for new teaching approaches, adapting existing sources, and planning and exercising flexibility in teaching and learning. **Fostering awareness to similarities and differences:** Teaching includes identifying what is different and what is similar among mathematical objects, classifying objects according to criteria and identifying relations between objects. **Coping with conflicts, dilemmas and problem situations:** Teachers must become problem solvers in the broadest sense of the term by enhancing their students' abilities to solve mathematical problems and cope with cognitive barriers and conflicts. **Selecting and using (appropriate) tools and resources for teaching:** In constructing mathematical tasks teachers must select and use appropriate tools and resources, such as professional literature, literature from other fields linking the task to everyday life and technology integration. **Identifying and overcoming barriers to students' learning:** Task construction must take the students' learning barriers into consideration and must give all students the opportunity to cope with these barriers. Learning barriers can include difficulty with various representations, missing prior knowledge, personal limitations and more.

Technological Multimedia Theory and Motivation Adapted to Cognitive Load

According to the second theory known as the cognitive load theory (Sweller, 2015), teaching should take the architectural structure of the human cognitive system into consideration. Human cognition resembles the concept of natural selection in that the strongest knowledge will always survive (be preserved) over the long term. The theory can be described using five principles: **Database principle**: In order to operate in a complex environment, natural information processing systems must store a large amount of information. In human intelligence, long-term memory serves as this storage space. **Borrowing and identification principle**: Information is stored by borrowing information from other sources of information in the learning environment and by identifying old knowledge and relating it to new knowledge. **Randomness and creativity principle**: When learners cannot borrow information from the environment, they create it. **Limited change principle**: Working memory is capable of containing a limited amount of information. Working memory makes sure that small changes take place while outside information is being processed. **Connectivity principle**: Working memory has no limitations when communicating with long-term memory, as manifested in organizing and retrieving large amounts of information.

The third theory (Mayer, 2014) discusses the relationship between multimedia, motivation and learning to solve problems. This theory calls for analyzing multimedia integration using three types of skills and abilities: **cognitive skills**, including learning, dismantling, assembling and organizing knowledge in order to process it; **metacognitive skills**, including the learner's personal strategies for solving mathematical or other problems; and **motivational skills**, including the learner's level of interest and self-direction as manifested in learning while solving problems.

According to Mayer (2009), we must distinguish three ways that learners process information while learning: The first is superfluous external processing deriving from faulty teaching design. The second is essential processing based on translating and analyzing mental representations resulting from complex content analysis. The third is generative processing whose goal is to create a feeling for the material. According to Mayer's approach, designing tasks using technological tools uses three basic principles: "less is more"—using a design that reduces superfluous processing to a minimum while facilitating essential processing; "more is more"—using design principles that motivate learners toward generic processing; and "focused more is more"—using design principles that motivate learners to cope with generative processing while at the same time focusing on reducing superfluous processing.

In this research, we combine the theories of Zaslavsky, Sweller and Mayer to create and design geometry tasks that use a technological tool for teaching teachers.

RESEARCH METHOD

The objective of the current research is to define principles for designing geometry tasks that integrate technological tools for the purpose of teaching teachers.

Research Population

The research participants included two groups of mathematics teachers studying in a master's degree program in teaching mathematics at a college of education. Each group comprised 12 participants. The participants were enrolled in a course in which they learned to solve geometry problems using technological tools.

Research Questions

1. What characteristics mark teachers' learning as they solve research problems in geometry designed according to the theoretical principles of task design?
2. What are the principles for designing research tasks in geometry for teachers that integrate work with a technological tool?

Research Instruments

The research instruments included research tasks in geometry designed to be solved by using technological tools as well as lessons that focused on solving research tasks using technological tools, solving problems using formal deductive proofs and discussions on teachers' learning processes. In addition, the researchers collected data in a reflective journal, including documenting and analyzing the stages task design prior to the lessons, teaching the tasks, and analyzing the results. Following is an example of an investigative task in geometry designed for the experiment.

What types of quadrangles do you get from connecting the midpoints of the sides of a quadrangle? The task was presented differently in the two groups of teachers. One group was given a table listing different types of external quadrangles, such as any quadrangle, a parallelogram, a rectangle, a rhombus, a deltoid, a square, a trapezoid and an equilateral trapezoid and asked to research the task for each of the quadrangles. The other group worked on investigating individual cases. An individual case refers to a focused task such as the following: **When do you get a rectangle inscribed inside a quadrangle after connecting the midpoints of the quadrangle's sides?**

Solving the task when the external quadrangle is any general quadrangle is based on drawing the diagonals of the external quadrangle and using the theorem about the midsections of a triangle. Investigation for specific external quadrangles is based on the mutual state of the diagonals and on their lengths. That is, if the diagonals of the external shape are equal, the result is a rhombus; if they are perpendicular, the result is a rectangle; if they are both equal and perpendicular, the result is a square; and so on. This graduated presentation of the task was designed according to the principles of task design so as to enable each teacher to investigate at his or her own pace and according to the way each chose to cope with the task. In addition, the design facilitated investigation while examining the similarities and differences between the inner and the outer shapes.

Work on the solutions was flexible in that teachers interested in solving the problem using formal deductive reasoning before attempting to use the technological tool. Designing the tasks for the teachers was influenced by the theories outlined in the research literature, which focused on designing tasks for teachers and designing tasks using technological tools while taking human cognition into consideration.

DATA ANALYSIS

For purposes of analysis, we divided the data into two types: data representing the designed tasks and data referring to the process of learning from these designed tasks. We observed both data groups in view of the above theories. We constructed the tasks according to principles from the research literature. Nevertheless, after we used these principles we again analyzed the designed tasks and re-examined their components in accordance with the theoretical principles we chose as the analysis framework. To illustrate this, we describe our analysis of the following designed task: **When do you get a rectangle inscribed inside a quadrangle after connecting the midpoints of the quadrangle's sides?**

The teachers coped differently according to their levels of geometric content knowledge. Thus, in accordance with Zaslavsky's design principle for matching task to learners, we adapted the task to the learners, some of whom knew how to draw a quadrangle using the technological tool and others did not. Those who knew how to use the tool solved the problem directly according to the quadrangle's attributes. Those in the second group used geometric construction to solve the problem and while doing so learned the critical attributes of the shape.

Retrospective analysis of the design indicates that the teachers who chose to solve the task according to the focused formulation can be divided into two groups. During the construction one group stressed the quadrangle's attributes, while the other focused on the critical attributes of the technological tool and of rectangular polygons.

In accordance with Mayer's theory, before the teaching task design considerations were based on the "less is more" principle on the assumption that focusing on one polygon would increase in-depth learning. Retrospective analysis indicates that both groups—those who already knew how to construct a quadrangle and those who learned during the lesson—increased their in-depth learning regarding the case of the inner quadrangle. From the individual case they managed to derive additional benefit and draw further conclusions regarding other polygons, even those they had not built.

According to Sweller's cognitive principle of borrowing and identification, design considerations prior to the teaching focused on task design that called for "borrowing and identifying" prior knowledge and the ability to create new knowledge. The task directed the learners to two geometric shapes—square and rectangle. These two shapes were constructed using the technological tools by borrowing and identifying the critical features of each of the shapes. The technological tool enabled the learners to represent their prior knowledge by means of trial and error, and by making conjectures they were able to construct new knowledge of their own creation.

The data point to two different aspects. The first reveals the behavior of the student teachers during problem solving. The second reveals their opinions and ideas regarding the advantages of designing a research task in geometry to be solved by technological tools in the context of their own learning experience.

FINDINGS

This section focuses on answering the two research questions. The first question refers to the characteristics of teachers' learning, including evidence found in the lesson documentation. We provide several examples of evidence from the teachers' reflective documentation, the researchers' research journals and the lesson documentation. We analyze how the teachers coped with the task based on the three outlined theories.

Examples of Teachers' Learning Events According to the Principles of Zaslavsky

The teachers' learning events seem to imply Zaslavsky's themes. The task presented to the teachers was based on selecting and using (appropriate) teaching tools and resources. We chose to present the task in combination with the technology, among other reasons because we assumed that the teachers' geometric knowledge was not sufficiently established to solve the problem using deductive proofs and that the technological tools could serve as an additional aid in coping with the task. Indeed, the teachers reported that the technological tool contributed to their research. "This tool helped me very much in my investigation. … I saw which internal quadrangles were generated, and by moving the vertices and the sides I checked whether I always got the same quadrangle." "Looking for an assumption using the technological tool saved us a great deal of time." "The technological tool always results in success."

The task was developed and adapted to the environment of teaching teachers (developing adaptability). In addition to being challenging, the task solution was based on geometric knowledge that some of the teachers had to refresh as they went along. All reported that the task was relevant to them in renewing their geometric knowledge and in expanding their pedagogic knowledge: I became convinced that group research using the technological tool contributes much more to learning than frontal teaching."

The task fostered the teachers' awareness of similarities and differences. The investigative process, which included different quadrangles and generalizations regarding the resulting inner quadrangle, helped teachers identify similarities and differences between the quadrangles. For example, all the resulting inner quadrangles were parallelograms, while certain outer quadrangles yielded an inner square or rectangle or rhombus. "Organizing the data in a table helped me work systematically so I could identify what happened in each case relative to prior cases."

Examples of Teachers' Learning Events According to Sweller and Mayer

According to the multimedia principle, people learn better from words and pictures than from words alone. When the teachers were asked to research the task for different quadrangles, the technological tool provided them a variety of relevant examples to help formulate a conjecture regarding what inner quadrangle they would get. The teachers reported that the technological tool helped them understand the task better visually. "There is nothing like seeing the geometric shapes with your own eyes and 'playing' with them. That is the power of the technological tool, which is superior to any lesson taught and demonstrated by the teacher." "The technological tool helped me see the shapes in different positions and that is very important to the research."

According to collaboration principle, people learn better through collaborative activities. For example, the collaboration between the teachers created a "safety net" of mutual inspiration and shared knowledge that that helped them feel more confident. "I always knew there was someone who could help." "Each of us has strong points and weak points, and collaboration helps us share information and overcome difficulties, as in how to construct different quadrangles using the technological tool."

According to the prior knowledge principle, the teachers constructed mediating questions that connected the new knowledge from the task to prior knowledge. In the current research, the task was formulated so that each teacher could begin the research based on prior knowledge. "The task refreshed my mathematical knowledge by learning with colleagues and by using the technology."

What are the principles for designing research tasks in geometry for teachers working with a technological tool? In connecting the three theories, we found that Zaslavsky's five themes for task design support the cognitive load learning theory and the multimedia theory as follows:

Developing adaptability: Zaslavsky states that teachers must consider changing tasks and planning teaching to suit all types of learning teachers. By combining this with the principle of cognitive load learning, we can design a task that includes multimedia according to Mayer's principle of "more is less": fewer words, fewer pictures, focused on the main geometric concepts the teachers need to learn from the task.

Fostering awareness to similarities and differences: Zaslavsky's design principle joins with Sweller's teaching principle that teachers should design tasks that challenge learners to identify new knowledge based on prior knowledge and encourage them to ask questions regarding similarities and differences between the current case and cases studied in the past. Sweller ties this concept to the structure of human cognition, claiming that the human brain learns while mapping similarities and differences. In the tasks designed for the current research, the teacher learners naturally mapped the similarities and differences between the tasks because the tasks were designed and worded in a similar manner and only one or two components were changed.

Coping with conflicts, dilemmas and problem situations: According to Zaslavsky, problems should be designed to elicit dilemmas and conflicts. This principle can be combined with Mayer's principle of cognitive load in learning through multimedia design, for example by implementing the "less is more" principle. In the current study the learners had two learning barriers: Some were unable to construct geometric shapes. In addition, some were not sufficiently proficient in constructing deductive geometric proofs. In the current study, the task was broken down into sub-tasks to help learners focus on drawing a particular case while discussing the relations between the features of a family of quadrangles in the context of the given problem.

In summary, we were able to identify two central characteristics of how teachers learn while solving the research tasks. The first relates to the teachers' knowledge. The teachers renewed, expanded and enhanced their geometric content knowledge and their pedagogic knowledge in a relatively short period of time. The lesson documentation shows that while solving the investigative task using the technological tool, each teacher remembered geometric knowledge on different levels, ranging from remembering relevant geometric theorems to remembering how to write deductive proofs. The second attribute was manifested in the teachers' collaborative work in the two research groups without their having specifically asked to work collaboratively. The collaborative research work together with the use of technology served as a catalyst for expanding and enhancing their geometric content knowledge and their pedagogic knowledge in the context of teaching geometry.

CONCLUSION

The current study attempted to construct a framework recommending principles for designing research tasks in geometry for teaching teachers. Based on Zaslavsky's five themes and the theories of Sweller and Mayer, the study suggested an integrated framework of principles that teacher educators should adopt when designing research tasks in geometry for teachers in a technological environment.

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