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Editors | Berinderjeet Kaur, Weng Kin Ho, Tin Lam Toh, Ban Heng Choy

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REVERSE FRACTION TASKS REVEAL ALGEBRAIC THINKING

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The University of Melbourne

Year 8 students from a Victorian secondary school completed two paper and pencil tests designed to ascertain their understanding of fraction concepts, their competence and fluency with fraction operations and evidence of algebraic thinking. This paper will examine their responses to three tasks which would be aided by algebraic thinking. Students were given the fraction, and the number of objects representing the fraction, and then asked to find the number of objects in the whole group. Analysis of the Year 8 students' responses highlights both the range of methods used, and the difficulties experienced, by many students as they attempted these fraction tasks. While some students successfully applied algebraic thinking others were struggling to do so.

INTRODUCTION

Expanding students' mathematical reasoning beyond arithmetic to generalised algebraic thinking is one of the key challenges for mathematics teachers (Mason, Stephens & Watson, 2009). Jacobs, Franke, Carpenter, Levi, and Battey (2007) emphasise the need to "facilitate students' transition to the formal study of algebra in the later grades (of the elementary school) so that no distinct boundary exists between arithmetic and algebra" (p.261). Many researchers argue that a deep understanding of fractions is important for a successful transition to algebra. The National Mathematics Advisory Panel (NMAP, 2008) stated that the conceptual understanding of fractions and fluency in using procedures to solve fractions tasks are central goals of students' mathematical development and are the critical foundations for learning algebra.

Year 8 students (13-14 years of age and in their 9th year of compulsory schooling) from a large secondary school in Victoria, Australia fractions competence and thinking was assessed using two paper and pencil tests: the Fraction Screening Test (Pearn & Stephens, 2015) and an Algebraic Thinking Questionnaire (Pearn & Stephens, 2016). This paper reports on evidence for students' development of algebraic thinking in their responses to three reverse fraction tasks (Figure 1) where they were given the fraction and the number of objects representing the fraction, and then asked to find the number of objects in the whole group. Responses to these three tasks were analysed to answer the following research questions:

- What strategies do Year 8 students use to solve the reverse fraction tasks?
- When a fraction, and the number of objects representing that fraction, change do the students' solution methods or strategies also change?

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- Do Year 8 students' responses to the reverse fraction tasks show evidence of a successful transition to algebraic thinking?

PREVIOUS RESEARCH

Researchers such as Kieren (1980) and Lamon (1999) believe that much of the basis for algebraic thought rests on a clear understanding of rational number concepts and the ability to manipulate common fractions. According to Wu (2001) the ability to efficiently manipulate fractions is "vital to a dynamic understanding of algebra" (p. 17). Particularly relevant to our research is the work of Lee and Hackenburg (2013) who showed that fractional knowledge appeared to be closely related to establishing algebra knowledge in the domains of writing and solving linear equations. They highlighted the importance of multiplicative operations to transform a known fraction to the whole. Such use of multiplicative methods will later be fundamental for solving algebraic equations. Empson, Levi and Carpenter (2011) suggest that some strategies students use to solve fraction problems "are motivated by the same mathematical relationships that are essential to understanding high-school algebra" (p. 410).

Our research extends the research of Empson et al. (2011) by using reverse fraction tasks to investigate students' capacity to establish an equivalence relationship between a given collection of objects and the fraction this collection represents of an unknown whole. In addition, we are investigating how students track successive transformations of the given fraction and the quantities represented.

THE AUSTRALIAN CONTEXT

According to the rationale given for the Australian Curriculum: Mathematics (ACARA, 2016) the mathematics curriculum:

> … focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, reasoning, and problem-solving skills. These proficiencies enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently.

The two content descriptors from the Australian Curriculum: Mathematics (ACARA, 2016) in Table 1 suggest that Year 8 students are expected to use the four operations with rational numbers using "efficient mental and written strategies" and "simplify algebraic expressions".

Fractions & Decimals	Patterns and Algebra
Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies (ACMNA183)	Simplify algebraic expressions involving the four operations (ACMNA192)

Table 1: Australian Curriculum: Mathematics Content Descriptors (ACARA, 2016)

These descriptors do not contain any reference to the links between fractional knowledge and algebraic thinking as suggested as being important by researchers such as Kieren (1980), Lamon (1999) and Wu (2001).

THIS STUDY

As part of a current research project the Fraction Screening Test (Pearn & Stephens, 2015) has been administered to more than 600 students from Years 5 – 8. The test is divided into three parts. Part A includes 12 routine fraction tasks such as equivalent fractions, ordering fractions and recognising simple representations. Part B includes five number line tasks. Part C includes tasks that require students to order four fractions from largest to smallest; match four fractions to their equivalent decimals; and two tasks which ask students to circle the one that does not belong e.g. in ¼, 25%, 0.4 0.25. The three reverse fraction tasks require students to find the whole using less common fractions (see Figure 1).

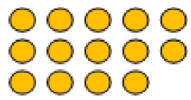
Reverse Fraction Task 1	Reverse Fraction Task 2	Reverse Fraction Task 3
This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? b. Explain how you decided that your answer is correct.	Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? _____ *Show all your working.*	This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.  a. How many counters did I start with? b. Explain how you decided that your answer is correct.

Figure 1: The three reverse thinking fraction questions

Based on the research of Lee and Hackenburg (2013), these three fraction tasks specifically require students to relate a given fraction to an equivalent number of objects, and when transforming the fraction to make a whole to carry out corresponding operations on the number of objects. These tasks would be unfamiliar for most of these students and so test their conceptual understanding of fractions.

In this paper results will be discussed for 117 Year 8 students who completed the Fraction Screening Test (Pearn & Stephens, 2015). However, the focus is on their responses to the three reverse fraction tasks given in Figure 1. Using a thematic analysis approach (Braun & Clarke, 2006) students' responses were classified according to the specific methods they used in their written solutions to each of the three fraction tasks.

RESULTS

The number, and percentage, of correct responses for the three reverse fraction tasks is included in Table 2. Four students did not attempt to answer Reverse Fraction Task 1 and six gave an incorrect response. For Reverse Fraction Task 2, which has no diagram, 24 students did not attempt to answer the question and 30 gave an incorrect response. Fourteen students did not attempt Reverse Fraction Task 3 while 18 responded incorrectly.

Reverse Fraction Task 1	Reverse Fraction Task 2	Reverse Fraction Task 3
107 (91%)	63 (54%)	83 (71%)

Table 2: Number of students with correct responses (n = 117)

Solution method classification framework (SMCF)

The Year 8 students' written responses to the three reverse fraction tasks (Figure 1) were analysed to determine the types of methods these students were using to complete these fraction tasks. Six categories were established using the six step process of the thematic analysis approach suggested by Braun & Clarke (2006). These categories were classified as: Not Clear, Visual, Additive/Subtractive, Partially Multiplicative, Multiplicative, and Advanced Multiplicative.

Not Clear refers to written responses that were incomplete or not attempted. *Visual* refers to the explicit partitioning of diagrams. The response given in Figure 2 shows how the student divided the drawing of the ten counters into two groups and then drew another five counters to get the answer 15.

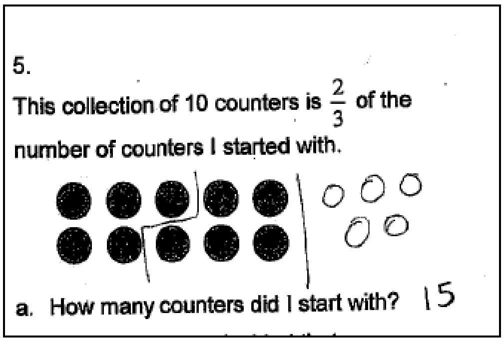


Figure 2: A visual method used to solve Reverse Fraction Task 1.

Additive/Subtractive refers to the use of additive or subtractive methods without explicit partitioning of the given diagram, or creating a new diagram. Students find the number of objects needed to represent the unit fraction and then use counting or repeated addition to find the number of objects needed to represent the whole. In Figure 3 the student keeps track of the parts e.g. 5 = 15 means that five-sevenths of the original group would be 15 counters.

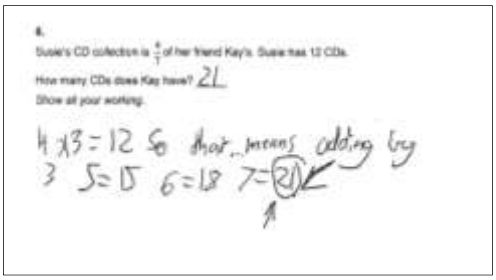


Figure 3: An additive method to solve Reverse Fraction Task 2

Partially multiplicative refers to the use of both multiplicative and additive methods. The student in Figure 4 found the number of objects for one-third then added these to the two-thirds to find the appropriate number of objects needed to make the whole.

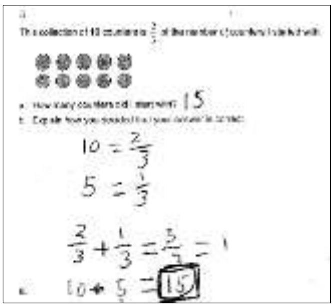


Figure 4: A partially multiplicative method to solve Reverse Fraction Task 1

Multiplicative refers to the use of fully multiplicative methods. Students find the quantity represented by the unit fraction using division and then multiply the quantity of the unit fraction to find the whole. Although the student in Figure 5 uses no verbal elaborations, and the recording is unconventional, the solution method is clear. The fraction four-sevenths is not stated explicitly but is implied in the division by four. In the second line the student states the equivalence between the number of objects and the fraction one-seventh while the third line shows the transformation needed to go from one-seventh to a whole without needing to refer to the fraction. This method anticipates how students would solve $\frac{4}{7}x = 12$

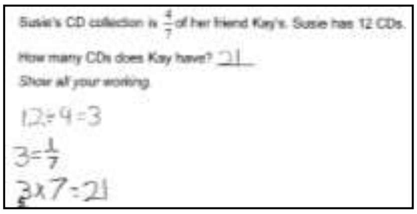


Figure 5: A fully multiplicative response to Reverse Fraction Task 2.

Advanced multiplicative describes the more advanced multiplicative methods students use to solve the reverse fraction questions. These include the correct use of appropriate algebraic notation to find the whole, or a one-step method to find the whole by dividing the given quantity by the known fraction. In Figure 6 the student used division by the numerator and multiplication by the denominator to solve Reverse Fraction Task 3.

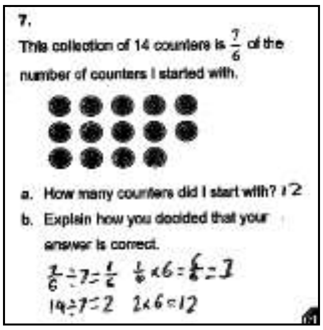


Figure 6: An advanced multiplicative response to Reverse Fraction Task 3

Table 3 includes the number and percentage of correct responses for each of the six categories described above: Not Clear, Visual, Additive/Subtractive, Partially Multiplicative, Multiplicative, and Advanced Multiplicative.

Response Type	Reverse Fraction Task 1 (n = 107)	Reverse Fraction Task 2 (n = 63)	Reverse Fraction Task 3 (n = 83)
Not clear	9 (8%)	1 (2%)	17 (20%)
Visual	4 (4%)	4 (6%)	9 (11%)
Additive/subtractive	11 (10%)	4 (6%)	0 (0%)
Partially multiplicative	43 (39%)	5 (8%)	36 (43%)
Multiplicative	39 (36%)	47 (75%)	18 (22%)
Advanced Multiplicative	1 (1%)	2 (3%)	3 (4%)

Table 3: Number of students in each of the six categories.

Only 54% of these Year 8 students gave correct responses to Reverse Fraction Task 2 compared to 91% and 71% for Reverse Fraction Task 1 and Reverse Fraction Task 3 respectively. However, 17 students did not explain their solution method for Reverse Fraction Task 3 and nine students drew on the diagram to correctly solve the task. It is possible that without a diagram these 26 students may not have completed the task successfully.

Eleven students used the same strategies for all three reverse fraction tasks. Two consistently used partially multiplicative strategies and nine used fully multiplicative strategies for all three reverse fraction tasks. Seven students used multiplicative strategies for Reverse Fraction Tasks 1 and 2 but used partially multiplicative strategies to solve the improper fraction in Reverse Fraction Task 3. Six students initially added the required fractional part for Reverse Fraction Tasks 1 and 2 then used the fully multiplicative method for Reverse Fraction Task 3.

DISCUSSION

According to the Australian Curriculum: Mathematics (ACARA, 2016) Year 8 students should be solving rational number tasks and simplifying equations written in algebraic form. This research has demonstrated that there are many students who did not attempt any or all of these reverse fraction tasks, gave incorrect responses to those they attempted, could not explain their solution methods or needed a diagram in order to use a 'guess and check' method to solve one or more of the reverse fraction tasks.

Students' written responses for the three reverse fraction tasks have been summarised according to the SMCF which has allowed a comparison the strategies used for each task. Students who used partially multiplicative strategies demonstrated conceptual understanding by moving from the given fraction to the unit fraction. Scaling up the unit fraction and its related quantities to find a whole was then achieved additively by relating successive fractions to the quantities they represent. Some of these students struggled with the task involving the improper fraction.

Students who used fully multiplicative thinking transformed a given fraction to obtain the corresponding unit fraction. These students then successfully scaled up the unit fraction and its related quantities multiplicatively to find the whole. These multiplicative methods, which most clearly mirror the thinking needed to solve the corresponding algebraic equations, can be seen as evidence of algebraic thinking. Students using the advanced multiplicative methods were able to generalise. The size of the fraction and the size of the given number of objects representing that fraction appear to be irrelevant as their solution methods or strategies remain the same regardless of the fraction or the size of the group the fraction represents.

CONCLUSION

All responses were able to be classified using the SMCF. These Year 8 students' responses have shown that the successful strategies varied from the concrete (visual), strictly arithmetic (additive, partly multiplicative) to the generalizable (multiplicative) and algebraic (advanced multiplicative). These students successfully responded to unfamiliar situations by employing a range of mathematical strategies as expected in the current curriculum (Table 1).

Some students' solution methods changed when a fraction, and the corresponding number of objects, changed. A few students using concrete or additive strategies were able to move to using multiplicative methods. Conversely, with a change of fraction and quantity some students were unable to successfully complete the task. Many students were successful with Reverse Fraction Task 1 because the diagram allowed them to proceed using visual methods but did not attempt, or gave an incorrect response, for the following fraction tasks. Students using multiplicative strategies for Fraction Task 1 used multiplicative strategies for all three fraction tasks. Some students were struggling to make the successful transition to algebraic thinking while others were able to generalise and use the same multiplicative strategies regardless of the size of the fraction or the number of objects the given fraction represented.

Previous researchers suggested that the relational thinking required for some fraction tasks is a precursor to algebraic thinking. We have tasks that both support this contention and reveal where students are on a continuum from concrete to algebraic thinking. Based on these results, we suggest that it is important for teachers to include reverse fraction tasks in their teaching and that it is important that these include a variety of fractions and quantities in order for students to develop and see the value of generalised processes. This would allow students to recognise the commonality of the problems, think algebraically, and not tackle each task as a completely new challenge.

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VALIDATING THEORETICAL SEEDING TO SUPPORT TRANSFORMATION OF MATHEMATICS TEACHING

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In this paper, the validity of "theoretical seeding" to support the transformation of mathematics teaching is studied in a case of professional development related to the design of learning activities supported by information and communication technologies (ICT). Theoretical seeding includes the use of a theoretical construct and teachers as agents of change of practice. The results suggest that the validity of theoretical seeding, with regard to the social consequences, is tentatively sufficient to consider it as purposeful and appropriate. Furthermore, the effects of theoretical seeding show a promising potential for future design efforts in terms of achieving theoretically underpinned and sustainable changes in teachers' practices with ICT.

INTRODUCTION

What seems to be an everlasting issue is the problem of integrating information and communication technologies (ICT) in mathematics teachers' everyday practice. One concern is that although teachers are both open to innovation and willing to use ICT, they tend to use ICT to sustain their current practices instead of beneficially transforming instruction (Monaghan, 2001; Ertmer, 2005; Trigueroz, Lozano, & Sandoval, 2014). These unintended outcomes suggest that in spite of the investments made and the efforts that teachers put into learning about new tools they may still be missing out on exploiting the full potential of ICT regarding support of learning processes.

In the current project, the author has specifically involved the use of dynamic geometry software (GeoGebra) as a technological tool and the use of a theoretical construct, for the purpose of achieving transformed teaching practices that may provide improved learning conditions for students. The theoretical construct provides "models-of-thought" that addresses didactical issues from multiple perspectives. First, the theoretical construct is introduced to the teachers by the researcher. Thereafter, the technological tool is introduced with focus put on connecting the theoretical construct with the affordances of the tool rather than only considering the technical aspects of the software. It is this specific use of the theoretical construct that is referred to as "theoretical seeding". In addition, theoretical seeding involves targeting teachers as agents of change and the adoption of a design methodology that supports collaboration between researchers and teachers. This approach allows the researcher to address complex educational problems in authentic settings and contributes to an understanding of how mathematical teaching practices may be improved.

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What is at stake in this paper is the validation of theoretical seeding with respect to achieving transformation, as the declared purpose of theoretical seeding. Before continuing with presenting the concept of validity together with the research question, the theoretical construct, utilized for theoretical seeding, will be presented.

THE THEORETICAL CONSTRUCT

In a previous case study the researcher worked with three mathematics teachers on the design of an ICT supported learning activity. In short, the problem encountered was that in spite of the researchers efforts of providing professional development, the teachers were still not able to exploit the full didactical potential of ICT as discussed or even as proposed by the teachers themselves. One particular aspect that contributed to this discrepancy was the communication patterns used by the teachers (including those mediated by different resources) that did not seem to fully support the intended learning objectives (Perez, 2015). The theoretical construct was originally developed by the researcher to provide feedback to the three teachers on this matter.

The theoretical construct (see Fig. 1) coordinates the three elements of the didactical relation, i.e. the teacher, the students and the knowledge taught. It is based on the IRE sequence (initiate, reply, evaluate), which is a typical pattern for teacher-initiated instructional communication. In a basic form, the teacher initiates communication by posing a question that he or she already knows the answer to, the student replies and the teacher evaluates by giving some feedback or evaluation (Mehan, 1979). Instructional communication can be interpreted in a wide sense to include orchestrations with technological tools as they can be used to both initiate and evaluate students' responses to questions and tasks that are supported by the tool.

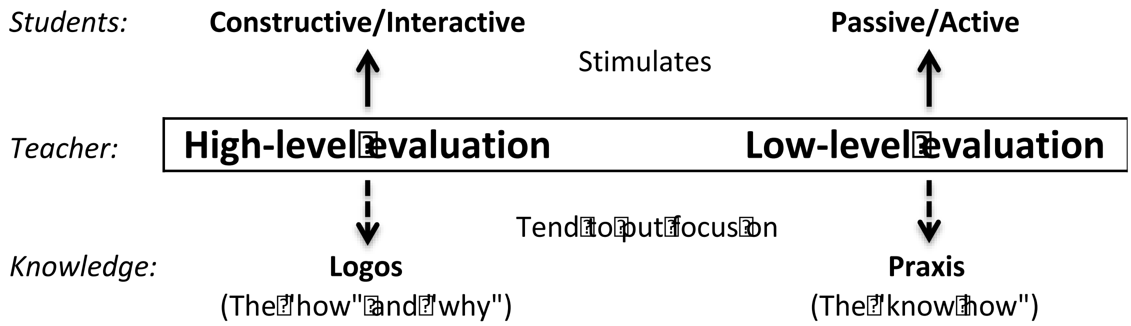

Figure 1: The three complementary components of the theoretical construct

Within the IRE sequence, some evaluations are called low-level evaluation because they tend to engage students in a discourse where the attention is basically on the routine itself and the teacher's pre-planned agenda of questions (Nystrand & Gamoran, 1991). Calling upon other student or simplifying elicitation until the expected reply is provided are examples of low-level evaluations. Low-level evaluation can also be provided by the teachers through ICT, e.g. by using the computational affordances of a technological tool to help students to move on with a task but at the same time significantly reducing the complexity of the task (Perez, 2014). Furthermore, discourses that are dominated by low-level evaluations are likely to engage students procedurally in the learning process as the routine is primarily used to control limited aspects of knowledge among students. In contrast, students may be engaged in a more substantial exchange. This is more likely to happen when teachers e.g. use follow-up questions, asks for clarifications, or gives hints in order to stimulate students' thinking. Evaluations that stimulate a discourse where the unfolding conversation is built on students' replies are called high-level evaluations (Nystrand & Gamoran, 1991).

Additional support for the construct, but from the perspective of the students, can be found in the framework of Chi (2009). According to this framework, students can be engaged overtly as either *passive*, *active*, *constructive* or *interactive* when learning. In short, being *passive* is learning by paying attention or receiving instruction. Being *active* is learning by engaging with the learning material. When being *constructive* the student is, in addition to being *active*, producing relevant outputs that go beyond what is already available in the learning material. Finally, when being *interactive* the student is being constructive with others in a joint dialogue. Students being *passive* or *active* are comparable with what Nystrand and Gamorand (1991) refers to as procedural engagement and the teacher's use of low-level evaluation. Students being *constructive* or *interactive* correspond to substantial engagement and high-level evaluation. Furthermore, Nystrand and Gamorand (1991) as well as Chi (2009) provide evidence that being *interactive/constructive* is better than being *passive*/*active* in terms of students learning outcomes. In this theoretical construct high-level evaluation is defined as teacher actions that afford students to be *constructive* and *interactive*. In comparison, low-level affords students being *passive* or *active*. This does not suggest that all evaluation should be high-level. In fact, an over-use of high-level evaluation may result in negative effects such as distal issues being addressed at the expense of the intended learning objectives (Perez, 2014).

The last part of the theoretical construct is the notion of *praxis* and *logos*, which are two inseparable aspects of knowledge or a praxeology (Chevallard, 2007). In short, *praxis* represents the "know-how" while *logos* provides a discourse with the purpose to describe and justify the *praxis* (ibid.). Low-level evaluations, especially those who only put focus on right and wrong, tend to be unsupportive of the creation of a comprehensive mathematical praxeology with a well-developed *logos* discourse. In contrast, high-level evaluations stimulates mathematically valuable activities that more likely to support discussions on how and why "things" work as they do and the connection between them (Perez, 2015).

THE CONSEQUENTIAL ASPECT OF VALIDITY

The theoretical construct was originally an analytic tool, developed to assess certain aspects of teachers' practices (Perez, 2015). In the current project it has advanced to be the basis of strategic action, i.e. theoretical seeding. The theoretical construct functions as a reference model, utilized together with teachers in the design of mathematical learning activities supported by ICT. As a reference model it tells how good or bad the ICT-tool is used, for the purpose of achieving transformation of the teachers' practices.

In this sense, the theoretical construct resembles what psychometrics would call a test or a measure (Messick, 1987).

Validity needs to be systematically addressed when any kind of qualitative or quantitative summary of performance or product is made and the results or scores are used for specific purposes (Messick, 1993). What needs to be validated is the meaning or interpretation of test scores along with the implications for actions that this meaning encompasses. Thus, validation means validating the use of the test in a specific context relative a specific purpose. Validity can therefore be quite high for one application but low for a different one. It can also change over time as new findings, new interpretations, or new inferences are made. In this sense, validity is an evolving property and a matter of degree (Messick, 1987). The basic validity question that needs to be answered is “To what degree if at all, on the basis of evidence and rationales, should the test scores be interpreted and used in the manner proposed?” (Messick, 1993, p. 14). In particular, the social values and social consequences cannot be ignored in considerations of validity as the appropriateness, meaningfulness, and usefulness of score-based inferences is dependent on the social consequences of the testing. One way to consider the value implications and social consequences of test interpretation and legitimate test use is to study both the intended outcomes and the unintended side effects (Messick, 1987).

The research question that will guide the judgement of validity of theoretical seeding for the purpose of achieving transformation is the following:

What are the social consequences of theoretical seeding and how may these consequences inform future design efforts.

RESEARCH SETTING AND METHOD

Educational design research refers to a family of design methodologies with similar features. Although teachers may be involved in different way in the design process the typical situation is that “a research team assumes responsibility for a group of students’ learning” (Cobb and Gravemeijer, 2008, p. 68). In the current project the teachers are the ones responsible for designing and implementing principle-based teaching activities in their own classrooms as part of their professional development. This approach aligns well with the current Swedish context where teachers as professionals are expected to make use of both formal knowledge and practical knowledge when orchestrating lessons.

The project encompassed three formal meetings. Many practical issues such as choosing and inviting participant were dealt by a representative of the central school administration and not by the researcher. The purpose of the three meetings, communicated to the teachers, was to 1) create opportunities for challenging routinized instructional behaviour, 2) plan and implement a learning activity supported by ICT, and 3) reflect on and share the experience. The project lasted for approximately one semester and seven mathematics teachers from lower secondary and upper secondary schools participated in all three meetings.

The theoretical seeding took place in the first meeting where the theoretical construct and the technological tool were introduced to the teachers. In the second meeting, the teachers continued by designing learning activities supported by the technological tool. The teachers were instructed to choose and prepare the learning objective for the second meeting according to their own individual needs. These two meeting lasted for 3 hours each and were audio recorded and transcribed.

In the beginning of the second meeting the teachers were interviewed (semi-structured group interview) in order to study the effects of the theoretical seeding on teachers' practices (i.e. the effects between the first and second meeting). No specific instructions were given to the teachers prior to the second meeting in order to capture the teachers' spontaneous responses. The second meeting was held approximately seven weeks after the theoretical seeding. In this paper, focus will be put on the empirical data from this interview.

SUMMARY OF THE THEORETICAL SEEDING

The theoretical construct was presented to the teachers as three interconnected tools that could be used to describe the actions of the teacher, what students do when learning, and the mathematical knowledge created in the classroom. The presentation also included a discussion on how the three tools could be used to plan, implement and evaluate teaching regardless of whether the technological tool is used or not. Furthermore, the researcher used several practical activities to exemplify the three interconnected tools. For example, when the teachers worked with a specific set of mathematical tasks, implemented in the technological tool, they also discussed different kinds of initiations (the I in IRE) and evaluations that could stimulate *constructive* and *interactive* processes, assuming their students were working with the same tasks. The purpose was to show how the technological tool could be used to support high-level evaluation. The introduction of the technological tool and the corresponding tasks were therefore used to further contextualise the theoretical construct. In addition, the teachers were provided with materials including a summary of the interconnected tools and several examples of learning activities implemented in the technological tool. The notions provided by theoretical construct were new to the teachers. However, some of the teachers had some experience in using the technological tool.

RESULTS FROM THE INTERVIEW

One of the teachers, Sue (all names are fictitious), described how she, a couple of days after the first meeting, conducted a lesson with a content that she was familiar with. The lesson was based on a classic optimization problem where you want to find the largest area of a rectangular shape when given the total length of three of its sides. She described her previous way of conducting the lesson as closed-ended in the sense that it was she who was responsible for taking the first steps towards the solution. This time she decided to transfer the responsibility for some key activities to the students. By doing this, she observed that the students made new valuable experiences that she had not seen before. The researcher asked the teacher if she could use the notions provided by the tools to describe and compare her new way of conducting the lesson:

> Sue: Previously they [students] have been passive or active maybe because I have been in control and I have formulated the [mathematical] expression rather quickly. But I am not even sure that all students understand that different areas [rectangular shapes] are possible. /…/. Then it is not possible for the students to be constructive because they fail to grasp this very first thing. But by making them draw figures everyone understood, as a start, that there actually are different possibilities.

A second teacher, Mark, mentioned his use of re-voicing, i.e. repeating a student's reply as if the student was only talking to him and not to peers. He felt that he should not do that but at the same time he did it anyway in order to clarify to everyone else. Mark also described how he during the last three years has begun to change his pedagogical approach. On a regular basis he gives his students e.g. three tasks that the students first try to solve individually, then in pairs and after that in groups. He reflected that this work allows students to be, if not constructive, at least more active.

Oliver, a third teacher, commented that he believed that it is necessary to combine different approaches because students are different. According to him, students are accustomed to a "low-level" discourse and that they need time to get used to doing things differently.

A fourth teacher, Jane told that she has become more aware of her own teaching and the purpose her different lessons. She described that she previously, e.g. when making a whole-class presentation, used to pose more or less trivial questions only to make sure that all students where following her in every step she made. Jane recognized this as a very time consuming procedure and has decided to stop doing it. Furthermore, she described that she has told her students about the different learning modes in order to better communicate her objectives with different lessons and allowing her students to prepare for the demands of different activities that address different aspects of knowledge. In addition, Jane commented what Sue (her colleague and one of the participants) said about how easy it was to make important changes in the classroom.

> Jane: /…/ letting students draw different rectangles, which you do not do but this is actually the key. This is what helps students to gain knowledge of what is asked in the task /…/

A fifth teacher, Carol, reported that she also has begun to reflect about how she poses questions and how she evaluates students' replies.

Finally, a sixth teacher, Kevin, initiated a discussion about how to make use of students' solutions to a task besides only presenting them on the board. The reason was that he recently had experienced a learning situation where he wondered what he could do differently. It that situation he eventually decided to do as usual.

SUMMARY OF THE EFFECTS OF THEORETICAL SEEDING

The interview with the teachers can be summarized in the following way. The teachers reported on an increased level of awareness of their actions and performance with respect to the IRE sequence. They were also able to use the notions provided by the tool to analyse their current practices and to suggest improvement. This is consistent with the expected outcomes. What is more unexpected is the extent of self-initiated classroom experimentation that particularly two of the teachers (Sue and Jane) reported on. Their experimentation can be summarized as a change in distribution of responsibilities between the teacher and the students. Activities such as drawing figures are examples of constructive activities that the teachers initially did not consider as part of solving the mathematical task but what had to be done in order to organize students' work with the task. Now the teachers had started encouraging students to draw pictures and reasoning about their mathematical features. Both teachers valued the impact that this transfer of responsibility had on their students' understanding of the task. For one of the teachers, the transfer of responsibilities also included a meta-discussion with her students about learning processes. Finally, a concern of the researcher was if the dual nature of the tool, especially the words "high" and "low", was going to be interpreted by the teachers as good and bad teaching but no such side effects were found. Instead there were clear indications that the teachers were selective in the way they implemented change in their existing practices.

After the interview, the teachers continued by designing learning activities supported by the technological tool. In particular, the teachers were able to use the notion of passive/active and constructive/active to discuss didactical issues with the researcher and to negotiate about design decisions. The design of the learning activities were not completed during the second meeting but the work continued in additional meetings.

CONCLUSION

The results suggest that theoretical seeding has provided the teachers with increased possibilities to transform instruction. Two of the teachers were motivated to identify, implement, and assess some significant changes in their own regular practices. Such changes that are initiated by the teachers themselves are more likely to be sustainable (Guskey, 2002). In general, the teachers appreciated the theoretical construct as models-of-thought and utilized it for analysing their regular instruction as well as for designing mathematical instruction supported by the technological tool.

So far the social consequences are both supportive of the purpose of theoretical seeding of achieving transformed teaching practices, and at the same time consistent with the teachers' values. Thus it can be concluded that the validity of theoretical seeding, with regard to the social consequences, is tentatively sufficient to consider it as purposeful and appropriate.

The common language that had been rapidly established within the group facilitated the design process where the researcher was simultaneously involved in the design of six learning activities. Although the designed learning activities may be considered to be less innovative from a research perspective the effects of theoretical seeding on regular instruction show a promising potential for future design efforts in terms of achieving theoretically underpinned and sustainable changes in the teachers’ practices with ICT.

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BEYOND THE "MOVE": A SCHEME FOR CODING TEACHERS' RESPONSES TO STUDENT MATHEMATICAL THINKING

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To contribute to the field's understanding of the teachers' role in using student thinking to shape classroom mathematical discourse, we developed the Teacher Response Coding Scheme (TRC). The TRC provides a means to analyze teachers' in-the-moment responses to student thinking during instruction. The TRC differs from existing schemes in that it disentangles the teacher move from the actor *(the person publically asked to consider the student thinking), the* recognition *(the extent to which the student recognizes their idea in the teacher move), and the* mathematics *(the alignment of the mathematics in the teacher move to the mathematics in the student thinking). This disentanglement makes the TRC less value-laden and more useful across a broad range of settings.*

Researchers (e.g., Fenemma et al., 1996) have found that teachers' use of student thinking during mathematics instruction supports student learning of mathematics. Both researchers (e.g., Franke, Kazemi, & Battey, 2007; Van Zoest, Peterson, Leatham, & Stockero, 2016) and recommendations for mathematics teaching (e.g., National Council of Teachers of Mathematics [NCTM], 2014) assert that teachers' use of student thinking undergirds features of effective mathematics instruction, such as classroom mathematical discourse. While the field benefits from research identifying how teachers may plan for and use written records of student work to facilitate whole-class mathematical discourse (Stein, Engle, Smith, & Hughes, 2008), less is known about how teachers respond in-the-moment to instances of students' mathematical thinking. We report here on a coding scheme designed to capture teachers' in-the-moment responses to instances of student mathematical thinking. Such a scheme could contribute to better understanding the role of the teacher in shaping meaningful mathematical discourse in their classrooms.

THEORETICAL PERSPECTIVES & RELATED LITERATURE

Current thinking about effective teaching and learning of mathematics as put forth by NCTM (2014) suggests fundamental ideas related to productive use of student mathematical thinking. As discussed elsewhere (Van Zoest et al., 2016), we see embedded in this document four core principles of quality mathematics instruction: (1) mathematics is at the forefront, (2) students are positioned as legitimate mathematical thinkers, (3) students are engaged in sense-making, and (4) students work

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collaboratively. These four principles form the basis of our theoretical perspective. As such they both provided a lens for examining existing research related to in-the-moment teacher responses to student mathematical thinking during whole-class interactions and informed the development of our coding scheme.

We found three themes in the literature related to teacher responses to student thinking: (1) student engagement in classroom communication, (2) responsiveness, and (3) attention to mathematics. These themes suggest important components to attend to in teacher responses, yet existing research seems to foreground only one of these components at a time. For example, Franke et al. (2009) foregrounded engagement by analyzing the way particular types of teacher questioning moves engaged students' in classroom communication. Bishop, Hardison, and Przybyla-Kuchek (2016) coded teachers' moves and student contributions to analyze teachers' responsiveness—the degree to which mathematical ideas in students' contributions were attended to by teachers' subsequent responses. Conner, Singletary, Smith, Wagner, and Francisco (2014) coded teachers' actions (moves) in support of collective argumentation, foregrounding the mathematics in the teacher responses. In general, existing research measures teacher responses against the particular component the researchers are foregrounding by incorporating that component into their definition of "move." In order to develop a more nuanced coding scheme, we disentangled these three components of a teacher's response from the teacher move. This disentanglement allows us to measure teacher responses against the core principles of our theoretical perspective, and provides a structure for other researchers to investigate teacher responses from their theoretical perspectives.

METHODOLOGY

Data for this paper come from a larger project (see LeveragingMOSTs.org) and included 278 instances of high-potential student mathematical thinking during whole-class interactions identified in 11 videotaped mathematics lessons from 6-12th grade classrooms that reflected the diversity of teachers, students, mathematics, and curricula present in US schools (Van Zoest et al., 2017). In addition, we analyzed 43 *Scenario Interviews* (Stockero et al., 2015) that involved teachers responding to a common set of eight instances of student thinking.

First, we articulated the student mathematics and mathematical point for each instance of student mathematical thinking. Student mathematics (SM) is defined as "a clearly articulated statement of an inference of what a student has expressed mathematically in the instance" (Van Zoest et al., 2017, p. 36). A mathematical point (MP) is "the *mathematical understanding* that (1) students could gain from considering a particular instance of student thinking and (2) is most closely related to the SM of the thinking" (Van Zoest et al., 2016, p. 326). We define a *mathematical understanding* (MU) to be *a well-specified statement of a mathematical truth*.

Next, we identified the teacher response to each instance of student mathematical thinking in our data. We define a *teacher response* as *the collection of observable* *teacher actions that begins as a given instance of student mathematical thinking ends and ends when that instance of student mathematical thinking is no longer the focus*[1] *of the observable teacher actions*. For coding purposes, a teacher response may be subdivided into a series of teacher moves, each serving different instructional intents.

The resulting teacher responses in the videos and Scenario Interviews were the data for, and from, which our coding scheme was developed. We used constant comparative analysis (Glaser, 1965) to revisit and refine the codes until each response was authentically captured by the coding scheme.

RESULTS

Our disentanglement of the three themes in the literature from teacher moves led to the *Teacher Response Coding Scheme (TRC)*. Figure 1 lists the TRC coding categories and their relation to the literature themes. In Figure 2 we provide an illustrative instance of student thinking, the inferred student mathematics (SM) and the related mathematical point (MP) of the instance, and four possible teacher responses to this instance. In the following subsections we make connections between the TRC coding categories and literature themes and use the teacher responses in Figure 2 to illustrate these categories and their codes.

Category	**Coding Category Description**	**Literature theme**
Actor	Who is publically asked to consider the student thinking	Engagement
Recognition	The extent to which the student who contributed the thinking is likely to recognize their idea in what is being considered	Responsiveness
Mathematics	The extent to which the focus is on improving students' understanding of the MP of the instance of student thinking	Attention to mathematics
Move	What the actor is doing or being asked to do with respect to the instance of student thinking	

Figure 1: TRC coding categories and their connections to the literature.

Context: While working on a problem that related the amount of money accumulated by saving both a one-time gift and babysitting money that was earned weekly, a student said during class discussion, "I put the money on the bottom and weeks on the side."
Instance: "I put the money on the bottom and weeks on the side."
Student Mathematics (SM): I put the money on the x-axis and weeks on the y-axis.
Mathematical Point (MP): The placement of the variables on the axes of a graph is determined by what makes the most sense in the problem situation given the established convention of the x-axis representing the independent variable.

	Teacher Response	**Actor**	**Recognition: Actions**	**Recognition: Ideas**	**Math**	**Move**
1	"Remember, we always put the independent variable on the x-axes."	Teacher	Not	Peripheral	Peripheral	Correct
2	"Did anyone label the axes a different way?"	Whole class	Implicit	Core	Cannot Infer	Collect
3	[To same student] "Why is the amount of weeks dependent on the amount of money which you put on the bottom?"	Same student	Explicit	Peripheral	Core	Justify
4	[To another student] "And what do I like to do first when I make a graph?"	Other student	Not	Other	Cannot Infer	Literal

Figure 2: Coding for teacher responses to an instance of mathematical thinking.

Actor

To capture *who* is likely to be engaging in the intellectual work in response to student mathematical thinking, the Actor category answers the question, "Who is publicly invited or allowed to consider the instance of student thinking?" It does this with the following four codes: *teacher, same student(s), other student(s),* and *whole class*. To illustrate distinctions among these codes, consider the sample teacher responses (TR) to the instance in Figure 2. In order to respond the teacher is likely to first privately consider the instance on some level. However, in TR1, "Remember we always put the independent variable on the x-axes," only the *teacher* engages in publicly considering the instance of student thinking. In contrast, TR2, "Did anyone label the axes a different way?" publically invites the *whole class* to consider the instance.

Recognition

To operationalize the *responsiveness* of teachers' responses to student thinking, we considered the extent to which the student who provides the instance would recognize their idea in the teacher's response. Through our iterative work in the data we noticed two distinct ways in which this recognition might occur in a teacher response: through attention to Student Action and attention to Student Ideas. The subcategory Student Action encompasses the exact, unique words a student has used (verbal), as well as any gestures or work provided by the student (non-verbal). The codes (*explicit, implicit,* or *not*) for student action capture the degree to which the teacher response uses the student action. To explore the subtle distinction between a response coded as *implicit* and one coded as *explicit*, consider TR2 and TR3. In TR3, the teacher uses language unique to that student instance ("on the bottom"). In contrast, in TR2 the teacher does not use this unique language, replacing "put," "money," and "weeks" with the verb "label," and replacing "on the bottom" and "in the side" with the term "axes." Hence, TR3 *explicitly* uses the student's actions while TR2 *implicitly* uses the student actions. Responses that do not use the student's unique actions or clear replacements (such as TR1 and TR4) are coded as *not*.

The subcategory Student Ideas focuses on the mathematical idea(s) the student is putting forth in the instance. The codes (*core, peripheral, other, cannot infer*, and *not applicable*) for this category capture the extent to which the student is likely to recognize their idea in the teacher response. For example, TR2 focuses on the labelling of the axes, which is the *core* idea in the instance of student thinking. On the other hand, TR1 and TR3 start to veer from this main idea to a *peripheral* or related idea—the connection between the labels of a graph and the independent-dependent relationship between the variables. In contrast, TR4 focuses the actor on considering what the teacher likes to do first when they make a graph. This focus is not related, even peripherally, to the student's idea and hence, we code TR4 as going towards an idea *other* than the core idea in the instance of student thinking. Responses that do not engage with the instance of student thinking (e.g., "Let's not talk about that now.") are coded *not applicable*.

Mathematics

In order to gauge the extent to which a teacher response focuses on improving student understanding of the mathematical point (MP), the Mathematics category documents the alignment between the mathematical understanding (MU) that is the focus of the teacher response to an instance of student thinking and the MP of that instance. The codes are: *core, peripheral, other, cannot infer*, *non-mathematical* or *not applicable.* For example, TR2 and TR4 illustrate responses that are coded as *cannot infer*. In both of these responses, it is not yet evident what MU the teacher is pursuing. In contrast, in TR3 the MU seems to be the MP (see Figure 2) of the student thinking and hence the mathematics of the teacher response is *core.* TR1 focuses on the mathematical conventions of labelling axes, thus the MU may be articulated as, "By convention, the x-axis represents the independent variable and the y-axis represents the dependent variable." Though this MU is contained in the MP, it focuses on following the convention rather than on deciding which variable is independent and which is dependent. Thus this MU is *peripheral* to the MP. When a teacher response has an MU that is not even peripherally related to the MP of the instance, it is coded as *other.* Teacher responses that do not address mathematics (e.g., "Nice work David!") are coded as *non-mathematical*. An instance of student thinking for which an MP cannot be articulated (see Van Zoest et al., 2016) receives the code *not applicable*, because a match or alignment cannot be determined.

Move

We use *move* to capture what the actor is doing or being asked to do with respect to the instance of student thinking. We identified 14 moves (see Figure 3). Although many of these moves are recognizable from other literature, they differ in that their descriptions do not include the three components captured in our other categories.

Move	Description
Adjourn	The teacher either explicitly or implicitly indicates that the instance(s) will not be considered publicly at that time, but suggests the instance may be considered later.
Allow	The teacher invites or leaves space for students to respond to the instance.
Check-in	The teacher elicits students' self-assessment of their reaction to or understanding of the instance.
Clarify	The teacher provides an interpretation and asks for verification that it reflects what the student meant, or asks the student to say what they meant (about a specific piece of the instance) without asking for elaboration.
Collect	The teacher requests or provides additional ideas, methods, or solutions.
Connect	The teacher asks for or makes a connection between or among representations, methods/strategies, solutions, or ideas that includes the instance.
Correct	The teacher describes or asks for a correct way of approaching, or thinking about, the instance.
Develop	The teacher provides or asks for an expansion of the instance that goes beyond a simple clarification.

Dismiss	The teacher either explicitly or implicitly indicates that the instance(s) will not be considered publicly.
Evaluate	The teacher asks for or provides a determination of the correctness of the instance.
Justify	The teacher asks for or provides a justification of the instance.
Literal	The teacher asks for or provides brief factual information related to the instance.
Repeat	The teacher (verbally or in writing) repeats or rephrases the instance without changing the meaning or asks a student to repeat the instance.
Validate	The teacher says something about the instance to affirm its value and/or encourage student participation (e.g., thank you, good).

Figure 3: Teacher moves and their descriptions.

Figure 2 provides examples of four of these 14 moves. In TR1, the teacher *correct*s the student's labelling, reminding the class of labelling conventions. In TR2, the teacher is *collect*ing additional methods for labelling the axes from the class. In TR3, the student who generated the instance is asked to *justify* their choice of money as the independent variable and weeks as the dependent variable. TR4 asks a *literal* question to engage a different student in providing factual information about what the teacher likes to do first when they graph.

DISCUSSION & CONCLUSION

Our initial coding scheme was based on moves drawn from the literature, but as it was applied to the data, it was quickly seen that focusing only on those moves was insufficient to characterize teacher responses given our four principles of effective teaching. For example, in examining a teacher move such as *develop*, the nature of the move is very different if the teacher is expanding on what a student has said as compared to asking other students to expand on the instance of student thinking. This difference led to the development of the actor coding category, which describes who is being publicly invited to engage with the instance of student thinking. This category provides the means for measuring the degree to which students are being engaged in classroom communication across all moves.

Another important aspect of the coding scheme is the degree to which the teacher response honors the mathematical ideas in what the student has said. For example, when a teacher uses an instance of student mathematical thinking as a launching point to discuss what they feel the students need to hear, the student who contributed the thinking in the instance might wonder what their idea has to do with the line of reasoning the teacher is now pursuing and feel that their thinking was not valued. The Student Actions and Student Ideas subcategories measure the degree to which the teacher appears to view students as legitimate mathematical thinkers.

Franke, Kazemi and Battey (2007) suggested that a focus on student mathematics should be a critical element of mathematics classroom practice. The Mathematics category, by assessing the degree to which the MU of the teacher response is aligned with the MP of the instance of student thinking, provides a way to characterize teacher responses relative to this focus on student mathematics.

The TRC has several notable strengths. It is applicable across grade levels and mathematics content. It has descriptive power because it disentangles the teacher move from the actor, the degree to which the student thinking is honored, and the extent to which the response focuses on the mathematics of the student thinking. As a result, it paints a rich picture of the teacher response without being evaluative in nature. The researcher who applies this coding scheme decides which combination of codes might be more or less productive based on their own perspective. The flexibility of the TRC makes it useful for a broad range of researchers interested in better understanding the teacher role in shaping meaningful mathematical discourse in their classrooms.

Our long-term goal is to better understand the teaching practice of building on high-potential instances of student mathematical thinking in the moment they occur during whole-class interactions. Such complex teaching practices are difficult to study and often require the practice to be decomposed in order to "articulate, unpack and study" (Boerst, Sleep, Ball, & Bass, 2011, pg. 2859) it. We anticipate that decomposing and studying teacher responses using the TRC will provide insight into teachers' current responses to high-potential student thinking and contribute to better understanding the broader teaching practice of productively using student mathematical thinking.

Endnote

[1]Focus on an instance typically ends when the next instance of student thinking occurs. However, a teacher's response may end prior to the next instance of student thinking—that is, prior to the end of the teacher's conversational turn.

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WHICH MATHEMATICAL PREREQUISITES DO UNIVERSITY TEACHERS EXPECT FROM STEM FRESHMEN?

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In STEM degree programs, high dropout rates are observed and mainly attributed to missing mathematical preparation. However, in contrast to the normatively specified educational goals of secondary education systems, the prerequisites from the perspective of tertiary education are far less clear – especially in countries without entrance examinations like Germany. The aim of our study with 36 university teachers is to explore which mathematical prerequisites are considered necessary for STEM degree courses in Germany. Moreover, we analyze whether university teachers differ with respect to their specified prerequisites.

INTRODUCTION

The transition from high school to university is known to be difficult, in particular for freshmen taking mathematics courses in their first semester. Such difficulties are often due to fundamental differences between the learning of mathematics at school and at college (e.g. Hoyles, Newman, & Noss, 2001). Whereas school mathematics curricula in many countries focus on content, which is relevant for the application of mathematics as a tool, higher education courses often take a scientific perspective on mathematics and emphasize proving and formalism (e.g. Hoyles et al, 2001). These challenges are enhanced by other circumstances such as the increasing number of students entering higher education, which is accompanied by an increase in diversity of student (social) backgrounds, expectations and ability ranges (e.g. Appleby & Cox, 2002).

Concern about the mathematical knowledge of college entrants, especially in the STEM subjects, can be observed in several countries. For example, at universities in Germany, programs with STEM subjects face rather high drop-out rates compared to programs with other subjects. Often missing mathematical skills are reported as the main drop-out reason (Heublein et al., 2014). Similarly, in the UK, the National Audit Office (2007) found low retention rates in STEM subjects mainly caused by missing mathematical preparation of the freshmen. Cox (2001) compared the expectations of different university departments (STEM subjects) with the students' actual capabilities and found a significant mismatch. Overall, the mathematical learning prerequisites of STEM freshmen seem to be a major challenge in the transition from high school to college, for both individuals and institutions.

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THEORETICAL BACKGROUND

An important question is which mathematical prerequisites should freshmen in STEM degree programs have in order to successfully master the transition from high school to university. In contrast to the educational goals of upper secondary education, which in many countries are normatively specified, the necessary prerequisites from the perspective of tertiary education are far less clear. Whereas in some countries this problem is partly covered by additional university entrance tests, in other countries like Germany the upper secondary school leaving certificate qualifies for all bachelor programs at all universities. Therefore the perspective of universities or rather the expectations of the university teachers concerning necessary mathematical prerequisites have to be explicitly described. This would be helpful for research on the transition from school to university.

Until now only a few empirical studies investigated university teachers' views on the required mathematical prerequisites for STEM programs. Sutherland and Dewhurst (1999) investigated the mathematical knowledge expected from undergraduates at the transition from secondary to higher education in the UK. For every subject area (STEM plus economics and business studies) the authors surveyed three departments representing high, medium and low average entrance qualifications. The questionnaire used represented the majority of common core AS and A-level mathematics including algebra, functions, geometry, proof, equations, differentiation, integration, vectors and matrices, complex numbers, statistics and mechanics. The authors reported the expected mathematical content and related mathematical skills for every department and subject area. Whereas the mathematics department representing high average entrance qualification requires all AS and A-level content but additional numerical methods and discrete mathematics (both A level), the Biological Science departments requires "only" basic mathematical knowledge and some AS-Level Pure Mathematics. In general, the expectation decreased with decreasing average entrance qualification and decreasing role of mathematics in the subject (from mathematics to chemistry and biological science degrees).

Already more than 30 years ago, Heldmann (1984) conducted a study in Germany to identify individual characteristics for general (i.e. subject-unspecific) higher education readiness. The authors asked more than 1000 teachers from German universities for the relevance of pre-selected characteristics such as the willingness to learn or resilience. Additionally, a few subject-specific aspects for a wide range of subjects were included in the questionnaire. Concerning necessary mathematical skills for higher education readiness, skills and knowledge of some mathematical content areas were reported as relevant: computation, elementary functions, differentiation, integration, geometry, and statistics. However, these mathematical skills and knowledge aspects were not described in a more detailed way and remained quite unspecific.

Besides the above empirical studies, there are also results of several working groups which aim to provide lists of mathematical core knowledge and skills needed for the STEM university programs. For example, in the U.S. the College Readiness Standards list mathematical content and processes like reasoning and problem-solving as well as additional student attributes like paying attention to detail (Transition Math Project, 2006). Similarly, the German project Cooperation High-School/College (COSH, 2014) provides a catalogue with mathematical content areas, and illustrates the expected skills and knowledge by a list of sample tasks. This catalogue is supplemented by a few student attributes such as perseverance. A third example is the catalogue of the European Society for Engineering Education (SEFI, 2013). It defines the mathematical content (algebra, analysis and calculus, discrete mathematics, geometry and trigonometry, statistics and probability) as well as mathematical skills (e.g. thinking mathematically, problem solving, symbolism and formalism) that should be studied before entering an undergraduate engineering degree program. Moreover, SEFI emphasized the importance of adequate attitudes towards mathematics.

The empirical studies and the theoretical work described above illustrate the importance university freshmen's mathematical learning prerequisites have in different countries. Comparing the results of these studies, it becomes clear that there is no consensus of what kind of mathematical knowledge and skills university teachers expect from STEM freshmen. Whereas all studies point to several mathematical content areas which are relevant for STEM degree programs, there is a far less consensus concerning the extent to which knowledge and skills in these content areas are necessary. In addition, mathematical skills like problem solving, reasoning, or modeling are sometimes included and sometimes not. Similarly, the catalogues differ with respect to mathematics-related student attributes and aspects of subject-unspecific personal characteristics (intelligence, conscientiousness etc.) which are included in some but not all reports. For example, prerequisites concerning the perception of mathematics are mentioned only in the catalogue of SEFI (2013).

RESEARCH QUESTIONS

Based on the existing studies and reports, it is not clear whether there is a consensus on mathematical prerequisites university teachers expect from STEM freshmen. Taking into account the heterogeneity of mathematical bridging courses at different universities (for an overview about the German situation, for example, see Biehler et al., 2014) and the different expectations of different departments in the UK (Sutherland & Dewhurst, 1999), the question arises if such a consensus among university teachers for mathematics is possible at all. Accordingly, the present study addresses the following research questions:

1) Which mathematical prerequisites are necessary for a successful transition into higher education in STEM degree programs from the perspective of university teachers for mathematics?

2) If university teachers do not agree: Which different types of university teachers can be identified with respect to necessary mathematical prerequisites for a successful transition into STEM degree programs?

METHODS

To analyze the expected mathematical prerequisites of higher education STEM subjects, we carried out an online survey with university teachers. The survey was anonymous to reduce group-effects. The participants' views were collected through three narrative questions which addressed the same topic from different perspectives and which should stimulate detailed and comprehensive responses. First, the participants were asked to describe the characteristics of mathematical university readiness or rather what mathematical knowledge and skills STEM freshmen should acquire before starting the first semester. In the second question, the participants were asked which mathematical aspects should be included into a mathematical entrance examination for STEM programs. In the third question, they were asked about the differences between successful and unsuccessful STEM freshmen with a special focus on the mathematical prerequisites the students acquired before entering the university. In addition, background information on the teaching experiences was collected (e.g. university vs. university of applied sciences; years of teaching experience; course programs taught in; etc.).

Sample

The study involved university teachers from Germany who taught first year mathematics courses in STEM degree programs at universities and universities of applied sciences during 2010 and 2015. Based on a list of more than 2000 German university teachers for mathematics, 82 participants were identified and invited to participate in the study. The invited participants were selected to equally represent German federal states, STEM programs, teaching experience and type of university. Overall, 36 of the 82 university teachers participated in the survey representing the majority of German states and all subject groups of science, technology, engineering and mathematics.

Analysis

In order to approach the first research question, a qualitative content analysis was performed (cf. Mayring, 2014). First, the collected responses of all three questions from the questionnaire were split into single text passages. These passages were grouped with respect to their meaning, and categories of necessary mathematical learning prerequisites were inductively determined. For every identified category, an explicit definition, examples and coding rules were developed. All passages were assigned a category. 10% of the material was rated independently by two coders in order to investigate and ensure interrater reliability of the coding system. Sometimes, the participants referred to educational documents (e.g. the catalogue of COSH, 2014, mentioned before). In these cases we included all mathematical prerequisites listed in the corresponding documents as well. This approach led to a catalogue of all mathematical prerequisites referred to by this sample of university teachers. To ensure the quality of the procedure with regard to completeness and correctness, the final catalogue has been compared to the original responses of the participants once more.

To answer research question 2, whether there is a consensus among university teachers or whether different types of university teachers with respect to the expected mathematical prerequisites exist, we ran a cluster analysis (e.g. Aggarwal & Reddy, 2015). The aim of the cluster analysis was to aggregate the participants to homogenous groups with regard to their expected mathematical prerequisites. For each mathematical prerequisite a dichotomous variable (1: stated; 0: not stated) was created for the 36 participants. If a participant referred to an educational document, all mathematical prerequisites listed in this educational document were taken into account. An agglomerative hierarchical approach was used to cluster the university teachers. Applying the Ward's agglomeration with the Euclidean distance the resulting dendrogram and distance parameters have been analyzed to determine a satisfactory cluster solution (e.g. Aggarwal & Reddy, 2015).

RESULTS

Based on the qualitative content analysis, we identified 152 mathematical prerequisites as stated by the participants. These were structured into the four main categories *mathematical content*, *mathematical processes*, *conception of mathematics* and *personal attributes*. For the mathematical prerequisites in each main category we had satisfying to good inter-coder reliability (percentage agreement between .85-.97 and Cohen's kappa between .60-.94).

1) *Mathematical content*

This largest category covers knowledge on several areas like basic concepts (in the area numbers, algebra, elementary function and geometry), calculus, vectors and matrices as well as statistical concepts including combinatorics, statistical distributions, and general concepts like propositional logic. In each sub-category there are several aspects. In the sub-category "calculus" for example, there are intuitive understanding of sequences and limits, continuity, differentiation, integration etc.

2) *Mathematical processes*

Freshmen are expected to have skills concerning mathematical processes of six different categories: basic skills (calculation, use of representations and technology), reasoning and proof, mathematical communication, mathematical definition, problem solving, and mathematical modelling. Each category comprises sub-processes addressing different cognitive levels, for example, "understanding and verification of a given proof" and "generation and formulation of a proof".

3) *Conceptions of mathematics*

This category covers meta-knowledge about mathematics. Examples are knowledge about the fundamental importance of proof in mathematics or the fact that there is an axiomatic structure of mathematics.

4) *Personal attributes*

This area includes attitudes and subject-unspecific cognitive and social skills. Examples are perseverance when faced with mathematical problems, openness concerning the learning of university mathematics and ability to work on his/her own.

For research question 2, a hierarchical cluster analysis with Ward's method indicates three different types of university teachers with respect to their expectations (categorial two-step cluster analysis using log-likelihood also result in a three cluster solution). *Type 1* teachers (N = 8) expect solid knowledge of basic mathematical content (in terms of lower secondary mathematics covering functions, elementary algebra, elementary geometry, and functions). Every university teacher of this type stated all identified prerequisites of this basic mathematical content category as necessary for STEM freshmen. These teachers rarely mentioned advanced school mathematical content (including calculus, matrices and vectors, probability), mathematical processes and conceptions of mathematics. Also, only three personal attributes were expected as a prerequisite by four or more participants assigned to this type. *Type 2* teachers (N = 8) expect knowledge of most basic mathematical content and a lot of prerequisites of advanced school mathematical content. In addition, they mentioned skills of many mathematical processes and ask for some specific personal attributes. *Type 3* teachers (N = 20) consider only few prerequisites necessary. Even of the prerequisites of basic mathematical content only some were mentioned by the *type 3* teachers. Similarly, these teachers consider only some of the aspects in the main category conceptions of mathematics as prerequisites for STEM degree programs.

DISCUSSION

The aim of this study was to (1) explore mathematical prerequisites for STEM freshmen which university teachers consider necessary and (2) analyze whether university teachers concur in these necessary mathematical prerequisites. By means of an anonymous online survey with 36 German university teachers for mathematics we identified 152 mathematical prerequisites in total. Comparing these with other catalogues of mathematical prerequisites (e.g., COSH, 2014; SEFI, 2013), we found a partial overlap of the prerequisites addressing mathematical content. However, our sample stated additional mathematical prerequisites for example the formal concept of differentiation and propositional logic. Besides, the responses of our sample covered skills for six types of mathematical processes which to this extent are only listed in the document of SEFI (2013). In contrast to the other catalogues, the university teachers in our study mentioned several concrete conceptions of mathematics the STEM freshmen should exhibit (the SEFI framework just emphasized the importance of appropriate conceptions). Finally, the university teachers mentioned several ”personal attributes” which indicates the significance the teachers attach to this aspect. In the debate about a general university readiness a lot of these attributes are viewed as requirements. With respect to mathematics the TMP (2006) and COSH (2014) listed a few aspects of these attributes. The comparison of existing catalogues of prerequisites to our results implicates a common ground of necessary mathematical prerequisites for freshmen in STEM degree programs with respect to mathematical knowledge and skills. Differences are mainly found for prerequisites related to personal attributes and conceptions of mathematics.

Based on a cluster analysis we identified three types of university teachers with respect to necessary mathematical prerequisites. Whereas teachers of the first type seem to be satisfied by solid knowledge of basic content, teachers of the second type additionally consider a lot of advanced school mathematics as necessary. A large proportion of participants of the third type stated only a few necessary mathematical prerequisites. This result could be due to missing motivation when working on the questionnaire. However, this interpretation seems rather unlikely because there is a lively discussion about insufficient mathematical prerequisites of freshmen within the scientific community and the participation in the survey has been voluntary. An alternative explanation might be that these university teachers were not used to explicitly and coherently describe the mathematical prerequisites they consider necessary. Discussions about freshmen’s mathematical prerequisite are often restricted to some examples and to the overall comment that the prerequisites are insufficient.

The data basis of the 152 mathematical prerequisites identified in the present study is limited. Accordingly, there is need for a confirmation by a survey with a much larger sample. In the end such a survey can yield a catalogue of necessary mathematical prerequisites for STEM degree programs which represents the view of the universities. Moreover, a larger data base allows an analysis of the relevance of the identified different types of university teachers.

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MATHEMATICS TEACHERS' KNOWLEDGE AND COMPETENCES MODEL BASED ON THE ONTO-SEMIOTIC APPROACH

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This paper aims at developing a model that intends to articulate diverse categories of mathematics teachers' knowledge and competences that are necessary for the appropriate teaching of mathematics, based on the theoretical notions of the Onto-Semiotic Approach to mathematical knowledge and instruction (OSA) and its many contributions to the fields of teacher training.

INTRODUCTION

The study of the didactic and mathematical knowledge and competences that a teacher should have to appropriately manage the students' learning process is a matter that has been largely researched, thus, generating several model designs that aim at characterizing such teachers' knowledge and competences (e.g., Shulman, 1987; Rowland, Huckstep & Thwaites, 2005; Hill, Ball & Schilling, 2008; Schoenfeld & Kilpatrick, 2008). Based on the theoretical notions of the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction (Godino, Batanero & Font, 2007) and its many contributions to the field of teacher training, this work develops a model (called Didactic-Mathematical Knowledge and Competences model or DMKC) that intends to articulate the diverse categories of teachers' knowledge and competences that are necessary for the appropriate teaching of Mathematics, and at the same time, refines the DMK model presented in Pino-Fan, Assis & Castro (2015).

DIDACTIC-MATHEMATICAL KNOWLEDGE AND COMPETENCES MODEL

A theoretical model of teachers' mathematical knowledge (Pino-Fan, Assis & Castro, 2015; Pino-Fan, Godino & Font, 2016) within the framework of the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction (Godino, Batanero & Font, 2007) has already been designed and is known as the DMK model. As stated by its authors, one of the perspectives of development of this model is the fitting of the notion of teachers' knowledge and teachers' competences. On the other hand, also within the framework of OSA, there have been other studies regarding Mathematics teachers' competences (Rubio, 2012; Giménez, Font, & Vanegas, 2013; Seckel, 2016; Pochulu, Font & Rodríguez, 2016), which have also exposed the need of having a model of teachers' knowledge to evaluate and develop their competences. These two research agendas have come together, thus generating the mathematics teachers' *didactic-mathematical knowledge and competences model* (DMKC model) (Breda, Pino-Fan & Font, *in press*).

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The notion of competence

Mathematics teachers are expected to be able to address basic didactic problems related to the teaching of this subject through the use of theoretical and methodological tools, giving way to a series of specific competences. Thus, the two first key questions that arise, in order to be able to develop the DMKC model, are: What is understood by the notion of competence? What are the key competences that Mathematics teachers should have? According to Weinert (2001), competency-based approaches can be classified into three large groups: a) Cognitive approach; b) Motivational approach; and c) integral approach or action competence. According to this, the conceptualization of competence that we use in this model comes from the action competence perspective, considering it as a combination of knowledge, dispositions, etc., that allows an effective performance, within typical contexts of the profession, of the actions aforementioned in its formulations. In an Aristotelian way, it is about a potentiality that is updated in the performance of effective actions (competences).

This formulation of the term "competence" has to be developed in order to be operational, and for that purpose it is necessary to characterize competence (definition, levels of development and descriptors) that allows its development and evaluation. According to Seckel (2016), we consider that the starting point for the development and evaluation of a professional competence has to be a task that generates the perception of a professional problem that needs to be solved, and for this purpose, the prospective teacher or in-service teacher has to mobilize skills, knowledge and attitudes in order to develop a practice (or action) that intends to solve the problem. Furthermore, we can expect such practice to be performed with more or less success (achievement) and, at the same time, it can be considered as evidence that the person can perform practices that are similar to the ones described by some descriptors of the competence, which is often associated to a certain level of competence.

Mathematical competence and competence in analysis and didactic intervention

Students' mathematical competences are developed through the solving of mathematical tasks and, at the same time, evaluated through the mathematical activity performed in order to solve the assigned task. In the case of evaluation, the teacher assigns a task to the student, who solves it by performing a certain mathematical activity. Then, the teacher analyses the student's mathematical activity and finds evidence of a certain level of development of one or several mathematical competences. In Rubio (2012), it is stated that, in order to evaluate their students' mathematical competences, teachers must have mathematical competence. However, it is also stated that this is not enough, since the teacher must also be competent in the analysis of mathematical activity. While the first competence is not specific to the teaching profession (it is common in several professions that use mathematics, although each profession gives it its hallmark), the second one, as a matter of fact, is.

The DMKC model considers that the two key competences of Mathematics teachers are *Mathematical competence* and *Competence in analysis and didactic intervention*, which, at its core (Breda, Pino-Fan & Font, *in press*) consists of: *Designing, applying and assessing sequences of one's own learning and others', through techniques of didactic analysis and quality criteria to establish periods of planning, implementation, assessment and outline suggestions for improvements*. In order to be able to develop this competence, the teacher needs, on the one hand, knowledge that allows to describe and explain what is happening in the process of teaching and learning (didactical dimension of the DMK model, one of the components of the DMKC model), and on the other hand, needs knowledge to assess what has already happened and outline suggestions for improvements in future implementations −meta didactic-mathematical dimension of the DMK model, one of the components of the DMKC model (Pino-Fan, Assis & Castro, 2015). In this work, we will focus mainly on the latter.

CHARACTERIZATION OF THE COMPETENCE IN ANALYSIS AND DIDACTIC INTERVENTION

This general competence is formed by different sub-competences (Breda, Pino-Fan & Font, *in press*): 1) sub-competence in analysis of the mathematical activity; 2) sub-competence in analysis and management of the interaction and its effect on students' learning; 3) sub-competence in analysis of norms and meta-norms; and 4) sub-competence in assessment of the didactical suitability of the process of instruction.

Sub-competence in analysis of mathematical activity

Rubio (2012) describes the design and implementation of a training period in the Secondary School Teachers Training Master Program of Universitat de Barcelona, in which teachers are first taught the technique for the analysis of practices, objects and processes proposed by OSA, and then, a technique for the evaluation of mathematical competences. The objective of this study was to corroborate (or not) the following hypothesis: the professional competence of teachers in the analysis of mathematical practices and mathematical objects and processes activated in such practices, is "in-depth knowledge" that allows to evaluate and develop the students' mathematical competences. Rubio (2012) concludes that after all the experiments conducted, such hypothesis can be confirmed. Furthermore, it is stated that if teachers are not competent in the analysis of mathematical practices, processes and objects, they will not be competent in the evaluation of mathematical competences. Thus, the results of Rubio's thesis (2012) point out a sub-competence of the competence in analysis and didactic intervention that mathematics teacher have to develop in order to develop and evaluate their students' competences: competence in analysis of the mathematical activity, in other words, the analysis of the mathematical practices, objects and mathematical processes activated in them.

This first sub-competence enables teachers to analyze mathematical activity. This type of analysis is important in the training of teachers and is a type of analysis that is somehow difficult for teachers and future teachers. For example, Stahnke, Schueler and Roesken-Winter (2016) carry out a revision of the empirical research conducted on mathematics teachers and conclude that teachers have difficulty analyzing the mathematical tasks (and its educational potential) assigned to their students.

As mentioned before, the lack of consensus over a paradigm that defines how should the analysis of mathematical activity be done in the field of mathematical education is a very problematic aspect. The DMKC model assumes that the theoretical tools of OSA (practice, primary and secondary objects emerging from the practices, meaning of a mathematical object in terms of practices, partial meanings, mathematical processes) allow such analysis in terms of practices, mathematical objects and processes. With these theoretical notions, when the meanings are understood pragmatically in terms of practices, one can, firstly, answer questions such as: What are the partial meanings of the mathematical objects that are intended to be taught? How are they articulated together? Later, an analysis of the primary mathematical objects and processes activated in such practices can be conducted. The identification by part of the teacher of the objects and processes involved in mathematical practices allows to comprehend the progression of the learning process, to manage the necessary processes of institutionalization and to evaluate the students' mathematical competences. Thus, it is possible to answer the questions: What are the configurations of primary mathematical objects and processes involved in the practices that constitute the diverse meanings of the intended contents (epistemic configuration)? What are the configurations of primary objects and processes used by students when solving problems (cognitive configurations)? Mathematics teachers have to know and comprehend the idea of configuration of objects and processes activated in a certain mathematical practice and be able to use it in a competent manner in the processes of teaching and learning mathematics (Pino-Fan, Godino & Font, 2016).

Sub-competence in analysis and management of the interaction and its effect on students' learning

The notion of didactic configuration has been introduced in OSA as a tool for the analysis of the interactions in instruction processes (Godino, Contreras & Font, 2006). It is about a theoretical construct to model the articulation of the performance of teachers and students regarding a specific task and content (a configuration of primary objects and processes) of teaching and learning, where knowledge arises from the interaction itself. Mathematics teachers have to be competent in the design and management of didactic configurations. It intends to answer the following question: What type of interactions between people and resources will be implemented in instructional processes and what are the consequences in the learning process? How can interactions and conflicts be managed in order to optimize learning? The teacher, therefore, should know the many types of didactic configurations (dialogic, etc.) that can be implemented and their effect on students' learning, and also, how to design and manage these types of didactic configurations in specific instruction processes.

Sub-competence in normative analysis

The different stages of the process of design and implementation are supported by and depend on a complex net of norms and meta-norms of different origin and nature (Godino, Font, Wilhelmi & Castro, 2009) that need to be explicitly recognized in order to comprehend the development of instruction processes and direct them towards optimal suitability levels. For example, when studying equations, there are rules regarding the way these should be written or the way these should be solved. Also, there are non-mathematical norms, such as the use (or not) of calculators, the method of evaluation, the way of participating in class, etc. Mathematics teachers have to become competent in the normative analysis of the processes of mathematical instruction in order to answer questions such as: what norms determine the development of instructional processes? Who, how and when are the norms established? What and how can these be changed in order to optimize mathematical learning? Etc.

Sub-competence in the assessment of the didactical suitability of instruction processes

The characterization of the competence in analysis and didactic intervention proposed above, needs tools for the description and explanation, as those described in Rubio's research study (2012), for the analysis of mathematical activity and also tools for assessment, as those presented in the research studies conducted by Seckel (2016) and Breda, Pino-Fan and Font (*in press*). These research studies show that, even when the teachers do not know the didactical suitability criteria with all their components and indicators, if they are exposed to a situation in which they have to assess a proposal of didactic innovation that could somehow affect them, then they use them in an implicit way to organize their positive or negative assessment.

For the assessment of instruction processes, OSA proposes didactic suitability as the essential tool. Once a specific topic has been selected in a certain educational context, the notion of didactic suitability (Breda, Font & Lima, 2015) helps to answer questions such as: what is the degree of didactical suitability of the teaching and learning processes implemented? What changes should be made in the design and implementation of the instruction process in order to increase its didactic suitability in future implementations?

Didactical suitability of an instruction process is defined as the degree to which such process (or a part of it) gathers certain characteristics that enables it to be assessed as suitable (optimal or ideal) to attain the adaptation between the personal meanings achieved by the students (learning) and the intended or implemented institutional meanings (teaching), taking into account the circumstances and available resources (environment). The notion of didactical suitability can be separated into six specific suitabilities: 1) Epistemic suitability that makes reference to the mathematics taught as ideally be "good mathematics". For that purpose, apart from taking the prescribed curriculum as reference, it also considers the institutional mathematics that have been transposed into the curriculum; 2) Cognitive suitability, that expresses the degree to which the intended or implemented learning is within the students' zone of potential development, and also the proximity of the attained learning to the learning intended or implemented; 3) Interactional suitability, that refers to the degree to which the modes of interaction allow to identify and solve conflicts of meaning and favor autonomy in learning; 4) Mediational suitability, the degree of availability and adaptation of the material and time resources necessary for the development of the teaching and learning processes; 5) Affective suitability or degree of implication (interest, motivation) of students in the process of study; and 6) Ecologic suitability, degree of adaptation of the process of study to the school comprehensive education plan, the curricular guidelines, the environment, etc.

For each of these criteria, there is a system of components and indicators that can be rated on a scale (of 1–3, for example). It is about a system of rubrics that allows to rate (or auto rate) in a complete or balanced way, the elements that, together, make up a process of quality instruction in the field of mathematics.

KNOWLEDGE OF MATHEMATICS TEACHERS

Professional competences have to be developed in the training of teachers. For that purpose, the teacher trainer has the capacity of analyzing the professional practices of teachers (future teachers or in-service teachers) when they solve professional tasks assigned to them in a training period, and the didactic-mathematical knowledge activated in them, in order to be able to find indicators that justify the assignation of degrees of development of the professional competence that is being evaluated. However, a problem that we have in the field of mathematics education is that there is not a single model that allows us to analyze the professional practice and there is no consensus over a paradigm for the analysis of the didactic-mathematical knowledge activated by teachers in their professional practices.

As discussed in the first section, there are several models and views worldwide regarding the knowledge that mathematics teachers should have in order to appropriately manage their students' learning. Pino-Fan, Assis and Castro (2015) propose a model for characterizing didactic-mathematical knowledge (DMK) of teachers, which considers, among other aspects, the contributions and developments of several models of mathematics teachers' knowledge, and the theoretical and methodological development of OSA. Thus, the DMK model suggests that teachers' knowledge is organized into three dimensions: 1) mathematical; 2) didactical; and 3) meta didactic-mathematical. The first dimension, mathematical, refers to the knowledge that enables teachers to solve mathematical problems or tasks that are typical of the educational level in which they will teach (common knowledge), and link the mathematical objects of such level to mathematical objects that will be studied at higher levels (extended knowledge) (Ibíd., p. 1433).

The didactical dimension of DMK proposes six subcategories of teachers' knowledge (Ibíd., p. 1434-1436): 1) epistemic facet, that refers to the specialized knowledge of mathematical dimension (use of diverse representations, arguments, procedures, partial meanings for a specific mathematical object…); 2) cognitive fact, that refers to the knowledge about cognitive aspects of students (difficulties, errors, conflicts, learning...); 3) affective facet, that refers to the knowledge of affective, emotional and attitudinal aspects of students; 4) interactional facet, knowledge of the interactions that occur in the classroom (teacher-student, student-student, student-resources...); 5) mediational facet, knowledge of the resources and means that can foster the students' learning process, and the time assigned for teaching processes; and 6) ecologic facet, knowledge of curricular, contextual, social, political, economical aspects that may have influence on the students' learning process.

The third dimension of DMK, meta didactic-mathematical dimension, refers to the knowledge needed by teachers to: reflect on their own practice, identify and analyze the set of norms and meta-norms that regulate the teaching and learning processes of mathematics, and assesses the didactic suitability in order to find potential improvements in both, design and implementation stages of such processes (Pino-Fan, Assis & Castro, 2015; Pino-Fan, Godino & Font, 2016).

The three dimensions described above are involved in the different phases of the design of processes of teaching and learning of specific mathematical topics: preliminary study, planning or design, implementation and assessment (Pino-Fan, Godino & Font, 2016).

FINAL CONSIDERATIONS

This work has presented a theoretical model, the mathematics teachers' Didactic-Mathematical Knowledge and Competences model (DMKC model), which is based on a series of empirical research studies that, on the one hand, have allowed its development and refinement and, on the other hand, have tested its theoretical constructs. Although the work that has been presented is basically theoretical, it is important to highlight that there have been a number of empirical research studies on the diverse components of the model, as can be seen in the section "Formación de profesores" (Teacher training) on the OSA website: http://enfoqueontosemiotico.ugr.es/. The DMKC model opens, therefore, a strong research program and development focused on the design, experimentation and evaluation of formative interventions that promote the professional development of mathematics teachers, taking into account the different categories of knowledge and didactic competences described in this work.

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THE ANTIDERIVATIVE UNDERSTANDING BY STUDENTS IN THE FIRST UNIVERSITY COURSES

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In this article, we present the results of a questionnaire designed to evaluate college students' understanding of the antiderivative. Specifically, by civil engineering students when answering the questionnaire' tasks, in order to identify and characterize the meanings on the antiderivative that are mobilized by them. In order to analyse the answers given, we used some theoretical and methodological notions provided by the theoretical model known as the Onto-Semiotic Approach (OSA) of mathematics cognition and instruction. The results show knowledge of antiderivative by the Civil Engineering students. Furthermore, the comparison between the mathematical activity of students provides information that allows concluding that the meanings that they mobilized might be shared among their communities.

BACKGROUND

In recent years, the mathematical education of engineering students has gained more attention from researchers in the field of mathematics education (Bingolbali, Monaghan & Roper, 2007). The reason lies in the fact that, nowadays, as pointed out by Gnedenko and Khalil (1979), mathematics has become more than just a calculus tool; it has become a powerful and flexible method for both science and engineering.

In this regard, there have been several studies that have dealt with the issue of how to address different mathematical notions in engineering contexts (Sonnert & Sadler, 2014). The suggestions given by these studies focus on the type of problems used to introduce mathematical notions, the impact of technological resources and textbooks for the teaching of mathematics to engineers, and even motivational factors. Other studies, focus on the study of the differences in the way of thinking mathematics between mathematics and engineering students (Jones, 2015).

This article aims at identifying and characterizing the meanings that civil engineering students, mobilize in their mathematical practices in connection to certain tasks assigned to them. For this purpose, we applied a questionnaire to two groups of civil engineering students, one from a Mexican University and another from a Colombian University. The questionnaire was designed as part of another study (Gordillo, Pino-Fan, Font & Ponce-Campuzano, 2015), to assess the aspects of comprehension that university students have of such mathematical object. The analysis of the answers to the questionnaire show the meanings and preferences that future civil engineers assign to the antiderivative, and how these relate to the partial meaning that make up the holistic meaning of this mathematical notion (Gordillo & Pino-Fan, 2016).

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THEORETICAL AND METHODOLOGICAL ASPECTS

In order to conduct this study, we considered the theoretical model known as the Onto-Semiotic Approach (OSA) of mathematical cognition and instruction. This theoretical approach arises in the field of the research of Mathematics Education in order to articulate the diverse dimensions that are present in the processes of teaching and learning of mathematics (Godino, Batanero & Font, 2007). In OSA, the notion of *systems of practices* (or *mathematical practices*) plays and important role in the teaching and learning of mathematics. Godino and Batanero (1994) define a system of practices as "any performance or manifestation (linguistic or not) done by someone in order to solve mathematical problems, communicate the solution to others, validate the solution and generalize it to other contexts and problems" (p. 334). These practices can be personal or institutional, depending on whether these are done by one person or shared within the core of an institution.

Besides, OSA assumes certain pragmatism when considering mathematical objects as entities that emerge from the systems of practices conducted in a field of problems (Godino & Batanero, 1994). In OSA, the meaning of mathematical objects is conceived from a pragmatic-anthropological perspective which considers the relativity of the context in which these are used. In other words, the meaning of a mathematical object can be defined as the system of operative and discursive practices that a person (or an institution) develops in order to solve certain type of situations-problems in which such object intervenes (Godino & Batanero, 1994). Thus, the meaning of a mathematical object can also be considered from two perspectives, institutional and personal.

In order to conduct a 'finer' and more systematic analysis of the mathematical practices developed regarding certain problems, OSA introduces a typology of primary mathematical entities (or primary mathematical objects), that intervene in the systems of practices: situations-problems, linguistic elements, concepts/definitions, propositions/properties, procedures and arguments. These primary mathematical objects are related among themselves forming nets of intervening objects that emerge from the systems of practices, which in OSA are known as *configurations*. These configurations can be epistemic (nets of institutional objects) or cognitive (nets of personal objects).

In this document, we use the notion of *cognitive configuration* to analyse the mathematical practices performed by civil engineering students regarding the solutions to the tasks of the questionnaire.

METHOD

This study uses the methodology of the mixed methods research (Creswell, 2009), since it is an exploratory study that considers the observation of quantitative variables (answers' degree of accuracy: correct answers, partially correct answers and incorrect answers) and qualitative variables (the type of cognitive configuration connected to the practices on antiderivative). For the study of the qualitative variable we adopted a technique of analysis known as *semiotic analysis* (Godino, 2002), which allows to describe in a systematic way the mathematical practices of students as well as the elements of cognitive configuration (linguistic elements, concepts/definitions, propositions/properties, procedures and arguments) which are activated in such practices, and their respective meanings.

The questionnaire

The questionnaire that we used to gather data was designed to evaluate the comprehension of the notion of antiderivative of university students and is composed of five tasks (Gordillo, et al., 2015). Each of these tasks is closely related to one of the four partial meanings of the antiderivative that were identified through a historic-epistemological study that aimed at reconstructing the ‘holistic meaning of reference’ for such mathematical object (Gordillo & Pino-Fan, 2016). Chart 1 shows a summary of the characteristics and goals pursued by each of the tasks.

Chart 1. Summary of the characteristics of the tasks of the questionnaire

Task	Objective	Representation activated	Partial meaning activated
Task 1: Meanings of the antiderivative	To explore personal meanings and definitions given to the antiderivative	Verbal/Written	Global
Task 2: Graphic exploration of the antiderivative	Treatment of the graphic representation of the antiderivative	Graphic	Tangent- squaring
Task 3: Calculation of the primitive function (parts A and B)	Construction of a family of functions from a derived function	Symbolic, graphic and tabular	Differential-sum
Task 4: Difference integral-derivative	To explore if there are conceptual differences between the notions of integral and derivative	Verbal, Written and symbolic	Elementary functions
Task 5: Solving of ordinary differential equations	Use of the antiderivative for solving differential equations	Verbal, Written and symbolic	Fluents- Fluxions

The questionnaire was applied to two groups of Civil Engineering students. The first group was composed by 23 students of the Civil Engineering of the Universidad Distrital in Colombia. The second group was composed by 23 students of the Civil Engineering of the Universidad Autónoma de Querétaro in Mexico. An essential requisite for the selection of the students that participated in the study was that, at the moment of taking the questionnaire, they had taken Integral Calculus courses.

ANALYSIS OF DATA

In this section, we present the analysis of the answers given by the students of the two groups, Mexican and Colombian. For the analysis of the quantitative variable (‘answers’ level of accuracy). The first study that we conducted with the variable level of accuracy was to determine if there were significant differences between the Colombian group and the Mexican group.

For the analysis of the qualitative variable we used the notion of *cognitive configuration*, which allowed us to describe in a systematic way the primary mathematical objects (linguistic elements, concepts/definitions, propositions/properties, procedures and arguments) that form the mathematical practices of the students, in connection to the tasks of the questionnaire.

Analysis of the answers of the Mexican and Colombian engineering students

In this section, we present the results of the quantitative and qualitative analysis of each of the tasks of the questionnaire.

Task 1: Meanings of the antiderivative

Given the general nature of this first task, only correct answers (answers in which at least one of the partial meanings of the antiderivative was expressed in verbal/written form) and incorrect answers (answers in which any of the partial meanings of the antiderivative were enunciated) were considered. The students did not have difficulties for solving the task, answering 82,6% correctly.

A high percentage of Mexican students (13) as well as Colombian (11), answered that the antiderivative is “the inverse process of derivation”. This first general approach to the conceptions that students have of the antiderivative show that more than half of them (52,2%) think of the antiderivative as a *procedure* (operation) that allows to find the “original function” from which certain derived function comes from. Out of the 46 students, only one student from Mexico answered that the antiderivative is a “family of functions”. The solutions that we have labelled as ‘absence of meaning’, that refer to incorrect answers from the point of view of the level of accuracy, are answers in which the students did not give any meaning to the antiderivative, providing answers of the type “the antiderivative is the area below the curve”, “the antiderivative is obtained from the fundamental theorem of calculus”, “the antiderivative is a function f of $f=f'$ ”, “the antiderivative is a mathematical form through which some real life problems can be solved”.

Task 2: Graphic exploration of the antiderivative

For this task, we only considered correct answers (in which the elements that belong to the family of the antiderivative were correctly identified and the way of finding them was justified), and incorrect answers (in which the graph provided did not correspond with the elements of the family of antiderivative for the function provided graphically). Task 3 has a higher level of difficulty for the students, with only 41,3% answering correctly. Among the mathematical practices that the students performed as part of their answers, we could identify three types of cognitive configurations.

Of the three configurations identified, the most used by the students was the ‘particular function’ (34,8%), in which a symbolic expression for the function is obtained from the graph of the function, and through algebraic procedures, it is possible to identify (or try to identify) which are the graphs of the elements of the family of antiderivatives. The second more used type of configuration was the ‘tabular interpretation of the graph’ (30,4%), which refers to the answers in which a table of values that describe the function given originally is constructed from the graph of the function provided; from the table constructed (and the relations and properties that are established with it) it is possible to try to identify the elements that belong to the family of antiderivatives. The configuration that we have identified as ‘advanced’ was activated in answers which were characterized by the use of procedures and justifications centred on the properties/propositions of derivation, specifically the criterion for the analysis of the characteristics and construction of graphs of functions, in order to identify graphically the member that belongs to the family of antiderivatives of the function provided.

Task 3: Calculation of the primitive function

Task three was composed of two parts. For the first part, part A, we considered as correct all the answers in which a valid symbolic expression was provided for $f(x)$; while incorrect answers were all the answers that did not provide valid symbolic expressions for $f(x)$. For part B, all the answers which provided a second expression for $f(x)$, different from the one given in part A and with valid justifications, were considered as correct. All the answers in which it was explicitly or implicitly mentioned that it was not possible to find a second expression for $f(x)$ were considered incorrect.

The students did not have problems to provide a symbolic expression for $f(x)$ in part A of the task, with 87% of them giving a correct answer. However, the students had more difficulties to answer part B of the task, with 50% (23) of them giving a second valid expression for $f(x)$ different to the one provided in part A.

We could identify two types of cognitive configurations from the answers provided by the students to part A of the task. The first type 'graphic-technical', refers to the answers in which, from the data given in the table, a graphic representation is provided from which the algebraic expression is obtained (graphic and symbolic linguistic elements, respectively) for the derived function. Subsequently, an expression for $f(x)$ is found from the argumentations and procedures centred on the "rules" (properties/propositions) of derivation. The second type of cognitive configuration, "numeric-technical", refers to the answers in which a pattern (property) that allows establishing the rule of correspondence that defines the derived function (concept/definition) is determined from the combination of the data provided in the table. Later, from the argumentations and procedures centred on the "rules" of derivation, an expression for $f(x)$ is found.

Regarding the cognitive configurations connected to the answers in part B of the task, we found three types. The first type, 'wrong interpretation of the uniqueness of the derivative', are answers in which the students show a wrong conception about the uniqueness of the derivative at a point and the derived function, providing answers of the type "it is not possible to find another expression for $f(x)$ because for $f'(x)$ there is one and only one $f(x)$, and vice versa". The second type of configuration, 'equivalent functions' is related to the answers in which, explicitly or implicitly, by means of the use of equivalent functions (concept/definition), some algebraic operations are developed (procedures that serve as arguments) to show that it is not possible to find another different function. The third type of cognitive configuration, 'advanced solution', was activated in answers in which the procedures and their justifications explicitly establish a connection among concepts such as antiderivative, the fundamental theorem of calculus, rules of integration, etc., to point out with the proposition "another expression for $f(x)$ can be any member of the family of functions $f(x) = x^2 + C$", that it is, indeed, possible to find another expression for $f(x)$. As we can observe, 50% of the students (12 Colombian and 11 Mexican), mobilized the third type of configuration to provide their answers. Regarding the antiderivative, the third type of configuration brings associated the meaning of inverse process of derivation.

Task 4: Difference between integral and antiderivative

Task 4 aimed at exploring whether the students conceived the integral and the antiderivative as different notions or not.

The correct answers were those in which the students pointed out and justified which were the differences between both notions. Partially correct answers were those in which the students mentioned that there were differences, but, the differences were not pointed out, or no justification was given, or the justification was not valid (from the institutional point of view). Only 26,1% of the students pointed out that the antiderivative and the integral were the same notion and that the terms were synonyms (Hall, 2010).

As shown above, the most activated cognitive configuration in the answers was 'definitions for the notions', used by 67,4% of the students. Such configuration was activated in answers in which there were arguments regarding the difference between the concepts of antiderivative and integral, providing definitions (personal or institutional) for both notions. For example, "…are different because the integral is a number, while the antiderivative is another function". The configuration 'examples of use' was the second most activated configuration (2 Colombian students and 6 Mexican), and was activated in answers in which there were arguments regarding the difference between both notions by means of concrete examples (situations/problems) of their use or application, for example, "the integral serves to calculate the area below the curve while the antiderivative serves to obtain a function". It is important to point out that the examples of use that were provided in this second configuration, made reference to the notions involved as process (or procedure) and not from a conceptual point of view. The third type of configuration activated was 'particular-general' (4 Colombian and 3 Mexican students), in answers in which the arguments were oriented towards the distinction of the antiderivative as a general case of the definite integral, in other words, the antiderivative was seen as indefinite integral, which is similar to what was found by Hall (2010).

Task 5: Solution of ordinary differential equations

The main objective of this task was to explore the process followed by the students in order to find the antiderivative, by means of a problem in which they needed to describe how they obtain the solution of a first order differential equation. Additionally, by means of the descriptions of the students, it was also intended to explore the meaning that they give to the constant *C*, known as constant of integration, in order to see if they comprehend the "inverse process" that finding an antiderivative implies.

Needless to say that the students had serious difficulties to solve the task presented. Only 5 of them were able to describe, from a correct mathematical point of view, the process that they follow in order to find the solution to the differential equation presented. Twelve of them (26,1%) omitted the constant of "integration" in their solutions, so we labelled their answers as partially correct. 63% of the students did not answer or answered something 'incongruent' (not valid or senseless from a mathematical point of view). The main cause mentioned by this 63% of the students, either orally at the moment that the questionnaire was given or written in the box intended for the answer to the task, was that they did not remember or did not know how to solve a differential equation.

Regarding the types of cognitive configuration activated in the answers, these were of 3 types, and were classified according to the type of linguistic element used in their arguments. The first, 'verbal', is a configuration that was activated in answers in which the verbal-descriptive language to narrate the procedure that they had to follow in order to solve a differential equation, but without "developing" such procedures symbolically, in other words, there is a description of what should be done, but it is not actually performed. Only one student who activated this type of configuration gave a correct answer.

The second type of configuration, 'symbolic' was activated in answers that centred their arguments on the procedure itself of calculation of the solution, in other words, they solved the differential equation symbolically without describing with words the process they followed. The third configuration activated was a mixture of the two previous configurations. Four students (two Colombian and two Mexican) described the procedure and the properties/propositions used in the calculation of the solution, verbally. Three of the students, who mobilized the third configuration, 'verbal-symbolic', answered the task correctly.

FINAL REFLECTIONS

Partial meanings of the antiderivative such as *tangents-squarings* and *elementary functions* (Gordillo & Pino-Fan, 2016), were not activated in the answers of the students. Now the questions would be, why did the engineering students of our study activate, with difficulties, one of the four partial meanings of the antiderivative? The answer to this question leads us, on the one hand, to face one of the limitations of our study, the type of problems suggested, were they appropriate for engineers, for their practices and interests? Although the questionnaire was designed to activate the different partial meanings of the antiderivative, and it aimed at exploring the comprehension that university students have of such notion (Gordillo, et al., 2015). On the other hand, the question brings to our mind the role of the educator of engineers. For this purpose, the educator of future engineers should be aware, first of all, of the diversity of partial meanings of the mathematical object under study, in our case, the antiderivative (Gordillo & Pino-Fan, 2016). By comprehending the use of such partial meanings in the context in which he works, the educator would have opportunities to pose problems that mobilize such meanings and, at the same time, adjust to the real needs of the engineers in training.

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GENERALIZATION IN FIFTH GRADERS WITHIN A FUNCTIONAL APPROACH

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This article discusses evidence of fifth graders' (10-11 year olds') ability to generalize when solving a linear function problem. Analyzed in the context of the functional approach of early algebra, the findings show that students generalized both when solving specific problems and when asked to define the general formula. The results are described in terms of the type of questions in which students generalized their answers, as well as of the functional relationships identified and the types of representation used to express them. Most of the pupils who generalized did so based on the correspondence between pairs of values in the function at issue.

INTRODUCTION

Research interest is growing around elementary school students' understanding and expression of notions about algebraic concepts (Blanton, 2008). Algebraic thinking plays a key role in research on school algebra, for it entails the development of the ability to analyze relationships between quantities, deduce general patterns and use symbols to represent ideas, among others (Kaput, 2008; Kieran, 2004). Functional thinking is the component of algebraic thinking focused on in this study. In particular, elementary school students' ability to generalize is explored in the functional approach to algebraic thinking. Students may express generalization, the key to such thinking, in words or, given time, symbolically (Blanton, 2008).

Functional thinking addresses the relationship between two (or more) variables: specifically, it involves the types of thinking that range from specific relationships to the generalization of relationships (Smith, 2008). Although such thinking appears to be beneficial for students, its application in in the lower grades has received scant attention (Blanton & Kaput, 2011). In Spain, functional thinking is a fairly recent area of research that has yet to be fully explored, although some of the findings in connection with early schooling have merited international interest (e.g., Cañadas & Morales, 2016). This study was preceded by research on fifth graders' ability to generalize from contextualized problems and the systems of representation used to express such generalization (Merino, Cañadas, & Molina, 2013).

Those studies revealed a need for further exploration of fifth graders' ability to generalize when establishing relationships between variables. In addressing that need,

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this paper focuses on the generalization displayed by such students when solving a problem involving a linear function.

GENERALIZATION AND REPRESENTATION

According to some researchers, generalization, the key element in algebra, is present when students intuitively perceive a certain underlying pattern, even though they are unable to represent it clearly (Mason, Burton, & Stacey, 1988). Generalization implies deliberate reasoning that builds on specific cases to identify inter-model, inter-procedural or inter-structural relationships (Kaput, 1999). Krutestskii (1976) identified two levels of generalization: (a) seeing what is general and known in a specific instance; and (b) seeing something general and still unknown in an isolated instance.

Algebraic symbolism has been directly associated with generalization in different grades. Moreover, other types of representation, including verbal, numerical, pictorial and manipulative, are of interest in the context of early algebra (Kaput, 2008; Merino, et al, 2013). Stacey (1989) distinguishes two kinds of generalization: (a) *near generalization*, for questions "which can be solved by step-by-step drawing or counting", and (b) *far generalization*, for questions "which goes beyond reasonable practical limits of such a step-by-step approach" (p. 150).

In a recent study, Blanton, Brizuela, Gardiner, Sawrey and Newman-Owens (2015) explored lower grade students' ability to generalize in problems involving linear functions. Their findings distinguished between students who identified a specific and those who detected a general relationship between variables, and related the distinction to the ability to symbolize. Students who established the relationship between variables for specific cases "did not yet have a representational means to compress multiple instances into a unitary form that could symbolize these instances" (p. 542).

FUNCTIONAL THINKING

Functional thinking is a "component of algebraic thinking based on construction, description and reasoning with and about functions and their constituents" (Cañadas & Molina, 2016, p. 210) that ranges from specific relationships to generalizing the relationships between two (or more) variables (Smith, 2008). In most countries, students are not introduced to functions, which comprise the core content of this type of thinking, until secondary school. The present study used the linear function $f(x) = ax + b$ (with the domain and codomain limited to natural numbers) as a port of entry for early algebra to afford students the opportunity to explore variations in quantities (Blanton, Levi, Crites & Dougherty, 2011).

The study focused on bivariate functions. Smith (2008) defined the functional relationships involving two quantities that co-vary to be: (a) correspondence, or the relationship between the pairs of values for the two variables (a, $f(a)$); and (b) covariation, or the relationship that describes how changes in one variable affect the other.

METHOD

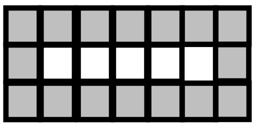
This study forms part of a broader teaching experiment on functional thinking in fifth graders in which the contextualized problem posed in each session revolved around a linear function. This article discusses the results of the fourth and final session, when student progress was greatest because they had already worked on a number of problems involving functions.

Subjects and tools

The 24 subjects were fifth graders (10- to 11-year-old) enrolled in a school in Granada, Spain, who were deliberately chosen on the grounds of school and teacher availability. In the first three sessions of the teaching experiment, the students worked with contextualized problems for which the functions were: f(x)=2x, f(x)=3x and f(x)=3x-7. The students had not worked on problems involving functions prior to these sessions.

The research team consisted in the teacher-researcher who led the sessions and two researchers who recorded the videos and helped answer students' questions. In the tiles problem posed to all students, the implied function was f(x)=2x+6. The problem and related questions are reproduced in Figure 1.

A school wants to re-pave its corridors because they are in poor condition. The faculty decides to use a combination of white and grey tiles, all square and all the same size. They are to be laid as in the drawing.



The school contracts a company to re-pave the corridors on all three floors. We want you to help the workers answer some questions before they get started.

Q1. How many grey tiles will they need for a corridor with 5 white tiles?

Q2. Some corridors are longer than others. So the workers will need a different number of tiles for each corridor. How many grey tiles will they need for a corridor with 8 white tiles?

Q3. How many grey tiles will they need for a corridor with 10 white tiles?

Q4. How many grey tiles will they need for a corridor with 100 white tiles?

Q5. The workers always lay the white tiles first and then the grey tiles. How can they figure out how many grey tiles they need if they have already laid the white ones?

Figure 1: The tiles problem

The questions posed involve: (a) specific instances (Q1, Q2, Q3 and Q4) and (b) the general case.

The information gathered included the session videos and the students' answers to the questionnaire. This article describes the results deduced from the students' written responses.

Analytical categories and data analysis

Category construction was based on grounded theory, which deems that phenomena are not conceived statically (Corbin & Strauss, 1990). The theoretical framework, background and characteristics of the contextualized problem were applied to define some of the categories. Generalization was identified based on its presence or absence in students' replies to the questions, with a focus on the answers where it was detected. Drawing from the ideas on generalization relevant to the conceptual framework of this study, a preliminary analysis of the data revealed two types of questions in which students exhibited generalization: (a) in Q1, Q2, Q3 and Q4, where they were asked to reply to specific (near or far) questions; and (b) in Q5, where they were (directly) asked to generalize. These two types of generalization were respectively labelled *spontaneous* and *prompted* generalization.

Students' generalization was described in terms of the functional relationship generalized (correspondence or covariation) and how it was represented (verbally, with algebraic notation or combinations of one or the other or both with other systems).

Students were labelled as S_i where $i = 1, \ldots, 24$.

RESULTS AND DISCUSSION

Of the 24 students, five gave direct answers only (i.e., only the numerical result), described how they counted the tiles or simply repeated the problem: no generalization could be attributed to these pupils. The other 19 answered at least one of the questions in a way that attested to generalization. Two profiles were identified: (a) three students exhibited both spontaneously and prompted generalization; and (b) 16 students generalized only when prompted (when replying to Q5).

Two of the students who generalized spontaneously and when prompted (S5 and S8) used algebraic notation to represent their replies. S8's answer to Q1 was: "formula: $(x \cdot 2) + 6 = 16$; x = number of white tiles." Figure 2 shows how this student related the pairs of values (number of white tiles-number of grey tiles), given five white tiles.

White tiles (x)	Grey tiles (y)
5 $\xrightarrow[(x\cdot 2)+6]{}$	16

Figure 2: Example of correspondence, S8 in Q1

Figure 2 also illustrates S8's use of algebraic symbols "(x·2)+6" to express the general relationship. In Q2, Q3 and Q4, this student simply answered the questions. That was interpreted to mean that the student used the same functional relationship for 8, 10 and 100 tiles, relating the pairs of values (a, f(a)) for a = 8, 10 and 100 and correctly finding that the number of grey tiles needed would be 22, 26 and 206, respectively.

S6, the third student who generalized spontaneously and when prompted, described the generalization in Q1 in the following words: "they need 16 grey tiles. For every white tile, there are 2 grey tiles, except on the sides, where there are 6. All the whites x2 + 6 on the sides". Hence S6 identified the relationship between variables as well as the constant number (six white tiles on the sides). This student used both verbal and numerical notation to express the relationship.

This student's answer to Q5 was: "multiplying the number of white tiles times 2 plus 6 on the sides: x·2+6=x". In other words, S6 used two types of representation: verbal and algebraic, exhibiting a transition from natural to a more general and abstract language.

Note that the three students who generalized spontaneously deduced the general formula by identifying the correspondence relationship in the function f(x)=2x+6.

Most of the 16 students who generalized when prompted (in Q5) expressed the general relationship between the pairs of values (correspondence) verbally. A few representative examples follow.

The students identified the pattern from which they deduced the general formula in a number of ways. In one, eight students described generalization in terms of a rule that in algebraic notation would be represented as f(x)=2x+6. S14, for instance, answered "you get the answer by multiplying the white tiles times 2 and then adding 6". In this case, as in the other seven, generalization was expressed verbally. Student S3, in turn, replied "multiplying the white tiles by two and adding three at the beginning and three at the end". The pattern detected by this student would be represented in algebraic notation as f(x)=2x+3+3. S24 adopted a third approach, identifying the pattern to be f(x)=2(x+2)+2.

One of these students, S1, used primarily verbal representation, although in conjunction with algebraic symbols. In Q5 the answer was "you need to use 2x white tiles +6"; i.e., verbal representation predominated, although with some elements of algebraic symbolism. The implication would seem to be that this student, who used some algebraic symbols sporadically when answering the previous questions, was en route to attaining a more natural and spontaneous use of algebraic symbolism to represent the relationship between variables.

Lastly, the relationship was incorrectly identified by six students in a way that translated to algebraic notation would yield f(x)=2x+2. One representative example of this relationship between variables was provided by S9, whose answer to Q5 was "multiply the top and bottom rows by 2 and add 2". Like the other five students, this pupil established a general, albeit mistaken, relationship between the variables.

CONCLUSION

This research supplements other studies focusing on lower grade students' ability to generalize in the context of classroom algebraic functions (e.g., Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015). Here the emphasis was on generalization as deduced by fifth graders.

The tiles problem affords the opportunity to explore students' functional thinking, as it enables fifth graders to progress beyond recursive sequences. In fact, they generalized on the grounds of correspondence and covariation relationships that involved the values of a set of variables.

The overall finding was the existence of two situations in which students generalize: (a) when answering questions about particular (near or far) instances; and (b) when specifically prompted to generalize. Three students generalized spontaneously, i.e., where the question could be answered without doing so. They consequently used generalization as a strategy to reply to questions involving specific circumstances. All the students who established a general relationship between the variables (spontaneously or when prompted) based their deduction on the correspondence relationship.

The students who generalized spontaneously used algebraic notation and verbal representation to express the general relationship between variables. Representation was primarily verbal in students who generalized only when prompted. In line with Blanton, Brizuela, Gardiner, Sawrey, and Newman-Owens (2015), the present authors venture that using algebraic notation would enable students to visualize generalization in fuller detail. That is consistent with the fact that the students who used notation in addition to verbal representation to express relationships did so in questions where generalization was not necessary (spontaneous generalization).

Moreover, the different ways in which students detected patterns in a problem involving a linear function (f(x)=2x+6; f(x)=2x+3+3; f(x)=2(x+2)+2; f(x)=2x+2) afforded the opportunity to interpret and understand their thought process when identifying a general relationship between variables.

Lastly, the present findings are related to earlier research results on Spanish fifth graders' ability to generalize (Merino et al, 2013), in which verbal representation was also observed to prevail. This paper describes the general functional relationships detected by students and the questions in which they were identified by functional thinking. These findings support the application of this approach to mathematics teaching in the lower grades, for its favors and enhances algebraic thinking (Blanton, 2008).

Acknowledgments

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UNPACKING YOUNG CHILDREN'S FUNCTIONAL THINKING

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Based on a synthesis of the literature, a model for young children's (Grades 1-3) functional thinking was formulated. The major constructs incorporated in this model were recursive patterning, correspondence relationships and covariational thinking. The study involved three hundred and forty five students. Data analysis validated the hypothesized model and suggested a sequential effect between the three factors. Recursive patterning had a direct effect to correspondence relationships and the latter had a direct effect to covariational thinking.

INTRODUCTION

Initiatives worldwide have underlined the significance of early algebra in mathematics education and pointed out that to meet the goal of "algebra for all", students in elementary school should be involved in activities that prepare them for algebra in later grades (National Council of Teachers of Mathematics, 2000). Thus, students in early grades should be encouraged to develop early algebraic thinking which will be expanded later. Developing such thinking could contribute to bridging the gap between arithmetic and algebra. This relatively new perception of early algebra has been gaining ground in mathematics education. A number of researchers (Drijvers, Goddijn, & Kindt, 2011; Blanton & Kaput, 2011) actually proposed the introduction of algebraic reasoning to students even before primary school.

Functional thinking is considered to be a fundamental dimension of early algebraic thinking (Blanton & Kaput, 2011; Carraher & Schliemann, 2007; Drijvers, Goddijn, & Kindt, 2011; Smith, 2008) and an important unifying strand across K-12 curriculum. Functional thinking involves generalizing relationships between covarying quantities and representing and reasoning with these relationships through natural language, algebraic (symbolic) notation, tables, and graphs. Most studies on functional thinking were conducted in secondary education since it is believed that functions require abstract thinking. However, research on functional thinking with younger students is growing due to the significant role it appears to play algebraic thinking (Blanton, Brizuela, Gardiner, Sawrey & Newman-Owens, 2015). Thus, there is a strong need to further investigate the nature of young children's functional thinking.

In this study, we examined the structure of young children's (Grades 1-3) functional thinking by empirically examining the validity of a theoretical model that synthesized research findings regarding the nature of functional thinking. In addition, we traced groups of students who exhibited different patterns of responses to functional tasks and

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 4, pp. 57-64. Singapore: PME.

finally we investigated possible relations between different components of functional thinking.

THEORETICAL BACKGROUND

Curriculum developers, policy makers, and researchers explored the way in which elementary students should be involved in early algebraic activities (Carraher & Schliemann 2007; NCTM 2000). In most of these cases, early algebraic tasks are considered the tasks that bridge arithmetic to algebra by promoting (a) understanding of the function of operations, (b) generalization and justification, (c) extension of the number system and (d) notation with meaning. It is suggested that these types of activities could help students make the transition from arithmetic to algebra and also empower computational fluency. In addition, research studies showed that elementary school children, primarily in Grades 3 to 5, manipulate functions and explore recursive, covarying, and correspondence relationships (Blanton, Stephens, Knuth, Gardiner, Isler & Kim, 2015). Blanton and Kaput (2004) found that even kindergarten to Grade 2 students can be successful in activities that require the interpretation of functional relationships, express covariation and correspondence among quantities. Research indicated that children of this age can investigate relationships between quantities, and not only simple recursive patterns. They achieve this by creating t-charts and other representations as well as with the use of variable notation (Blanton, et al., 2015; Cooper & Warren, 2011).

Functional thinking has been described as the specific type of thinking that focuses on the relationship between varying quantities and representing these relationships through natural language, symbols and appropriate representations. Smith (2008) proposed a framework to discuss the kinds of functional thinking found in classroom data. This framework includes three types of functional thinking: (a) recursive patterning: finding variation within a sequence of values, describing the pattern rule in words and using a rule to predict near data; (b) “correspondence relationship”: identifying a correlation between variables, using the function rule to predict far function values, finding the value of the independent variable, given the value of the dependent variable (c) covariational thinking: analyzing the way in which two quantities vary simultaneously and keeping that change as an explicit, dynamic part of a function’s description and. Research findings in the upper elementary grades showed that the study of recursive patterns was a necessary bridge for the development of children’s functional thinking since a student cannot understand a relationship between two quantities without first understanding variation in a single sequence of values (Blanton, et el., 2015; Cooper & Warren, 2011).

Research findings in early childhood education underlined the importance of early patterning skills and the early development of structure in mathematical thinking (Papic, Mulligan, & Mitchelmore, 2011). By the term patterning skills we mainly refer to the capacity of conceptualizing replicable regularity. Patterns may occur within a single object, within an ordered set of objects or between two ordered set of objects in the form of spatial structure patterns, repeating patterns and growing patterns (spatial or numeric) (Papic, et al., 2011). Recognizing the structure of a pattern is central to the notion of unit of repeat and the development of composite units and may contribute to the development of structure and generalization.

THE PRESENT STUDY

The purpose of the present study was to describe young children's (Grades 1-3) capacity in functional thinking, through empirically validating a proposed theoretical model. In an attempt to provide a comprehensive, functional, flexible and dynamic description of functional thinking, we hypothesized that young children's functional thinking captures three distinct, but correlated dimensions, as proposed by Smith (2008) and adopted by Blanton and Kaput (2011). We hypothesized that the first dimension can be conceptualized by a latent factor that models the key parameters of recursive patterning (Blanton & Kaput, 2011; Papic, et al., 2011; Smith, 2008). In particular, we hypothesized that recursive patterning is a second-order latent factor that consists of children's capacity to extend repeated patterns, express in symbolic form the rule of a repeated pattern and extend growing patterns (geometric and numeric) by recognizing and isolating the repeated action. The second dimension can be modelled by a latent factor that corresponds to children's capacity to notice the correspondence relation between corresponding pairs of variables, while, the third dimension by a latent factor that involves functional thinking; stating the rule based on which two quantities covariate with an emphasis on coordinating the underlying rate of change and the corresponding changes in the individual variables, when x increases by 1, y increased by 2 (Smith, 2008).

Measures

Four types of tasks were used to measure recursive patterning: (a) extending geometric repeated patterns (Tasks 1-3), (b) identifying the rule of a repeated geometric pattern (Tasks 4-6), (c) extending geometric growing patterns (Tasks 7-9), and (d) continuing numeric patterns (Tasks 10-12). In Tasks 1-3 students were asked to find the next three terms of repeated geometric patterns of the form ABAB, AABB and AAB. In Tasks 4-6 students were asked to express with letters the rule of repeated geometric patterns. In Tasks 7-9 students were asked to extend geometric growing patterns (Papic, et al., 2011). In Tasks 10-12 students completed the next three terms of a numeric pattern (e.g. 4, 8, 12,…). Nine tasks were used to measure correspondent relationships. Three of them asked students to express with symbols the correspondence relationship of the input and output value of three function machines (Tasks 13-15) and to find the input value of the machine, given the output value (Tasks 16-18). The rules of the function machines were simple additive and multiplicative relations (e.g. add 2, multiply by 3). In the other three tasks students were asked to express in words the rule of function machines (Tasks 19-21). Five tasks were used to measure covariational thinking. In Tasks 22-24 we adopted an activity suggested by Blanton and Kaput (2011) in which students were asked to investigate the number of body parts of a growing snake and express the change in the length of the snake in terms of varying number of days. Finally, in Tasks 25-26 students were asked to analyse the change of the length of a plant compared to the change of days (presented in a graph).

Participants, Procedure and Data Analysis

Three hundred and forty five young children (171 males and 174 females) were the subjects of the study from two urban primary schools in Cyprus. One hundred and seventeen students were first graders, 115 were 2nd graders and 113 were 3rd graders. The tasks of the study were randomly split into two parts. Each part was administered in the form of a written test during one school period. The two parts were administered in two successive days. The instructions were provided in written and verbal form.

Confirmatory factor analysis was used to examine the validity of an a priori model, based on past evidence and theory. CFA was conducted by using MPLUS (Muthén & Muthén, 2007). Latent class analysis was used to trace categories of students reflecting different patterns of responses in the factors of functional thinking. To evaluate model fit, three widely accepted fit indices were computed: The chi-square to its degrees of freedom ratio (x^2/df should be <2); the comparative fit index (CFI should be >.9); and the root mean-square error of approximation (RMSEA should be <.08). The Cronbach's alpha index of internal consistency was very good (a=.87).

RESULTS

Confirmatory factor analysis (CFA) was used to evaluate the construct validity of the model, by validating that the a-priori model matched the data set of the present study and determined the "goodness of fit" of the hypothesized latent construct. The Comparative Fit Index (CFI) of the three-factor model was .97, the ratio of χ^2 to the degrees of freedom was 1.49 (χ^2 =437.58 and df=292) while RMSEA did not exceed .04. CFA showed that the factor loadings of the tasks employed in the present study were statistically significant and most of them were rather large (see Figure 1). The factor loadings ranged from .40 to .97, giving support to the assumption that all latent factors were adequately measured by the observed variables. Thus, in accordance with our theoretical assumption, all functional thinking measures were clustered into one second order and two first-order factors in the expected factor loading pattern. These factors served as the latent structure of the functional thinking model. In particular, the second-order latent construct "recursive patterning" could accurately model students' variances in extending and describing the rule of repeated geometric patterns, extending growing geometric and numeric patterns. The factor loadings of the four first-order factors (repeated geometric patterns extension, repeated geometric patterns rule, growing geometric pattern, number patterns) to the second order factor "recursive patterning" ranged from .53 to .72 (see Figure 1), showing almost equal contribution to the hypothesized higher order construct. In addition, the construct "correspondence relationship" could accurately explain students' variances in identifying the relation between variables and use the function rule to predict far function value, and finally the theoretical construct "covariational thinking" could adequately model students' variances in stating the rule in which two quantities vary simultaneously. Significant correlations were found between the three factors. The correlation between recursive patterning and correspondence relationships was .71 (p<.05), the correlation between covariational functioning and correspondence relationships was almost identical (r=.72, p<.05), while the correlation between recursive patterning and covariational thinking was .78 (p<.05).

Latent class analysis was used to trace categories of students that reflect categories of students with different patterns of responses. We applied a stepwise method to validate the model under the assumption that there were one, two, three, four or five categories of students. The best fitting model with the smallest AIC and BIC indices and the largest entropy was the one involving three categories of students. Table 1 presents the mean and standard deviation of the three categories of students. The first category of students (n=56) reflected students that fail in the correspondent relationship and covariational thinking tasks and just over exceed 50% in recursive patterning tasks. The second category of students (n=204) had a very good performance in recursive patterning tasks ($\bar{x} = .88$) and failed in the other two types of tasks. The third category of students had almost a perfect performance in recursive patterning tasks ($\bar{x} = .96$) and satisfactory performance in correspondence relationship tasks ($\bar{x} = .64$) and covariational tasks ($\bar{x} = .69$). Thus, the first category can solve successfully just half of the recursive patterning tasks, the second category can solve successfully almost 90% of the recursive patterning tasks, while the third category can additionally solve sufficiently the correspondence and covariational tasks. It seems that the distinguished characteristic of this category is the fact that they can manage both the correspondence and covariational tasks. All categories consist of students from all grades (e.g. 13% of category 1 are 3rd-graders while 5% of category 3 are first graders).

	Category 1		Category 2		Category 3	
	Mean	SD	Mean	SD	Mean	SD
Recursive Patterning (n=56)	.52	.12	.88	.10	.96	.05
Correspondence relationship (n=204)	.10	.12	.21	.16	.64	.19
Covariational thinking (n=85)	.05	.10	.16	.20	.69	.32

Table 1: Means of the three categories of students.

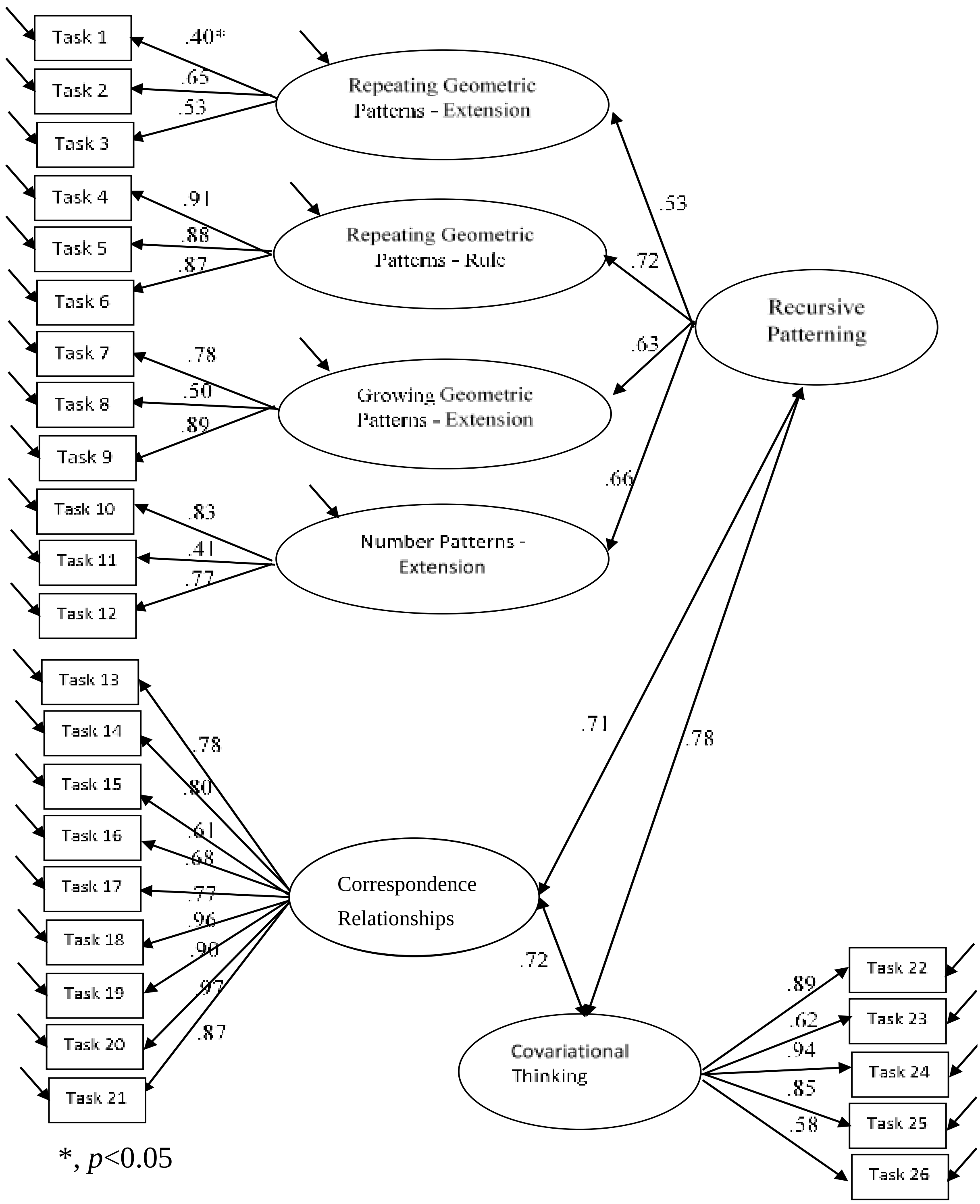

Figure 1: The nature of young children's functional thinking

Taking into consideration the distinguished characteristics of the three categories of students, the correlations between the three factors and the research findings suggesting that recursive patterning is a necessary bridge in the development of children's functional thinking, we examined the validity of a structural model to investigate the relations between the three factors of functional thinking, This model hypothesized the existence of direct effects between the three factors, representing individual differences in the three functional thinking factors. Thus, we tested the validity of a model where correspondence relationship factor was regressed on recursive patterning factor and covariational thinking factor was regressed on correspondence relationships factor. The fit of this structural model was very good (CFI=.96, χ^2=467.78, *df*=293, χ^2/df=1.59, RMSEA=.04). Figure 2 presents the sequential path between the three factors. The standardized solution of the model showed a statistically significant regression coefficient of recursive patterning to correspondence relationship (*r*=.79, *z*=10.63, *p*<.05) and almost an equal regression coefficient of correspondence relationship to covariational thinking (*r*=.76, *z*=20.76, *p*<.05). Thus, the adopted structural model supported the existence of a sequential effect between the three factors (Recursive Patterning→ Correspondence relationship→ Covariational thinking).

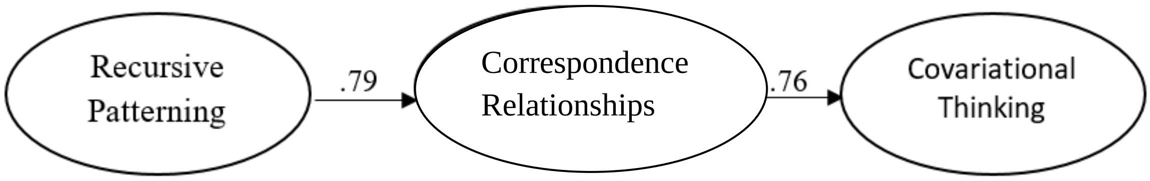

Figure 2: The relation between the functional thinking factors.

DISCUSSION

The contribution of the study lies on the empirical evaluation of a theoretical model that unpacks the dimensions of young children's functional thinking, based on a synthesis of the literature (Papic, et al., 2011; Smith, 2008). The results of the study showed that young children's variances in functional thinking situations can be modelled by three distinct and interrelated latent factors. The first factor involves children's capacity in recursive patterning tasks, the second factor in correspondence relationship situations, while the third factor reflects covariational thinking. In addition, the results showed that there is a sequential effect between the three factors. Young children's capacity in recursive patterning directly predicts their capacity in correspondence relationships and the latter directly affects covariational thinking. This finding reaffirmed research findings suggesting that recursive patterning is a necessary bridge of children's functional thinking development (Cooper & Warren, 2011). Thus, students' advancements in recursive patterning might enhance their further development in correspondence relationships and covariational thinking by enhancing awareness of the structure of patterns and applying generalizations strategically. In addition, the study suggested that finding the correspondence relationship facilitates children's covariational thinking. Children's developments in recursive patterning and correspondence relationships make possible the qualitative change in functional thinking that could explain the successful manipulation of covariational tasks.

Defining the components of young children's functional thinking is important because mathematics teachers should have a deep understanding of the components of functional thinking and the specific type of tasks that can been taught to young students from Grade 1 to Grade 3. The results also suggest the existence of a possible learning trajectory for functional thinking from recursive patterning, to correspondence relationships and finally to covariational thinking. However, future studies can actually explore the impact of different teaching approaches on these types of functional thinking tasks.

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MULTILINGUAL MATHEMATICS TEACHING AND LEARNING: LANGUAGE DIFFERENCES AND DIFFERENT LANGUAGES

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The reported research provides findings from the study of lesson episodes during student group work in multilingual mathematics teaching and learning. The theoretical lens of language-as-resource is taken to examine how certain uses of language and representations of speakers are voiced in ways that positively mediate the emergence and restoration of mathematics learning opportunities. This is illustrated with an example of analysis applied to an episode where language concerns are built up close to the communication of a mathematical idea for the resolution of a task. Overall, language is framed as a powerful resource in the classroom, whose resourcing for mathematics learning implies a multiplicity of languages (and hence discourses and voices) about language modeling and group identification.

WHAT KIND OF RESOURCE IS LANGUAGE?

In this report, the notions of social language and mathematics learning opportunities are examined under the theoretical lens of language-as-resource (Planas, 2014). This lens presupposes the ontological stance that language is not an actual resource unless someone uses it in a context of activity with particular tasks intended for some learning opportunity to emerge. It is therefore in the use that the potential quality of resource for a certain purpose can be realized and recognized (Remillard, 2013). Once language is put to use in context, it is also presupposed that numerous possible directions for the development of activity are present in a number of ways, each of them of value within specific discourses and for the resourcing of particular directions. In this framework, the issue of how teachers and students use language as they do during multilingual mathematics teaching and learning becomes fundamental.

To examine this issue, in my classroom-based research I look at how social languages and mathematics learning opportunities work together in the understanding of small group work and whole class discussion. In the research completed so far with group work episodes in four lessons, the analysis shows discourses about uses of language and representations of speakers in interaction with the course of student mathematical activity. At this stage a case can be argued regarding the relationship between the creation and restoration of mathematics learning opportunities, on the one hand, and what is discursively built up and voiced in classroom discourse with the support of a variety of languages and their speakers, on the other.

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Putting language into the social

Previous work on students' difficulties with languages has been decisive in setting a rationale for pursuing a profound understanding of language (diversity) in mathematics education research (Phakeng, 2016). Forms of speaking, varieties of languages, discourses and voices constitute a family of notions that have preceded and prepared adoption in the field of the notion of social language and, with it, the progressive move away from deficit-based arguments. In this respect, Barwell (2016) draws attention to how "natural" languages (i.e. normative vocabularies tied to abstract grammatical systems) weave discourses and voices together, and to how discourses and voices in turn weave representations of certain languages as "natural." Mathematics teaching and learning is rarely about language in the broad sense in which we talk about using, e.g., Spanish, but rather about the "social languages" that recognizable groups of people use to carry out and voice their social practices. When we enter the mathematics classroom, we all navigate within, between and across the different social languages through which participants express their views and worlds.

The notion of social language calls into question developmental views of language proficiency as a variable of time and individual effort, as well as essentialist views of language diversity that equate, e.g., one bilingual classroom with two distinct languages. The illusion of measuring language proficiency and labeling and counting languages indicates the underlying conceptualization of language as material, pure and unitary. Even though this may not be the standpoint taken in some research, exclusive expressions like bilingual classrooms and bilingual students are not rare. Languages may look more alike if they exist within a "single" labeled language, and indeed this is relevant in the understanding of the actual diversity of a classroom, but there is a more complex reality across the countless social languages – that often go under the rubric of a language – with a role in representing student (language/activity) proficiency.

Situating language in mathematics learning

The question of the social underpinnings of mathematics learning is not new. In their seminal work, Yackel, Cobb and Wood (1991) presupposed the availability of opportunities for the learner to learn mathematics; that is the existence of mathematical ideas and social conditions more or less ready to be grasped for the development of mathematics learning in a context of activity. The research design experiments that followed from that work aimed to introduce changes in the social conditions of teaching and learning in mathematics classrooms. These experiments were substantiated by three inseparable claims: learning cannot take place without learning opportunities being available, these opportunities are created by people, and they are made available in accordance with the social conditions and not only the personal insights of individuals. Since not all the opportunities created in a context of activity are tackled as such by everyone all of the time, a separate issue was whether or not they are exploited in activity conducive to individual learning.

Together with the readiness of favorable social conditions, Saxe (2012) relates the discussion of mathematical ideas to the creation, exploration and development of mathematics learning opportunities. He refers to the travel of ideas in his research into the ways in which mathematical ideas are produced and transformed over the course of discussion-rich interaction, hence enabling new ideas to emerge. The availability of opportunities to learn mathematics is thus posed in relation to the availability of resources to allow ideas to surface and travel. Making mathematical ideas travel implies in turn the availability of resources for participation in discussion-rich interaction, as well as the availability of opportunities to utilize these resources in classroom activity. This formulation supports the relationship between opportunities to learn mathematics and opportunities to resource mathematics learning. Given that language is critical for participation in discussions, we finally come to the connection between availability of opportunities to learn mathematics and availability of (social) language(s) in the communication and discussion of mathematical ideas.

From a developmental approach, language availability is a long-term product that, once achieved by someone, implies durability. Accordingly, some learning opportunities are thought of as diminished or postponed in contexts of activity where there are participants who do not "own" such a product and, for this reason, are expected to contribute less than others. Within the framework of social languages, language availability – and the corresponding facilities to allow ideas to be expressed, collected and commented upon by multiple people – implies a different understanding (Planas, 2014). This availability is variously high or low for a number of reasons other than preconceived levels of proficiency in a given language. Thus, it is not a product to be achieved by individuals, but a dynamic feature of the context in which language is put to use by various participants to voice different discourses in the interaction. Under the basic assumption that language is not available all the time for all participants (Makoni & Pennycook, 2005), the opportunities to use it to make ideas travel can be (dis)encouraged by infused processes of assessment and (dis)placement of speakers who do not conform to standardized forms of speaking and acting.

APPLYING LANGUAGE-AS-RESOURCE TO THE STUDY AND VIEW OF STUDENT MATHEMATICAL ACTIVITY

The data in this report draw from a Grade 8 classroom of a school in a low-income zone of Barcelona, the capital city of Catalonia, a north-eastern region of Spain with its own language in education policy – Catalan is the official language of teaching and learning, although it is not necessarily the language of learning and thinking for all students. The teacher was a Catalan-dominant speaker who occasionally used a variety of Castilian Spanish in her lessons. Fourteen students were children from Latin American (Colombian, Ecuadorean and Peruvian) families who declared Spanish to be their home language, nine of whom were raised abroad; five students were children of Castilian Spanish-dominant families, two of whom were raised in Castilian-speaking parts of Spain; and four were Catalan-dominant speakers raised in Barcelona. Varieties of Colombian, Ecuadorean, Peruvian and Castilian Spanish, or combinations of these, are not typical of the varieties of Spanish spoken by people raised in Catalan-speaking regions. There are differences in the sounds of some letters and in the conjugation of some verbs, among others. Students who begin to learn the language of instruction at school – mostly due to histories of immigration – are located in special lessons for some time to learn this language. When they finally enter the regular classroom they tend to speak varieties of Catalan with sounds, conjugations and words borrowed from their home languages; such varieties are marked as "poor Catalan" by groups who claim ownership of the language of instruction in the region.

Local educational debates that occupy much of the current public discourse generally address the merits of the parallel system of special lessons for "latecomers" (Planas & Civil, 2008), rather than debating about how children learn and teachers teach in either system. Other important debates are motivated by ideological stances regarding the politics of language use at school. All these debates inform curricular decisions and pedagogic practices that often adhere to rigid conceptualizations about what counts as language – in theory and in classroom practice – and how language use is seen in relation to mathematics teaching and learning. It is thus significant to explore how language is shaped by discourses and voices that make some mathematical ideas more likely to travel (and thus some mathematics learning opportunities more likely to emerge) when they are expressed in the standardized language of instruction by speakers who are represented as (more) competent in this language. Even if a student has the necessary school mathematics knowledge to discover and value a learning opportunity, she may fail to do so because there is limited access to certain forms of speaking and speakers in the context in which the opportunity arises.

Lesson, task and methods

Student work in three small groups was video-taped during a problem-solving unit of four lessons devoted to algebra. The groups had one or two Catalan-dominant speakers each and remained the same throughout the sequence. For this report, I take lesson four and the group with Maria and Ton, from Catalan-dominant families, and Ada and Leo, who were raised in Peru and attended classes for latecomers during Grade 6. The problem was a representation of the Fibonacci numbers starting at 1 and 2:

> In a house there is a staircase with ten steps. If we can go down the steps one or two at a time, in how many different ways can we go down the staircase?

Transcripts of lesson data were produced regardless of shifts between the two labeled languages involved. Group work was coded under three main types: Language Modeling (LM), Group Identification (GI) and Mathematical Ideas (MI). LM was assigned to turns with visible references to vocabulary (LM-V), grammar (LM-G) or pronunciation (LM-P), whereas GI was assigned to turns with mentions of or allusions to speakers as members of groups, in some of which issues of vocabulary (GI-V), grammar (GI-G) or pronunciation (GI-P) were mentioned. Each coded turn was interpreted as a component part of the activity that was constituted during mathematics teaching and learning. The segmentation of talk into turns was followed by segmentation into instances. In this study an instance is defined as a range of spoken turns, which are sequential but may not be consecutive, from a single sentence all the way to a lengthy interaction. After the detection of LM and GI turns and the construction of paired instances, the analysis went on with the focus on student mathematical activity. This stage was guided by the search for turns and paired instances with literal comments and implied allusions regarding ideas of relevance for the understanding and resolution of the problem. MI codes were named after topic identification in accordance with the central mathematical content to be kept for the development of the idea. From here, LM, GI and MI instances were related in the construction of episodes with one MI and at least one LM/GI instance as well as sufficient before-and-after-turns to understand what was being mathematically developed and how some language issues had been voiced in-between.

For each episode and when feasible, relationships were elaborated between the availability of language and the availability of mathematical ideas. This was done by imagining "figured worlds" (Holland, Lachicotte, Skinner & Cain 1998) under two basic phenomenological standpoints: 1) any account of reality requires imagination, and 2) imagination is necessary to make any inference out of what appears. In the application of imaginative variation to each episode – any variation is a possibility –, the following dual question was posed: 'Can language modeling and group identification be imagined as obstacles to/resources for mathematics learning?' Answers originated from the examination of possible directions in the ways that discourses and voices (could) put language to use. This process served to imagine language within a cycle of diverse resourcing directions. Since many interpretations can be imagined, the variations considered were only those in which the new episodes lacked the coded language turns but kept the mathematical idea. The varying of language modeling and group identification was undertaken to explore how realistic the development of the corresponding mathematical idea was in the worlds imagined.

"Minor" details taking on "major" importance

In this section, we analyse an episode of group work in lesson four. The creation of certain opportunities to learn from the study of a simpler version of the given problem is associated with the voicing of some concerns about language modeling (LM-V) and group identification (GI-V). Language is interactionally worked out turn by turn in ways that model forms of speaking, represent groups of speakers, and allow the emergence of new ideas to be discussed. The transcript below shows how apparently "minor" details in language use can take on "major" importance in interaction. Language modeling and group identification are voiced, respectively in relation to the right meaning of the Catalan word for going down ("baixar" in [5]) and the name of the student at risk of interpreting the word wrongly ("Ada" in [6]). When it is later said that there is "too much to go down and jump" [9], mentions of the distinction between the two verbal actions are not taken up and a relevant mathematical idea for an approach to the resolution of the Fibonacci problem emerges instead.

1 Maria: Per què tens tot uns i aquí tot dosos? [Why do you have all ones and here all twos?]

2 Leo: Puedes bajar sempre o saltar sempre. [You can always go down or always jump.]

3 Maria: Sempre es baixa, no t'estàs parat. [You always go down, you don't stand still.]

4 Leo: Pero a veces no bajas, saltas. Y a veces solo bajas. [But sometimes you don't go down, you jump. And sometimes you go down only.]

5 Ton: Baixar no vol dir d'un en un. Mira, baixar és un a un, dos a dos, tres a tres, tot és baixar. [Going down does not mean one by one. Look, going down is one at a time, two at a time, three at a time, all this is going down.]

6 Maria: Ada, tu ho tens clar? [Ada, is this clear to you?]

7 Ada: Sí, baixar. [Yes, going down.]

8 Ton: Així et deixes de barrejar uns i dosos. [This way you miss combinations of ones and twos.]

9 Leo: He empezado pero hay mucho que bajar y saltar. Al menos treinta. Si la escala fuera más corta… [I began but there is too much to go down and jump. At least thirty. If the staircase was shorter…]

10 Ton: Umm… Si fos tres, seria: u, u, u; dos, u; u, dos… i dos, dos impossible. Ara ve quatre. [Umm… If it was three, it would be: one, one, one; two, one; one, two… and two, two impossible. Now four comes.]

There is some difficult mathematics involved in the resolution of the problem in this lesson. One can always determine the possibilities by counting them one by one, but this is not very manageable, as suggested by Leo in [9]. While it is easy to represent the extreme cases [1], when the combinations of ones and twos are considered in a classroom with students who are not familiar with combinatorial formulas and binomial coefficients, a process to represent the total of 89 possibilities is not easy to discern; it may occur that one possibility is counted twice or that some possibilities are missed during the counting. Nonetheless, there is a pattern embedded in the resolution whose exploration can be strategically approached by starting with staircases which have smaller numbers of steps (the 3-step and the 4-step staircases in [9-10]). Although the students from the group did not see a pattern, they foresaw the option of examining reductions of the problem and, hence, approached the challenge of solving the problem without adding up the total number of ways of going down ten steps. At the end of the lesson, the teacher presented the recursive pattern that relates the number of ways to get the 10^{th} step to the number of ways to reach the 8^{th} and the 9^{th}, and successively until the dependence of the 10-step on the 1-step and the 2-step staircases.

The mathematical idea introduced by Leo in [9] and taken up by Ton in [10] is preceded by two moments in which activity moves away from the task resolution toward language concerns. Maria and Ton model the acceptable meanings for a term in Catalan when Leo equates the movements of one step at a time with "baixar" (going down) and two steps at a time with "saltar" (jumping). In [3] and [5] "baixar" is given an extended meaning that includes jumping, which is the common meaning in mainstream Catalan. Another mathematically critical moment comes when group identification is voiced in [6]. The suggestion that Ada may experience the same confusion with vocabulary, as she may be interpreting "baixar" like Leo, can be seen as an allusion to the qualities attributed to the group of people that Leo and Ada are seemingly placed in. References to shared backgrounds are echoed and the account of these students as poor users of the official language of teaching and learning is made visible. However, when the distinction between going down and jumping comes again in [9] during the proposal of the idea about shorter staircases, discourse moves away from the focus on language issues and goes back to mathematics.

What we see in this episode is that language modeling and group identification are voiced in ways that momentarily interrupt the mathematical discussion in student group work. However, it is from here that another discourse emerges with the option for the students to unvoice vocabulary and group differences and explore a more sophisticated approach to the resolution of the problem. A primary representation of language as obstacle is thus difficult to imagine in this episode, as is a primary representation as epistemological resource for mathematics learning. By varying [5] and [6] and imagining an alternative episode without these turns, the possibility of the same mathematical idea emerging and traveling with the same intensity in student interaction is feasible. This said, important learning opportunities arise from the fact that the participation of Leo is facilitated and recognized in a discourse that follows the anticipation of language difficulties and differences among students.

DIRECTIONS IN THE RESOURCING OF LANGUAGE

In this report, by means of a short transcript, I have tried to illustrate the diversity of directions in the resourcing of language. The finding regarding the plurality of directions in the resourcing of language during student mathematical activity provides further understanding about the role of language in mathematics teaching and learning, especially as it intersects with a multiplicity of languages in the multilingual mathematics classroom. The discussion of social languages suggests that, far from assuming that language and mathematical difficulties reside in students, it is reasonable to consider that some of these difficulties as well as the possibilities of overcoming them primarily reside in discourse. In the data presented, Leo and Ada are immigrant children from lower income homes who have been taught a differentiated mathematics curriculum during their school year in the parallel system of special lessons for "latecomers." They have little practice at home with school-based forms of language and interaction, and they are actually represented as poor speakers of the official language of instruction. Nonetheless, we have seen how Leo participates with a relevant mathematical idea in the middle of a discourse in which language difficulties and differences are voiced. Both the relevance of the distinction between 'going down' and 'jumping' and the relevance of Leo's mathematical idea reside in discourse.

The learning of mathematics cannot be understood separately from the process of learning the social language that is characteristic of those who do well in the school mathematics of the local educational system. All children can effectively be introduced to the dominant social language of a given classroom. Nonetheless, this should not be done at the expense of reducing their participation in the creation of mathematics learning opportunities during the process of learning *a* language. In the teaching and learning it is not easy to juxtapose the primary languages with the newer languages so as to allow students to smoothly navigate across them for the primary purpose of mathematics learning. Hence, the importance of intentionally integrating in mathematics teaching the issue of focusing on the many mathematical ideas that can be communicated and discussed at the intersection of the social languages that students and teachers bring with them. It cannot be forgotten that the resourcing of language for multilingual mathematics teaching and learning rests not only upon the possibility of resourcing the emergence, exploration and development of mathematical ideas, but also upon the possibility of postponing the discussion of these ideas.

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CONTENT- AND LANGUAGE-INTEGRATED LEARNING: A FIELD EXPERIMENT FOR THE TOPIC OF PERCENTAGES

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Supporting language learners requires content- and language-integrated instructional approaches to coordinate conceptual learning trajectories with systematically structured language learning opportunities, so-called macro-scaffolding approaches. This paper provides empirical evidence for their effectiveness under field conditions in regular mathematics classrooms. For this purpose, a field experiment was conducted with n = 108 students based on a macro-scaffolding intervention for learning percentages. The ANCOVA shows that after 15 sessions of intervention, the intervention group significantly outperformed the control group (with comparable pre-knowledge) and medium effect sizes. This shows that teachers can foster language learners' conceptual understanding when supporting their learning by necessary language means.

BACKGROUND: FOSTERING LANGUAGE LEARNERS BY CONTENT- AND LANGUAGE-INTEGRATED APPROACHES

Academic language proficiency in the language of teaching and testing has repeatedly been shown to influence achievement in mathematics (see Barwell et al. 2016 for overviews). As a consequence, current design research activities have been focusing on developing and investigating content- and language-integrated instructional approaches for supporting students with low language proficiency (Gibbons 2002; Moschkovich 2013). In this paper, we contribute to these efforts by developing an instructional approach based on the design principle of macro-scaffolding (Gibbons 2002; Smit et al. 2013). The main idea of macro-scaffolding is to coordinate a conceptual learning trajectory with well-structured language learning opportunities in lexical and discursive dimensions. Although the general structure of language trajectories from students' everyday resources to academic registers and formal technical registers has been well described, its topic-specific realization for different mathematical topics is still an urgent need of research as the general lines do not sufficiently guide teaching practices (Smit et al. 2013).

Previous research has especially shown the high relevance of the discourse practice of explaining meanings of the mathematical concepts in view (Moschkovich 2013; Prediger & Wessel 2013). For this, macro-scaffolding approaches can be supported by the design principle of relating registers and representations, according to which the graphical and symbolic representations should be systematically related to the different verbal registers (the everyday registers, the academic school register, and the technical

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 73-80. Singapore: PME.

register) forward and backward in order to achieve a deep and well-connected conceptual understanding (Prediger & Wessel 2013). As some first empirical findings exist on the effects of topic-specific realizations of these instructional approaches and design principles (Gibbons 2002; Smit et al. 2013; Prediger & Wessel 2013; and others), further research can widen the approach to other mathematical topics and especially provide a wider base for quantitative empirical evidence for their efficacy. Since many of the existing studies on content- and language-integrated learning in mathematics education have taken place in laboratory small-group settings, it is time for the next step of research: investigating the functioning and effectiveness in whole-class settings with regular teachers in field experiments, as requested by Burkhardt and Schoenfeld (2013). That is why the current study continues the research on a content- and language-integrated macro-scaffolding intervention on percentages which has so far only been investigated qualitatively in laboratory conditions (Pöhler & Prediger 2015). The current step comprises a quasi-experimental field experiment in three classrooms with regular teachers and matching control students from other classes.

REALIZING MACRO-SCAFFOLDING FOR PERCENTAGES

In order to foster the conceptual understanding of students with diverse language backgrounds, an intervention was designed in several iterative design research cycles (described in detail in Pöhler & Prediger 2015). Based on general literature on students' difficulties with and teaching approaches for percentages (Parker & Leinhardt 1995), the intervention follows the design principles of macro-scaffolding and relating registers and representations. It coordinates on six levels a conceptual learning trajectory towards conceptual understanding and flexible use of percentages with well-structured language opportunities in a lexical learning trajectory (see Fig. 1).

The intended *conceptual learning trajectory* towards percentages (see Fig. 1) was adapted from previous design research on percentages in the context of Realistic Mathematics Education (van den Heuvel-Panhuizen 2003). It starts with students' everyday experiences and proceeds to constructing meaning for percentages. Students' informal strategies for determining rates, amounts, and bases are then elicited and later elaborated into calculation strategies for standard problem types. The conceptual learning trajectory finally aims at the ability to also flexibly use learned concepts and strategies in more complex and non-familiar situations.

The intended *lexical learning trajectory* (see Fig. 1) sequences the discourse practices and language means required for the conceptual learning processes. It starts from students' everyday resources by discussing intuitive ideas, establishes the discourse practice of explaining meanings and supports it using the *basic meaning-related vocabulary* for rates, amounts, and bases (e.g., old price, new price, rate to be paid), and then introduces formal vocabulary (base, amount, rate) and relates it to the basic meaning-related vocabulary for reporting and justifying formal procedures. Finally, the vocabulary is widened to the so-called extended reading vocabulary necessary to crack more complex percentage problems in non-familiar contexts.

	Conceptual learning trajectory towards mathematical concepts	Lexical learning trajectory for different discourse practices
Level 1	Constructing meaning for percents by representing and estimating rates	Intuitive use of *students' everyday resources* for discussing first ideas in percent bar
Level 2	Developing informal strategies for determining rates, amounts, later bases	Establish *basic meaning-related vocabulary* for explaining meanings
Level 3	Calculating amounts, rates, and bases	Introduce *formal vocabulary* in the technical register
Level 4	Widening to other problem types: change and comparison	Enrich the *basic meaning-related vocabulary* to more complex problem types
Level 5	Identifying problem types of (also non-)standard problems	Explicit use and training of formal and basic meaning-related vocabulary
Level 6	Cracking more complex context problems flexibly (non-familiar contexts)	Introduce extended reading vocabulary for non-familiar contexts

Figure 1: Dual learning trajectories towards percentages

Both trajectories are mediated by a structure-based scaffold, the percent bar (van den Heuvel-Panhuizen 2003). The function of the percent bar changes on the levels of the dual learning trajectory, first functioning as a model *for* problem situations in the contexts of download bars and shopping, then as a model *of* the abstract mathematical concepts of percentages. Later, it serves as a strategic scaffolding tool for mathematizing complex word problems.

A sequence of 21 instructional tasks was developed for realizing the intended dual learning trajectory. Two exemplary tasks (in Fig. 2) illustrate how conceptual and lexical aspects are intertwined and how the percent bar can serve as a mediator. Students not only solve percent items, but are often encouraged to verbalize their structure. The vocabulary offered for these discussions is always bound to the percent bar, which allows students to relate the vocabulary to its meaning.

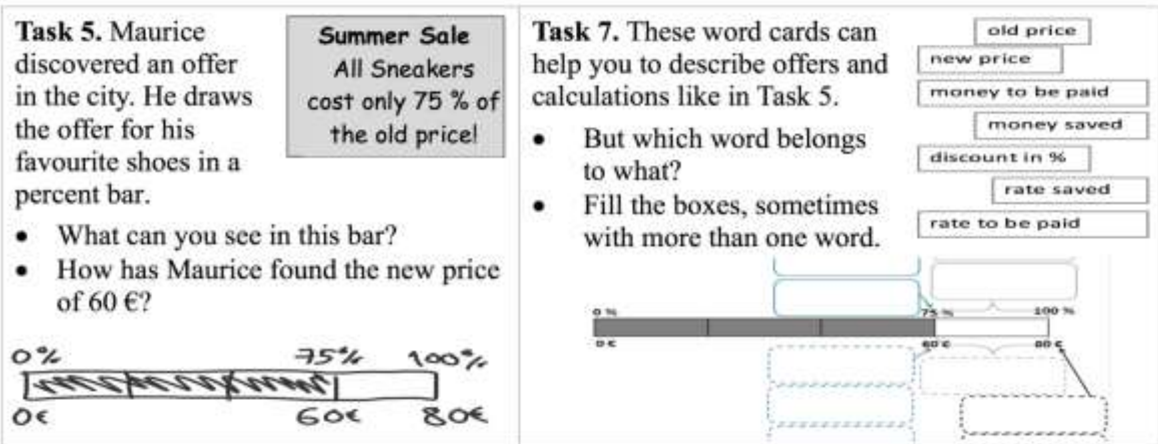

Figure 2: Tasks exemplifying the intertwinement of conceptual and language aspects

The functional use of these successive vocabularies for the different discourse practices (explaining meanings, reporting strategies, justifying strategies) is scaffolded by language frames, word banks, and repeated teacher prompts initiating students' rich discursive practices. As the previous qualitative analysis of small-group teaching has reconstructed challenges for teachers to adaptively provide micro-scaffolding, a field experiment is required to test how regular math teachers can meet these challenges.

RESEARCH QUESTIONS

Previous design research case studies have qualitatively shown that the designed intervention on percentages was beneficial for low-achieving students with limited academic language proficiency in small-group settings (Pöhler & Prediger 2015). In order to also extend the scope to regular teachers and to provide quantitative empirical evidence for the effectiveness, the current study tested the effects of the intervention in a field experiment in whole-class settings with two research questions:

Q1. What are the learning outcomes of the intervention group compared to the control group?

Q2. What are the learning outcomes of intervention and control group for the three problem types (Find the base, find the amount, and find the base after reduction)?

METHODS OF THE FIELD EXPERIMENT

Research design. The research was conducted as a quasi-experimental field experiment with a pre- and post-test design with seventh graders in urban schools in whole-class settings, taught by their regular teachers, all of whom had been sensitized to language issues in classrooms by professional development.

Intervention forms. The three intervention classes were taught in approximately 15 sessions of 45 minutes each by means of the described macro-scaffolding intervention program on introducing percentages with the dual learning trajectory. The control classes were taught according to the traditional program for percentages, which included the same mathematical content but did not follow the lexical and conceptual learning trajectory and the principle of relating registers and representations.

Measures for control variables. For achieving comparability between intervention and control group, the following control variables were taken into account:

(1) German *language proficiency* was assessed using a C-Test, offering economical and highly reliable measures, with Cronbach's α = .774 (N = 1,122),

(2) *Mathematical pre-knowledge* that is relevant for learning percentages (fractions, parts of whole, bar representations, etc.), measured before the intervention by a standardized test (from Prediger & Wessel 2013) with Cronbach's α = .83 (28 items, N = 1120), and

(3) General cognitive ability was assessed using BEFKI, with a focus on figural dimensions of *fluid intelligence*, with Cronbach's $\alpha = .76$ ($N = 1122$).

Measure for the learning outcome. The learning outcomes of the interventions were assessed using a standardized test on percentages that was optimized to assess conceptual understanding and flexible use of percentages. It consists of open items of the problem types "Find the amount," "Find the base," and "Find the base after reduction." For each problem type, items varied in three formats: "pure format," "text format," and "visual format," with percent bar representations (examples in Table 1).

Example items for problem type "Find the base"	
Pure format	5 % are 250 €. Find the base.
Visual format	What is unknown here? Find the missing value. 
Text format	Potatoes consist of 75 % water. How much water (in g) is contained in 1000 g potatoes?
Example item for problem type "Find the amount"	
…	
Text format	When buying a new kitchen, Family Mays receives a discount of 250 €, that was 5 % of the regular price. What is the normal price of the kitchen?
Example item for problem type "Find the base after reduction"	
…	
Text format	Mrs. Schmidt pays 30 € for a dress in the summer sale. The dress was reduced by 40 %. How much did the dress cost before?

Table 1: Items in the percent test in different problem formats

Sample. In order to achieve comparability in the quasi-experimental field experiment, each student of the intervention group was matched to a student from the control classes with respect to the control variables (see Table 2). In the variance tests, no significant differences appeared between the intervention group ($n = 54$) and the control group ($n = 54$) for the three control variables: language proficiency, mathematical pre-knowledge, and fluid intelligence (with $p > .05$ in the t-tests for all three variables).

	Language proficiency (max. 60) m (SD)	**Mathematical pre-knowledge (max. 19) m (SD)**	**Fluid intelligence (max. 16) m (SD)**	**Socio-eco nomic status (max. 5) m (SD)**	**Age in years m (SD)**
Intervention group (*n* = 54)	41.17 (6.95)	11.89 (4.96)	9.11 (3.02)	2.69 (1.12)	12.53 (0.74)
Control group (*n* = 54)	40.89 (7.2)	11.61 (4.28)	9.13 (3.82)	3.24 (1.12)	12.57 (0.66)

Table 2: Description of the comparable subsamples

Methods for data analysis. As the pre-test measured a wider repertoire of pre-knowledge than the post-test did, with its specific focus on percentages, the analysis of the effects of intervention were calculated using a covariation analysis (ANCOVA). This allows comparison of the differences in the learning outcomes of the intervention and control groups by taking into account the control variables. The ANCOVA was conducted for the complete percent test as well as the subscales of different problem types (Find amount, find base, find base after reduction; see Table 1). While the qualitative data analysis of the videotaped teaching-learning processes will be documented in further publications, this paper focuses on the quantitative results.

EMPIRICAL RESULTS AND INTERPRETATION

Research question Q1 asks for differences in the learning outcomes between the intervention group (who followed the content- and language-integrated intervention on percentages) and the control group (who followed a traditional percentage course). The descriptive data for this research question presented in the second column of Table 3 shows that the whole sample achieved an average score of 36.7. The intervention and control group (which were comparable with respect to the relevant control variables of language proficiency, mathematical pre-knowledge, and fluid intelligence) performed differently in the percent test after the intervention: The intervention group achieved an average score of 45.4, whereas the control group only achieved 28.0 (with similar standard deviations).

	Complete percent test (max. 100) m (SD)	**Subscale Type “Find amount” (max. 33.3) m (SD)**	**Subscale Type “Find base” (max. 33.3) m (SD)**	**Subscale Type “Find base after reduction” (max. 27.8) m (SD)**
Whole sample	36.7 (25.7)	14.2 (8.9)	15.0 (11.0)	6.1 (9.1)
Intervention group (*n* = 54)	45.4 (26.4)	15.8 (8.5)	18.4 (10.0)	9.7 (10.2)
Control group (*n* = 54)	28.0 (22.3)	12.6 (9.0)	11.5 (10.8)	2.5 (5.8)

Table 3: Group differences in the learning outcomes

The covariation analysis for the complete percent test (reported in Table 4) shows the anticipated result that mathematical pre-knowledge is a significant predictor for learning outcomes ($p < 0.01$). When controlling for the language proficiency, mathematical pre-knowledge, and fluid intelligence, the independent variable of belonging to either the intervention group or the control group shows a significant difference between the two groups (with $F\,(4{,}103) = 14.7497$, $p < 0.0000$). The effect size is captured by a partial eta squared of $\eta^2 = 0.141$, which is considered a medium effect.

Variables	Regression coefficient	Standard error	*p*-value	Partial η^2 (proportion of variance in percent test explained by variable)
Intercept	-1.454	2.212	0.512	0.004
Language proficiency	0.096	0.057	0.097	0.027
Mathematical pre-knowledge	0.343	0.097	0.001	0.109
Fluid intelligence	0.177	0.126	0.165	0.019
Intervention/control group	-3.018	0.733	0.000	0.141
$R^2 = 0.364$, $R^2_{Adj} = 0.34$, F (4,103) = 14.7497, $p < 0.0000$				

Table 4: Results of the ANCOVA for the complete percent test

Research question Q2 asks for differences between both groups for the subscales of different problem types (Find the base, find the amount, and find the base after reduction). As the different means in Table 3 (columns 3 to 5) show, the intervention group outperforms the control group in all subscales. The differences are smallest for the most elementary problem type, "find the amount," for which the intervention group reached a score of 15.8 and the control group of 12.6 (a difference of 3.2). For the inverse problem type, "find the base," the scores of 18.4 and 11.5 differ by 6.9 points. For the most complex problem type, "find base after reduction," which requires two-step thinking, the scores of 9.7 and 2.5 differ by 7.2 points. The ANCOVAs for all three subscales provide evidence for significant group differences when controlling for language proficiency, fluid intelligence, and mathematical pre-knowledge:

$F_{\text{Find amount}}$ (4,103) = 9.0898, $p < 0.05$, $\eta^2 = 0.04$ (small effect size)
$F_{\text{Find base}}$ (4,103) = 10.6239, $p < 0.001$, $\eta^2 = 0.105$ (medium effect size)
$F_{\text{Base after red}}$ (4,103) = 9.5627, $p < 0.0001$, $\eta^2 = 0.187$ (large effect size).

To sum up, the group differences in problem types of different varying and familiarity might be interpreted as indicating that the percent is a fruitful strategic tool for mathematizing, especially for complex problem types (find base after reduction) and for avoiding over-generalizations. These interpretations are supported by previous qualitative analyses of small-group settings (Pöhler & Prediger 2015).

DISCUSSION

The larger design research project in which this field experiment is embedded aims at designing and investigating the functioning of content- and language-integrated approaches based on the design principles of macro-scaffolding and relating registers and representations. This research is specifically important in classes with diverse language backgrounds (Gibbons 2002; Smit et al. 2013; Prediger & Wessel 2013). Previous qualitative analysis of the intervention has shown the principal transferability of these approaches to mathematical topic percentages (Pöhler & Prediger 2015). However, it has also shown the high relevance of micro-scaffolding by teachers, so it has been an open question as to whether the intervention would also be effective under regular classroom conditions.

In this paper, we provided empirical evidence that the intervention has had better effects on students' conceptual understanding and flexible use of percentages than traditional courses. Many other studies have shown that such evidence is more difficult to provide under field conditions than under laboratory conditions (Burkhardt & Schoenfeld 2003): Whole-class settings with students' regular teachers have a higher complexity and typical constraints for implementing research-based designs. Nevertheless, the ANCOVA results in a significant difference with high effect sizes: Students who acquire conceptual understanding of percentages in a content- and language-integrated intervention based on principles of macro-scaffolding and relating registers outperform students learning in a traditional course with the same content. The comparability between intervention and control group was controlled for the control variables of language proficiency, mathematical pre-knowledge, and fluid intelligence. The detailed analysis of learning outcomes on different problem types provided insights into the specific strength for non-routine problems. However, further qualitative analysis of the classroom video data will be necessary to understand the chances and limits in more detail.

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CONFIDENCE AND COLLABORATION IN TEACHER DEVELOPMENT OF DIGITAL TECHNOLOGY TASKS

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This paper discusses the results of an intervention designed to enable richer digital technology (DT) task design by secondary school teachers. Working in groups of three the teachers designed their own tasks. They were then introduced to some theoretical design principles, following which they further developed their tasks. The results show that the intervention produced richer, more student-centred tasks. Some factors contributing to this improvement include a confident attitude to teaching with DT and the collaborative nature of the groups they worked in.

BACKGROUND

Many mathematics teachers claim to support the use of digital technology (DT) in their teaching, yet the extent to which it has been implemented in the classroom remains variable (Zbiek & Hollebrands, 2008). More recently, there has been a focus on supporting teachers in the design and implementation of their own DT tasks rather than those designed by researchers. This paper describes an intervention experiment of how groups of teachers were guided in their DT task production and implementation. The intention was that a well-designed DT task would provide opportunities for mathematical thinking, formulation and communication of new ideas, justification of procedures, and defence of reasonableness of answers (Cennamo, Ross & Ertmer, 2014). Although a great deal of thought and effort is often put into researcher-designed tasks, carefully considering the constituents of the milieu and the pragmatic and epistemic value of the DT tools, etc, the teacher still has to adapt these tasks to fit their own pedagogical and epistemological perspectives and the needs of their students (Leung & Bolite-Frant, 2015). For example, a teacher with a participationist orientation may encourage students to participate in the construction of mathematical knowledge through shared experiences or discourses, while one with an acquisitionist view could favour exploration and discovery of established mathematical knowledge (*ibid*). Along with the purpose of the task, or the reason for engaging with it, tasks utility should be considered (Ainley & Pratt, 2002), whereby the designer may perceive future uses, but these may differ for the teacher-designer. In addition, the issue of teacher instrumental orchestration, comprising didactic configuration, exploitation mode and didactical performance (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010) is highly relevant to DT task design and suggests it is important for the teacher to reflect on the configuration of artefacts in the teaching setting, how they will be employed to achieve goals and the decisions that will be made during

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implementation. Thus, the relationship between the researcher, teacher, students and task development is complex and critical.

Our framework employs the theoretical model of Pedagogical Technology Knowledge (PTK) (Thomas & Hong, 2005) as it places the enhancement of mathematical thinking at the centre of the task design and implementation processes. It presents intrinsic factors that influence teacher use of DT in their classroom, including a teacher's orientations and the goals these give rise to (Schoenfeld, 2010), along with their instrumental genesis of DT tools and their mathematical knowledge for teaching (MKT). Our hypothesis is that to design rich tasks with epistemic value using DT, a teacher needs to use their MKT and align it with positive orientations and strong instrumental genesis. There is some research evidence that one crucial orientation is teacher confidence, with Thomas and Palmer's (2014) research finding a significant correlation between measures of teacher confidence in using technology and their PTK, as well as their belief in the value of technology in teaching mathematics. This provided an impetus for a consideration of the role of teacher confidence and beliefs in task design in the study reported on here.

METHOD

The participants comprised 12 Sri Lankan secondary school teachers, eight females and four males, divided into four groups, diverse in age and experience. None of the teachers had completed a Master's level degree and all of them had very limited, or no, experience in using DT for teaching mathematics. Their students were Grade 12 (17-18 years old) studying Advanced Level (A-level) combined mathematics from boys, girls or mixed schools in urban or semi-urban areas. In this paper, we focus on Group C, from the Western province, whose members were similar in age and predominantly less experienced than the other groups (see demographics in Table 1).

Teacher	Gender	Age	Mathematics Qualification	Years of teaching	Use of DT in teaching
C1	F	31-40	BSc	<5	Seldom
C2	F	31-40	BSc	<5	Seldom
C3	M	31-40	BSc	5-10	Never

Table 1: Group C Teacher Demographics.

For their task, which was video- and audio-recorded, Group C used GeoGebra to work on the sign of the graph of $f(x) = ax^2 + bx + c$ when the discriminant is negative. The research design involved three stages. The first stage began with a Likert-style questionnaire about attitude with five subscales: confidence in teaching mathematics; confidence using with DT; the value of DT in teaching/learning mathematics; attitude to teaching mathematics with DT; and confidence in task development with DT. Some examples of the questions addressing confidence in DT task development included: "I prefer to use digital technology tasks developed by other people"; "It is worth devoting time to task development with digital technology": "I feel more comfortable in designing tasks using digital technology with other teachers who are good at it". Semi-structured interviews with the teachers followed to seek further clarification and then each group attempted their initial design of the task. The second stage, a task development intervention by the lead author, discussed some theoretical principles of rich DT task design (e.g., Kieran & Drijvers, 2006), what the criteria for a rich mathematical task comprised (Cennamo, Ross & Ertmer, 2014) and an example of a quality DT algebra task. The intervention also included instruction on planning a lesson using DT, based on Schoenfeld's (2010) theory of the role of resources, orientations and goals (ROG). To assist the designing of a suitable task emphasising student thinking, a three-point framework for lesson planning, delivery and review (Choy, 2013) was introduced. This focused on key concepts, possible points of difficulty, and the proposed course of action. In the third stage, the teachers were given an opportunity to modify their task based on what they just learned and a group interview followed focusing on how they had planned their task, how it worked in practice, modifications they made and what factors influenced the process. One teacher implemented it in their classroom, observed by the other two and the researcher. A final discussion with the researcher was held and further modifications made if necessary. Copies of the tasks were collected for analysis and the teachers completed a final questionnaire.

RESULTS

Group C chose to design a task to help students understand how the variation of a graph of a quadratic function of the form $f(x) = ax^2 + bx + c$ is related to the values of *a*, *b*, *c* and the discriminant, Δ, and these constituted the key points of their task. In their initial attempt designing the task, the teachers did not really consider DT. Instead, it read like a set of teacher notes comprising the details of completing the square, to end up with $f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$. They concluded "When $b^2 - 4ac < 0$, $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$ is always positive. Then the sign of $f(x)$ will be the sign of a." They did not present the case when $\Delta > 0$. This was followed by the direction "Divide the class in 2 groups and guide them to observe the behaviour of the graph. Each group will get one condition given below [namely $a < 0$ or $a > 0$]." and two sets with three explicit functions in each. No indication of how the students were to be guided or how they would "observe the behaviour of the graph" was given. The group's final task, after the intervention and modification, is in Figure 1. Changes are apparent from the first line, specifying the use of GeoGebra and directing students to observe the graph as *a* varies. Although not mentioned in the written task the lesson plan showed that they expected students to use sliders to change the variable values to get the different graphs. Their final task now focused on understanding the mathematical ideas of the relationship between the sign of *a* and the concavity of the graph, the effect of the discriminant on the number of zeros, the effect of *a* on the sign of the function when $\Delta < 0$ and use of the axis of symmetry.

Worksheet

1. Draw a rough sketch of the graph of the function $y = ax^2 + bx + c$ using GeoGebra
2. Observe the variation of the graph when the value of *a* changes.
3. How does the maximum and the minimum of the graph changes with the sign of *a*?
4. Get the value of $b^2 - 4ac$ for the values of a, b, c in the "Algebra view".
5. Change the values of *a, b, c* and observe the sign of the discriminant and observe whether the graph cuts the *x*-axis or touches the *x*-axis or neither cuts nor touches the *x*-axis.
   What is the sign of the discriminant when the graph cuts the *x*-axis at two distinct points?
6. What is the value of the discriminant when the graph touches the *x*-axis?
7. What is the sign of the discriminant when the graph neither touches nor cuts the *x*-axis?
8. (a) Using the method of completing the square rearrange the equation of the function $y = ax^2 + bx + c$ to get the above results algebraically.
   (b) Draw the axis of symmetry using the input bar of Algebra view.
   (c) What is the sign of $\left(x + \frac{b}{2a}\right)^2$ for all real values of x?
   (d) What is the sign of $\left(\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a^2}\right)$ for all real values of x when the sign of $(b^2 - 4ac)$ is negative?
   (e) Then, the sign of *y* changes according to the sign of *a* as:
   When *a* is positive the sign of *y* is _____________.
   When *a* is negative the sign of *y* is _____________.
   (f) Write down how the graph changes with 'a' when Δ is negative. (Change the values of *b* and *c* to get negative values for Δ).
   When Δ is negative:
   - Graph lies above/below the *x*-axis when *a* is positive.
   - Graph lies above/below the *x*-axis when *a* is negative.
9. Fill the blanks using the observations of the graph and the results obtained from rearranged function.
   a. If '*a*' is positive and $b^2 - 4ac$ is negative then the function is __________ for all real values of x.
   b. If '*a*' is negative and $b^2 - 4ac$ is negative then the function is __________ for all real values of x.

Figure 1: The final task produced by the group.

A key difference post-intervention was that the instructions now explained how to investigate different graphs, observe any changes, draw conclusions and generalise them. Further, by hand work was carefully and meaningfully integrated with the DT work in the task. In Table 2, the scores for each factor in the Task Richness Framework (developed from Kieran and Drijvers, 2006, Leung and Bolite-Frant, 2015 and our own ideas) are presented with brief explanations of the reasons. The richness metric increased from 5/36 for the first task to 28/36 post-intervention. A paired sample *t*-test of the factor scores shows that there was a highly significant improvement in richness of the final task ($t = 1.796, p < 0.0001$). We now consider factors that may have contributed to improved quality of the DT task the teachers designed.

Seven factors were considered for analysis: confidence in teaching mathematics (CTM); confidence in using DT (CUDT); value of DT in mathematics teaching/learning (VDT); attitude to teaching with DT (ATDT); confidence in task development with DT (CTD); a teaching focus on the three key points; and group dynamics. The changes in the aggregate scores of the group's attitudes measured using

Likert scales (1-5) are shown in Table 3. A Wilcoxon signed-rank test was applied to the individual scores in the five attitude subscales above.

Principles of Rich Tasks	First Task		Final Task	
	Evidence	Score (0-3)	Evidence	Score (0-3)
Focuses on mathematical ideas, e.g. epistemological obstacles	Behaviour of the graph when Δ is negative, completing the square, sign of the function	2	Good: Variation of the graph with the sign of *a*. Sign of the graph when Δ is negative. Completing the square.	3
Considers the role of language & discourse	Very little: 'behaviour' without support	1	Words such as 'discriminant'; 'completing the square', 'zeros'	2
Students written interpretations	No evidence for students' interpretations	0	Students' are asked to fill the blanks in statements by observing the graphs	3
Goes beyond routine methods	Observe behaviour of the graph	1	Sufficient. Students are guided to think logically about the sign of '*y*' when Δ is negative and when a>0 and a<0	3
Encourages student investigation	No evidence	0	Students identify the sign of the graph of function by themselves.	2
Multi-representational aspects	Involves graphs and algebra	1	Very good. Graphs, algebra, filling the blanks. Observe the changes in the algebra view	3
Appropriate for student instrumental genesis	No evidence	0	Students use sliders to get different graphs under the given conditions.	3
Instrumental feedback	No evidence	0	Observe the graph's concavity and relative position to axes and relate to *a* and Δ.	3
Integration of DT and by-hand techniques	Not mentioned that students use any DT	0	Good. Use GeoGebra to draw the graphs and observe the changes of Δ. Complete the square and fill the blanks by hand.	3
Aims for generalisation	No evidence	0	Good. Sign of '*y*' when Δ is negative depends on *a*.	3
Students think about proof	No evidence	0	No evidence	0
Develops mathematical theory	No evidence	0	No evidence	0

Table 2: Pre- and post-intervention task scores using the Task Richness Framework.

Two of these increased significantly, namely, confidence in teaching mathematics (CTM) ($N = 7$, $W = 0$, $p<0.05$) and attitude to teaching with DT (ATDT) ($N = 8$, $W = 0$, $p<0.05$). Although the increments in confidence in using DT (CUDT) and confidence in task development with DT (CTD) were not statistically significant, it should be noted that Group C had the highest confidence scores of the four groups in these two areas, both initially and at the end. Our results suggest that confidence is likely a critical factor for teachers designing and developing rich DT tasks and using them to teach mathematics.

	CTM	CUDT	VDT	ATDT	CTD
Pre-intervention	22.1	26.0	27.6	26.4	24.5
Post-intervention	26.4	26.8	28.4	29.6	25.0

Table 3: The groups' attitudes scores pre- and post-intervention.

Additionally, the teachers' use of the three-point framework (the key, difficult and critical points) assisted them in making their final task more student-centred. For example, they put themselves in the students' position to consider difficulties they might face engaging with the task and the measures needed to overcome or minimise these. One example of a difficult point was the challenge of getting a graph that touched the *x*-axis by changing the sliders for *a, b, c*. We see their discussion of this from a student perspective in this interchange, along with the critical point of increment size needed to overcome the problem:

C2: Yes they will get the sign of the discriminant when the graph cuts the *x*-axis at two points. Then, 'what is the value of the discriminant when the graph touches the *x*-axis?' Then s/he can do that.

C3: Either change *c*. Yes, and should get when it exactly touches.

C2: It's difficult.

C3: Yes its bit difficult to get that point. That's the problem. Little bit more. Down. The students might get the idea I guess. Because when the graph changes from positive to negative there should be a place it becomes zero. Students might get that idea.

C2: Yes, it is bit difficult to get that. If we change this to 0.5s.

C3: Ah yes the increments isn't it? [C2 & C1: Yes.]

C3: Yes we have to design it. If it will be in small increments it will work.

They also considered the difficulty that the graph might look as if it were touching the axis but in reality was not. Having tried it first, the teachers realised that the students would need to zoom out on the graph to understand this critical point.

C2: But if a student sees this s/he will consider this as it is touching.

C3: Yes but not it's not touched exactly.

C2: Yes. But it looks like it's touching…

C3: It can be seen if we zoom out this. …

C1: Yes now you can see. Look it cuts the axis. If a student asks about the graph in this situation we can help them. We can show them it is not yet touching.

Another difficult point the teachers raised was how students might think there should be a case when $\Delta = 0$, since the graph changes from positive to negative and so must pass zero. Their strategy to overcome this was to have the values $a = 1, b = 2 \text{ and } c = 1$ ready during implementation as they had checked that $\Delta = 0$ for these.

Group C was homogeneous for age, experience, instrumental genesis and mathematical knowledge for teaching, which might have positively influenced their confidence in presenting their ideas to each other. Unlike the other groups, they easily worked collaboratively, with all three teachers contributing equally. While sometimes C1 played the role of an unofficial leader, at other times C3 took over that role. In the transcripts, the voices of all three teachers can be heard contributing equally to the design process. Thus the group dynamic of equal contribution by all three members may have helped them to develop a richer task.

DISCUSSION

The analysis above suggests that it is possible to improve the nature of the DT tasks that teachers construct through an intervention that stresses theoretical features of a rich DT task, lesson planning for implementation of DT tasks and a student-centred framework focussing on difficult and critical points. We have also identified some of the crucial factors that appear to be behind this richer task development. In a partial confirmation of the results of Thomas and Palmer (2014), two positive factors that significantly increased after the intervention were the teachers' confidence in teaching mathematics coupled with their attitude to teaching with DT. We propose that the supportive nature of the intervention along with its theoretical basis contributed to an increased level of confidence and a more positive attitude to the DT. Secondly, the three point framework has been shown to be a practical means to promote productive noticing by directing teacher attention to the relevant mathematical details of critical incidents (Choy, 2013). Indeed, introducing the teachers to this framework encouraged them to develop a student-centred approach to their task. This took into account possible student difficulties that might arise in the implementation and helped them to generate targeted teaching strategies that could be employed to overcome them. Thirdly, the approach of forming small communities of inquiry, where teachers work collaboratively on task construction may be more productive in terms of rich DT task production if the group makeup is such that it enables interactions that give rise to equal contributions by each of the members.

Finally, although the task the teachers developed after the intervention was richer they were still primarily employing DT as a partner (Goos, Galbraith, Renshaw, & Geiger, 2000). Used in this way, DT provides access to new ways of approaching existing tasks to develop understanding and mediate mathematical discussion, but the teachers had not reached DT use as extension of self (*ibid*), where it is seamlessly integrated into the teacher's mathematical and pedagogical activity. The group's task design process also incorporated operational and pedagogical themes: Ambience enhanced; Restraints alleviated; Engagement intensified; Routine facilitated; Activity effected; Feature accentuated; Attention raised and Ideas established, as described by Ruthven and Hennessey (2002). We feel confident that further iterations of the design experiment would help the teachers to identify other themes and achieve use of DT as an extension of self.

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EXPLORING THE POSSIBILITIES OF ONLINE ASSESSMENT OF EARLY NUMERACY IN KINDERGARTEN

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The aims of the study are to develop an easy-to-use online test for early numeracy, to empirically validate the instrument and to examine the effects of ICT familiarity on early numeracy achievements. The research design comprised of online and face-to-face measures with 30 children from age five to six. The newly developed early numeracy and ICT familiarity tests were administered online. Face-to-face measures included basic counting and numeracy, and relational reasoning. Results revealed that the online early numeracy test was reliable and it was strongly correlated with the face-to-face measures. There was ceiling effect on the ICT familiarity test although it also correlates with early numeracy results. Findings of our study indicated that our online test can become a useful, easy-to-use educational tool to assess early numeracy and provide valuable information for teachers to design their teaching process.

INTRODUCTION

Mathematics achievement highly depends on the successful acquisition of early numerical skills therefore assessment of these skills in kindergarten is inevitable to diagnose difficulties in time and prevent children falling behind. However, carrying out testing on a regular basis in kindergarten is cost and time consuming since it has to be done by traditional face-to-face assessment methods. Technology-based assessment could be the solution to overcome these difficulties by providing the possibility of developing easy-to-use assessment instruments for kindergarten teachers (Csapó, Molnár, & Nagy, 2014).

EARLY NUMERACY

The construct of early numeracy comprises several basic skills and concepts (Jordan, Kaplan, Locuniak, & Ramineni, 2007). Number word sequence skills are an important basis of other early mathematical skills. The knowledge of the correct order (forward or backward) of number words is essential to the development of enumeration skills, and it has significant role in solving basic additions and subtractions (Aunio & Rasanen, 2015). Enumeration is also an important component of the early numerical skills. It is related to the cardinal meaning of numbers when children identify the last number of the sequence with the number of the element they counted (Aunio & Niemivirta, 2010; Aunio & Rasanen, 2015). Basic counting skills develop swiftly after children are aware of number word sequences and understand the cardinal number

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concept. They are able to solve additions, subtractions and later they understand the part-part-whole concept as well (Fritz, Ehlert, & Balzer, 2013; Resnick, 1992). Another essential component of early numerical skills is the knowledge of number symbols. Numeral recognition and number identification are the two segments of this factor. At the beginning children learn Arabic numerals then they will be able to identify and read numbers. Establishing connections between Arabic symbols and quantities are also important basic skills of numeracy. Studies have shown the numeral knowledge as a strong predictor of later formal mathematical achievement (Purpura & Napoli, 2015). There are several standardised mathematical test batteries to measure the numerical skills of children from age four to eight (e.g., the Utrecht Test of Early Numeracy – ENT; Early Numeracy Test – WENT) (Aunio & Rasanen, 2015). In Hungary a diagnostic test battery called DIFER (Diagnostic System for Assessing Development for four- to eight-year-old children) is widely used to assess key skills for school readiness (Nagy, Józsa, Vidákovich, & Fazekasné Fenyvesi, 2004). These instruments share the same characteristics: they require resource and time consuming face-to-face test administration and the educators' proper qualification is also necessary otherwise the objectivity of the measurement can be compromised.

TECHNOLOGY-BASED ASSESSMENT IN EARLY CHILDHOOD

Technology-based assessment is an umbrella term and it refers to use any technological solutions during the testing process. Online assessment is a narrower term where test administration and data processing is carried out on computers through internet (Jurecka & Hartig, 2007). Over the past decades there have been a growing interest for technology-based assessment in educational context due to its advantages over traditional assessment formats such as the opportunity to present more stimulating innovative items (e.g. using sounds, pictures and videos, interactivity), to apply automatic feedback and to manage data processes more effectively (Csapó, Ainley, Bennett, Latour, & Law, 2012). All of these features contribute to the implementation of testing young students and carrying out large scale assessments as well (Csapó, Molnár, & Nagy, 2014; Molnár & Pásztor, 2015). For instance, we can apply pre-recorded instructions and design items with the possibility of manipulation which is essential in the development of skills in young ages. In contrast with traditional face-to-face methods technology opens the way for testing more children at the same time. To conclude if the infrastructure is available (e.g. tablets and internet connection) we can provide easy-to-use assessment instruments for kindergarten teachers to identify children difficulties and to improve the quality of their teaching (e.g. fitting their teaching methods for the actual level of students' knowledge and skills). However, there are many concerns as well regarding the validity and reliability of these technology-based instruments in early childhood. For example the relationship has to be explored between the results from using face-to-face assessment tools and online tests to ensure the validity of the online instruments (Csapó, Molnár, & Nagy, 2014). In addition, the level of children ICT familiarity may influence the achievement scores in the targeted construct. Therefore we have to ensure that children have the necessary ICT skills to provide the answers for the tasks (Molnár & Pásztor, 2015).

AIMS OF THE STUDY

The aims of the study are (1) to develop an easy-to-use online test for early numeracy and to analyse the psychometric properties of the test; (2) to empirically validate the instrument and (3) to examine the effects of ICT familiarity on early numeracy achievements.

METHOD

Participants

Our assessment took place in two kindergartens with the participation of 30 children. The sample consisted of 15 boys and 15 girls between the age of five and six (M age=5.7 years SD=.22). Informed consent form about the research was given to the parents.

Instruments

In our newly developed online test children could solve the tasks through manipulation: they had to drag and drop objects or select the right solution by tapping on it. Items were designed with respect of range of interest of the targeted age cohort (see figure 1). Children listened to instructions through headphones, which were reviewed by experienced Kindergarten teachers. There were no letters to read on the task pages. Children could listen to the instructions as many times as they wanted by tapping on the speaker icon. The test comprised of 40 items and included five subtests; Basic counting, Number word sequence, Numeral recognition, Magnitudes and numerals and Relations. The *Basic counting* subtest contains manipulative tasks, addition, and subtraction of magnitudes, and tasks related to the part-part-whole concept. Within these tasks children need to add, take away or sort the right amount of magnitudes. The *Number word sequence* subtest measures whether children can recognize a correct forward or backward number word sequence. They hear a sequence of three numbers, then they can listen to three possible conclusions and they need to decide which is the correct one (Figure 1). In the *Numeral recognition* subtest children need to recognize Arabic numbers with one, two and three digits. They select the right card out of four that shows the number what they hear. In the *Magnitudes and numerals* subtest children manipulate magnitudes based on the number they hear or see. Tasks with smaller amounts are solved by drag and drop technique (Figure 1) but tasks with larger amounts can be solved by the selection of the right picture of three different magnitudes. Tasks of the *Relations* subtest measured whether children can compare number sets and find the larger, largest or smaller, smallest quantities.
In order to provide possibilities for practising the tapping and drag and drop operations and to familiarize children with the test environment they also completed an ICT familiarity test before the early numeracy assessment. The instrument consisted of 16

items (Figure 1). To maximize the training effects children had a second chance to solve these items in case of failure.

Figure 1: Sample items for the ICT familiarity and the early numeracy test (from the left to the right). ICT familiarity, instruction: '*Drag the matchboxes to the shelf and the balls to the carpet*.' Number word sequence, instruction: '*Help Pete decide which animal continues the counting correctly. Pete starts the counting every time. If you click on the speakers next to the animals you can hear how they continue the counting. Click on the animal which continues the counting correctly. Click on the speaker next to Pete and he starts the counting'*. Magnitudes and numerals, instruction: '*You can see a number on the card. Drag as many ducks into the lake as the card shows!*'

To validate our online assessment we used two tests (counting and basic numeracy, relational reasoning) of the Hungarian DIFER test battery. The counting and basic numeracy skills test included 38 items. 14 items were intended to assess the knowledge of the number word sequence forward and backward (e.g., count up to 21), 11 items were related to manipulative counting skills (e.g., 'Here are six sticks. Make it ten.'), 9 items aimed to assess counting number sets (e.g., 'Show me the card with five drawings.'), 4 items assessed the ability to read one, two and three digit numerals (e.g., 3, 22, 118) (Nagy et al., 2004). The test was reliable (Cronbach's alpha = .80).

The relational reasoning test assessed the understanding of words which stand for relations between different objects, attributes or processes. It has four equivalent test versions; each of them contains 24 items of 24 relation words. We used the first variant of the test which had eight words connected to spatial relations (e.g., inside, in front of), four items determining quantity (e.g., few, many), four words indicating actions (e.g., step in, step on), four items related to time (e.g., night, afternoon) and four relational expressions (e.g., the longest, the same length) (Nagy et al., 2004). Verifying the reliability of the test two items were excluded from further analyses. For the remaining 22 items Cronbach's alpha = .62.

Procedure

Data collection by the online early numeracy test and ICT familiarity test were administered through Internet via eDia (Electronic Diagnostic Assessment) online assessment platform (Csapó, Lőrincz & Molnár, 2012) on tablet computers. The tests were carried out in the kindergartens in groups of four or five children supervised by kindergarten teacher candidates. Their assistance was expected only in case of technical difficulties. Face-to-face assessments of the DIFER test were also organized in separated rooms in the kindergartens. The DIFER test is usually administered by kindergarten and primary school teachers but in our research we trained kindergarten teacher candidates to carry out the assessments. 15 children completed the online tests first, the other half of the sample started with the face-to-face measurements in order to precede effects of which test they complete first. Assessments were carried out during the first two weeks of December 2016. Beside the quantitative measures we used video observation as well, the analysis of the data is still in progress.

RESULTS

Reliability and the average performance on the online early numeracy test and its subtests are listed in *Table 1*. Due to the low reliability value (Cronbach’s alpha < .3) of the Relations subtest we excluded its 6 items from further analyses. The test with the remaining 34 items proved to be reliable (Cronbach’s alpha=.88). The reliability of the subtests was still acceptable apart from the value of Numeral recognition. The average achievements on the online early numeracy subtests were over 50% except the Number word sequence where the mean score was the lowest (31.82 %p; SD=24.32; *Table 1*). Large standard deviations indicate that the test had good differential power and they also refer to large individual differences (*Table 1*).

Subtests	Number of items	Reliability (Cronbach’s alpha)	%p (SD)
Basic counting	11	.75	74.85 (20.21)
Number word sequence	11	.75	31.82 (24.32)
Numeral recognition	6	.66	60.00 (27.19)
Magnitudes and numerals	6	.77	77.22 (28.19)
Early Numeracy Total	34	.88	58.73 (19.21)

Table 1: Reliability and the average performance on the early numeracy test and its subtests

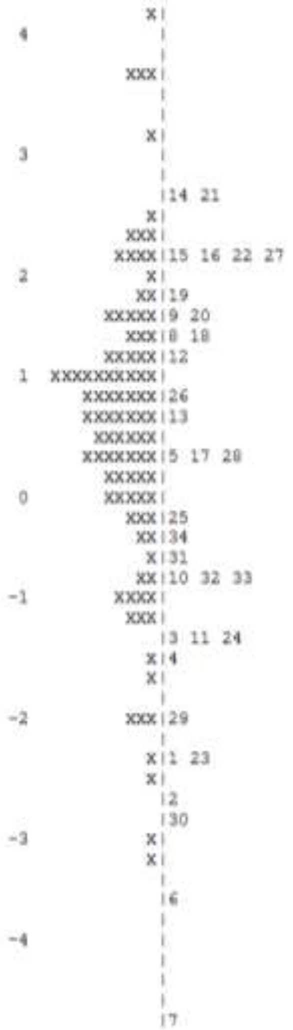

Figure 2: Person item map for the early numeracy test. Each 'X' represents .3 cases.

One-parameter RASCH analyses was also carried out in order to gain a more detailed picture about the behaviour of the items. The EAP/PV reliability was good: .91. *Figure 1* shows that in general the items were covering the different skill levels. However, the pattern is not balanced. In case of some skill levels further item analysis would be necessary.

We found significant correlations between the achievements on the subtests which provide some empirical evidence for construct validity (*Table 2*). Further support for validity is the strong correlation between the face-to-face Counting and basic numeracy test and the online early numeracy test (r=.84). In addition, the online subtests are also connected to the face-to-face test results.

The reliability of the ICT literacy test was low (Cronbach's alpha= .40). Possible reasons for this might be the training (i.e. the second chance for solving the items in case of failure) or the ceiling effects (M=91.5% SD=7.6%; the lowest score was 75% and the highest was 100% in case of 9 items out of 16). The high achievements indicate that children had no difficulties in handling the tablets and providing answers for the tasks. However, there are positive correlations between the early numeracy test results and the ICT familiarity scores. But this result has to be interpreted in the light of the finding that ICT familiarity also correlated with face-to-face test results.

Measure	1	2	2a	2b	2c	2d	3
1 ICT familiarity	-						
2 Early Numeracy	$.47^{**}$	-					
2a Basic counting	$.48^{**}$	$.82^{**}$	-				
2b Number word sequence	.32	$.79^{**}$	$.46^{*}$	-			
2c Numeral recognition	$.51^{**}$	$.80^{**}$	$.65^{**}$	$.45^{*}$	-		
2d Magnitudes and numerals	.18	$.75^{**}$	$.49^{**}$	$.45^{*}$	$.57^{**}$	-	
3 D. Counting and basic numeracy	.34	$.84^{**}$	$.63^{**}$	$.49^{**}$	$.81^{**}$	$.84^{**}$	-
4 D. Relational reasoning	$.48^{**}$	$.60^{**}$	$.45^{*}$	$.51^{**}$	$.39^{*}$	$.51^{**}$	$.46^{*}$

Table 2: Correlations between the measured constructs. *Note.* ** *= p<.01;* * *= p<.05; D.= DIFER*

DISCUSSION

The current study is part of a longer test development process. Our intention is to create an easy-to-use online test environment which grabs and maintains young childrens' attention while provides reliable and valid information about the current state of the early numerical skills. The present version of our online early numeracy test proved to be reliable, even the subscales had acceptable reliability. However, items of important subscales such as Relations and Numeral recognition need to be revised. IRT analyses also showed the potential for further item development.

The correlations between the subscales and also the relation of our online test results to the face-to-face test performances provided empirical evidence for validity. The strong correlation between the face-to-face counting and basic numeracy test and the online early numeracy test indicates that we succeeded to measure a nearly equivalent construct of early numeracy compared to the one in the face-to-face assessment. However, further item analyses and item to item comparisons between face-to-face and online measures are necessary to strengthen these claims.

In addition, we have some concerns related to ICT familiarity which raise some questions regarding validity. In spite of the ceiling effect on the ICT familiarity test we still found positive correlations between early numeracy and ICT familiarity test achievements. This finding can lead to a conclusion that we have validity problems. However, the ICT familiarity test scores were also positively correlated with the face-to-face test results. This relation pattern could be interpreted in a way that ICT familiarity test measures not only ICT familiarity but something else as well. It might be related to social background or also attention or motivation of the students. Further research should focus on investigating these assumptions. In addition, the examination of the role of ICT literacy could be integrated with qualitative measures such as the analyses of the data from video observation. Nevertheless, our findings clearly represented the necessity of a prior ICT familiarity testing in early childhood in case of any online assessment.

To conclude, by further improvements our online early numeracy test can become a useful, easy-to-use educational tool to assess early numeracy and to provide valuable information for teachers to design their teaching process. Even our current version is suitable for everyday use if the infrastructural background is given (e.g. tablets and internet connection). Our findings indicate that online assessment even in early childhood has many advantages. However we need to be very careful when carrying out the measurements and interpreting the results.

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MANIPULATING FRACTIONS: EFFECTS OF IPAD-ASSISTED INSTRUCTION IN GRADE 6 CLASSROOMS

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Research suggests that hands-on activities may support students' acquisition of mathematical concepts. Understanding the use of iPads, which offer specific options for manipulating objects, was addressed in a research project on fractions. The project aimed at defining and evaluating a learning environment (iBook) based on students' active manipulation of different representations of fractions. During a four-week intervention 474 grade six students participated in one of three groups. Both treatment groups (iPad, paper-based) scored significantly better in a posttest than the control group. However, there was no significant difference between the treatment groups. The results provide evidence that students could benefit from a manipulation-oriented learning environment but were not additionally fostered by using the iPad.

INTRODUCTION

There are numerous aspects which have proven to be meaningful for successful learning. Defining a learning environment asks for choosing among them in order to implement a coherent pattern. The learning environment for fractions presented here relies primarily on an approach, which takes students' active participation in the mathematics classroom and their individual knowledge base into account.

Understanding mathematics is seen to be based on students' active engagement in mathematical activities. In particular, for young students concrete manipulations seem to be beneficial when they are asked to acquire mathematical concepts. However, there is a variety of ways how to perform concrete manipulations. In the last few years, mobile electronic devices gained importance as they allow for focused and systematic hands-on activities. Moreover, successful learning presupposes that concepts are integrated in a student's prior knowledge. Accordingly, adaptive systems that regard this status should lead to better learning results than conventional textbooks.

Fractions in the Classroom: Characteristics of a Learning Environment

It is difficult for students to understand the concept of fractions. In the last decades, many research studies provided insights in students' problems with this topic. The idea of a "conceptual change" turned out to be particularly fruitful: Students need to revise their concept of numbers – well-established with respect to natural numbers – in order to prevent overgeneralization. So Vamvakoussi and Vosniadou (2004) investigated ninth graders' understanding of density as a structural property of rational numbers:

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 97-104. Singapore: PME.

They could show that some students' mistakes were based on their limitation of the concepts to discrete subsets of rational numbers. Furthermore, Obersteiner, Van Hoof and Verschaffel (2013) found out that a natural number bias did not only affect novices in comparison tasks. Their results show that even expert mathematicians needed more time to judge fractions with identical nominators ("incongruent" with respect to natural numbers) than with identical denominators ("congruent"), though in the end they gave the correct answers.

Moreover, Winter (1999) identified misconceptions about rational numbers which could easily be assigned to concepts of natural numbers as they are provided in primary school mathematics, such as *cardination* ("numbers are answers to the question: how many?"), *uniqueness of number and symbol* ("one symbol represents exactly one number."), and *discrete order* ("each number has a successor").

When educators wish to make students revise their established concepts, four conditions are beneficial according to Posner, Strike, Hewson, and Gertzog (1982): (1) students must struggle with tasks which cannot be solved using their established ideas, (2) the new concept must be intelligible, (3) using the new concepts must lead to solutions of not-yet solvable tasks, and (4) the new concept should be usable in a broader context.

The conceptual change approach can be helpful in general to gain insights in how students' difficulties with fractions can be assessed. However, theoretical considerations lead to specific aspects of fractions ("mental models"), which need to be taken into account. According to Padberg (2009) developing knowledge on fractions presupposes that students establish the aspects *part of the whole* (especially the equivalent meaning of "¾ of 1" and "¼ of 3"), *expanding and reducing* (in the meaning of: to refine or to coarsen a given division), and *comparing the size of fractions* (regarding an understanding of density on an elementary level, such as a fraction "having no successor"). According to Charalambous and Pitta-Pantazi (2005), in particular *part of the whole* plays a fundamental role in developing an understanding of fractions. Their study provided empirical evidence that fostering other mental models of fractions related to *part of the whole*.

Moreover, fractions in the classroom should address the aspect of *visualization* (creating and modifying representations as well as changing between different forms of representation), and the aspect of *calculation* (applying mathematical definitions, rules and algorithms as well as solving mathematical tasks on a symbolic level) to foster the understanding of rational numbers (Padberg, 2009).

The iPad as a Teaching Tool

There are only few empirical findings on mobile electronic devices in mathematics education, however, some results encourage their use in the classroom. So, Blair (2014) reports on a better learning of the natural number concept in primary school. As another example, Black, Segal, Vitale, and Fadjo (2012) could show that first and second graders obtained better results in addition tasks after learning the rules of addition on the iPad compared to students who worked with the same software on a computer. They argue that the difference in the learning outcome might be justified by how students handle the two different digital devices: Students operate the iPad by using natural gestures while the input via a mouse is more or less fixed. This argumentation follows the *embodied cognition* theory (Gangopadhyay & Kiverstein, 2009).

Besides that, students may benefit from diverse features when using digital devices in general: Software can be designed to give short and accurate *feedback* referring to the assignment immediately after completing the task (Hattie & Timperley, 2007), and to *adjust difficulty of tasks adaptively* to students' individual skills, which has shown to be more effective than a non-adaptive increase of difficulty in computer games (Sampayo-Vargas, Cope, He, & Byrne, 2013).

Study Overview and Research Questions

The characteristics mentioned above were taken into account while implementing a learning environment for fractions (Hoch, Reinhold, Werner, Reiss, & Richter-Gebert, 2016). In particular, this learning environment takes into account that students gain a deeper understanding of fractions when their need for revising established concepts about natural numbers is addressed directly. Furthermore, the mental models of fractions (*part of the whole*, *expanding and reducing*, and *comparing the size*) are integrated in this environment.

The research described above suggests that the iPad is a good teaching tool to aid this process, as it allows to stimulate both the cognitive and the sensomotoric area of the brain simultaneously, to give students detailed feedback on their mistakes, and to change the difficulty of tasks adaptively. With this in mind, an interactive textbook for the iPad (iBook) was developed. The following research questions were addressed:

1. Will the use of an iPad as learning environment and/or equivalent paper-based material enhance the students' knowledge on fractions if compared to regular classroom instruction?
2. Will the use of an iPad as learning environment and/or equivalent paper-based material support the development of visualization and calculation in sixth graders differently?

It was assumed that both the iPad-assisted learning environment and the paper-based material would have a positive effect on the learning outcome. Furthermore, it was hypothesized that working with the iPad has an additional positive effect on the learning outcome in general and the development of visualization specifically.

METHODOLOGY

Participants

A total of 474 sixth grade students (214 female, 260 male) from 19 classrooms took part in the study. Students were assigned to two treatment groups – the *iPad* group (n =

155) working with the learning environment on the iPad, and the *book* group (n = 182) working with equivalent paper-based material – and to a *control* group (n = 137) working with conventional textbooks.

Material

To investigate the influence of digital media in teaching fractions, an interactive textbook was created using iBooks Author (Apple Inc., 2017) as framework and CindyJS (The CindyJS Project, 2016) as programming language for the interactive content. The part of the book used during this study contained six topics, analogous to the curriculum for sixth grade mathematics: representing fractions with pictures and mathematical symbols, understanding fractions as a part of a whole, expanding and reducing fractions, fractions as marks on the number line, conversions between fractions and mixed numbers, and comparison of the size of fractions. Each topic consisted of an introductory part, a summary of the new content, and exercises to reinforce the material. Illustrative examples were used in the introductions, e. g. distributing pizza to three persons. New tasks were created via an adaptive algorithm. Levels of difficulty were defined for each exercise based on difficulty-generating characteristics (Eichelmann, Narciss, Schnaubert, & Melis, 2012). If a certain number of tasks in one stage was solved correctly the participant proceeded to the next level. In addition, mistakes were not only corrected, but revised with detailed feedback, stating *what* was wrong and *how* it would have been done correctly. Whenever possible, this feedback addressed mistakes not only at a symbolical level, but also at a graphical level. A total of 88 interactive exercises were used within the iBook. The paper-based version was created as an exact copy. For this softcover book, two to three tasks from each level of difficulty were chosen randomly for each of the 88 exercises.

Two tests were designed and used in this study, both had been piloted before in a small study: the pretest made use of some items from a former test (cf. Padberg, 2009) and was presented to 142 students at the end of grade five, the posttest was presented to 257 students at the end of grade six. The pretest on prior knowledge of fractions was found to be highly reliable (10 items; Cronbach's α = .82). Posttest items were created according to the mental models presented above (*part of the whole*, *expanding and reducing*, *comparing the size*) and the mathematical competencies of *visualization* and *calculation*. The posttest as measure for basic knowledge on fractions was also found to be highly reliable (20 items, α = .82). For the analysis, this test was divided into two subscales on the *competency* dimension: the visualization subscale consisted of eleven items (α = .68), and the calculation subscale consisted of nine items (α = .72). As the items for these two complex constructs were designed on a profound theoretical base, these α-values can be interpreted as acceptable for this study (Schmitt, 1996). Test scores and scores for the two constructs visualization and calculation are reported as solution rates between 0 and 1.

Procedure

The study took place during the first four weeks of the school year in grade six. Referring to the curriculum a total of 16 lessons was targeted. In fact, the teachers held around 15 lessons during the intervention (M = 15.2, SD = 1.1). Teachers in the treatment groups (iPad and book) were requested to use only the learning environment provided in their classrooms. Furthermore, teachers in the iPad group were asked to use the iPad as their main teaching tool. Each participant in the iPad group worked on his own device during the intervention. Students and teachers in both treatment groups received a printed version of the teaching material. Students and teachers from the control group did not receive any additional material.

Pretests (15 minutes) were executed at the beginning of the first lesson, posttests (55 minutes) were executed after the last lesson. Both tests were presented as paper-and-pencil tests for all participants. Both pretest and posttest were coded independently by two persons.

Different analyses were conducted using analysis of covariance (ANCOVA). The data sufficiently met statistical assumptions. Only homogeneity of variance was found to be violated, but as this assumption loses its relevance for big sample sizes and roughly equal sized samples (Glass, Peckham, & Sanders, 1972), it can be assumed that this will not alter the results of the ANCOVA to be described now.

RESULTS

There was a moderate, positive correlation between pretest scores and posttest scores using Spearman's rank correlation, r_s (472) = .57, p < .01. Therefore, the pretest score may be used as a covariate in statistical evaluations.

The first question was whether the teaching material had a positive influence on the knowledge gained during classroom instruction on fractions. In particular, students from both treatment groups (iPad, book) should have higher posttest scores than students from the control group. As Table 1 reveals, this could be confirmed. A one-way ANCOVA was conducted to determine differences between participants who worked with the material on the iPad, with the material in a printed version, or with conventional material in a regular textbook in the posttest score. Pretest scores were used for control. A main effect of the treatment was found, $F(2,470) = 7.95$, $p < .01$, $\eta_p^2 = .033$. This indicates the estimated effect of the teaching material. As can be seen in Table 1, participants from the iPad group had slightly lower posttest scores than participants from the paper-based book group. A post-hoc Tukey test showed that both treatment groups, iPad and paper-based, differed significantly from the control group, p = .01 and p < .01, though there was no significant difference between the two treatment groups, p = .70. So, it may be assumed that the teaching material supports sixth graders' knowledge acquisition on fractions better than conventional textbooks. However, these results seem to contradict the hypothesis that students might profit better from using the iPad than from using equivalent paper-based material.

Since there was no significant difference in the overall outcome after four weeks of learning between the two treatment groups, it was controlled whether the iPad had the desired effect on fostering the competency of visualization in terms of fractions. Table 1 shows that the paper-based treatment group had the highest scores. A one-way ANCOVA showed that there was a significant main effect of the treatment on visualization controlling for the pretest score, $F(2,470) = 18.24$, $p < .01$, $\eta_p^2 = .072$. A post-hoc Tukey test showed that both treatment groups, iPad and book, differed significantly from the control group, $p < .01$ and $p < .01$. Although there was no significant difference between the two treatment groups, $p = .86$. It may be assumed that the advantage of both treatment groups over the control group in terms of visualization arised from the teaching material, and not from the iPad used as a teaching tool.

Group	Pretest	Posttest	Visualization	Calculation
iPad	.50 (.29)	.67 (.17)	.72 (.18)	.62 (.20)
book	.50 (.29)	.69 (.19)	.73 (.19)	.64 (.24)
control	.45 (.29)	.60 (.21)	.60 (.22)	.59 (.25)
total	.48 (.29)	.66 (.20)	.69 (.21)	.62 (.23)

Table 1: Mean scores (and standard deviations) for different analyses

This result leads to the question if working on the iPad has a negative effect on solving basic fractional computation tasks. As Table 1 reveals, the iPad group in fact had a slightly lower mean solution rate in calculation than the paper-based treatment group. However, a one-way ANCOVA showed no significant effect of the treatment on calculation controlling for the pretest score, $F(2,470) = 0.63$, $p = .53$. Hence, it may be concluded that neither the teaching tool nor the teaching material had an influence on the development of calculation. In particular, there is no negative influence of working with the iPad on calculation.

DISCUSSION

The findings suggest that an enriched conceptual change framework, where the necessity to revise established conceptions about natural numbers fostered in primary school is addressed by building up mental models for fractions step by step is a profound framework to teach basic knowledge about fractions, as students of both treatment groups achieved significant higher scores in the posttest than students of the control group.

Furthermore, it is evident that these higher scores are due to significant better results in visualization tasks, but that there is no significant difference between the three groups when only calculation tasks are considered. Since these visualization tasks were designed to address mental models directly these findings indicate that students working with the teaching material have gained deeper insights into fractions during the intervention than students that were taught using conventional material. Further investigation of this aspect will be done by taking two additional items from the posttest into account, in which the participants were asked for explanations. These two items were not yet used.

However, the findings reveal that students did not benefit additionally from the iPad as a teaching tool. This result leads to new questions that need further investigation: as appropriate feedback as well as adaptive difficulties of tasks have shown to be beneficial for the learning process in different studies, one must ask why these aspects had no effect in the experiment. Regarding this, it can be asked whether students use feedback or whether they chose the next task immediately. It is planned to address this by looking at the data captured by the devices, as the period of time that feedback was displayed.

Another aspect that might have influenced the outcome is the lack of experience both teachers and students had with iPads as an instrument for learning. Since none of the attending schools had their own devices, teachers as well as students worked with iPads in classroom for the very first time. Interviews with the teachers after the study and an evaluation of these interviews might give deeper insights in how the teachers used the devices and whether they were satisfied with their first teaching approaches using iPads in the classroom.

Repeating the study in the upcoming school year with the same teachers in the iPad group and an initial lesson for the sixth graders about how to use the device right before the intervention could be one way to approach these open questions.

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WHO OWNS A THEORY? THE DEMOCRATIC EVOLUTION OF THE KNOWLEDGE QUARTET

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The Knowledge Quartet is a theoretical framework for the analysis and development of mathematics teaching. It focuses attention on classroom situations when the teacher's knowledge of mathematics and of mathematics-related professional knowledge comes to the fore. The Knowledge Quartet was first developed some 15 years ago in empirical research in the context of pre-service elementary mathematics teaching in the UK. Some details of the theory have evolved since that time, in response to its application and testing by researchers in other contexts, and through communication between ourselves and those researchers. In this paper we describe that process of evolution and pose related questions about the ownership and development of theories in general.

INTRODUCTION

The Knowledge Quartet is a practice-based framework for the identification and discussion of mathematics teachers' mathematics-related knowledge as evidenced in classroom practice. It originally emerged in 2003 from intensive scrutiny of 24 videotaped lessons, taught by novice teachers. This research, which began in 2002, was conducted in Cambridge UK by a team which included the two authors. The Knowledge Quartet is a 'theory' in the sense that it proposes a way of thinking about mathematics teaching in the usual institutional settings (lessons/classes), with a focus on the disciplinary content (mathematics) of the lesson. A prototype of the theory was outlined in Huckstep, Rowland and Thwaites (2003), and progressively more definitive versions announced in Rowland, Huckstep and Thwaites (2003, 2005). Since that time, researchers in several countries have used the Knowledge Quartet as a framework for the analysis of their own classroom data, and some have corresponded with us about their findings, and shared their experiences of the comprehensiveness of the theory in relation to their own data. In this paper, we describe a slow process of evolution of the Knowledge Quartet in response to this feedback, and conclude by posing some questions about the ownership and development of theories in general. But first, we outline the 'state of the art' where research into mathematics teacher knowledge is concerned, and the Knowledge Quartet in particular.

THE KNOWLEDGE QUARTET

The origins of the Knowledge Quartet were in observations of primary mathematics teaching, and grounded theory methodology (Glaser and Strauss, 1967), in the context of one-year graduate primary teacher preparation in England. A programme of

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empirical research at the University of Cambridge, UK, investigated teachers' mathematics-related knowledge, and the ways that this knowledge is activated and made observable both in their planning and in their teaching in the classroom. From this research, a practice-based framework for the observation, analysis and improvement of mathematics teaching was developed, with a focus on the contribution of the teacher's mathematics-related knowledge. While Shulman's distinction between subject matter knowledge and pedagogical knowledge underpins this consideration of mathematics teaching (Shulman, 1986), the Knowledge Quartet identifies three *categories of situations* in which teachers' mathematics-related ('foundation') knowledge is revealed in the classroom: these categories, or dimensions, of the Knowledge Quartet are named 'transformation', 'connection' and 'contingency'.

Dimension	Contributory codes
Foundation: knowledge and understanding of mathematics per se and of mathematics-specific pedagogy; beliefs concerning effective mathematics instruction, the nature of mathematics, and the purposes of mathematics education.	awareness of purpose; adheres to textbook; concentration on procedures; identifying errors; overt display of subject knowledge; theoretical underpinning of pedagogy; use of mathematical terminology
Transformation: the presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations	choice of examples; choice of representation; *(mis)use of instructional materials*; teacher demonstration (to explain a procedure)
Connection: the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks	anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness; making connections between procedures; making connections between concepts; *making connections between representations*
Contingency: the ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events	deviation from agenda; responding to students' ideas; use of opportunities; *teacher insight during instruction; responding to the (un)availability of tools and resources*

Table 1: The Knowledge Quartet – dimensions and contributory codes

Table 1 outlines these dimensions and their contributory codes, which arose from analysis of primary mathematics classroom data (Rowland et al, 2005). Each dimension is composed of a small number of cognate subcategories (codes) that we judged, after extended discussion, to be related. These codes capture observed teacher-actions in preparation and/or in classroom instruction. The 17 shown in 'normal' font emerged in our analysis of 24 lessons in 2002-013. Those shown in italics are the subject of this paper.

In the years since 2002, there has been a process of refinement of the conceptualisation of the Knowledge Quartet, and enhancement of the constituent codes, both in response to additional classroom data and in the process of application (Weston et al., 2013).

Conceptualising the Knowledge Quartet

The concise conceptualisation of the Knowledge Quartet which now follows is a synthesis of the characteristics of its four dimensions.

Foundation

The first member of the Knowledge Quartet is rooted in the foundation of the teacher's theoretical background and beliefs. It concerns their knowledge, understanding and ready recourse to what was learned in preparation (intentionally or otherwise) for their role in the classroom. Both empirical and theoretical considerations have led us to the view that the other three units flow from a foundational underpinning. We take the view that the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way. The key components of this theoretical background are: knowledge and understanding of mathematics *per se*; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt.

Transformation

The remaining three categories focus on knowledge-in-action as *demonstrated* both in planning to teach and in the act of teaching itself. At the heart of the second member of the Knowledge Quartet is Shulman's observation that the knowledge base for teaching is distinguished by " … the capacity of a teacher to *transform* the content knowledge he or she possesses into forms that are pedagogically powerful" (Shulman, 1987, p. 15, emphasis added). This category, unlike the first, picks out behaviour that is directed towards a pupil (or a group of pupils), and which follows from deliberation and judgement informed by foundation knowledge. The choice and *use of examples* has emerged as a rich vein for reflection and critique (Rowland, 2008).

Connection

The next category concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons. Our conception of connection includes the *sequencing* of topics of instruction within and between lessons, including the ordering of tasks and exercises. To a significant extent, these reflect deliberations and choices entailing not only knowledge of structural connections within mathematics itself, but also awareness of the relative cognitive demands of different topics and tasks, and the implementation of strategies to remove (or lessen) obstacles to learning.

Contingency

Our final category concerns the teacher's response to classroom events that were not anticipated in the planning. This dimension of the Knowledge Quartet is about the ability to ‘think on one’s feet’: it is about *contingent action.* Whilst the teacher’s intended actions can be planned, the students’ responses cannot. The teachers’ response to students’ unexpected contributions is one of the most low-inference codes of the Knowledge Quartet.

We have found that many moments or episodes within a lesson can be understood in terms of two or more of the four units; for example, a **contingent** response to a pupil’s suggestion might helpfully **connect** with ideas considered earlier. Furthermore, the application of content knowledge in the classroom always rests on **foundational** knowledge.

THE EVOLUTION OF THE KNOWLEDGE QUARTET

Since its initial development, the Knowledge Quartet has been put to the test as an instrument for mathematics lesson observation and analysis. This testing has taken a number of forms, including its application to various extents in doctoral studies, including the second author’s longitudinal study of the knowledge, beliefs and practices of early career elementary teachers (Turner, 2010). Although our experience to date indicates that the fundamental anatomy of the Knowledge Quartet is complete, we take the view that the details of its component codes, and the conceptualisation of each of its dimensions, are perpetually open to revision. This fallibilist position seems to us to be as appropriate for a theory of knowledge-in-mathematics-teaching as it is for mathematics itself. In grounded theory methodology, it is also inherent in the notion of ‘theoretical sampling’ (Glaser & Strauss, 1967), whereby the application of the theory exposes some shortcoming, and thereby lays it open to refinement, modification and possible improvement until (perhaps) it achieves saturation.

As a consequence of this process, and the possibility of electronic global communication, four additional codes have been added to the original 17: these new codes are shown in italics in Table 1. Either the teacher behaviours captured in these codes were absent in our 2002-03 video data, *or else* we failed to note it in our analysis of that data. The names of the four new codes, the Knowledge Quartet dimension which they enrich, the year in which they emerged, and the researchers who brought them to our attention, are as follows:

- *teacher insight during instruction* (Contingency). 2005, Dolores Corcoran - Ireland
- *(mis)use of instructional materials* (Transformation), 2006, Marilena Petrou - Cyprus
- *responding to the (un)availability of tools and resources* (Contingency). 2009, Libby Jared et al. - UK
- *making connections between representations* (Connection). 2015, Abraham de la Fuente - Spain

In the light of space limitations, we will proceed now to elaborate the first and last of these codes. Accounts of the other two are given in Petrou (2010) and in Rowland et al. (2011), respectively.

Contingency: teacher insight during instruction

The first instance of this incremental process of enrichment is the case of Máire, a prospective teacher participant in the study of Dolores Corcoran, located in Ireland. Máire was observed teaching a lesson on whole-number division to a class of girls aged 9-10 years (see Corcoran (2007). She had written worksheets on division, set in a fantasy Harry Potter scenario. The first problem for one of the groups was as follows:

> Ron has 18 Galleons and a pack of cards costs 3 Galleons. How many packs can he buy?

[Note: The Harry Potter novels by J. K. Rowling are well-known in Ireland. Galleons are the fictional currency in use at Hogwarts - Harry Potter's school]

There are two principal division problem structures, variously called partition (or sharing) and quotition (or measurement, or grouping). In the problem under discussion, the problem structure is *quotition*. Máire had provided butter beans as manipulatives, to represent the Galleons. One pupil, Rosin, read out the problem, while Megan volunteered to count out 18 butter beans.

Máire offered a few words of explanation about the "wizard money", then she asked:

Máire: How many groups does she [Megan] need to break it into and can you tell me why? Hannah, what do you think?

Hannah: Into three groups.

Máire: Into three groups. Well done, and why? You can read the question again if you want.

Máire's query here about the *number of groups*, and not to their size, points inappropriately to a *partition* structure, and this is picked up by Hannah. Máire congratulates the child ("Well done") on her inappropriate suggestion. Máire asks Hannah to explain ("and why?"), and the interaction then takes a different direction.

Hannah: Because there's three packs of cards.

Máire: It's *not* that there's three packs of cards. But what *is* it about the cards?

Hannah: It costs three galleons.

Máire is pulled up short at this point. She knows that there are *not* three packs of cards. Máire has inadvertently directed the pupils to the wrong division structure, she *realises* that this is so, and she resolves to find a way out:

Máire: It costs three galleons. […] You've got 18 and what are you doing?

Máire is attempting to alter the direction of the discussion, but the child who answers has not altered course:

Child: Splitting them up into three groups ...

Maire responds with a direct correction, and her language is now correctly aligned with quotition/grouping

Máire:	Ahh …? Into groups of three [she nods]. And how many groups do you have?
Child:	Six.
Máire:	So *how many packs of cards* could Ron buy?
Child:	Six.
Máire:	He could buy six packs of cards. Can everybody follow that? What sentence would you write to explain what we just did?

This ability to change course as a result of reflection had not been noted in the lessons that were the data for our original study. We see an instance of reflection-in-action Schön (1983) in this episode, and in what we would call a ‘contingent moment’. Máire could not have prepared (in her planning) for what she did at that moment, but what she did say and do brought about a significant and pedagogically important shift in the discourse and the cognitive content of the lesson. This was possible because Máire seems to have experienced a pedagogical *insight* of some kind. The Contingency dimension of the Knowledge Quartet was rooted, as it arose from the data in our original study, in the teacher’s response to *children’s* insights and misconceptions. In this instance we seem to have a moment where Máire herself suddenly realises that the problem, the child’s suggestion, and her approval, are in contradiction. Máire’s moment of insight is an instance where theoretical sampling found the existing Knowledge Quartet theory to be wanting, and caused it to be rethought and enhanced. Consequently, we added an additional code - *teacher insight during instruction* (TII) - to those previously associated with Contingency.

Connection: making connections between representations

Our second instance of theory evolution draws on the doctoral study of Abraham de la Fuente in Spain. The participants were mathematics teachers working in the first two years of the secondary stage. Specifically, this research aimed to understand how the teachers used their knowledge to help students to learn to use algebraic language in a problem-solving environment. In the episode described here the intention was that students would learn to solve simultaneous linear equations by engaging with iconic, algebraic and tabular representations of key information. For example, the equation $3a+3b=12$ was represented initially in a picture of 3 slices of pizza and 3 drinks costing 12 euros. Various student responses included listing various prices of drinks and slices that would satisfy each of the two equations.

After working on several problems like this, the teachers devised a ‘test’ for the students, consisting of three simultaneous linear equations problems. The second gave $3x+y=55$ and $2x+2y=62$ in precisely that symbolic form; the first was isomorphic to it, but with an iconic form (involving two different types of ‘Star Wars’ figures and total costs in euros). The third was a different pair of equations in conventional symbolic form only.

After the students had spent 20 minutes or so working on the problems, the teacher led a whole-class discussion about solving the first two problems, drawing out the fact that problem 2 is the 'same' problem as problem 1.

For further details and a Knowledge Quartet analysis of the lesson, see de la Fuente et al. (2016). On the basis of this analysis, de la Fuente proposed the additional code *making connections between representations* in the *Connection* dimension of the Knowledge Quartet.

Once this code had been brought to light, Fay Turner (the second author) was able to identify instances of it in the data in her own doctoral study. For example, in a lesson on the comparison (or 'difference') subtraction structure with a Year 2 (pupil age 6-7) class, Kate began by comparing the heights of two towers of interlocking 'multilink' cubes. She then displayed two-dimensional representations of pairs of towers of cubes on the classroom interactive whiteboard, before showing pairs of lines numbered, respectively, to 5 and to 9, on the interactive whiteboard. In this way she made connection between enactive, iconic and symbolic representations (Bruner, 1974) of the difference between 5 and 9.

Discussion and Conclusion

The Knowledge Quartet is a theory in the sense that it provides a way of thinking about mathematics teaching in the usual institutional settings (lessons), with a focus on the disciplinary content (mathematics) of the lesson. In that sense it offers a framework for focusing in the relationship between what the teacher 'knows' – about mathematics and about mathematics didactics in particular – and what transpires when they set out to enable students to learn mathematics. At the outset, we did not know what kind of 'theory' might emerge from our close scrutiny of our lesson videotapes. It might have been an explanatory theory of the kind "Because this teacher knew this, he or she did (or did not do) that in the lesson". Alternately, it might have been a 'lens' type of theory – a new way of seeing classroom events from the perspective of teacher knowledge. In the event, the theory that materialised was more of the second kind.

While the Cambridge-based originators of the Knowledge Quartet cannot possibly claim 'ownership' of the theory, we take a deep interest in proposals to develop it, of the kind described in this paper. Having said that, we think it only proper that such proposals be empirically-based outcomes of theoretical sampling as opposed to the result of speculations about what codes could be added to those (21 so far) that have emerged from focused analysis of classroom data, and the debate that followed from it. Altogether, this has given us rich and fascinating insight into generosity and collaboration in the worldwide mathematics education community.

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REQUESTS FOR MATHEMATICAL REASONING IN TEXTBOOKS FOR PRIMARY-LEVEL STUDENTS

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In Germany, language competencies in mathematics lessons have received increasing interest in recent years. On the basis of national curriculum standards, argumentation should also be strengthened in primary school mathematics classes (KMK, 2005). Reasoning can be seen as a key issue in mathematical argumentation. Mathematics-textbook analyses are used to gather information about the importance of reasoning in primary schools. This study reports the results of the textbook analyses. It presents comparative data on requests for reasoning in textbooks for grade 3 and 4 pupils. Quantitative results are represented by the frequency of tasks with requests for reasoning in 2010 and 2016. Qualitative results discuss the different kinds of requests.

REASONING IN EARLY MATHEMATICS LEARNING

Early mathematical argumentation can be divided into four steps: detecting mathematical regularities, describing them, asking questions about them and giving reasons for their validity (Bezold & Ladel, 2014; Meyer, 2010; Bezold, 2009; Wittmann & Müller, 1990). The content base of argumentation is achieved through description of the detected structures or by reference to common knowledge (Krummheuer, 2000); reasoning, therefore, is necessary to acknowledge the described regularities as true (Toulmin, 2003/1958; Schwarzkopf, 1999). As a consequence, German curriculum standards require competencies in all four steps even at the primary level (KMK, 2005).

The didactical value of reasoning in mathematics learning is to gain deeper insights into mathematical structures, which, in turn, develop one's mathematical knowledge. In this sense, reasoning leads to questions about mathematical statements to ensure their correctness and to develop new mathematical connections (Steinbring, 2005). Two intertwined processes can be distinguished: one's own understanding and the process of sharing this understanding with others. In most cases, these two processes do not occur separately, but are the response to cognitive–social needs (Harel & Sowder, 2007; Hersh, 1993). It follows that in its epistemic function, mathematical reasoning may be monologic in leading to deeper individual understanding; in its communicative function, where mathematical structures are explained and justified, it is dialogic and dependent on other people (Neumann, Beier, & Ruwisch, 2014; Ruwisch & Neumann, 2014).

In primary classrooms, mathematical reasoning usually takes place in the oral communication between pupils, as well as in the interactions with the teacher. These

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communicative processes have been studied extensively; from an epistemological perspective, the emergence of shared knowledge and its structures are described (e.g., Steinbring, 2005), and from a more interactionist perspective, the types and structures of argumentations in classroom interactions are traced (e.g., Krummheuer, 2015).

Mathematical reasoning, in this sense, must be distinguished from reasoning in language classes, especially at the primary level. While both are considered as concepts that develop out of situated everyday ("vernacular") speech (Elbow, 2012), reasoning in language learning focuses more on self-evident facts and personal meanings than on provable structures in special content areas. It follows that argumentation in language learning leads to more addressee-oriented cognitivization (Krelle, 2007) because reasoning of this kind is much more about persuasion than about proving. Nevertheless, typical linguistic forms of reasoning are learned in everyday situations, and students must learn how to use these in different content areas (e.g., Wellington & Osborne, 2001; Lemke, 1990). In mathematical reasoning situations, typical linguistic clues show the demand for reasoning, and special linguistic markers and forms should also be used by pupils to give reasons for the validity of the detected regularities.

While most age-related studies on primary-level students focus on oral communication, experts in language learning emphasize writing as an important instrument to deepen individual understanding (Becker-Mrotzek & Schindler, 2007; Pungalee, 2005; Galbraith, 1999; see also Wellington & Osborne, 2001; Morgan, 1998; Miller, 1991). Although primary school children are not yet experts in writing, fourth-graders are capable of constructing expository texts with a relevant number of causes in elaborating a topic (Hayes, 2012; Krelle, 2007). Looking at their written argumentations, especially at how they offer reasons for mathematical regularities, may therefore be fruitful (Ruwisch & Neumann, 2014; Fetzer, 2007). At the same time, we need more information about the status of reasoning in mathematics lessons. Especially if we are interested in written reasoning, examining the textbooks of students might be relevant: Qualitative content analysis may give information about the linguistic forms of requests for reasoning. The frequency of tasks with a request for reasoning may serve as an indicator of the importance of reasoning in mathematics learning.

REQUEST FOR MATHEMATICAL REASONING IN TEXTBOOKS

Textbooks play a major role in mathematics classrooms, so they may serve as a source for gaining information about the frequency and type of reasoning in primary mathematics. In implementing the national curriculum standards in Germany (KMK, 2005), reasoning should be given more attention in mathematics classrooms.

Textbooks and teacher's guides serve as important materials when teachers prepare their lessons. If reasoning in primary schools needs more emphasis, this could be seen in both books. Whereas textbooks are written with the intention of addressing students directly, teacher's guides play an indirect role. Teachers may be asked to consider more reasoning in their lesson discussions. In this sense, we first focus on students' textbooks to analyze direct requests for reasoning.

RESEARCH QUESTIONS

1. Do German mathematics textbooks support mathematical reasoning through specific demands? Do textbook series differ in their demand for reasoning?
   ⇒ Textbooks that contain more text ask for more mathematical reasoning than textbooks that contain only little text.
2. Do mathematics textbooks differ in the numbers and kinds of requests for reasoning over time or across different ages?
   ⇒ The textbook series for fourth graders ask for more reasoning than those for third graders. Newer textbook series ask for more reasoning than older ones.

DATA AND METHOD

Data sources

In 2016, about 30 different primary mathematics textbook series were licensed in Germany. Because of the federal system, even different issues of the same textbook series were licensed in different states. For our study, we focused on 10 different series, which were selected through the following procedure. Publisher–consortia were asked to name and rank their most common textbook series. In consideration of the size of the publishing house, the named and ranked series, and the number of states in which the textbook series is licensed, 10 textbook series were identified. We also want to focus on new data at the time of analysis, so the books included in the first analysis differ from those in the second one; five series are the same in both corpora, whereas five are different.

A primary textbook series in Germany normally includes four packages, one for each grade. Aside from the main textbook or several smaller issues for every pupil, exercise books and worksheets, exercise books for special needs, and materials, such as cuisenaire rods, can be bought. Teachers will also find a teacher's guide, additional digital materials, and diagnostic and testing materials, among others. The additional materials offered differ from series to series, so the only materials included in our analysis are the main textbooks for grades 3 and 4 of the 10 selected textbooks series. We always used the version of the series that is licensed in north Germany.

Method of analysis

Qualitative categorization: Demands for reasoning in mathematics textbooks are made through language or specific symbols that reflect the meaning of a "reasoning request." In the German language, requests for reasoning are often made through interrogative sentences. These sentences start with a so-called "W-question-word", followed by a verb. Typical examples are "Warum gilt …?" (Why is … true?) and "Warum ist das so?" (Why is it like this?). Aside from interrogative sentences, imperative sentences also ask for reasoning, such as the following: "Begründe" (Give reasons) or "Erkläre, warum …" (Explain, why). In mathematics classrooms, we also ask for reasoning through more implicit questions and imperatives: "Woher weißt du das?" (Where do you know it from?) or "Wie bist du auf deine Antwort gekommen?" (How did you get your answer?). These questions could be answered in a linguistically correct way by stating a name or saying "by thinking". Questions, such as "Are you sure?" or "Can this be true?", can also be answered by "yes/no". "What did you recognize?" can be interpreted as asking for a description instead of giving reasons for the detected regularities. Nevertheless, in all these cases in mathematics lessons, we expect children to give logical reasons.

All the questions and imperatives mentioned above are considered indicators for reasoning. We categorized them as explicit if "why", "give reasons", or "explain, why" is used. Otherwise, we categorized them as implicit.

Quantitative indicators: The quantity of text is recorded indicated by three indicators: total number of words, total number of pages, and total number of problems. The frequency of requests for reasoning is the relation of requests to the number of problems. We also differentiated between explicit and implicit requests for reasoning, as well as the content areas, e.g., arithmetic, geometry, stochastics, and the rest.

RESULTS

Results and further questions of the first textbook analysis

In 2012, we already presented an analysis of 10 German third-grade mathematics textbooks (Ruwisch, 2012). The main results can be summarized as follows:

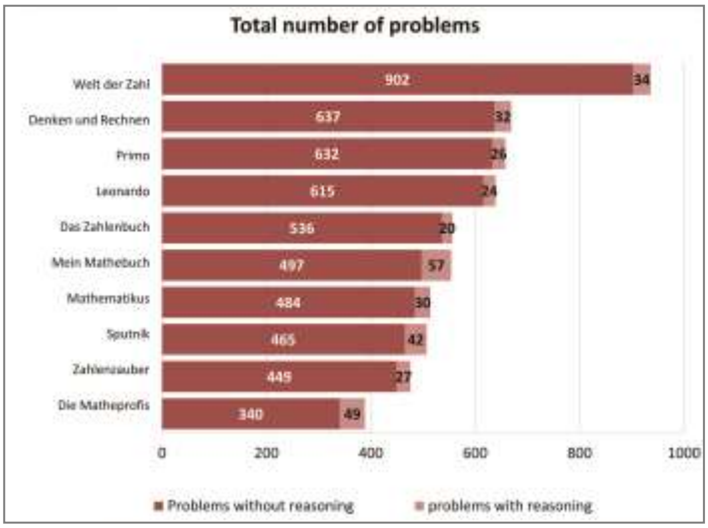

Figure 1: Total number of problems without and with the request for reasoning

Overall, the number of requests for reasoning was very low (see Figure 1). On average, 6% of the tasks ask for reasoning. Only in two textbooks was the required reasoning more than 10%; in five textbooks, less than 5% of all textbook tasks asked for reasoning.

The request for reasoning seems independent of the number of text. A textbook that seems to strengthen linguistic issues in mathematics lessons does not provide more opportunities for reasoning.

Since the conduct of this first analysis, several questions have emerged. First, the textbooks that we analyzed were authorized for use in mathematics classrooms between 2004 and 2010. The national curriculum standards came into force in 2005, so finding any requests for reasoning in these textbooks as answers to the standards might have been too early. Second, was this low frequency of requests for reasoning caused by the fact that we analyzed grade 3 books? In Germany, reading and writing skills are assumed to be acquired at the end of the second grade. Analyzing the textbooks of older students and hopefully finding more requests for reasoning may therefore be of more interest. These two questions lead to the assumption that nowadays, there might be more tasks that request for reasoning in fourth-grade textbooks. However, the requirements for inclusion during the past years (children with learning difficulties, as well as those with different language backgrounds) seem to strengthen another line of development, which can be characterized by more restrictive and directive mathematics classroom learning. The second question can be further divided into the following two sub-questions: 1) Do recent mathematics textbooks have more requests for reasoning? To answer this question, we will compare the previous third-grade textbooks with the new ones. 2) Do grade 4 textbooks have more requests for reasoning than grade 3 textbooks? This question will be answered by comparison of actual third- and fourth-grade textbooks.

Results of the recent textbook analysis

Textbook series	Grade 3				Grade 4			
	Problems with reasoning/total number of problems	Relative amount of reasoning	Explicit request for reasoning	Implicit request for reasoning	Problems with reasoning/total number of problems	Relative amount of reasoning	Explicit request for reasoning	Implicit request for reasoning
Denk. u. Rech.	61/598	10.20	4.18	6.02	66/616	10.71	4.87	5.84
Einstern	36/679	5.30	2.21	3.09	40/561	7.13	3.03	4.10
Flex & Flo	84/777	10.81	2.83	7.98	59/650	9.08	4.62	4.46
Mathebuch	22/663	3.32	1.06	2.26	45/580	7.76	2.76	5.00
Mathefreunde	45/720	6.25	0.97	5.28	46/603	7.63	1.82	5.97
Mathematikus	37/503	7.36	2.98	4.37	58/511	11.35	3.91	7.44
Nussknacker	32/443	7.22	1.35	5.87	32/401	7.98	2.24	5.74
Welt der Zahl	44/795	5.53	0.13	5.41	52/810	6.42	0.25	6.17
Zahlenbuch	72/577	12.48	5.20	7.28	63/607	10.38	5.60	4.78
Zahlenzauber	49/472	10.38	3.60	6.78	57/451	12.64	5.10	7.54
Average		7.89	2.46	5.43		9.11	3.41	5.70

Table 1: Requests for reasoning

Comparison between the textbooks: Overall, about 8.5% of the tasks require reasoning. The textbooks differ considerably. In the third-grade textbooks, we found that 3.32% to 12.48% of all problems ask for reasoning, whereas in the fourth-grade textbooks, the range is slightly smaller at 6.42% to 12.64% (see Table 1). Although the textbooks seemed to show an increased demand for reasoning over time, the increase is so negligible that it could instead be an effect of the different textbooks.

Explicit and implicit requests for reasoning: In nearly all textbooks, more problems ask implicitly for reasoning than explicitly. On average, more than double of the requests in third-grade textbooks are implicit (5.43%) than explicit (2.46%). Examining these two groups, we find a slight increase in explicit demands for reasoning from third to fourth grade; whereas, on average, 2.46% of all tasks ask explicitly for reasoning in the third grade, 3.41% of all tasks do so in the fourth grade. The average in implicit reasoning remains almost the same: 5.43% in the third grade and 5.7% in the fourth grade.

Comparison between grades: The number of requests in the third and fourth grades does not differ significantly, although a small increase is shown in the fourth grade, especially in the textbook series, indicating a very small number of requests in the third grade. As already mentioned earlier, the range of requests in the fourth grade is slightly smaller than that in the third grade. No fourth-grade textbook contains less than 6% of tasks asking for reasoning, whereas three third-grade textbooks do so. In both grades, only four textbooks contain slightly more than 10% of tasks asking for reasoning.

DISCUSSION AND CONCLUSION

Analysis of textbook tasks that require reasoning was used to obtain information on the importance of reasoning in mathematics classrooms.

Qualitative analysis of textbook problems showed two different kinds of requests: explicit requests that directly ask pupils to give reasons and implicit requests that might be answered without giving reasons. In the latter, argumentation must be initiated in the classroom discourse, probably directly required by the teacher. We considered more implicit than explicit requests for reasoning, and we observed a small increase in explicit demands for reasoning across the ages, but what does this mean? If teachers and textbook authors think that implicit requests are not as formal as explicit requests are, they might prefer implicit forms for younger students. From a linguistic point of view, one might argue that explicit requests are much clearer and easier to follow, so these forms are necessary, especially in initiating reasoning. The data do not tell us anything about these assumptions. This might not be a conscious decision neither of the teachers nor of the authors. We also do not know how students of this particular age react to explicit demands versus implicit ones. The initial results of written reasoning in geometry show that the same students produce much more substantive sentences that indicate linguistic forms of reasoning if they are explicitly rather than implicitly asked for reasoning (Schmid, 2016). These results are only initial ones, so the question on whether children understand implicit demands as a requirement for reasoning remains; whether they need explicit demands and perform better with these demands has yet to be investigated.

The quantitative comparison between 2010 (Ruwisch, 2012) and 2016 shows no significant differences in the frequency of demands for reasoning in response to national standards. The identified amount of less than 10% of textbook problems does not show that teachers are supported by the textbooks if they want to strengthen pupils' competencies in argumentation. Examination of the differences in the textbook series licensed in the same state shows that reasoning is seemingly not a key focus. Maybe textbook authors consider it only as a nice addition to a textbook series, or perhaps reasoning is seen as too demanding for most pupils, particularly the linguistic demands of writing down argumentation. Students are seemingly required to reason by their textbooks only by chance. Nevertheless, this does not mean that no requests for reasoning exist in mathematics lessons. Our own results on the written reasoning of pupils in the fourth grade show that most of them are capable of understanding explicit requests and of starting written argumentation even if they do not fully grasp the concept yet (Ruwisch & Neumann, 2014).

If we want to deepen our understanding of the processes of reasoning and argumentation, which are expected as adequate answers to formulated tasks, we need to expand our research area. Even if teachers' materials show much focus on reasoning, we need to know how students learn reasoning in both written tasks and classroom discourses. In focusing on written reasoning—requests of textbooks and teachers, as well as interpretations and implementation of the students—we hope to gather information for future and better formulations, which foster reasoning competencies (Krelle, 2007; Ruwisch & Neumann, 2014).

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TEACHERS' BELIEFS AND HOW THEY CORRELATE WITH TEACHERS' PRACTICES OF PROBLEM SOLVING

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In this report, a case study with three teachers from Indonesia is presented. The purpose of this present study is to understand if and how teachers' beliefs correlate with their practices of problem solving. The teachers were asked to teach the topic "problem solving" and the corresponding lessons were observed. Additionally, the participating teachers were interviewed to capture their beliefs regarding mathematics and problem solving. The analyses of the interviews and the observations show a correlation between the teachers' beliefs and their actions in those lessons.

INTRODUCTION

Due to the bad performances of Indonesian students in all PISA studies since 2000, The Ministry of Education and Culture of Indonesia aimed at reforming the mathematics education by publishing a new curriculum. A closer look into these documents (Safrudiannur & Rott, 2016) reveals that the changes do not only affect mathematics contents but also the process-related standards: In the new curriculum, the learning standards emphasize more on problem solving. In contrast, solving problems was only an implicit goal of the mathematics education in the old curriculum.

The reformation of the learning standards raises the issues of how Indonesian teachers implement problem solving in their teaching and, especially, how they teach to solve problems. As we know, many researchers have revealed that teachers' beliefs affect the way how they teach in their classes (Thompson, 1992; Phillips, 2007). Furthermore, Rott (2016) has shown that teachers' style in practicing problem solving often match with their beliefs of the nature of mathematics. Therefore, we are especially interested in the Indonesian teachers' beliefs about mathematics and problem solving.

To better understand how the beliefs correlate to teachers' practices, we have carried out a qualitative study by asking teachers to involve problem solving in their lessons. The underlying research question of this present study is "how do their beliefs correlate their practices of problem solving in the Indonesian educational context?"

THEORETICAL BACKGROUND

Beliefs

Beliefs play an essential role in learning and teaching of mathematics (Thompson, 1992; Philipp, 2007). Philipp (2007) defines beliefs as psychologically held

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 121-128. Singapore: PME.

understandings, premises, or propositions about the world that are thought to be true. A belief does not stand isolated from the other beliefs. They are interconnected. For example, beliefs about the nature of mathematics will affect teachers' beliefs about mathematics teaching and learning.

Ernest (1989a) states that there are three beliefs regarding the nature of mathematics: the instrumentalist view, the Platonist view, and the problem solving view.

> First of all, there is the instrumentalist view that mathematics is an accumulation of facts, rules, and skills to be used in the pursuance of some external end….. Secondly, there is the Platonist view of mathematics as a static but unified body of certain knowledge…… Thirdly, there is the problem-solving view of mathematics as a dynamic, continually expanding field of human creation and invention, cultural product (Ernest, 1989a, p. 250).

Furthermore, he describes how those three views influence teachers' practices.

> For example, the instrumental view of mathematics is likely to be associated with the instructor model of teaching, and the strict following of a text or scheme. It is also likely to be associated with the child's compliant behaviour and mastery of skills. Similar links can be made between other views and models, for example: Mathematics as a Platonist unified body of knowledge – the teacher as explainer – learning as the reception of knowledge; Mathematics as problem-solving – the teacher as facilitator – learning as the active construction of understanding, possibly even autonomous problem-posing and problem-solving. (Ernest, 1989a, p. 251-252)

Ernest's theory will serve as a fundamental theoretical basis for interpreting the relations of teachers' beliefs from the interviews and the observations in this present study.

METHOD

The method of our preliminary study is observation and semi-structure interview. We asked three mathematics teachers to conduct a lesson involving one or more math problem(s). Few days before those teachers conducted the lesson, a particular problem was chosen and discussed together with the teachers, but they had options to add or pose their own problems. The lesson of each teacher was observed and filmed. The coding system from TIMSS Videotape Classroom Study 1999 was used for interpreting the underlying lessons. One of the reasons for this choice was that the TIMSS video study was conducted in several countries worldwide, including Indonesia in 2010.

We categorized the learning process of the entire lesson by applying the coding such as the classroom interactions (public, private, or mix) and the content activities (problem or non-problem segment). In this paper, we present only the problem segment.

The problem segment is a segment containing the discussion of a mathematical problem. The segment starts when a teacher states or assigns the problem. The segment ends when solutions have been found or the discussion around the solution has been finished, whichever occurs in the final stage of the segment (NCES, 1999). The segment is characterized by the problem statements, the processes of searching for a solution, discussion including the checking of the solutions, and others (transitions between them).

At the end of each lesson, we interviewed the teachers in order to acquire more information about their beliefs of mathematics and problem solving. The analyses of the interviews were discussed with two colleagues from the mathematics education department of the Mulawarman University in Samarinda, Indonesia.

This pilot study was implemented in three junior high schools in Samarinda. Three teachers (A, B, and C) voluntarily participated in this present study. They each have been mathematics teachers for more than ten years. Schools of teacher A and C are implementing the old curriculum (2006 curriculum) and school of teacher B is implementing the new curriculum (2013 curriculum).

RESULTS

Teacher A

Observation: Table 1 represents the problem discussed in the lesson of teacher A and the activities during the problem segment. The duration of his lesson was 82 minutes and 54 seconds (82:54). The problem segment lasted for 42:40 (51.5%).

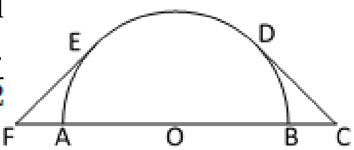
<table>
<tr><th>Activities</th><th>Duration</th></tr>
<tr><td>Problem Statement</td><td>08:52</td></tr>
<tr><td>The problem: O is the centre of the semi-circle. D and E are the tangent points of CD and FE respectively. $|\angle BCD| = |\angle AFE| = 45^{o}$. If the length of OC is $\sqrt{392}$ cm, find the total length of CD, DE, and EF.</td><td>
</td></tr>
<tr><td>Process to find the solution</td><td>32:43</td></tr>
<tr><td>Interactions during the process in chronological order:
1. Public Interaction 1 (09:55)
2. Private Interaction 1 (01:18)
3. Public Interaction 2 (03:15)
4. Private Interaction 2 (01:36)
5. Mix interaction 1 (00:56)
6. Private Interaction 3 (00:49)
7. Public Interaction 3 (04:51)
8. Private Interaction 4 (08:27)
9. Mix interaction 2 (01:00)
10. Private Interaction 5 (00:36)</td><td></td></tr>
<tr><td>Discussion (Checking) on the solution</td><td>00:07</td></tr>
<tr><td>Other</td><td>00:58</td></tr>
</table>

Table 1: Activities during the problem segment of teacher A's lesson.

Table 1 shows that in this lesson, public interactions dominates the interactions during the process to find an answer. In the public interactions, teacher A helped his students by explaining not only the step how to solve the problem, but also the necessary concepts and formulas required to solve the problem. For example, in the first public interaction, he told his students to draw lines connecting points *O* and *D* as well as *O* and *E*. He guided them to recognize that *CDE* was an isosceles right triangle, and he reminded them of the appropriate formula to find the length of *CD* and *OD*.

All private interactions, in which the students worked on their own, were initiated by teacher A by instructing the students to do calculations. For example, in the first private interaction, teacher A asked the students to find the length of *CD* and *DO* by using the Pythagorean Theorem. Time of the first private interaction was limited due to increasing difficulties encountered by some students. Afterwards, teacher A continued to guide students applying the theorem in the second public interaction.

Interview: For teacher A, a problem is a task that is difficult for students to solve. A student is successful in problem solving if the student gets the correct answer. To ensure that, he guided his students step by step in his lesson. He said that without his guidance, his students would not be able to find the answer. He assessed that his students in the current lesson had low mathematical abilities.

The way he guided his students is harmonious with his beliefs about mathematics. He expressed that he somehow believes that mathematics is an accumulation of useful facts, rules, and skills. Students should know which mathematical formulas are appropriate to solve a problem and know how to apply them. Those expressions indicate that teacher A holds the *instrumentalist view*. Thompson (1984) argued that if a teacher holds this belief of mathematics, it is important that students are able to recall what the teacher taught and then apply it to obtain the correct answer.

Teacher B

Observation: Table 2 visualizes teacher B's activities during the problem segments. The total time of his lesson was 80:58. He posed his own three problems regarding mean values. Time of the first, second, and third problem segment were 10:22 (12.8%), 13:42 (16.9%), and 21:38 (26.7%), respectively, or 45:42 in total.

Table 2 shows that private interactions and mix interactions dominate the process of obtaining a solution in all problem segments. The public interaction in second problem segment has occurred because teacher B explained the text of the second problem. He wanted to ensure that his students understood the problem.

In contrast to teacher A, he did not give any clues to his students. In each problem segment, he repeatedly encouraged his students to create their own strategies. He emphasized that his students could use their own formulas.

Interview: The problems posed by teacher B reflect what his beliefs about a mathematical problem. For teacher B, a problem should be the application of mathematics in the real world.

The way he guided his students in problem solving is also influenced by his beliefs about the nature of mathematics. He released his students to create their own formulas because he strongly disagreed that mathematics is an accumulation of facts, formulas, or skills. It is not obligatory for students to memorize formulas. For him, mathematics contents are not fixed but can change and are open for revision. From the interview, teacher B seems to hold the *problem solving view*.

Activities	Duration
First Problem Segment:	
Problem Statement	01:25
The problem: The average height of eight volleyball players is 176 cm. After two players leave, the new average is 175 cm. Find the average height of the two players!	
Process to find the solution	05:56
Interactions during the process in chronological order: 1. Private Interaction 1 (02:42) 2. Mix interaction 1 (03:14)	
Discussion on the solution	01:02
Other	01:59
Second Problem Segment:	
Problem Statement	01:55
The problem: The average weight of six futsal [a ball game] players is 65 kg. After a substitution, the new average weight is 63.5 kg. If the weight of the player who left is 64 kg, find the weight of the new player.	
Process to find the solution	09:15
Interactions during the process in chronological order: 1. Public interaction 2 (01:09) 2. Private Interaction 2 (02:48) 3. Mix interaction 2 (05:18)	
Discussion on the solution	01:37
Other	0:55
Third Problem Segment:	
Problem Statement	01:17
The problem: The math test average score of a group of students is 63. If a student whose score is 80 is included to the group, the new average score is 64. Find the initial number of the students in the group.	
Process to find the solution	14:54
Interactions during the process in chronological order: 1. Private Interaction 3 (11:20) 2. Mix interaction 3 (03:34)	
Discussion on the solution	03:07
Other	02:20

Table 2: Activities during the problem segments of teacher B’s lesson.

Teacher C

Observation: Duration of his lesson is 73:41 and the problem segment lasted for 24:38 (33.5%). Before the problem segment, teacher C announced that his students are allowed to use their own strategies but the strategies should follow mathematical rules. Then he gave them two mathematical tasks. Both tasks were based on two mathematical facts respectively: the total size of three angles forming a straight angle is 180^o and the total size of three angles of a triangle is 180^o.

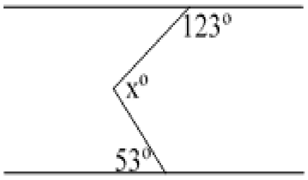
Activities	**Duration**
Problem Statement	00:40
The problem: Find *x*! 	
Process to find the solution	23:43
Interactions during the process in chronological order: 1. Private Interaction 1 (14:41) 2. Mix interaction 1 (07:16) 3. Private Interaction 2 (01:46)	
Discussion (Checking) on the solution	00:00
Other	00:15

Table 3: Activities during the problem segment of teacher C's lesson.

In the process to find a solution, private interactions dominated the process. During the private interactions, students worked individually and teacher C walked around and motivated them to show their works on the whiteboard. To assist his students, he gave a clue that students needed to draw an auxiliary line to solve the problem.

In the mix interaction, there were two students showing their works on the whiteboard. After they finished writing their solutions, teacher C clarified their works by asking some questions. For example, one of the two students wrote the equation $123+x+53=180$ and teacher C asked him to explain how he got it.

Interview: For teacher C, a mathematical problem is a task that is difficult for students to solve. To succeed in solving it, he said that his students needed his help. He gave clues consisting of concepts or ideas related to the problem, not how to solve it.

For him, concepts are crucial for solving problems successfully. He said that the two tasks before the problem segments were his clues since the concepts in the two tasks would be useful and related to the problem. In addition to the two task, he also gave the students an idea by telling them to draw an auxiliary line which could help them to apply the concepts. He said that if there was enough time, he would draw it.

He believes that mathematics contents are not fix but dynamics, can change over time, and are open for revision. He strongly disagrees that students should memorize formulas. Thus, he does not care how his students solve a problem as long as the approaches follow mathematical rules. From this interview, teacher C seems to hold the *problem solving view*.

DISCUSSION

The results show that the observed teachers' beliefs of the nature of mathematics correlate with the way in which they involve problem solving in their lessons. We interpret this in the following way: the beliefs influence the teaching style.

Teacher B who holds the *problem solving view* does not require his students to memorize and apply formulas that he taught. He encouraged his students to create their own strategies or even their own formulas to solve problems. The coding of his lesson shows that private and mix interactions dominate the process to get an answer. In the private interactions, students work privately and he encouraged them to create their own formulas. In the mix interactions, one or two students show their works and other students can look at the works or still work on problems. These interactions indicate that he gave his students a lot of time to work. He did not disturb his students by giving clues what formulas are appropriate. He gave his students opportunities to work on their own.

Contrastingly, teacher A shows a different style on teaching problem solving since he has a different view. He holds the *instrumentalist view* which mathematical formulas are very important. His view influences how he guides his students on problem solving. He tried to direct his students to solve the problem by reminding them of appropriate formulas and concepts and also guiding them how to apply the formulas or concepts. He did not encourage students to create their own strategies. He also did not introduce alternative ways to solve the problem. He believed that it would confuse his students. Apparently, it is enough for his students to copy his procedures. In the mix interaction, students' products on the whiteboard were the applications of his clues.

Teacher A and B's actions show that their beliefs about the nature of mathematics and their style in teaching problem solving are related to each other. But, compared to teacher A, there is a gap between the way teacher C guided his students in problem solving and what he believes about mathematics.

The interview indicates that teacher C holds the *problem solving view*. At the beginning of his lesson, he told his students that they can develop their own strategies to solve problems. But during the process to find the answer, he did not encourage them to express their own ideas. He gave clues which could lead his students to apply his clues. This can be seen from the example of a student who wrote his answer during mix interaction. This student tried to apply teacher C's clues but did so incorrectly. He wrote down the equation $123+x+53=180$. With this equation, he tried to follow the equations to find the answers of the two tasks that were posed before the current problem segment. In those two tasks, the equations needed to find the answers used the fact that the total size of three marked angles was equal to 180.

Teacher C insisted that he wanted to free and encourage his students to create their own strategies but he was sure that they could not do that. He gave clues due to the low mathematical performance of his students, which led to the assumption that his students would not be able to develop strategies for problem solving.

CONCLUDING REMARKS

This present study shows that in the Indonesian educational context, teachers' beliefs about the nature of mathematics correlate with their practices of problem solving. What they believe about mathematics can be matched with their style in teaching problem solving. However, there are factors which can make a gap between teachers' beliefs about the nature of mathematics and their way to teach problem solving in their lesson. One of them is students' low abilities in mathematics.

In the next study, we are going to develop a quantitative instrument to collect beliefs of a large sample of Indonesian teachers. This present study has shown that teachers' beliefs of the nature of mathematics are an important factor to understand how teachers practice problem solving. Therefore, the theory about teachers' beliefs about the nature of mathematics will serve as basis to develop the instrument.

We also consider counting on students' mathematics abilities in our instrument. This study has found that students' mathematical abilities can explain a gap between teachers' beliefs and their observed actions. We suppose that students' mathematical abilities can be one of the social contexts which according to Ernest (1989a) can cause disparities between a teachers' espoused and enacted model of teaching mathematics. By considering students' mathematical abilities, we can better understand the disparity between teachers' beliefs and their actual teaching especially in problem solving.

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HOW THE TEACHER PROMOTES AWARENESS THROUGH THE USE OF RESOURCES AND SEMIOTIC MEANS OF OBJECTIFICATION

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This article presents some preliminary results from a wider research that analyzes the practice of expert and novice teachers from a socio-cultural approach and the use of resources through the networking of theories. The data analyzed here correspond to the transcriptions of video recordings of two physics teachers of grade 11 (expert and novice), in original teaching settings. The discussion focuses on determining how the two teachers' practices are different from the other, based on the use of semiotic means of objectification and resources (physical and conceptual). The aim is to promote the awareness of conceptual meanings. The results show significant differences between the way in which the teachers produce meanings in the classroom and the use of resources.

INTRODUCTION

The motivation to carry out this research stemmed from placing ourselves in line with the interest of conceptualizing mathematics teaching and learning in a different way from those individualistic approaches that place the student in the center of the knowledge production process as well as from the constructivist paradigm in which students only learn what has been constructed by them and the teacher is relegated to play an auxiliary role. One of the theories based on the constructivist approach, focused on analyzing learning, is that of conceptual change (Appleton, 1997). According to this approach, learning takes place when previous beliefs and knowledge are changed by new [scientific] ones to be learned (Rolka, Rösken & Liljedahl, 2007). However, change does not come easily and resistance that hampers learning the correct ideas often arises (Bayraktar, 2009). Then, the teacher's role is to find didactic teaching strategies to promote such change in the students. Indeed, it is important for the teacher to identify the students' false ideas even though the reason or the importance of change is not identified (Gomez-Zwiep, 2008).

BACKGROUND AND RESEARCH QUESTION

Research regarding teaching practice has become the main focus of interest in the mathematics education field since the first years of this century (Sfard, 2005). Among these works is research that contrasts the practice of novice and expert teachers (e.g., Robinson, Even & Tirosh, 1994). Regarding the theoretical approaches, Vygotsky's

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work—and his notion of semiotic mediation—permeated and evolved in several research lines dealing with teacher's practice, mainly since the 90s (da Ponte & Chapman, 2006). Two relevant characteristics in such notion are the ones related to the issue of meaning and the influence culture has on the production of those meanings. Thus, Moreno-Armella & Sriraman (2010) consider that the access to [mathematical] objects is through mediators (as language). Then, the relationship object-subject does not take place directly. This epistemological relationship is opposed to the one considered, since Descartes and Kant, in several approaches as a direct relationship without intermediaries (Radford, 2000). Therefore, we are able to develop our knowledge and culture through cultural mediators (artifacts and signs) that they have an historical development. Considering the role culture plays in the development of systems of scientific ideas is bringing forward the relevance of the historical process those systems have gone through. Particularly, Karam (2015) points that the fruitful and close relationship between physics and mathematics arises from a historical process. However, the picture is quite different in educational contexts. Students "have a hard time understanding where mathematical concepts come from and why physics has little to do with their experiential world." (Karam, 2015, p. 487).

We revisit the interest on the practice of expert and novice teachers and use a sociocultural approach to analyze the teachers' practices. Therefore, we turn to the theory of objectification—TO— (Radford, 2008; 2014a) as a theoretical framework that provides us with an epistemology of the characteristics of knowledge (as awareness). The TO also allows us to identify the teacher's role during the processes of meaning production. Thus, we pose the following research question: How does each teacher promote the objectification of physics concepts while teaching through semiotic means and the use of resources?

CONCEPTUAL FRAMEWORK

This research is supported by the TO (Radford, 2008; 2014a), which incorporates the notion of semiotic mediation by Vygotsky and the importance of the use of artifacts and signs in the processes of knowledge production. In the TO, knowledge mediation occurs through social labor; artifacts and signs are part of such labor (Radford, 2014b). Then, the concept of labor is the fundamental principle of the TO (Radford, 2014a). Unlike the approaches that consider knowledge as something that individuals possess, acquire or construct, knowledge is collectively produced by students and teachers in the TO as: "a dynamic and evolving implicit or explicit culturally codified way of doing, thinking, and relating to others and the world." (Radford, 2014b, p. 7); that is, it is sheer possibility. It is the possibility of ways of doing and thinking. "Objects of knowledge [mathematical objects] (…) are social-historical-cultural entities" (Radford, 2015, p. 134). Therefore, learning takes place through the awareness of such ways of thinking and doing in the systems of scientific ideas. Then, according to the approach of the TO, learning is defined as a problem of awareness. Conscience is something concrete; it is a subjective reflection on the world and can be perceived through its manifestations: discourse, gestures, and the rest of sensual actions (Radford

& Roth, 2011). For awareness to be achieved, knowledge must take concrete (singular) shapes through specific activities.

An important part of data analysis, corresponding to the use of resources by each teacher to promote awareness, takes place through the *networking of theories*. Kidron and Bikner-Ahsbahs (2015) state that such concept [*networking of theories*] is "essentially a methodological approach for theoretical and empirical research that connects different theories to broaden and deepen insight into problems" (p. 221). So, the approach on the use of resources through documentational genesis (Gueudet & Trouche, 2009) is added to the research to analyze the teachers' practice. In documentational genesis, one of the aims is to consider the teacher's activity as goal-oriented and conceptualize it as a social activity. Gueudet & Trouche (2009) use the term *resource* to emphasize the variety of artifacts they consider teachers use. These researchers regard an artifact (whether physical or psychological) as a socio-cultural means provided by human activity (e.g., computers and language) and produced with specific purposes (e.g., problem solving). We consider that using resources is relevant for the teacher to intentionally use artifacts and gestures to promote awareness in the students.

METHODOLOGY

The research is qualitative and is conducted through a case study. The pilot study was carried out in a high-school (grade 11); with two physics teachers (expert: who has taught over 20 years and novice: 2 years). The instrument for data collection was the non-participative observation of the Physics I classes of each teacher where mechanics topics were addressed. Each class lasted two hours twice per week and one hour once per week. The observation time was different with each teacher since the objective was to obtain data of the topic dynamics, where the concepts of force, movement and cartesian graphs interpretation are included. We observed to expert teacher for 10 sesions (16 hours); and to novice teacher for 12 sesions (20 hours). The data collection was carried out simultaneously with both teachers in different schedules. We used two cameras controlled by one of the researchers; one remained fixed and focused on the board while the other was moved to record the interactions during the students' participations. We also used a voice recorder placed on the teachers to have another audio element of the classes. After the data were collected, we watched the videos from the classes to identify moments when key concepts had been addressed. Once the moments (class segments-excerpts) were identified, we transcribed what occurred in those segments. Our analysis is based on those transcriptions.

ANALYSIS AND DISCUSSION OF RESULTS

To carry out the data analysis of each teacher's practice (while they promote scientific concepts related to object movement), we defined two categories of analysis: (1) artifacts and cultural signs (gestures) as elements of the activity during teaching practice, and (2) use of resources. In addition, for the purposes of data analysis, we named the novice teacher "Edgar" and the expert teacher, "Peter".

Analysis of Edgar's practice

From the video recordings of Edgar's practice, we did not identify the use of resources. During the observation period, Edgar's classes focused on having the students (organized in teams) solve kinematics exercises. We therefore directed our attention to the analysis of cultural signs used during his practice, particularly to the use of language and gestures as meaning carriers. Here, we analyze excerpts of the transcription of a moment during class in which a student (S1) has a question regarding the sign of a vector quantity [the sign of gravity acceleration, *g*] in a free fall problem where the students are asked to find the final velocity of an object. It is then that Edgar explains.

L1 S1: Teacher, is gravity negative? (…)

L2 Edgar: Gravity, gravity will always be negative, right? … I'd told you that acceleration was a vector, right? Then, for example, if you want to speak in, let's say, a vector manner, you must express gravity with its negative. Because it will always point down [*makes a gesture; see Figure 1-Photo 1*], right? But in this case, if you place it like this, in a scalar manner … we're only looking at the magnitude of the gravity. Which would be 9.8.

The free fall exercise set by Edgar demands the use of a reference system to find the solution. However, he does not address the concept of reference system in his discourse before the students start solving the exercises. Edgar establishes the sign of *g* from the type of movement and not as a mathematical object that allows the analysis of a phenomenon. The gesture he makes when pointing down using his finger reinforces this concept (Photo 1). The meaning given to the gesture is such that *g* will always be negative for Edgar because: "it will always point down" (L2) due to the characteristic of the phenomenon. Gestures are expressed as semiotic means when they are added to Edgar's discourse to convey a meaning. They do not merely appear as an aid element in the discourse but are incorporated to promote awareness regarding the sign of *g*. In consequence, language and gestures expose the way in which Edgar makes the existence of the meaning of *g* sensible. However, Edgar's explanation is ambiguous to the student. On one hand, Edgar is adamant when saying that *g* will always be negative and, on the other, he says the students might work with the positive sign, 9.8. Later, Edgar speaks again because the students were still uncertain.

L3 Edgar: I'm telling you "*g*" will always be negative, right? Now, you will take a point of reference … If you take a point of reference here [*see Figure 1-Photo 2*], guys. Here, it would be *y*, *x*, right? … we are only acting on *y*, then the value will always be zero at *x*. Then, if this [*the stone*] is falling towards here [*simulates the fall of the object with respect to the diagram; see Figure 1-Photo 3*]. That is why we have a negative value in *y*. Because *y* that goes down is negative.

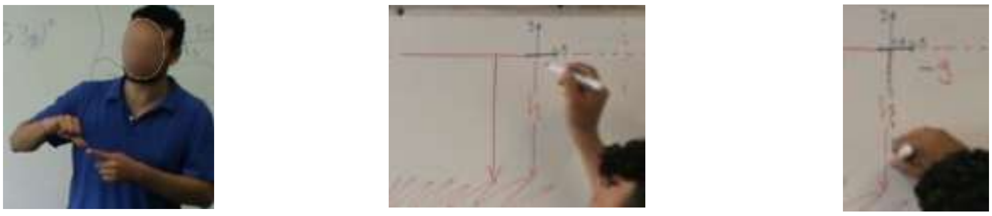

Figure 1: From left to right, photos of gestures used by the teacher to represent: the sign of gravity (Photo 1), the origin of the system of reference (Photo 2), the motion of the object with respect to the system of reference (Photo 3).

Edgar considered that the sign of *g* was determined by the nature of the phenomenon they were dealing with. However, he had yet to clarify the meaning of the negative [or the positive] sign to his students. Thus, Edgar had to incorporate the concept of reference system (L3). Still, the reference system is evidently subordinated to the free fall physics phenomenon and not as mathematical object to analyze object movement of which the students have to be aware. The way in which Edgar works with the concept of reference system makes the students unaware of the meaning of relativity regarding the sign of *g*. then, Edgar expresses the reference system is subordinated to the movement when he says: "Then, if this [*the stone*] is falling towards here. That is why we have a negative value in *y*. Because *y* that goes down is negative". Still, Edgar fails to promote awareness regarding he meaning of the sign of *g*.

Analysis of Peter's practice

Meanwhile, Peter's practice was carried out in a different way, in the use of language and gestures and in the use of resources to promote awareness in the students. Peter even focused the content of his classes on the topic of dynamics. Here we present excerpts of the discussions that arose when talking about linear momentum in the context of collision of two objects.

L4 Peter: Remember that velocity has a sign, right? On what does the sign of velocity depend? [It depends] on where it moves with respect to your frame of reference. (…) So, you'll have to pay attention to several things: which the system is; but also which signs you're going to associate from the frame of reference you're providing.

Linear momentum, a vector magnitude, is defined: ***p***=m***v***, where ***v*** is the velocity Peter refers to (L4). Peter is aware of the relationship between the sign of ***v*** and the frame of reference. That is clear when he points that the sign depends: "… On where it moves with respect to your frame of reference" (L4). So, he sets the conditions for a later analysis of movement in which the relativity in the signs of the variables ***v*** and ***p*** must be taken into account. The sign will not be determined by the direction of the object movement since the reference system is not static; that is, it does not depend on the nature of the phenomenon nor is it restricted to the usual orientation of the system of coordinate axes (see what Edgar says in L3). As Radford (2014b) states on the concept as sheer possibility, the production of meanings concerning the reference system and the movement interpretation must be set in motion through the *Activity*. So, Peter adds another semiotic registry, a Cartesian graph, to the Activity on collision analysis. His students will discuss about this the following class.

L5 Peter: Well, you’re going to have a graph like this one [*see Figure 2-Photo 1*]. Here, what are you plotting? Time versus location. So, this point [*see Figure 2-Photo 1*] is the collision. Why? Why do I say this point is the collision?

L6 S2: Because the velocity changes.

L7 Peter: Because the velocity changes. How do you know the velocity of this graph? [*points at the graph on the board*] From the slope.

When referring to the graph, Peter adds another element to the process of awareness, a sign that is a meaning carrier at the same time. Peter explains these meanings: “And then, this point is the collision.” (L5). However, Peter is aware that, for the students to achieve awareness on the meaning of this point in the graph, the meaning of the point must be sensible to them. Then, in the following class, Peter takes the analysis deeper by incorporating the use of software to analyze a video recorded by the students. In the video, two pellets collision.

L8 Peter: Let’s see, where do they collision? [*a student plays the video*] Stop! [*he says when the collision in the video is about to happen; see Figure 2-Photo2*] Rewind [*he asks the students to rewind the video a little*] There! Let’s see if you detect that the change of the momentum in the graph corresponds to this moment [*he focuses his attention on the graph*; see Figure 2-Photo3]. Do you see it? Then, this here is the collision [*referring to the point in the graph he had spoken about the class before; Figure 2-Photo1*].

Figure 2: From left to right, photos of: graph representing the collision phenomenon (Photo 1), software analysis of a video recorded by the students on the collision phenomenon (Photo 2), graph obtained using software (Photo 3).

The students have tried to provide a meaning to a graph made by Peter (L5), yet he considers this is not enough. He then incorporates the use of a resource (software). We can state that Peter uses the software as a resource., because this resource represents an artifact that influences the students’ knowledge production when trying to provide a meaning for the movement analysis during its *use*. The software is also a resource because of the intentionality and scheduling of its use. Using the software, the students manage to observe the phenomenon (Figure 2-Photo 2) and provide a meaning to the change of slope in the graph (Figure 2-Photo 3). The change of slope represents the change in velocity and, therefore, the moment of collision, as Peter said (L5). This is how Peter promoted awareness related to velocity change when two objects collide.

CONCLUDING REMARKS

In this article, we analyzed how a novice teacher and an expert one promote awareness in their students regarding the analysis of object movement. The analysis, due to the cultural and historical nature of knowledge, reveals the importance of mediation though Activity –where the use of artifacts and gestures is placed–; as well as the use of resources in teaching practice to promote awareness in students concerning the meaning of concepts. We observed differences in the practices of the two teachers. Each addressed a different content: while Edgar focused on kinematics exercises, Peter dealt with dynamics phenomena where the concept of force (not discussed in this work) is essential. Piaget (1979) considers that, in ancient times, there was a lack of analysis on the notion of force (the core of Newtonian dynamics) to build rational kinetics and mechanics. We also observed that Peter kept on developing the Activity across several sessions. He revisited concepts previously discussed, which points to a continuity when promoting awareness, and even more when directed to awareness on the meaning of concepts. In contrast, Edgar focused on a more numerical matter by solving problems that were discusses only in one session. Finally, we must consider Peter's use of software [artifact]. Radford (2014c) notes that digital artifacts are complex objects that carry historical meanings; they are more than providers of places to experiment. These artifacts deeply and distinctively affect the meanings the students create by suggesting defined ways of action and reflection. They also affect the potential lines of social and cognitive development.

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MATHEMATICAL COMMUNICATION AND NOTE-TAKING IN DYADS DURING VIDEO-BASED LEARNING WITH AND WITHOUT PROMPTS

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18 pairs of novice teacher students from a German university learned with video tutorials in two different study settings, either with or without accompanying prompts. This paper focuses on the commonalities and differences in mathematical communication and note-taking behavior within the dyads while learning with the videos which dealt with descriptive statistics, i.e. measures of center and spread. The analyses of the coded videos reveal differences between the two groups with regard to the 'density' of mathematical communication and the taking of notes in different processing phases.

LEARNING WITH MEDIA IN DYADS

In the last years, more and more instructional media like video tutorials, podcasts, presentations were created to facilitate the transition from secondary school to university for students in mathematical courses and courses that need mathematics during the first semesters (psychology, business administration, engineering, …) (Biehler, Fischer, Hochmuth & Wassong, 2014). Apart from their mere design, the successful implementation of such formats depends on several factors like the social form, supporting impulses and the learners' prior knowledge and learning strategies, etc. (e.g. Lou, Abrami & d'Apollonia, 2001).

Research has shown that the communication between the learners — when learning in dyads — is crucial for their learning benefit, e.g. that some dialogue-patterns are more beneficial for learners than others (Chi & Menekse, 2015; Lavy, 2006; Teasley, 1995).

The role of prompts in collaborative learning sessions is not extensively investigated so far. Hausmann, van de Sande & VanLehn (2008) have shown in a worked-example setting that groups which are prompted to self-explain perform better than prompted single learners. Furthermore, an analysis of the protocols showed that pairs focus more on specific self-explanations in their communication than single learners (Hausmann, Nokes, VanLehn, & van de Sande, 2009). Further studies analyze the effects of prompts in video learning contexts for metacognitive processes (Moos & Bonde, 2016) or the outcome with respect to argumentation skills and knowledge (Schworm & Bolzert, 2014); however, the concrete learning processes are investigated seldom.

Thus, it remains mostly unclear in how far prompts affect the *communication behavior* — that is the 'when', 'how' and 'what' of the mathematical communication processes — of groups.

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Another part of the collaborative learning process that is rarely investigated is the learners' *note-taking behavior* — that is when, how and what learners write down during their learning phases. When students take notes, they write down information and filter, comprehend, organize, restructure and integrate it into their knowledge (Makany, Kemp & Dror, 2009; Anderson & Armbruster, 1986). Investigating the process of note-taking and working with notes in dyads has rarely been looked into so far. The few qualitative analyses underline the importance of inscriptions and materialities for face-to-face interactions (Streeck, Goodwin & LeBaron, 2014) and learning processes (Salle, Schumacher & Hattermann, 2016). It is not known how the learners' note-taking behavior is structured when they learn with instructional media. Furthermore, the impact of prompts on this behavior is unknown. Therefore, our selected research questions are the following:

- *How long and when do the students of the prompt- and no-prompt groups communicate about mathematical aspects during their learning processes?*
- *How long and when do the students of the prompt- and no-prompt groups take notes?*
- *What are the differences concerning these aspects between the dyads working with prompts and those working without?*

METHODS

Procedure

The 36 teacher students were asked to work in dyads with video tutorials dealing with descriptive statistics in order to pass a post-test after having learned with the videos. Half of the dyads received prompts that are questions or short tasks asking for aspects of the instructional material, the other half did not. The questions addressed the common use of measures of center and measures of spread and the answers were typed with a computer keyboard.

The media-intervention-phase with the instructional material lasted about 70 minutes during which the investigator left the room. The computer screen was captured in the meantime; the sound and the image of the two learners were videotaped. Moreover, students were allowed to take notes which were collected directly after the intervention and scanned for further analyses.

Sample & Learning Material

The whole study is conducted with five different instructional media formats (e.g. verbally annotated scripts, video tutorials) at four German universities (total: N = 300). Because of the still ongoing analyses and our interest in the collaborative learning part, we focus on the dyads of one location in this paper exemplarily. From the 42 students working in dyads at Bielefeld University, the video material of 3 dyads could not be further processed due to unusable data. The remaining 36 teacher students (32 female, 4 male) who worked in dyads during the intervention are in their first semester.

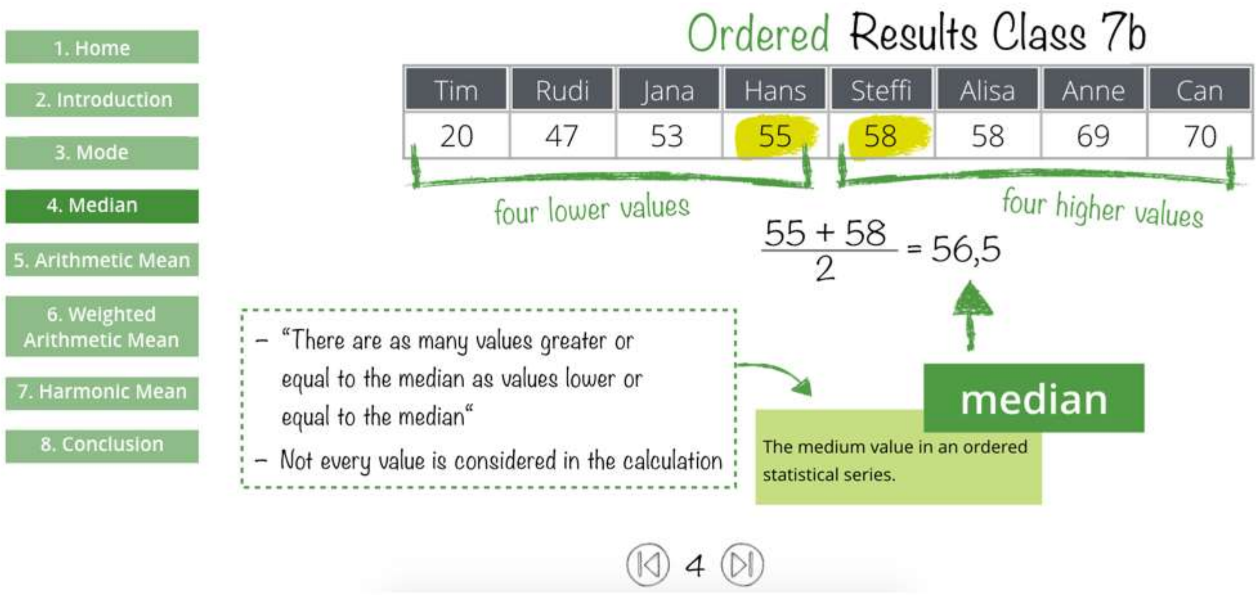

Figure 1: Screenshot of the video explaining the median (translation by authors).

The 36 students learned with two educational video presentations, both about 13 minutes long. The first one focuses on measures of center (i.e. median, arithmetic mean, harmonic mean), the second one on measures of spread (i.e. range, variance, standard deviation). The video presentations explain statistical terms and concepts embedded in the context of an invented smartphone typing competition at school. Both presentations, which start with an introduction and end with a summary in each case, were organized into several subsections. On the left-hand side (Fig. 1), there was a menu item for each of those subsections. By clicking on a particular subsection, a new slide appears and a video concerning the topic could be started by clicking on the play-button. Each video builds up its content step by step, accompanied by matching verbal annotations. The students could pause, rewind and fast-forward the media.

Analysis

The learner interaction and the computer screen were captured and analyzed using video recordings. For the analysis, a time-sampling method was used and the videos were segmented into 10 seconds-units (Bakeman & Gottman, 1997; Petko, Waldis, Pauli & Reusser, 2003; Seidel, Prenzel & Kobarg, 2005). A coding scheme with three focal points was created based upon data from a pilot study and a literature review. This scheme was applied for each 10 seconds-unit:

(1) *Communication* (Do the students communicate with each other about mathematical aspects of the intervention?): categories are "communication about mathematical aspects" and "communication not relevant for the study".

(2) *Medium* (i.e. which thematic section of the video presentation is present on the screen): exemplary categories are 'Introduction', 'Arithmetic Mean', 'Median', Harmonic Mean', 'Prompt template', etc.

(3) *Notes* (whether or not students are taking notes; differentiated by students): categories are “Left student takes notes”, “Right student takes notes”.

A 10-second-segment for (1) and (2) was coded, if 5 or more seconds could be identified as the particular category. (3) was coded, if the student used his/her pen/pencil on the paper even if only very short during the 10-second-segment.

Intercoder reliability

The coding of the 18 videos was conducted by four persons. To determine the intercoder reliability, two of the 18 videos were coded by all four persons (Lombard, Snyder-Duch & Bracken, 2002). The overall Krippendorff’s alpha was 0.77 for the coding of communication behavior and 0.91 for the coding of note-taking behavior, which are satisfying respectively excellent reliability values (Krippendorff, 2013).

RESULTS

The lengths of the intervention phases were nearly equal. On average, the 9 pairs of the no-prompt group learned for 3883 sec, sd = 341 sec (64:43 minutes, sd = 5:41), the 9 pairs of the prompt group learned for 3906 seconds, sd = 665 sec (65:06 minutes, sd = 11:05). On average, 1073 sec, sd = 452 sec (17:53 minutes, sd = 7:32) of this time was coded as ‘prompt template’.

The analysis of the coding of communication about mathematical aspects shows one outlier in both groups – one dyad (in the prompt group) communicated about 2 and half times more than all other dyads and another one (in the no-prompt group) about 2 and a half times less. So, in the following analyses, 8 pairs in every group — without the outliers — will be compared.

1. Differences between both groups in overall coding results

A comparison of the average coding results concerning communication and note-taking behavior of both groups shows differences in the note-taking behavior (Tab. 1). Table 1 shows the percentages of coded segments compared to all possible segments.

Category: Notes	**Prompts** (n = 8)	**No Prompts** (n = 8)
Left student takes notes	M = 31.6 % (SD = 11,5 %)	M = 34.5 % (SD = 7.5 %)
Right student takes notes	M = 31.9 % (SD = 16.3 %)	M = 36.3 % (SD = 7.8 %)
Category: Communication		
Mathematical communication	M = 18.9 % (SD = 4.0 %)	M = 25.4 % (SD = 7.9 %)

Table 1: Coding results for note-taking and communication.

Overall, there is a non-significant difference between the note-taking behavior of the two groups: The no-prompt group took more time to create notes (Mann-Whitney-Wilcoxon Test, p=0.44). The no-prompt group communicated not significantly (Mann-Whitney-Wilcoxon Test, p=0.65) longer about mathematical aspects than the prompt group.

The reported results give only hints concerning the coding for the whole intervention time of both groups. It remains unclear, how the prompt time is distributed over the intervention phase and if several patterns can be identified there. Concerning the communication about mathematical aspects in the prompt group, it is not clear, if there are differences during ‘prompt template’-time and the time the video presentation is opened. The same remains unclear for the note-taking behavior.

To give some insight into these uncertainties, we will investigate the coding of the intervention phase more detailed in the next paragraph.

2. Codeline differences between both groups – mathematical communication

Figures 2 and 3 show codelines from dyads in the prompt and no-prompt group.

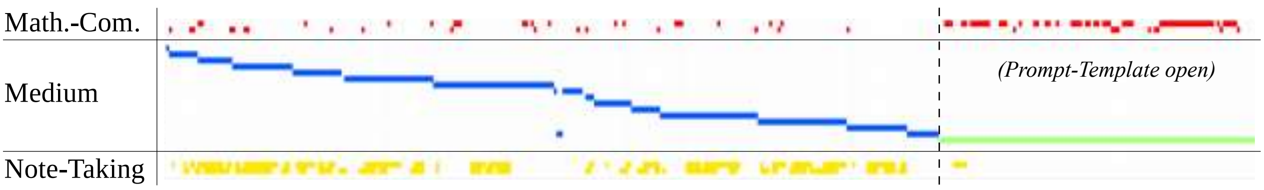

Figure 2: A codeline of a dyad in the prompt group (H5Z120).

The rectangles in the upper corridor depict all segments coded as “communication about mathematical aspects”. The corridor in the middle depicts the medium categories. In this case, the prompt template is opened — visualized as a horizontal bar after the vertical dashed line. The bottom corridor shows the coding of “left person takes notes” and “right person takes notes”.

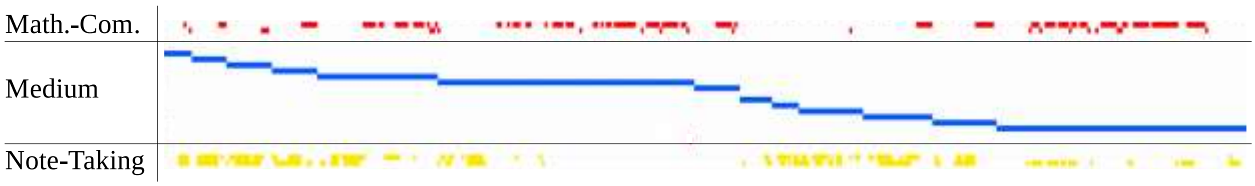

Figure 3: A codeline of a dyad in the no-prompt group (H5Z008).

All dyads in the prompt group showed the structure from the upper codeline (Fig. 2): First, they learned with the medium and then they answered the prompt questions. Two of the dyads opened the medium again during the prompt phase. In conclusion, a real prompt phase can be detected.

A comparison of the codelines showed different distributions for the category “communication about mathematical aspects”. In the prompt group, the codings seem to clot during prompt phases. An analysis of the coding during different phases of the intervention revealed that the *density* of communication about mathematical aspects during prompt phases — intervals coded as ‘communication about mathematical aspects’ divided by intervals coded as ‘prompt template’ — is very high compared to other phases (Fig. 4). Further, the no-prompt phases in the prompt group are comparatively free of communication about mathematical aspects.

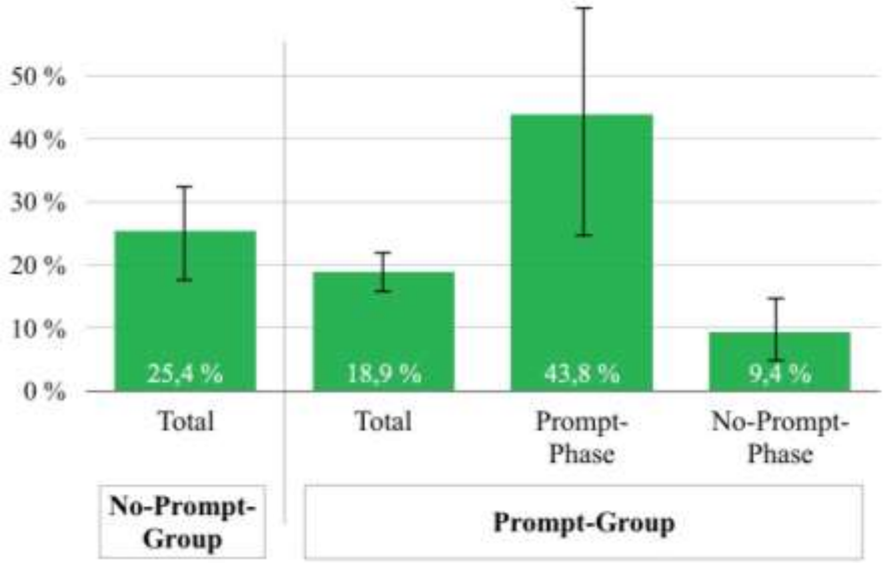

Figure 4: Average density of communication about mathematical aspects during phases of the intervention with standard deviation.

Different structures could be observed in the codelines of the no-prompt group. Codings of some pairs' communication showed clotting, others were spaced out evenly. A further, more detailed analysis cannot be given in this paper.

3. Codeline differences between both groups – note-taking behavior

All students of both groups take notes eagerly on most of the slides of the presentation. Dyads in the prompt group almost solely take notes when they do not work on the prompts themselves. Only 6 segments in total (of 2012 units coded as „person takes notes") were coded during prompt-phases. That does not only mean that no new notes were created. Moreover, no notes were altered or corrected during the prompt phases either.

Dyads working without prompts also show periods in which no note-taking is coded. This especially takes place during the "summary-slides" in the first and second video (see e.g. Fig. 3, sixth bar in the middle corridor). Due to lack of space, these and further analyses cannot be presented here in detail.

CONCLUSION & PERSPECTIVES

It could be shown that there are differences regarding the mathematical communication and note-taking in dyads with or without prompts.

It seems as if the no-prompt group is more active regarding the mathematical communication, although prompt phases seem to foster communication about mathematical aspects in the current setting. We found a clear structuring of the intervention phase in the prompt group.

One reason for this clear structuring in the prompt group may be the concrete questions. To answer them, both video presentations must have been watched. It remains unclear if different questions, which e.g. require only one video presentation, would lead to a different structuring. In how far characteristic phases may be identified in the no-prompt group must be revealed by deeper analyses of communication behavior. Taken together with the note-taking behavior, one can identify phases with a lot of communication about mathematical aspects, but no note-taking — maybe some kind of review phases during that students use their notes to sum up what they have learned.

During prompt phases, no notes were created, corrected or altered. One reason is the use of the computer to answer the prompts. Nevertheless, it is interesting to recognize that no matter what is discussed during the prompt phases does not influence the notes.

The presented findings are only a partial result of a larger study so far. Analyses are still ongoing and will be complemented by the samples from other universities with their specific instructional material. This should allow us to check if the higher girls-ratio in this course of study influenced the communication or note-taking behavior. Next steps will lead to looking into the corresponding contents and possible relations between the amount of time spent discussing or noting with regard to the topics covered at that moment. Moreover, looking into the data of the other samples in order to find similar schemes or patterns of communication and noting will be another issue to cover. With this knowledge, it may be possible to make some more general statements about the pros and cons of video learning with or without prompts.

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CONCEPTION TO CONCEPT OR CONCEPT TO CONCEPTION? FROM BEING TO BECOMING

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Previous approaches to mathematics knowing and learning have attempted to account for the complexity of students' individual conceptions of a mathematical concept. Those approaches primarily focused on students' conceptual development when a mathematical concept comes into being. Recent research insights indicate that some students give meaning not only to states/objects that have a being but also to states/objects that are yet to become. In those cases, conceptual development is not meant to reflect an actual concept (conception-to-concept fit), but rather to create a concept (concept-to-conception fit). It is argued that the process of generating a concept-to-conception fit, in which ideas that express a yet to be realized state of the concept are created, might be better referred to meaning-making than sense-making.

INTRODUCTION

Consideration of mathematical concept formation has a long history in, and is certainly an important branch of, cognitive psychology in mathematics education (see Skemp, 1986). Previous research has focused on the complexity of students' conceptions and their conceptual development when a mathematical concept comes into being. Students have been regarded as active sense-makers in mathematical concept formation (von Glasersfeld, 1995), that is, students actively seek comprehensibility of a mathematical concept. Students might, in this process, develop conceptions (from Latin *concipere*, 'to conceive') of a mathematical concept that are construed by a researcher (or educator) as a way a mathematical concept is perceived (or regarded) as it seems to be (for a discussion on conception and concept, see Simon, 2017). Recent research, however, suggests that students not only activate conceptions to make sense of how they perceive (or regard) a mathematical concept that comes into being in a certain context but also to imagine (or envision) a mathematical concept that is yet to become. In those cases, conceptual development is not meant to reflect an actual concept, but rather to create a concept.

The purpose of this paper is to clarify in which respects this act of creation differs from sense-making construed as an act of comprehension. In doing so, a theoretical background is briefly outlined that orients the general discussion of concept formation and sense-making. Afterward, key insights from recent research are summarized that foreground the act of creation in concept formation. Then, critical differences between two different states that a mathematical concept can have ('making it being' and

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 145-152. Singapore: PME.

'making it becoming') are discussed which allow to conclude that the act of creation might be better understood as meaning-making than sense-making.

THEORETICAL BACKGROUND: ON CONCEPT AND CONCEPTION

The work presented here is framed in theoretical assertions made by Scheiner (2016) with regard to mathematical concept construction. In Scheiner's (2016) view, the meaning of a mathematical concept comes into being in the ways that an individual interacts with the concept; or more precisely, in the ways that an individual interacts with objects that in a Fregean (1892a) sense fall under a concept. (A mathematical concept might be best described as an organic, multidimensional, structured gestalt, whose dimensions emerge from an individual's interactions with it.) As such, a concept does not have a fixed meaning. Rather, the meaning of a concept is relative (a) to the senses$_F$ that are expressed by representations that refer to objects coming under a concept and (b) to an individual's system of ideas$_F$ (the subscript F indicates that these terms refer to Frege, 1892b). Frege (1892b) revealed the fundamental distinction between reference and sense$_F$ as two semantic functions of a representation (an image, sign, or description): a reference of a representation is the object to which a representation refers, whereas a sense$_F$ of a representation describes a certain state of affairs in the world, namely, the way that some object is presented. Thus, it seems to follow that we may understand Frege's notion of an idea$_F$ in the manner in which we make sense of the world. Ideas$_F$ can interact with each other and form more compressed knowledge structures, called conceptions. A general outline of this view is provided in Fig. 1.

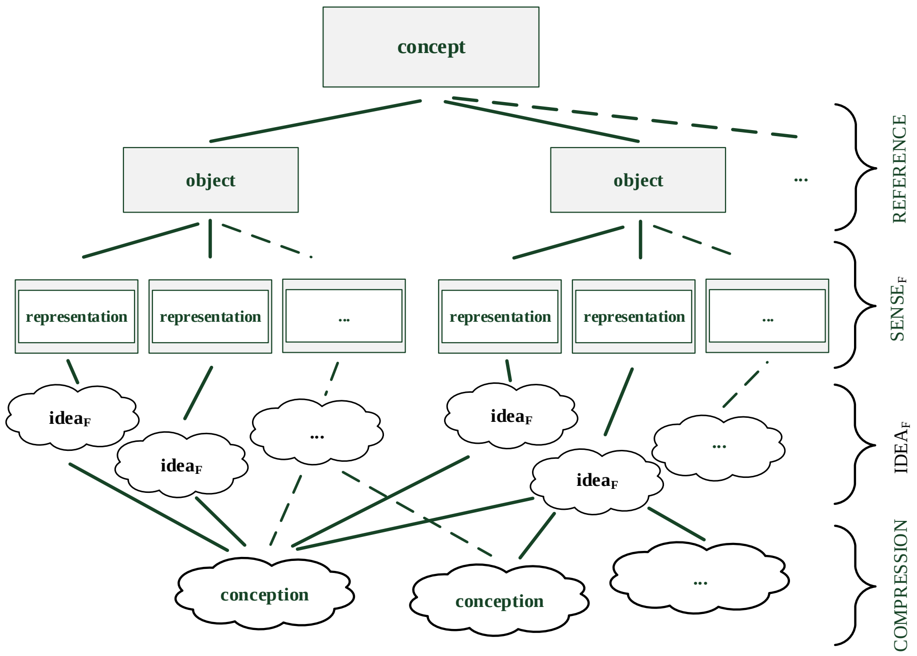

Fig. 1: On reference, sense$_F$, idea$_F$, and compression

(reproduced from Scheiner, 2016, p. 179)

There are several ways that individuals can make sense of a mathematical concept; the focus here is on extracting meaning and giving meaning (Pinto, 1998). Pinto and Tall (1999) stated with respect to sense-making of a formal concept definition,

> "Giving meaning involves using various personal clues to enrich the definition with examples often using visual images. Extracting meaning involves routinizing the definition, perhaps by repetition, before using it as a basis for formal deduction." (p. 67)

Tall (2013) explicated that these two approaches are related to a 'natural approach' that builds on the concept image and a 'formal approach' that builds formal theorems based on the formal definition. Scheiner (2016) broadened the original conceptualization provided by Pinto (1998), emphasizing that individuals can extract meaning from objects and give meaning to objects; or more precisely, extract meaning from their interactions with objects and give meaning to their interactions with objects. Further, extracting meaning was linked to the manipulation of objects and reflections of instances that appear in senses$_F$ when objects are manipulated – a phenomenon often discussed in terms of reflective abstraction, that is, abstraction of actions on mental objects (see e.g., Dubinsky, 1991). Giving meaning was related by Scheiner (2016) to attaching meaning to instances of objects that appear in senses$_F$ – a phenomenon that has been considered in terms of structural abstraction, that is, abstraction of "the richness of the particular [that] is embodied not in the concept as such but rather in the objects that falling under the concept […]. This view gives primacy to meaningful, richly contextualized forms of (mathematical) structure over formal (mathematical) structures" (Scheiner, 2016, p. 175). Scheiner (2016) offered a theoretical grounding for coordinating extracting meaning and giving meaning by putting in dialogue reflective abstraction and structural abstraction. Earlier, Tall (2013) discussed the relations of structural and operational abstraction and the natural and formal approach that evolve into a wider framework of the long-term development in mathematical thinking. (Structural abstraction focuses on the structure of objects, and operational abstractions on actions that become operations that are symbolized as mental objects (Tall, 2003).) The research presented in this paper has built on these theoretical interpretations of extracting meaning and giving meaning, and the assumed relationship between them.

RESEARCH BACKGROUND: GIVING MEANING REVISED

Recently, Scheiner and Pinto (2017a, 2017b) reanalyzed students' reasoning and sense-making of the limit concept of a sequence using theoretical innovation that involved contextuality, complementarity, and complexity of knowledge, plus knowledge development, and knowledge usage when giving meaning.

In their case study, Scheiner and Pinto (2017a) discussed giving meaning as a sense-making strategy in which ideas$_F$ are activated to give meaning to instances of an object that are actualized in certain, or even new, contexts. They described that the context in which an object is actualized might trigger the activation of ideas$_F$; however, it seems that it is not the context but the knowledge system that determines what is activated. (This does not mean that a knowledge system determines the meaning of a mathematical concept nor the form of interaction with objects that fall under a concept.) This is to say, it is not the context that determines the interpretation or meaning of an object, but the ideas$_F$ that are attached to instances of an object that orient an individual in giving meaning when making sense of certain contexts. As such, individuals do not construct a mental image of an 'external reality' that appears in the senses$_F$, but rather they give meaning to a sense$_F$ of an instance by attaching an idea$_F$ to it. Scheiner and Pinto's (2017a) analysis also suggests that this attachment is highly context dependent, that is to say, individuals might attach different ideas$_F$ to the same object that is actualized in different contexts.

In a cross-case analysis, Scheiner and Pinto (2017b) foregrounded that the attachment, however, seems to take place in such a way as to create and maintain coherence in a student's reasoning. However, the authors did not interpret coherence within the meaning of an established body of knowledge, but rather in the meaning of a student's usage. As such, coherence is not so much an attribution of the interconnectedness of the pieces of a created knowledge system, but of activity: students, who give meaning, activate ideas$_F$ that are coherent with their reasoning. This suggests that what seems to matter are coherence in reasoning and functionality of an individual's knowledge system, rather than any sort of correctness that mirrors a pre-specified 'reality' of the mathematical concept. This leads one to suppose that students are not concerned with creating a knowledge system that best reflects a given reality, but they are concerned with creating a reality that best fits with their knowledge system.

The most remarkable issue, however, is that Scheiner and Pinto's (2017a, 2017b) analyses point to the idea that students might even give meaning to states that are yet to become. This means though an object does not appear in a sense$_F$, an individual might create an idea$_F$ of a potential instance of that object. That is, students might give meaning beyond what is apparent. It is proposed that the creation of such ideas$_F$ is of the nature of what Koestler (1964) described as 'bisociation', and Fauconnier and Turner (2002) elaborated as 'conceptual blending'.

Koestler's (1964) central idea is that any creative act is a *bisociation* of two (or more) unrelated (and seemingly incompatible) frames of thought (called matrices) into a new matrix of meaning by way of a process involving abstraction, analogies, categorization, comparison, and metaphors. More recently, Fauconnier and Turner (2002) elaborated and formalized Koestler's idea of bisociation into what they called *conceptual blending*. The essence of conceptual blending is to construct a partial match, called a cross-space mapping, between frames from established domains (known as inputs), in order to project selectively from those inputs into a novel hybrid frame (a blend), comprised of a structure from each of its inputs, as well as a unique structure of its own (emergent structure).

The point to be made here is that unrelated ideas$_F$ can be transformed into new ideas$_F$ that allow 'setting the mind' (see Dörfler, 2002) not only to actual instances but also to potential instances that might become 'reality' in the future. In those cases, conceptual development is not merely meant to reflect an actual concept, but rather to create a concept (see Lakoff and Jonson (1980) on the power of (new) metaphors to create a (new) reality rather than simply to give a way of conceptualizing a preexisting reality:"changes in our conceptual system do change what is real for us and affect how we perceive the world and act upon those perceptions" (pp. 145-146.)). It is reasonable to assume that students transform ideas$_F$ to express a yet to be realized state of a concept.

DISCUSSION: ON 'MAKING IT BEING' AND 'MAKING IT BECOMING'

The research insights outlined in the previous section assert construing two different states that a mathematical concept can have: (1) a mathematical concept is given and comes into being in the dialogue of extracting meaning and giving meaning (in short, *making it being*) and (2) a mathematical concept is created and comes into becoming in the process of transforming ideas$_F$ (in short, *making it becoming*).

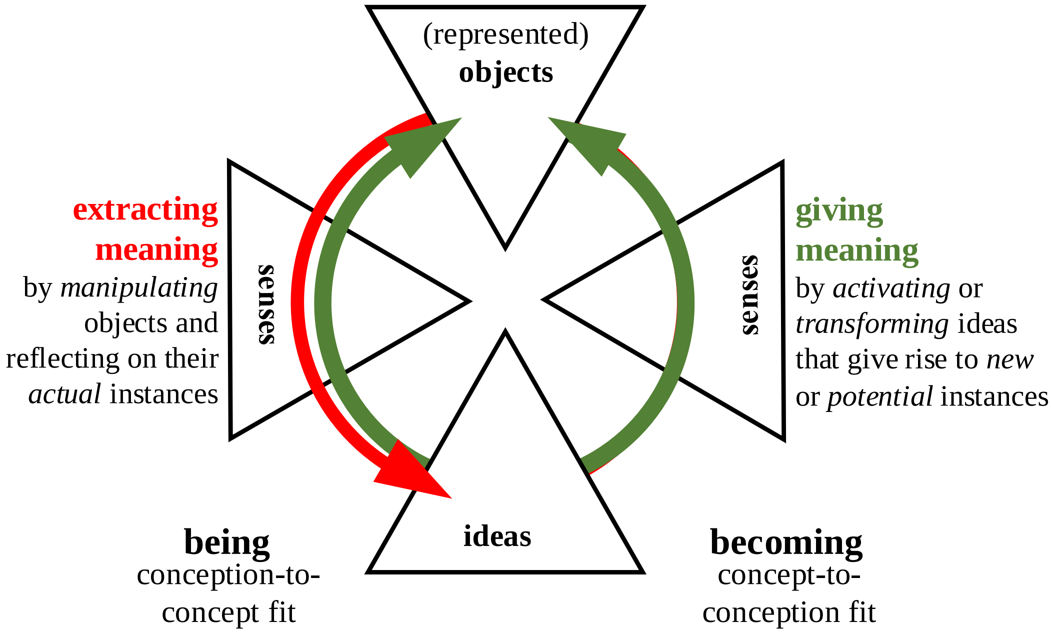

Fig. 2: From being to becoming

In making it being, extracting meaning and giving meaning can occur simultaneously: an individual might extract meaning by manipulating objects and reflecting on the actual instances of such objects, while at the same time an individual gives meaning to the instances that appear in the senses$_F$ by activating and attaching ideas$_F$ (see Fig. 2). With respect to giving meaning, an individual might either activate already available ideas$_F$ to attach meaning to instances or an individual might create new ideas$_F$ in the moment by transforming ideas$_F$ to gain new insight that allows attaching new meaning to an object of consideration.

In making it becoming, giving meaning means not only attaching ideas$_F$ to actual instances of an object but also creating new ideas$_F$ for potential instances. As such, ideas$_F$ can also be transformed in order to give meaning to instances that are yet to become (see Fig. 2). This means an individual might set her or his mind to future possibilities in which the object might be realized. In such cases, the mind would shape the future in a way that individuals might work to move the present to an intended future. That is, rather than creating conceptions that reflect a seemingly given concept, individuals might create a meaning of a concept that best reflects their conceptions of the concept. That is, individuals might create new forms of meaning, suggesting that the meaning of a mathematical concept varies on its actual use and intentions, rather than having an inherent meaning.

The differences between "making it being" and "making it becoming" can be discussed around at least three related issues:

(1) *Different states of the meaning of a mathematical concept*

In making it being, students treat objects that fall under a concept as states that have a being. Here students seem to understand the meaning of a mathematical concept as given. As such, an individual might extract meaning from manipulating objects and give meaning to actual instances of such objects. The meaning of a concept, then, emerges (from Latin *emergere*, 'to become visible') in the dialogue of extracting meaning and giving meaning.

In making it becoming, students create new ideas$_F$ by transforming previously created ideas$_F$ that are directed to objects that are yet to become. They transform ideas$_F$ to create future possibilities. Here the meaning of a mathematical concept is created that is to say, the meaning evolves (from Latin *evolvere*, 'to make more complex') in transforming various ideas$_F$.

(2) *Different functions of senses$_F$*

In making it being, senses$_F$ are construed as bearers of actual instances of an object that seems to have a being prior to students' attempts to know it. That is, the seeming 'objectivity' of an object appears in such senses$_F$.

In making it becoming, objects are not seen as preceding students' attempts to know them. Senses$_F$ are not construed as bearers of instances of an object but rather as triggers to transform ideas$_F$ to create new, potential instances of an object.

(3) *Different directions of fit*

Making it being is meant to reflect the concept as it is actualized, suggesting a conception-to-concept direction of fit: students extract meaning that reflects the concept and give meaning that fits the concept as it is assumed to be.

Making it becoming is meant to create the concept, suggesting a concept-to-conception direction of fit: students express a yet to be realized state of the concept, that is, they express a way that the concept can, or should, be. Students create the meaning of a concept that fits their conceptions.

CONCLUSION: ON SENSE-MAKING AND MEANING-MAKING

Sense-making was discussed in this paper in terms of extracting meaning and giving meaning. Extracting meaning and giving meaning were construed as interactions with objects to seek comprehensibility of a mathematical concept when it is actualized. Individuals can make sense if their conceptions fit the concept as it is assumed, or pre-specified, to be. As such, sense-making is an act of comprehension that consists of creating conceptions that best reflect a given concept.

Recent research, however, prompts one to rethink how students give meaning in the immediate context. In addition to attaching activated ideas$_F$ (already existing in the knowledge system) to actual instances of a mathematical concept, ideas$_F$ can also be transformed to attach new meaning to potential instances of a mathematical concept that, in this process, comes into becoming.

While with respect to the former it is assumed that students might make sense of the objects that fall under a particular concept primarily within their existing knowledge system, the latter allows an individual to journey toward a new meaning of a concept. It is asserted that this might be better referred to as *meaning-making*.

In consequence, sense-making is here understood as an act of comprehension, while meaning-making is construed as an act of creation. In a nutshell:

> (1) A student might intend to *comprehend* a meaning of a mathematical concept in a way that best *reflects* the concept as it is. The meaning of a concept *emerges* (comes into *being*) by a continuous dialogue of the *sense-making* of extracting meaning and giving meaning.
>
> (2) A student might intend to *create* a meaning of a mathematical concept that best *fits* student's conceptions. The meaning of a concept *evolves* (comes into *becoming*) by *meaning-making* via transforming ideas$_F$.

It is hoped that this distinction better brings to light critical issues and underlying cognitive processes in students' sense-making and meaning-making. The research insights outlined above and the theorizing provided here allow one to sharpen the distinction between making sense when the meaning of a mathematical concept comes into being and making meaning when the meaning of a mathematical concept comes into becoming. This nuance of sense-making and meaning-making might better highlight the critical differences of 'making it being' and 'making it becoming' with respect to the different states of the meaning of a mathematical concept, the different functions of senses$_F$, and the different directions of fit.

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EYE-TRACKING AND ITS DOMAIN-SPECIFIC INTERPRETATION. A STIMULATED RECALL STUDY ON EYE MOVEMENTS IN GEOMETRICAL TASKS

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Eye-tracking offers various possibilities for mathematics education. Yet, even in suitably visually presented tasks, interpretation of eye-tracking data is non-trivial. A key reason is that the interpretation of eye-tracking data is context-sensitive. To reduce ambiguity and uncertainty, we studied the interpretation of eye movements in a specific domain: geometrical mathematical creativity tasks. We present results from a qualitative empirical study in which we analyzed a Stimulated Recall Interview where a student watched the eye-tracking overlaid video of his work on a task. Our results hint at how eye movements can be interpreted and show limitations and opportunities of eye tracking in the domain of mathematical geometry tasks and beyond.

INTRODUCTION

Eye-tracking—the process of capturing eye movements of persons when they are looking at stimuli at hand (Chen, 2011)—is a technology and research method increasingly gaining popularity over the last decade (Andrá et al., 2015; Salvucci & Goldberg, 2012). In the mid-1970s, commercial eye-tracking devices started to become available and since then eye-tracking became more accessible than ever before (Holmkvist et al., 2011). In particular, the recent advent of affordable portable head-mounted devices revived the promise of eye tracking and fuels an increased interest in this technology—also in the PME community as could be seen at PME 40.

Eye-tracking offers various possibilities for mathematics education research (e.g., Andrá et al, 2015; Obersteiner & Tumpek, 2016), in particular it lends itself to "visually presented cognitive tasks, [where] eye movements are assumed to correspond to mental operations" (p. 257). However, even in geometrical or otherwise suitably visually presented tasks, the interpretation of eye-tracking data is non-trivial. It typically rests on the so-called "eye-mind" hypothesis (Just & Carpenter, 1980), which posits that a person's eye movements are tightly related to their cognitive processes (Jang et al., 2014). The interpretation of eye-tracking data is challenging because (1) the eye-mind hypothesis does not always hold (Holmkvist et al., 2011) and (2) the interpretation of eye-movement data is not bijective (Hayhoe, 2004), and (3) is furthermore context-sensitive, in particular conditioned on the task (ibid.). The inherent ambiguity and uncertainty, which comes with the context-sensitivity can be reduced by narrowing down the interpretation of eye movements to a particular domain: "Although the mere presence of gaze at a particular location in the visual field does not reveal the variety of brain computations that might be operating at that

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moment, the experimental context within which the fixation occurs often provides critical information that allows powerful inferences” (p.267).

We therefore see the need to approach the methodological question of how to interpret eye-tracking data in the domain of mathematics education and its sub-domains. In particular, we focus on geometrical tasks, where eye-tracking data are perceived as especially beneficial (e.g., Schindler et al., 2016; Muldner & Burleston, 2015). We refer to a task set (geometrical creativity tasks, so-called Multiple Solution Tasks (MSTs)) and their corresponding entities (figures, lines, corners, etc.). Instead of relying on eye movement measures as common in mathematics education research (e.g., Muldner & Burleston, 2015; Obersteiner & Tupek, 2016), we focus on raw data—eye-tracking overlaid videos—thus avoiding a dependency on the eye-tracking device used or the actual computation of eye movement measures. This paper presents results from a qualitative empirical study in which we analyzed a Stimulated Recall Interview where a student watched the eye-tracking overlaid video of his work on a Multiple Solution Task and described and explained his according thoughts and strategies in detail. Results from the qualitative SRI data analysis hint at how and in what (different) ways eye movements can be interpreted (e.g., fixations on small areas, rapid eye movements). Beyond the directly considered domain of geometrical MSTs, our analysis also sheds light on opportunities and limitations of eye-tracking as a research method in the domain of mathematical geometry tasks and beyond.

THEORETICAL BACKGROUND

Eye-tracking

First methods for eye tracking date back to the beginning of the 1900s and initial methods were obtrusive or even invasive (Jacob & Karn, 2003). Nowadays video-based systems dominate the market for eye trackers; either in the form of head-mounted devices such as eye-tracking goggles (as used in our empirical study) or remote devices attached to a computer screen to display the visual stimuli (Holmkvist et al., 2011). Eye-tracking offers various possibilities for mathematics education research. It is used, for instance, for analyzing students' strategies when comparing fractions (Obersteiner & Tumpek, 2016), for identifying highly creative persons working on geometrical creativity problems (Muldner & Burleston, 2015), and for investigating students' strategies when working on geometrical creativity problems (Schindler et al., 2016). In particular, in geometrical settings researchers focus on “how and which information students are attending to” (Andrá et al., 2015, p. 241).

Interpreting eye-tracking data

In order to reduce the effort for analyzing eye-tracking data, events—computed from raw eye-tracking data—are typically analyzed instead of the raw data itself (Holmkvist et al., 2011). This holds also true for eye-tracking research in the domain of mathematics education. In particular, fixations and saccades are used (Salvucci & Goldberg, 2012). Fixations are moments when the eye remains relatively still and focuses—consciously or not—stably on certain focus point or a small area. Saccades are fast eye movements in between fixations (Chen, 2011).

However, the interpretation of eye-tracking data is non-trivial—and this is one key reason that prevents eye-tracking technology to fully live up to its potential (Jacob & Karn, 2003). Interpreting eye-tracking data typically draws on the so-called “eye-mind” hypothesis, which “posits that there is no appreciable lag between what is being fixated and what is being processed” (Just & Carpenter, 1980, p. 331), meaning that “what a person looks at is assumed to indicate the thought ‘on top of the stack’ of cognitive processes” (Jang et al., 2014, p. 318). However, the eye-mind hypothesis does not always hold: people can, e.g., look at an object without registering it in their working memory and, conversely, they may also recall non-fixated objects (Holmkvist et al., 2011). A second difficulty is that the mapping of students’ eye movements to their attention and their cognitive processes is *not bijective*: “Although a given cognitive event might reliably lead to a particular fixation, the fixation itself does not uniquely specify the cognitive event” (Hayhoe, 2004, p. 268). Fixations can, for instance, indicate difficulty of information extraction and interpretation (Jacob & Karn, 2003) or cognitive attention on the aspect of a task looked at (Andrá et al., 2015). We hypothesize that it can even indicate other processes, such as staring because of tiredness or boredom, or else. Finally, the interpretation of eye movements needs to be *context-sensitive*: conditioned on the task, the internal state of the participant, and their “cognitive goals” (Hayhoe, 2004, p. 268). A comprehensive theory about how to interpret eye-tracking data is thus limited to rather general relationships, for instance, that “saccades are preceded by an attentional shift to the target location” (p. 267) and that “shifts in attention made by the observer are usually reflected in the fixations” (p. 268). Notably, these general relationships do not relate to the semantics of the entities that caused visual attention. In order to reduce the inherent difficulty and ambiguity that comes through context-sensitivity, we suggest to investigate domain-specific interpretation (focusing on geometrical tasks) and take into account the corresponding, known semantics of visual entities in this domain (figures, lines, corners, etc.). Accordingly, we ask the research question: *How can students’ eye movements be interpreted domain-specifically?* We approach this question through a Stimulated Recall study (see below), which will also shed light on the questions of *What opportunities does the analysis of eye movements offer over the analysis of simple videos in our domain?* and *What limitations does the analysis of eye movements entail?*

METHOD

Setting the scene

This study took place in the Swedish research project KMT (“kreativa matteträffar”), where mathematically interested upper secondary school students worked on multifaceted mathematical problems and were fostered in their mathematical creativity over one year. This paper focuses on a students’ work on a particular MST (Fig. 1)—a geometrical MST, which had revealed itself rich and suitable for addressing mathematical creativity in prior work (Schindler et al., 2016).

Task: Solve the following problem. Can you find different ways to solve the problem? Show as many ways as you can find.
Problem: This figure is an equilateral hexagon: How big is the angle ε?
Remember: In an equilateral hexagon, all sides have the same length and all angles have the same size, which is 120°.

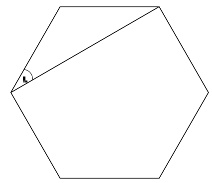

Figure 1: The hexagon-problem (Multiple Solution Task)

The student participating in this study was an 18-year old Swedish student in his last school year, David. David was very interested, talented, and dedicated to mathematics; he read mathematics books in his spare time and furthermore went on studying mathematics six months after this study had taken place. David worked on the MST wearing eye-tracking goggles and then an additional SRI was conducted using the eye-tracking overlaid video of his work on the MST with a length of 17:30 min.

The study described in this paper was carried out with the headset Pupil Pro (Kassner, Patera & Bulling, 2014). Though remote eye-trackers measuring eye movements on a computer screen can be advantageous in terms of accuracy (see Muldner & Burleston, 2015), goggles allow portable, unobtrusive eye-tracking and are easy to set-up. They can be used in a natural setting (the student worked on the MSTs with pen and paper) and in a familiar room, avoiding biases through an artificial surrounding.

Stimulated Recall Interview (SRI) based on eye-tracking overlaid video

In our endeavor to illuminate what eye movements may indicate and how they can be interpreted, we conducted a Stimulated Recall Interview (SRI) using the eye-tracking overlaid video of his work on the MST. Stimulated recall is a research method that is to be understood as an introspection procedure “through which cognitive processes can be investigated by inviting subjects to recall, when prompted by a video sequence, their concurrent thinking during that event” (Lyle, 2003, p. 861). In our case, we wanted the student to describe and explain his thinking using the eye-tracking overlaid video. SRI avoids the disadvantages that a thinking aloud method may have (e.g., high levels of interaction, time constraints, or emotive contexts, see Lyle, 2003). However, it also comprises weaknesses that have to be taken into account when planning an SRI study (Lyle, 2003): For instance, we reduced anxiety through (a) creating a trustful personal relation between the interviewer and the teacher over the project time span, (b) conducting the SRI in an environmental context well-known to the student (the room regularly used in the project), and (c) avoiding judgmental utterances by the interviewer, who rather indicated interest in the student’s thought.

In the SRI, the student and interviewer jointly looked at the eye-tracking overlaid video arising from the student’s work on the MST. The interviewer asked the student to comment on his eye movements. Both the student and the interviewer were able to stop the video and to go back. Also the student took the opportunity to explain his eye movements and thoughts. The SRI was taped by two cameras.

One important concern regarding SRI is about incomplete memories leading students to react to what they see on the video and accordingly rather re-construct their thoughts than recalling them (Lyle, 2003). David's utterances (in which he mostly used present tense when talking about his proceeding) indicate that he could recall his original thoughts impressively clearly. We further think that the eye-tracking overlaid video, a clear and strong stimulus, helped the student recalling his thoughts.

Data analysis

This paper focuses on the video data from the SRI with David (approx. 76 min). In a first step, we transcribed the largest parts of the video: Passages were left out when the discussion did not address the student's eye movements. We transcribed the student's and interviewer's utterances as well as the eye movements were addressed.

The data analysis was conducted in an inductive manner, which was suitable as our research questions are explorative and descriptive in their nature. Following Mayring's (2014) qualitative content analysis (focusing on the techniques of summarizing and inductive category development) and Beck and Maier's (1994) category developing text interpretation, we conducted the following analysis steps (see Tab. 1 for an example) that aimed at handling the comprehensive transcript and at inductively working out categories (e.g., special patterns of eye movements and their interpretation). In a *paraphrasing step*, we paraphrased the content-bearing semantic elements in the transcript relevant for our research questions. In a *transposing step*, we generalized these entities to the defined level of abstraction and transposed them to a uniform stylistic level (see Tab. 1). In a *category development step*, we went through all data (transposes) and inductively assigned categories and according descriptions/definitions. In a *category revision step*, we revised the category system after having categorized all data and—based on the revised category system—went through all data again, partially re-categorizing if necessary. In a *subsumption step*, for every category we collected all instances matching this category. Thus, we found, for instance, for the category "looking outside the task sheet" all interpretations of this eye movement arising from our data.

Transcript	**Paraphrase**	**Transpose**	**Category**
(D. looking outside the task sheet (saccade)) D: Now I'm just thinking and trying to remember how you calculate an interior angle in a regular polygon.	Looking outside the task sheet (saccade): thinking and trying to remember a calculation.	Looking outside the task sheet (saccade) indicates that he is thinking and trying to remember a calculation.	Looking outside task sheet

Table 1: Data analysis steps—examples

RESULTS

The analysis of the data from the case study gives hints on how eye movements can be interpreted and shows limitations and opportunities of eye tracking in the domain of mathematical geometry tasks. Below we summarize the categories of eye movement patterns and their interpretation and illustrate them with according instances.

Interpretation of eye movement patterns

David's SRI hints on how to interpret eye movements in different instances. In many cases, his visual attention matched his cognitive attention: When he was looking back and forth between two corners, he, for instance, thought "how can I use the fact that these two (angles, authors' note) are equal to start determining how big they are?" When encircling a triangle with his eyes, he "was thinking if I should do something with this right triangle over here" [see link for both]. Here, the cognitive attention largely agreed with the visual focus of attention; however, it was not inferable in what way the visual focus was relevant for the ongoing cognitive process. Fixations on a corner of the hexagon indicated, that David was summing up the adjacent angles in this corner [link]. The attention was on the corner (in line with the eye-mind hypothesis), however, the process of calculating was hardly inferable from the eye movement. As this clip furthermore illustrates, some fixations indicated that David noticed a mistake he had made before: "After I calculated that, then I realized that my final answer down here [which he then fixated, authors' note] was also wrong and so I must have made a mistake." Here, the eye-mind hypothesis holds: the focus of visual attention was the focus of cognitive attention (the mistake). The SRI revealed further ambiguities in the interpretation of eye movements. E.g., the eye movement of looking along a line can indicate both: envisioning this line in his mind [see link for several instances] or taking into account or comparing the two adjacent areas [link] with peripheral vision.

In other instances, the eye-mind hypothesis did not hold. In one instance, David explained that while fixating a point in the figure, he was calculating something else in his mind that did not have any connection to the fixated area [link]. Another eye movement pattern that David commented on seven times in the SRI was a quick saccade where he looked outside the task sheet [e.g., link]. This pattern always indicated that he was thinking or reflecting: how to proceed next, trying to remember a calculation, or conducting a mental calculation. He argued that this helped him to focus on a certain thought: "Then I look up and just think for a second. Because if I look at this (the task, authors' note), I get distracted a little bit. So I just wanna follow the exact same trail of thoughts for a couple of seconds." Furthermore, David commented on two instances where his eyes wandered around in the task rather "hectically", with quick saccades, without fixating meaningful entities (such as corners or lines). He described that in these instances, he was "panicking a little bit", because he had realized that he "had made a quite big mistake". The eye-mind hypothesis held insofar as the saccades were interrupted by a fixation on the mistake on the task sheet [see link]). A related observation was that accelerated eye movements, where saccades and fixations get shorter, can indicate excitement, e.g. induced by time pressure or a new discovery.

Opportunities and limitations of analyzing eye-tracking data

Order of approaches. The SRI revealed that David's eye moments reflected the order of strategies used even when this was not always reflected in his drawings, gestures, and writing. This indicates that the analysis of eye-tracking data has an analytical advantage over the analysis of pure, simple video data.

Discarded approaches. Our analysis indicated that when working on the geometrical task, David went into several "dead ends" that he finally discarded. He describes such approaches five times in the SRI commenting on his eye movements; however, none of them expressed themselves in any gesture, drawing, or writing.

Up-to-dateness. David's eye movements in many instances preceded his writing, drawing, and gestures, e.g. pointing. The SRI indicated that he was already thinking of and envisioning a line-to-be-drawn 10 sec or even 30 sec before he then drew it. The up-to-dateness is a further advantage of eye-tracking analysis; which especially becomes significant if researchers or educators want to interact with students and immediately react to their problem-solving (giving feedback, support, or similar).

Ambiguity. As outlined above, our results indicate that a bijective relation between eye movements and cognitive processes solving a geometrical task cannot be assumed.

Emotional arousal. In the SRI, it appeared that in all instances where David mentioned emotional arousal (excitement, panicking), the reliability of the tracking was reduced. It is currently not clear whether this is an artefact of the eye tracker used in the study.

DISCUSSION

There is no doubt that eye-tracking offers various opportunities for mathematics education research. However the so-called "eye-mind" hypothesis is a rather vague guiding principle for analysis, especially because—as pointed out— eye movement interpretation can only be valid if the specifics of the domain and context are taken into account. Accordingly, we see the need to address the question of how to interpret eye movements in the context of mathematics education and of how closely eye movements are in fact related to students' cognitive attention and processes.

The results from our study confirm the power that eye-tracking data analysis holds in geometrical MSTs and relates to previous research (e.g., Muldner & Burleston, 2015). We found that in many instances cognitive attention agreed well with visual attention, eye movements indicated approaches that were not perceivable in gestures or drawings, and eye movements often allowed for immediate access to students' cognitive attention. This relates to the perceived merit of eye-tracking "that we can examine how and which information students are attending to" (Andrá et al., 2015, p. 241). However, in other instances, the cognitive attention did not go along with the visual attention. In these cases, the eye-tracking data are misleading and can thus easily be misinterpreted. Furthermore, many cognitive processes were not perceivable in the eye movements. What a student is thinking while fixating a point or looking along a line is not visible and is—in an analytical viewpoint—subject to interpretation. This confirms the bijectivity of the mapping of students' eye movements to their cognitive processes (Hayhoe, 2004) also for the domain of geometrical tasks.

It is a challenge for future research to deal with the ambiguities and possible misinterpretations that our paper gives a glimpse on. We believe our case study to be a springboard for further discussion and research on the interpretation of eye-tracking.

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ARE VALUES RELATED TO STUDENTS' PERFORMANCE?

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Values are believed to be important for students' affect and achievements. In the present study, I administered task-unspecific and task-specific questionnaires to investigate a connection between students' values and performance in solving problems with and without a connection to the real world. 192 ninth graders were randomly assigned to group 1 or group 2. In group 1, students reported their values after task processing; and in group 2, they reported their values before task processing. The main result was in line with expectations: Students who achieved higher scores on the performance test reported higher values, and students who valued mathematics and problem solving activities performed better on the tests.

INTRODUCTION

Values are an important part of affect. However, they are the least studied of the affective measures in mathematics education (Zan, Brown, Evans, & Hannula, 2006). Prior research on values has focused mostly on values in mathematics teaching, which are reflected, for example, in school text books (Bishop, Seah, & Chin, 2003) or on case studies that have demonstrated the importance of values for changes in students' affect (Hannula, 2002). In education, a number of studies have been conducted to investigate the relation between values and students' achievement. However, only some of them have focused on mathematics, and in doing so, they have often included course choices or grades but not students' performance as indicators of achievement. In the present study, I examined whether students' performance on problems with and without a connection to the real world would be found to be related to students' task-unspecific and task-specific values, measured before and after task processing.

THEORETICAL BACKGROUND

Values and their relation to performance

Values have been investigated in cultural, social, and psychological contexts (Bishop et al., 2003) and refer to the subjective importance of objects (e.g., mathematics), actions (e.g., problem solving), or outcomes (e.g., grade in mathematics) for human beings. Values are believed to be valuable appraisals of motivation and emotions. Research on values has often been embedded into motivational and emotional theories such as the control-value theory of achievement emotions. For example, students' values were hypothesized to trigger their enjoyment, and positive changes in students' values were found to be related to positive changes in their enjoyment (Buff, 2014).

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Values can be traditionally categorized as intrinsic or extrinsic (or utility) values (Pekrun, 2006). Whereas persons with high intrinsic values ascribe high valence to mathematical activities per se, persons with high utility values do it because of the usefulness of these activities for their career, grades, or other indicators of success.

Students' high values are believed to influence their career-related choices, efforts in learning, persistence in achievement-related activities, and thus also to predict their learning outcomes such as performance (Guo, Marsh, Parker, Morin, & Yeung, 2015). In turn, the feedback students receive from their learning outcomes influences their responses to affective variables. Thus, higher performance in mathematics can trigger higher values with respect to mathematical activities (Simpkins, Davis-Kean, & Eccles, 2006). The hypothesized positive relation between students' values and their achievements in mathematics has partly been confirmed in empirical studies. High values in mathematics were found to be related to higher mathematics grades in the 10^{th} grade but not to higher mathematics grades in the 5^{th} grade (Simpkins et al., 2006). Intrinsic and extrinsic values were found to be positively connected to mathematical performance in Grade 8 (Guo et al., 2015).

Characteristics of measures of values

Solving mathematical problems is a complex process that is accompanied by different affective phenomena. Efklides (2006) distinguished between prospective, current, and retrospective affect measured before, during, and after problem solving activities, respectively. Students' prospective values indicate the importance they ascribe to problem solving before they start the solution process. Students' retrospective values indicate the importance they ascribe to task processing after it is completed. Both the prospective and retrospective valuing of problem solving activities are important for students' performance and achievements (Schukajlow & Krug, 2014).

Several calls in mathematics education have demanded that a variety of instruments be used to assess affect and to take into account the domain-specificity of affect (Zan et al., 2006). One way to heed these calls is to complement the well-known task-unspecific affective scales with a novel task-specific approach (Schukajlow et al., 2012). The application of two different measures of affect further allow researchers to examine the stability of the correlations between performance and affective measures. A main difference between the task-unspecific and task-specific approaches is the level of object specificity (Schukajlow, 2015). Whereas task-unspecific measures describe the object more generally, in task-specific questionnaires, the object of interest is specified in more detail. For values, task-unspecific questionnaires typically refer to the value of learning mathematics, whereas task-specific questionnaires refer to the value of solving a sample problem such as $2x + 4 = 9$. In the present study, I expected that the relation between students' values and performance would be similar for task-specific and task-unspecific measures because the two types of questionnaires assess the same constructs.

Problems with and without a connection to reality

Mathematical problems can be divided into two types of problems: problems with a connection to reality and problems without a connection to reality (or intra-mathematical problems) (Rellensmann & Schukajlow, in press). Problems with a connection to reality include modelling and "dressed up" word problems (Blum, Galbraith, Henn, & Niss, 2007). To solve modelling problems, students need to construct a situation model, which they then simplify and idealize before constructing a mathematical model. Further, students need to interpret and validate their results at the end of the solution process. "Dressed up" word problems present a simplified situational model, and thus, students can proceed with the mathematizing process directly after the task comprehension process. Moreover, they do not need to perform sophisticated interpretation and validation activities after calculating the mathematical results. Both types of real-world problems are important for learning mathematics (Schukajlow et al., 2012).

As problems with and without a connection to reality are essential parts of the curriculum in different countries, we chose these problem types to investigate the connection between values and performance. In a previous study, we found that students valued very similar problems with and without a connection to reality (Schukajlow et al., 2012). However, to the best of my knowledge, no studies have previously compared the relation between values and performance for these two types of problems. As both types of problems are mathematical problems, I did not expect that there would be a significant difference in the correlation between performance and values when comparing problems with and without a connection to reality.

PRESENT STUDY: RESEARCH QUESTIONS AND EXPECTATIONS

The present study was embedded in a research project aimed at investigating task-specific affect and its relation to performance (Rellensmann & Schukajlow, in press; Schukajlow, 2015). In the present paper, I addressed the following questions:

1) Is students' performance in mathematics positively connected to their task-unspecific and task-specific values measured *after* problem solving? Are students' task-unspecific and task-specific values measured *before* problem solving positively connected to their performance in mathematics?

2) Is students' performance connected more strongly to their *task-specific* than to their *task-unspecific* values measured after problem solving? Are students' *task-specific* values measured before problem solving connected more strongly to their performance than their *task-unspecific* values are?

3) Are correlations between performance and values measured after problem solving different for problems *with and without a connection to reality*? Are correlations between students' values measured before problem solving and performance different for problems *with and without a connection to reality*?

On the basis of theoretical considerations, a positive relation between performance and values was expected; students' performance was expected to be similarly related to task-specific and task-unspecific values; correlations between performance and values were expected to be comparable for problems with and without a connection to reality.

METHOD

One hundred ninety-two ninth and tenth graders from German middle track and grammar school classes (53.6% female; mean age=16.1 years) were randomly assigned to group 1 or 2. In group 1, students solved the problems first and afterwards filled out task-specific and task-unspecific questionnaires that assessed their values. In group 2, students first filled out both types of questionnaires and then solved the problems (Fig. 1).

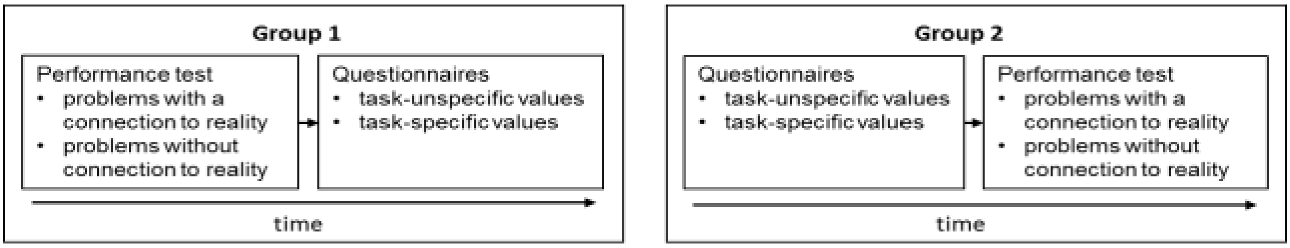

Fig.1: An overview of the study

Example of problems with and without a connection to reality

Sixteen problems with a connection to reality and seven problems without a connection to reality that could be solved by applying Pythagoras' theorem and linear functions were selected for this study. These problems were used to assess students' performance and their task-specific values. Sample tasks on the topic of Pythagoras' Theorem are presented below (for more sample tasks, see Rellensmann & Schukajlow, in press; Schukajlow et al., 2012).

Maypole

Every year on Mayday in Bad Dinkelsdorf, there is a traditional dance around the maypole (a tree trunk approx. 8 m high). During the dance, the participants hold ribbons in their hands, and each ribbon is fixed to the top of the maypole. The participants dance around the maypole with these 15-m-long ribbons, and as the dance progresses, the ribbons produce a beautiful pattern on the stem (such a pattern can already be seen at the top of the maypole stem in the picture).

At what distance from the maypole do the dancers stand at the beginning of the dance (the ribbons are tightly stretched)?

Fig. 2: Problem with a connection to reality "Maypole"

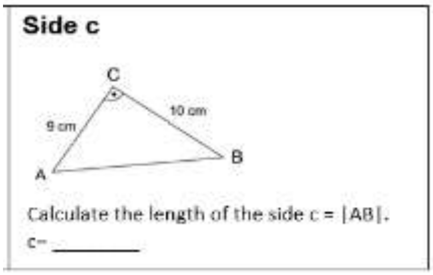

Fig. 3: Problem without a connection to reality "Side c"

Performance

Students' performance in solving problems with and without a connection to reality was measured with 16 and 7 problems, respectively. Cronbach's alpha as a measure of reliability for the test of the ability to solve problems with a connection to reality was satisfactory (.77). The reliability for the test of performance on problems without a connection to reality was low (.52) but acceptable for the small number of items and the diversity of mathematical procedures needed to solve the problems.

Task-unspecific and task-specific values

Task-unspecific values were assessed via the intrinsic component with scales that were taken from other studies and consisted of 5 statements that were answered on 5-point Likert scales ranging from (1=strongly disagree) to (5=strongly agree). A sample item is "Mathematics is my favorite subject." Cronbach's alpha was .85. To assess task-specific values, each of the 23 problems was followed by a statement about the extent to which the students valued the processing of the task. The instructions were: "Read each problem carefully and then answer some questions. **You do not have to solve the problems!**" After task processing, the students in group 1 were presented the problems again and were asked to rate the extent to which they agreed or disagreed with the statement "I think it is important to be able to solve this problem." Students in group 2 were asked before task processing to rate the same statement. A 5-point Likert scale was used to record their answers (1=not at all true, 5=completely true). One scale measured task-specific values for problems with a connection to reality and was formed across 16 problems (Cronbach's alpha=.96). Another scale measured task-specific values for problems without a connection to reality and was formed across 7 problems (Cronbach's alpha=.91).

An implementation check indicated that students in group 1 solved the problems significantly more often than students in group 2 *before* they reported their values (Schukajlow & Krug, 2014).

RESULTS

The analysis of the relation between students' performance and values assessed after task processing confirmed my expectations. Students who achieved higher scores on the tests valued mathematics and solving mathematical problems higher than students who achieved lower scores (see Table 1).

		task-specific values		task-unspecific values
		problems with a connection to reality	problems without a connection to reality	
performance	problems with a connection to reality	.386**		.500**
	problems without a connection to reality		.240**	.233*

Note: ***p*<.01; **p*<.05; one-tailed; sample size N=100.

Table 1: Pearson correlations between performance and task-specific and task-unspecific values after task processing (group 1).

Similar correlations were found for the relation of values measured before task processing and performance (Table 2), indicating that students with higher values with respect to mathematics and problem solving activities before task processing achieved higher scores on the performance test. However, three of four correlations just missed the significance level of .05, and thus the results should be interpreted cautiously.

		task-specific values		task-unspecific values
		problems with a connection to reality	problems without a connection to reality	
performance	problems with a connection to reality	.157[a]		.332**
	problems without a connection to reality		.149[a]	.145[a]

Note: ***p*<.01; **p*<.05; [a]*p*<.10; one-tailed; sample size N=92.

Table 2: Pearson correlations between performance and task-specific and task-unspecific values before task processing (group 2).

To answer the second and third research questions, I compared the correlations with Fisher's *Z*-scores and two-tailed significance tests (Steiger, 1980). In group 1, the correlations between performance and task-unspecific values did not differ from the correlations between performance and task-specific values. For example, the *Z*-score for the comparison between the correlations of .386 and .500 in Table 1 was 1.147 and was not significant (*p*=0.252). Similar results were found for the comparison of correlations in group 2 for problems without a connection to reality (.145 and .149) and for problems with a connection to reality (.157 and .332). The latter difference in correlations was found to be marginally significant (*Z*=1.726, *p*=.084) and indicated that students' task-unspecific values tended to be more closely related to performance than students' task-specific values for problems with a connection to reality.

In investigating the third research question, I was interested in differences in correlations between two types of problems: problems with and without a connection to reality. As expected, correlations between performance and task-specific values measured after task processing (.386 and .240) did not differ significantly between the two types of problems (Z=1.507, p=.132). However, the correlation between performance and values for problems with a connection to reality (.500) was higher than the same correlation for problems without a connection to reality (.233, Z=2.969, p=.003). Similar results were found for the relation of values measured before task processing and students' performance. There was no significant difference between the two problem types in the task-specific correlations (.157 and .149), but there was a difference for the task-unspecific ones (.332 and .145, Z=2.057, p=.040).

SUMMARY AND DISCUSSION

The aim of the study was to investigate the relation between performance and values. To achieve this aim, students' values were assessed before and after task processing, using task-specific and task-unspecific questionnaires and using problems with and without a connection to reality. As predicted by motivational theories (Guo et al., 2015), performance and values were found to be related to each other. This result indicates that students' performance might be important for the development of values and vice versa. The reciprocal relation between the two measures is an open question for future longitudinal and interventional studies.

As expected, correlations between performance on problems with and without a connection to reality and values were comparable for task-specific and task-unspecific scales as task-specific and task-unspecific measures refer to the same affective construct. Similar results were found for assessments of how performance is related to boredom, enjoyment, and interest (Schukajlow, 2015; Schukajlow & Krug, 2014).

The analysis of differences in correlations between problems with and without a connection to reality revealed that the type of problem is a significant factor that should be taken into account in future studies. Correlations between students' performance on real-world problems and students' task-unspecific values measured before or after task processing were higher than the respective correlations for intra-mathematical problems. Note that modelling problems were a significant part of the problems with a connection to reality used in this study. As these kinds of problems are not typical in mathematics classrooms, solving them might require a greater transfer of abilities than curricularly valid intra-mathematical problems. Because of this, the extent to which students value mathematics might be more strongly related of their performance on tasks with a connection to reality than on tasks without a connection to reality. A similar tendency was found for the relation between interest and performance on modelling and intra-mathematical problems (Schukajlow & Krug, 2014). Future studies are essential to investigate these findings further.

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MULTILINGUAL STUDENTS' DEVELOPING AGENCY IN A BILINGUAL TURKISH-GERMAN TEACHING INTERVENTION ON FRACTIONS

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Multilingual students who are able to exercise agency – when they deliberately position themselves to contribute to shape the mathematical discourse – can activate their specific resources. In this case study, a five-session bilingual Turkish-German teaching intervention for promoting the conceptual understanding of fractions is investigated in regard to how students develop their agency. Whereas in the beginning of the intervention the students contribute to the development of the discourse, the students' opportunities for exercising mathematical agency decrease over the course of the intervention. This might be a result of the discourse becoming more teacher-centered and the nature of the tasks. More research is needed to better understand this phenomenon.

INTRODUCTION

Multilingual students can activate their multilingual resources in the mathematics classroom, when classroom activities take multilingual experiences as starting point (Dominguez, 2011). Building on available linguistic resources and incorporating multiple modes and representations open pathways for multilingual students to participate in rich discourse practices (Moschkovich, 2015). Research on multilingual mathematics learning has started to investigate under which conditions multilingual students can activate their available resources in the mathematics classroom, where opportunities for students to exercise agency have been identified as a relevant condition (Langer-Osuna, Moschkovich, Norén, Powell & Vazquez 2016).

Agency is associated with students contributing to the development of the mathematical discourse, for example with students taking the initiative for their mathematical understanding and for constructing meaning (Gresalfi, Martin, Hand, & Greeno 2009, p. 56), and with students developing their own ideas and extending existing ideas (Boaler, 2002). However, the affordances for exercising agency, as well as the development of agency have not yet been systematically investigated for multilingual students. This case study investigates the development of 7th grade multilingual students' agency who, over the course of five weeks, participate in a five-session bilingual Turkish-German teaching intervention on fractions.

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MULTILINGUAL STUDENTS' AGENCY AND ITS DEVELOPMENT

In general, the concept of agency is concerned with the ways individuals can act in a given social situation, and specifically, is concerned with how individuals act within this situation in independent and self-reflective ways in order to contribute to shaping the social situation at hand. In teaching-learning situations, students ideally participate as thinking agents, and this way contribute to the unfolding of the mathematical content (Boaler & Greeno, 2000). Accordingly, agency means more than being engaged in the classroom, it includes the ability to make deliberate choices about how to participate (or to not participate) (Gresalfi et al., 2009). Here, a student's *mathematical agency* is defined as his or her deliberate, self-conscious positioning to direct the ongoing discourse or to contribute to its development (Norén, 2015, Boaler & Greeno, 2000). Mathematical discourses are ways of "combining and integrating language, actions, interactions, ways of thinking, […] using symbols, tools and objects to enact a […] socially recognizable identity" as doers of mathematics (Gee, 2011, p. 201). They develop by the introduction of new definitions, routines, objects, tools, symbols etc.

Agency is operationalized with positioning theory (Davies & Harré, 1990). Positioning theory states that individuals act based on their in-the-moment position in the conversation (Harré, Moghaddam, Cairnie, Rothbart & Sabat, 2009). The individual's actions, and the associated positons, become intelligible and objectively and subjectively coherent through storylines. Storylines are culturally shared, but individually represented repertoires of how conversations develop, for example between nurses and patients (Herbel-Eisenmann, Wagner, Johnson, Suh & Figueras, 2015). Storylines can belong to certain genera, or frames, for example the frame of school learning which comprises storylines that organize conversations in teaching-learning situations.

Students exercise agency when they deliberately position themselves. When students deliberately position themselves in an ongoing conversation in the mathematical classroom, they stress their independency and their identity as thinking agent – in other words, they deliberately engage in the discourse to contribute to its development. For example, students can deliberately position themselves to present their perspective on the mathematics at hand and/ or what their course of action would be (cf. van Langenhove & Harré, 1999, p. 24). Deliberate self-positioning is complemented by forced self-positioning, where the initiative for a position lies with someone else. In mathematical classrooms, the teacher can force students to position themselves, as he/she acts as a representative of the institution school, and thus, his or her demands or his or her questions can exercise a strong force for students to take a certain position (cf. van Langenhove & Harré 1999, p. 26).

While there are case studies on multilinguals agency (Norén, 2015), or suggested situations where especially multilinguals might be able to exercise agency (Langer-Osuna et al. 2016), there is no development model for agency in the mathematics classroom, according to the author's reading of the current literature. A possible trajectory for the development of agency could be that students learn to position themselves deliberately based on their previous experiences from forced self-positionings, where agency first exists in the interpersonal plane, and only later on the intrapersonal plane (cf. Vygotsky, 1986).

Thus, in this study, the following research question is investigated:

How does multilingual students' mathematical agency develop over the course of a five-session multilingual Turkish-German teaching intervention on fractions?

METHODOLOGY OF THE STUDY

Research context of the study

The study is part of the larger research project MuM-Multi, which is a learning process study on fostering multilingual students' understanding of fractions in a Turkish-German teaching intervention, conceived as a randomized control trial with n = 139 seventh grade multilingual students. The teaching intervention consisted of five sessions with 90 min. each. The students participated in small groups of 2-5 students in the teaching intervention, each group was videotaped. The here presented study is one of several case studies within MuM-Multi.

Data corpus

From the data corpus that was generated in MuM-Multi, here the second and fourth teaching intervention sessions are investigated in regard to the development of agency over the course of the teaching intervention. The first and last sessions were not chosen as the students need time to adapt to the teaching intervention in the first session, and might participate differently in the fifth session knowing that it represents the last session of the intervention. As this study is of exploratory nature, only one teaching intervention group is investigated. This group of four students mostly worked in pairs in Session 2 (Ilknur and Akasya, Halim and Hakan) and as a whole group in Session 4. Accordingly, transcripts of 2x90 min. in Session 2 and 90 min. in Session 4 were analyzed. In their regular classroom, these students are educated monolingually in German.

Methods of qualitative analysis

The transcripts are analyzed employing qualitative content analysis (Mayring, 2016) with the categories of deliberate and forced self-positioning. Further categories were generated from the material. In a first passage through the material, segments were identified were students exercise agency based on their use of linguistic markers that indicate self-positioning (I, me, myself, my), where these linguistic markers allow a relatively good approximation of self-positioning ("lexical bundles", Herbel-Eisenmann, Wagner & Cortes, 2008). In the second passage, these segments were analyzed in regard to the nature of the positionings and their frames. For investigating development processes, all segments for a student are ordered in their original order of occurrence.

EMPIRICAL RESULTS

The multilingual students Akasya, Hakan, Halim and Ilknur exercise agency within different frames. Here I will only refer to the two dominant frames of *private relations* and of *mathematical understanding*. The frame of *private relations* encompasses storylines of ‘being brought up by a strict mother’ or of ‘solving conflicts in the friendship’. The frame of *mathematical understanding* includes storylines that organize teaching-learning situations, for example ‘asking the teacher for assistance’. Only deliberate self-positionings within the frame of *mathematical understanding* are considered as candidates for *mathematical agency*.

Ilknur’s agency in Session 2

In the following episode, the student Ilknur is exercising agency within the aforementioned frame of *mathematical understanding*. This episode is located in Session 2 of the teaching intervention, in its fourth task (of 9). Previous to this episode, Ilknur and Akasya try to find fractions that are equal to 2/6 within the fraction bar board (Fig. 1), with which they have difficulties.

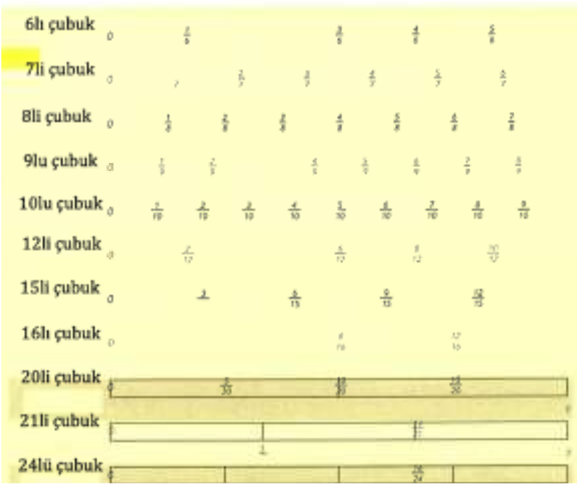

Figure 1: Fraction bar board of Akasya (written inscriptions were added in later tasks)

Turn Person	Original (Turkish in black, German in grey)	English Translation (from Turkish in red, from German in orange)
113 Ilknur	Şimdi bu zwei Sechstel ya	[points at the fraction bar for 1/6] Now, since this there is two sixth
	[…]	
119 Ilknur	Doch, das ist richtig! Guck zwei Sechstel, bak gel.	[looks at the fraction bar board und puts her ruler on it] After all, it is right. Look, two sixth, come over, look.
	Zwei Sechsteli aşağıya yaparsan bu zwei Sechstel	If you do two sixth downwards, this is two sixth [points at 7/21].
	Akasya, guck ma hier steht # Akasya, guck ma hier steht sieben Einundzwanzigstel. Das ist richtig.	[points at Akasya’s fraction bar board] Akasya, look, here stands, Akasya, look, here stands seven twenty-one. This is correct.
	Guck, das ist doch zwei Sechstel. Baksana. Bak şunu şu Strich. Ey, zwei Sechstel	[Ilknur compares Akasya’s fraction bar board with hers and puts her rules on Akasya’s board] Look, this is two sixth, after all. Look. Look at this line. Ey, two sixth. [inaudible]
	Ey, ben yapamıyorum! Çok zor!	Ey, I cannot do this. This is hard.

After a discussion with Akasya, Ilknur has an idea how to find equal fractions. She explains her thinking to Akasya, directly addressing her (“Look”), and explains that 7/21 has to be equal to 2/6. Above that, Ilknur comments on the correctness of her solution (“is correct”). In the end, Ilknur also addresses her difficulties with coming up with a solution.

Ilknur exercises agency in taking a deliberate and self-conscious position as a learner who struggles to understand the mathematics at hand, and this way self-positions herself complementary to her coworker Akasya. At the same time, she brings the discourse forward in proposing a strategy of how to solve the task, and assesses that her solution is correct and should be taken up by Akasya. In summary, Ilknur is exercising agency as an equal student in a shared struggle for understanding.

Ilknur’s Agency in Session 4

The following episode from the first task in Session 4, the students Ilknur, Akasya, Halim and Hakan try to determine 5/7 of 21. For that, they were given the fraction bar with seven fields by the teacher (Fig. 2), on which the students now distribute sunflower seeds.

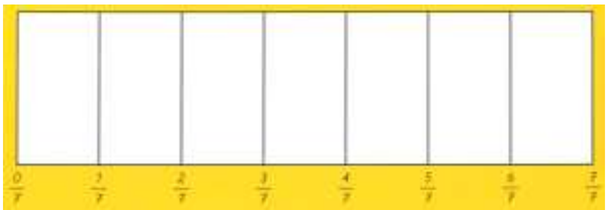

Figure 2: Fraction bar in Task 4

Turn	Person	Original (Turkish in black, German in grey)	English Translation (from Turkish in red, from German in orange)
346	Halim	Rechne ma richtig, nach Drei, Sechs, Neun, Zwölf, #vor Fünfzehn	Calculate right, after three, six, nine, twelve, # before fifteen [points at the respective field in the fraction bar for the denominator 7]
347	Ilknur	# Das war falsch. #Zwei	# That was wrong. Two.
348	Akasya	Zwölf, Fünfzehn, #Achtzehn, Einundzwanzig, und was steht hier?	Twelve, fifteen, #eighteen, twenty-one, and what is here? [points at the green card where 5/7 is printed on]
349	Halim	#Achtzehn, Einundzwanzig	#eighteen, twenty-one
350	Akasya	Einundzwanzig	Twenty-one
351	Ilknur	Ooh. Boah, dann hatte ich ja recht.	Oh. Wow, then I was right.

Halim and Akasya come up with a correct solution, and try to explain their solution by counting how much sunflower seeds are added with each field in the fraction bar. After that, Ilknur expresses her surprise that she had been right about the solution.

Ilknur takes up a position as learner who struggles to understand, but who is able to come up with a correct solution. In other words, she is positioning herself as competent learner. This time, however, Ilknur does not position herself deliberately, but in reaction to the others’ presentation of a solution. Also, she does not contribute to the development of the mathematical discourse, but instead claims a place in the ongoing discourse in the sense that she expresses her importance in the conversation. While this position is important for Ilknur in order to secure future opportunities for *mathematical agency*, and to collaborate with the other students, it does not allow for *mathematical agency* in this situation. More general, in Session 4, there are only few instances where learners exercise *mathematical agency*, which indicates that there might be fewer opportunities to do so.

Ilknur's developing agency

Over the course of the bilingual Turkish-German teaching intervention, opportunities to exercise *mathematical agency* seem to decrease (see Tab. 1). At the same time, the number of forced self-positionings increases. This goes hand in hand with the teacher taking more and more responsibility for the development of the discourse. In teaching intervention 2 students exercise *mathematical agency* deliberately and relatively independent of the teacher, and the students take positions as being equal to each other in a common struggle for understanding. In Session 4, the struggle for understanding remains, but the teacher takes the responsibility for the development of the mathematical discourse, e.g. by giving more direct instructions. At the same time, the students more often seem to take 'forced' positions, e.g. after being asked for help by the other students (as indicated in right column in Tab. 1), but also influenced by the teacher.

	Deliberate self-positionings		Forced self-positionings by other students	
	Session 2	Session 4	Session 2	Session 4
Frame of *private relations*	5	7	0	6
Frame of *mathematical understanding*	*8*	*3*	0	14

Table 1: Number of positionings of Halim and Ilknur

(numbers for *mathematical agency* in italics)

DISCUSSION

The results presented here are a first step towards an understanding of the mechanisms how multilinguals' *mathematical agency* can develop in a bilingual teaching intervention. In has been illustrated how a student's *mathematical agency* decreases over the course of the intervention, developing from Ilknur taking a deliberate and self-conscious position as a learner who struggles to understand the mathematics, towards a position which Ilknur seems to take up to secures future opportunities for mathematical agency.

The decrease of *mathematical agency* needs to be explained. The here presented results indicate that it cannot be assumed that *mathematical agency* develops 'automatically'. Presumably, the development of *mathematical agency* depends upon multiple factors. Two factors at play here are the intersubjective nature of agency and the nature of the tasks in the teaching intervention.

- Agency is intersubjective, that is, agency is always shared and dynamically moves between multiple subjects (cf. Vitanova, Miller, Gao & Peters, 2015). Accordingly, as a consequence of the teacher taking the responsibility for the development of the discourse in Session 4, the students have less opportunities to contribute to its development – to introduce their own mathematical definitions, routines, objects, symbols etc. (cf. Gresalfi et al., 2009).
- The main task of Session 4 limits the possibilities for the students to contribute to the development of the discourse, as it aims to develop a routine. Accordingly, the students cannot exercise *mathematical agency* to a greater degree.

The here presented study only shows first tendencies in the development of students' *mathematical agency*, as it only investigates one group of students. In future studies, the development of agency within the different teaching intervention groups within MuM-Multi has to be analyzed comparatively to uncover examples where the opportunities for *mathematical agency* increases. Understanding how to support the development of *mathematical agency* could then provide new insights into how to foster multilingual students in engaging with mathematics from their own perspective and their own understanding of the mathematics (Gresalfi et al., 2009; Noren, 2015), especially in regard to the roles of tasks and teachers. Furthermore, the role of the two languages Turkish and German has to be investigated more systematically.

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TEACHERS' DISCOURSE ON STUDENTS' CONCEPTUAL UNDERSTANDING AND STRUGGLE

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We employ a communicational lens on the discourse of elementary mathematics teachers, asked to identify themselves with relation to vignettes describing four teaching types: high/low student struggle and high/low attention to concepts. Our goal is to examine the narratives that support low struggle or low attention to concepts. Data included interviews with four experienced elementary school teachers. Findings show that teachers had a coherent story for why they adopted or rejected each teaching type and that support of other-than-optimal teaching types was related to their conceptualization of "learning with understanding" as well as the ways in which they identify students of different "abilities".

BACKGROUND

In a review of links between teaching practices and students' learning, Hiebert and Grouws (2007) pointed to the importance of two aspects in teaching: explicit attention to concepts (EAC), defined as "the public noting of connections among mathematical facts, procedures and ideas" (p. 383) and students' opportunity to struggle (SOS), that is: "students' expending effort to make sense of mathematics, to figure something out that is not immediately apparent" (p. 387). Optimal teaching, they claimed, combines both EAC **and** SOS. Schoenfeld (2014) also concludes that such instruction, which lends students authority, as well as exposes them to important mathematical ideas, is the best for achieving robust learning.

Yet studies show that such teaching, despite decades of curricular reform and professional development attempts, is still pretty rare (Resnick, 2015). Moreover, changing teachers' practice may prove to be a long and difficult process (Guskey, 2002). In the present study, we offer to view this as a result of teaching practices being a part of a *pedagogical* discourse or discourse about teaching and learning. Similar to any other discourse (Sfard, 2008), it is made of certain key-words, narratives, and meta-rules. These dictate *what* to teach students, *how* to teach them and, often not talked about but still very important, *who* can learn (or not learn). This view is anchored in Sfard's (2008) view of mathematizing as participating in a discourse about mathematical objects thus the pedagogical (the *how* and for *whom*) is closely intertwined with the *what*.

Participation in pedagogical discourse is very much a matter of constructing a certain *identity* of oneself as a teacher (Goos, 2005). Thus, the story a teacher tells about herself is constructed on the web of narratives she endorses about teaching and

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learning. Examining the ways in which teachers identify themselves with relation to certain prototypical practices holds the potential to unearth the web of narratives teachers endorse as part of their “pedagogical frame” – the set of meta-rules determining what is effective teaching for them.

Based on Hiebert and Grouws’ (2007) work, Stein and her colleagues (Stein et al., 2017) have come up with a framework that divides teaching into relatively simple “types”. These are named “quadrants” and typify teaching according to high or low levels of the two aspects identified as most important for students’ learning: explicit attention to concepts, and students’ opportunities for struggle (see Figure 1). Stein et al (2016) developed a survey based on vignettes of a “typical lesson” of these four quadrants.

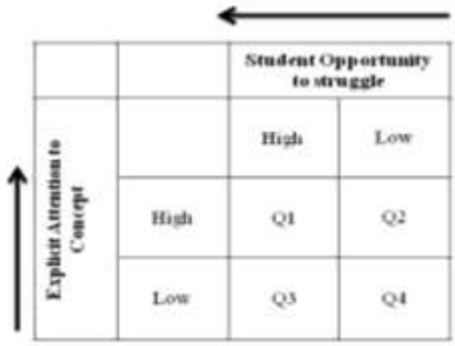

Figure 1: The Teaching Quadrants (according to Stein et al., 2017)

These vignettes, which present relatively simple but sufficiently informative typification of the four teaching approaches, offer teachers an opportunity to identify with alternative forms to the “optimal” view of high EAC/high Struggle. The understanding of these alternative approaches is important for disrupting them and moving teachers towards instruction that is high both in EAC and in student struggle. Our question in this research was thus: what may the discourse of teachers around vignettes of “typical quadrant teaching” reveal about teachers’ identity and their reasons for adopting or rejecting high EAC and high SOS?

METHOD

Since our goal was to compare and contrast narratives about teaching, we chose four teachers that the first author, from her professional role as district instructor of mathematics, knew to be quite different in their teaching practices. All teachers held teaching certificates ranging from a B.Ed to M.Ed and had experience ranging from 11 to 25 years of teaching mathematics. They were asked to answer Stein and her colleagues’ survey. The survey includes 6 vignettes depicting different types of teaching practices. Each vignette is constructed to describe a “typical” quadrant, without, of course, hinting which teacher is better or that there are, in fact, such “quadrants” underlying the vignettes. Following is a short description of each of the vignettes (the originals are around half a page):

All four vignettes describe a lesson dealing with the same subject: connecting fractions, decimals and percents. The Q1, Q2, and Q3 teachers all use a similar task, which affords connections between an area diagram, factions, decimals and percents.

The Q1 teacher - presents the topic for the lesson, then hands out the task for students to work on in groups. She then walks around and assists with specific questions tailored to advance students' thinking. At the end of the lesson she invites two students who have found different solution paths to present and discuss their work. She then draws students' attention to the equivalence of the three representations (fraction, decimal and percent) as seen in the different diagrams.

The Q2 teacher – starts the lesson similarly to Q1 but after most students apply one solution, she points to the equivalence of the diagram and the fraction, then elicits from students the equivalence to percents. She concludes by explaining the meaning of equivalence.

The Q3 Teacher - starts the lesson similarly to Q1 and walks around students monitoring their work. When students make mistakes, she asks them to: "think harder" but does not guide their thinking. At the end of the lesson she invites a group to present their correct solution. Connections between fractions, decimals and percents are not made explicit at the end of the lesson.

The Q4 teacher chooses a different task, which would provide opportunity for targeted practice on an efficient procedure for converting fractions, decimals and percents. She demonstrates the procedure and then gives students similar tasks to work on individually.

After teachers completed the survey, they were individually interviewed on it by the first author. Each semi-structured interview lasted around 30-40 minutes and was designed to elicit teachers discourse around key pedagogical words such as "conceptual/procedural understanding" as well as teachers' identity narratives in relation to the vignettes.

Interviews were fully transcribed and first analysed in search of common issues and statements. Next, we paid attention to particular sentences and word use, for example, around "understanding" and "students".

FINDINGS

Only one of the teachers identified herself with the Q1 teacher. The rest identified themselves as either between Q1-Q3, Q2, or "eclectic". In what follows, we first describe the self-identifications of the teachers and their relation to their definition of "understanding". We then move to present some commonalities in their discourse about students.

Hadar: Identifying with Q3

Hadar hesitated at first between the Q1 and Q3 vignettes. She said: “I feel that at work I zig-zag between the two”, explaining: “one (Q1) gave complete freedom to students. Counted on them. The other moderated them a bit”. Eventually, she leaned towards Q3, explaining that that the Q1 teacher “gives too many hints”. She, in contrast, likes “that they (the students) struggle themselves and then I create a conflict, and only facilitate the discourse”. In that sense, it is clear that Hadar picked up the vignettes’ depiction of the Q3 teacher as letting students “struggle themselves”. However, she did not appreciate the Q1’s teacher explicit attention to concepts. Rather, she interpreted that as “giving too many hints”. We found Hadar’s discourse around “conceptual understanding” linked to this neglect of EAC:

> Conceptual understanding is when I know what can belong to a concept and what does not belong to it. Like, a square, I can define what is a square and what is not a square, so I have a definition of the square concept.

We found this to be a rather constrained conceptualization of conceptual understanding. It does not mention relations between objects, between procedures or between different representations (graphical, numerical, etc.). Hadar was missing in the Q1 vignette an indication of “cognitive conflict”. Thus, for her, “understanding” seemed to be only facilitated by “conflict” not by other means of relating between different representations and procedures.

Hila: Identifying with Q2

Hila identified herself with the vignette of the Q2 teacher. She explained:

> She works in a gradual way. She doesn’t send them straight into the lion’s den … (she) works with them step by step. She makes them understand together the complex task while not giving up on the difficulty of the task, like Sharon (Q4 teacher).

Even from this short excerpt, one can see Hila views student struggle very differently than Hadar. For her, such struggle is a threatening experience (“lion’s den”) that students should be protected from. “Student understanding” is achieved through “working with them together, step by step”. What such “understanding” means for Hila is revealed in her example of how she ensures her students will “understand”:

> For instance, when they study long division. So it’s important that they learn the way (procedure). And understand why and how to do each step. I connect it to DMSB (explains a Hebrew mnemonic for memorizing the steps of long division)… That way they have a good understanding of long division.

Thus, Hila equates “good understanding” with the correct memorization and execution of procedures. She does not make links to mathematical objects or to connections between routines.

Dana: I'm a little bit of everything

Dana insisted she could not identify herself with any particular type of teaching. She explained:

> My lessons really change a lot from lesson to lesson, according to what I feel. I really really think that I use everything. I can use the methods of Nitza (Q1), needing to give directions and hints, and there's a lesson that I would actually use Sharon's (Q4) technique. And there are periods, or days, or a year, that I would act otherwise.

The insistence of Dana that she could not identify with any types of teaching represents, in our view, a narrative of itself: by claiming she uses eclectically different "methods", Dana resists the idea that she should adopt a certain type of coherent instructional practice. She continues:

> So to say that I only do technique, that's the least correct. To say that I only use tasks that are explored independently, that's totally incorrect, and to say that I facilitate all the time – also incorrect. In short, it really really depends.

Though she rejects identifying herself with any specific teaching style, the ways in which each of the vignettes is interpreted by Dana is pretty clear: Q1 "gives directions", Q2, "constantly facilitates", Q3 gives "tasks for exploring alone", and Q4 "does only technique". These rather shallow labels clarify that for Dana, none of the vignettes signifies a coherent teaching approach. Further explaining her choice of "method" Dana explains:

> When I open a subject and introduce students to a subject, I give them (the students) a task, and I say 'take your time, and work alone, and inquire, and check'. It could take one day or two or three or even a week, and (I tell them) 'explore on your own' … I can open with such tasks that will lead them to insights, but at a certain point, you turn to technique and you direct (them).

Thus, for Dana, "insights" are not connected with "technique". "Independent exploration" is reserved for the slow process of "gaining insights" and is the luxury of "beginning a subject". Once that luxury is over, she has to step in and "teach the technique". This is connected, again, to her conceptualization of "understanding". Explaining her insistence on "understanding", especially in lower grades, she gives an example:

> I am now starting (with my 2nd graders) the numbers in the domain of 100. And I'm supposed to start in a short while long addition and subtraction. And since September I've been working on Digi (base ten) blocks, on units and tens, on composition, and on tens. And really put effort into their understanding. What composition actually means.

Thus, Dana mostly equates understanding with a slow process whereby students engage with manipulatives to be able to "understand", and eventually follow a certain procedure (composition). She never mentions connections between mathematical objects or a relation to a wider web of mathematical ideas.

Nira: Identifying with Q1

Nira expressed her self-identification very clearly:

> I'm like Nitza (Q1), period. I give them challenging tasks, and then assist as much as needed and according to the difficulty that arises. And I give space for independent inquiry.... I don't tell too much and I don't give students unrealistic work.

Thus, similar to the other teachers, Nira located herself between two extremes: "too much telling" and "unrealistic work" (or challenge). However, unlike the other teachers, Nira had a pretty clear vision of how this type of instruction connects to "understanding", and in particular, to understanding of low-achieving students:

> She (Q1 teacher) can help them (the students). She can take them from the place they're at and make them fully understand, deeply, any subject. She will work on connections to other subjects, and on different representations.

Nira's clear view of how students "independent inquiry" can lead to "understanding" was connected to her description of "conceptual understanding" which was, by far, the richest we received from our interviewees:

> (Conceptual understanding) is understanding the subject in any form it can be represented and also the relations between the concepts in that subject, … For instance, multiplication – understanding the relation to the area model, understanding the relation to repeated addition, understanding that it also belongs to proportional reasoning, and that it can be described by repeated jumps on the number-line.

Teachers' discourse about students

There was one interesting commonality to all three teachers, except Nira: they all differentiated between their practices with students who have "different abilities". Importantly, this issue was not raised by the interviewer, neither was it a part of the survey. Hila (Q2) talked about matching her regular instruction to the abilities of the "middle group" and about working differently with "low ability" (or "weak") students, who "needed something more technical". A similar narrative was told by Dana (Eclectic), who said: "If it's a student with difficulties that I know that has no choice, then I work on the technique".

Hadar (Q3) did not explicitly label students as being of a particular type, yet she still referred to students that deserve "other" types of instruction:

> There are kids that I know, for example, that showing them the algorithm, or explaining the procedure, the solution… I know that they won't succeed in understanding, and I do want them to know, so I use it (Q4 instruction).

Given the issue raised by the three interviewees, about differentiating teaching according to students' "abilities", we went back to Nira (Q1) and asked her how she would teach "students with difficulties". Nira reacted with some puzzlement to the question, answering immediately: "(I teach) regularly, why?" When hearing that other teachers thought it was an important factor, she added:

> Look, for students with difficulties, it's important to give scaffolds, manipulatives or anything that would help them work on the same tasks that are learned in the classroom, so they don't feel behind.

Still, she insisted that the Q1 teaching is the best for these types of students and in fact, "can help them the most".

SUMMARY AND CONCLUSION

Our goal in this study was to expose the narratives underlying choices of teachers to identify with particular types of teaching, according to high/low EAC and high/low SOS. The findings show that each teacher had a coherent set of narratives for explaining her choice (or avoiding the choice) of a particular teaching type. Thus, the vignettes were highly effective in eliciting teachers' identity narratives and in helping them reflect on their teaching practice.

Our findings also point to a possible relationship between teachers' choice of which vignette to identify with, and their discourse on students' "understanding". Except for Nira (Q1), the three teachers' discourse about "conceptual understanding", or "understanding" more generally, was quite limited, and mostly referred to being able to follow and explain a given procedure correctly. There also seemed to be a disconnect between building on what students already know (seen in words such as "inquiry" and "gaining insights") and having students carry out mathematical procedures ("the technique"). These differentiations went often together with the identification of who can "understand" and who can "only do the technique".

These findings, as initial and embryonic as they are, point to the possibility of there being a relationship between narratives about mathematics – being a set of rules to be followed or being an interconnected web of relations between mathematical objects – and narratives about students. In other words, it seems the narrative that certain students are "simply not able to understand so they need do the technique" is easier to endorse when "understanding" and "technique" are differentiated.

Unfortunately, this pedagogical discourse may be, in part, responsible for the construction of learning difficulties to begin with. Previous research (e.g. Heyd-Metzuyanim, 2013) has shown how a teacher and a low-achieving student "co-construct" the students difficulties by both sticking to ritual rule following, in the face of the students' ever-growing gaps vis-à-vis the curriculum. However, attempts to disrupt the common belief that low-achieving students should engage with cognitively demanding tasks are still rare.

Another insight we gained is that none of the teachers interviewed on the vignettes actually related to the *explicit attention to concepts* in them. In fact, teachers judged the appropriateness of the practices almost solely based on the *struggle* aspect of the story, essentially placing all vignettes on one 'struggle scale' (roughly Q3, Q1, Q2 and Q4, from highest to lowest). This finding hints at the ubiquity of teachers' discourse around students' struggle (good or bad), at the price of discourse on attention to concepts, or mathematical narratives, more generally. It also echoes Chazan & Ball’s (1999) well-known lament about teachers only being “told not to tell”, while *what* to tell (or not to tell) is not being explicated.

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THE COMMOGNITIVE FRAMEWORK LENS TO IDENTIFY THE DEVELOPMENT OF MODELLING ROUTINES

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Modelling abilities.are considered important for everyday life. The current study aimed at using the communicational framework to monitor the development of modelling abilities through constructing models. To this end, we monitored a group of five future teachers as they worked on two model-eliciting activities. Their work process was videotaped and transcribed. The participants' discourse was analysed to identify changes in their routines while they worked on the two model eliciting activities. We were able to trace changes in the participants' routines through eliciting models. Specifically, we identified a change from using a non-systematic routine to using a systematic routine and from routines focusing on choosing specific cases to routines focusing on eliciting criteria for making choices.

INTRODUCTION

Model-Eliciting Activities [MEAs] offer students opportunities to confront mathematical as well as everyday challenges (Lesh, Hoover, Hole, Kelly & Post, 2000). Engaging in MEAs requires the learner to develop models to describe, explain, of real-word situations and refine those models in other situations (Doerr & English, 2003). Several researchers (e.g., Shahbari & Peled, 2017) have described the modelling abilities involved in eliciting a model. Yet, more rigorous analysis is still needed to monitor the processes and abilities involved in eliciting models, extend and generalized in another situations. A good candidate for such an analysis is the commognitive framework (Sfard, 2008), which has been used to study mathematical learning on different mathematical topics at the micro level. The aim of the current study is to monitor the development of modelling abilities through constructing models at the micro level among a group of prospective teachers. Using the communicational perspective enabled us to closely monitor how their working processes unfolded and changed while they engaged in MEAs.

FRAMEWORK

Model Eliciting Activities

MEAs involve partial, unclear or undefined information about a situation that needs to be mathematized in ways that are meaningful to learners as they work in small groups (English & Watters, 2005). These activities are designed according to six principles of Lesh et al. (2000): model construction, reality, self-assessment, construct documentation, construct/ shareability/ reusability, and effective prototype. Engaging in MEAs lead to development of significant mathematical constructs through iterative

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cycles, in which learners use mathematical and communicational competences, such as translation, simplification, construction, justification, conjecture, representation, quantification, organization, and prediction of outcome data and solution path (Lesh & Doerr, 2003).

The Commognitive Framework

The communicational framework (Sfard, 2008) is a socio-cultural perspective for studying the learning process. The framework suggests that mathematics is a type of discourse and that thinking is a certain form of self-communication. Sfard proposes four characteristics of mathematical discourse: 1) *Words and their uses:* Each discourse is characterized by its own keywords. 2) *Visual mediators*: As mathematics is not about physical objects, in many cases communication is fostered by referring to visual realizations that are part of the communication. 3) *Narratives:* Narratives are sequences of utterances framed as descriptions of objects, relations between objects or processes with or by objects that can be endorsed or rejected. 4) *Routines*: Routines are repetitive discursive patterns characteristic of a specific discourse. According to the communicational framework, learning is a change in the individual's discourse, that is, a change in words and how they are used, in narratives endorsed or in routines used. Sfard suggested two types of learning: learning at the object level and learning at the meta-level. Object-level learning involves expanding the discourse by using new routines and expanding the assortment of endorsed narratives for a mathematical object. Meta-level learning involves changes in the meta-rules of the discourse.

Research questions

1. What routines can be identified in the participants' work on model-eliciting activities?
2. What changes in routines can be identified while the participants worked on a sequence of two model-eliciting activities?

METHOD

We monitored one group comprising five prospective mathematics teachers in their second year of studies in the mathematics education track at a college of education. They had no previous experience with modelling activities.

Model eliciting activities

The authors designed two MEAs —the *camp* activity and the *good teacher* activity—based on the six principles designated by Lesh et al. (2000). In the first activity, the *camp activity*, the participants were asked to choose the most suitable camp/camps and to suggest a means of choosing suitable camps for the coming years. The *camp activity* was represented via four tables providing information about six camps, with each table referring to several components. The first table included the dates of each camp, as well as information on transportation, food and cost. The second table included types and number of entertainment activities at each camp. The third table consisted of data from the previous year about number of participants and number of counsellors at each camp. The fourth table presented the parents' evaluations and rankings of the camps from the previous year, with the rankings ranging from one to five stars.

In the second activity, the *good teacher activity*, the participants were asked to choose the most suitable candidate/candidates and to suggest a means of choosing suitable candidates for the coming years. The *good teacher activity* also comprised four tables that describe ten candidates for a teaching position. The first table included the candidates' age and their average grades in their B.Ed. studies. The second table included the candidates' ranking by their pedagogical instructors for their practicum work in the schools, with the ranking ranging from A+ to F over three years. The third table included the ranking of the candidates' performance in an interview, with the ranking ranging from "not at all acceptable" to "widely acceptable." The fourth table included the candidates' ranking on social initiatives, ranging from "did not participate at all" to "participated to a large extent." For each of the two activities, the participants were required to write a letter explaining their decisions.

Research procedures and data sources

The participants first worked on the *camp activity* and a week later worked on the *good teacher activity*. The main data sources were two video recordings of the group working on the two MEAs. These recordings were transcribed verbatim. We also used the group's reports and working drafts for the two activities as an additional data source.

We analysed the data derived from the video recordings based on Sfard's commognitive perspective (2008). Of the four discourse characteristics, we searched for participants' uses of routines. We parsed the transcript into episodes according to sub-tasks the participants performed while working on the two MEAs. If during one sub-task the participants enacted two routines, we separated the participants' utterances into two episodes. In each episode we searched for participants' actions in order to identify routines employed while working on each MEA. For each identified routine, we searched for two defining parts: the *when* of the routine—when it began and later when it ended, and the *how* of the routine—the utterances between the opening and the closing of the routine. These utterances were classified as the routine procedure.

FINDINGS

Modelling routines in the first activity (camp activity)

The participants' engagement in the first modelling activity can be separated into two phases. In the first phase, the participants tried to accomplish the activity by choosing specific camps among those listed [lines 1-200]. In the second phase of the camp activity, the participants tried to elicit a model for choosing specific camps [lines 201-458]. In the following, we describe the first phase and then the second phase.

Choosing specific cases (camps)

During the first phase, the participants discussed the components of the first table and chose three camps. They then tried to check their choice based on the information provided in the other three tables but without integrating the information from the four tables. The participants' engagement during this first phase was captured by Routine 1, which included two main sub-routines: 1.1 - Non-systematic comparison and 1.2 - systematic comparison that focused on subsets of cases. Each sub-routine consisted of two nested sub-routines: looking at all cases (LAC), looking at a subset of cases (LSSC), using average (UA) and using estimation and ratio (UER). Figure 1 shows the parts of each routine. These sub-routines appear in Episodes 1-5. Due to space limitations, we provide only the participants' discourse from the first episode.

	1.1 Non-systematic comparison		1.2 Systematic comparison subset of cases	
Sub- Routine	Looking at all cases [Episode 1]	Looking at a subset of cases [Episode 2]	Using average [Episode 3]	Using estimation and ratio [Episodes 4&5]
Opening: Choose "perfect camp"	Consider first table	Consider second table	Consider third table	Consider third and fourth tables
Procedure: Comparing the camps in each table separately	Non-systematic comparison focusing on all cases	Non-systematic comparison focusing on the three chosen camps	Using average to compare the three chosen camps	Using ratio and estimation focusing on the three chosen camps
Closing: Choosing camps	Choosing three camps	Endorsed three chosen camps	Endorsed three chosen camps	Endorsed two chosen camps

Figure 1: Routine 1 - Searching for specific cases (camps)

Episode 1 presents the participants' discussion immediately after reading the camp activity.

Episode 1: Non-systematic comparison with focus on all cases

1 S1: [Reads the activity]

2 S2: Let's first consider the dates of the camp. The first one is from 2-10 July.

3	S3:	All the dates are suitable.
7	S2:	Let's compare camp A and camp C. What is the difference between them? Both of them offer transportation. The first one provides food and costs 750. The third costs 900.
8	S4:	The first is better.
9	S2:	True, C offers one more day but does not provide food and is more expensive. So, A is preferable to C.
10	S4:	We can choose both A and H.
23	S4:	So the camps are A, D and H.

Episode 1 shows the participants' discussion of the four components in the first table: dates, transportation, food and cost. (The participants' engagement in this episode is described by the LAC routine.) Opening the LAC routine was triggered by the call to choose a favourite camp\camps [1]. The procedure includes direct and non-systematic comparison [2-22]. The routine closed with choosing three camps—A, D and H [23]. As an example of the procedure, S3 [3] determined that the dates of all the camps are suitable without explicitly relating to the features of each camp, such as number of days or other components. S4 [8] determined that camp A is better but did not provide any justification. She considered only two components in the first table and did not consider the others.

To summarize, analysis of the participants' discussion from utterances [1] to [200] indicates that they worked unsystematically. The participants focused on specific cases—three camps—and did not provide a model that satisfies the *camp activity*.

Modelling routines for eliciting a model for choosing cases (camp activity)

After about half of the total time allocated to the activity, the participants started thinking about the need for eliciting general criteria for choosing between the cases rather than choosing individual cases. The need to write a letter about their considerations and decisions triggered this change. By the end of the working time, after 92 minutes, the participants provided only a partial model and did not manage to write a letter about their suggestion. Based on their work in this second phase, we identified a second routine—Routine 2—that has two main sub-routines: Routine 2.1 - integration between the components, which includes the sub-routine for assigning relative weighting [ARW], and Routine 2.2 - systematic comparison, which consists of three sub–routines: quantification of numerical data [QND], defining range and grading [DRG], and quantification of qualitative data [QQD]. Figure 2 shows the need for using each sub-routine, the details of the procedures and the closing. These sub-routines appear in Episodes 6 – 9 of the participants' discourse.

	2.1 Integration between the components	2.2 Systematic comparison		
Sub-Routine	Assigning relative weighting [Episode 6]	Quantification of qualitative data [Episode 7]	Quantification of numerical data [Episode 8]	Defining range and grading [Episode 9]
Opening: choosing "perfect camp"	The need for weighing tables by their importance	The need for assigning values to quantities	The need for assigning values to quantities	The need for weighing numerical quantities
Procedure: searching for general model	Assigning relative weighting	Quantification of qualitative data	Quantification of numerical data	Defining range and grading
Closing: arriving at a partial Model	Assigning values to the tables: 40%, 30%, 20% and 10%	Assigning quantity to each value	Assigning quantity to each value	Assigning point for selected range: 15-20 get 18 points

Figure 2: Routine 2- Eliciting a model for choosing specific cases (camps)

Modelling Routines in the second activity (*good teacher* activity)

The participants' engagement in the *good teacher activity* was different than in the *camp activity*. The participants did not choose a specific candidate. Rather, they elicited a model and then weighed all the candidates according to their model. The participants adapted most of the sub-routines from Routine 2 (quantification of qualitative data [QQD], quantification of numerical data [QND] and assigning relative weighting [ARW]) and one sub-routine from Routine 1 (using average [UA]). In addition, the participants identified a new routine in the *good teacher activity* and implemented it in the elicited model [IEM]. We show the participants' discourse from utterances [1] to [201] parsed into six episodes, 10-15, in accordance to each routine. (Due to space limitations, we only show the participants' discourse from Episodes 10 and 15.) Episode 10 shows the participants' discussion about ways to scale numerical quantities. The participants used the QND sub-routine at the beginning of their work on the *good teacher activity*, while in the *camp activity* they used this sub-routine only in the second phase of their work.

Episode 10: Quantification of numerical data in first table in the *good teacher activity*.

1	S1:	[Reads the good teacher activity]
5	S2:	We can start with the candidates' averages.
7	S1:	The lowest is, 78, so the candidates' average is between 78 and 96… so we can say, whoever has 78 will have lower scores. Do you

remember how the scores are used? For example, 78 will get 5 points, and if the average is higher, it will get more points.

16 S4: That means 70 gets one point, 71 gets 2 points, 72 gets 3 points…100 gets 30 points. We should make better use of the range.

28 S1: There is no problem with the use of numbers, 95 gets 26 points, 82 gets 13…

Episode 10 demonstrates the QND routine. The routine opens with reading the activity and understanding the goal of choosing a good teacher [1]. S2 then suggests [5] starting with the candidates' averages. The procedure in the QND routine involves transforming the average to other quantities, as S1 [7] explains to her classmates. She asks them if they remember a past routine "Do you remember how the scores are used?" S4 [16] suggests using the DRG routine by saying "We should make better use of the range." Routine QND closes by assigning numerical values to the averages component [28].

After the participants elicited a model for choosing a candidate, they began implementing this model. Episode 15 presents the participants' discussion about implementation of the elicited model.

Episode 15: Implementation of the elicited model.

193 S2: Now, we calculate the general score for each one.

194 S1: You calculate for these two candidates.

197 S3: The general score for candidate Ⅰ is 78.8 and for the other it is 63.88

201 S4: The highest score is for candidate Ⅲ, then candidate Ⅶ...

Episode 15 depicts Routine 3, IEM—implementing the elicited model. The routine opens with S2 [193] requesting that they calculate the scores for each candidate. The procedure involves substituting values in the model [194-195]. The routine closes [197-201] with participants' answers about each candidate's scores.

DISCUSSION AND CONCLUDING REMARKS

Our findings indicate that through eliciting a model in the first activity the participants worked with Routines 1 and 2, but when they got to eliciting a model in the second activity they worked with Routines 2 and 3. We interpret this change in their use of routines as an indication that learning took place. Moreover, we consider this learning to be at the meta-level learning, because the participants learned the rule of how to work on MEAs beyond the particular activity. Let us elaborate: The participants noticed that working with Routine 1 in the camp activity did not provide a solution. Then they began working with Routine 2 in response to the need to develop a tool for choosing appropriate cases. Routine 2 was part of a modelling discourse that the participants previously lacked. During the second activity, the participants are able to work with the activity by replicating something they did in the first activity. After the opening, they decided to use the procedure of Routine 2. This indicates that the participants changed their own rules about how to start working on this type of activities. Routines 2 and 3 involved dealing with multiple data tables; creating, using, modifying, quantifying and transforming quantities; and coordinating, organizing data and representing findings in visual and textual forms. As we mentioned earlier, these routines are considered an elaboration of modelling abilities.

The communicational framework was used in this study as a tool for identifying the development of modelling abilities through eliciting models at the micro level. Describing the participants' actions according to routines served us as a magnifying glass for the modelling processes, allowing us to monitor when and how the modelling abilities developed. In addition, the communicational framework provided a tool for identifying when the learning occurred at the meta-level, as reflected by the learners' discourse. Finally, the routines emphasized the operationalization of the modelling abilities, while the description of the procedures for each routine allowed us to focus these procedures through the learning process. In conclusion, we recommend using the communicational framework to describe participants working in MEAs. To the best of our knowledge, this framework has not been used thus far to examine modelling abilities through eliciting model.

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RECONSTRUCTING A LESSON SEQUENCE INTRODUCING AN IRRATIONAL NUMBER AS A GLOBAL ARGUMENTATION STRUCTURE

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The study clarifies some argumentative characteristics of a square roots lesson sequence introducing an irrational number in a ninth-grade classroom. It focuses on local and global argumentations in order to reconstruct classroom processes. There are two main findings: one is concerned with the transition between different argumentation streams, and the other is related to the types of argumentation structure. Implications for teachers' practice are also discussed.

INTRODUCTION

The notion of argumentation has often been emphasized in relation to "reasoning and proving" in school mathematics (e.g., NCTM, 2000), and to "competencies" in international assessment (OECD, 2002). However, it is internationally well known that many students have serious difficulties with reasoning and proving. Therefore, argumentation, reasoning, and proving are crucial mathematical processes at all grades of school mathematics and have been studied as such. In the field of mathematics education, Toulmin's model (Toulmin, 1958) has been widely used for describing how argumentation may develop through classroom interactions (Krummheuer, 2007; Yackel, 2001), analysis of interview data with postgraduate school students' arguments (Inglis et al., 2007), analysis of the relationship between argumentation and proof in individual proving processes (Pedemonte, 2007), and reconstruction of classroom proving processes (Knipping, 2008; Reid & Knipping, 2010). Since the model has been used in various ways for various purposes, it is important to mention how to use this model and for what purpose in the present study.

The present study, based particularly on Knipping (2008), describes a model that allows us to understand some implicit but essential aspects underlying a series of mathematics lessons. Analysing the development of argumentation over time is a significant issue (Knipping, 2008). The target of the study is a lesson sequence on square roots that intended to introduce an irrational number in a ninth-grade classroom. As will be shown later, reconstructing the lesson sequence for the development of argumentation allows us to obtain a deeper understanding of the process by which students come to an understanding of the meaning of a new concept of number. The research questions in this paper are as follows: 1) *How can the method of reconstructing local and global arguments reveal the characteristics of a lesson*

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sequence on square roots? and 2) *What characteristics of the lesson sequence can be found through the analysis of the development of argumentation over time?*

THEORETICAL PERSPECTIVE

Toulmin's model of argument

Toulmin's model of argument is developed to be applicable to probable arguments rather than logical arguments in different fields (Toulmin, 1958). This model has three basic types of statements that investigate the function and structure of arguments. The *claim* (**C**) is a statement which is made to convince someone. The support or evidence one might give for the conclusion (claim) is the *data* (**D**). The *warrant* (**W**) refers to the rationale that may justify the connection between the data and the conclusion. Based on Knipping (2008)'s usage, in this study, the author also uses "claim" in cases "where data and warrants have not yet been provided, and "conclusion" when they have been" (p. 430). Although Toulmin (1958) mentions three other types of statement - the *backing*, *qualifier*, and *rebuttal* - only one of them will be considered in this paper. In case the warrant is doubted, the backing (**B**) is needed for providing further justification to support the warrant. Figure 1 shows Toulmin's model of argument, which contains four elements to be considered in the study.

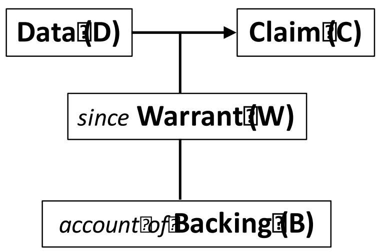

Figure 1: Toulmin's model

Local and global argumentation

Based on previous studies (Knipping, 2008; Reid & Knipping, 2010), there are three key notions that will be considered in this paper. First, *local argumentation* refers to each argumentation step where three or more elements of the argument are related to each other in the structure. Second, *global argumentation* allows us to describe the gross structure of the arguments. Third, "as the conclusions of some steps are recycled as data for others, these steps join up into [the] *argumentation stream* (AS)" (Reid & Knipping, 2010, p. 180). In terms of global argumentation, Knipping (2008) shows the scheme of reconstructing a global argumentation (Figure 2) and an example of the global argumentation structure (Figure 3).

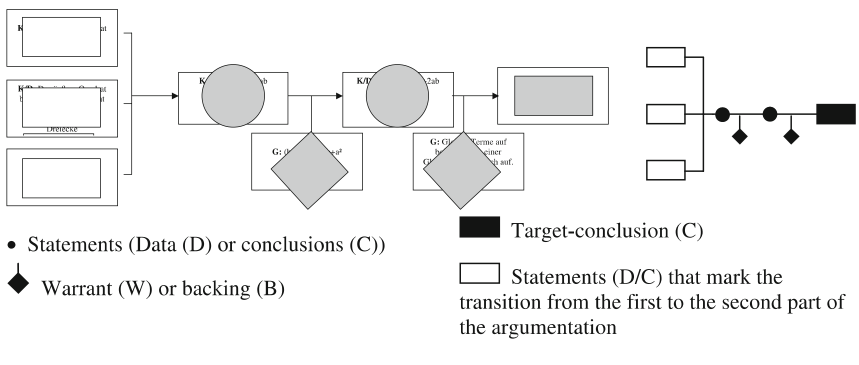

Figure 2: Global structure

(Knipping, 2008, p. 435)

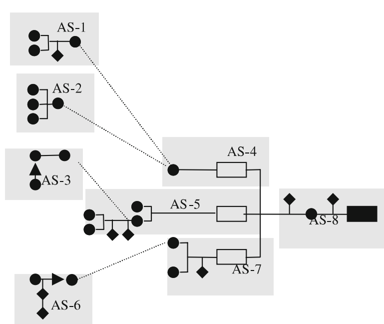

Figure 3: A source-structure

(Knipping, 2008, p. 437)

Further, Knipping (2008) discusses two types of global argumentation structures: *source*-structure and *reservoir*-structure. Although Reid & Knipping (2010) also describe two other structures, these are not suitable for the present study. A source-structure is illustrated by Figure 3, which means that "arguments and ideas arise from a variety of origins, like water welling up from many springs" (Knipping, 2008, p. 437). Meanwhile, the reservoir-structure means that arguments "flow towards intermediate target conclusions that structure the whole argumentation into parts that are distinct and self-contained. The parts that make up the argumentation are like reservoirs that hold and purify water before allowing it to flow on to the next stage" (ibid., p. 437). Knipping (2008) also recognizes the necessity of further research for identifying significant argumentation steps in both structures. Because global argumentation in classroom processes can be quite complicated phenomena, there is a room for further methodological and empirical elaborations on this topic. By reconstructing classroom processes as a global argumentation structure, the study aims to identify some argumentative characteristics involved in the structure.

METHOD

According to Knipping (2008), there is a three-stage process to the method for reconstructing arguments in classrooms (p. 431):

- Reconstructing the sequencing and meaning of classroom talk (including identifying episodes and interpreting the transcripts)
- Analysing arguments and argumentation structures (reconstructing the steps of local arguments and short sequences of steps that form "streams"; reconstructing the global structure)
- Comparing local argumentations and global argumentation structures, and revealing their rationale.

Although each stage is included in this study, the focus is on the second stage. Therefore, in this section, I will briefly illustrate the first stage of the observed lessons on square roots in a Japanese ninth-grade classroom. In the teaching of square roots, the teacher is responsible for introducing terms and symbols such as "square root," "rational number," "irrational number," "radical sign," and "$\sqrt{\ }$." More often than not, proof by contradiction might be treated in this grade, for example, by showing the irrationality of $\sqrt{2}$, but this proof method is not the official teaching content of this grade (although it is the official content of the tenth grade).

The sequencing of the 14 lessons of the square roots was observed (e.g., "L1" indicates the first lesson). The 14 lessons (each lasting approximately 50 min.) were videotaped by the author. This lesson sequence was designed by the teacher, who has more than 20 years of experience of teaching in lower secondary schools in Japan. There were 39 students who were 14 to 15 years old in the classroom.

L1: Quadratic equation

L2: Existence of square root

L3: A number that cannot be expressed as a ratio of integers (simple fraction)

L4: Radical sign ($\sqrt{\ }$)

L5: Ordering the square root of numbers

L6: Magnitudes of the square root of numbers

L7: Prime number and prime factorization

L8: Multiplication and division of square roots (Session 1)

L9: Simplifying expressions, including square roots

L10: Rationalizing the denominator

L11: Multiplication and division of square roots (Session 2)

L12: Addition and subtraction of square roots (Session 1)

L13: Addition and subtraction of square roots (Session 2)

L14: Various calculations including square roots

It is important to note that the term “square root” was introduced in L2, but other new terms and symbols (“rational number,” “irrational number,” “radical sign,” and “$\sqrt{\ }$”) were introduced in L4. I chose to develop and analyse the transcripts of the first four lessons, which are concerned with the existence of the square root of numbers (L1 and L2) and with introducing an irrational number (L3 and L4). Although the previous study conducted by the author (Shinno, 2016) focused on the latter parts of the same lesson sequence (L8 to L14), the theoretical and methodological approaches were quite different from the present study.

In the next section, students’ and teacher’s utterances during the lessons were analysed by means of Toulmin’s model to reconstruct a local and global argumentation, which was developed in the classroom processes. In this analysis, like Knipping (2008), only utterances (and teacher’s writings on the backboard) that are publicly presented as a statement in the classroom were identified. This means that analysing the role of individual utterances and determining how they may influence the development of argumentation are beyond the scope of this paper.

RESULTS AND ANALYSES

In this section, I will first illustrate each argumentation stream in terms of local argumentations with Toulmin’s model, although the skeleton diagram and some of the transcripts that corresponds to argumentation streams will not be shown because of the limited space available. Then, these streams will be integrated into a global argumentation structure to reveal some significant characteristics of the lesson sequence.

Reconstructing local argumentations

Six argumentation streams (AS1 to AS6) are reconstructed from the episodes in the four lessons (L1 to L4). In Table 1, the references (e.g., <L1_38:28-38:52>) indicates the time elapsing within the lesson. I will illustrate AS1 and AS3 with the transcripts but describe AS2, AS5, and AS6 with other data such as pictures of students' or teachers' writing on the blackboard.

AS1: Solving the quadratic equation x2=10 <L1_38:28-3852>	*AS4: Representing a repeated decimal number as a simple fraction* <L3_23:47-38:00>
D: The solution of *x*2=10 is 3.1622777 **W1** (objection to a datum): Since the square of the last digit is not 0 but 7×7=49, 3.1622777 is incorrect **W2**: "Even if the same numerals from 1 to 9 are multiplied by each other, it does not make 0" (utterance of S2) **C**: the decimal representation of the square root of 10 can be an infinite decimal number	**D**: 0.1=1/9, 0.01=1/99, 0.001=1/999, … **W**: calculations by using the given patterns (e.g., 0.121212…=12/99=4/33) **C**: any repeated decimal numbers can be expressed as simple fractions
AS2: Constructing the geometric square having the area 10 <L2_23:44-25:16>	*AS5: Proving the irrationality of the square root of 10* <L3_44:45-48:20>
D: the area of the geometric square is 10 **W**: the way of constructing the geometric square having the area 10 (explained by S4) **C**: the solution of *x*2=10 exists as the length of a square with the area of 10	**D**: the decimal representation of the square root of 10 can be an infinite decimal number **W**: proof by contradiction (explanation by the teacher) **C**: the square root of 10 is a number that cannot be expressed as any simple fraction
AS3: Representing a simple fraction as a repeated decimal number <L3_13:26-17:02>	*AS6: Introducing rational and irrational numbers* <L4_05:14-09:22>
D: 2/7=2÷7 (a form of long division) **W**: if the same remainder can be found, the quotient will start to be repeated **C1**: 2/7 can be expressed as repeated decimal numbers **C2**: any simple fractions can be expressed as finite or repeated decimal numbers	**D**: the square root of 10 is a number that cannot be expressed as any simple fraction **W**: classification of rational and irrational numbers **B**: the domain of numbers, which the students already learnt, is rational numbers **C**: the square root of 10 is an irrational number

Table 1: Argumentation streams in the classroom processes

AS1 is identified when students are addressing the task "Find the solution of x^2=10" through successive approximations with the calculator. A student gave "the solution is 3.1622777" as a conjecture, but this will be refused by another student. S2 explains the reason that the decimal representation of the solution continues infinitely.

L1_38:28 T: Can you explain why it continues infinitely?

L1_38:42 S2: Even if the same numerals from 1 to 9 are multiplied by each other, it does not make 0.

L1_38:52 T: If it stops somewhere, the product should be 0.

Figure 4: AS2

In L2, the mathematical context of the lesson changes from an algebraic equation to geometric construction. The claim of the next argumentation stream (AS2) is that the solution of x^2=10 exists as the length of a square with the area of 10. After students work individually, a student (S4) explains the reason that the constructed geometric square has the area of 10 by referring to the drawing of the blackboard (Figure 4). In this situation, the term "square root" is officially introduced by the teacher.

The main task of L3 is to explore the question "what is the difference between the square root of 10 and numbers that the students have already learnt?" The students have already learnt decimal fractions and simple fractions, although they have not yet learned the expression "rational numbers." Therefore, it is meaningful for them to clarify the conceptual relationship between decimal fractions and simple fractions. This process is reconstructed as AS3 and AS4.

Data, warrant, and conclusion 1 in AS3 are identified in the following transcripts, but conclusion 2 is a generalized statement made by the teacher. S8's explanation is supported by the teacher's writing of the long division (Figure 5). Subsequently, the teacher suggests a conjecture that repeated decimal numbers can be expressed as simple fractions and encourages student to examine it by using facts such as $0.\dot{1}=1/9$, $0.0\dot{1}=1/99$, and $0.00\dot{1}=1/999$; for example, $0.\dot{1}\dot{2}=12/99=4/33$. AS4 is the result of such a justification.

L3_13:26	S7: In the case of 7, there are the seven (remainders) from 0 to 6. If it is a repeated decimal number, the six numbers (remainders) would be found repeatedly, I think.
[…]	
L3_16:43	T: Let's consider a concrete example, say, 2/7, which is 2 divided by 7. 2÷7. In writing this, in the form of long division …
L3_16:55	S8: We get 20…
L3_16:57	T: We get 20, then 7×2=14, so then…
L3_17:02	S8: Then, keep going…

Figure 5: AS3

Toward the end of L3, the irrationality of the square root of 10 is eventually proven by the teacher as shown in Figure 6. This proof is reconstructed as AS5. Because of the introduction of the new terms for rational numbers and irrational numbers in L4, the teacher enables the summarization of the relationship between number concepts as Figure 7 and finally introduces the radical sign ($\sqrt{\ }$). This process is reconstructed as AS6.

We suppose that the square root of 10 is expressed as a simple fraction b/a. (b/a is irreducible)

$$3.1622\cdots = \frac{b}{a}$$

Square both sides.

$$10 = \frac{b^2}{a^2} = \frac{b \times b}{a \times a}$$

Because $\frac{b^2}{a^2}$ is irreducible, $10 = \frac{b^2}{a^2}$ is not true.

Therefore the supposition $3.1622\cdots = \frac{b}{a}$ is wrong.

Therefore <u>the square root of 10 is not expressed as a simple fraction</u>.

Figure 6: Proving of the irrationality of √10

Decimals
- Finite
- Infinite
  - Repeated — Rationals: It can be expressed as a simple fraction
  - Non-repearted — Irrationals: It cannot be expressed as a simple fraction

Figure 7: Relationships between numbers

Reconstructing a global argumentation structure

Here, it is possible to lay out the structure of the six argumentation streams as a whole. Figure 8 shows a global argumentation structure in the classroom processes. In this gross structure, there are two main findings: one is concerned with the transition between different streams, and the other is related to the types of the structure.

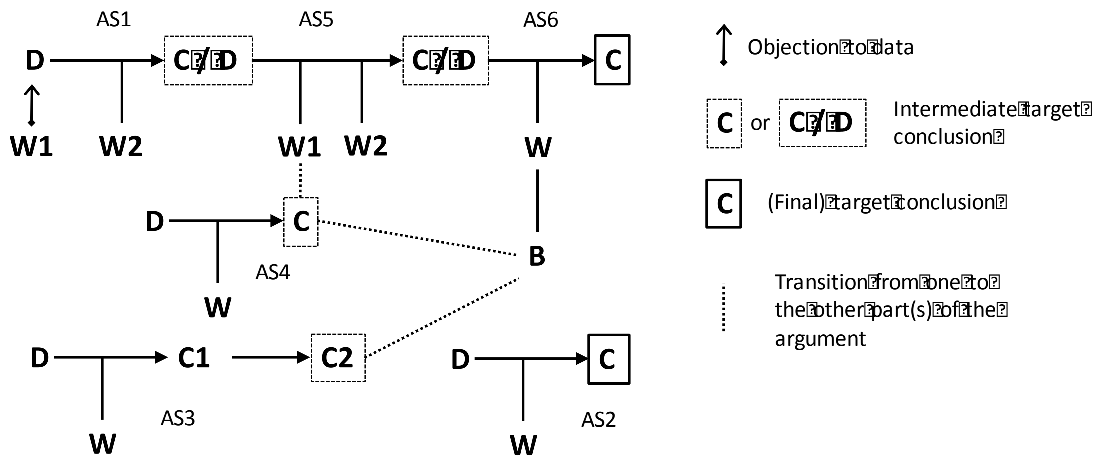

Figure 8: A global argumentation structure

Previous studies (Knipping, 2008; Krummheuer, 2007) have demonstrated that arguments can be chained in a way that an accepted conclusion can function again as data for a subsequent new argument. This characteristic is also illustrated in Figure 8 (i.e., the transition between AS1, AS5, and AS6). In addition, Figure 8 also shows that an intermediate conclusion can function as a warrant or backing for a new argument. In other words, AS3 and AS4 can be considered as mathematical underpinnings for AS5 and AS6. These statements are essential for students to understand the meaning of the proof of irrationality of $\sqrt{10}$. Although this proof is the teacher's construction, these statements that are have been acquired through students' explanations and examinations can be used for underpinning the proof.

In terms of the types of structure, it seems that Figure 8 may well correspond to the reservoir-structure rather than the source-structure. As far as AS3 and AS4 are concerned, it seems that the reconstructed argumentation flows forward towards, and backward from, an intermediate and final target conclusion in AS5 and AS6. However, it seems that AS2 is distinct from any other streams, but it also contributes to the development of the global argumentation because it provides the geometric context. Therefore, the status of AS2 could be interpreted as a part of a source structure. The conclusion of AS2 might convince some students of the existence of an irrational number even after arriving at AS6.

FINAL REMARKS

Further research is needed for analysing the nature of the justification, because some ASs in the observed lessons are based on inductive generalization (i.e., AS3 and AS4) and some on deductive reasoning and proving. Moreover, students' knowledge system also needs to be taken into consideration (cf. Pedemonte & Balacheff, 2016), because the number domain (see AS6) heavily relies on what they already know or do not know. Finally, let me mention an implication for teachers' practice. The results of the reconstruction of local and global arguments may suggest that an intermediate conclusion occurring in a lesson could be taken up in later lessons when it fits into a warrant or backing of the other argument. This implies that taking a chain of arguments into account may help teachers to design and reflect on their instructions from a long-term perspective.

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MATHEMATICS AND SCIENCES TEACHERS COLLABORATIVELY DESIGN INTERDISCIPLINARY LESSON PLANS: A POSSIBLE REALITY OR A WISHFUL THINKING?

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The increasing interest in STEM education raised awareness of issues related to interdisciplinary approach to teaching. In secondary schools, such teaching is rare, mainly because teachers are often certified in one specific discipline. This implies that implementation of interdisciplinary approach to the teaching of mathematics and science necessitates the cooperation and collaboration of teachers from both disciplines in writing suitable learning materials. Our study aimed at examining the various aspects associated with such collaboration. The results indicate that mathematics teachers acknowledged the advantages of interdisciplinary teaching; however, they questioned the feasibility of the collaborative lesson planning and were concerned about their capability to implement the approach.

INTRODUCTION

The mathematics, science, and technology education communities are undergoing a major reform in curriculum design, instructional approaches, and assessment practices, thinking of how to integrate among disciplines (Berlin & White, 2010). These changes are reflected in the American national standards for teaching the disciplines: The Principles and Standards for School Mathematics (2000) recognizes the need to connect mathematics to other disciplines, particularly to science: "*The opportunity for students to experience mathematics in a context is important. Mathematics is used in science, the social sciences, medicine, and commerce. The link between mathematics and science is not only through content but also through process. The processes and content of science can inspire an approach to solving problems that applies to the study of mathematics*" (p. 66). Similarly, the National Science Education Standards (National Research Council, 1996) suggest that "*The science program should be coordinated with the mathematics program to enhance student use and understanding of mathematics in the study of science and to improve student understanding of mathematics*" (p. 214). However, school reality is different. Although there are problems that require the perspective of multiple disciplines, in practice, teachers rarely integrate mathematics and sciences [thereafter M&S], because they do not have sufficient didactic and content-related knowledge to integrate among these disciplines. Therefore, there is a need to identify ways to join forces in order to respond curricular

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challenges that concern with integrated teaching and learning, and support teachers in taking a literacy stance to teaching (Berlin & White, 2010),

Given the increasing interest in STEM education, which among others entails interdisciplinary approach teaching, and recognizing the benefits and challenges embedded in implementing an integrated approach to the teaching of M&S, we initiated an academic specialized course for secondary mathematics teachers and science teachers [thereafter MSTs] who attended M.Ed. program in a college of education. In the framework of this course, teachers worked in mixed teams to design, collaboratively, interdisciplinary lesson units. In our country, there is a mandatory mathematics curriculum, and to our best knowledge, there is no official program for teaching M&S in an integrative manner. Therefore, we were interested in examining the feasibility of such collaboration, and the study described in this paper was set in order to explore its benefits and limitations as perceived by the study participants.

LITERATURE BACKGROUND AND CONTEXTUAL FRAMEWORK

The natural philosophers of the Renaissance did not draw an explicit distinction between mathematics, the sciences and to some extent the arts (Sriraman 2009). In his paper, Sriraman explores the "*connections forged by the thinkers of the Renaissance between mathematics, the arts and the sciences, with attention to the nature of the underlying theological and philosophical questions that call for a particular mode of inquiry*" (p. 75), and concludes that today's education should move away from separating between disciplines and create bridges among them for the benefit of tomorrow's innovators. Indeed, when confronting with problems in daily life, people seek out helpful information from all sources, which means that conventional school format operates in opposition to the brain's natural way of thinking (Ronis, 2007).

Our paper discusses issues related to integrative approach to the teaching of M&S. The research review carried out by Ann (2016) indicates that such integration has a long history in modern education. Previous studies examined the nature of the connections between M&S, related learning objectives, models for teaching, classroom practices, and more. Moreover, there is a large variety of terminologies, meanings and approaches. Among them: Integration can be modelled as multidisciplinary, interdisciplinary, or transdisciplinary; It can be classified according to 10 levels of integration (fragmented, connected, nested, sequenced, shared, webbed, threaded, integrated, immersed, and networked) or dimensions of integration (discipline specific, content, process, methodological and thematic) integration. Additionally, as indicated by Samson (2014), teachers face great challenges while trying to implement and integrated approach to the teaching of M&S, and most of the lesson plans [thereafter LPs] they develop present low level of integration and a lack of conceptual ground.

Obviously, all these are only the 'tip of the iceberg'; however, they indicate that the integrative approach to teaching involves a great complexity, even before discussing its benefits and limitations in terms of learning outcomes. Recognizing this complexity, and acknowledging that secondary teachers' knowledge is generally limited to a specific discipline, we were curious about the feasibility of designing integrative lesson units by mixed teams of MSTs. To that end, we planned an academic specialized course for secondary MSTs who attended M.Ed. program in a college of education. In designing the learning environment, the *first step* was to find out which terminology best serves are purposes. The research literature often relates to the integration in terms of "multidisciplinary" or "interdisciplinary". The distinction between the two terms is well explained by Lederman and Niess's (1997) metaphor of chicken noodle soup vs. tomato soup. According to the metaphor, multidisciplinary integration is like a bowl of chicken noodle soup, where each ingredient preserves its own identity, and together they constitute a whole; While tomato soup is comparable to the interdisciplinary approach, in which all ingredients merge into one another and cannot be separated or discern. For our purposes, we chose the **interdisciplinary approach**. In the case M&S, we also found it is useful to use Brown and Wall's (1976) scale of **'continuum of integration of M&S'** that concerns the relations between the disciplines as falling on a 5-point continuum: 1. Mathematics activities not involving science; 2. Mathematics concepts of primary importance with science concepts supporting mathematics concepts; 3. Balanced M&S, with no clear boundaries of each discipline; 4. Science concepts of primary importance with mathematics concepts supporting science concepts; 5. Science activities not involving mathematics. Point 3 signifies a "true integration" between the disciplines, and thus reflects the interdisciplinary approach. The *second step* in designing the learning environment, concerned with selecting the methodology for teaching M&S in an integrative manner. For this purpose, we built on Nikitina and Boix Mansilla's (2003) specification of three central approaches- **essentializing** (identifying core concepts that are central to two or more disciplines and bridging among them); **contextualizing** (connecting a particular discovery or theory to the history of ideas of that time); and **problem-centered approach** (recruiting the knowledge and modes of thinking in at least two disciplines to address problems, develop specific products, or propose a course of action).

THE LEARNING ENVIRONMENT

As mentioned, the learning environment was part of a specialized one-semester course. During Sessions 1-5 MSTs were exposed to a wide variety of information and research findings that concern with integrative and interdisciplinary teaching and learning of M&S, and to examples of suitable learning materials. Teachers were then divided into mixed teams according to the grade and level they teach. Each team included at least one mathematics teacher and one science teacher. During Sessions 6-11 teachers collaboratively designed integrative learning units and interdisciplinary LPs. Each group was allowed to formulate its own working procedures, while keeping the following guidelines: (i) The unit should include: description of learning objectives, issues and key concepts in M&S and their interrelations, and relevant prior knowledge; 3-5 integrative LPs, where at least one of them should reflect the idea of "true integration", interdisciplinary, as implied by Point 3; (ii) All the three approaches, essentializing, contextualizing, and the problem-centred, should be expressed in the unit. In Sessions 12-14 each group presented its unit, and engaged the classmates with activities that were included in the LP they indicated as “true integration”. Subsequent to receiving feedback from peers, the units were modified in accordance.

THE STUDY

The study followed the experience of forty teachers, who worked collaboratively in mixed groups of MSTs to design integrative lesson units.

Participant. Among the 40 teachers, 8 teachers teach only in middle school, 19 teach only in high school, and 13 teach in both grade levels. Twelve teach only mathematics, 11 teach mathematics and physics, and 17 teach sciences (7 biology, 4 science-technology, 2 chemistry and physics, 2 computer sciences, 1 chemistry, and 1 biology and chemistry). On average, the teachers had 11.9 years of teaching experience. None of the teachers had prior experience with interdisciplinary teaching, and most of them were not even familiar with this idea. The teachers were divided into 11 mixed teams.

Research questions. The study included three main questions: (1) What are the aspects and processes involved in a collaborative endeavour of designing integrative learning units and interdisciplinary LPs by MSTs?; (2) What is the effect of these processes on teachers' professional development in terms of pedagogical content knowledge and perceptions regarding interdisciplinary teaching and learning?; (3) How do the participants perceive the feasibility of ongoing collaborative work of mixed teams?

Methodology, research tools, and data analysis. The study was carried out through implementing qualitative research paradigm, focusing on processes and meanings as reflected in the eyes of the participants. Since we are not familiar with similar studies that took place in our constellation, the study bore the nature of a pilot study (Creswell, 1994). Data was collected by using the following research tools: (1) Pre- and post-test questionnaire that included open-ended questions related to teachers’ perceptions regarding integrative teaching of M&S, in general, and interdisciplinary in particular; (2) Teachers’ reflective journals in which they documented their experiences during the semester, described their insights, and reflected on them. One a week, the journals were e-mailed to the lecturers and received feedback and advices; (3) Open and participant observations carried out by the lecturers during the teams' work; (4) Open interviews were conducted after the end of the semester in order to deepen our understandings regarding participants’ perception of their experience; (5) The contents and approaches manifested in the learning used for figuring out teachers’ perceptions and interpretations of interdisciplinary teaching and learning. The data was analysed through a gradual process of content analysis (Krippendorff, 2013) in order to identify themes and typical patterns. Then, open and axial coding were employed to generate the main categories and sub-categories (Corbin & Strauss, 2008).

RESULTS

First, it is noteworthy that the collaborative teamwork had different effects on teachers who teach only mathematics and on those who teach physics and mathematics or sciences teachers, and different perspectives arose as a result. This could be anticipated, because while science teachers have to integrate mathematics in their lessons, as requested by the nature of the scientific disciplines (mainly as point 4 on the continuum suggested by Brown and Wall, 1976), mathematics teachers may satisfy with teaching pure mathematics, as implied by point 1 on the continuum. Due to space limitation, in what follows we describe only partial results, focusing on the group of 12 teachers who taught merely mathematics and present some of their typical assertions.

Teachers' main benefits from the collaborative process. As was evident from the teachers' journal entries and the interviews, experiencing the collaborative work had three main general effects: (1) Acknowledging the advantages embedded in interweaving scientific contents in mathematics lessons, both from the teacher's and the students' perspective. From the teacher's perspective: "*As a teacher, I can enrich my math lessons by the beauty of science*"; "*science has the power to breathe life into mathematics. It refreshes the teaching*". From the students' perspective: "*teaching math in its biological or chemical context makes it more relevant for the students*"; "*Explaining mathematical concepts through examples taken from life itself, turns the mathematics into something more 'approachable' for the students*"; "*The integration makes it easy to explain to the kids what mathematics is for*"; (2) Rediscovering the unique position of mathematics, thus feeling a 'professional pride': "*Although M&S are inseparable, obviously science cannot exist without mathematics. It makes me feel some kind of a pride*"; "*the relations among math and sciences are not two-way. Math doesn't need science approval for verifying statements!*"; (3) Acquaintance with teaching methodologies they were not familiar with: "*They* [science teachers] *are accustomed to problem-centered approach, and it is interesting to learn how they implement it.*"; "T*he science teachers use the Internet and YouTube in their classes. We don't do it in math lessons, and it is about time! It is so refreshing*";

Teachers' perceptions regarding the collaborative process. Teachers related to the advantages as well as to the limitations of the collaborative process. One of the prominent advantages was associated with the virtue of the **teamwork and the peer learning**. As for the teamwork: "*This was a real lesson in teamwork. Since every teacher had to take responsibility for a certain component according to his discipline, everyone had the opportunity to contribute and to learn from one another*"; "*In order to explain to Nora* [chemistry teacher] *the idea of the derivative, I had to simplify it. When I thought of a way to show it to her, it suddenly occurred to me that I have misconceptions about continuity and derivatives*". Learning from their peers, teachers noted with satisfaction that "*I found myself listening eagerly to the explanations of Shir* [biology teacher]. *I really enjoyed listening to her. I learned many new things I didn't even knew were existing*"; "*Not only students can benefit from integrative learning. This was also beneficial for me. I learnt a lot about the human body. If we could make* *it a habit, we would enrich our scientific knowledge*". Nonetheless, the mathematics teachers were unanimous about what they regarded as the "***imbalance of the mutual learning***", and perceived it as a limitation of the collaborative work: "*Because science teachers must use mathematics, they knew what they needed, and in a sense they were those who 'dictated' the topic and structure of the unit*"; "*I sometimes had the feeling they don't really need me in the group. While I was excited to learn more about the scientific aspects, they didn't seem to be interested in deepening their understanding of math*"*; "there were moments when I felt that the science teachers were merely glad that I was there to do the 'dirty math work' for them".* This dynamic is well reflected at the titles and contents of the learning units: "*as we could see at the presentations, the titles of all the units indicate the scientific content rather than the mathematical topic (The eye as a vision machine; Electricity in daily life; Genes and heredity…). This is probably a result of our kind of 'marginal status' in the group*"; "*After you mentioned it in class, I realized that actually the majority of the lesson plans correspond Point 4. For some reason we allowed this to happen*". Indeed, our analysis of the 11 lesson units designed (each included 3-5 LPs, on a total 48 LPs) indicated that 5, 6, 30, and 7 LPs were characterized as corresponding Point 2, Piont 3, Point 4 and Point 5, respectively. It should be noted that every lesson unit was examined at least twice in its draft mode. We commented on it, and directed the groups how to change some LPs to better match Point 3 on the continuum. However, apparently, it was not easy for most groups to adjust their LPs to the constraints of Point 3. Interestingly, despite teachers' difficulties to design Point 3 LPs, they had no trouble in recognizing their peers' LPs as not corresponding Point 3.

Teachers' concerns. By the end of the process, teachers expressed their concerns. Some of the concerns pointed out to **inadequate sense of self-efficacy** to teach scientific contents: "*Although I recognize the importance of teaching in integrative manner, still, for me, having to teach scientific topics would be like losing control*"; "*I don't feel I have the confidence to teach even our own unit, not to mention those of the other teams"; "What if I'll not be able to answer questions that relate to the scientific parts of our unit?*"; Teachers were also concerned about the **collaboration itself**: "*It seems that teachers who pertain to different disciplines are like people who come from different cultures. It is difficult to collaborate with someone who doesn't speak your language"; "how can I trust someone I don't even understand what he's talking about?*"; "*There were kind of arguments about many things. For example, how should students' worksheet look like? Should they start from a mathematical problem? Scientific problem? We could not agree about it, and it seemed that although all the disciplinary Standards have similar titles, there are no real connections*".

SUMMARY, DISCUSSION AND CONCLUSION

STEM education is gaining a momentum around the world. One of its implications is the need to develop new learning materials, to change teachers' patterns of work, and prepare for a partially interdisciplinary teaching. Given the considerable complexity embedded in preparing for and implementation of integrative teaching of M&S (e.g.

Berlin & White, 2010; Ann, 2016), we set up a study aimed at identifying the various aspects and processes associated with MSTs collaborative effort to design integrative lesson units and interdisciplinary LPs, and examine teachers' perceptions regarding its feasibility. The idea of interdisciplinary teaching was new to all the study participants; therefore, we assumed from the outset that a one-semester experience is too short for assimilating the full meaning of interdisciplinary teaching. Indeed, as indicated by the results, the study participants had difficulties in developing interdisciplinary LPs (Point 3 on Brown and Wall' (1976) continuity). Description of the nature and sources of these difficulties is not included in this paper. However, this is not surprising, since in order to be able to design such LPs, mathematics teachers should know enough science and science teachers should know enough mathematics (Samson, 2014). This is not the way teachers were taught in teacher training programs (Sriraman, 2009). Nevertheless, the teachers viewed positively the collaborative teamwork, and felt they had gained new knowledge in sciences, new didactic knowledge, and insights regarding the benefits of interdisciplinary teaching. This is the encouraging side of the coin. However, it is the feeling of *imbalance mutual learning* and the need *to blindly trust science teachers* that appear to bother mathematics teachers most, and perhaps cast doubt on the feasibility of such collaboration. These concerns were explicitly expressed by one of the teachers in the following manner: *"I'm quite exciting about interdisciplinary teaching…It is only a matter of teachers' good will to cooperate…there was something in the unbalanced group dynamics that bothered me, and I kept asking myself whether such teaching is a possible reality or merely a wishful thinking…I believe that either you add scientific contents to the various programs of math teachers or instead of just co-designing integrative lesson plans we should also co-teach*". Since in the near future, it does not seem to us likely that in our country teacher training programs will modify and become integrative, it appears that in order to enable mathematics teachers to teach in various levels of M&S integration (as well as integration of math with another disciplines), and provide them with tools that will allow them to independently develop teaching materials, professional development programs should enrich mathematics teachers' scientific knowledge, and support their ability to design interdisciplinary LPs by themselves. Obviously, this necessitates the collaboration of mathematics and science teacher educators. Further studies are needed in order to examine math teachers' needs for this purpose, and the required support.

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DECOMPOSITION CONSIDERATIONS IN GEOMETRY

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We motivate and display primary considerations for analysing and designing geometry tasks, with respect to the utilization of the heuristic of decomposition and recomposition. The considerations are relevant for both proof tasks and calculation tasks. They involve an aspect of viewing geometrical configurations as compositions of generic structures in the forms of: concatenation, inclusion, and interleaving; and an aspect of fluency and flexibility notions in examining decomposition directions and resource manipulations. We display a study of 7th and 8th graders, and underline the importance of the close link between the stages of decomposition and recomposition of the heuristic, in solving geometry problems.

INTRODUCTION

One of the more dominant problem-solving heuristics in mathematics in general, and geometry in particular, is the heuristic of decomposition and recomposition. One facet of this heuristic is the decomposition of a task into subtasks and the recomposition of the subtasks' solutions. Another facet is the decomposition of a task configuration into subparts and the reconfiguration of the subparts. The heuristic is relevant in both proof and calculation tasks in geometry.

Consider the following proof task and two calculation tasks.

Given: AC=BD, AD=BC **Prove:** $\angle C = \angle D$	Given a rectangle to which a semicircle has been attached. **Calculate** the grey area size.	Three squares are placed one on top of the other. **Calculate** the size of the total area in the configuration.
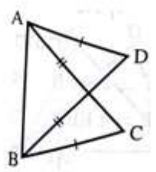	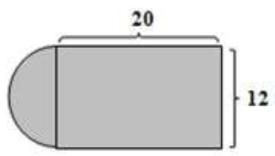	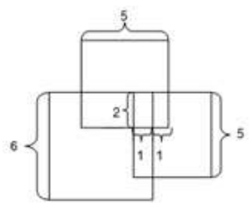
Fig 1. An elementary proof task	**Fig 2.** An elementary calculation task.	**Fig 3.** An additional calculation task.

The proof task in Figure-1 involves two related stages: 1. Decomposition of the given configuration into the two triangles that are originally interleaved; 2. Recomposition of the obtained triangles into a congruence theorem, which yields angle equality. The calculation task in Figure-2 involves two related stages as well: 1. Decomposition of the configuration into a semicircle concatenated to a rectangle; 2. Recomposition of the computed areas of these two parts. The task in Figure-3 involves these stages in a somewhat subtler way (as will be shown below).

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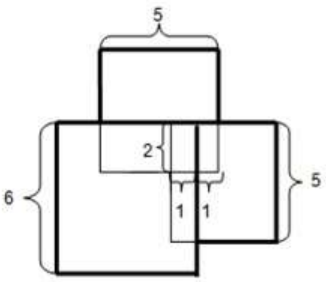
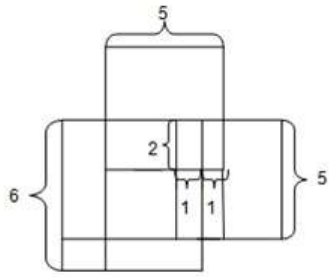
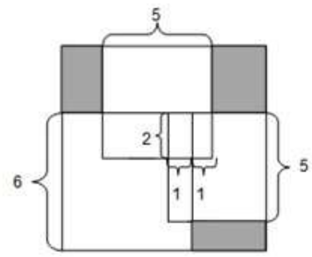
In our experience with 7th and 8th graders, quite a few students demonstrate difficulties with the required decompositions and recompositions. In the proof task, quite a few 8th graders focus in their decompositions on the small "*atomic*" triangles, rather than on the large interleaved ones. This leads them to dead-end directions in the recomposition stage. Their difficulties seem to stem from the less evident *interleaved* triangles.

In the computation task of Figure-2, 7th graders recognize the two parts into which the configuration should be decomposed, probably since the two parts are *concatenated*, rather than interleaved. They properly conduct the decomposition, but face difficulties in the recomposition stage, in calculating the area of the semicircle. The calculation requires a series of manipulations: recognition that the rectangle width is the circle's diameter, division of this diameter by 2, calculation of the circle area, and division of this area by 2. The required *flexible manipulations* are challenging for some students.

The computation task of Figure-3 illuminates an additional challenge to students. The configuration involves three interleaved squares, and may be decomposed in various ways. Some decompose it into these three squares and subtract *overlapping* sub-areas, but their subtractions are often erroneous. Others attempt decompositions into *concatenated* areas, which do not overlap, like those shown in the figures below.







The left figure displays division into an elegant *concatenation* of a square and two rectangles; and the middle figure shows *brute-force* division into many "*atomic*" *fragments*. Many attempt divisions similar to that in the middle figure, and then yield erroneous summation. The right figure offers an *inclusion* perspective (of the original configuration in a large square), which is obtained from an auxiliary construction. All in all - a variety of decomposition alternatives from which to choose; and the selected alternative affects the complexity of the recomposition calculations.

The above illuminates several essential geometry considerations: **1.** *Structural forms* of geometric configurations - concatenation/inclusion/interleaving - are significant; **2.** *Concatenation* is the simpler form; it is more natural to decompose into "*atomic*" substructures (with no lines inside them); **3.** *Interleaving* is the subtler form; it may "mask" the relevant substructures on which to focus, in both proof and arithmetic tasks; **4.** Decomposition and recomposition go *hand-in-hand*; in particular, one may need to carefully choose among the *fluency* of decompositions a suitable one, which yields an effective recomposition; **5.** Recomposition may require *flexible* manipulations.

In this paper, we elaborate on the notions italicized above, and extend the perspectives of previous studies on problem solvers' behaviours in geometry.

Previous studies in geometry offered a variety of perspectives, such as: Van Hiele's (1986) levels of geometrical thinking, Fischbein's (1993) study of internal tension with figural concepts, Gal and Linchevski's (2010) study of difficulties in recognizing substructures in a structure, and Duval's observations of "operative apprehension" (1995, 1998) and visualization aspects (2006).

Van Hiele's (1986) levels focus on the linear progress in geometric thinking, from visualization to deduction and rigor. Fischbein's (1993) terms of internal tension with figural concepts are primarily focused on visualization, but without much emphasis on decomposition-recomposition processes. Such is also the case with Gal and Linchevski's (2010) study, which focus more on difficulties in identifying particular substructures and tendencies to relate unfamiliar structures to familiar ones.

Duval examined the process of decomposition in several studies, including some that involve operative apprehension (e.g., Duval, 1995, 1998). He characterized (among other things) cases of novice decompositions of given configurations, which were oriented toward different goals, in various activities. Some of the goals that directed decompositions were related to the application of geometrical theorems. He noticed that when novices think of a theorem that was necessary, their operative apprehension, of progress with decomposed elements, was more successful. Nevertheless, novices' still tended to focus on the easier-to-identify subparts; and often followed a single decomposition path into subparts, even when it was unsuitable.

Operative apprehension was more apparent and conversely less apparent, possibly due to triggering and inhibiting elements. These elements are related to convexity and complementarity of (sub)configurations, as well as to additional Gestalt aspects. We believe that Duval's perspective may be enhanced with elements such as those we indicated earlier, related to explicit structural forms, such as interleaving and concatenation, as well as to aspects of multiple decomposition alternatives and flexible recomposition manipulations.

ENHANCED DECOMPOSITION/RECOMPOSITION CONSIDERATIONS

The notions of decomposition and recomposition are relevant not only in mathematics. One additional domain is that of computer science. In computer science, solution approaches such as Top-Down design and Divide-and-Conquer encapsulate decomposition and recomposition of subtasks' solutions. In addition, a computer program may be viewed as composed of *generic algorithmic templates*, such as counting and searching, which are combined to form a whole program.

Computer science educators specify three common composition forms - concatenation, inclusion, and interleaving (Soloway, 1986). *Concatenation* involves "gluing" of templates; *inclusion* involves the enclosure of one template fully inside the other; and *interleaving* involves merging of templates, such that some parts (but not all) of one template are within the other. Concatenation is considered the simplest composition form, and interleaving is considered the more involved one (Muller et al., 2007).

We may adopt these terms and employ them in geometry, with *compositions of generic geometrical structures*, such as triangles and squares (as we have seen in Figures 1 to 3). This perspective may offer us a means for classifying *geometric configurational forms*, according to their compositions from generic structures. Such a classification may hint at less challenging and more challenging configuration decompositions, and may be useful for explaining students' tendencies and difficulties. As we have seen earlier, interleaved forms may "mask" relevant substructures, and possibly draw attention to irrelevant "atomic" (no-inner-lines) substructures.

In addition, multiple decomposition alternatives and non-immediate manipulations pose a challenge to problem solvers in geometry tasks. The multiplicity of options and the required manipulations may be tied to two essential notions in the domain of creativity - the notions of fluency and flexibility (Ginat & Spiegel, 2015). *Fluency* involves the number of responses, or ideas generated in response to a prompt; and *flexibility* involves shifts and alternations in the general responses (e.g., Torrance, 1988). Creativity is studied in mathematics and geometry, but primarily with respect to solving "rich", intriguing tasks (e.g., Levav-Waynberg & Leikin, 2012). We propose borrowing these notions and examining them here somewhat differently.

Thus, we tie the term *fluency* to the various, possible decompositions of a given configuration, which one debates upon examining a given configuration; and we relate the term *flexibility* to necessary manipulations that may be needed upon recomposition. Manipulations may include auxiliary constructions, segment addition/subtraction, arithmetic calculations, and more. Proper demonstration of both notions expresses the essence of suitable control in Schoenfeld's problem solving model (1992). In the next section we shed light on novices' geometry problem solving, through the enhanced decomposition/recomposition considerations displayed here.

METHODOLOGY

Sample

The study's sample included 151 7^{th} and 8^{th} grade students, from four junior-high schools. The 7^{th} graders were all acquainted with fundamental geometry terms and calculations, such as circumference, area, angle calculations, segment subtraction, and more. The 8^{th} graders practiced a variety of triangle-congruence proofs, using the very basic congruence theorems. In both years of geometry studies they have seen and practiced diverse tasks that required decomposition and recomposition.

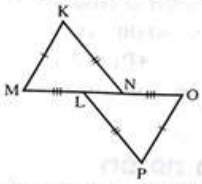
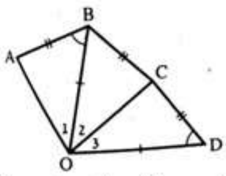
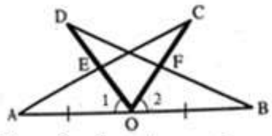
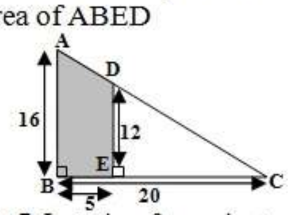
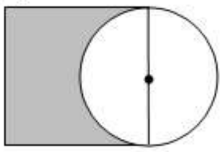
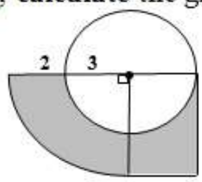
Tools

The study involved two questionnaires. A questionnaire with 11 calculation tasks was posed to 68 7^{th} graders; and a questionnaire with 12 proof tasks was posed to 83 8^{th}

graders. The calculations questionnaire involved circumference and area calculations; and the proofs questionnaire involved triangle congruence proofs. The questionnaires' tasks required different levels of competence, which were designed following our suggested considerations. In particular, we varied: the amounts of generic structures in a configuration, the forms of their compositions, the resources needed for solutions, and the kinds of relevant, flexible manipulations. The table below displays the tasks' classifications with respect to the forms of substructure compositions.

Proof tasks – **triangles congruence**	Calculation tasks – **circumference and area**
Q1-Q3: Concatenation, 2 generic structures	**Q1-Q3: Concatenation, 2 generic structures**
Q4-Q8: Interleaving, 2 generic structures	**Q4-Q6: Inclusion, 2 generic structures**
Q9-Q10: Concatenation, 3 generic structures	**Q7-Q8: Interleaving, 2 generic structures**
Q11-Q12: Compositions of 4 gen-structures	**Q9-Q11: Compositions of 3 gen-structures**

The table below displays six of the 23 tasks we posed to the students (three additional ones were displayed in the Introduction). They represent tasks of different levels of complexity, with respect to the elements mentioned above.

Given: KM=OP ,PL=KN, ML=ON **Prove:** $\Delta KMN \cong \Delta POL$  **Fig 4.** Concatenation, 2 generic structures	Given: CD=BC=AB , DO=BO, $\angle ABO = \angle CDO$ **Prove:** $\angle O_1 = \angle O_2 = \angle O_3$  **Fig 5.** Concatenation, 3 generic structures	Given: AO=OB ,CO=DO , $\angle O_1 = \angle O_2$ **Prove:** $\Delta FOB \cong \Delta EOA$  **Fig 6.** Interleaving, 4 generic structures
Given the two right triangles ABC and DEC, **calculate** the area of ABED  **Fig 7.** Inclusion, 2 generic structures	The square area is 144 cm². **Calculate: a.** the circle area; **b.** the grey area.  **Fig 8.** Interleaving, 2 generic structures	Given the interleaving of a rectangle, a circle and a quarter circle, **calculate** the grey area.  **Fig 9.** Interleaving, 3 generic structures

Process

The students in both groups were given 90 minutes to solve the written questionnaires. Some students requested a time extension, and we let them work as long as they needed. Following their written solutions, about 20% of the students, who demonstrated different levels of competence, were interviewed about their solutions. We analysed the solutions with respect to the considerations displayed in the previous section, while viewing fluency and flexibility difficulties as control obstacles in Schoenfeld's perspective of problem solving (1992).

FINDINGS

The overall statistics in both questionnaires were similar - the average number of solved tasks was: 5.7 of the 11 proof tasks (52%) and 5.9 of the 12 calculation tasks (49%). Closer examination reveals several aspects of difficulty with both decomposition and recomposition in the various tasks. We display them with the solutions of the six tasks presented in the previous page.

The task in Figure-4 was one of the simplest proof tasks. It involved two *concatenated* triangles that share a segment. The decomposition was trivial; but the recomposition was less so. Only 61% (51 of 83) of the students properly solved it. The solution is based on the Side-Side-Side congruence theorem; yet, in order to apply it properly one needs to add the mutual segment LN to those next to it (ML and NO). Students did select the relevant theorem, but some expressed "... a confusing thing here with the segments ...", and did not handle the little *flexibility* that was required.

The task in Figure-5 was subtler, as it involved *three* concatenated triangles. Students did recognize the three triangles, but struggled with the decomposition-recomposition *link*. Only 39% of the students properly solved the task. Their main difficulties stemmed from the need to first map two triangles to one congruence theorem, and then map two additional ones to another theorem. Some mentioned that: "... I see all the data together and it is hard to tell which way to go ...". It seemed that the *cognitive load* derived from having three triangles interfered in the progress of *hand-by-hand* decomposition/recomposition, while *mapping* triangles to theorems.

The task in Figure-6 was the hardest proof task. It involved the *interleaving* of four triangles. The main challenge was first and foremost with decomposition, and then with the decomposition-recomposition link. Only 11% of the students properly solved the task. There are several "*atomic*" triangles in the figure, and although two of them are necessary for the end result, one had to first turn to "*non-atomic*" ones. Many failed the latter. They indicated that: "... there are many parts and they are all mixed together ...". The interleaving yielded a blurred picture of *fluency* alternatives.

The task in Figure-7 was a calculation task that involved the *inclusion* of one triangle in another. Inclusion, like concatenation, is much simpler than interleaving, but still poses some challenge. It is suitable here to subtract the area of the "included" triangle; and 75% (51 of 68) of the students properly managed the subtraction. Yet, some did not handle the inclusion, and just argued that: "... we did not learn how to calculate the area of a trapezoid ...". Others attempted the inclusion, but failed the simple *flexibility* required to calculate the side of the included triangle. One student said that: "... it is always hard for me to obtain data that is not directly given to me ...".

The task in Figure-8 involves interleaving of two generic structures - a square and a circle. Yet, unlike the interleaving in Figure-6 (and earlier, in Figure-3) the interleaving here posed less of a decomposition challenge. The challenge was rather in the recomposition. Only 44% of the students solved the task. Many failed to properly calculate the area of the semicircle and subtract it from the area of the square. Some argued that: "... one half is inside and one half is outside ...", and "... it is complicated, you need to remove the piece of the part that goes in ...". The *interleaving* created a difficulty in *flexibly* manipulating the recomposition calculations.

The task in Figure-9 was the subtlest amongst the calculation tasks, as it involved three *interleaved* structures. The challenge was both in the decomposition's *fluency* and the recomposition's *flexibility*. One had to decompose the configuration into relevant subparts, and then recompose the calculated subparts data. Only 32% of the students provided the right calculation. Some said that: "... the structure is a mess, parts are one inside another ... and where is the rectangle? ...". Many failed to calculate the left subpart of the grey area. All in all, quite a few gave up already at the decomposition stage, and some who did advance further found the recomposition too challenging.

DISCUSSION

In examining the findings, we may notice that the kinds of structural forms - concatenation, inclusion, and interleaving - yielded different levels of challenge to the students. So was the amount of generic substructures within a given structure.

The less competent students did not demonstrate major decomposition difficulties with concatenated structures; yet they did struggle in the recomposition stage, even with simple flexible manipulations. When it came to interleaved structures, many more students were challenged, due to multiple decomposition alternatives, that often "blurred the picture" to many and yielded only "atomic" subpart views.

We regard the above difficulties as strongly tied to Scoenfeld's control component of problem solving (1992), with respect to the creativity notions of fluency and flexibility (Torrance, 1988). Decomposition requires careful fluency analysis, of possibly overlapping elements. Recomposition requires flexible manipulations with elements and resources (such as theorems and formulas). In addition, fluency and flexibility may be strongly tied together, as there are cases (such as in the task of Figure-3) in which the decomposition may directly increase, or reduce the recomposition complexity.

Clements and Battista (1992) argue that one of the obstacles of students in geometrical problem solving is the students' very limited familiarity with task and configuration diversity. This may have been a primary reason for the student difficulties that we observed. We believe that in order to develop geometrical competence, teachers should demonstrate, and practice with students, tasks like those in our questionnaires, which take into consideration the elements listed above. In particular, teachers should be aware of the explicit relevance of the various forms of generic structures, and of the notions of fluency and flexibility in geometrical problem solving. This awareness may enable teachers to plan the tasks they pose to students in an orderly fashion, according to suitable measures of challenge, in developing their students' competencies with the heuristic of decomposition and recomposition in geometry.

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POETIC STRUCTURE CHAINING IN A PROBLEM-SOLVING CONVERSATION

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Poetic structures are commonplace linguistic resources in which speakers repeat some of the grammatical structure and words of a previous comment. Poetic structure analysis allows us to trace key mathematical ideas through a conversation as they shift from speaker to speaker, to account for the growth of collaboratively-generated mathematical ideas. Analysing dialogicality with poetic structure chains reveals the level of collaboration, the robustness of seemingly trivial mathematical activities, and the complexity of referencing as speakers get closer to an answer.

INTRODUCTION

Students in a mathematical conversation must share bits of meaning through their discourse, but they must also deconstruct and reconstruct these meanings as they engage the task. Linguistically, one means of accomplishing this negotiation is through poetic structures. Poetic structures occur when a speaker repeats a prior comment, keeping some of the grammatical structure and some of the words. In a 2016 research report, I outlined a method for tracing the growth of mathematical ideas in conversation through poetic structures, by noticing the most recent prior statement that contains a grammatically and lexically similar statement about mathematics (Staats, 2016, 2008). These poetic structures can facilitate significant mathematical activities such as associating one variable with another, generalizing, and shifting from verbal to written mathematical forms. However, if we can identify the most recent prior, similar mathematical statement in a conversation, then we can recursively apply the method to identify the prior statement of the prior statement. The following report engages the idea that through poetic structure analysis, we can identify chains of related utterances in a mathematical conversation that track collaboratively-generated ideas.

THEORETICAL FOUNDATION

Analysis of poetic structure chains is based on the framework of dialogicality proposed by M. M. Bakhtin and V. N. Voloshinov (Bakhtin, 1981; Voloshinov, 1973). Dialogicality refers to the perspective that any linguistic expression carries with it the voices of others, voices from the past and those expected in the future: “Any utterance…makes response to something and is calculated to be responded to in turn. It is but one link in a continuous chain of speech performances” (Voloshinov, 1973, p. 72). This heteroglossic dialogicality is present in all frames of speech, from a word to a genre (Bakhtin, 1981). Poetic structures are one way in which this interpenetration of

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voice occurs. Repeating prior comments, with small changes, establishes a dialogic relationship with the past and offers a platform for future linguistic constructions.

Barwell (2015) discusses three Bakhtinian orientations of dialogicality, multivoicedness, multidiscursivity, and linguistic diversity, as resources for learning mathematics. This report uses poetic structure analysis to decompose a problem-solving transcript into strands of mulitivoicedness, thematically connected comments in which students revoice and transform prior mathematical statements as they move towards a solution to the task. By identifying poetic structure chains in a concrete manner, this paper examines multivoicedness as a resource for learning.

PARTICIPANTS AND TASK

Two undergraduate students who had recently completed a university class in precalculus participated in a paid, audio- and video-recorded problem-solving session outside of class. Together, they found a formula for the perimeter of a string of n adjacent hexagons. The task is given in Wilmot, et al, 2011 (p. 287). The key issue in the task is that by arranging hexagons in a string, some sides occur inside the figure and no longer contribute to the perimeter. A correct answer for the perimeter of a string of n hexagons is given by $p = 4n + 2$. The students' answer after 90 conversational turns was $\#H(6) - 2(N - 1) =$, in which both H and N stand for the number of hexagons.

METHOD

Each of the first 90 turns of the conversation were analysed to determine whether the turn included a repetition of a previous statement (Staats, 2016). A phrase within a turn at talk counted as a repetition of a prior phrase if the two phrases had syntax in common and at least one word in common, to insure some continuity of topic instead of producing a purely abstract, grammatical mapping of the conversation. Each repetition, along with its prior phrase, was recorded in a spreadsheet. A repetition could be "internal" to one speaker's turn at talk, or "across" two different turns at talk, either from the same or different speakers. These "across" repetitions most directly record the quality of dialogically in the conversation, and they are the focus of the current research report.

In the first 90 turns, each turn at talk was distinct, with very little overlap between speakers' voices. This fortunate happenstance means that each odd-numbered turn at talk was Joseph's speech and each even-numbered turn was Sheila's speech, which makes reading the summary table easier than it might be for other conversations.

Because the spreadsheet recorded the connections between a student's current comment and previous, similar ones, the relationships of repetition could be traced backwards through the conversation to reconstruct poetic chains of comments that pairwise share some grammatical structure and some words. Taken pairwise, there are similarities in the students' comments, but over a long chain, the comments can transform substantially as students improve or modify their shared mathematical ideas.

These chains of poetic structures involved temporary, informal verbal signs that students used and shared as they collaborated on the task.

For example, the following transcript shows a chain that starts at turn 14 with an echo or repetition of the word *6*, and ends at turn 52. In table 1, this is chain 14b to 52. The phrases that were identified as poetic structure repetitions of each other are highlighted in bold. This deconstructed transcript leaves out many turns of talk in order to focus on the poetic structure repetitions that grew from echoing the word *6*.

14b S: 1, 2, 3, 4, 5, 6. **6.**

15 J: **6.** So, this would be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. 10.

40 S: So **that would be 6L.**

41 J: So **our perimeter is 6L**. But how did we find 6L?

46 S: Well **perimeter is adding up all the sides**.

47 J: Right. But **we wouldn't need all the sides**. You know what I'm saying? Cause these, these are the, these are on the inside of the square. And then the bases of these triangles are on the inside so we wouldn't need those. We just need this hypotenuse of that triangle. So this is the inside of this triangle.

52 S: Complete the table showing the number of hexagons in 1 chain along with the perimeter. So then **we're counting all the sides**, so it'd be 6L. For 2 it'd be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. 10L.

In this chain, Sheila and Joseph both echoed 6 as they began to fill a table of values for the number of hexagons vs. the perimeter. The *6* was echoed as *6L*, and Joseph associated it with perimeter in turn 41. Sheila transformed his comment with another elaboration of *all the sides*, which she used again in turn 52 to establish agreement that they needed to compute the external perimeter rather than the area of the figure.

Space does not permit me to offer many examples of poetic structure chains, but they are all similar to this one (many turns at talk are presented in Staats, 2016). At any point in the conversation, we find the most recent prior phrase that has shared grammar and at least one word. Pairwise, the phrases are similar but the initial idea transforms gradually into a somewhat different idea. Here, echoes of 6 transformed into more generalized statements about perimeters. Poetic structure chains provide a view of the connectivity and referencing in a conversation as students transform prior mathematical comments during problem-solving.

RESULTS

This analysis identified 22 poetic structure chains in the first 90 turns of the conversation. If we consider the length of the chain as its number of turns, then there were 10 turns of the minimal length of two, two turns of length three, and 10 turns of length five to nine. Poetic structures chains occurred throughout the conversation, from the earliest turn to the final turn at 90. Turns 14 and 62 were the origin of six and five chains, respectively. These turns originated chains that were highly influential in the course of the conversation, and they are discussed below in more detail. Table 1 summarises the poetic structures and the transformations that they facilitated.

For the sake of readability, the table gives a slightly simplified view of poetic structure chains. Poetic structures were identified at the level of small phrases within a turn at talk, but the chains are identified in the table by their turn, not the phrase. When chains are identified with a letter, for example as 14a and 14b, it means that different phrases in turn 14 gave rise to different poetic structure chains. Underlining indicates turns that are shared in more than one poetic structure based on the same shared phrase. For example, in the transcript above, the chain 14b-15-40-41 was shared by three chains, but each took a different pathway after turn 41 by transforming different phrases into different, small, mathematical ideas.

Poetic Chains	**Repetitions and transformations in the chain**
1-3	Repetition about working in a girl scout camp
9-11	Repetition about benches that seat two people.
14a-15-17-19-21-24-26-27 14a-15-17-19-21-24-52-62 14a-15-17-19-21-24-56	The 14a chains started with counting the sides of one hexagon, 1, 2, 3, 4, 5, 6. Sheila's data summary at 24 was a branching point. Chain 14a to 27 completed the data summary. Chain 14a to 62 culminated in the first mention of minus for subtracting interior sides. Chain 14a to 56 involved reducing the interposed list into a list only of the y-values or perimeters.
14b-15-40-41-44-45 14b-15-40-41-46-47-52 14b-15-40-41-44-52-62-64	The 14b chains started with echoes of the number 6 after counting the sides of one hexagon. Chain 14 to 45 explored the perimeter of one hexagon in terms of phrases *6L* vs. *6 square*. Chain 14 to 52 developed a generalized discussion of perimeters and *all the sides*. Chain 14b to 64 involved a series of echoes and brief comments involving *6*, *6L*, *6 square*, and *6L*.
21-25	An echo of 18 after previous speaker counted to 18.
29-45	Repetition of *2L plus* while discussing how to calculate perimeter.
31-32	Repetition of *four triangles* while drawing triangles inside the hexagons.
42-43	Repetition of *for one* while discussing that one hexagon has 6 sides.
62-64	Turn 64 is a repetition and refinement of turn 62.
62-64-90a	Repetitions of 2, 4, 6, helped validate the final formula.
62-64-72-74-76a-79-80--81	Chain 62-(76a)-81 branched off at 74 with the phrase *hexagons minus* and involved discussion of whether there is a minus 2 or plus 2 in the formula. At 79, it rejoined other chains with a clarification about *times 2*

	or minus 2 at 79.
62-64-72-74-76b-78b--79-80-81	Chain 62-(76b-78b)-81 involved a consolidation of commentary at 64 and 74 about *total number of sides minus 2* and *4 hexagons, hexagons minus 2*. It also joined chains at 79.
62-64-72-74-76b-78b--90b	Chain 62-(76b-78b)-90b contributed to the validation of the n = 4 case through comments involving counting *1, 2, 3 ,4*, and about *times 2*.
62-64-72-75-78a	Chain 62-78a, involved comments on multiplying the number of hexagons times 6 ending with: *number of hexagons would be 4 times 6 minus n minus 2*.
68-70	Echoing *negative two*.
80-81-86	Repetitions alternating between *24 minus 8* and *24 plus 8*, exploring a formula for the four hexagon case.
84-85	Repetitions of *it'd be 18* and *it'd be 16*, exploring subtraction in the four hexagon case.
84-87	Repetition of *24 would be 18* and *22 would be 18,* exploring subtraction in the four hexagon case.

Table 1: Summary of poetic structure chains in the hexagon task.

DISCUSSION

Analysis of poetic structure chains reveals that the conversation was more collaborative than one might think from simply reading the transcript. Without close attention to shared discourse, it could seem as if Sheila was responsible for most of the insights. In turn 24, she articulated patterns linking the number of hexagons and the perimeter of the figure; in turns 62 and 64, she explored the number of interior sides that must be subtracted, and in turn 90, she conveyed the final method. Sheila used internal poetic structures extensively in each of these turns (Staats, 2016), but these internal repetitions do not show up in the current analysis of dialogicality. Both Sheila and Joseph spent about the same amount of time speaking, with an average of 12 words per turn for Sheila and 10 for Joseph, so both had opportunities to originate mathematical ideas that could become poetic structure chains. Accounting for duplicates chains in table 1, there were 27 unique repetitions that involved the same speaker, and 26 unique repetitions that involved a change in speaker. This suggests that speakers were voicing the other person's idea about as often as they were voicing one of their own prior ideas. Although Sheila seemed to be dominant, the conversation was quite collaborative.

Analysis of poetic structure chains exposes several structural features of the conversation that are not apparent when reading the transcript line by line. First, there is branching behaviour in the conversation. If a student's mathematical comment is later repeated in two different turns, then the chain breaks into more than one subchain.

This happens if different phrases in a comment need to be unpacked and discussed in different ways. These points of bifurcation in the conversation could indicate an influential moment, or a moment that raises several issues that need to be resolved.

Another conversational structure is that students' use of a prior mathematical idea can create reunions within the poetic structure chains. This happens if a later poetic structure coordinates two statements from the past into a new statement. A few types of poetic structures are likely to have this effect. Some poetic structures establish a contrast between two prior statements, some establish a comparison, and some consolidate two or more prior statements into a new statement (Staats, 2016). Circuits are possible, too, if a comment creates two issues that students wrestle with and then later coordinate, contrast, compare or resolve.

Turns 14 and 62 created long poetic structure chains that involve some of these structures and that were influential in the students' solution method of $\#H(6) - 2(N - 1) =$. I will use some of the terminology that I introduced in 2016 for different types of poetic structures, but I will incorporate more self-contained descriptions, too.

Table 1 suggests that turn 14 was influential in the later discussion, because six chains emerged from it. The 14a chains started with counting, or "listing," the sides of one hexagon: *1, 2, 3, 4, 5, 6. 6.* Sheila and Joseph created a table of data for *n*, the number of hexagons vs. perimeter of a chain of *n* hexagons. Turn 24 was a branching point, when Sheila coordinated the number of hexagons with their perimeters in an "interposed list:" *…we're just putting in the 1 to 6, 2 to 10, 3 to 14, uh, 5, 1, 2. Wait.* From here, chain 14a to 62 incorporated or "consolidated" more side-counting back into the interposed list and culminated with the first mention of *minus* for subtracting an interior side. Chain 14a to 56 involved reducing the interposed list at turn 24 into a list involving only the perimeters: *So this was 10L, 14L, 18L, 22L, right?* Generally, the 14a chains helped students understand that they needed to subtract interior sides, and it helped them focus on the perimeters for which they needed a formula.

The 14b chains emanated from revoicing or "echoing" the number 6 in *1, 2, 3, 4, 5, 6. 6.* Echoing *6* in various forms occurred deep into the conversation as the speakers returned to easier ideas in order to build harder ones. For example, echoes of 6 and 6L helped Sheila begin her turn 62, a turn which itself launched several new poetic structure chains. At turn 62, Sheila began to focus on how to subtract the interior sides of the hexagon chains (for 1, 2, 3 and then back to 1 hexagons) with her comment:

> 62 S: *Uh, so this would be 6L. 6. And then this would be 10L minus 2. Minus 2. This would be 2, 4 minus 4. This would be 6. 18L. So the total number of sides minus 2 on this side. So it'd be, uh, 6. 6, and then this would be, uh, 12 minus 2. So.*

Like turn 14, turn 62 was very influential in the problem-solving pathway because it gave rise to five poetic structure chains, all examining different facets of the issue of subtracting interior sides. Table 1 also shows us that the students' referencing of prior ideas—their own ideas and the ideas of the other speaker—became rather complex after turn 62, as they got closer to their formula. The five poetic structure chains that start at turn 62 include branches, reunions, circuits, and some contributed directly to the final method in turn 90. After turn 62, students had numerous issues to pursue, and they could not resolve them all in one or two speaking turns.

Turn 74, for example, was a branching turn. Sheila had already mentioned *total number of hexagons times six* (turn 72), but she could not fully coordinate this multiplication with subtraction in her turn 74: *So total number 1, 2, 3, 4 would be 4 hexagons, hexagons minus.* Turn 74 became a branching point because Sheila continued to count hexagons with the phrase *1, 2, 3, 4* in turns 76 and 78 as she tried to decide how many interior sides are generated by a 4 hexagon string; and also, because her phrase *hexagons minus* was repeated through various transformations and attempts to write a formula that circled around whether it should have a minus two or a times two (chains 62 to 81 and 62 to 90).

Turn 79 was a point at which two chains joined together. Joseph's comment of *Times 2 or minus 2?* was a "contrast" poetic structure in which he quoted Sheila's phrases from turns 76 and 78, asking for clarification.

For students to arrive at their solution *#H(6) – 2(N – 1) =* , they needed to establish at least four ideas: multiplying the number of hexagons by 6; subtracting interior sides; subtracting interior sides in pairs (the minus 2 of their formula); and that there are *N – 1* pairs of interior sides. Of these, all but the last item were facilitated rather strongly by poetic structure chains. The initial echoes of *6* in turn 14b led to echoes of *6L* in Sheila's critically important turn 62. The poetic structure chain 14a to 62 started with counting and ended with the first mention of subtraction, and many other chains involved commentary on subtraction. Considering the pairs of interior sides was handled through chains 62 to 90a; 62 to 90b, 62-(76a)-81, and 62-(76b-78b)-81. However, the precise moment of Sheila's final insight at turn 90 that there are *(N – 1)* pairs of interior sides did not directly reference poetic structure chains. Instead, she used internal poetic structures to arrive at this result (Staats, 2016). Poetic structure chains across different turns at talk did not contribute directly to this key achievement.

CONCLUSION

A transcript of a mathematical conversation is typically read in a linear fashion, as it unfolded over time. Analysis of poetic structure chains offers a different way of navigating a transcript, and as one might expect, it has disadvantages as well as advantages. Disadvantages include generation of a great deal of detail, of literally focusing on the trees—the chains and branches—rather than on the forest, or the overall flow of the conversation. Important insights sometimes occur while one person is speaking, even if their insight draws upon the accomplishments of several poetic structure chains. No one would wish to read poetic chains instead of a linear transcript, if they were required to make a choice.

Still, awareness of poetic structure chains can improve our understanding of collaboration in mathematical conversations in some important ways. It helps us judge the degree to which a dialogue is actually collaborative, because we can trace how often speakers shared phrases and ideas between them. This form of analysis yields a greater appreciation of how seemingly inconsequential statements like echoing or counting may be persistent and significant. Identifying poetic structure chains allows us to notice highly influential moments in a conversation: the initial statement of a long chain, a statement that branches into several issues that students grapple with and transform along separate chains, and moments in which different comments are coordinated into a comment that closes a circuit. Reading a transcript in a typical, linear manner obscures the complexity of collaborative student reasoning that poetic structure chaining reveals.

For researchers interested in collaborative learning, it is not necessary to account for all the poetic structure chains in a problem-solving conversation as I have done here. One could instead identify moments in a conversation that heuristically seem important, and reconstruct the poetic chains that formed this moment, and that emanated from it.

Perhaps most importantly, analysis of poetic structure chains allows us to connect the concept of heteroglossia to students' extended struggle to create detailed mathematical ideas. When students speak their way through a mathematical task, as Bakhtin predicted, their comments are densely inhabited with the voices of the past.

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MATHEMATICAL KNOWLEDGE IN UNIVERSITY LECTURING: AN IMPLICIT DIMENSION

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Mathematical knowledge in lecturing has an implicit dimension, which characterises the unarticulated mathematical knowledge. Interviews and lecture observations with eight research mathematicians revealed some of the reasons for withholding their mathematical knowledge in lecturing. Research mathematicians' mathematical and pedagogical reasons gave evidence for the implicit dimension of mathematical knowledge in lecturing. Further, the implicit mathematical knowledge is found context-specific.

INTRODUCTION

Most of the past research (Ball, Thames & Phelps, 2008; Schoenfeld, 2011) focussed on teacher's explicit mathematical knowledge in teaching and is conceptualised looking at real-time teaching practices. This is inadequate to represent teachers' disciplinary knowledge of mathematics (Davis & Renert, 2014). Researchers (Kersting et al., 2016) maintain that teachers' usable knowledge does not represent most of their mathematical knowledge in teaching. Teachers hold 'largely unarticulated' (Elbaz, 1981) knowledge, which is implicit (Brown & McIntyre, 1993). This study focuses on the implicit dimension of research mathematicians' mathematical knowledge in lecturing. Research mathematicians' mathematical knowledge and their approach to teaching mathematics are acknowledged in relation to school teaching (Ralston, 2004). Petrou & Goulding (2011) suggest the need to understand the functioning of mathematical knowledge in lecturing regarding wider systems of mathematical and pedagogical considerations. In this paper, I extend the conceptualisation of mathematical knowledge in teaching to university lecturing looking at research mathematicians' lecturing practices. The research question addressed here is what are the mathematical and pedagogical reasons underlying research mathematicians' implicit mathematical knowledge in lecturing?

THEORETICAL BACKGROUND

Mathematical knowledge in lecturing

Shulman (1986) proposed seven categories of teacher knowledge of which three relate to the content categories of teacher knowledge, namely Subject Matter Knowledge (SMK), Pedagogical Content Knowledge (PCK) and Curriculum Knowledge. According to Shulman (1986, p. 9), Subject Matter Knowledge (SMK) refers "to the amount and organisation of knowledge per se in the mind of the teacher". Pedagogical

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Content Knowledge (PCK) is "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9).

Following Shulman and colleagues, Hill, Rowan and Ball (2005) developed a practice-based theory of content-based teaching at primary levels. Ball et al. (2008) attempted to refine it with a model of Mathematical Knowledge for Teaching (MKT) based on empirical conceptualisation. The model has two major categories namely Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) with further subdivisions within each. SMK was subdivided into Common Content Knowledge (CCK), Specialised Content Knowledge (SCK) and Horizon Content Knowledge (HCK). PCK was subdivided into Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC).

The MKT framework is based on school teaching. Adopting MKT construct to university mathematics lecturing needs to take into account the main variation between school and university setting, namely in relation to teacher qualifications and mathematics subject knowledge. Most university mathematicians will not have formal teaching qualifications or training in teaching (Wood et al., 2011). University research mathematicians are stronger in their mathematics content knowledge (Speer & Wagner, 2009) because of their advanced mathematical qualifications and their research knowledge. The implicit dimension of university research mathematicians' mathematical knowledge is studied, considering these variations between school and university settings.

Implicit mathematical knowledge in lecturing

Researchers (Adler & Davis, 2006; Baumert et al., 2010) have been increasingly interested in the implicit dimension of teacher knowledge. Alder and Davis (2006) suggested that some parts of teachers' mathematical knowledge are not easily accessible for scrutinising, and that significant parts of teachers' mathematical knowledge are inert (Baumert et al., 2010). From a knowledge system perspective, Kersting et al. (2016) used teachers' analyses of teaching video episodes. They concluded teachers' usable knowledge represents "*rather the subset of knowledge that teachers deem most relevant and essential*" (p. 106). In a similar sense, research mathematicians' enacted mathematical knowledge represents the relevant contextual knowledge. This suggests that some knowledge remains implicit or not articulated. I call the unarticulated, not revealed and not explicit mathematical knowledge the 'implicit' mathematical knowledge. I follow Mason (1998) and conceptualise implicit knowledge as conscious. That is, a research mathematician may be conscious and may deliberately choose not to reveal some mathematical knowledge, which may have underlying mathematical and pedagogical reasons. I propose that the notion of implicit mathematical knowledge is context-specific.

METHODOLOGY

This paper presents data gathered from eight research mathematicians at a university mathematics department in New Zealand, using a combination of semi-structured interviews and lecture observations. The eight research mathematicians (RM1, RM2, RM3, RM4, RM5, RM6, RM7, and RM8) are all research mathematicians with more than 15 years of experience. None of them had any formal teaching qualification. Research mathematicians were observed in class with interviews before and after each lecture observation. Classes observed were in undergraduate mathematics including calculus, linear algebra, number theory, mathematical modelling and statistical computing. I collected the data through video recordings of the lecture observations and audio recordings of the interviews. The interviews and lecture observations had the aim of understanding research mathematicians' implicit mathematical knowledge and revealing possible reasons for it remaining implicit.

Interviews were open-ended and semi-structured (Bryman, 2012). The researcher was a non-participant observer who did not involve in the events in the lectures. Aspects of implicit mathematical knowledge were identified in lecture observations and followed up in post-observation interviews. In this way, the inferences from lecture observations were validated with the research mathematician. Interview questions were based on the context of the observed lecture; however, the common question centred on the researcher asking the research mathematician about aspects of unrevealed mathematical knowledge in lecturing. Inductive thematic analysis (Braun & Clarke, 2006) was used to analyse the data from the transcripts using 'data sets' and 'data items'. This paper reports the '*data set*' corresponding to the analytical theme of implicit mathematical knowledge. The '*data items*' that formed the theme of implicit mathematical knowledge provided the underlying mathematical and pedagogical reasons for research mathematicians' implicit mathematical knowledge.

RESULTS AND DISCUSSION

In what follows, I give instances of implicit mathematical knowledge. During interviews and lecture observations, research mathematicians talked about some of the mathematics content they will not explicitly use in lecturing. Brief quotes from research mathematicians illustrate this. Post-observation interviews uncovered some of the possible reasons why it remained not revealed in mathematics lectures. They are classified into (a) pedagogical reasons and (b) lecturer reasons.

(a) Pedagogical reasons

Ball et al. (2008) in their theoretical model describe pedagogical content knowledge consisting of curricular knowledge (KCC) and knowledge of student thinking and learning (KCS). In a similar way, pedagogical reasons include (i) curricular reasons and (ii) student learning reasons.

(i) Curricular reasons

In the post-observation interview, RM8 was asked why he chose to hold back the knowledge of the rule of matrix multiplication, and did not explicitly make it clear to students, apart from a brief comment.

RM8 replied:

> The rule of matrix multiplication is not in the syllabus. I will just give a hint because of the limited time and is not essentially in the syllabus. [RM8, Stage 2 course, Post-observation interview]

RM8 withheld this knowledge in the lecture, and did not reveal due to the contextual factor of limited time to cover the syllabus. Another reason for not revealing some of his knowledge relates to the content being out of syllabus.

In a similar sense, RM3 did not emphasise polar coordinates in the lecture '*because it's not part of the syllabus'*.

> The idea of polar coordinates is quite simple ... that's all I want to say, don't put too much stress on this because it's not part of the syllabus. [RM3, Lecture 14, Stage 2 course]

(ii) Student learning reasons

These reasons mostly have the aim of assisting student learning. An example is RM5's deliberate pedagogical intention of creating self-learning opportunities for students. RM5 is conscious of why he does that.

> For this class, I want them to go and think about other stuff. So I am deliberately not giving a whole series of examples. [RM5, Pre-observation interview]

Another example is when RM2 suggests to students in the lecture that he will give a '*kind of an obscure hint*' for solving the exercise. This is taken as an indication of some mathematical knowledge hidden from students for them to work out.

> I won't prove the second part of the exercise, I will give you kind of an obscure hint, which is the second bit, and you can actually follow it from the first bit. So the idea is that if you have some vector v, which is orthogonal to all of these u_i's, then it is also orthogonal to any linear combination of u_i's. [RM2, Lecture 1, Stage 2 course]

A further example where RM2 chooses to give an opportunity for students to think is:

> If we have a linearly independent subset of R^n, $\{u_1, u_2, \ldots u_n\}$ in R^n is spanned by these vectors. Maybe I will leave that one for you to think about why that's true ... Any basis for R^n has n vectors, so that's how the argument goes, I leave you to think about that one and work it out yourselves. [RM2, Lecture 1, Stage 2 course]

These examples are evidence of research mathematicians' implicit mathematical knowledge.

Pedagogical reasons are evident in RM4's choosing a simple example in line with the goals of the lecture on Partial Differential Equations (*PDEs*).

> Researcher: What do those two equations mean in the physical world, is it possible to explain what the terms represent in the physical world?

> RM4: Yeah, it is possible, those first order *PDEs* do come up within various models, but I just wanted to have it as a very simple *PDE* that they could see the method being used rather than anything else. Then I wanted to go onto a physical model with a more complicated *PDE*. So it was really just a very quick example just to show the mathematics. But there are other examples. I mean that is an example of a wave; you can get wave propagation from such *PDE's*. But usually, wave propagation physically comes from higher order *PDE's*. But there are complicated waves and traffic flows that arise from first order *PDE's* like that, and students learn about those a bit in MATHS 363, a different course. But that model would have taken too long to discuss really (laughs) ... Anyway, the goal was really to give just a quick example showing, a quick and a simple example illustrating how Laplace transforms could be used. [RM4, Post-observation interview 1]

This is another example illustrating the student learning reason. It also shows the context-specific nature of implicit mathematical knowledge in lecturing. For example, if RM4 did not have the goal of giving students a simple and quick example in this context, he would have made his knowledge of complicated PDE's explicit in the lecture.

Another possible reason for withholding some mathematical knowledge is that it might not be suitable to the current level of student learning. For example, RM5's quote below:

> I will tell the class about the concept of infinity and things being bigger than other things. Infinite is an important concept, but I won't go into any detail. I want to try [to] get them to understand that's what I do [in my research] ... The whole idea of different sizes of infinity is an important one and so there are connections, but it can't be too explicit I suppose. It is indirect at this level. [RM5, Pre-observation interview]

RM1 shares the same line of thought:

> And it is not something that they would immediately connect with because it is undergraduate [lectures], you [students] don't know enough to even think properly about how that will work. So I often tell them that one as an example of how do you do that. I don't tell them the details, of course, that's sort of several PhD's, but I tell them the sort of things that are involved and why this is related to what they are doing. [RM1, Questionnaire]

Again, this is a clear conscious decision of RM1.

RM5 gives another student-related pedagogical reason to keep some mathematical knowledge implicit: so as not to distract students' attention away from the key idea he was lecturing on, and avoiding unimportant details for their learning.

> I decided not to prove that because I think that would have obscured the point I was trying to make ... I wanted to keep the class focused always on the thought of these different kinds of numbers and how big are they and how many are they focusing on the big picture. And I didn't want them getting distracted by exactly how I was doing the decimals or why that I put that there, why that I add on a 5 and 6 here. That's all the details, which are not important. [RM5, Post-observation interview 1]

In summary, curricular reasons for withholding knowledge included ‘content not in the syllabus and ‘limited time to cover the syllabus’. Pedagogical reasons included ‘creating a self-learning opportunity for students’ (this was sometimes done by giving an obscure hint to students), ‘to provide a simple and quick example’ thus avoiding complex examples, and ‘omitting mathematical content that is not suitable for student’s current learning levels’ (so not to distract student attention from the main point of the lecture).

(b) Lecturer reasons

Lecturer reasons for keeping mathematical knowledge implicit address flexibility and gaps in research mathematicians’ mathematical knowledge.

RM5 mentions the flexible use of mathematical knowledge in lecturing when he was making a point about advanced mathematical knowledge.

> Yes, I think that is the point. You can use it in a flexible manner if the time arises and if the time doesn’t arise, you won’t necessarily use it, but at least it is there if you need to which I think is good. [RM5, Pre-observation interview]

Mathematicians’ are aware of some gaps in their mathematical knowledge; for example, RM6 says that it is hard to find the purpose of certain lecturing topics and struggles when thinking about what to tell students.

> And over the years, I have struggled quite a bit with that fact that I don’t actually see the point in some of what we do. I struggled because you know; yesterday’s examples on the random walk, what can you use it for really... I have always found that a problem in my teaching [is] I can appreciate the results for what they are, but I can’t really see how much further or if indeed, they can bring us that much further. [RM6, Post-observation interview]

RM6 says that such an awareness of missing knowledge is itself a great thing because it provides opportunities for further development of knowledge. If someone is ignorant of gaps in their mathematical knowledge, they will miss out further opportunities for further developing knowledge.

> Sometimes I want to clarify details in my own mind about why we have to worry about the other properties of continuous functions or limits, or that sort of thing. It’s not something I know a lot about, I guess I only know enough to know when I don’t know something, does that make sense? I think, that would be the real difference if you did not know very much, you wouldn’t even know that you are missing things, whereas I know enough to know that I am missing things (laughs), it can sometimes be helpful. [RM6, Pre-observation interview]

RM6 is conscious of the way students see things different from their understanding, which sometimes might provide them with new insights into the topic. This is evidence of another aspect of gaps in RM6’s mathematical knowledge.

> I am perfectly sure that if I start teaching another new course, it will happen again that students will see things in a different way from anything that I have seen before and I will have to broaden my understanding. I am positive being in front of the class with the type of

> material first time, there would definitely be times when I suddenly get taken off, I just haven't thought about that before. [RM6, Post-observation interview]

There are also certain mathematical topics for which explicit lecturing is difficult. For example, RM7 made the following comment.

> Error messages is another one, again it is very hard to explicitly teach the stuff. [RM7, Pre-observation interview]

Comparing this to Ball et al.'s (2008) MKT construct, pedagogical reasons are part of PCK. In particular, curricular reasons align with Knowledge of Content and Curriculum (KCC), and student learning reasons connect to Knowledge of Content and Students (KCS). Further, lecturer reasons of 'flexibility in knowledge use' and 'gaps in mathematics content knowledge' relate to Subject Matter Knowledge (SMK) in mathematics.

CONCLUSION

This study gives evidence for the implicit dimension of research mathematicians' mathematical knowledge in lecturing. During lecture observations, research mathematicians were seen not to articulate some parts of their mathematical knowledge. Lecture observations and interviews revealed the underlying mathematical and pedagogical reasons for research mathematicians' implicit mathematical knowledge. Pedagogical reasons included both curricular reasons and student learning reasons. Curricular reasons included 'mathematics content not in the syllabus' and 'limited time to cover the syllabus'. Student learning reasons included 'creating self-learning opportunities for students', 'omitting content not suitable for student's current learning levels', and 'to give a quick and simple example'. Pedagogical reasons thus align with the PCK of the MKT construct. Lecturer reasons for implicit mathematical knowledge included flexibility in using mathematical knowledge according to the context of the lecture. Further, lecturer reasons included research mathematicians' gaps in mathematical knowledge. These become mathematical reasons underlying implicit mathematical knowledge in lecturing. Mathematical reasons relate to SMK of the MKT construct. Research mathematicians are conscious of their implicit mathematical knowledge and can deliberately choose not to reveal it. They are also conscious of gaps in their mathematical knowledge. Further, implicit mathematical knowledge is context-specific. More study is needed to understand about other underlying reasons for the implicit mathematical knowledge. It has implications in designing professional development programmes, which can address the concern of gaps in mathematical knowledge in lecturing.

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MATHEMATICAL LANGUAGE EXPOSED ONLINE

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Online discourse draws attention to students' use of a "transitional mathematical register" prompted by the dialogic nature of language development. A Bakhtinian lens is used to examine a case study of the use of informal and formal mathematics registers by primary school students engaged in mathematical problem solving while working in a Computer Supported Collaborative Learning (CSCL) environment

INTRODUCTION

Communicating mathematical ideas requires students, over time, to gain facility with the use of a formal mathematical "register" (FMR) of words. Within teaching and learning the role of language has been acknowledged as both social and semiotic (see for example (Maybin, 1994). This social construction of knowledge takes place in a variety of contexts. It may occur in the physical classroom, or as is the case in this study, in an online environment that has become known as a Computer Supported Collaborative Learning (CSCL) environment.

The purpose of this study is to examine evidence of students' mathematical language development as they work on solving mathematical problems in a CSCL environment. We consider whether Halliday's (1978) notion of informal and formal language registers is sufficient to guide teaching at the primary school level. The research is framed by Barwell's (2012) application of the Bakhtin's (1981) dialogic perspective on language development. Two excerpts from online discussions will be used as a case study to illuminate aspects of upper primary students' language development.

In the sections below we first provide some details of the 'mathematics register' then the theoretical framework that underpins this paper is outlined. Next details of the study are described followed by results, discussion, and implications for teaching.

LITERATURE REVIEW

Language use in Mathematics Education

Halliday (1978, p. 195) describes registers as "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings". He lists various ways in which words come to be accepted and agreed upon for usage within a given register. For young students the informal register (IMR) encompasses everyday words such as "going" or "pointy," while "multiply", "equation," and "median" lie in the FMR.

The acquisition of appropriate language is a part of learning mathematics. Doerr and Lerman (2010) describe communication as the driving force behind all learning. Their

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four year study provides insights into the role of speaking, writing and reading within mathematics teaching and learning. They draw on Herbel-Eisenmann's (Herbel-Eisenmann, 2002) concept of a *Transitional Mathematical Language* (TML) when referring to teacher or student developed idiosyncratic language specific to a particular classroom and note the need, in time, for the use of official mathematical words. They emphasise that, if mathematical language is to be appropriated it is important for students to have opportunities to discuss mathematics with peers, to connect words to each other and to contexts meaningful to them.

Language use in CSCL

(Turvey, 2006) notes that when students share their ideas via virtual exchanges this creates an opportunity for learning to be observed. Sharing mathematics asynchronously online removes communication through facial expressions and gesture but enhances the opportunities for visual, graphic and tabular representations. (Lemke, 2003) describes the difficulty some students experience trying to communicate their mathematical meaning when describing a pattern or relationship. Nason and Woodruff (2005) suggest that the use of multiple representations, some offering dynamic manipulation, "enable young children to communicate meaning via showing and telling rather than by merely telling (p.119)"(Nason & Woodruff, 2005, p. 119).

However researchers point to the, superficial online talk that can be a result of lack of structure and lack of instruction with regards to what productive talk might constitute (Hong & Jacob, 2012; Mercer & Wegerif, 1999; Pifarré & Staarman, 2011). These authors, equate productive talk with critical thinking, reflective thinking and creativity. They believe well prepared student online interaction, with access to multiple representations, makes CSCL ideally placed to positively impact current pedagogies (Pifarré & Staarman, 2011; Wegerif, 2007).

THEORETICAL FRAMEWORK

This study draws on Barwell's (2012) application of the Bakhtinian (1981) dialogic perspective as a means to expose the tensions that exist between informal and formal mathematical language. Barwell demonstrates that the FMR is privileged throughout international curricula by pointing to a tendency of these documents to accept informal mathematical language in the earlier years of schooling, whilst working towards the use of the FMR in later years. He argues that IMR and FMR are always required and always in tension.

Barwell (2012) suggests that privileging the FMR is not ideal, because it places great importance on the 'correct' use of mathematical language at the potential expense of meaning making. Bakhtin's (1981) view of language was that it is situated, dynamic and dialogic. He saw languages as being either unified (unitary) or to use his term, in a state of 'heteroglossia'. The theoretically complete FMR can be seen as a unified language. Tensions exist within curricula, schools and society advocating for this unified mathematics language. This centripetal force may inhibit students' trialling of new words.

In describing mathematical language Barwell identifies four inter-related tenets in Bakhtin's work. We tag these B1, B2, B3 and B4.

B1. Language is dialogic: This is observed when students engage in group conversations. Bakhtin suggests that utterances made in the present are also in dialogue with those made in the past.

B2. Language precedes us: Bakhtin claims that utterances of the past inform present and future utterances.

B3. Tensions exist between the unitary language and heteroglossia: For any given moment, a range of alternative modes of communication are possible. These various registers or languages compete with each other. The selection of these languages causes tension as for example, between the IMR and FMR.

B4. Language is not unidirectional: Bakhtin emphasises the variety of routes that discourse and language take. This dialogic perspective conflicts with views promoting the movement from informal to formal mathematical language use as unidirectional.

METHOD AND CONTEXT OF RESEARCH

The present study forms part of a larger project in an Australian primary school. It focuses on excerpts from two separate problem discussions in a mixed ability group of 1 girl and 3 boys (Year 5 students, aged 10-12 years, pseudonyms used). Over ten weeks the group investigated nine mathematical problems incorporating aspects of each content strand of the Australian Curriculum. The students were being introduced to the language of the FMR so we are particularly interested in the words that they chose to use in their online discussions. We anticipated that the asynchronous CSCL environment may encourage students to trial this new mathematical vocabulary. No online facilitator took part in the CSCL. This decision was taken in order to avoid discussion between students being prompted by an 'expert'.

The students did receive support. For each of the first 7 weeks an hour of standard classroom time was led by the first author of this paper: discussing appropriate approaches to online collaboration; reviewing the previous week's solutions and discussing challenges that students perceived; and finally examining the following week's problem. In the final two weeks students were expected to work independently. The excerpts analysed in this paper occurred in weeks three and six.

FINDINGS

Example One: Wallpaper Symmetry

In 'Wallpaper Symmetry' students were required to demonstrate their understanding of symmetry by creating a piece of 'digital wallpaper'. Students were asked to represent line/ mirror symmetry, rotational symmetry and translational symmetry and then describe, in their online discussion, how symmetries were used in their wallpaper.

In Figure 2 we see examples of students wanting to express their mathematical thinking, but struggling to do so. This is may be a result of an inner tension (B3) due to a desire to express mathematical thinking in the FMR but their understanding that they lack the words to do so. An example of this occurs in Olivia's opening utterance when she says she does not have the specific formal mathematical words to describe the geometric feature to which she refers. Olivia's use of language is disjointed, far removed from the FMR. However, when read carefully, we realize that Olivia is developing an understanding that when considering rotational symmetry, a shape will look identical when rotated on its axis the number of degrees corresponding to its order of rotation. She does not have the words to describe this accurately, however is able to make her meaning known. Olivia's mix of formal and informal language to represent mathematical thinking evolves as a shared language through dialogue with other students in her group. Zander, for example, takes up her words when he says "… rotational symmetry means, depending on how many pointy sides …" This development of shared language and meaning making exhibits aspects of B1, B2 and B4.

Note also Zander's statement, *after you* ***move it 4 times****, it* ***goes back to the same spot****.* If read literally this statement would make little sense but, given the context, we can see that Zander is making the accurate assertion that when rotated 4 x 90 degrees a cross will return to its original position. Again, mixed formal and informal language allows him and his group to make sense of his mathematical thinking.

Olivia: i think to **work this out** we would need to **choose a shape** with the **pointy sides** (dont really know how to say it) so it would be easier with for us to do it does anyone agree with me?

Olivia: when i mean the **point sides** something like the example Mr. Symons showed us a **shape** simillar to that.

Chris: What **shape** is everyone deciding on. i was thinking of a **hecsigon**

Olivia: I'm think of an **shape** that has **a pointy side**. ?? it also would be much easier if we do a **shape** that **has a pointy side** in my opinion . Does anyone agree with me??

Olivia: I'm also thinking about doing little key box and say what you did and also saying what we used and explain how we done it??and why we **came up with the shape** we are going to use ?? Anyone agree with me ?

Olivia: i **changed it** i have **done a triangle** i **created something like a fan** so when it **spins** you could see **the pattern** and also **it would never changes** i have uploaded mine to edmodo.

Zander: **Mirror/Line Symmetry- line symmetry** means when you have a **shape** or anything, and you **cut it in half, it looks exactly the**

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same size and lining on every single thing as the other side.

Zander: **Rotation Symmetry- rotational symetry** means, depending on **how many pointy sides they have**, say for example, i had a **plus sign +,** it has **4 pointy sides**. So then, after you **move it 4 times**, it **goes back to the same spot**.

Zander: **Reflection Symmetry- reflection symmetry** means if you have a **picture of your face, you keep drawing that, making look the same height, the same lengh**, and etc.

[Olivia and Igor list names of attached files]

Zander: Good job. But there was a slight mistake.

Zander: This is Week 3 homework. It isn't really my work. I just edited Igor's so it's better and it has more **symmetry**. [name of attached file]

Figure 2: Uncorrected Discussion of Wallpaper Symmetry Problem

Example Two: Shapes

In 'Shapes' students investigated alternative shapes that could be made with four straight sides. They then provided discussion about the identity of the shapes they represented based on a definition that they developed and provided.

Zander: This is **the table** that he showed us how to do it. If you have any comments please, I actually advise you to reply or comment on this. I have not finished it but if you have anything you would like me to change please reply. Haven't finished but please get some more **shapes** so I can finish it off. This is not my computer so please say anything if there is something wrong. [names of attached files]

Olivia: I'm think of a **square shape** that have **straight sides** and a **rectangle** anyone else have a idea ?!

Olivia: And they also have a **straight length** for **the sides**

Olivia: And the **square is 90 degree**

Olivia: And also the spyware [an image downloaded from the internet] is a **right angle** anyone agree with me?

Zander: Sorry something going on with me and my dad important. So i have read your replies, I think we should do what you said. I agree with you.

Olivia: With the **table** what **shape** are you planning to do on I know that a **square** and **rectangle** could be one what else do you agree???

Olivia: With the **table** that you have done it was good didn't it also have to **only be with straight sides** ?? **Like you added a circle** but it was still really good

Zander: Don't know, but I'll **add another bar** so it will **make for that one** if we were not supposed to do it.

Olivia: Okay that is a good idea

Figure 3: Uncorrected Discussion of Shapes Problem

Figure 3 depicts further discussion between Olivia and Zander. Using conversation in a subsequent task between these two students shows some indication of how language use changed over time within the online space.

Zander initiates the discussion by stating *This is **the table** that he showed us how to do it*. This is evidence of B2. The utterances and discussion that occurred within the classroom prior to students working in the online environment are being drawn upon to move discussion and thinking forward.

Olivia responds by stating, *I'm think of a **square shape** that have **straight sides** and a **rectangle** anyone else have a idea?!* Her grammar is poor but her use of mathematical language allows us to infer that she understands that both squares and rectangles are examples of four sided shapes. The difficulty Olivia has communicating her thinking is evidence of tension (B3). A desire to communicate her ideas is in tension with her understanding that her use of language is disjointed and may cause difficulties for her peers' understanding of her contributions.

She continues; *And they also have a **straight length** for **the sides…** And the **square is 90 degree.*** Her language use here, appears to fit between the IMR and FMR. We infer from the context that when Olivia refers to ***straight length** for **the sides*** she is attempting to convey her understanding that squares have sides of *equal* length. She uses the word *straight* rather than *equal.* We infer from her statement; *the **square is 90 degree*** that she is aware that the four interior angles of a square are each 90 degrees. She does not yet have the FMR that allows her to express her understanding without interpretation, however by expressing herself in this way she does show her developing mathematical understanding. She uses language in a way that sits in between an IMR and FMR: evidence of the Transitional MR. The language here is representative of B4.

The discussion between Zander and Olivia continues:

Olivia: With the **table** what **shape** are you planning to do on I know that a **square** and **rectangle** could be one what else do you agree???

Olivia: With the **table** that you have done it was good didn't it also have to **only be with straight sides** ?? **Like you added a circle** but it was still really good

Zander: Don't know, but I'll **add another bar** so it will **make for that one** if we were not supposed to do it.

Olivia: Okay that is a good idea

Here we see confirmation from Olivia that she understands that a rectangle and a square both fit the task of identifying shapes that have four straight sides.

She provides feedback to Zander when she states; *With the **table** that you have done it was good didn't it also have to **only be with straight sides** ?? **Like you added a circle** but it was still really good.* Olivia, correctly advises Zander that the inclusion of a circle in his table does not satisfy the requirements of the task. She exemplifies her understanding by highlighting a criterion that a circle fails to meet; the shape must only have straight sides. A level of tension is evident here. Tension is a product of Olivia's understanding that Zander's work contains a 'mistake' and highlighting this within the group may cause some degree of embarrassment for Zander. Olivia attempts to mediate this tension by phrasing her feedback as a question and also by adding that it *was still really good.* Olivia's collaboration is evidence of the dialogic nature of language (B1). Tension is largely averted as Zander acknowledges that it may not have fitted the guidelines for the task and that he will update the work to reflect Olivia's observations.

DISCUSSION AND IMPLICATIONS

The constraints of the CSCL discussion created a need for students to use mathematical words to describe their solution processes. The dialogue was not simply with each other (B1) in the online space but also with participants from the earlier classroom discussions (B2). The students showed benefit from being in dialogue with the language and understanding of their teacher, who in turn had acquired this knowledge through dialogue with their past peers and teachers (B2). Students' written discussion, required in the CSCL environment exposed their struggles to use new terms along with common language (B3) allowing us to observe the to-and-fro between the IMR and FMR (B4).

Year 5 is a 'tipping' or 'bridging' point in students' language development. In mathematics, no longer are they only required to use the language of basic place value, the four operations, position and shapes. They must begin, at this juncture, to develop an understanding of more sophisticated notions; for example, those that require geometric, algebraic, proportional and relational thinking. Articulating these concepts requires the acquisition of aspects of the unified FMR. However, students can only make sense of their newly emerging understanding of this language through appropriating familiar informal language in combination with the newly discovered formal vocabulary. Like Doerr and Lerman (2010) we see that, together, the informal and formal use of language allows students to reason and communicate their emerging understandings. As students struggle to appropriate the unified FMR, they utilise elements of the FMR and along with words from their everyday register with some of these being assigned special meaning. We see this amalgamation as the 'transitional mathematical register' with features distinct from the IMR and FMR.

The importance of the dialogic nature of language is highlighted in this study as we see evidence of students who have been exposed to, and contribute to, the co-definition of various established and 'new' mathematical terms try to use this vocabulary within the CSCL environment. The language is used with various degrees of precision. This data supports Barwell's (p. 279), contention that curricula-based and other societal tensions placed on students to use the FMR may be counterproductive in this context. We see the use of the transitional mathematical register as both being important to students' meaning making necessary in the problem solving process, and also as forming a bi-directional bridge between the IMR and the FMR. Requiring students to work together in a CSCL environment supported the developmental TMR bridge by challenging them to communicate their evolving mathematical understanding.

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CAN A REGION HAVE NO AREA BUT INFINITE PERIMETER?

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In this report we present an analysis of 10 graduate mathematics students who were individually interviewed about the area and perimeter of the Sierpinski triangle (ST). The ST is a paradigmatic example of a paradoxical situation involving infinite limit processes. We use conceptual blending as a theoretical and methodological tool for analysing students' reasoning. Our analysis documents the diverse ways in which the students reasoned about the situation. Results suggest that conceptualizing an infinite perimeter is more accessible to these students than is zero area and that when running the conceptual blend they resolve the paradox by considering the ST as 'only a concept' or by using metaphors for the perimeter. The analysis contributes to what we know about how students think about infinite limit processes and furthers the theoretical/methodological framing of conceptual blending as a useful tool for revealing the structure and process of student reasoning.

INTRODUCTION

The notions of area and perimeter of a geometric shape are learned extensively in elementary school, and are re-visited throughout middle and high school years. Yet, when encountering fractals some counter-intuitive situations regarding these two well-known notions might occur (Sacristan, 2001). An example of one such puzzlement of a region with zero area and an infinitely long perimeter was encountered by a class of master degree students in a chaos and fractals course when investigating the Sierpinski Triangle (ST). Several weeks after a lengthy discussion of the area and perimeter of the ST in small groups and whole class forums, ten students were individually interviewed. Based on the interview data, we address the following two related research questions: (1) How do students perceive area and perimeter of the Sierpinski triangle? (2) How do they "defend" their reasoning when confronted with a paradoxical situation? We use conceptual blending theory (Fauconnier & Turner, 2002) as a theoretical and methodological tool for analysing students' coordination of two infinite processes, one increasing and one decreasing, as seen in the construction of many fractals.

THEORETICAL BACKGROUND

The theory of conceptual blending (Fauconnier & Turner, 2002) is based on the notion of mental spaces, which are "small conceptual packets constructed as we think and talk, for the purposes of local understanding and action" (p. 40). According to the

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 241-248. Singapore: PME.

theory, these mental spaces "organize the processes that take place behind the scenes as we think and talk" (p. 251). Conceptual blending is defined as the conceptual integration of two or more mental spaces to produce a new blended mental space. An important feature of this new blended mental space is that it develops an "emergent structure" that is not explicit in either of the input mental spaces (p. 42). This theory has been applied to the learning of mathematics by a number of researchers. For example, Lakoff and Núñez (2000) propose that most of the important ideas in mathematics are metaphorical conceptual blends. Alexander (2009) notes that conceptual blends in mathematics offer a distinct cognitive advantage in the mental compression that occurs when the blended objects are given their own identity and the inputs are "relegated behind the scenes" (Fauconnier & Turner, 2002, p. 13).

To represent an analysis based on conceptual blending theory, a diagram with conceptual blending conventions may be used: circles represent mental spaces, the upper circles represent the input mental spaces, the lower circle represents the blended mental space and the lines show mappings between the spaces (see Figures 1 and 2). Inside the circles the researcher represents his or her interpretation of a person's mental representation of those items. We use the same conventions in this paper.

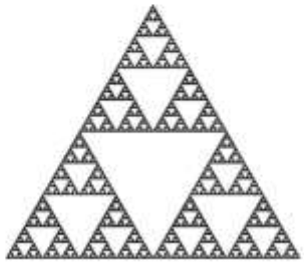
Given the nature of the ST, we expect that the input and blended spaces will involve infinite limit processes. Prior research (e.g., Quine, 1966; Mamolo & Zazkis, 2008) has investigated the historical and epistemological challenges that infinite limit processes pose. For example, Fischbein, Tirosh, and Hess (1979) claim that the concept of infinity (and specifically of infinite divisibility) is intuitively contradictory because finite interpretations tend to prevail and interfere with infinite processes. Moreover, Fischbein et al. reported that formal mathematics teaching does not modify students' conceptions and intuitions of infinity. Paradoxical statements regarding the infinite stem from the seemingly impossible attributes of mathematical infinity, and tend to expose preconceptions that were once believed to be fundamental.

METHODOLOGY

The study took place in a graduate level mathematics course with 11 students (10 of whom agreed to participate in individual interviews). Students were either part-time or full-time secondary school teachers or community college instructors. Their master's degree program required a substantial mathematics component, and the chaos and fractals course studied here fulfilled part of that requirement. The course was taught by one of the research team members. Data collected as part of a larger study included video-recordings of each class session, individual problem solving interviews conducted at the middle and end of the semester, and copies of all student work. The methodological approach for the larger study involves both Abstraction in Context and Documenting Collective Activity (Tabach et al., 2014), but in this report we focus only on an analysis of the 10 mid-semester individual problem-solving interviews.

The following question from the mid-semester interview is the focus of this analysis:

In class, we discussed the Sierpinski Triangle (ST). How do you think about what happens to the perimeter and the area of the ST as the number of iterations tends to infinity? We followed up by asking students about their confidence regarding what happens to the area and perimeter (separately) and why, and we asked them to tell us what they thought about the following claim of a fictitious student, "Fred:"



The computation shows that the perimeter goes to infinity because the perimeter is given by $3\times(3/2)^n$ *which increases to infinity as n tends to infinity. But, the perimeter can't really be infinitely long, because there is nothing left to draw a perimeter around, since the area goes to zero.*

The question was structured so that we would first gain insight into the students' reasoning about the area and perimeter of the ST, followed by an opportunity for them to engage in the hypothetical reasoning of another student. The basis for Fred's reasoning was actually expressed by one of the students during a whole class discussion that took place several weeks before the interview, and so was not entirely unfamiliar.

The transcripts and student work produced during the interviews were coded and mapped as follows. *Input mental spaces* contain a student's understanding of area process and perimeter process to be coordinated; A *Generic mental space* maps onto each of the inputs and contains infinite iterative process characterizations; *Blend space*, or "the blend," is where the input spaces are "put together" to create a coordinated scenario. It does not contain every element of the input spaces, and contains some things that neither input space has; and finally, *Running the blend*, develops emergent structures that are not in the inputs. The blend can be "run" or "elaborated," modifying it imaginatively and arriving at new conclusions which remain tied to the original input spaces. As expected, ideas related to infinite limit processes were prominent in students' reasoning.

FINDINGS

We start by an overview of responses from all ten students, followed by a detailed analysis of four cases. All students agreed that the perimeter tends to infinity, but only six of them claimed that the area tends to zero. In fact, three students claimed that the area will not tend to zero (and a fourth one did not commit). The students were quite confident about their responses, always giving 7 or above. Also, with the exception of Carly, students were equally or more confident in their answer about the perimeter than about the area. For these students, we found that it is fairly intuitive to grasp the infinity of the perimeter whereas the area being zero was a counter-intuitive idea for some. All students were willing to engage in Fred's reasoning. For some this involved hypothetical thinking and elicited thoughts about the "essence" of the ST – is it "real" or an "idea"? The four students selected for in-depth presentation were chosen as clear examples of the range of approaches taken by all 10 students.

Curtis

We begin by discussing Curtis, whose response and conceptual blend (see Figure 1) are the most straightforward to describe. He takes a quantitative approach to reasoning about the infinite processes describing the area and perimeter of the Sierpinski triangle. While some students took decidedly different approaches to each, Curtis in both cases identified a multiplicative sequence and took its limit, reaching the conclusion that the area of the Sierpinski triangle is zero while the perimeter is infinite. His explanation of his thinking about the perimeter is particularly clear:

> After one [step], you've got another, half, so. If this [perimeter of outer triangle] is one whole, then this [perimeter of inner triangle] is half of that. And then, so, it's increasing by three-halves. And then, since each of these [three remaining triangles] is the same thing, the whole thing increases by three-halves at each stage. So, you can just say that's three-halves to the *n* gives you the perimeter at *n*, since you're looking at the limit as *n* goes to infinity that's equal positive infinity because that number is greater than 1. So, that's perimeter.

The elements of Curtis's mental space for the ST perimeter include the recursive nature of the Sierpinski triangle, the stepwise nature of iterative processes, a multiplicative conception of the rate of change (increase) of the perimeter as *n* goes to infinity, and how to deal with the limiting behavior of an infinite sequence. The recursive and stepwise nature of the infinite process by which the Sierpinski triangle is created are used by Curtis to justify the symbolic expression he writes down to quantify the perimeter at each step *n*, and then uses prior knowledge to explain that when the factor is greater than one, that sequence diverges and the limit equals infinity.

Curtis's explanation is less articulate when he describes the area of the Sierpinski triangle, but his mental space for ST area is constructed in a parallel way. Again, he touches on the recursive nature of the ST, the stepwise nature of iterative processes, a multiplicative conception of the rate of change (decrease) of the area as *n* goes to infinity, and how to deal with the limiting behavior of an infinite sequence. Again, he uses the recursive and stepwise nature of the infinite processes to quantify the area at each step *n*, and pulls from the same prior result to determine that the sequence converges to zero, this time because the factor is less than one.

The blended mental space that Curtis develops for the ST as a whole contains many elements from each input space. This includes the idea that the perimeter increases to infinity at the same time that the area is decreasing to zero, and the fact that there is a final state at which these results co-exist. In running this blend, Curtis encounters the pseudo-paradoxical situation of an object with zero area and yet an infinite perimeter, which at this time he resolved by declaring that the ST is not bound by conventions for triangles, because one is "not physically drawing something like a perimeter, it's kind of just a concept."

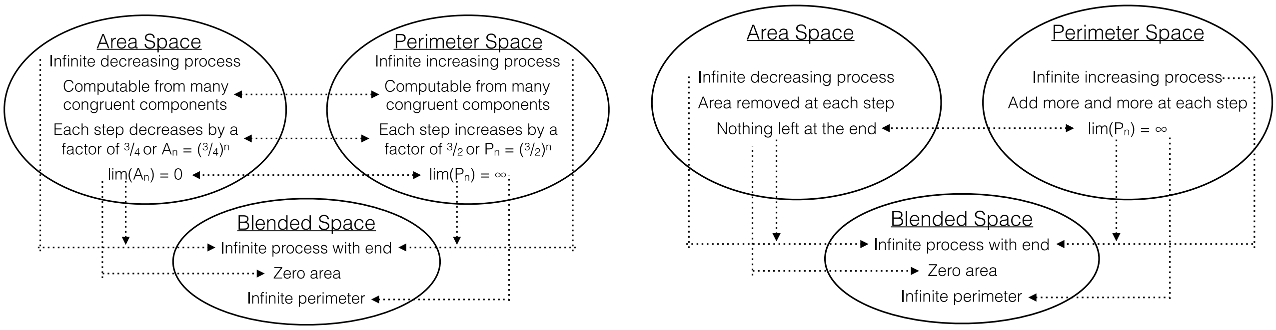


Figure 4. Conceptual blend diagrams for Curtis (left) and Elise (right).

Elise

Elise argues that the area of the ST goes to zero and that the perimeter goes to infinity. Her reasoning for both of these conclusions is strongly grounded in thinking of both area and perimeter as the result of a step-wise process of creation with an end at infinity. Unlike Curtis, however, who expressed a multiplicative relationship between stages, Elise thought of the relationship between stages in a more figural, additive manner. For example, she begins her argument for why the area is zero by stepping through the first three iterations in which chunks of area are taken out. She then imagines carrying out the process indefinitely, stating that "…if I keep doing this to infinity like, I'm looking at this little tiny black piece in here and like I'm going to get in that and then take out everything, so if I'm going to infinity I'm going to eventually take out everything so I don't think there is anything left." The elements of Elise's mental space for the ST area include an infinite decreasing process of area removal and a verbal limit of the process reaching zero.

Similar to area, Elise's reasoning about perimeter is grounded in what happens to the figure and even more strongly additive in nature. For example, she explains that in "my first iteration I've got my three side lengths [of] whatever, we'll call them *a*, so I have 3*a* as my perimeter. And then in the next iteration I'm adding I think three-halves of *a*. Because each of these is a half of *a* and I'm adding three of them. And then the next iteration I'm adding now a fourth of *a*. But I'm adding nine of them." As we see in this excerpt, one element of Elise's mental space for perimeter is an infinite increasing process of adding more and more sides. As she continues her explanation we see two other elements of her mental space for perimeter, an increase at each stage of more than that at the previous stage, and a verbal limit of the process going to infinity. Elise proceeds to calculate, assuming the initial length of each side is *a*, that after the first step one is adding three-halves *a* (or 1.5*a*), then at the next step one is adding $^{9}/_{4}\,a$ (or 2.25*a*), and so on. She concludes that "every time after the first iteration I'm adding more perimeter than I added before. So if I keep adding more then I think it's going to keep going to infinity because I'm just going to keep adding bigger and bigger."

The blended space (see Figure 1) that Elise develops for the ST as a whole includes an infinite process that finishes, decreasing and increasing amounts, and an end result of zero area and infinite perimeter. In running the blend Elise resolves the paradoxical situation tendered by Fred by describing the ST as just a “skeleton.” As a metaphor, thinking of the ST as a skeleton is quite interesting if one thinks of area as flesh and bones as perimeter.

Cathy

Cathy (see Figure 2) was the only one of our four examples who did not bring “zero area” into her blended space, and so did not encounter the paradox with the same force. The elements of Cathy's mental space for area include the creation of the ST as an infinite process in which at each step some area is removed, and hence the remaining area decreases. The amount of area removed at each step becomes smaller and smaller, so much so that the removed bits become miniscule, and hence the area converges to some finite non-zero value. The elements of Cathy's mental space for perimeter include an infinite process, in which at each step additional line segments are added to the perimeter. Hence the perimeter increases, and therefore, according to Cathy, tends to infinity.

The elements of Cathy's blended space are two infinite processes (potential infinity), one increasing to an infinite perimeter and the other decreasing to a non-zero area. Hence, when running the blended space, she does not run into the same paradox most other students do and explains that the perimeter can run infinitely tightly around the remaining area. However, she is capable of hypothetical thinking by assuming that the area tends to zero; even then, though, she considers Fred's reasoning as incorrect, explaining that the perimeter could "be just kind of wrapping around itself." While this is in response to the paradoxical situation, it is not part of her mental space surrounding the ST and so is not the result of running her blend.

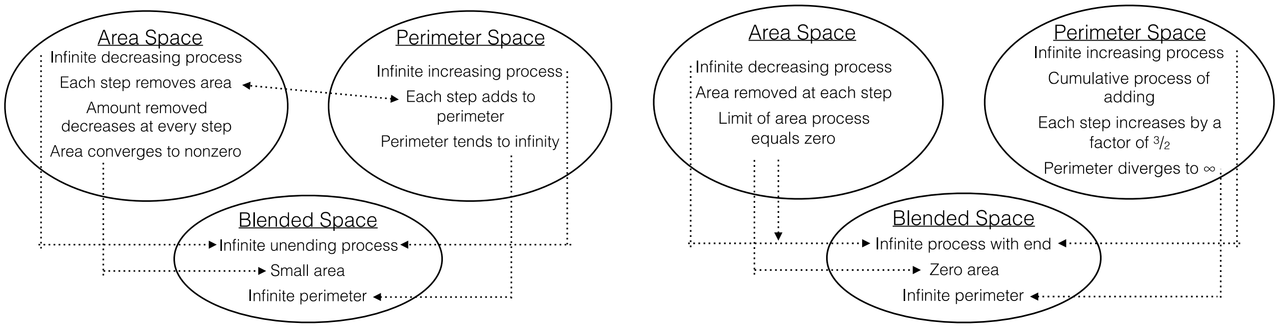


Figure 5. Conceptual blend diagrams for Cathy (left) and Carly (right).

Carly

The elements of Carly's mental space for area include the ST as the result of an infinite process of repeated removal of area, as a result of which the area tends to zero. Carly distinguishes the process from the final state, the ST, which has area zero (actual infinity): “so ok eventually the area gets to zero, but that's if you could do it infinitely many times.” In order to make the point that her conclusions are not outrageous, Carly makes an analogy to the rectangles in a Riemann sums and the notion of lower bound: “I was thinking about how there's analogies to calculus … Like, oh, what were those called? Is it the.... lower, lower upper bounds? Superior upper? Sups and something. That kinda reminds me of that, like you get really really really close, like the biggest thing you can get to it without being there at the smallest thing. So kind of that type of thing with the area."

The elements of Carly's mental space for perimeter include a cumulative process of the perimeter growing to infinity by repeated addition of 3, then 9, then 27 etc. additional segments. Carly asks herself whether the totals of the added bits are increasing or decreasing, but argues convincingly that even if they are decreasing, their sum might still tend to infinity as in $\sum \frac{1}{n}$. She tends to think they are decreasing but is uncertain and later rejects decrease on the basis of Fred's growth by a factor of $^{3}/_{2}$.

The elements of Carly's blended space (see Figure 2) are two infinite processes that eventually complete (actual infinity), one increasing to an infinite perimeter and the other decreasing to zero area. When running the blended space, Carly expresses her empathy with Fred: "I totally get Fred, because I felt the same way. And I was like, you can't have a fence around nothing. It just doesn't work." Carly relates the paradox in her own terms: "those triangles are still drawn. They're still drawn there, like blocking off the not-there spaces". She then goes about solving the paradox by starting from the calculus analogy: "there's nothing really left to draw a perimeter around" [pointing to Fred's argument], no there's not, but we're saying that there is, kind of in that, limits argument". This leads her to make use of self-similarity: "I think once we started thinking about self-similarity, that helped a bit … we could like zoom in and keep seeing ok you can keep going, … eventually we're saying the area gets to zero, but where's this number going to stop if we can keep zooming in, there would be a spot where like ok we've zoomed in enough, so that's kinda why it can't be a number." In our interpretation: Because of self-similarity, zooming can go on indefinitely, and hence, perimeter segments of any size, however small, will be added at the same time as more and more bits of area are removed.

CONCLUDING REMARKS

The four students we have discussed here represent a range of approaches to handling the creation of the ST through an infinite recursive process which changes its area and perimeter. Returning to our research questions, we note that all four students understood the infinite process of creation as one where the perimeter iteratively increases to infinity and the area iteratively decreases, and all four students included these elements in their blended space. Curtis was the only one of the four who perceived the area and perimeter of the Sierpinski triangle changing multiplicatively, while the others took an additive approach to the stepwise changes. Cathy, who believed that some amount of area must remain, did not encounter the same paradoxical situation as the others. When prompted, she provided a rationale for the possibility of infinite perimeter with no area, but this did not result from running her own blend. Curtis, Carly, and Elise all encountered the pseudo-paradox when developing their blended space due to their understanding that the area becomes zero, leading to an end state "at" infinity where an object exists with infinite perimeter and no area, but defended this result in different ways. Curtis, the only student who approached area and perimeter multiplicatively, ran his blend and determined that the paradox could be resolved by declaring the ST to be a "concept" and not a real object. Carly and Elise encountered the paradox, but resolved it differently by stating that the "skeleton" or "fence" built by adding pieces of perimeter must remain, as at no step was it removed. Their additive approach may have contributed to this outcome. It is interesting to note that Carly's conceptions of ST area and perimeter appear based on mathematical intuition rather than Elise's more quantitative approach, and yet reached comparable conclusions when running their respective blends. The analysis contributes to what we know about how students think about infinite limit processes and furthers the theoretical/methodological framing of conceptual blending as a useful tool for revealing the structure and process of student reasoning. In particular, our work is significant in that we work in the context of fractals, an underutilized avenue for probing students' underlying beliefs about mathematical objects.

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TOWARDS A CONCEPTUAL FRAMEWORK FOR ASSESSMENT LITERACY FOR MATHEMATICS TEACHERS

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As part of a mathematics teacher's skillset, knowing what kinds of assessment, when and how to assess, and for what purposes, are important. Although there have been frameworks outlining the various facets of teacher assessment literacy, recent literature suggest that gaps exists on the theoretical underpinnings as well as in psychometric claims of the measures. A framework of assessment literacy is proposed in this paper, informed by the mathematics education literature: (1) knowledge about assessment concepts; (2) skills in applying the knowledge in actual assessment practices; (3) communication and action in providing feedback or changing instructional practices based on assessment information; (4) attitudes and beliefs about assessment and its role in mathematics teaching and learning; and (5) meta-cognition and self-regulation of teachers' own assessment literacy.

INTRODUCTION

As a platform to provide feedback for follow-up instructional activities, it has long been recognised that assessment plays a pivotal role in the teaching-learning process. Although assessment literacy of teachers is an established field within the general education academic community (see Kahl, Hofman, & Bryant, 2013), it is still developing in mathematics education. Indeed, Santos and Cai (2016) provided a timely reminder on this in their chapter on Curriculum and Assessment in the Second Handbook of Research on the Psychology of Mathematics Education because there was no chapter on assessment in the first Handbook. Additionally, Lin and Rowland (2016) who reviewed several models of teacher knowledge in the studies presented in PME conferences and forums discovered that knowledge about what, when, and how to assess students was not raised as aspects of pedagogical content knowledge or mathematics knowledge for teaching. Whilst it is important for teachers to be literate on generic assessment issues such as validity and reliability, it is also crucial to help mathematics teachers develop a more in-depth understanding of assessment within the sensitivities of the subject matter knowledge. This paper presents a preliminary conceptual framework for mathematics teachers' assessment literacy to help anchor future discussions on mathematics teachers' assessment literacy. Implications of the framework for the wider international mathematics education community will be discussed.

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LITERATURE REVIEW

The word *assessment literacy* has been attributed to Stiggins (1991), who averred that assessment literates are able to “know what constitutes high-quality assessment” and to seek and use appropriate assessment methods “that communicate clear, specific, and rich definitions of the achievement that is valued” (p. 535). Knowledge of the importance of the sampling of performance information, the extraneous factors that can interfere with assessment, and the usability of the assessment data are critical aspects of assessment literacy (Stiggins, 1991).

Assessment Literacy: Definition of Terms and Key Aspects

Assessment literacy is multidimensional; its meaning changes with context (Stiggins, 1991). Several issues and debates in concepts about assessment and measures of assessment literacy were raised in previous research. Taras (2010) argued that the dichotomy made between formative assessment (view of assessment as process and part of pedagogy) and summative assessment (view of assessment as product, with negative connotations) is problematic. An assessment strategy (a single process) can be used for formative or summative purposes (function), and formative assessment necessitates an initial summative evaluation followed by pedagogical action (for example feedback). In addition, Gotch and French (2014) conducted a systematic review of 36 assessment literacy measures from 1991 to 2012, and found that the psychometric support for the measures was weak.

In the mathematics education community, related components of assessment literacy such as the use of formative assessment, providing quality feedback, the choice of tasks, and the statistical literacy of interpreting summative assessment data are on-going topics of concern and research in mathematics teacher education.

Formative assessment, one in which the assessment information is used to improve teaching and learning, usually includes a myriad of assessment strategies (e.g., journal writing, diagnostic tasks). Santos and Cai (2016) described three aspects when considering assessment strategies: the conditions for creating a supportive environment for using an assessment that contributes to teaching and learning, characteristics of assessment strategies themselves, and the effectiveness of such strategies for mathematics learning. We summarise some main ideas of these aspects to highlight that they are important considerations for inclusion in our proposed framework for teachers’ assessment literacy.

In creating the appropriate conditions of assessment for learning, a classroom culture that treats errors as “natural and inherent to the learning process (only one who is learning errs) and a fundamental source to access the different types of students’ reasoning” (Santos & Cai, 2016, p. 158) should be encouraged rather than abhorred. Moreover, as part of the classroom assessment culture, the assessment criteria need to be appropriated by students. This demands the teacher not just to declare the criteria, but also engage in a continuous process of communication and negotiation to create student ownership of the criteria. Past literature revealed that teachers found it challenging to clearly define the assessment criteria themselves and create student ownership of assessment criteria. Teachers' beliefs about nature of mathematics and mathematics learning are possible barriers to this (Santos & Cai, 2016).

In considering the main characteristics of assessment strategies, Santos and Cai (2016) cautioned that regardless of the quality of feedback, its effectiveness is not guaranteed and a challenge is for teachers to assure that feedback is viewed and used as part of a dialogical process rather than a one way communication. Although formative and self-assessment can be used to promote student self-regulation, metacognition and problem solving skills, teachers would also need to be mindful of students' cognitive, affective, and emotional readiness in order to create a positive environment for assessment to be effective. Furthermore, teachers' appropriate use of technology in the assessment process is another avenue for further research as part of their assessment literacy.

In the era of accountability driven education, research-based education practices are important and drive policy and practice. Within this context, student outcomes in large-scale assessment globally (e.g., Trends in International Mathematics and Science Studies) and nationally (e.g., Singapore's Primary School Leaving Examination [PSLE], Australia's National Assessment Program: Literacy and Numeracy [NAPLAN]) are often scrutinised and interpreted and results discussed in the public domain. Thus, to a certain extent, school leaders and teachers need some degree of statistical literacy to understand and interpret the results and findings from assessment practices. This means an understanding about sampling and assessment, descriptive statistics of assessment data (mean scores and standard deviation), and inferential statistics. Further use of statistics as part of teachers' assessment literacy may involve comparing performance between different groups of students to investigate equity issues in teaching and learning.

Studies on Mathematics Teachers' Assessment Literacy

Assessment literacy can be considered as one component of a teacher's pedagogical content knowledge. Indeed, Kahl et al. (2013) called for the inclusion of assessment literacy in pre- and in-service teacher education courses. Recent research conducted within the PME community concur with the findings found in the general education community that teachers in general show a lack of declarative and procedural knowledge of assessment (Santos & Cai, 2016). Hoch and Amit (2013), on 139 pre-service and beginning mathematics teachers in Israeli elementary and secondary schools, found that teachers lack knowledge even in basic concepts of assessment. The teachers sampled in their survey rated themselves moderately on their declarative knowledge of assessment concepts but answered poorly (4.4 out of 11 questions correct) on the extent they made use of assessment concepts in their work. Teachers' extent of use of assessment concepts in their educational practice also correlated positively with their extent of declarative knowledge. Beginning teachers in secondary schools significantly used less alternative assessment tools and more tests and quizzes, compared to beginning teachers in elementary schools.

In Singapore, Koh (2011) compared the assessment literacy of teachers teaching elementary years 4 and 5 (aged 10 and 11) in two kinds of professional development courses on designing authentic classroom assessment and rubrics: those in a short-term, one-session workshop versus those in an ongoing sustained programme. Focusing on two aspects of assessment literacy (quality of classroom assessment tasks and teachers' conceptions about authentic assessment), it is not surprising that Koh found that teachers with on-going professional development sustained increased assessment literacy and better understanding of authentic assessment. Among the English, Science, and Mathematics teachers in his sample, Koh found that the quality of mathematics assessment only improved slightly compared to other subjects after the sustained professional development programme. He explained that "many mathematics teachers still believe that students' mastery of factual and procedural knowledge is important for their conceptual understanding" (p. 272) and this is ingrained in their assessment practices.

The studies cited above suggest that teachers' beliefs about and attitudes towards the use of certain assessment tools (such alternative assessment or assessment technologies) are important issues impacting their assessment practices. Education context may well play a critical role shaping teachers' beliefs and attitudes.

Koh's (2011) study indicated that on-going professional development and support helps teachers in developing their assessment literacy. However, the nature of professional development programme on teachers' assessment literacy depends on the assumptions and approaches teachers educators take too.

The two teacher educators Lee and Son (2015) had different expectations on the nature of pre-service teacher education on assessment literacy. One of them believed that it is important for pre-service teachers to learn from experience of designing their own assessment items from scratch, whereas the other emphasised the importance of critically selecting and modifying assessment items to suit the educational context. Lee and Son had made references to Kahl et al.'s (2013) generic framework on teachers' assessment literacy to springboard their research. From analysis of their pre-service students' written responses to a survey on both their beliefs about assessment as well as an item requiring them to critique a set of mathematics tasks, the teacher educators found that their students believed that assessment for learning (formative and diagnostic assessment) was the most important purpose of assessment, and that assessment requiring higher cognitive demand were better than those requiring routine computations. When asked to choose their most preferred mathematics task from a list of five different tasks on fractions, the sample in both groups led by the two educators were able to justify their choice based on cognitive aspects of assessment items in the critiquing task, citing cognitive demand, clarity, personal mathematical and pedagogical preference, mathematical complexity, format, and other issues. However, there were some differences in the pre-service teachers' responses. In one sample, more attention was paid to visual representation of one of the mathematics tasks whereas the other revealed more cases of personal preference as a criterion for evaluating assessment items. Lee and Son (2015) conceded that more research needs to be done to investigate the impact of teacher education courses on developing teachers' assessment literacy and also on the professional development of teacher educators in order to improve their own practice in this area.

Although documents from various teacher education institutes or government agencies (e.g., Australian Institute for Teaching and School Leadership, 2014; National Institute of Education, 2012) have outlined levels of teacher competencies in assessment practices expected from pre-service, beginning, and experienced teachers, these still need to be translated for mathematics teachers as part of their pedagogical content knowledge because of demands on the subject matter. Existing research in mathematics teacher education surfaced the lack of a coherent conceptual framework defining and connecting the various aspects of assessment literacy to guide mathematics teacher educators in their professional development programmes. Hence, we attempt to synthesise findings from existing studies to conceptualise a framework to describe teachers' assessment literacy.

TOWARDS A CONCEPTUAL FRAMEWORK FOR MATHEMATICS TEACHERS' ASSESSMENT LITERACY

Based on the literature reviewed, four aspects of assessment literacy were found: (1) knowledge about assessment concepts; (2) skills in applying the knowledge in actual assessment practices; (3) communication and action in providing feedback or changing instructional practices based on assessment information; and (4) attitudes and beliefs about assessment and its role in mathematics teaching and learning. Additionally, in reflecting about their own teaching and assessment practices for developing pre-service teachers' assessment literacy, Lee and Son (2015) demonstrated the importance of reflection as a strategy for professional development (in their case professional development as teacher educators). Santos and Cai (2016) pointed to teacher reflection and teacher collaboration as professional development strategies for teachers to improve their assessment knowledge. From this, a fifth aspect can be considered: (5) meta-cognition and self-regulation of teachers' own assessment literacy.

Inspired by the Singapore mathematics curriculum framework (Ministry of Education, 2012), the authors proposed a conceptual framework that includes five inter-related aspects to assessment literacy, shown in Figure 1.

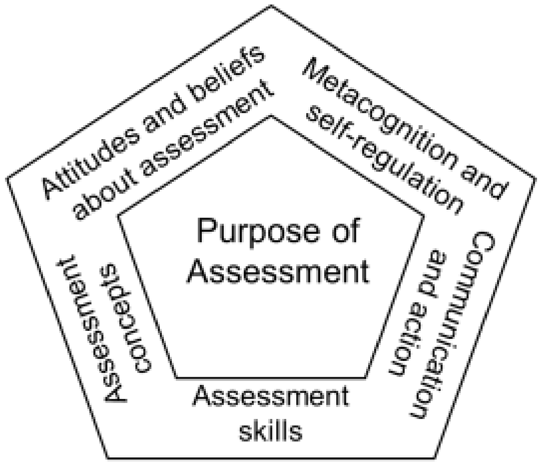

Figure 1. Conceptual framework for assessment literacy.

At the heart of the framework is the ***purpose*** of assessment. This can be pedagogical purpose, such as using assessment as part of classroom teaching and learning processes, or accountability purpose, such as the NAPLAN tests in Australia. In order to engage in the assessment process effectively, teachers need knowledge about the ***concepts*** in the field of education assessment, such as the different assessment strategies and their strengths and weaknesses, the suitability of tasks and questions for the mathematics topics assessed and possible student responses. The ***skills*** involved the assessment process includes knowing how a task can be used to promote mathematical understanding, when to implement a task, and the ability to design fair and inclusive assessment (Australian Association of Mathematics Teachers, 2008). ***Communication and action*** are important aspects of assessment literacy in order to create a positive assessment climate in the classroom (e.g., viewing errors as part of learning, providing feedback as part of dialogical process) and implementing appropriate and effective follow-up mathematical instructional activities. This includes communication about the mathematical focuses in the assessment tasks and students' mathematical performances in them with various stake-holders such as students and parents. Teachers' ***attitudes and beliefs*** about assessment play crucial role in driving their assessment practices. For example, an entrenched belief that mathematical knowledge is about mastery of skills may lead teachers to employ more of quick paper and pencil forms of assessment. Finally, ***metacognitive*** aspect includes self-awareness about assessment literacy, self-evaluation of own assessment processes and their impact on students' attitudes and mathematics learning including on-going monitoring and regulation of their decision making process in selection or design of assessment tasks, and self-reflection of issues relating to assessment processes such as ethics and equity. The first three aspects of concepts, skills and communication relate to the cognitive and social dimensions, and the last two aspects relate to the affective and critical dimensions of assessment literacy. These aspects are interrelated and inextricably intertwined, and are closely related to teachers' educational context and pedagogical content knowledge. For example, a secondary school teacher teaching in an environment with high stakes examinations might "feel an obligation to prepare their students for success" (Hoch & Amit, 2013, p. 71) and choose to use more tests or quizzes and less alternative assessment strategies, despite having knowledge and skills in various forms of assessment. Alternatively a teacher may choose to carry out peer and self-assessment tasks to facilitate a culture of collaboration and encourage students to improve their own performance over time, believing that these strategies will lead students to become better mathematics learners and achieve better results in examinations. In another example, whether teachers choose to use technology in assessment as one of their strategies will depend on their beliefs and professional knowledge.

CONCLUSION

The paper attempts to propose a conceptual framework for mathematics teachers' assessment literacy grounded in a critical review of current literature on assessment literacy by the mathematics education community. It offers a lens by which assessment literacy of mathematics teachers might be explored for future professional development courses and in research. It also presents a more cohesive structure connecting the key aspects of assessment literacy for further examination of the role of assessment in a teachers' pedagogical content knowledge. Future research may involve the validation of this framework in terms of its comprehensiveness in preparing pre- and in-service teachers in assessment literacy as well as using the components of the framework to develop instruments for the measurement of teachers' assessment literacy in mathematics.

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ENCHANCING TEACHERS' REFLECTION THROUGH LESSON STUDY: IS IT FEASIBLE?

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This study aimed to explore the content and levels of teachers' reflection as they engaged in Lesson Study (LS). However, this article only focuses on changes in teachers' reflection from a LS group. This LS group was made up of six primary mathematics teachers and four knowledgeable others. They carried out five LS cycles. Qualitative data were collected through reflection sessions, participatory observation, collection of artefacts and interviews. Analysis of data revealed that there were changes in the teachers' reflection. These changes included improvement in the depth of reflection about pupils' learning, shift from teacher's perspectives to pupil's perspectives, anticipation of pupils' responses and reflection from several perspectives. Thus, enhancing teachers' reflection through LS is feasible.

INTRODUCTION

Reflection practices have increasingly been used to support teachers' professional development (Suratno & Iskandar, 2010) because teachers would be able to recognize their own weaknesses and strengths through reflection (Boon, 2002). Furthermore, through reflection, teachers would be able to understand better the complex nature of their own teaching and their pupils' learning (Zeichner & Liston, 1996).

In Malaysia, the practice of reflection was first introduced to in-service teachers in 1999 (Ministry of Education, 1999). Teachers were required to reflect on to what extent they have achieved their teaching and learning outcomes. However, this requirement did not really encourage the teachers to reflect critically and deeply. Therefore, it was not surprise that a review of local studies (e.g. Siti Mistima Maat & Zakaria, 2010; Tee, 2007) reported that Malaysian teachers' reflection was still descriptive and not in-depth. Reflection that is descriptive will not help teachers to fully understand, and thus improve their teaching. Hence, there is a need to enhance reflection practices among Malaysian teachers.

To date, only a few studies have done on teachers' reflection in LS. Review of literature (Tosa, 2014; Myers, 2013; Posthuma, 2012) showed that the teachers generally reflected about LS process, teaching, learning and physical set up of lesson. Tosa (2014) discovered that most of the teachers' reflection which were at higher level focused on teaching strategy and pupils' thinking. Posthuma (2012) reported that the teachers became more aware of the pupils' needs after involving in LS.

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Other studies (e.g. Chiew, 2009; Fernandez & Chokshi, 2002) found that LS could enhance teachers' reflection. However, they did not study in detail how LS could impact teachers' reflection. Likewise, studies from Tosa (2014), Myers (2013) and Posthuma (2012) did not show substantial evidences that LS improve the teachers' reflection. Thus, this study aimed to explore changes (if any) in the teachers' reflection as they engaged in LS process.

THEORETICAL FRAMEWORK

Two theories that underpin this study were Situated Learning Theory by Lave & Wenger (1991) and the framework of teacher reflection practices of LS (Suratno & Iskandar, 2010). According to Situated Learning Theory (Lave & Wenger, 1991), learning occurs through the learners' legitimate peripheral participation in the activity of the community of practice. There are experts and novices in the community of practice. As the novices participate in the practice of community, they interact and collaborate with experts and other novices in the community. After an extended period of time, the novices internalise the culture of the community, change their beliefs and behaviour, and ultimately change to become experts of the community.

Reflection was defined as "active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusion to which it tends" by John Dewey (1933, p. 9). According to Suratno and Iskandar (2010), reflection is the heart of LS. Teachers reflect when they are preparing the lesson plan (prospective analysis), teaching or observing the research lesson (situational analysis) and reflecting on the research lesson (retrospective analysis). However, in this study, only teachers' reflection during reflection sessions were studied. During the reflection sessions, the teachers analysed the relationship between their teacher teaching and their pupils' learning. They also compare the learning trajectory design (LTD) with the actual learning trajectory (ALT). They framed and reframed the problem analysed and developed alternative LTD for future lessons.

Therefore, in this study, "reflection" was defined as the activity carried out by a group of teachers, who looked back into their pupils' learning during the research lesson, analysed their pupils' learning, identified the reasons of the incidents happened and explored alternatives to improve their pupils' learning. The community of practice was the LS group. The teachers were not familiar with reflection when they first conducted the LS. As they engaged in LS, they interacted with the knowledgeable others and other teachers in the group. After several LS cycles, they internalised the way of reflecting and they became able to reflect like expert. The teachers were expected to attain a fruitful understanding and the ability to frame and reframe problem after several LS cycles (Suratno & Iskandar, 2010).

METHODOLOGY

This paper discussed the teachers' reflection in a LS group, which was set up by six primary mathematics teachers. Besides, four knowledgeable others, who were comprised of two university lecturers, a postgraduate student and a School Improvement Specialist Coach, also involved in the LS cycles. The LS group carried out five LS cycles. Each cycle consisted of four steps, namely, (1) identify and formulate goals; (2) plan research lesson collaboratively; (3) teach/ observe research lesson; and (4) reflect and refine lesson plan.

Qualitative data were collected through participatory observation, reflection sessions, interviews and collection of artefacts. The artefacts collected were observation sheets written by the observing teachers and knowledgeable others during research lessons, research lesson plans, as well as pupils' worksheets. All the research lessons observed and reflection sessions were video-recorded, and transcribed verbatim for data analysis. Then, the transcripts were divided into segments. A segment refers to part of the transcript which was related to a topic or theme of reflection. The segment ended when the topic changed. The length of the segment ranged from a phrase from a person to several utterances from different persons. Next, the segments were coded to the themes of reflection, like pupils' learning, teaching strategy and instructional content. Triangulation of reflection transcripts, observation sheets and field notes were also carried out. Lastly, the coding was compared across the five LS cycles to explore any changes in the teachers' reflection.

FINDINGS AND DISCUSSION

The findings revealed that there were four changes in the teachers' reflection as they progressed from the first to the fifth LS cycles. These changes included:

Improvement in the depth of reflection about pupils' learning

The teachers' reflection about the pupils' learning became more in-depth as they progressed to the fifth LS cycle. At the beginning stages, the teachers' reflection about the pupils' learning were superficial and general. They merely described whether the pupils "understand" or "able to calculate". They did not elaborate further with evidences to show the pupils' understanding or learning. For instance, during the first reflection session, a male teacher, John articulated,

> I think they did not understand the 'Golden Hour'. Then, when talking about 72 hours, golden 72 hours is equivalent to how many days, they were able to calculate it. Next, one week is equal to how many hours, they need to know there are seven days in one week, so they faced problem in solving that question. After that I asked two weeks is equivalent to how many hours, they were able to calculate, but they need some guidance. (Reflection Session LS1)

Comparatively, at the later stages, the teachers' reflection about the pupils' learning became more in-depth, as they were able to point out exactly the related pupils' misconceptions. Figure 1 displays a question which most of the pupils answered wrongly during the fifth research lesson. The pupils were expected to write the improper fraction of the picture, which was $\frac{10}{6}$. Teacher, Sophy found that the pupils were not able to answer this question correctly because:

> The pupils did not know the way of identifying the denominator, they counted [the total number of portions], for example, there were three circles… the denominator should be six, but [the pupils] added up all the portions, [so their denominator became 18]. The pupils have not mastered the concept of denominator yet. (Reflection Session LS5)

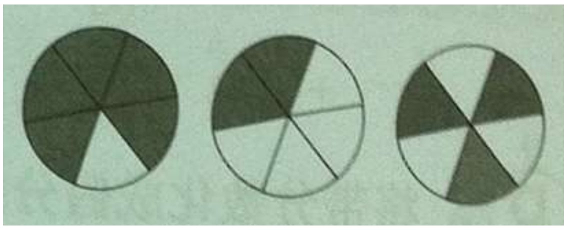

Figure 1. The question posed during the fifth research lesson

This finding supported the findings reported by Hart and Carriere (2011) that the teachers' reflection about the pupils became deeper after the LS process.

Shifting of teachers' reflection from teachers' perspectives to pupils' perspectives

During the first two reflection sessions, the teachers tended to reflect and comment from the teachers' perspectives. Their comments focused on the observed teacher's teaching strategy, his/her personality and behaviour based on their own perceptions. For instance,

> I want to praise him for admitting his mistakes… because we, as teachers, after we design an idea, sometimes we change the idea on the spot, so got mistakes, he changed immediately, this is correct [attitude]. (Betty, Reflection Session LS1)

> The [induction set] was interesting, because teacher used a big dice. (John, Reflection Session LS2)

> Only the teacher was talking, the pupils did not talk, this is my weakness, I always forgot to give my pupils chances to talk. (Betty, Reflection Session LS1)

These teachers gave comments based on their own perceptions of effective teaching. Betty perceived that the teacher should change immediately if there are any mistakes done during the research lesson and also get the pupils to involve actively. Whereas, John deemed that using the big dice made the lesson interesting.

At the later stages of LS, the teachers started to reflected from their pupils' perspectives. They reflected based on the pupils' misconceptions, behaviours and problems faced during the research lessons. For example, during the fifth reflection session, the teachers discovered that the pupils have misconception in getting equivalent fraction. The pupils identified fractions with same denominator as equivalent fractions. Based on the pupils' misconception, the teachers refined the lesson plan by giving example and non-example. As suggested by Sophy, "making comparison, $1\frac{1}{2}$ and $\frac{3}{2}$ are equivalent, then show $\frac{4}{2}$, [ask the pupils], are they equivalent? Although their denominators are the same, they are not equivalent" (Reflection Session LS5).

In addition, Betty also commented on the teaching strategy based on the pupils' behaviour during the fourth reflection session, she articulated that "actually we don't have to discuss the answers one by one, the pupils already felt bored" (Reflection Session LS4). Then, Fanny made suggestion based on the pupils' behaviour, she said, "get the [four groups of pupils] to exchange their answers, and ask them to mark the answers written by other groups" (Reflection Session LS4).

Furthermore, the teachers also reflected based on the pupils' problems. The problem was raised up by John, "some pupils did not understand Mandarin" (Reflection Session LS4). But, the lesson was carried out in Mandarin. Thus, some of the pupils did not understand the lesson. Based on the pupils' problem in understanding Mandarin, Betty suggested, "sometimes, teacher should speak some English in order to help the pupils to understand" (Reflection Session LS4).

In sum, the teachers reflected purely from teachers' perspectives at the beginning stages of LS. As they progressed to the later stages, they became more aware of their pupils' learning problems and needs. So, they moved to reflect from the pupils' perspectives.

Anticipation of pupils' responses

The teachers became more aware of the pupils' possible responses at the later stages of LS. They could anticipate their pupils' responses when they were refining the lesson plan and the mathematical tasks. During the fifth reflection session, the teachers tried to change the mathematical task of matching mixed number with improper fraction which they found it was not suitable. John started by suggesting to give all the pupils a fraction. Then, ask them to draw the picture of the fraction and find another friend who holds a fraction equivalent to their fraction (Line 1 in the transcript). Sophy anticipated the pupils' answer, where they might be drawing different shapes (2). Betty also predicted that the pupils would draw different shapes and sizes. But, she perceived that the differences in shapes and sizes might cause confusion among the pupils (3).

So, Sophy suggested to change the task to colour the boxes based on the fractions given and find another fraction equivalent to theirs. She proposed to provide more boxes than needed to make the task more challenging (7). But, Ashley predicted that the extra boxes would confuse the pupils (8). As the result, Betty suggested to refine the task by giving only the number of boxes as needed based on the fraction (9).

In sum, the teachers became more aware of the pupils' responses. They anticipated the pupils' possible answers and confusions, and tried to eliminate the pupils' possible confusions when they were refining the lesson. Previous literature (Fernandez & Chokshi, 2002) reported that anticipation of pupils' responses occurred during the preparing lesson plan stage, but in this study, it was found that anticipation of pupils' responses could also happen when the teachers were refining the lesson plan. It is important for the teachers to anticipate the pupils' responses as this activity encourage the teachers to think in term of the pupils, which supports the teachers to develop knowledge of mathematics and pupils (Meyer & Wilkerson, 2011).

1 John: What about the teacher poses a fraction of 3/2 and asks the pupils to draw?

2 Sophy: They draw whatever shapes they like, then their friend [holding equivalent fraction] might be drawing different shape. But the main point is, both of them, have one full piece, and one half piece.

3 Betty: I think we should not ask them to draw by themselves. They might be confused if their shapes are different. I think, the pupils have not mastered the concept yet, they don't know… because this is bigger, that is smaller, they are not equivalent, they will think like this?

4 Ashley: It's possible.

5 Sophy: Or we ask them to colour.

6 John: Colouring also can.

7 Sophy: For mixed number, let's say like this; give them three boxes without dividing lines and two boxes with dividing lines. Then $1\frac{1}{2}$, they would colour, one whole box and half of the box. This is the colouring of mixed number. For improper fraction, give them all the boxes with dividing lines, give them more boxes than needed, like this, at least they can colour like this. Then, they would see one whole box and one half of the box, so they are equivalent.

8 Ashley: I worried that the remaining boxes will make them confused.

9 Betty: More confused. You give them two, all with dividing lines. Then they colour by themselves.

Reflecting from several perspectives

The teachers reflected from several perspectives at the later stages of LS. During the fifth reflection session, the teachers pointed that the pupils faced problem in determining the denominators for mixed number and improper fraction. Sophy elaborated that it was because

> [the pupils] counted [the total number of portions], for example, there were three circles… the denominator should be six, but [the pupils] added up all the portions, [so their denominator became 18]. (Reflection Session LS5)

When they were analysing the causes of the pupils not being able to determine the denominator correctly, they viewed the problem from three perspectives, namely the pupils' prior knowledge, the pupils' learning during previous lesson and the instructional content delivered during the research lesson. Betty perceived that the problem was caused by the pupils' prior knowledge. She suspected that "the pupils have not mastered the basic concept of the fraction" (Reflection Session LS5).

However, her comment was rejected by Ashley and Sophy. Ashley argued that, "no, [the pupils] have mastered the basic concept of fraction" (Reflection Session LS5). Then, Sophy linked the problem with the pupils' learning in the previous lesson which taught about the basic concept of proper fraction. She expressed, "could it be because the pupils confused with the concept taught in the previous lesson? Because in that lesson, we taught them to count all the portions" (Reflection Session LS5). But, again, this statement was rejected by Ashley, the teacher who taught the previous lesson. She clarified that “I only used one paper at that time, I drew all the portions on the paper” (Reflection Session LS5).

At the end of the discussion, the teachers believed that the misconception was caused by the instructional content delivered during that particular research lesson. Ashley explained that “[the teacher] did not emphasize that there are many pieces, but you should not count all the portions, you only count the number of portions in one piece” (Reflection Session LS5). The discussion among the teachers during the fifth reflection session showed that the teachers reflected the lesson from several perspectives.

This result showed that the teachers have attained the ability of framing and reframing the problem discussed after engaging in LS (Suratno & Iskandar, 2010). Reflection from several perspectives is categorized as high level reflection (Lee, 2005; Ward & McCotter, 2004; Jay & Johnson, 2002), because by reflecting from several perspectives, the teachers would be able to understand the complex nature of teaching and learning in a holistic manner.

CONCLUSION

We acknowledge that analysing data from only a case of LS group which involved five LS cycle may not be sufficient to render the claim that LS process can enhance teachers’ reflection. However, in this case study, at least four observable changes in teachers’ reflection were noticed as they progressed from the first to the fifth reflection sessions. These changes include improvement in the depth of reflection about pupils’ learning, shifting the reflection from teacher’s perspectives to pupil’s perspectives, anticipation of pupils’ responses, and reflecting from multiple perspectives.

Although the impact was not very obvious within the five cycles of LS, but there is definitely some gradual improvement in the teachers’ reflection observed as they conducted multiple LS cycles. Hence, enhancing teachers’ reflection through LS is feasible. Nevertheless, future studies are needed to explore the factors that could make teachers’ reflection in LS more effective, for instance, the role of knowledgeable others and the anticipation of pupils’ responses during the planning stage.

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PSYCHOMETRIC EVALUATION OF A QUESTIONNAIRE MEASURING TEACHER BELIEFS REGARDING TEACHING WITH TECHNOLOGY

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Teacher technology related beliefs play an important role when integrating technology into the mathematics classroom. However, there are no empirically validated questionnaires available for quantitatively measuring teacher beliefs regarding technology use. In this paper the empirical validation of a questionnaire to measure teacher's beliefs regarding technology use in the mathematics classroom is presented. Psychometric properties like reliability and factor structure are assessed with a sample of 180 in-service teachers in Germany leading to some adaptions of the item set. The resulting questionnaire provides researches with an instrument to more reliably and validly assess teacher's technology related beliefs.

INTRODUCTION

Research shows, that the introduction of technology into the mathematical classroom can support student learning and that teacher beliefs play an important role in this integration process. Hence, research studies try to investigate teacher's technology related beliefs and how, for example, these beliefs are linked to classroom practice. Whereas qualitative studies rely for example on interviews to determine teacher's beliefs, quantitative studies usually assess teacher beliefs using questionnaires. However, so far there are no profoundly developed questionnaires to quantitatively measure technology related beliefs of teachers. Studies to this point use single-item measures (i.e. likert – items) to assess teacher beliefs regarding a specific topic. However, the problems associated with such types of measures are tremendous. As McIver & Carmines (1981) note: "*The most fundamental problem with single item measures is not merely that they tend to be less valid, less accurate, and less reliable than their multi-item equivalents. It is rather, that the social scientist rarely has sufficient information to estimate their measurement properties. Thus their degree of validity, accuracy, and reliability is often unknowable.*" *(*McIver & Carmines 1981, p. 15). Hence the validity of results gained from studies that are using single-item measures are highly questionable. As Blalock (1970) pointed out researchers may "*remain blissfully unaware of the possibility of measurement [error], but in no sense will this make his inferences more valid*" (Blalock 1970, p. 111).

Therefore, there is a need for well-developed multi-item scales to measure teacher's technology related beliefs and *"more studies should focus on how to develop methods or instruments that can help in the rigorous identification and evaluation of teacher beliefs."* (Chen 2008, p. 74),

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 265-272. Singapore: PME.

In this paper a detailed multi-item scale questionnaire assessing teacher beliefs when teaching with technology is presented. The questionnaire is based on the work of Rögler (2014, 2015) who developed a set of items by cycles of cognitive interviews with in-service teachers. In this paper, the psychometric properties and factor structure of the questionnaire are investigated using a sample of n=178 in-service-teachers in upper secondary school in Germany. The resulting questionnaire provides researchers with an instrument in order to reliably assess teacher beliefs. This may support future research in reliably answering the many questions that are associated with teacher's technology related beliefs.

THEORETICAL BACKGROUND

Research shows that, digital technology can play an important role in enhancing the learning of mathematics (e.g. Zbiek et al. 2007). This can be achieved for example by supporting discovery learning, problem solving, modelling and more interpretative tasks in the mathematics classroom and by enabling a shift from a focus on procedures to more conceptual understanding. In particular, by providing easy access to different forms of representation and to multiple linked representations students can explore relationships between these forms of representations which is crucial in developing conceptual understanding.

In the following the term "technology" is used for all digital tool affording the described benefits. These tools comprise for example graphing calculators (GC) or computer algebra systems (CAS).

The successful integrating of such technology into the mathematics classroom depends on many factors and research has identified teacher beliefs as one such critical factor (i.e. Ertmer 2005, Chen 2008) suggesting beliefs to be *"the final frontier in our quest for technology integration"* (Ertmer 2005, p. 25). When using the term belief, we refer to the definition of Phillip (2007, p. 259) who characterized beliefs as "*Psychologically held understandings, premises, or propositions about the world that are thought to be true".*

However, despite the importance of understanding teacher's technology related beliefs there are no psychologically profound developed empirically scrutinized questionnaires to measure such beliefs. For example, the studies of Dewey et al. (2009), Fleener (1995), Duncan (2010), Tobin et al. (1999), Milou (1999), Pierce & Ball (2009), Tharp et al. (1997) and Molenje (2012) all use single-items measures where each item is referring to a different aspects of teacher beliefs.

Although the use of such items is easy, fast and provides a first access to teacher's technology related beliefs it is well known that single-item measures are very problematic and that multi-item scales should be used instead. The reasons are that single-items measures lack reliability, precision and scope (Gliem & Gliem 2003). Reliability is generally low since single-item measures have a high measurement error. On the contrary, in multiple-item scales *"Measurement error averages out when* *individual scores are summed to obtain a total score"* (Nunnally & Bernstein 1994, p. 67). Precision of single-items measures suffers from the fact that single item measures can only roughly categories people into groups. For example, a question with a 5-point likert response format can only categorize people in five groups. In contrast, multi-item scales with summated rating scales can discriminate on a much finer level. Finally, scope is lacking in single-item measures, since *"It is very unlikely that a single item can fully represent a complex theoretical concept or any specific attribute for that matter"* (McIver & Carmines 1981, p. 15).

Hence there is a need to develop reliable and valid multi-item measures to assess teacher's technology related beliefs. Since single-items measures *„should not be used in drawing conclusions"* (Gliem & Gliem 2003, p. 1) only by using multi-item measures we can expect to gain more valid answers to the important questions associated with teacher's technology related beliefs.

DEVELOPMENT OF THE QUESTIONAIRE

The items of the questionnaire were developed in the work of Rögler (2014,2015) who used semi-structured interviews with in-service teachers to develop items covering different aspects of technology use. This items could broadly be categorized in seven distinct categories with each category being covered by 4-6 items:

***Beliefs that computations should be shifted to technology* (S – Shifting):** Items in this category refer to the belief that computations should be shifted more towards technology. This is an often mentioned benefit of technology since *"Shifting the burden of computation to [technology] makes time available for students to concentrate on how to approach a problem, to delineate subproblems, and to consider alternatives, rather than spending most of the time routinely following one algorithm."* (Small 1986, p. 145).

Beliefs that technology supports discovery learning (D -Discovery learning): Items covering this category refer to beliefs about the support of discovery learning by the use of technology. This can be achieved for example when students use technology to generate many examples to explore the relationships between them.

Beliefs that technology support multiple representations (R – Multiple Representations): Items in this category focus on teacher beliefs regarding the support of multiple representation by means of technology.

***Beliefs that technology is too time consuming* (T – Time consuming):** Items in this category cover the belief that technology use is too time consuming. This belief is for example reflected in the statement of one teacher in the study of Coffland & Strickland (2004): *"We don't have time to teach the current curriculum; much less add time with technology."* (Coffland & Strickland 2004, p. 358).

Beliefs that technology has a negative impact on computational skills (S – Skill loss): This category of items covers the common concern that pen-paper-skills may be lost in the presence of technology.

Beliefs that technology leads to mindless working (M - Mindless working): Items in this category covers the belief that technology leads students to *"mindless button pushing"* (Mackey 1999, p. 3) and that working with technology is just a *"substitute for thinking"* (Mackey 1999, p. 3).

***Beliefs that students must master concepts and procedures prior to technology use*. (Procedures first – P):** Items in this category cover the belief that *"calculators should be used only after students ha[ve] learned how to do the relevant mathematics without them"* (Ballheim 1999, p. 4). This belief is strongly linked to the Black-Box / White-Box principle where the order of instructional use of technology is discussed (Drijvers 1995).

Overall the seven categories cover many of the most relevant and widely researched aspects of teacher beliefs regarding technology. Not covered by these categories are for example affective components of technology integration like effects on student motivation.

EMPIRICAL EVALUATION OF THE QUESTIONNAIRE

Method

To investigate the psychometric properties of the questionnaire the set of items developed by Rögler (2014, 2015) was administered to 167 in service teachers in Germany. Data collection took place within a large larger research study that was carried out in the federal state of North Rhine-Westphalia in Germany in November 2014 (Thurm et al. 2015). In this German federal state, the use of technology (GC or CAS) is compulsory since the schoolyear 2014/15.

The questionnaire was administered as pen & paper questionnaire with responses given from 1="strongly disagree" to 5="strongly agree". Confirmatory factor analysis was used to test the seven-factor structure of the questionnaire. Overall model fit was evaluated by the chi-square-fit index ($\chi 2$/df), the root mean square error of approximation (RMSEA), the standardized root mean square residual (SRMR) and the comparative fit index (CFI). To evaluate the local model fit, indicator reliability, factor reliability, cronbachs alpha, coefficient omega, average variance extracted (AVE) as well as the Fornell-Lacker criteria (FLC) were used. The FLC is a measure for discriminant validity of latent variables. It holds if the AVE is higher than every squared correlation of the scale with other scales.

Results

The confirmatory factor analysis yielded a good overall model fit. However not all local fit measures were good. In particular, all items of the scale named "Shifting" had a very low AVE (<0.5) and hence could only explain less than 50% of the variance. Furthermore, one item of the scale "D" and one item of scale "E" had very low indicator reliability. Finally, the scales "S" & "M" did not fulfill the FLC.

These results led to the following adaptions: The scale “A” had to be complete discarded because of the low AVE of the items in this scale. In addition, the two items with low reliability from the scale “A” and “E” were discarded as well. Afterwards the model fit was reassessed using the truncated questionnaire. The model fit was still good with RMSEA=0.050, SRMR=0.046, CFI=0.961, and $\chi 2/df=1.445$ as literature reports the following as sufficient for a good fit: $\chi 2/df < 3$; RMSEA < 0.08; SRMR < 0.11 and CFI > 0.9. All items now show good indicator reliability and the reliability of the scales are high with Cronbach’s alpha ranging from 0.85 to 0.93 (Table 1). However, the scales “S” & “M” still do not fulfill the FLC, due to the high correlation (0.83) between these two scales.

	α	ω	AVE	M	SD
R – Multiple Representations	.85	.86	.61	3.8	0.81
D – Discovery learning	.87	.87	.57	3.3	0.83
T – Time consuming	.91	.91	.78	2.6	1.22
S – Skill loss	.86	.86	.61	3.8	0.90
M – Mindless working	.88	.88	.60	3.4	0.93
P – Procedures first	.93	.93	.77	3.2	1.24

	R	D	T	S	M
D	.66	1			
T	-.54	-.67	1		
S	-.37	-.57	.70	1	
M	-.45	-.62	.71	.83	1
P	-.28	-.43	.44	.49	.57

Table 1: Local model fit (left) and correlation table (right)

(α=Cronbach’s alpha, ω= coefficient omega, AVE=average variance extracted, M=Mean, SD=Standard deviation)

Since the scales “D” and “R” refer to benefits of technology use whereas the scales “T”, “S” and “M” refer to problems associated with technology use one would expect these scales to be negatively correlated which is indeed the case as seen in Table 3 providing support for the validity of the questionnaire.

SUMMARY

Reliable and valid instruments to measure teacher’s technology related beliefs are not yet available and studies so far use unreliable single-item measures. This paper presents the empirical evaluation of a multi-item questionnaire to assess teacher technology related beliefs. Psychometric properties like reliability and dimensionality of the questionnaire are scrutinized leading to some adaption of the questionnaire. The resulting questionnaire comprises six scales covering a broad range of teacher beliefs. It shows very good statistical properties supporting reliability and validity of the instrument. The questionnaire allows researchers and educational leaders to measure teacher’s technology related beliefs in different areas much more accurately than single-item measure can afford. The questionnaire can be used in many settings and can be easily adapted to focus on a specific technology (like GC, CAS or Geo-Gebra)

However, the questionnaire should be further scrutinized in future research. Since modifications were made due to the results of the statistical analysis “*one must realize that the analysis has moved from confirmatory to exploratory.*“ (Schreiber et al. 2006, p.330). In this case the psychometric properties of the questionnaire should be evaluated with a new sample of teachers. Work may also be done on extending the questionnaire by including scales covering affective topics like student motivation. Only if reliable and valid questionnaires are used empirical based research can generate reliable results.

APPENDIX:

In the following the final questionnaire is given. The items were translated from the German version presented in Rögler (2015). When using the questionnaire, it is recommended replacing the word [technology] by the appropriate type of technology used in the given context (i.e. CAS or GC) to avoid ambiguity.

	Multiple Representations
R1	An important advantage of [technology] is the opportunity to quickly change between forms of representations like algebraic expression, graph and table.
R2	[Technology] helps to link the different types of representations (i.e. Graph, table, algebraic expression).
R3	By the use of [technology] students can use different types of representations to solve problems or tasks.
R4	The use of [technology] helps students to better understand the link between algebraic expression, table and graph of a function.
	Discovery learning
D1	By using [technology], it is possible to generate many examples, so students can realize relationships and structures (i.e. symmetries of a graph of a function).
D2	[Technology] supports tasks where students can explore new content on their own.
D3	[Technology] enables students to explore mathematical concepts (i.e. meaning of parameters) on their own.
D4	The use of [technology] leads students to actively acquire particular content on their own.
D5	The use of [technology] particularly enables students to explore open problems on their own.
	Time consuming
T1	The use of [technology] costs valuable time which is subsequently missing in the mathematics classroom.
T2	[Technology] should be avoided in the mathematics classroom since otherwise too much time is lost.
T3	The introduction of [technology] costs so much time that its use does not pay off.
	Skill loss
F1	By the use of [technology] students forget procedures and algorithms (or do not learn them at all).
F2	The use of [technology] leads to students mastering arithmetic techniques worse or not all.
F3	By the use of [technology], students loose essential basic skills (i.e. mental calculation skills, methods of fractional arithmetic or precise drawing skills).
F4	Essential skills (i.e. solving systems of equations, calculating matrices or differentiation of functions) are less mastered by students due to the use of [technology].
	Mindless working
U1	If [technology] is used, students think less and rely blindly on the output that technology provides.
U2	[Technology] misleads students to work on every task without reflection.
U3	If students have access to [technology] they think less.
U4	When [technology] is used, there is the danger that students just type command sequences without understanding.
U5	The output that [technology] provides is accepted uncritically as correct by students.
	Procedures first
P1	[Technology] may only be used if the mathematics is mastered by pen & paper.
P2	Students should know the mathematical procedures thoroughly before they are provided access to [technology].
P3	Within an instructional sequence students should not work too early with [technology], but rather only if they understood the mathematics sufficiently.
P4	[Technology] may only be used to ease students procedural work if the procedures are already mastered without [technology].

Table 2: Overview of the items of the final questionnaire

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THE RELATIONSHIP BETWEEN LANGUAGE COMPLEXITY AND MATHEMATICS PERFORMANCE

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We investigated the extent to which number word structure might offer an insight to children's development of early number sense. In conjunction with TIMSS Grades 4 and 8 mathematics scores of 60 countries, we selected the transparency base-ten structure as a criterion for linguistic complexity of the 33 spoken languages in translating numeral symbols into number words. Our findings showed, to some extent, that languages with no transparent base-ten structure had an advantage over those with transparent base-ten structure at the initial stage of mathematics learning but not as significantly at the later stage.

INTRODUCTION

International studies comparing mathematics achievements of children in the U.S. and those in other countries, especially in East Asia, indicated a persistent significant difference (Stevenson, Chen, & Lee, 1993; Torney-Purta, 1990). Researchers attributed to such discrepancy a number of affective and cultural qualities including beliefs, motivations, and study habits (Fuligni & Stevenson, 1995; Leung, 2006; Newman et al., 2007; Wang, 2004). Little was known regarding the role of number word acquisition on young children's mathematical understanding (Miller, Kelly, & Zhou, 2005).

In this study, we examined the extent to which different languages might influence the development of early number sense. From a list of official languages of the countries participating in the Trends in International Mathematics and Science Study 2011 (TIMSS 2011; Mullis, Martin, & Arora, 2012), we surveyed the number word structure that linked numerals and word names. We were interested in answering the question of whether learning a particular spoken language might be a predictor of a child's success in mathematics learning.

THEORETICAL BACKGROUND

Kindergarten was one of the most prominent grade levels where students brought with them to school the most heterogeneous mathematics skills from informal contexts (Cross, Woods, & Schwingruber, 2009). Prior to formal education, young children's counting skills in particular ranged widely from little exposure to word names for numerals to simple visual manipulations of objects via number decomposition or place value approach (Cross, Woods, & Schweingruber, 2009; Syrett, Musolino, & Gelman, 2012).

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Such a considerable variation had been seen in mathematical understanding, particularly the understanding of number words and number sense, of not only young children across the U.S., but also those in the U.S. in comparison with those in other countries, especially in East Asia (Stevenson, Lee, Chen, Lummis, Stigler, Fan, & Ge, 1990). While the latter on the whole were more successful in a number of mathematical assessments than the former, some researchers pointed out that the substantial difference in mathematical competence might not be a direct result of formal schooling or the superiority of the curriculum (Geary, Bow-Thomas, Fan, & Siegler, 1993). Miller, Smith, Zhu, and Zhang (1995) indicated that this difference in mathematical performance might be more related to the differences in the number-naming system.

The acquisition of word names using a certain language had been shown to accelerate particular counting strategies. Ho and Fuson (1998) observed that Chinese-speaking children were more adept than English-speaking children at operating the number ten as a benchmark when performing addition tasks. Geary and colleagues (1993) found that when unable to immediately recall addition facts, Chinese-speaking children applied more abstract fallback strategies such as the verbal counting method, while English-speaking children applied more concrete fallback strategies such as the finger counting method.

To the extent that Chinese-speaking children were perceived more mathematically proficient than English-speaking children, earlier researchers, in addition to cultures (Leung, 2001), student self-beliefs (House, 2006), and parental beliefs and practices (Miller, Kelly, & Zhou, 2005), credited the persistent difference to the transparency of base-ten structure in Chinese language that was not immediately apparent in English language (Miller & Paredes, 1996). Unlike English language, Chinese language was noted to convey a more direct relationship between Hindu-Arabic numeral symbols and their corresponding number words as one might read them in a literal translation.

Chinese language consisted of only ten unique number words associated with numerals from 1 to 10, the combinations or variations of which constructed all number words associated with numerals from 11 to 99. These 99 constructions were done using the literal translation of base-ten structure. For example, the number word “shí-yī” was a combination of the number words “shí” and “yī,” which literally translated into “ten” and “one.”

On the other hand, English language introduced unique number words for numerals 11 and 12, in addition to the ten unique number words associated with numerals from 1 to 10. For example, the number word “eleven” for the numeral 11 was neither a combination nor a variation of the number words “ten” or “one.” In total, English-speaking children would need to memorize not only two additional unique number words compared to their Chinese-speaking peers, but also another seven number words, whose constructions made use of a modified word “teen,” for numerals from 13 to 19.

These seven number words were mostly learned through rote memorization by English-speaking children, resulting in many not fully comprehending how those number words might be connected to their base-ten constructions (Fuson, Richards, & Briars, 1982). Evidently, by the age of five years old, compared to Chinese-speaking children, English-speaking children were less capable of recognizing that the numeral 13, for example, might be equivalently viewed as "10 + 3" (Ho & Fuson, 1998). Such lack of understanding of the base-ten structure in the Hindu-Arabic numerals might be considered to be comparable to the lack of association between number words and number concepts of any typical two-year-old children (Wynn, 1992).

More generally, the base-ten structure was considered to be more consistent and transparent in Chinese language than in English language (Miller, Kelly, & Zhou, 2005). For example, the numerals 16 and 60 were read in Chinese language as "shí-lìu" and "lìu-shí," which literally meant "ten-six" and "six-ten," or equivalently, "10 + 6" and "6 × 10," respectively. On the other hand, these numerals were read in English language as "sixteen" and "sixty," which were comparable to "6 + 10" and "6 × 10," respectively. While Chinese language clearly operated under the natural base-ten structure sequencing each number word with its place value, English language reversed the natural order, especially, for some numerals from 11 to 19.

METHODOLOGY

We determined the classification of each official language of TIMSS participating countries according to the transparency of its base-ten structure in the manner in which numeral symbols were associated with number words. Out of the 50 and 42 TIMSS Grades 4 and 8 participating countries, respectively, 60 were identified as distinct countries that utilized 35 unique languages. With the exception of Danish and Georgian that did not follow either pattern, we identified two number-word-naming patterns from the 33 different languages.

We recognized the evolution of number words from 1 to 10 to from 11 to 19 and from 20 to 99. If the place value and word naming association of the Hindu-Arabic numeral symbols was naturally translated into a literal manner, then a language was categorized as the first word-naming pattern (e.g., Chinese language). If the place value and word naming association of the Hindu-Arabic numeral symbols was not naturally translated into a literal manner, it was categorized as the second word-naming pattern (e.g., English language).

Based on the word-naming pattern of its language, each of the 60 distinct TIMSS Grades 4 and 8 participating countries was associated with either the first or the second group. We performed the Grubbs' test at 0.05 level of significance to recognize any extreme value of the TIMSS Grades 4 and 8 mathematics mean scores of the two groups in each group. We then analyzed the mean scores of these two groups using the t-test for independent means at 0.05 level of significance.

FINDINGS

We identified 42 and 16 countries that corresponded to the first and second group, respectively (see Table 1). Examples of TIMSS participating countries in the first group were Hong Kong–CHN and Yemen whose official languages were Chinese and Arabic languages, respectively. Examples of TIMSS participating countries in the second group were the U.S. and Spain whose official languages were English and Spanish languages, respectively.

In the first group, 33 countries reported a TIMSS Grade 4 mathematics mean score of 474.63 (SD = 80.32), and 33 other countries reported a TIMSS Grade 8 mathematics mean score of 465.09 (SD = 67.14). In the second group, 15 countries reported a TIMSS Grade 4 mathematics mean score of 525.07 (SD = 35.80), and eight other countries reported a TIMSS Grade 8 mathematics mean score of 480.25 (SD = 80.84).

	First group	Second group
Language complexity	Transparent base-ten structure	Non-transparent base-ten structure
Number of distinct countries	42 countries (e.g., Hong Kong–CHN and Yemen)	16 countries (e.g., the U.S. and Spain)
Number of unique languages	29 languages (e.g., Chinese and Arabic languages)	6 languages (e.g., English and Spanish languages)
Number of participating countries in TIMSS Grade 4	33 countries	15 countries
TIMSS Grade 4 Mathematics mean score	474.63 (SD = 80.32)	525.07 (SD = 35.80)
Number of participating countries in TIMSS Grade 8	33 countries	8 countries
TIMSS Grade 8 Mathematics mean score	465.09 (SD = 67.14)	480.25 (SD = 80.84)

Table 1: Language complexity and TIMSS Mathematics performance

The Grubbs' test showed that there was no outlier in each group at 0.05 level of significance. Children in the first group performed worse than those in the second group at Grades 4 and 8. The t-test for independent means showed that at 0.05 level of significance, there was a significant difference between the groups at Grade 4 but not at Grade 8.

ANALYSIS

Our findings demonstrated that to some extent, children who learned number words using a language with a transparent base-ten structure did not necessarily have a linguistic advantage over their peers who learned number words using a language without a transparent base-ten structure. Indeed, the former did significantly more poorly than the latter at Grade 4. The trend persisted at Grade 8, although not significantly.

To a certain degree, this result contradicted the findings by Miller and colleagues (2005), who suggested that Chinese-speaking children benefited from the consistency and transparency of base-ten structure in Chinese language. This result also indicated that being born in a certain country that used a more transparent base-ten structure language might not effectively guarantee an early mathematics performance.

Restricting the TIMSS participants to include only Chinese-speaking countries (e.g., Hong Kong–CHN and Chinese Taipei–CHN) and the U.S. as selected in the study by Miller and colleagues (1995) did show that the former group was at a greater advantage than the latter group in mathematical competence, confirming their findings. Nevertheless, the hypothesis by Miller and colleagues (2005), to some extent, did not appear to apply in general to children in the countries outside East Asia whose primary language was not Chinese language but had a similar transparent base-ten structure as Chinese language.

One counterexample was Arabic language. Arabic language was the official language of nine out of 50 and 11 out of 42 TIMSS Grades 4 and 8 participating countries, respectively. These 13 distinct countries were among the very lowest performers in TIMSS Grades 4 and 8 mathematics assessments. For instance, children in Yemen, who used Arabic language as their primary language, scored 248, which was the lowest score in TIMSS Grade 4 mathematics. This score was significantly lower than that of Chinese-speaking countries (e.g., 602 and 591 for Hong Kong–CHN and Chinese Taipei–CHN, respectively).

Likewise, their hypothesis did not generalize to children in the countries outside the U.S. whose primary language was English language. For instance, English language was the language of instruction for students in Singapore, which scored 606, the highest score in TIMSS Grade 4 mathematics. This score was significantly higher than that of the U.S. (541).

In contrast to the studies by Miller and colleagues (1995; 2005), the present study suggested, to some degree, that children in the countries whose primary language had no transparent base-ten structure performed significantly better than their peers in the countries whose official language had a transparent base-ten structure.

That is, English language, despite its lack of transparent base-ten structure, might not, or at least not mainly, be the primary reason as to why children in the U.S. consistently fared less satisfactorily than their peers in the East Asian countries in early mathematics performance. Still, it should be noted that the dominance in mathematics performance, which was significant at the initial level (e.g., Grade 4), became slightly diminished at a later stage of mathematical development (e.g., Grade 8).

CONCLUSION AND DISCUSSION

The present study examined the relationship between language complexity and mathematics performance. It sought to understand the extent to which number word structure of a certain language might shape children's trajectory in mathematics learning.

To some extent, the findings showed that: (a) with the exception of its proximate variations (e.g., Japanese and Korean languages as the official languages for Japan and Korea, respectively), Chinese language was the only language with a transparent base-ten structure that delivered a positive advantage of early mathematics competence in comparison with English language or other languages with no transparent base-ten structure, and (b) languages with no transparent base-ten structure more generally had an advantage over those with a transparent base-ten structure at the initial stage of mathematics learning but not as significantly at the later stage.

It could therefore be inferred that language did not appear to play as a critical role in the process of acquiring mathematical understanding. The substantial difference in the mathematical development of young children in East Asian countries and the U.S. might have been confounded by affective and cultural qualities as opposed to with linguistic aspects.

PEDAGOGICAL IMPLICATIONS

Pedagogical implications that examined young children's various counting mechanisms and proposed an alternative counting framework that connected the concepts of numbers and numerals might be reconsidered based on the findings of the present study. For example, despite having been considered to be of superior quality, Singapore mathematics textbooks, when blindly approached, might not necessarily find an identical success in its adoption in the U.S. as one in its own natural implementation in Singapore (Ginsburg, Leinwand, Anstrom, & Pollock, 2005). Similarly, concrete presentations of the base-ten structure of number words, for instance, using manipulatives such as base-ten blocks, might not be as well grounded, or even counterproductive, in an attempt to ameliorate the perceived weaknesses of languages with no transparent base-ten structure as previously suggested (Miller et al., 1995).

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EXAMINING MATHEMATICAL SOPHISTICATIONS IN COLLABORATIVE PROBLEM SOLVING

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This paper reports on efforts to characterise levels of mathematical sophistication for students in collaborative mathematics problem solving. Using a laboratory classroom in Australia, data were captured with multiple cameras and audio inputs. Students worked individually, in pairs, and in small groups (4 to 6 students). We focused on investigating collaborative work, with the goal of studying the mathematical sophistications of students' reasoning when solving problems. Drawing from two analytical frameworks to document the mathematical sophistication in students' exchange, levels of cognitive demands and mathematical practices, this research highlights different aspects of students' reasoning in solving these tasks.

SOCIAL ENVIRONMENT FOR LEARNING

In social settings, learning involves complex processes including teacher-student and student-student interactions. Research designs in such settings need to be sensitive to the multifaceted nature of learning (Clarke et al., 2012). During collaborative problem solving of open-ended tasks, students have to negotiate approaches to a task as a group, which obliges the students to articulate their thinking overtly and this can make visible the learning processes. This study is part of a bigger project that investigates social interactions in learning through a research design that focuses on collaborative problem solving in mathematics. Available research facilities in Australia capture different sources of data including videos and audios records, as well as student artefacts. This paper specifically focuses on applying two approaches for documenting mathematical sophistication in students' reasoning in the classroom setting. The analysis reported in this paper addresses the research question: What are the levels of mathematical sophistication (in written product and in spoken interaction) displayed by individuals and groups in the social unit (pair and small group) as they solve open-ended mathematical tasks?

RELATED LITERATURE

Given the focus of this paper is on students working collaboratively on real-world mathematical problems, we have examined related works drawing on research on problem solving prior to 1990 including problem difficulties and characteristics of problem solvers (cf. Lesh & Zawojewski, 2007). One line of research exclusively focuses on features of tasks for students to solve in school. According to Lester and Kehle (2003), these task features include content and context, structure, syntax, and heuristic behaviour variables. Lesh and Zawojewski commented that still missing in

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this line of research is the consideration of the interactions between task difficulties and the characteristics of the problem solver. In other words, how students respond to tasks as a result of their personal characteristics matters. When solving problems, tasks alone do not account for how problem solvers interpret the same task differently. Students' interpretation of tasks depends not only on task characteristics (e.g., mathematical content, figurative task context, levels of cognitive demand), but also on characteristics of the learner and the class (i.e., cognition and affect) (Lesh & Zawojewski, 2007). A second line of research distinguishes between good and poor problem solvers. Lester and Kehle (2003) summarized that, (a) good problem solvers know more than the poor ones and their knowledge is well organised, not in discrete form but as a structured and connected network, and (b) the attention of good problem solvers is on the structure of the problems, while poor problem solvers focus on irrelevant information and the surface features of the problems. In this study, we did not aim to document task difficulties or to distinguish types of problem solvers in term of the novice-expert paradigm. Instead, we focused on what students do in the setting as they solve the problems and we documented evidence of their mathematical sophistication. Arguably, students' responses are dependent on the task variables, therefore the focus of this paper lies in the interaction between the two lines of research identified above.

Researchers such as Stein and Lane (1996) have emphasized the role of instructional tasks as catalysts for student learning. Stein and Lane conceptualise tasks as passing through three phases: (a) as represented in curriculum/instructional materials, (b) as set up by the teacher in the classroom, and (c) as implemented by students during the lesson. This study focused on what students do when they are working on problem solving tasks, therefore it could be considered as addressing the third phase of task implementation. Furthermore, tasks can be examined for their cognitive demand—the kinds of thinking processes that are required in solving each task. Stein and Lane found that the cognitive demand required by tasks influence student learning because they determine the ways students think about, develop, and use mathematics. Their framework presents four levels of cognitive demand: *memorization*, *procedure without connection*, *procedure with connection*, and *doing mathematics*. This framework was adopted in this study to focus on what students do when facing such tasks. From the perspective of the cognitive demand of tasks we could deduce what was required of students by each task and compare this to what the students actually did when attempting the tasks. For example, in the high level of *doing mathematics*, tasks require complex and non-algorithmic thinking to provide the opportunity for students to execute such thinking in the setting.

An alternative way to look at mathematical sophistication is through documenting how different mathematical practices (CCSSI, 2010) or mathematical habits of mind (Cuoco, Goldenberg, & Mark, 1996) are performed when students solve mathematical problems. Building on mathematical proficiencies (Kilpatrick, Swarfford, & Findell, 2001) and National Council of Teachers of Mathematics (NCTM) process standards

(2000), eight standards of mathematical practices were formulated representing the process that mathematicians and students carry out when they are doing mathematics. These practices include: (a) Make sense of problems and persevere in solving them. (b) Reason abstractly and quantitatively. (c) Construct viable arguments and critique the reasoning of others. (d) Model with mathematics. (e) Use appropriate tools strategically. (f) Attend to precision. (g) Look for and make use of structure (h) Look for and express regularity in repeated reasoning. Further details for the practices could be found in CCSSI (2010). In this study, these eight mathematical practices were used to investigate what features of the practices are evident when the students were interacting with each other when attempting the problem solving tasks. Together, the two analytic frameworks provide complementary perspectives to capture the nuances of mathematical sophistications in students' reasoning.

METHODOLOGY

The Setting

The recent development of a laboratory classroom, the Science of Learning Research Classroom (SLRC) at the University of Melbourne has made possible research designs that provide a better approximation to natural social settings, while allowing researchers to retain some control over aspects of the setting. In the Social Unit of Learning project, which utilised the SLRC for data collection, students work individually, in pairs, or in groups with their usual teacher. Yet, researchers could control task characteristics, the level of intervention from teachers, and possible forms of social interactions. With 10 built-in video cameras and up to 32 audio channels, the SLRC has the capability to capture classroom social interactions with a rich amount of detail. The facility was purposefully designed to allow simultaneous and continuous documentation of classroom interactions. The Social Unit of Learning project collected multiple forms of data including student written products and high definition video and audio recordings of every student and the teacher in the classroom. Intact Year 7 classes were recruited with their usual teacher for the project in order to exploit existing student-student and teacher-student interactive norms. Each class participated in a 60-minute session in the laboratory classroom involving separate problem solving tasks that required them to produce written solutions.

Problem Solving Tasks

To make the meaning negotiation process of the students visible for observation, open-ended tasks were chosen to allow students to have multiple entry points and require students to interact. Such tasks also call for different representations including numerical, symbolic, and graphical. In addition, the tasks afforded connection to contexts outside the classroom in order to facilitate discussion. These tasks were drawn from previous research (e.g., Sullivan & Clarke, 1992) and have been found to create opportunities for students to reason and to articulate their thinking. In the session analysed in this study, the three tasks included content foci that were disconnected to avoid carry-over effects between tasks. Task 1 focused on students' abilities to make sense of information from an incomplete graphical display – a bar graph. Students need to interpret what it is about and create the story from the graph. In Task 2, students were given an average age of the people in a household for which one person's age is constrained but requires interpretation and were asked to figure out the age of the other people in the household as well as the relationship between them. For the last task, students were required to work out the plan for a five-room apartment, which has a total area of 60 square metres. The students attempted the first task individually (10 minutes), the second task in pairs (15 minutes), and the third task in groups of four to six students (20 minutes).

The wording Task 2, used in this study is as follows:

Task 2: "The average age of five people living in a house is 25. One of the five people is a Year 7 student. What are the ages of the other four people and how are the five people in the house related? Write a paragraph explaining your answer."

Data Analysis

Two frameworks were used for coding data: one related to levels of cognitive demands, and another to mathematical practices. The cognitive demand framework (Stein & Lane, 1996) was adapted to describe what students do when facing the cognitive demands of tasks. Next, specific observations related to eight mathematical practices (CCSSI, 2010) were undertaken to help guide the coding of transcript and student artefact data. Videos of students working on the three problem solving tasks and the associated transcripts were used as a primary source for data analysis. After watching the videos and reading the transcripts, we created a mathematical story line to document their problem solving process. The students' written work was referenced occasionally to help explain their talk in the transcript. After creating the story lines, we then mapped the students' actions onto the two frameworks: levels of cognitive demand and mathematical practices. For levels of cognitive demand, which are hierarchical in nature, we observed what was going on in the discussion and how mathematical reasoning was developed during problem solving. A level was considered to have been attained if the student(s) illustrated at least one of the criteria appropriate to that level. Furthermore, when several levels were observed, the highest level was coded. For the coding of mathematical practices, each of the mathematical practices was documented when performed together with the time that the practice occurred.

PRELIMINARY FINDINGS

Initial observation suggests that when students work individually, barely any conversation happened and students rarely talked aloud. The main source of data for Task 1 (individual work) was their written work. Therefore, in this paper, we will illustrate how the coding was employed for a pair of students when they worked on Task 2 (pair work). This paper illustrates its key points by drawing on the written solutions, transcripts, and video record from one pair, two male students. John and Arman, working on Task 2. First, a mathematical story line, a narrative of student's mathematical reasoning when solving the task, was constructed by one of the researchers for John and Arman. John, an English language learner, had some difficulty understanding mathematical and non-mathematical words in the written task. This seemed to restrict his entry into the task. Notwithstanding, he asked about the meaning of the words *related* and *average* and strived to make sense of the task. The teacher explained to him the meaning of related, but not the meaning of average. When John approached his peer Arman, Arman provided different unclear descriptions of average.

Arman: You know what average is? Average is - average age of five people living in a house is... It's like the maximum.

John: Huh?

Arman: Maximum age.

John: Oh. What's it mean? Okay, okay. Is not - important but …

Arman: Okay. Average is like the most likely so most of the people in the - so most of five people living in a house is 25.

John was able to identify the age of the Year 7 student in the house as 12 years old, and tasked himself to find the ages of other four people. As he still had problems with understanding the concept of average, he also had difficulty elaborating what he was looking for: "Is five people the - which is together is 25 or each person is 25?" (John). He persisted with solving the task as he got more information from his peer.

John: Yeah. So you just guess the person of - no, it's a - how to say it? Just bigger than 13.

Arman: Yeah. So 12 and 25, lower - lower - younger than 25 and older than 12.

John: Are they same age or different?

Arman: Okay. So one can be 17 …

John: Yeah.

Arman: … yeah? Another one can be 14, 15.

John: I don't think so.

Arman: Or it can be older.

John: Oh… (moaning and groaning helplessly) Just - just don't like to [inaudible] (laughs).

Arman re-read the problem and picked out critical information from the task: what was the given information (one person was in Year 7), the household average age of 25 years and that the Year 7 student was not 25 years old, "Year 7 student is not 25 years, right?"), and what was being asked (age of each of the five people and their relationship). Arman interpreted average as maximum and typical, "most likely" in the sense of mode – "most of the five people living in a house is 25, close to 25". He then moved forward with the misinterpretation of average as maximum, and tried to find three numbers between 12 (or later 13) and 25. In the end, he said the numbers cannot be more than 27 or 28. He tried to generate five numbers from 12 to 25, with the same gaps between two consecutive numbers, starting with a gap of three: 12, 15, 18, 22, 25; he realised that this did not work and then revised and proposed alternatives of 17, 19, 21, 23; then 15, 21, 13, 18, 21 ("three years difference from two consecutive numbers, except for this one, it's 7 years, right?") – which was not consistent with the different interpretations of average that he had. As the time was running out, he rushed to the answer, and said to John, "just write something." The pair ended up with 13, 15, 18, 21, 28, and worked out the relationships between the five people as brothers and sisters. John jumped in to help figure out the relationships between the people based on their ages: "It's a brother or father or brother or brother or friends?" Arman seemed to have created a mathematical model for the problem as finding five consecutive numbers with equal gaps knowing the minimum and maximum and assign the numbers to ages of people in a family and figure out the relationship. The focus of their attention was on their interpretation of the mathematics requirements. They then worked backwards to reconstruct the context.

This line of reasoning was coded as a *High level of doing mathematics* as the pair engaged in several actions at that level, including:

- Use of complex and non-algorithmic thinking
- Explore and understand the nature of mathematical concepts, processes, or relationships
- Self-monitor and self-regulate their own cognitive processes
- Access relevant knowledge and experiences and make appropriate use of them
- Analyse the task and actively examine the task constraints that may limit possible solution strategies and solutions (Stein & Lane, 1996).

In terms of mathematical practices, we can observe that both students were involved in:

- *Making sense of the problems and persevering in solving them*. Both students started by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analysed givens, constraints, relationships, and goals for the task.
- *Constructing viable arguments and critiquing the reasoning of others*. They understood and use assumptions, definitions (average), and established results in constructing arguments. They justified their conclusions, communicated them to others, and responded to each other's arguments.
- *Modelling with mathematics*. They applied the mathematics they know to solve problems. They were able to identify important quantities in a practical situation and mapped their relationships using such tools as diagrams. They

routinely interpreted their mathematical results in the context of the situation and reflected on whether the results make sense, possibly improving the model (CCSSI, 2010).

As can be seen, the two frameworks (cognitive demands and mathematical practices) are conceptually disjoint, addressing entirely different aspects of mathematical sophistication with one framework hierarchical in nature and the other one does not assume any particular order. The juxtaposition of the two frameworks informs a more nuanced reading of the data.

DISCUSSION

The paper reports the use of two analytical frameworks to document levels of mathematical sophistications of students' reasoning during collaborative problem solving. The creation of the story lines appears to be useful for tracing the reasoning process of the students. It is a novel approach to apply the cognitive demands framework to document students' mathematical sophistication when reasoning during collaborative problem solving rather than focusing only on task features. The framework appears to be useful for capturing the nuances of the students' reasoning when working on the problem solving task. In addition, the application of the mathematical practices standards as a classificatory framework draws attention to student actions that are valued when solving mathematical problems. The applications of the two analytic frameworks could help advancing ways to examine collaborative problem solving. Furthermore, by applying these two frameworks, the researchers could examine the connections between each level of cognitive demand and the eight mathematical practices. The analysis is descriptive but not explanatory. It represents the first step in a research process directed towards the development of theory in relation to student collaborative problem solving and learning. Using the combined frameworks to identify student pairs or groups engaged in sophisticated mathematical activity, the video and transcript records of their activity can be examined to identify forms of interaction characteristic of such mathematically successful social groups.

The analysis reported here focused on providing an overall evaluation of the reasoning processes of the students. Further analysis is anticipated to examine the finer-grained patterns in mathematical sophistication of the students' reasoning during the pair discussion. Chunking the transcript data into smaller units could reveal patterns of mathematical sophistication evident when the students were negotiating during problem solving. A possibility is to chunk the transcript data into the unit of negotiative events (cf. Clarke, 2001) as a further step to document the levels of mathematical sophistication at this grain size. Furthermore, it could be useful to associate the coding at this grain-size with the coding of other aspects of the student interactions (e.g., student dialogic talk and affect) and use other variables as a way to account for mathematical sophistication. The Social Unit of Learning project concerns the identification of regularities in the negotiative interactions of students and how the social interactions influence the mathematical sophistications of student reasoning during collaborative problem solving. Future work will involve combining the analysis just described with other analyses of student affect, intersubjectivity, and discursive practice to identify factors of potential value to explain or account for students' mathematical sophistications.

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CONCEPTION OF NUMBER AS A COMPOSITE UNIT PREDICTS STUDENTS' MULTIPLICATIVE REASONING: QUANTITATIVE CORROBORATION OF STEFFE'S MODEL

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This study[1] provides statistical analysis that corroborates a prediction implied by Les Steffe's model: the strength of children's conception of number as a composite unit predicts their ability to reason multiplicatively. In individual clinical interviews, 33 fourth graders (age ~10) correctly solved a 1-digit addition word problem (8+7). Students spontaneously used one of three strategies: counting-on, doubling, or break-apart-make-ten (BAMT). Our statistical analysis revealed that students' spontaneous use of BAMT largely predicted their ability to reason multiplicatively, counting-on predicted poor ability, and doubling fell in between. We discuss implications of these findings for research and practice.

INTRODUCTION

We examine how a central element of Steffe's (1992) model of children's mathematical thinking—children's conception of number as a composite unit—might help predict their extant ability to engage in multiplicative reasoning (MR). Like Ulrich (2015, 2016), our study addresses Lamon's (2007) call for research linking students' additive and multiplicative structures. It sheds light on a novel aspect of this link—the vital role students' conception of number may play in developing more advanced concepts, such as fraction and ratio (Hackenberg, 2013).

We argue that a child's spontaneous use of an additive strategy (counting-on, doubling, BAMT) to solve a 1-digit addition word problem indicates the strength of the child's conception of number. Our focus is not on students' *potential to learn* to reason multiplicatively. Rather, we aim to predict students' current ability to use multiplicative reasoning based on the additive strategy they spontaneously use. We follow Kilpatrick's (2001) assertion of the need for statistical corroboration of the predictive power bestowed by conceptually sound models. Drawing on constructivist theory, researchers have used qualitative methods to develop conceptual models of students' additive and multiplicative reasoning. However, little work has been done in the field to test and corroborate such qualitative models. Our study follows Norton and Wilkins' (2009) lead, by providing new quantitative analysis to corroborate a central element of Steffe's (1992) model of children's mathematical thinking.

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CONCEPTUAL FRAMEWORK

The core of the constructivist framework for this study is the depiction of children's conception of number as an abstract, symbolized composite unit (Steffe, 1992). This conception allows a child to operate on a numerical symbol as a single "thing" and to decompose it into sub-parts. When reasoning additively, a child can mentally coordinate the *same type of unit* (e.g., 8 grapes + 7 grapes = 15 grapes). In contrast, when reasoning multiplicatively, a child can simultaneously coordinate *different levels of units*: items in each composite unit (1s), number of composite units, and a total number of 1s (Ulrich, 2015). For example, consider the problem: "Sarah wants to put 8 grapes into each of 7 baskets. How many grapes does she need?" A child reasoning multiplicatively can distribute items of one composite unit *(grapes per basket)* over another composite unit (*baskets*) to find the total number of items (1s) in a collection of composite units (*total of grapes*). Such a coordination requires conceiving of number as composite unit (Steffe, 1992; Ulrich, 2016).

We sharply distinguish a child's solution to a problem from conceptions that underlie her solution (Tzur et al., 2013; Ulrich, 2016). A student may correctly solve a 1-digit addition problem (e.g., 8+7) by spontaneously using additive strategies such as: counting-all (1, 2, …, 14, 15), counting on (8; 9, …, 14, 15), doubling (7+7=14; 14+1=15), BAMT (8+2=10; 10+5=15), or fact retrieval. If we focus on the correct solution, any of these strategies would suffice. Instead, we contrast them based on the strength of a child's conception of number that we infer to underlie each strategy.

Steffe (1992) used a criterion of number as a composite unit to claim counting-all does not indicate a conception of number. We also claim that counting-on indicates a weak conception; doubling an intermediate conception; and BAMT a strong conception. In each of those latter three strategies, a child could conceive of one addend as a composite unit. Yet, a child using counting-on does not decompose numbers into sub-parts other than 1s. Rather, she accrues, one after another, units of 1 that constitute the second addend. In doubling, the child could decompose one addend to create easy fact retrieval (8 is 7+1), then add two composite units in their entirety (7+7=14), and finally 'call-back' the decomposed 1 (14+1=15). In BAMT the child could both *decompose* one addend into units larger than 1 (7 is 2+5) and *integrate* them into another unit (8+2=10) as a means to add two composite units in their entirety (10+5=15). A child's use of decomposition indicates that she can operate on a number as a unit in and of itself, without constantly reconstituting it from 1s. Because a child using BAMT uses decomposition into and integration of sub-parts that are themselves composite units, we argue that spontaneous use of BAMT indicates a stronger conception of number than does doubling.

METHODS

This study was part of a larger project focusing on promoting and studying upper elementary teachers' shift toward a student-adaptive pedagogy (AdPed), and how such a shift impacts students' learning and outcomes. To this end, we developed and validated a written measure for assessing students' multiplicative reasoning (MR) (Hodkowski et al., 2016). The measure contains five word problems: one screener (1-digit addition) and four problems through which we intended to measure students' multiplicative reasoning. Our team includes language experts who helped design word problems appropriate for students learning English as an additional language.

In Problem #1 (screener), we intended for students to spontaneously use a strategy to add two 1-digit numbers (8+7). In Problem #2, we intended for students to iterate a composite unit (e.g., a tower of 5 cubes) to determine if it could constitute a larger composite unit (e.g., a tower of 24 cubes). In Problem #3 (MR), we intended for students to distribute items of one composite unit (3 cubes per tower) over another (6 towers) to find the total number of items in a collection of composite units (total cubes). In Problem #4, we intended for students to keep track of composite units (4 teams of 5 players each). We asked them to determine the correctness of a hypothetical student's (Joy) statement that, through 'skip-counting' by 5, she found there are 35 players in all. In Problem #5 (MR), given a total number of items (28 cookies), we intended for students to iterate one composite unit (4 cookies per bag) to determine the total number of composite units (bags) needed. In each of the MR word problems (#2-5), we included sub-questions that required students to fill in blanks with key, given information. For example, in Problem #4, students had to fill a given in the blank: "In each team there are ___ players." We included these sub-questions, in part, to assess students' comprehension of problem statements.

Our initial analysis of the interviews revealed a novel correlation that extended beyond our initial design: the spontaneous additive strategy students used to solve a 1-digit addition word problem (8+7) seemed linked with their score on the MR measure. Thus, we designed a follow-up, quantitative study (reported here) to collect and analyze data to examine this novel correlation.

Setting and Participants.

Participants were 4th graders (age ~10) at an elementary school in a large urban school district in the western USA. A total of 43 students—roughly 50% of all 4th graders in that school—completed the MR measure during an individual, clinical interview conducted by the first author. We excluded ten students from our analysis: 3 who used counting-all, 4 with no consent, and 3 who incorrectly responded to the sub-questions assessing students' comprehension of the word problems. The study sample thus consisted of 33 students (15 girls), all mainstreamed for math instruction. Most (85%) participants identified as non-white, including 17 (52%) Latino/a and 11 (33%) African-American students. Of the 33 participants, 45% were designated as English Language Learners, and three had Individual Educational Programs (IEPs).

We have established three important commonalities for this sample. First, all 33 students correctly solved Problem #1, using either counting-on, doubling, or BAMT. Second, 32 of them correctly responded to the sub-questions (fill givens in the blanks) intended to assess their comprehension of the problem statements. Third, the actions and time lapse between each student's reading and answering Problem #1 (from 4 to 30 seconds) indicated none solved 8+7 through fact retrieval.

Data Collection and Analysis.

The first author administered the MR measure to individual students, during clinical interviews lasting about 30 minutes. Each problem was first read out loud by the student or the interviewer. The student then solved the problem on her or his own, without assistance. After a student finished solving a problem, the interviewer asked follow up questions to gather further evidence of students' thinking. As each student solved Problem #1, to increase the likelihood of accurate inference of the additive strategy each student used, during the interview, he made notes of the child's actions and utterances—and the inferred additive strategy.

Students used two forms of doubling: (a) 7+7=14; 14+1=15 and (b) 8+8=16; 16-1=15. No statistically significant difference could be found between those two sub-groups. We thus combined them into a single category (doubling). We classified three ordinal levels of the independent variable: 1=incipient/weak (counting-on), 2=developing/intermediate (doubling), and 3=developed/strong (BAMT) conception of composite unit. We then conducted three tests. First, we used ANOVA to test whether means in solutions to MR problems (Problems #2-5) were significantly different for those three groups. Next, we used Kendall's Tau-b, a test of correlation that does not assume normal distribution or equal interval scaling. Finally, we used a t-test to compare between every pair of groups, supposing (based on the ANOVA) the comparison between counting-on and BAMT is the imperative one.

RESULTS

In this section, we present data analysis to substantiate our claim that the strength of a child's concept of number as a composite unit, inferred from her or his spontaneously used additive strategy (independent variable), can help predict the child's ability to reason multiplicatively (dependent variable). We begin with statistical analysis of all participants (N=33), followed by between-group differences.

Multiplicative Reasoning – All Participants.

In Table 1 we provide percentages of students who correctly solved each MR problem. We observe two important results. First, despite all 33 students solving 100% of Problem #1 correctly, they collectively solved less than 40% of each MR problem correctly. We interpret students' success on Problem #1 to indicate their ability to use additive reasoning and their difficulty with Problems #2-5 (MR) to indicate their lack of multiplicative reasoning. This contrast between additive and multiplicative reasoning lends support to researchers' theoretical predictions of a conceptual leap involved in shifting from additive to multiplicative reasoning (Hackenberg, 2013; Ulrich, 2015).

MR Problem	2	3	4	5
Percentage of correct solutions	33%	18%	36%	39%

Table 1: Percentages of students who correctly solved each MR problem.

Second, students' success rate was lowest (18%) in solving Problem #3. A paired-samples t-test comparing all students' solutions to Problem #3 and to the three other multiplicative problems shows non-significant difference with Problem #2, a statistically significant difference with Problem #4 (t=2.25, df=32, p=.032), and nearly statistically significant with Problem #5 (t=1.88, df=32, p=.07).

Between-Group Differences in Multiplicative Reasoning (MR).

In Table 2 we show percentages of students who correctly solved each MR problem, disaggregating the data by the spontaneous additive strategy students used to solve Problem #1. We found statistically significant differences among students when disaggregating by their spontaneously used additive strategy. The percentages of students who solved all MR problems correctly were highest for BAMT (56%), midway for doubling (34%), and lowest for counting-on (17%). ANOVA shows these differences are statistically significant (F=8.25, p=.001). Further t-tests on success rates for the four MR problems showed nearly statistically significant differences between counting-on and doubling (t=2.04, df=22, p=.053) and highly significant between counting-on and BAMT (t=4.29, df=23, p<.0005), but *not between doubling and BAMT*, possibly due to the smaller n of these two groups.

MR Problem	2	3	4	5	Across all 4 MR Problems
Counting on	13%	6%	19%	31%	17%
Doubling	50%	13%	38%	38%	34%
BAMT	56%	44%	67%	56%	56%

Table 2: Percentages disaggregated by students' spontaneous additive strategy.

A Kendall's Tau-b (KTb) test of correlation further demonstrates the linkage between students' additive strategy and the success rate on problems involving multiplicative reasoning (KTb= 0.5, p=.001). Data in Table 3 further highlight this: a child's spontaneous use of counting-on predicts a very low success rate on MR problems. Of students using counting-on, 94% correctly solved *at most* one problem. In contrast,

No. of MR problems Solved Correctly	0	1	2	3	4 (All)
Counting-on (N=16)	37.5%	56.3%	6.2%	-	-
Doubling (N=8)	12.5%	62.5%	-	25.0%	-
BAMT (N=9)	11.1%	11.1%	33.3%	33.3%	11.1%

Table 3: Percentages of students who correctly solved 0, 1, 2, 3, or 4 MR problems.

a child's spontaneous use of BAMT predicts a much higher success rate on MR problems: 78% of students spontaneously using BAMT correctly solved *at least two* MR problems (33.3%, 33.3%, 11.1%)

To examine the impact of spontaneous additive strategy on student success rate for each MR problem separately, we conducted ANOVA for between-group differences. Table 4 presents these results. Group contribution to this variance, calculated using S-N-K post-hoc statistics and t-tests (Table 5), showed statistically significant differences between counting-on and doubling (Problem #2), and between counting-on and BAMT (Problems #2, #3, and #4).

Problem No.	2	3	4	5
ANOVA	F=3.42 (.046)	F=3.25 (.053)	F=3.146 (.006)	-

Table 4: ANOVA of between-group differences for each MR problem.

Problem No.	2	3	4	5
Count-on vs. Doubling	t=2.1 (.048)	-	-	-
Count-on vs. BAMT	t=2.49 (.021)	t=2.47 (.021)	t=2.61 (.015)	-

Table 5: Independent samples t-test values of between group-pairs differences on each problem (equal variance **not** assumed; p-values in parentheses).

Results presented in Tables 4 and 5 indicate two main points that, combined, support our claim that the strength of a child's conception of number as composite unit holds predictive power for her current ability to reason multiplicatively. First, we focus on responses to Problem #4, on which students who used counting-on were most successful. To solve Problem #4 correctly, students needed to (a) determine that Joy's response is wrong (in 4 teams of 5 players each, there are not 35 players), (b) select an appropriate reason for Joy's mistake, and (c) figure out the correct number of *teams* that Joy counted (35 players would make 7 teams of 5 players each). Only three students (19%) who used counting-on could solve this problem correctly, seemingly by their ability to skip-count by 5s to arrive at 20. The other thirteen (81%) students were unsuccessful. Among those thirteen, ten students (63%) incorrectly selected "35" as the number of teams that Joy counted. We interpret the students' error to provide empirical evidence to support Ulrich's (2015) claim that such students rely on operating on 1s—a reliance that may *often be masked* by their successful performance when iterating familiar numbers, such as 5.

Students' performance on Problems #2, #3, #4 provides further support of Steffe's (1992) model. In each of these problems, a child would have to carry out the simultaneous, coordinated monitoring of the accrual of *both* 1s and composite units. Among the 16 students who spontaneously used counting-on to solve Problem #1, only two (13%) could solve Problem #2, only one (6%) could solve Problem #3, and only three (19%) could solve Problem #4. These data corroborate the prediction that students with weak composite unit—solving addition tasks by adding 1s—are unlikely to reason multiplicatively. In contrast, among the nine students who spontaneously used BAMT, five (56%) could solve Problem #2, four (44%) could solve Problem #3, and six (67%) could solve Problem #4. These data corroborate the prediction that students with strong composite unit—solving addition tasks by decomposing the second addend—are more likely to reason multiplicatively.

DISCUSSION

We examined how an element of Steffe's (1992) model—children's conception of number—might help predict their ability to reason multiplicatively. We provided analysis of the conceptual foundations of students' spontaneous use of three additive strategies—counting-on, doubling, and BAMT. Importantly, students' use of any of these strategies provides evidence that they have constructed a conception of number. Our study corroborated Steffe's model: the strength of a child's conception of number, as evidenced by their spontaneous use of an additive strategy, can help to predict their extant ability to engage in multiplicative reasoning. Specifically, a child who spontaneously uses counting-on is highly unlikely to engage in multiplicative reasoning. In contrast, a child who spontaneously uses BAMT is likely to do so. Keeping with Kilpatrick's (2001) assertion, our study thus contributes to the field's knowledge base by testing a long-known and well-articulated conceptual model that links, developmentally, students' additive and multiplicative reasoning.

Implications for Research.

We note three implications of this study. First, it opened the way for identifying 1-2 tasks that can indicate a child's likelihood for advanced ways of reasoning based on observable, lower-level solutions. A future, larger N study may confirm the predictive power of a child's additive strategy. Second, a related measure to the one we used could be developed to examine the linkage between a child's comprehension of a realistic word problem and her or his conception of number and/or multiplicative reasoning. Third, this study implies the need to carefully examine the design and findings of studies intended to determine the impact of an instructional intervention on student learning and outcomes. Lack of impact of such interventions may be rooted not in the intervention per se (Woodward & Tzur, in press), but in students' lack of a cognitive prerequisite that affords the intended learning (e.g., lack of strong enough composite unit; see Tzur, Xin, Si, Kenney, & Guebert, 2010).

Implications for Practice.

For practice, our study implies the possibility to use a quick measure (screener Problem #1) to assess the strength of each student's conception of number. In our current project, *teachers* are learning to use it so they: (a) link between a child's additive strategy and her MR, (b) can conduct short, task-based interviews to elicit students' strategies, (c) document the results of their assessments, and (d) *adapt their subsequent instruction to meet the needs of students in each group*. Teachers with whom we work seem to deeply appreciate the main goal for each student who uses counting-on and doubling: strengthen her or his conception of number as composite unit by learning to decompose addends into sub-composite units.

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ARE YOU JOKING? OR IS THIS REAL? INCREASING REALISTIC RESPONSES TO WORD PROBLEMS VIA HUMOR

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Research has shown that children have a strong tendency to exclude real world considerations when solving word problems. In this study, we investigated whether children would adapt their behavior when solving word problems in which realistic considerations are required (P-items) when these problems are embedded in a humoristic context as compared to when they are offered in a typical word problem solving context. 148 sixth graders solved 4 P-items in a humor condition versus a word problem condition. It was found that overall significantly more realistic responses were given in the humor condition, and this was the case for 3 of the 4 problems. Implications of these findings for further research and for classroom instruction are discussed.

THEORETICAL AND EMPIRICAL BACKGROUND

A major role for including word problems in the curriculum is that children should develop the skill to know when and how to apply mathematics in everyday life. For a long time, word problems have played this function without much critical concern, but during the last two decades, it has been shown that the current school practices do not at all foster in children a genuine mathematical modeling disposition. Apparently, as a result of their year-long participation in traditional mathematical word problem solving lessons, many children approach word problems in a superficial and artificial way. They just search or the mathematical operation(s) to perform with the given numbers, with little or no attention to the meaningfulness of their solution (Lave, 1992; Reusser & Stebler, 1997; Schoenfeld, 1991; Verschaffel, Greer, Van Dooren, & Mukhopadyay, 2009).

For instance, in a seminal study by Verschaffel et al. (1994) fifth graders solved so-called problematic word problems (P-items), in which the correct solution cannot be found by simply performing the mathematical operation. Instead, the reality of the problem situation has to be taken into account. An example is the rope item: “A man wants to stretch a rope between two poles that are 12 metres apart. He has only pieces of rope that are 1.5 metres long. How many pieces does he need?” From a genuine modeling perspective, the solver has to conclude that 12 / 1.5 = 8 pieces are insufficient, because the man also needs to tie the pieces together and also needs extra rope to stretch around the poles.

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Verschaffel et al. (1994) found around 15% of realistic answers to their set of P-items, a finding that was replicated many times across the world (for an overview, see Verschaffel et al., 2009). During the past 15 years several researchers have tried various manipulations to better understand the origin and development of this tendency, but also to counter it. Examples are studies in which pupils are alerted at the beginning of the test that some problems need careful consideration (Yoshida, Verschaffel, & De Corte, 1997) or are provided illustrations that represent the situation that is described in the word problem (see e.g., De Wolf, Van Dooren, Ev Cimen, & Verschaffel, 2014). At best, such manipulations had only minimal effects. The only significant improvements are obtained when children are confronted with more authentic formulations of the P-items (e.g., Palm, 2008), or when these P-items are transformed into performance tasks involving real-life goals and concrete materials (e.g., DeFranco & Curcio, 1997). However, these improvements did not transfer to word problems with realistic modelling complexities presented in a maths classroom context. So, improving children's inclination to react realistically when solving word problems in a maths classroom context remains a great challenge (e.g., Van Dooren, De Bock, Janssens, & Verschaffel, 2005). In the current study, we tried to improve children's tendency to react realistically to P-items that were used in previous research by means of a new manipulation, namely embedding them in a humoristic context rather than in the context of a typical word problem test.

The literature on humor (e.g., Ivy, 2013; Nicewonder, 1994) points to many possible advantages of using humor in educational settings: It can reduce stress and tension, it attracts attention, it can motivate and change attitudes, and it may stimulate children to see the problem situation from a different perspective. It is this last advantage that seems most relevant for the current study: Children need to see the situation described in the word problem from a different perspective. This aspect of humor is elaborated in the Incongruity Theory of Humor (e.g. Attardo, 1997; Feyaerts, 2008), which focuses on the cognitive processing of humoristic situations. More specifically, it focuses on humor that originates when situations allow for two possible interpretations of the same situation which are incongruent. Only one of these two interpretations of the situation – namely the more plausible one – occurs in the listener/reader. The joke's clue, however, is that the alternative, less plausible interpretation – the one the listener/reader did not think of – ultimately turns out to be true. The reaction to this unexpected experience of incongruity is one of laughter.

RATIONALE AND RESEARCH GOALS

The mechanism of incongruity leading to humorous experience as pointed out by Attardo (1997) and Feyaerts (2008) was used in our study. We specifically looked for jokes presented in the form of word problems involving two incongruent interpretations with respect to the acceptability and desirability to include real-world considerations into their solution. Our expectation was that by surrounding the P-items by jokes that specifically addressed the incongruity between sticking to the rules and norms of the mathematics classroom (= the more plausible interpretation) and using real-world considerations (= the less plausible interpretation) children would be more inclined to reason realistically when solving the P-items.

METHOD

Design and materials

148 sixth graders from four different schools in Flanders, Belgium were randomly assigned to a Humor Condition or a Word Problem Condition. Assignment to conditions happened on a random basis within classrooms. In both conditions, pupils solved four P-items adapted from Verschaffel et al. (1994). These 4 P-items, which can be found in Table 1, were offered in exactly the same way in both conditions. Figure 1 illustrates how this was done for one P-item, i.e. the swimming item.

The sheets containing these 4 P-items were embedded in a different booklet, depending on the condition children were assigned to:

- In the Humor Condition, the pages containing the P-items were alternated with pages containing jokes showing cartoon characters from the Calvin and Hobbes series. Each of these pages contained two jokes addressing the incongruity between sticking to the rules and norms of the mathematics classroom versus using real-world considerations. Children had to indicate which joke they found most humoristic (see Figure 2).
- In the Word Problem Condition, the pages containing the P-items were alternated with pages where pupils were asked to solve two word problems that would typically appear in a sixth grade mathematics textbook (except that they also contained the Calvin and Hobbes characters, see Figure 3).

Coding

In line with earlier studies using P-items (e.g., Dewolf et al., 2014; Verschaffel et al., 1994), pupils' reactions were coded as realistic (RR) whenever pupils gave some indication of making realistic considerations in their solution, either in their calculations or in their comments (e.g. by taking the realistic aspects into account in their calculations, by indicating that there is no correct numerical answer, or that the word problem is somewhat strange). When none of these elements were present in the answer and only straightforward arithmetical calculations on the given numbers were conducted, reactions were coded as non-realistic (NR). Interrater reliability of this scoring (based on a subset of 20% of all answers by the first and second author) was 100%. (Further subcategories referring to specific types of RRs were distinguished while coding the answers, but these are not discussed due to space limitations.)

A repeated measures logistic regression analysis was conducted to model the probability that a RR occurred. This was done using the GEE module in SPSS20, which allowed to correct for possible correlations due to the fact that four items were administered per pupil. We analysed the main effects of Condition (Humor Condition versus Word Problem Condition) and Item (Item 1, 2, 3, or 4), as well as their interaction.

Name	Formulation	Example of NR and RR
Item 1 Rope	Calvin and Hobbes play badminton in the garden. As a net, they use a rope that is stretched between two poles that are 12 m apart. They only have pieces of rope that are 1.5 m long. How many pieces to they have to tie together to stretch a rope between the poles?	NR: "*12 / 1.5 = 8 pieces of rope*" RR: "*More than 8 pieces because of the knots*" / "*You cannot know but more than 8*"
Item 2 School	Calvin and Inge attend the same school. Calvin lives at a distance of 17 km from school, and Inge lives at a distance of 21 km. How far do they live from each other?	NR: "*21 – 17 = 4 km*" / "*21 + 17 = 38 km*" RR: "*It is not clear where they live*" / "*Between 4 and 38 km*"
Item 3 Swimming	When Calvin goes to school, Hobbes sometimes takes a swim. His best time to swim 25 m is 20 seconds. How long does it take Hobbes to swim 500m?	NR: *"20 * 20 = 400 seconds"* RR: "*Probably more than 400 seconds*" / "*You cannot know as he will get tired*" / "*He can't keep that pace*"
Item 4 Christmas	It is almost Christmas. Each year, Calvin's mom buys a large Christmas tree. The man who is selling the Christmas trees told Calvin that he sold 243 Christmas trees in December. How many do you think he will sell in January, February and March altogether?	NR: *"243 * 3 = 729 trees"* RR: "*Probably less than 729*" / "*Impossible to tell*" / "*Not many!*"

Table 1: The P-items with examples of non-realistic (NR) and realistic (RR) reactions (items adapted from Verschaffel et al., 1994)

Figure 1: Illustration of the presentation of P-items (c.q. the swimming item)

Figure 2: Exemplary worksheet included in the Humor Condition test booklet

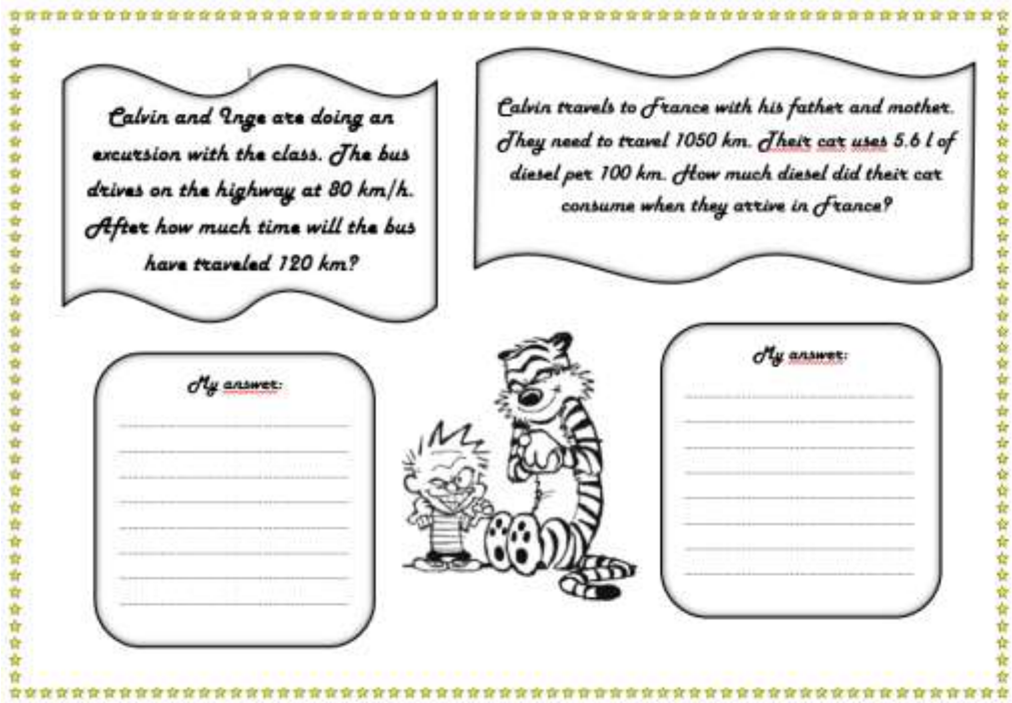

Figure 3: Exemplary worksheet included in the Word Problem Condition test booklet

MAIN RESULTS

The number of RRs to the various P-items is given in Table 2. The GEE analysis indicated a main effect of Condition, *Wald ChiSquare* (1, N = 148) = 6.903, p = .009. While in the word problem condition, only 16.00% realistic answers were given, this was 28.08% in the humor condition. The interaction between Condition and Item was also significant, *Wald ChiSquare* (3, N = 148) = 11.504, p = .006, indicating that the effect of the condition was not identical for the four P-items. Additional pairwise

	Item 1	Item 2	Item 3	Item 4	Total
Humor condition (n = 73)	6.85	13.70	28.77	63.02	28.08
Word problem condition (n = 75)	7.99	1.33	9.33	45.33	16.00
Total	7.44	7.44	18.93	54.60	21.97
p-value for pairwise comparison	.789	.004	.002	.041	.009

Table 2: Percentage of realistic responses to the four P-items, by condition

comparisons for Condition per item indicated that only for Items 2, 3, and 4 there was an effect of condition.

So, as expected, embedding the P-items in humoristic learning materials had a beneficial effect, as pupils gave almost twice as many realistic reactions compared to the word problem condition. This is one of the first studies wherein such a large effect was found in a short term intervention wherein P-items were used in their traditional formulation, and within the actual mathematics classroom. The finding that pupils showed a significant adaptation of their word problem solving behavior in the humor condition, suggests that humor can be productively used to address their beliefs about the role of realistic considerations in mathematical word problem solving.

CONCLUSION AND DISCUSSION

In several previous studies, attempts were made to enhance pupils' tendency to make real-life considerations when solving word problems with realistic modeling complexities (P-items) (e.g., Dewolf et al., 2014; Yoshida et al., 1997). Typically, short-term interventions that stuck to offering such word problems without embedding them in more authentic contexts and/or offering concrete materials had little or no effect. In the current study, however, we found that eliciting humor by alternating the P-items with jokes was a succesful strategy in eliciting realistic reactions in pupils. Given the disappointing results of previous research, this is an important and promising finding.

However, there are still several unclarities that should be addressed in further research. First, the impact of humor was different for the various P-items, and for Item 1 there was even no effect at all. The present study does not allow to understand why embedding the P-items in a humoristic setting seems to have a differential effect on different types of P-items. Further research could shed light on this, for instance by asking children to think aloud while solving the problems and by probing the possible realistic considerations they make during the different stages of the solution process.

Second, this study investigated the momentaneous effect of humor on the number of realistic reactions, and did not look at effects in the long run. Further research could investigate whether transfer would occur, in the sense that also after working with the humor materials, when confronted with mathematical word problems involving realistic modeling complexities in a regular school setting, pupils would continue to give more realistic reactions. Further, it is worth investigating whether children's attitudes towards solving word problems would change as a consequence.

The current study focused on only one specific mechanism of humor that may be beneficial for stimulating children's word problem solving. Various other potentially beneficial effects are suggested in the literature. If future research supports these results, the inclusion of humor at some points in classroom practice may become part of the range of instructional strategies for stimulating children to make use of their real-world knowledge when solving word problems.

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NUMBER LINES: OBSERVED BEHAVIOURS AND INFERRED COGNITIONS OF 8 YEAR-OLDS

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A number line is a mathematical representation that is widely used in early mathematics education, yet the mathematical meanings of its structure and conventions are ambiguous. The purpose of this paper is to explore how 8 year-old children use filled and empty number lines in the representation of their addition and subtraction strategies. Task-based interviews were used to observe mathematical behaviours and infer cognition. The findings highlight the challenges young students face in understanding multiple forms of number lines and integrating them with their own internal reasoning.

INTRODUCTION

The use of number lines is foundational in early mathematics education. Although ubiquitous in classrooms, there is not just one standard form of number line, but rather a variety of forms, with different conceptual bases, semiotic structures and conventions for their use, as presented in the analysis conducted by Teppo and van den Heuvel-Panhuizen (2014). Although various forms of number lines are potentially powerful tools for teaching and learning mathematics (Rivera, 2014; Steenpass & Steinbring, 2014; van den Heuvel-Panhuizen, 2008), developing children's conceptual understanding across a variety of forms has been shown to be problematic (Deizmann, Lowrie, & Sugars, 2010). The depth of teachers' understanding of the different forms of number lines has also been questioned (Bobis & Bobis, 2005).

During the first several years of schooling, it is common to expect that children will understand two or more different forms of number line. Insufficient research has been conducted into the depth of young children's understanding of these forms of number lines and the challenges they face in transitioning to new forms. The results of such research have the potential to assist in the development of improved pedagogy, and therefore improved student understanding.

The purpose of this paper is to analyze the representational and conceptual ways in which 8 year-old students utilize both structured and empty number lines for basic addition and subtraction operations.

THEORETICAL FRAMEWORK

The epistemological framework of this study draws on the moderate constructivism espoused Goldin (1990, 1998) in which mathematics is not completely extrinsic - there being an element that exists mentally, through insightful processes unique to the

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 305-312. Singapore: PME.

individual (Cobb, Wood & Yackel, 1991; Goldin, 1990). Representation plays a critical role for abstract thinking. Therefore, an understanding of the way students' construct their own *internal* representations in relation to their mathematical thinking, is necessary to fully comprehend the formation of knowledge (Goldin & Shteingold, 2001). Within this theoretical perspective it is helpful to think of 'representation' as a process rather than a product. A component of the representation process is the interplay between *internal* and *external* representations– including the external representations created by the child, and interactions with representations created by others (such as the teacher). The term *interplay* implies a two-way dynamic interaction between internal and external representation, which blurs the boundary between the two (Goldin, 1998).

FORMS OF NUMBER LINES

Teppo & van den Heuvel-Panhuizen (2014) explored the external form of the number line and constructed a rationale to categorize its various forms, arguing that the various number line structures depict different meaning. Their research identified a large variety of number lines and with each representation they compiled a classification framework categorizing; visual features, types of numbers used and how the numbers were represented within the number line. Their research addresses the semiotics of number lines and used Duval's (2014) definition of "figural units" where understanding and conceptual knowledge is composed based on visual features within the representation. Of particular interest in this study are the categories of *filled number lines* and *empty number lines*, because these two forms are typically present in the mathematics curriculum for the first several years of schooling.

Filled number lines (also called structured number lines) consist of a line, often including an arrow at either end, equidistant markings, the start point labelled with 0, followed by the sequence of whole numbers. Epistemologically, this particular form works as an external representation of the cognitive act of counting, and for young students acts as a visual catalyst for mathematical thinking and early additive calculation (Teppo & van den Heuvel-Panhuizen, 2014). Even though this form is a ubiquitous, traditional graphic, its mathematical meanings are ambiguous, resulting in individualized interpretations by children (Steenpass & Steinbring, 2014). For example, when children attend only to the point marks and not the interval between them (referred to as measurement-based thinking), conceptual confusion arises (Diezmann, Lowrie & Sugars, 2010).

Empty number lines begin with no structural features, as their purpose is for children to translate knowledge from their own internal representations of mental arithmetic strategies into an external representation. In doing so, they have to create their own structure to order and position numbers, and use symbols such as arcs to depict operations. Children's use of empty number lines has the potential to both support and reveal arithmetic reasoning (Bobis & Bobis, 2005; van den Heuvel-Panhuizen, 2008), but only if the child engages in the dynamic interplay between internal and external representations through mathematical thinking.

METHODOLOGY

Task-based interview was selected as the most suitable approach for exploring children's internal/external representation related to number lines. As internalised representation is enigmatic, research depends on the premise that it can be cautiously inferred from externalised representation processes, such as drawing and verbalising. Task-based interview has the dual purposes of observing mathematical behaviour and drawing inferences about the interviewee's cognitions (Goldin, 1993), making it well suited to the aims of this study.

The study took place in an Australian school in the suburbs of a major city. Six children, aged between between 7 and 9 years, from the same Year 3 class (4th year of primary school), where interviewed individually. Participant selection involved the teacher's recommendation regarding comfort level in talking about number tasks, and aimed to a cover a range of mathematics achievement levels. The age level was chosen because, according to the mathematics syllabus (BOSTES, 2012), the children would have had substantial exposure to both structured and empty number lines, and would be beginning to use number lines to locate basic fractions.

Five tasks were designed, requiring the addition or subtraction of one and two digit numbers and the use of structured/filled, empty and partially structured number lines. The interviews were digitally recorded for both vision and sound. The students were asked to explain each solution as soon as it was completed, and probe questions were used to elicit further explanation when needed.

Analysis involved repeated viewing of the videos in conjunction with the students' drawings, guided by three questions: a) What calculation strategy was used?; b) What number line structures and conventions were used?; and, c) What was the relationship between the calculation strategy and the external representation?

This paper reports the results for only three of the children and for only three of the tasks (See Table 1). As all six children responded differently on every task, the three participants do not represent any commonality in the cohort, but rather serve to illustrate the diversity of responses.

RESULTS

Tables 1, 2 and 3 present the responses of three students to the three tasks, and include a commentary drawn from the videos of the children's actions and audio of the verbal explanations. A discussion of the results occurs after all three cases have been presented.

The case of Harry

Harry was considered by his teacher to be working beyond syllabus-level expectations in mathematics. Overall, his responses indicate harmony between his mental strategies, internal representations and external representations on number lines (see Table 1).

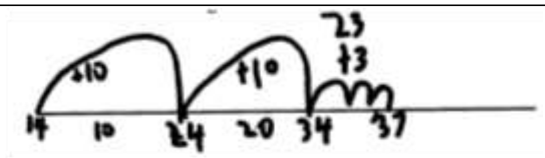
Student's drawing	Commentary derived from video/audio
 Figure 1a: Task 1. (7 + 15), blank paper	"Start at 15, then I'm breaking up the 7" Had a mental strategy that seemed to be leading (but almost simultaneous with), the steps in the drawing construction. Correct answer 22.
 Figure 1b: Task 2. (25 – 8), fully structured pre-drawn number line, 0 to 30	Started at 25, marked jumps backwards by ones to give correct answer 17. "I wanted to make sure what I worked out in my head was correct". Then started at 25 again and drew jumps of 2,3,3. "I was just looking for another way to do it."
 Figure 1c: Task 3. (14 + 23), blank line	Started at 14. Labelled jumps and positions on number line as he went. "I'm breaking up the 23 on my number line, using tens…..". Correct answer 37.

Table 1: Results for Harry

Harry had knowledge of the structure and conventions of number line use, though he did not bother to draw the actual line in Task 1 (Figure 1a). He was comfortable with only drawing the segment of the number line that was relevant to performing the operation – so didn't need to mark 0, and only marked the numbers that were subtotals and totals in his calculation methods. Arcs were used to indicate the addition or subtraction of an amount. The size of the arcs (jumps) suggested the magnitude of the number being represented, that is, a large jump for 10, small jump for 1 (Figure 1c).

Harry demonstrated flexible mental strategies in addition, using 'bridging to the decade' in Task 1, and place value knowledge in Task 3. He illustrated his mental strategies on the empty number lines as he progressed through the steps. For the subtraction (Task 2), Harry used a less sophisticated method of 'counting back by ones', but immediately looked for a more efficient method. Unlike the empty number line tasks, Harry began solving Task 3 using the number line as as a calculation tool, rather than reasoning with a mental strategy. It appeared he may have been influenced by the presence of the fully structured and labelled line.

The case of Mindy

Mindy was considered by her teacher to be achieving syllabus-level expectations in mathematics. In her responses we see a disconnect between her mental calculation strategies, mental representations and the external representations of number lines (see Table 2). She does not comprehend how the external number line structures can be used to model her strategies for addition and subtraction.

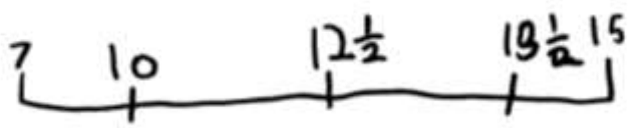
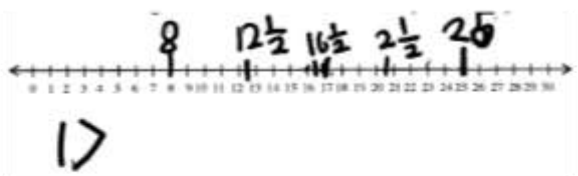
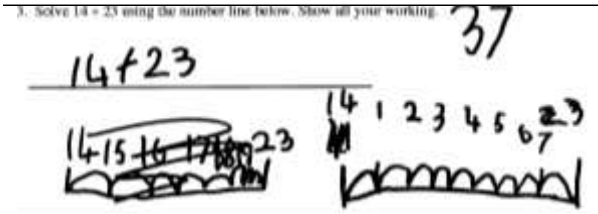
Student's drawing	Commentary derived from video
 Figure 2a: Task 1. (7 + 15), blank paper	Locates end points of 7 and 15, then marked the midpoint, marked another two points either side. "I found out where the middle point was." Thinking time, then labelled the midpoint 12½, then labelled 10 and 13½. Described "jumps" of 3, 2½, 1½ to "land on" 15. Did not calculate the addition.
 Figure 2b: Task 2. (25 – 8), fully structured pre-drawn number line, 0 to 30	Read out the task said, "15 – 8, (pause) *to* 8". Put a mark on 8 then 25. Gestured counting back by ones from 25, then forward by ones from 8, to locate 16½ in the middle. Repeated similar procedure to locate 12½ and the midpoint between 16½ and 25, incorrectly labelled as 2½. When asked for the final answer, she counted back from 25, gesturing jumps of one, to get the correct answer 17.
 Figure 2c: Task 3. (14 + 23), blank line	Drew and interval from 14 to 23. Marked in some jumps. Crossed it out, "there's not enough room". Redrew the interval, and drew 9 jumps. Labelled the landing points from 1 to 7 (not including the final one at 23). Paused then wrote 37. When asked why 37, "I knew that 23 + 7 = 30, then I just added another 7".

Table 2: Results for Mindy

Mindy has some knowledge of number line structure, recognising number sequence, and equal distance between numbers. She also understands that halves can be located between numbers. Mandy is focused on the length of the interval between the two numbers presented in each question and does not work beyond these upper and lower limits on her number line. In Tasks 1 and 2 she is fixated on halving and quartering the distance (Figures 2a & 2b).

Mindy did not use the number lines to reflect her calculation strategies. She did not calculate the addition at all in Task 1, apparently being completely distracted by constructing her number-line interval. In Task 2, when pressed for an answer, Mindy ignored her complicated number-line labelling and simply counted back from 25 by ones. Similarly, in Task 3, when asked for an answer, she abandoned her drawing and quickly used the efficient mental strategy of 'bridging to the decade'.

The case of Nina

Nina was considered by her teacher to be working below syllabus-level expectations in mathematics. She appeared to lack mental calculation strategies and so used the external number line as a tool for reaching a solution. However, her limited understanding of the structures and conventions of number lines prevented her from effectively constructing her own number lines. (See Table 3).

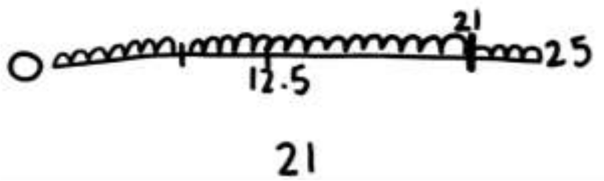
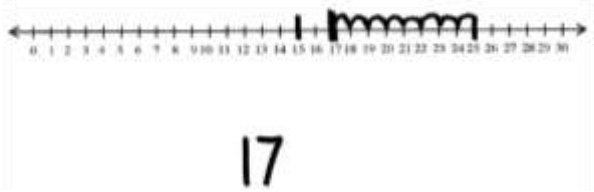
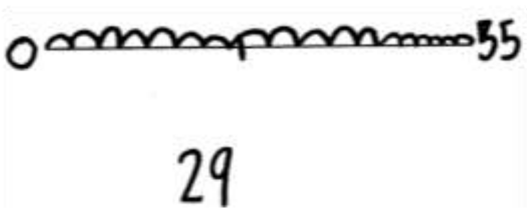
Student's drawing	Commentary derived from video
 Figure 3a: Task 1. (7 + 15), blank paper	Drew line and labelled ends 0 and 25. "I knew the answer would be somewhere between 1 and 25". Marked (but not labelled) the midpoint. Represented the 7 and 15 by drawing 7 jumps from 0, then drew 15 more. Counted 4 jumps back from 25 then labelled 21. Silently calculated the midpoint to be 12.5 and labelled it.
 Figure 3b: Task 2. (25 – 8), fully structured pre-drawn number line, 0 to 30	Marked 15, then 25. "I wanted to find the middle between 0 and 30 so I could … I don't know" (shrugged and smiled). Then counted back from 25 with jumps of 1, to give correct answer 17.
 Figure 3c: Task 3. (14 + 23), blank line	Marked midpoint then labelled ends 0 and 35. "Because I thought the answer would be between 0 and 35". From 0, drew 7 jumps, paused, drew 5 more. Then drew 6 smaller jumps back from 35. Wrote 29. "I counted up to 14, which is about there (points to midpoint). From there I counted up to 23. Then I counted back 6 (from 35) and found myself at 29".

Table 3: Results for Nina

Nina comprehended a number line as an interval, starting at 0, with an upper limit labelled at the other end, and the midpoint marked. She found locating other numbers on a blank number line challenging. In Tasks 1 and 2, her jumps represented counting-by-ones (Figures 3a & 3b). However, in Task 3 the jumps represented counting-by-twos at first, then smaller jumps for counting-back-by-ones (Figure 3c).

Nina did not demonstrate any mental addition strategies, although it is possible that the request to utilise number lines may have distracted her from attempting mental calculation. In Tasks 1 and 3 she represented the two separate addends, but did not combine them by 'counting-through' or 'counting-on', instead landing on an unknown position on the number line. To find out what the number at this position was, Nina counted back from her upper limit until she physically arrived on the spot where she had ended her forward count – ignoring the spatial/measurement inaccuracies of her drawing. She could not articulate a mathematical reason for doing this. Nor could she give the mathematical reason for locating the midpoint of her selected interval. Nina relied on her drawings of number lines to work out the answers, but was only successful when provided with the fully structured and labelled number line (Task 2, Figure 3.2). On an empty number line, Nina was unable to effectively visualise the missing equidistant, labelled points.

DISCUSSION AND CONCLUSION

On the whole, the findings of this study reinforce a conclusion reached by other researchers – that the number line is only a useful tool for a child's mathematical activity if the child fully comprehends the meaning of its structural components and its function as a model for mathematics concepts (Steinpass & Steinbring, 2014). Harry's task responses exemplify the dynamic interplay between internal and external representation of operations described by Goldin (1998).

The findings of this study also highlighted that a child can possess the mathematical knowledge for effective mental (internal) calculation strategies, but be unable represent them externally using a number line, as illustrated by Mindy's task responses. Mindy was able to externalise her mental strategies through speech, but the number line was not part of her representational system for operations.

The preoccupation of two of the children with defining the structure of the number line as an interval with a midpoint, requires further investigation. This approach has received little attention in previous literature. A possible explanation is that the class may have begun learning about placing sequences of halves and quarters (as specified in the syllabus) on a number line. Mindy and Nina appeared to perceive their interval as being 'a whole', and mistakenly applied iterative halving to set up the structure of their number line. The implication is that teachers need to be aware of the conceptual pitfalls of changing to a new form of number line, and the complexity of reconciling the new form with 'old' forms.

This study has highlighted the need to more thoroughly investigate children's understanding of, not only structured and empty number lines, but also rational number lines. More specific advice to teachers about effective pedagogy is needed to maximise opportunity for children's comprehension of different forms of number lines. The extensive confusion of children revealed in this small sample of 8 year-old students calls into question the wisdom of introducing a third form of number line at such an early stage in mathematics education.

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ASPECTS AND BASIC MENTAL MODELS ("GRUNDVOR-STELLUNGEN") OF BASIC CONCEPTS OF CALCULUS

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This paper discusses Aspects and Basic Mental Models ("Grundvorstellungen") in the development of basic concepts of calculus, the concepts of limit, derivative and integral. The focus is put on perspectives that are relevant, when these concepts are first introduced in high school or college. We distinguish between mathematically motivated Aspects and Basic Mental Models or "Grundvorstellungen" which are associated with the conceptual interpretation of these concepts that give them meaning. We will discuss (and clarify) both the differences and the relationships between these perspectives, which are central to the students' understanding of the respective concept. Finally, we integrate these perspectives into a framework of a theory of concept understanding.

ASPECTS AND BASIC MENTAL MODELS (BMMS) OF MATHEMATICAL CONCEPTS

With respect to the concepts of calculus, there are numerous didactic ideas, suggested teaching methods, empirical studies and practical investigations within the framework of curricula and mathematical textbooks. Nevertheless, Rasmussen et al. (2014) come to the following conclusion:

> "While the past several decades of research in calculus has contributed to better understanding of mathematical thinking, learning, and teaching in areas such as limit, derivative, and integral, too much research remains isolated and uncoordinated." (p. 508)

In the paper, we strive for a foundation to overcome this problem. To this effect, mathematical Aspects of limit, differentiation and integration are identified, together with associated BMMs or "Grundvorstellungen" (Hofe, v. et.al 2005), and the relationships between them. This structures the complex relationship, enabling these central concepts in calculus to be considered and studied from multiple perspectives with respect to mathematics education.

The concepts of Aspects and BMMs

Students' BMMs of mathematical concepts have been discussed in German-language pedagogy and the didactics of mathematics for more than 200 years, for example, by Pestalozzi, Herbart or Kühnel. BMMs give meaning to content-based *Aspects* of a mathematical concept, providing relations to meaningful contexts. This is a crucial

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prerequisite for working meaningfully with a concept. We define two expressions that are basic for the present article:

An *Aspect* of a mathematical concept is a subdomain of the concept that can be used to characterize it on the basis of mathematical content.

A *Basic Mental Model (BMM)* or *Grundvorstellung* of a mathematical concept is a conceptual interpretation that gives meaning to it.

The relation *"Aspect – BMM*" of a given mathematical concept is not one-to-one. An Aspect of a mathematical concept can provide a basis for several BMMs. Vice versa, a specific BMM can be developed with respect to several Aspects and give them meaning.

The concept of BMM can be used in both a prescriptive and a descriptive sense (see Hofe, v. et al. 2005): BMMs as a *prescriptive* notion are the answer to the question: How should students generally and ideally think of a given mathematical concept? Supporting students in developing these BMMs is one of the objectives of mathematical teaching. Thus, they provide teachers with guidance for organizing lessons. In contrast, BMMs as a *descriptive* notion answer the subject-matter didactic question of how a given student actually thinks about a given mathematical concept. BMM in this sense are the result of individual learning processes.

Relation to the idea "Concept Image – Concept Definition"

BMMs of mathematical concepts can be considered within the theoretical framework of "Concept Image – Concept Definition". These terms have been used in the didactics of mathematics since the early 1980s to distinguish between technical issues of a concept and the associated mental images (e.g. Vinner & Hershkowitz 1980; Tall & Vinner 1981; Bingolbali & Monaghan 2008, p. 31). "Concept Definition" refers to the formal or explicit definition of a particular concept. "Concept Image" refers to all individual mental images identified with the concept. A recurring problem is that the Concept Image associated with a given Concept Definition is very narrow. In addition, students are in danger of drawing conclusions about the Concept Definition by generalizing a Concept Image that focuses exclusively on certain special cases (Vinner 2011, p. 248). This danger is particularly present in the basic concepts of calculus, given that in specific classroom environments and especially in exam assignments, there is a bias towards calculation-oriented exercises, which are easily practiced beforehand on a formal, symbolic level.

The relations *"Concept Image – BMM"* and *"Concept Definition – Aspect"* can be described as follows: A Concept Image may contain several individual BMMs that conceptualize different perspectives of that concept. Individual BMMs are central components of a valid Concept Image. These BMMs give meaning to mathematical concepts that may be studied with respect to various Aspects. Each of these Aspects may be expressed by one of the various Concept Definitions that one reads in textbooks. Thus, a Concept Definition is a specific realization of an Aspect. These linkages are shown in Figure 1, which illustrates that the relation between Concept Definition and Concept Image is highly non-trivial. This relation may become more tangible by taking BMMs and Aspects and their relations into account.

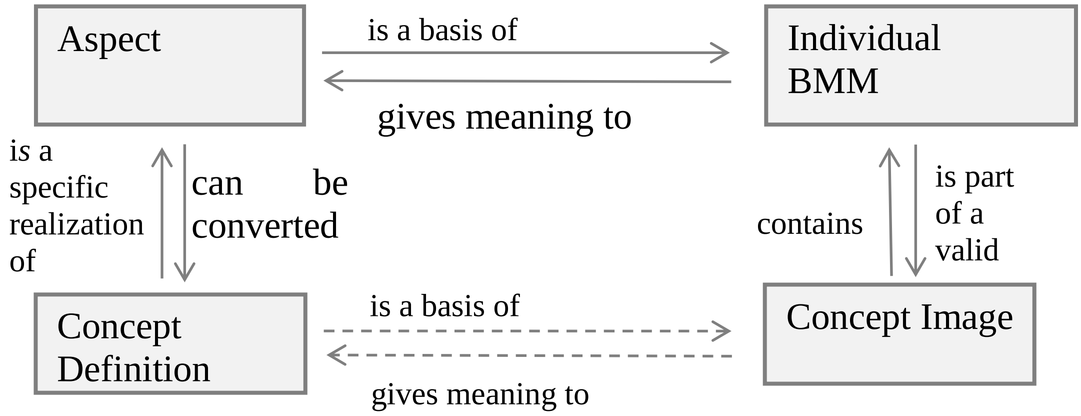

Figure 1: Relations between Aspect, BMM, Concept Definition and Concept Image

In an educational process that is based on understanding the central mathematical basics of calculus, it is the aim to develop Aspects and BMMs of calculus in a way that makes it possible to proceed – later – to strongly formal notions.

THE LIMIT CONCEPT

(The concept is presented here only in a snapshot. For more details see Greefrath et al. 2016).

Dynamic and static Aspects and the BMMs "approach" and "neighborhood"

Up to the 1960s, calculus lessons at German (and many European) high schools had been developed in close association with university mathematics. The sequence concept served as a basis for the limit concept and therefore, the "ϵ-n_0-definition" for the limit of a sequence. This strict approach to the limit concept in relation to the sequence concept, had been widely criticized and some different alternatives concerning the approach of calculus in high schools were developed; especially the "intuitive limit concept", which is based on the college lessons of Serge Lang (1927-2005) und Emil Artin (1898-1962). Nowadays, the formal definition of the limit is only of minor importance in European high school curricula and the conception of the "intuitive limit concept" is emphasized (see Törner et al. 2014). The intuitive limit concept builds on dynamic ideas with alterable variables or values seen as "fluent" or successively discretely changing. Especially the gradual discrete approach to "infinity" is, in terms of "walking along" the natural number line, an intuitive and basic concept and a key term in the development of the limit concept. Perceiving the *approach* of the sequence values to a fixed value while the "n-values" are converging towards infinity, is the basic intuitive perception of a limit. We associate the dynamic Aspect of limit with the BMM *approach*. The BMM *neighborhood* associated with the static or formal Aspect of Limit is based on the idea that all values of a sequence (or function) – starting with a special n_0-value – are located in even the smallest environment of the limit value.

THE CONCEPT OF DERIVATIVE

Differentiation can be defined in various ways and we elaborate on this by defining two Aspects of this concept.

The Aspects "limit of difference quotient" and "local linearization"

There are two main Aspects of the concept of derivative. One Aspect is differentiation as the limit of a difference quotient, another Aspect stems from the fundamental idea of approximation and is called local linearization. (These two Aspects are well-known and we do not explain here in details. See Greefrath et al. 2016.) From a didactic perspective, these two Aspects of differentiation correspond to different concept images for students. The Aspect of differentiation as the limit value of a difference quotient supports concepts of speed and rates of change. The interpretation of differentiation as a local linear approximation helps students to understand the error between the optimally approximating linear function and the original function and about the opportunity of describing the function as linear in a small neighborhood—its graph appears as a straight line when zoomed in at one particular point.

The BMMs "Local rate of change", "Tangent slope", "Local linearity" and "Amplification factor"

(We refer to these well-known perspectives quite generally. For more details see Greefrath et al. 2016.)

Local rate of change: The rate of change is an interpretation of the difference quotient. The *local (or instantaneous)* rate of change can be obtained as the limit of the difference quotient (first Aspect). The interpretation of the content of this Aspect (particularly by means of dependent entities), in case the differences are regarded as changes, is the BMM "local rate of change".

Tangent slope: Tangents are understood as lines that are *locally* tangential to the graph. In order to judge whether a line is a tangent or not, it is sufficient to look (graphically in the verbal sense, algebraically in a metaphorical sense) at an arbitrarily small interval around the point in question. The connection between this BMM and the associated Aspects is quite intricate. From the point of view of the developed mathematics, the derivative is the primary object and the tangent is defined only in terms of it. However, the genetic perspective of how mathematics develops, one may say that the tangent is an intuitive concept that can be modeled by either of the Aspects described above.

Local linearity: Curves can be approximated by piecewise-linear curves. A comprehensive, explicit BMM of local linearity includes e.g.: 'when zooming in very close to a point of the graph of a differentiable function, one sees an almost straight line segment' or 'for small changes in argument, the function is essentially linear, so that it can be approximated with a linear model'.

Amplification factor: If there is a functional relationship between (two) parameters, changes or uncertainty, such as errors in measurement, in the independent parameter induces changes in the dependent parameter. The BMM of amplification factor for small changes is that they are proportional.

Overview: Aspects and BMMs of the concept of derivative

The two Aspects and four BMMs are summarized in Figure 2, including the relations between the different Aspects and BMMs. The connecting lines indicate that the Aspect is a basis of the related BMM and that the BMM gives meaning to the Aspect.

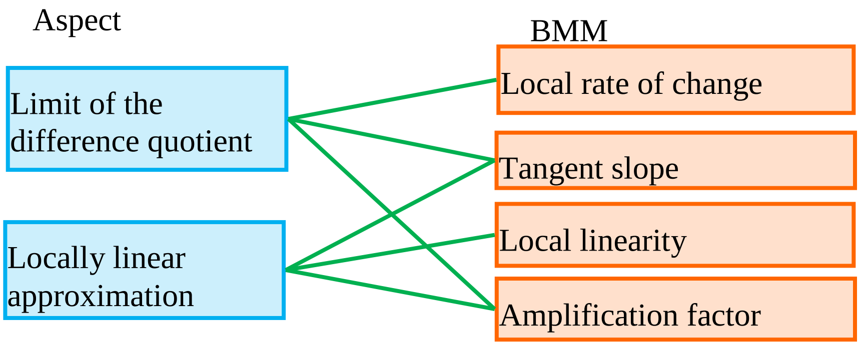

Figure 2: Aspects and BMMs of differentiation

ASPECTS AND BMMs OF THE CONCEPT OF DEFINITE INTEGRAL

The Aspects "Product sum", "Antiderivative" and "Measure"

Product sum: To define the definite integral in the sense of Riemann, you have to build product sums $\sum_{i=1}^{n} M_i \cdot (t_i - t_{i-1})$, with t_{i-1} and t_i points of a partition of a given interval [a, b] and M_i the supremum or infimum of f on the subinterval $[t_{i-1}, t_i]$.

Antiderivative: Given a function $f: I \longrightarrow \mathbb{R}$ defined on an interval I, we say that $F: I \longrightarrow \mathbb{R}$ is an *antiderivative* of f, if F is differentiable and $F' = f$. By defining definite integrals as antiderivatives, it becomes clear that the operations of differentiation and integration are mutually opposing.

Measure: Definite integrals are used to measure length, area and volume in the context of their measure Aspect, and are interpreted as a measure for certain representatives of these physical quantities. Measure theory and Lebesgue integral can be considered as the mathematical foundation of the measure Aspect of integration.

These Aspects provide interpretations of three separate subthemes of the concept of definite integral. Whereas the product sum Aspect primarily emphasizes developing the concept of a definite integral from the Riemannian approach, the antiderivative

Aspect highlights the Fundamental Theorem of Calculus and thus underlines the link between integration and differentiation. The measure Aspect, on the other hand, illustrates the application of integration in making measurements and the link to the Lebesgue integral. Narrowing down viewpoints to one or more specific Aspects of integration is generally seen as questionable (Huang 2012, p. 167).

The BMMs "Area", "(Re)Construction", "Average value" and "Accumulation"

Area: The BMM of definite integrals as area, emphasizes one of the applications of definite integrals within the framework of its measure Aspect.

(Re)Construction: By construction or reconstruction in the context of integration, we mean both the (re)construction of a quantity from the given information about rates or speed and the (re)construction of one of the antiderivatives of a given function. Examples: the reconstruction of distance travelled from velocity data or the reconstruction of the net amount of water left over, using data on inflow and outflow for a given container.

Average value: The technical basis of the BMM of average values is the mean value theorem. This BMM is therefore associated with the idea of forming a rectangle with the same area as a given region delimited by a curve.

Accumulation: The BMM of accumulation builds on a suitable interpretation of product sums that tend towards the definite integral as their limit. The intended meaning is the aggregation or cumulative summation of partial products to form a product sum. Example: the physical work as the scalar product of force and distance vectors product sum: $= \vec{F_1} \cdot \vec{s_1} + \vec{F_2} \cdot \vec{s_2} + \vec{F_3} \cdot \vec{s_3} + \cdots + \vec{F_n} \cdot \vec{s_n}$.

Overview: Aspects and BMMs of the concept of definite integral

Figure 3 gives the links between Aspects and BMMs, which together describe the concept of definite integral from both technical mathematical and subject-matter didactical perspectives.

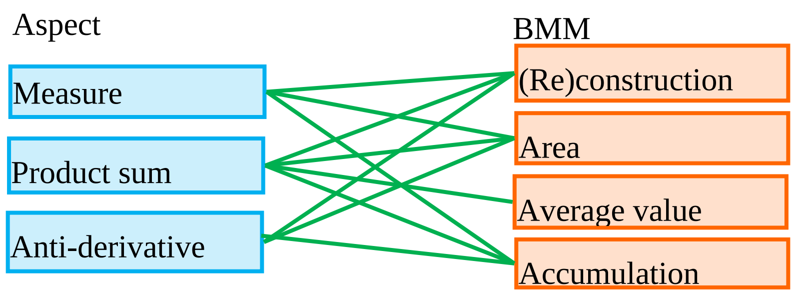

Figure 3: Aspects and BMMs of integration

The characterization of Aspects and BMMs for integration are not used consistently across the literature. In particular, there is often no differentiation between Aspects and BMMs.

ASPECTS AND BMMs IN THE CONTEXT OF CONCEPT UNDERSTANDING

The *Aspects* and *BMMs* of limit, differentiation and integration are now classified according to the process of concept development. To describe the level of proficiency in more detail, we add four categories or levels of Concept Understanding based on Vollrath (1984). The level of *intuitive concept understanding* enables us to use simple examples and to compare different representations. The level of *subject matter concept understanding* allows a concentration on properties and characterizations – in different representations – of the concept. The level of *integrated concept understanding* enables seeing relations to other concepts, as well as relations between properties of the concept. The level of *critical concept understanding* enables the indication of formal definitions, argumentations and proofs.

Figure 4 shows the relation between Aspect and BMM (Grundvorstellung) in differentiation. This can be represented by a 2 x 4 matrix (with 6 non-empty cells). If the "dimension" of Concept Understanding is added for each of the 4 categories, this can be represented by a 3d-model.

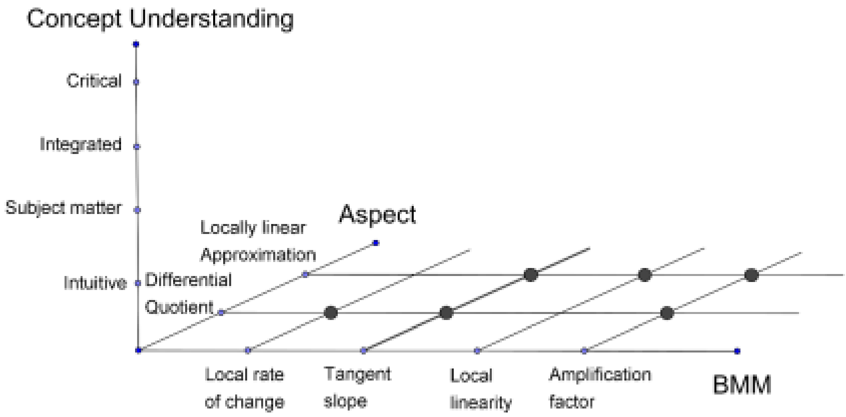

Figure 4: 3d-matrix representing Aspect, BMM and Concept Understanding for the concept of derivative

Using this representation, the 6 x 4 "cells" can be characterized by the triple (Aspect, BMM, Concept Understanding). To emphasize a triple-perspective in classroom activities, special examples for each of these 24 cells have to be found. For instance, at the level of *intuitive concept understanding*, the characterization of the cell with the BMM of local rate of change and the limit Aspect encompasses the conception that the average rate of change stabilizes numerically, as the interval declines in size.

CONCLUSION

The aim of this article was to specify the concepts of Aspects and BMMs for basic concepts of calculus and to embed it into a model of understanding. The result is a representation in the form of a three-dimensional matrix, whose cells are described by a triple (Aspect, BMM, level of concept understanding). This 3d-model can be used in several ways. On the one hand, it could support teachers in introducing this important topic of derivative. Thus, one has an overview of the BMMs and Aspects at the addressed level of concept understanding. On the other hand, this model can also be used for diagnostic intervention. Thus, one also has an orientation, while constructing examples for a certain test on–here–derivatives.

The model also leads to a description of the expected competencies for each of the "cells" of the 3d-matrix. This 3d-matrix can be interpreted as the basis for a competency model, in this case of the concepts of limit, derivative and integral. Furthermore, this can be a first step towards an empirical evaluation of models of understanding on the basis of Aspects and BMMs.

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ZOOMING IN AND OUT - ASSESSING EXPLORATIVE INSTRUCTION THROUGH THREE LENSES

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We examine the implementation of an identical task by two middle-school teachers. Both teachers underwent short professional development training according to Smith and Stein's (2011) 'Five Practices for Orchestrating Productive Mathematics Discussions'. The different levels of implementation are examined through three analytical lenses: Instructional Quality Assessment tool, Accountable Talk coding, and commognitive analysis. Each of these three lenses provides a different resolution of the opportunities for explorative participation students had in the two classrooms. We discuss implications both in terms of benefits for research and in terms of usefulness for teacher training.

Instruction that supports deep, meaningful learning is multifaceted: it involves certain mathematical activities, as well as a particular social structure to support it. As such, it provides a challenge for any researcher who wishes to examine the extent to which such instruction is implemented in the classroom. The need for such an examination has become urgent in the face of ever-growing attempts to improve mathematics instruction. Most assessments of effective teaching are usually done using a scale designed to measure certain "best practices" (e.g. IQA, Boston, 2012; TRU math, Schoenfeld, 2014). The use of a single scale may be unhelpful for teachers wishing to improve their instruction. It may either be too general, leaving teachers wondering what to change, or too specific, neglecting the whole picture of the lesson. We suggest to take a different approach: we first define the type of teaching we wish to see in the classroom – explorative instruction. Then, we examine what different analytical tools may tell us about the extent to which this instruction was observed in the classroom.

THEORETICAL BACKGROUND

We define explorative instruction as instruction that supports explorative participation in mathematical learning. Explorative participation (Sfard & Lavie, 2005) is participation for the sake of producing mathematical narratives to solve problems or to describe the world. Such participation is contrasted to *ritual* participation, which main goal is pleasing others and which is characterized by rigid rule following and endorsement of results as "correct" according to external authority. Explorative participation is linked more broadly to the view of mathematical learning as the process by which students gradually become able to communicate about *mathematical objects*. These discursive objects are a result of the "saming" of different *realizations* (Sfard, 2008). For example, connecting between different *visual mediators* such as:

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 321-328. Singapore: PME.

graphs, tables and algebraic expressions, is central to the "saming" of the function object. A mathematical object can be visualized as a "realization tree" where complex objects are made of simpler ones.

Given this view of explorative participation, instruction that supports it is characterized by several features. First, it provides tasks that afford multiple opportunities for "saming" different realizations, producing narratives based on different routines and enacting mathematical meta-rules such as conjecturing and proving. In the words of Stein, Grover and Henningsen (1996), such tasks are "cognitively demanding". Second, explorative participation can benefit from instruction that constructs certain *participant frameworks* in the classroom (O'Connor & Michaels, 1993). Such participant frameworks allocate students and teachers appropriate roles and duties to carry out the construction of mathematical narratives, in the face of the initial uneven status where students are less experienced than the teacher in doing so. Accountable Talk (Resnick, Michaels, & O'Connor, 2010) provides a set of talk moves that can create such participation frameworks. In particular, it suggests talk moves for holding students accountable to each other (the community) and to rigorous reasoning. Both types of accountability are important for explorative participation. Accountability to reasoning encourages building narratives based on formerly established narratives; accountability to the community moves the authority structure from being solely based on the teacher, to being more equitably divided between him/her and the students.

In the context of a professional program aimed at offering practical tools for teachers to improve their instruction, we thus asked: to what extent did teachers succeed in giving students' opportunities for explorative participation? This question was divided into the following sub-questions: (1) Were teachers able to maintain the cognitive demand of a task? (2) To what extent did teachers encourage the accountability for reasoning and for the community during the lesson? (3) To what extent did teachers give students opportunities for saming different realizations of mathematical objects?

METHOD

Participants included four teachers of 7^{th} and 8^{th} grade mathematics. The teachers participated in PD training sessions that introduced the main components of the *5 Practices for orchestrating productive mathematics discussions* or 5Ps (Smith & Stein, 2011). The 5Ps framework suggests a set of instructional practices that support the maintenance of high-cognitive demand of a task. These include anticipating students' responses, monitoring their work, selecting solutions to be presented to the whole classroom, sequencing these solutions, and connecting between them. By giving teachers a road-map of steps that they can prepare in advance and during whole-class discussions, these practices have the potential for helping teachers to more effectively orchestrate discussions that are both responsive to students' emerging understandings and emphasize important mathematical ides. As part of the PD, the teachers were asked to implement a lesson prepared according to the 5Ps. They were asked to give their students an identical task, *the Hexagon Task* which asks students to describe the perimeter of a general "train" in a pattern of hexagon "trains" (See Figure 1):

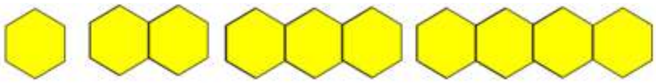

Figure 1: The Hexagons Pattern

The task was used since it had previously been rated as a high cognitive demand task that is productive for teachers' initial attempts to implement the 5Ps (Heyd-Metzuyanim, Smith, Bill, & Resnick, 2016). We observed, video recorded, and transcribed all lessons. In addition, the teachers were interviewed before and after the lessons, and the lesson planning sessions were recorded.

Data Analysis. According to the above explained conceptualization of explorative instruction, we used three analytical tools to examine: Cognitive demand, Accountable Talk, and opportunities for saming different realizations of mathematical objects.

Cognitive Demand: Measuring the general level of implementation of the task was done based on Implementation rubric (AR2) of the Instructional Quality Assessment tool (IQA) (Boston, 2012). This rubric evaluates the cognitive demand of the lesson based on an observation of the recorded lesson. The rubric includes a scale from 1 to 4 where 1 means students engage only in rote memorization and producing facts, 2 means they engage in the application of procedures explicitly taught, 3 means cognitive demand is not lowered but mathematical reasoning is not sufficiently explicated, and 4 means full maintenance of cognitive demand.

Accountable Talk: For achieving a higher resolution of the lesson, and in particular, the participant framework supported in it, we used the Accountable Talk coding scheme (AT) (Resnick et al., 2010). This scheme codes classroom transcriptions on a line-by-line basis. It includes eight codes for teacher moves, where four codes measure accountability to reasoning and knowledge (press for reasoning, challenge, say more and revoice) and four codes measure accountability to the community (add-on, restate, agree/disagree, and solicit additional viewpoints). These moves track the number of teachers' attempts to make students' thinking public, help students reason mathematically, and hold them responsible for attending to the reasoning of others. In addition, the scheme codes students' moves (students' agree/ disagree, students' justification, students' press for reasoning and students' challenge). The two authors achieved 84% agreement on 50% of the data reported in this paper.

Opportunities for objectification: For examining the opportunities for exploring mathematical objects given to students during the lessons we used the Realization Tree Assessment tool (RTA) (Weingarden, Heyd-Metzuyanim, & Nachlieli, 2017). The RTA can be used both to assess the potential of the task to afford the "saming" of different realizations of a mathematical object, as well as the way in which these opportunities actually play out during implementation. It depicts the different realizations of a mathematical object as nodes in a “tree” and then uses different shades to signify who articulated the realization – the teacher or students.

Close-up analysis of opportunities for objectification: Excerpts that have been found to be particularly telling during the scanning of the lesson for RTA analysis were examined on a close-up word-by-word resolution to determine the precise moves used by the teacher to encourage students to articulate certain mathematical narratives. In particular, we looked for teacher questions that encouraged connections between different realizations of mathematical objects and for narratives raised by students and taken up or missed by the teacher.

FINDINGS

Due to space limitations, we restrict our exemplification of the analysis to two teachers who implemented the same task in different ways: Dani, a 7th grade teacher with 3 years of experience and Sivan, an 8th grade teacher with 2 years of experience.

Maintaining cognitive demand during implementation

In Dani's lesson, cognitive demand was maintained and scored at the highest level (4). Scoring was based on observing that Dani did not lead the students towards any particular solution; multiple solutions were found and presented by the students; solutions were linked to each other both by the teacher and by the students; and there was no proceduralization of the task. In contrast, Sivan's implementation was scored as a 2. This, since she led students toward a particular solution (4x+2) that was not exemplified through the visual Hexagon’s representation, connections were not made with other algebraic expressions, and students seemed to be well rehearsed in producing a table, algebraic expression from it and a graph of that expression.

Accountable talk

Interestingly, the two lessons were quite similar in terms of Accountability to Reasoning as a whole, that is, aggregating “press”, “challenge” ,“say more” and “revoice” moves (Dani N=36, Sivan N=33). However, Dani’s talk included more “press for reasoning” (Dani N=23, Sivan N=14) requesting students to justify their claims. In accordance, student justifications in Dani’s lesson were higher than in Sivan’s (Dani 22, Sivan 11). Most important, moves encouraging Accountability to the Community were much higher in Dani’s lesson (12) than in Sivan’s lesson (1). Students’ talk aligned with these different demands. In Dani’s lesson, there were 20 instances of students' agreement/disagreement with their friends mathematical narratives, while in Sivan’s lesson there were no such instances (0). Thus, the AT counts point to the participation framework in Dani’s classroom being more conducive for students’ authority to propose mathematical narratives than Sivan’s lesson was.

Opportunities for saming different realizations of mathematical objects

Sivan's RTA (Figure 2a) and Dani's RTA (Figure 2b) show that as a whole, the classroom discussion made different realizations of the object “perimeter of a general train” accessible to students.

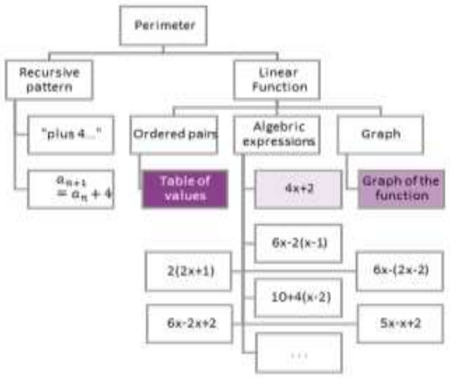

Figure 2a: Sivan's RTA

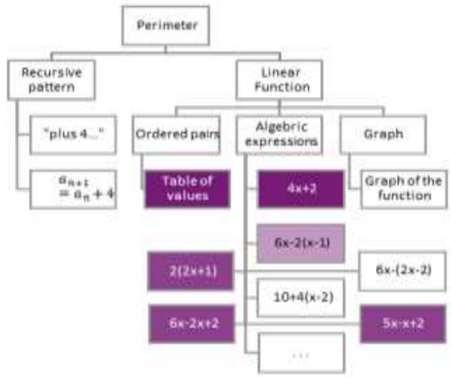

Figure 2b: Dani's RTA

4	The student's explanation was complete
3	The student's explanation was not complete. The teacher explicated the idea
2	The student did not articulate the realization, but the teacher did
1	The realization was mentioned, but neither the student nor the teacher explained it fully
0	The realization was not mentioned, but was hypothesized to be relevant to the task

Dani's RTA shows that his classroom discussion included more narratives about the perimeter object, especially of the algebraic type, than Sivan’s discussion. Moreover, the dark colours of Dani’s RTA show that his students were more active in forming the narratives around these different realizations, while Sivan was mostly responsible for the narratives in her lesson, eliciting them from the students or simply presenting them on the board. Though Sivan did refer to different realizations of the perimeter (table, algebraic expression and graph), the ‘branch’ explored in Dani’s lesson provided an excellent opportunity to connect different algebraic expressions to a single visual mediator (the Hexagons), as there are various different algebraic expressions that express the desired perimeter, based in how the hexagon-sides are counted and connected to the perimeter (referring to external sides only, all hexagon sides and then taking away inner sides, etc.). In contrast, Sivan took the Hexagons visual mediator only as a starting point and directed most of the discussion to realizations of a linear function, which were not directly connected to the Hexagon’s mediator. In that sense, Dani utilized the visual mediator to a much greater extent.

Close-up analysis of instructional talk supporting the saming of different realizations

The wide view of the mathematical narratives raised in the lesson, given by the RTA, show that Sivan’s lesson was more limited in its affordance for saming different realizations. However, this snapshot view is limited in its ability to lend any explanation as to *why* this restriction occurred, given the multiple affordances of the task. In fact, Sivan herself was aware of this restriction but did not know to explain why it occurred. In her post-lesson interview, she said: "If there was even one student who found another (expression), I would have asked him immediately to show it to everyone. But they just all came up with 4x+2". However, the examination of students' discourse revealed otherwise. See for example an instance where a student was trying to explain why he came up with the expression 4x+2:

51	Eitan	The four is like the difference between each perimeter. Let's say, train number one has 6 and train number two has 10.
52	T	O.K. I'll write like this, train number 1 is 6, train number 2 is 10… (writes up a table of values) and here, you saw what? (Draws arcs between the table rows)
53	Eitan	Plus four
56	T	O.K. So I found this (pointing to 4x) four… What's the '2'?

Taking Eitan's explanation that related to the visual mediator ("train number one", etc. [51]), the teacher transformed this narrative into a well-rehearsed visual mediator of a table with "arcs". The fact that this was well known to the students could be seen in the fact that Eitan easily fit the appropriate "blank" in the teacher's prompt "you saw what?"[52] with a "plus four" [53]. The teacher's next question, "What's the '2'?" [56] was more difficult for Eitan to see from the table. Thus, he hesitated while another student, Orit, asked to explain.

61	Orit	These two sides that seem to be connected and that we, like, don't consider them in the drawing, there are always these two sides that get connected to form one side… so they get reduced from the perimeter, so that is the '2'.
62	T	So you say there are two sides connected, so instead of taking them off, you add them? … Why not to do minus two? Aren't you taking them off?

The teacher's question [62], which pertained to whether the '2' should be added or removed from 4x, reveals that she was not seeing the visual mediator of the perimeter the same way Orit was suggesting. Thus, she missed a visualization that could have led to another narrative: 6x-2(x-1) which describes the visualization of all hexagon sides counted, then the double inner sides taken away. While Orit pondered around the teacher's question, Eitan offered an explanation according to the tables of values:

65	Eitan	To reach 4 (probably means 6) we had to add two then we added the two.
66	T	Oh. So you just substituted, you saw it's true. So is it always true?
67	Eitan	Yes.

With this, Sivan left the issue of where the '4' and '2' "come from". Once declaring the correctness of the 4x + 2 expression, she invited a student to draw on the board a graph depicting y=4x+2. The drawing of the graph was done according the algebraic expression by locating one point on the Cartesian plane according to the intercept ((0,2)), and another point according to the rate of change ((1,4)) and stretching a line between them. In fact, the only reference back to the Hexagon's occurred when Sivan asked the students "so when I have 0 hexagons, their perimeter is 2?!" which led to a discussion of the domain and range of the function but was not implicated on the graph itself. In contrast, Dani offered many opportunities for students to connect between the different realizations, including the Hexagon's perimeter, table of values and algebraic expressions. He pressed for these connections consistently. For example, when a student suggested the expression 2(2x+1) Dani said: "O.K., O.K., but why is this true, in relation to the trains?" and when a student suggested another expression (6x-2x+2), he asked: "but how do you explain it with respect to the drawing?" In addition to this insistence on relating expressions to the Hexagon's perimeter, Dani also invited other realizations. He asked a student, who he had seen working on a table, to present his work. After the student's presentation, he revoiced it:

> "Tom built a table… Tom built the placement and the number … once he did that, what did he actually do? He detached himself from the trains. He is no longer thinking about the trains. He only looks at the numbers and tries to find a pattern in the numbers, alright?"

By revoicing Tom's solution, Dani not only explained it in terms of "placement" and "number" (meaning number of sides), he also clarified that the table *could* have been related to the Hexagons ("he detached himself from the trains"). In addition to encouraging connections to the visual mediator, Dani also asked students to explain why 6x-2(x-1) was "true" "algebraically", encouraging students to use algebraic manipulation routines to prove that 6x-2(x-1) = 4x+2.

CONCLUSION AND DISCUSSION

Our goal in this paper was to examine the instructional practices of teachers introduced through a short PD to instructional practices that can support explorative participation. The two contrasting cases afforded us the opportunity to better understand what each of our three different analytical lenses shows about the lesson. These three lenses cohere in showing that Dani's lesson gave more opportunities for explorative participation than that of Sivan. The IQA implementation measure gave the first, rather coarse indication, that Dani maintained the cognitive demand of the task while Sivan lowered it; AT coding showed that Dani encouraged more accountability to the community and to reasoning than Sivan and that his students were more accountable for the community; the RTA showed that Dani's lesson included more mathematical narratives about different realizations of the "general perimeter" object (which in later grades would be called "the general term" of a sequence) than Sivan's lesson. Finally, the close-up analysis, of specific words and sentences, showed that Sivan missed some potential narratives relating the algebraic expressions to the Hexagon trains, and did not make attempts to links between representations and connect them to the visual mediators, while Dani persistently pressed for such links.

The tools we used in this research are useful not only for their varying resolutions and foci, they also have different potential to help teachers improve their instruction. Measures such as the IQA's general implantation rubric are good for assessing the successful implementation of a task. However, they are less effective in showing teachers where they actually could do things differently. Accountable Talk moves provide a more useful tool, in our view, since they give teachers specific suggestions for what they can say in the classroom and give them a clear image of what they need to look for in students' talk. Similarly, but focusing on the mathematical content, commognitive analysis can show teachers what specific talk moves may elicit the saming of different realizations. This, in addition to the RTA which can help teachers map the narratives they wish to elicit from students based on the mathematical task at hand. Our multifocal lens approach thus holds promise both for research purposes and for teacher training purposes. It is, however, only in its initial stages. Therefore, further studies will be needed to establish its coherence and usefulness.

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EXPLORING STUDENTS' APPROACHES AND SUCCESS WITH GROWING PATTERN GENERALISATION AT YEARS 7 TO 12

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Although figural growing pattern generalisation is one approach advocated for developing students' conceptions of a function, there is more to learn about how students' choice of approach and success in reaching a full algebraic (symbolic) rule might develop over time as they experience other representations and concepts in algebra. This paper discusses an investigation of 215 Australian students' responses to a pattern generalisation task to provide insight into their development across Years 7 to 12. Implications for secondary school teaching approaches are discussed.

A development in the past few decades, particularly in response to calls to reform algebra teaching and learning (Carraher & Schliemann, 2007; Kieran, 2007), has been the generalisation of figural growing patterns as a way of developing students' understanding of functional relationships. Moss, Beatty, Barkin, and Shillolo (2008) argued that patterns "offer a powerful vehicle for understanding the dependent relations among quantities that underlie mathematical functions" (p. 156). These experiences of counting quantifiable aspects of items in figural growing patterns with repeated addition provide a foundation for understanding the simplest type of function: linear functions (Smith, 2008).

Many studies have examined students' strategies and difficulties with figural pattern generalisation (e.g., Carraher, Schliemann, Brizuela, & Earnest, 2006; Kaput, 2008; Stacey, 1989) but little is known about their progression in generalisation ability throughout the secondary years of schooling. A recent comparative project is focusing on English and Israeli students' function concept development from Years 7 to 12, including pattern generalisation (Ayalon, Watson, & Lerman, 2015a, b). Such research is important for considering how curriculum and teaching approaches in different contexts might influence students' development of algebraic reasoning. This paper discusses one subset of data collected on Australian secondary students as part of the larger project and which focuses on their figural pattern generalisation at different year levels. It addresses the following research question: *What is the nature of Australian students' approaches and success with pattern generalisation throughout the secondary years of school mathematics?*

BACKGROUND

Two key theoretical perspectives were considered important for this study: covariation and correspondence approaches for learning about functions, and learning to translate

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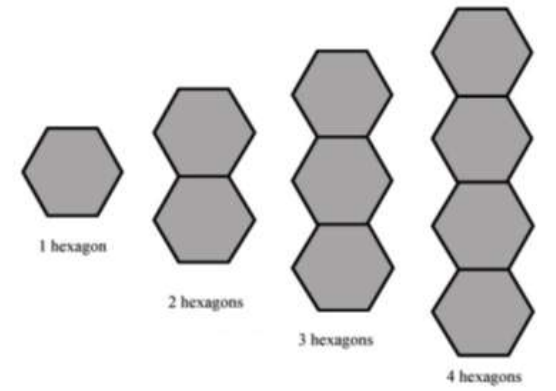
between different representations of functions. These are discussed in turn along with empirical findings from previous research in the literature.

An early perspective on a covariation approach to understanding a functional relationship and based on ratio concepts is that two sequences are generated independently through a pattern of data values and are juxtapositioned. It is contrasted with a *correspondence* approach where a function is described as an algebraic rule relating two variables (Confrey & Smith, 1994). Another definition of covariation useful for rate-of-change concepts is that it involves coordinating two varying quantities (variables) while attending to the ways in which they change in relation to each other (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). The relationship between a quantifiable aspect of figural items in a growing pattern and the position of items in the sequence lends itself to both covariation and correspondence approaches for understanding a function. Figure 1 illustrates these two approaches using the task discussed in this paper, which asked the students to generalise a hexagonal chain pattern (the whole task will be shared at the conference and can also be found in Ayalon et al. (2015a) since space is limited here).



Covariation – when the number of hexagons increases by 1, the perimeter always increases by 4 units. The starting value is 2 (for the first and last sides in the chain), so $t = 4n + 2$.
Correspondence – each hexagon has 4 sides that form the perimeter of the hexagon chain and then the first and last hexagons in the chain need a side each to complete the perimeter. To find the perimeter, multiply the number of hexagons by 4 and then add 2 ($t = 4n + 2$)

Figure 1: Approaches for generalising the figural growing pattern used in this study

Confrey and Smith (1994) found that students preferred using a covariational approach with tables of values to using a correspondence approach with algebraic equations since they were “easier and more intuitive” (p. 33). They suggested that covariation is more powerful and also emphasises rate of change clearly. Yet in the context of growing pattern generalisation, some studies have found that a correspondence approach is more likely to lead to successful generalisation and an explicit rule (e.g., Lannin, 2005; Wilkie, 2016). An empirical study of US upper primary students found that they progressed from term-to-term to position-to-term approaches (Markworth, 2010). Hershkowitz, Arcavi, and Bruckheimer (2001) found that colleagues and preservice teachers attempted a covariation approach when trying to generalise a quadratic matchstick growing pattern (by creating a table of values) but were unsuccessful. Younger students (13 to 14 years old) could use a correspondence approach successfully for the same pattern to create an explicit rule, even for the more difficult quadratic function. Nonetheless, Küchemann (2010) argued that a term-to-term approach for exploring functions ought not be banished, since it is closely allied to the notion of gradient in graphs of functions. Rather, he emphasised that students need "to see how term-to-term and position-to-term approaches can complement each other" (p. 242). This study provided the opportunity to examine if students' choosing to use a covariation or correspondence approach changes over time as they are exposed to these two views of function in other contexts at secondary levels of algebra.

Mathematical ideas are made meaningful through translations between different representations, such as real-world situations, spoken symbols, written symbols, pictures (static figural models), and manipulative models (Lesh, 1981). "A mathematical representation cannot be understood in isolation. The representational systems in mathematics and its learning have structure so that different representations within the system are richly related to one another" (Goldin & Shteingold, 2001, p. 2). Lesh (1981) also emphasised that the ability to use an idea depends on the way it is *linked* to other ideas and to processes within an appropriate cognitive structure that integrates ideas with a system of processes. Functional relationships can be represented by tables of values, figural growing patterns, descriptions, graphs, and algebraic (symbolic) equations. Despite the recent focus on pattern generalisation for learning about functions, there is little in the literature on if or how students might develop in their approach and success at secondary levels of schooling. One large study of Lebanese students found an increasing level of reasoning in pattern generalisation across clusters of grade levels from Years 4 to 11 (Jurdak & Mouhayar, 2014). This study provided the opportunity to investigate students at different year levels constructing or translating between different representations of the same functional relationship – a figural growing pattern, non-sequential pairs of values, description, and algebraic equation.

RESEARCH DESIGN

This article discusses one subset of data collected on Australian secondary students as part of a larger comparative project on function concept development (see Ayalon, et al., 2015a, b). Over 200 Australian secondary students in Years 7–12 from two middle-SES governments schools and their teachers participated – alternating year levels from each school to minimise school effects and reduce the burden on the teaching staff. The students completed a series of written tasks eliciting different types of functional knowledge and concepts, and their teachers completed a questionnaire on their expectations for different year levels and sequencing of concepts. This paper discusses the responses to one of these tasks in which the students were initially given the picture in Figure 1 and the perimeters for a chain of 1 hexagon and 3 hexagons, then asked to find the perimeter for some other (non-sequential) chains. A tabular representation was deliberately excluded from the task to examine what types of reasoning and approaches the students might use without elicitation of term-to-term or covariation approaches. The students were then asked to explain how they could find the perimeter for 100 hexagons, and to construct an algebraic rule for finding the perimeter of any number of hexagons. This was an explicit request for a new representation and no direct support was given, but the prior questions were designed to direct students towards an explicit rather than recursive generalisation. They were then asked to justify their answer in words to elicit further insight into their approach. A scoring rubric (Table 1) was developed and refined collaboratively by the initial research team from the overall project (Ayalon et al., 2015a) and adapted slightly for the Australian responses. For example, an *unclear approach* category was needed for the students who gave an algebraic rule but did not explain or show how they found it. Illustrative scoring from the Australian data reported on in this paper will be shared at the conference.

RESULTS AND DISCUSSION

The following discussion focuses on three aspects: the students' overall choice of approach and subsequent success in creating an algebraic rule for the hexagonal pattern (Table 1), their success at different year levels (Figure 2), and then students' choice of approach (correspondence and/or covariation) and success with creating an algebraic rule at different year levels (Table 2).

Table 1 presents the students' choice of type of approach in generalising the hexagonal growing pattern (column 1) and their subsequent success in finding an algebraic rule for the linear function.

		Level of generalisation				**Sub-total**
Score	**Approach**	**No generalisation**	**Descriptive rule**	**Algebraic rule**	**Algebraic rule explained**	
0	No response	3.7%	-	-	-	**3.7%**
1	Counting	21.9%	-	-	-	**21.9%**
2	Correspondence	9.8%	12.1%	2.8%	18.6%	**43.3%**
3	Covariation	11.2%	2.8%	0.5%	3.3%	**17.7%**
4	Correspondence then covariation	0.5%	-	-	-	**0.5%**
5	Covariation then correspondence	0.5%	1.4%	0.5%	0.9%	**3.3%**
unclear	Unclear approach	0.5%	0.5%	6.0 variation %	2.8%	**9.8%**
	Sub-total	**47.9%**	**16.7%**	**9.8%**	**25.6%**	**100%**

Table 1: Approach and level of growing pattern generalisation (n = 215)

Just over half of the cohort overall found a correct explicit generalisation for the growing pattern expressed descriptively or algebraically, with 35% able to find a correct algebraic rule. The use of a correspondence approach was clearly the most likely to lead to successful generalisation, as over 20% of the students used this approach and subsequently developed a correct algebraic rule. This resonates with earlier findings from research on Australian students (Stacey, 1989), but is different to Ayalon and colleagues' (2015a) finding that on the same task, the English students (n = 120) were more likely to be successful with a covariation approach (but similar to the Australian data, one third of the English students overall found a correct algebraic rule). They also found that the Israeli students (n = 110) were highly successful with either approach, with three quarters overall able to find a correct algebraic rule (Ayalon et al., 2015b).

Nearly 10% of the Australian students' responses could not be allocated to a particular approach because they did not give an explanation, but most actually successfully generalised the pattern algebraically. A few students had drawn their own table of values for the pattern but their choice of approach was not clear. Many students simply wrote the algebraic rule, and often with an *x* rather than an n as prompted by *p(n)* in the task.

Figure 2 presents the students' generalisation success at each year level.

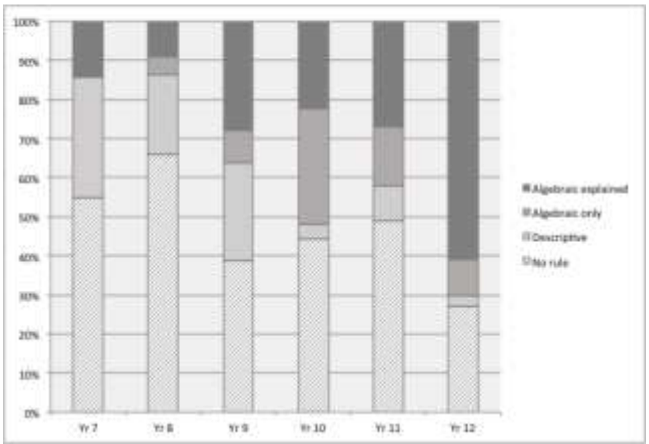


Figure 2: Percentages of levels of pattern generalisation within year levels (n = 215)

The graph in Figure 2 shows surprisingly little progression across Years 7 to 11 in the proportion of students able to generalise explicitly. Within a particular year level (excluding Year 12) around half of the students were able to find an explicit generalisation, expressed descriptively or algebraically. This lack of progression is different to the patterns of progression found in both the English and Israeli data for the same task (keeping in mind the smaller data samples), with nearly all Israeli students able to generalise from Year 8 onwards (Ayalon et al., 2015b) and the English students demonstrating increased competence over time (Ayalon et al., 2015a). It seems that these Australian students' exposure to different conceptions of function may not support their development of figural growing pattern generalisation and warrants further investigation into differences between the three countries' curriculum content and teaching approaches in this area. There is some sense of progression, however, in those students able to create a symbolic rule from Year 7/8 to Year 9/10/11 and then to Year 12, suggesting some development of the understanding of representing the variables from a growing pattern with pronumerals.

Table 2 relates the students' approach to pattern generalisation with their ability to find an explicit rule (no rule / descriptive rule / algebraic rule) within each year level.

		Level of generalisation: % (No rule), Descriptive rule, Algebraic rule					
Score	**Approach**	**Year 7**	**Year 8**	**Year 9**	**Year 10**	**Year 11**	**Year 12**
0	No response	(7), 0, 0	-	(3), 0, 0	-	(12), 0, 0	-
1	Counting	(12), 0, 0	(43), 0, 0	(17), 0, 0	(15), 0, 0	(18), 0, 0	(21), 0, 0
2	Correspondence	(19), 24, 14	(7), 11, 7	(5), 17, 25	(18), 4, 22	(6), 9, 22	(3), 3, 52
3	Covariation	(14), 2, 5	(16), 7, 0	(14), 5, 0	(8), 0, 7	(9), 0, 3	(3), 0, 9
4	Correspondence then covariation	(3), 0, 0	-	-	-	-	-
5	Covariation then correspondence	-	-	(0), 3, 0	(4), 0, 0	(0), 0, 6	(0), 0, 3
unclear	Unclear approach	-	(0), 2, 7	(0), 0, 11	(0), 0, 22	(3), 0, 12	(0), 0, 6
	Total	*100%*	*100%*	*100%*	*100%*	*100%*	*100%*

Table 2: Approach and levels of pattern generalisation across year levels (n = 215)

The data in Table 2 highlights that the students appear unlikely to change to a different approach to pattern generalisation as they get older, as there are no noticeable shifts in the proportions of students choosing a particular approach. At each year level, a correspondence approach is more popular than covariation, which is similar to the data on the Israeli student responses (Ayalon et al., 2015b). The teachers of the Year 7/8 students indicated that although they used pattern generalisation in their teaching, they expected that most students would most likely use recursive counting strategies and have difficulty creating an explicit rule. It is interesting that more of the younger students used a correspondence approach than expected, and more students generalised the pattern correctly than expected by their teachers, but with descriptive rather than algebraic rules. The teachers of the higher year levels expected most of their students to be able to complete the task successfully.

CONCLUSION

This study found that the students at each year level both preferred, and were more successful with, a correspondence approach to pattern generalisation than covariation. Unlike the Israeli and English students' responses to the same task (Ayalon et al., 2015a, b), a lack of progression was found in the proportion of these students able to generalise a figural growing pattern with increasing year levels. This finding is also different from Jurdak and Mouhayar's (2014) study of Lebanese students. Many of the Australian students, although not successful with creating an algebraic rule, were nevertheless able to use a correspondence approach to structure a descriptive explicit generalisation. Yet Radford (2006) argued that for thinking to be distinctively *algebraic*, it must: handle indeterminacy, in an analytic way, and designate its objects *symbolically*. Why did many in this cohort of students demonstrate descriptive rather than symbolic generalisation?

Several examples of tasks with growing patterns made with matchsticks can be found in typical Australian textbooks at lower secondary levels, yet they ask students to *describe* the generalisation rather than create a symbolic rule. It seems likely that this type of task has been experienced by many students, but does not support students' ability to develop algebraic representations in this context. This result resonates with recent research on Year 7 Australian students' pattern generalisation, which found that a majority of students adopted a correspondence approach and 44% were able to find an explicit rule, approximately half with descriptions, and half with algebraic rules (Wilkie, 2016). We suggest that if students are not taught to represent pattern generalisations symbolically, they are missing an opportunity to develop algebraic thinking (Radford, 2006). They are also missing another opportunity to link algebraic equations to other representations of functional relationships, and to learn how they are richly related to one another, as emphasised by Goldin and Shteingold (2001). Since Australian lower secondary students already explore the use of pronumerals to represent variables in other contexts, there is the potential for them also to experience their use with creating algebraic rules for growing patterns.

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SPECIALISING AND CONJECTURING IN MATHEMATICAL INVESTIGATION

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This paper introduces a new framework to model the interactions of the processes of specialising and conjecturing when students engage in mathematical investigation. The framework posits that there is usually a cyclic pathway alternating between examining specific examples (specialising) and searching for pattern (conjecturing), instead of a linear pathway as in many other theoretical models. The framework also distinguishes between observing a pattern and formulating it as a conjecture, unlike most models that treat an observed pattern as a conjecture to be proven or refuted. I will then use the framework to analyse and explicate a secondary school student's specialising and conjecturing processes while he attempted an open investigative task.

INTRODUCTION

There is quite a number of theoretical models developed by educators on the processes in mathematical investigation, e.g. Height (1989) and Bastow, Hughes, Kissane and Mortlock (1991). But many of these models remain theoretical in the sense that there are very few empirical studies on these processes in mathematical investigation despite a thorough search of past and current literature. Moreover, most of these models show a linear pathway from one process to another when in reality, based on empirical data such as those from Yeo (2013), many of the pathways are cyclic. Many theoretical models also oversimplify some of the processes, such as equating an observed pattern to a conjecture to be proven or refuted, when empirical data suggest that some students will go back to try more examples after observing a pattern in order to be more certain that there is indeed a pattern before formulating it as a conjecture. Therefore, there is a need for a more comprehensive framework to more accurately describe the interactions of the processes in mathematical investigation.

This paper will describe a new framework called the Model for Cognitive Processes in Mathematical Investigation (or the Investigation Model in short), which was developed as part of my doctoral study (Yeo, 2013) to analyse and explicate the cognitive processes when secondary school students attempted open investigative tasks. In particular, this paper will focus on two of the processes called specialising and conjecturing. Specialising is the process of examining special cases or trying specific examples to search for patterns in order to generalise, and conjecturing is the process of searching for patterns and formulating conjectures based on the patterns observed. Specialising and conjecturing, together with justifying (conjectures) and generalising,

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are the four main mathematical thinking processes identified by Mason, Burton and Stacey (2010).

Some researchers (e.g. Clement, 2000) believe that one of the most important needs in basic research on thinking processes is the need for insightful explanatory models of these processes. This type of explanatory models is often iconic in nature and the purpose of the model is to give satisfying explanations for patterns in observations (Lesh, Lovitts, & Kelly, 2000). Schoenfeld (2002) explained that the descriptive power of a model will be high if the model can capture the essence of the phenomenon. Therefore, this paper will illustrate how the Investigation Model can be used to analyse, describe and explicate the processes of specialising and conjecturing in mathematical investigation.

THEORETICAL FRAMEWORK

Based on existing theoretical models of mathematical investigation in literature, I had modified and designed an explanatory framework to model the types and interactions of cognitive processes in mathematical investigation (Yeo, 2013), which is reproduced in Figure 1. The left side of the Investigation Model shows the three phases and eight stages of mathematical investigation. The right side of the model shows the types of processes (indicated by unshaded boxes) and outcomes (indicated by shaded boxes), and their interactions. It is necessary to include outcomes in the model because the processes do not just interact among themselves but they also interact with the outcomes. Most of the stages are named after the main process(es) in that stage.

The process of specialising occurs in the stage of 'Specialising and Using Other Heuristics', but it is beyond the scope of this paper to examine 'other heuristics' such as deductive reasoning. The stage of 'Conjecturing' consists of the process of searching for patterns and two outcomes: 'Observed Pattern' and 'Formulated Conjecture'. Many theoretical models usually show a single pathway from specialising (or trying examples) to pattern searching. But the Investigation Model on Figure 1 allows for a cyclic pathway alternating between specialising and searching for patterns. Unlike other models, the Investigation Model also separates the formulation of a conjecture from the observation of a pattern because some students will go back to specialising some more after observing a so-called pattern because they are not sure whether there is really a pattern. Only after trying more examples and finding the same pattern will the students finally treat it as a conjecture to be proven or refuted.

Although the focus of this paper is on specialising and conjecturing, it is necessary to consider what follows after a conjecture is formulated because one of the processes in the Justifying Stage, called naïve testing, looks rather similar to the specialising process when students go back to try more examples to be more certain of a pattern. However, naïve testing of a conjecture is different in the sense that there must be a conjecture first, and the students are supposed to try to justify or refute the conjecture by using either a formal proof (Tall, 1991) or a non-proof argument based on the underlying structure (Mason et al., 2010). But very often the students are not able to think of a proof or argument, so some of them will test the conjecture by trying to find if there are counter examples to refute it: this is called naïve testing by Lakatos (1976).

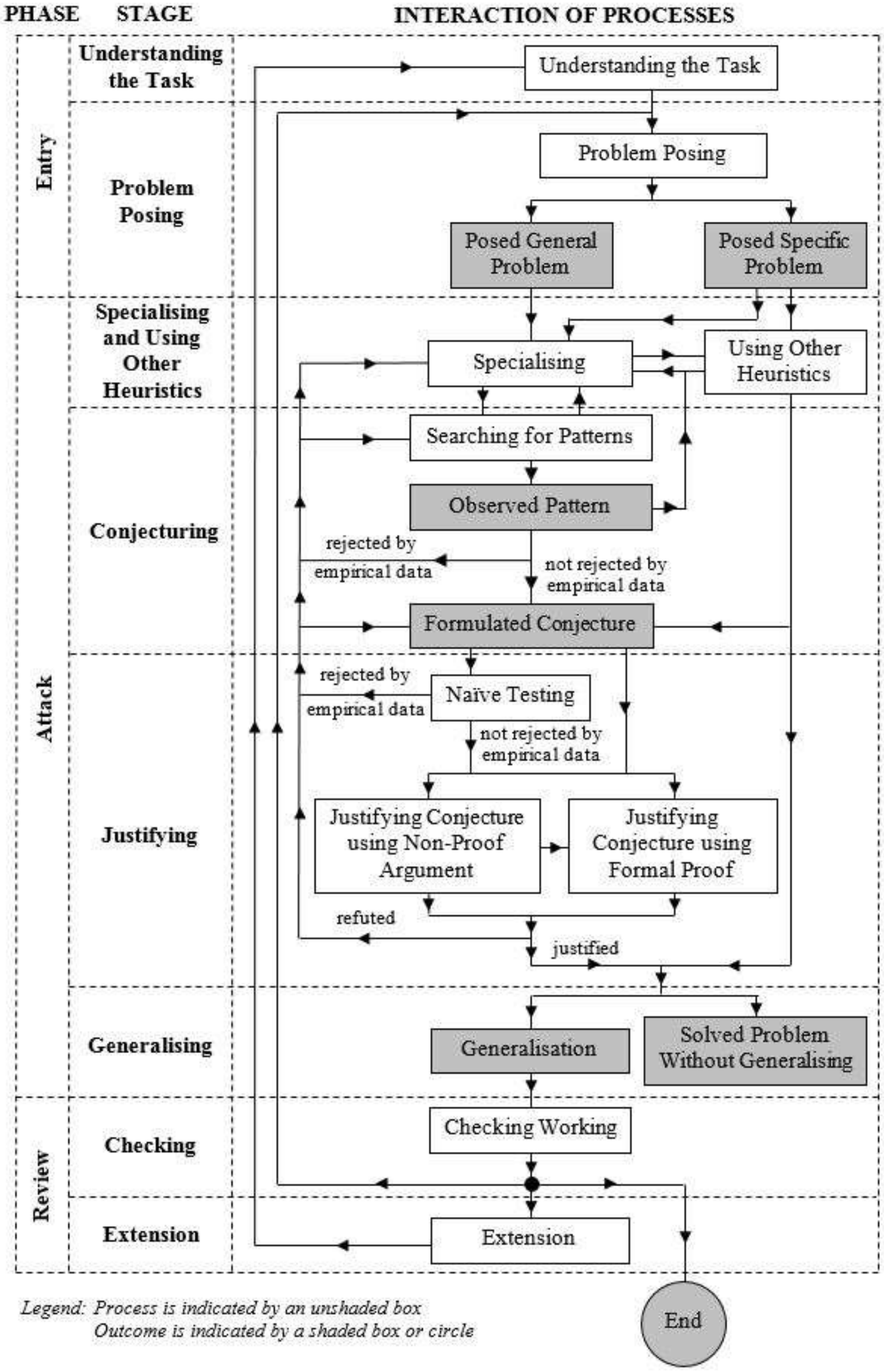

Figure 1: Model for Cognitive Processes in Mathematical Investigation

METHOD AND ANALYSIS

The Investigation Model will be used to analyse the cognitive processes of a secondary school student Ben (pseudonym) when he attempted the following posttest task:

> Choose any number. Add the sum of its digits to the number itself to obtain a new number. Repeat this process for the new number and so forth. Investigate.

Ben was one of the students in my doctoral study (Yeo, 2013) who had undergone a teaching experiment on mathematical investigation consisting of six two-hour lessons: he had been taught how and what to investigate when given open investigative tasks, including cognitive processes such as problem posing, specialising, conjecturing, justifying and generalising. Ben was videotaped thinking aloud while he attempted two pretest and two posttest tasks. The verbal protocols were then transcribed and coded to identify the types of processes and their interactions. The following episodes were chosen to illustrate how the Investigation Model can be used to describe and explain Ben's processes of specialising, conjecturing and naïve testing.

Episode 1: Interaction between specialising and pattern searching

Figure 2 shows the first portion of Ben's working for the task. He started with the number 12345 and added the sum of its digits to itself to obtain 12360. Then he made a serious mistake in misinterpreting the task: instead of repeating the process for the new number 12360, he repeated the process for a completely new random number 242 and obtained 250. Despite him not recovering from his error throughout the investigation, the Investigation Model is still able to capture his thinking processes.

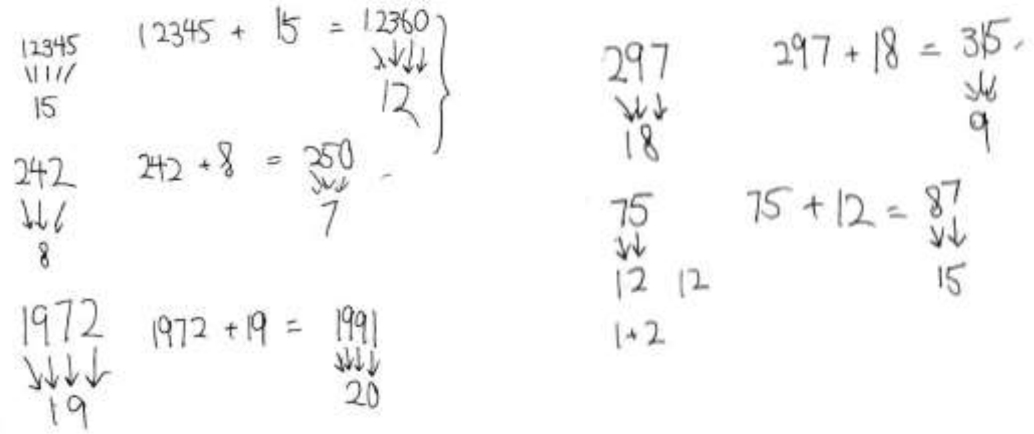

Figure 2: First part of Ben's working

The following protocols continued from when Ben started trying the fifth number 75 (this will be called Example 5) at the right end of Figure 2. Square brackets were used in the transcript to enclose the transcriber's comments such as what the student was writing. An ellipsis was used to indicate a short pause of three seconds or less.

20 03:43 If I try to use a small number like ... 75 [write: 75] ... 75 [draw an arrow from each digit of 75 downward] I add it together I have [write: 12] 12 … [continue writing in another line] 75 + 12 = 87 [stop writing] … 87 [draw an arrow from each digit of 87 downward] equals to 15 [write: 15] … 15 which is 1 …[start writing] 1 + 2 [stop writing] ... added to 12 [write: 12] …

21	04:20	So 19 [point pen at 19 in Example 3] … is ...
22	04:25	If I try with another, another two-digit number like [turn to new p. 2] … 27 [write: 27] [draw an arrow from each digit of 27 downward] 27 equals to, add together is 9 [write: 9] 9 … [continue writing in another line] 27 + 9 [stop writing] … will give me 36 [write: 36] 36 [draw an arrow from each digit of 36 downward]. If I add them together, I will still get 9 [write: 9].
23	04:50	There is no difference …
24	04:54	If I try 50 [write: 50] … 50 [draw an arrow from each digit of 50 downward] if I add it together, I will get [write: 5] 5 … [continue writing in another line] 50 + 5 = 55 [stop writing] ... 55 [draw an arrow from each digit of 55 downward] you add it up, you get a maximum of 10 [write: 10].
25	05:15	[Pause 4 seconds]

In Line 20, Ben was trying Example 5 (specialising), and in Line 21, he was searching for patterns when he pointed his pen at the number 19 in his previous Example 3. After failing to find any patterns, Ben tried a new Example 6 (Line 22) and continued to search for patterns but to no avail (Line 23). Then Ben tried Example 7 (Line 24) and paused for four seconds, presumably to search for patterns. Thus Ben was alternating between specialising and pattern searching as illustrated by the Investigation Model in Figure 3. The numbers in the figure represent the line numbers in Ben's transcript but with a difference. For example, Line 20 is coded as 'Specialising' and Line 21 as 'Searching for Patterns, but in Figure 3, it is more helpful to use the line numbers to indicate the pathways so that we can see that Ben moved from 'Specialising' to 'Searching for Patterns' (indicated by 20), then back to 'Specialising' (indicated by 21), and then to 'Searching for Patterns' again (indicated by 22), and so forth.

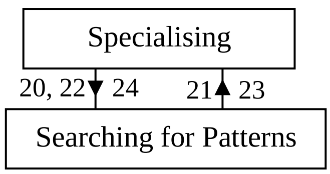

Figure 3: Alternating between specialising and searching for patterns

Episode 2: Difference between observed pattern and formulated conjecture

The following protocols picked up from when Ben started trying Example 10 using the number 5000.

40	09:05	If I try to add ... if I try example of an even number [write: 5000] 5000 ... 5000 [draw an arrow from each digit of 5000 downward] if I add it together, I will have 5 [write: 5] ... If I try ... so [continue writing in another line] 5000 + 5 = 5005 [stop writing] —
41	09:32	— which is an odd number [write: (odd no.)] ... So from an even number [point pen at 5000], I obtain an odd number ...
42	09:44	Let me try an odd number now. [Write: 5001] 5001. [Draw an arrow from each digit of 5001 downward] If I add it all together, I will have 6 [write: 6] ... [continue writing in another line] 5001 + 6 [write: 5001 + 6 = 5007].
43	09:56	It will still give me an odd number: 5005, 5007.

44	10:02	[Pause 5 seconds]
45	10:07	I try five thousand [write: 5] and ... 5222 [write after the digit 5: 222]. [Draw an arrow from each digit of 5222 downward] It will give me [write: 11] 11 [continue writing in another line] 5222 + 11 = 5233 [stop writing]
46	10:28	— which is still an odd number ...
47	10:32	Every time I add them together [draw a big brace from 5005 to 5233], I get an odd number ... Is it the same for every single pattern? ...

In Line 40, Ben was trying Example 10, and in Line 41, he observed a pattern that the next number was an odd number. Sometimes a student may observe a pattern immediately after trying an example, so it is not easy to distinguish the exact juncture between searching for patterns and observing a pattern. However, Ben was not sure whether there was really a pattern because he continued to try two more examples (Example 11 in Line 42 and Example 12 in Line 45) and he observed that it was still the same pattern (Lines 43 and 46). So he said, "Is it the same for every single pattern?" (Line 47) What he probably means is whether the pattern is the same for every single example. At this moment, Ben was more certain that there was a pattern and this was coded as when he formulated his conjecture. In other words, there is a difference between 'observed pattern' and 'formulated conjecture': Ben did not treat the pattern as a conjecture when he first observed it, but he went back to try more examples to see if the pattern could withstand the test of a few more examples before formulating it as a conjecture. Figure 4 shows the pathways of Ben's processes and outcomes in Episode 2 as modelled by the Investigation Model.

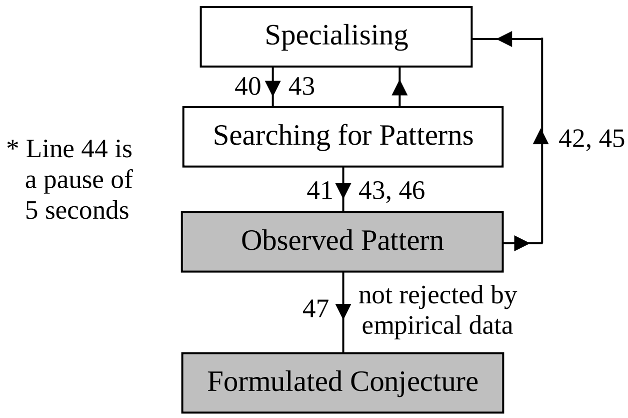

Figure 4: Observed pattern vs. formulated conjecture

Episode 3: Difference between specialising and naïve testing

The following protocols showed what happened immediately after Ben formulated his conjecture in the previous episode. Instead of trying to think of a non-proof argument or a formal proof to justify his conjecture, Ben decided to test his conjecture by trying to find if there are counter examples to refute it.

48	10:40	Write: 2987] 2897. [Draw an arrow from each digit of 2987 downward] If I add it all together, I get ... 26 [write: 26] ... [continue writing] 2897 + 26 = [stop writing] ... 2 ... 2903 ...
49	11:12	2923 [write: 2923].

50	11:14	[Pause 4 s]
51	11:18	What if I try to... The main reason why I'm getting an odd number is because this is odd [circle 7 in 2897] ... and this is even [circle 6 in 26] ... If I put another odd number and odd number, or even number and even number like ...
52	11:31	[Write: 2572] 25 ... 72. [Draw an arrow from each digit of 2572 downward] If I add them all up, I'll get a total of ... [write: 16] 16 ... [continue writing in another line] 2572 + 16 [stop writing] —
53	11:47	— will, should give me an even number 2588 [write: 2588].
54	11:55	So, therefore, this is an even number [write: (even no.)]. So it doesn't have to be odd number all the time ...

In Line 48, Ben tried Example 13 and found the same pattern (Line 49). Then he paused for four seconds. During this time, he was able to deduce the reason behind his conjecture, which he articulated in Line 51. The deductive argument led him to think of a counter example (Example 14 in Line 52) to ensure that the next term in the sequence was even instead of odd (Line 53), thus refuting his conjecture (Line 54). This kind of naïve testing in the Justifying Stage is different from trying more examples in the Specialising and Conjecturing Stages to be more certain that there is indeed a pattern first because naïve testing happened after the formulation of a conjecture at the end of the Conjecturing Stage. Figure 5 shows the pathways of Ben's processes and outcomes in both Episodes 2 and 3 as modelled by the Investigation Model. It is beyond the scope of this paper to discuss what happens after a conjecture is rejected by naïve testing.

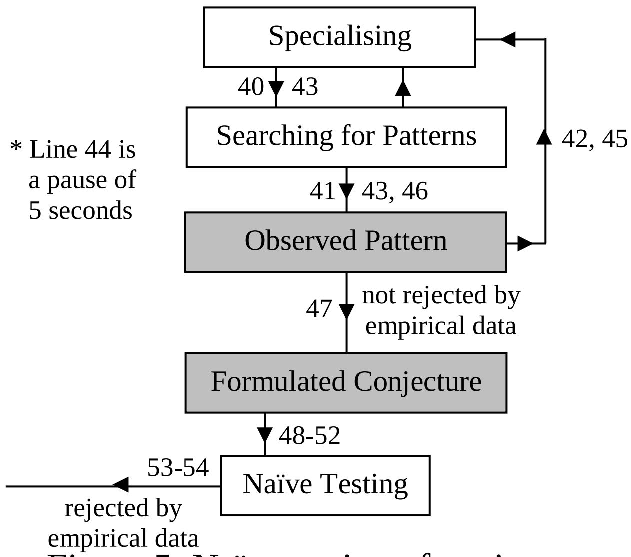

Figure 5: Naïve testing of conjecture

DISCUSSION AND CONCLUSION

The power of an insightful explanatory model on thinking processes lies in its ability to capture the actual processes and explain the interactions accurately (Schoenfeld, 2002). The analysis presented in the previous section has demonstrated how the Investigation Model is capable of faithfully depicting the interactions of the processes of specialising, conjecturing and naïve testing during mathematical investigation. The empirical data from my doctoral study have suggested that the processes and their interactions are much more complex than those modelled by most theoretical frameworks. For example, some students often alternate between specialising and searching for patterns, and a more robust model should capture such phenomenon. Also, such a model should be able to discern between processes or outcomes that look similar, such as an 'observed pattern' and a 'formulated conjecture'. Researchers can then use the framework to analyse other students' cognitive processes at a fine-grained level while teachers can use the model to teach students how to think when engaging in mathematical investigation. Therefore, the Investigation Model can help to make students' thinking processes visible to researchers and teachers.

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