PME 41

Proceedings

Singapore | July 17-22, 2017

Of the 41st Conference of the International Group for the Psychology of Mathematics Education

Volume 1

Editors | Berinderjeet Kaur, Weng Kin Ho, Tin Lam Toh, Ban Heng Choy

Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education

Editors
Berinderjeet Kaur
Weng Kin Ho
Tin Lam Toh
Ban Heng Choy

Volume 1

PME 41, Singapore, 17-22 July 2017

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Singapore

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PREFACE

We are pleased to welcome you to PME 41. PME is one of the most important international conferences in mathematics education and draws educators, researchers, and mathematicians from all over the world. This is the first time such a conference is being held in Singapore at the National Institute of Education. The National Institute of Education is an institute of the Nanyang Technological University in Singapore. It is the nation's only institute dedicated to teacher education. Working closely with the Ministry of Education and the schools, it ensures that teacher education programmes are relevant and delivered with rigour. The institute's engagement in cutting-edge educational research also provides an important voice to inform the policy formulation and practice of education in Singapore.

"Mathematics **E**ducation **R**esearch – **L**earning, **I**nstruction, **O**utcomes & **N**exus?" has been chosen as the theme of the conference. The theme offers opportunities to reflect about what we have learned in the past, investigate the present issues, and more importantly, project the future directions in mathematics education research. The theme is inspired by the iconic Singapore mascot, MERLION, which reflects the past and the present. The "Mer" or fish part indicates Singapore's origin as a fishing village; while the "Lion" part comes from the word "Singa-pura", which means Lion city.

Mathematics Education is a relatively young field of research. Over the last century researchers have progressively adopted multiple and integrated perspectives of *learning*, *instruction* and *outcomes* in mathematics. This theme is apt as it provides opportunities for the community to take stock of our past and present perspectives while exploring new ones in the theory-practice nexus of mathematics education.

The papers in the four volumes of these proceedings are organised according to the type of presentation. Volume 1 contains the presentations of our plenary speakers, Research Forum activities, Discussion Group activities, Working Session activities, the Seminar and the Oral Communication presentations. Volume 2 contains the Poster presentations and Research Reports (A-G). Volume 3 contains Research Reports (H-O) and Volume 4 contains Research Reports (P-Y).

The organisation of PME 41 is a collaborative effort involving all the academic and support staff of the Mathematics and Mathematics Academic Group at the National Institute of Education. They are all members of the Local Organising Committee. The organisation of the conference is also supported by the International Programme Committee, the PME Administrative Manager and the Association of Mathematics Educators in Singapore. We acknowledge the support and effort of all involved in making the conference possible. We thank each and every one of them. Finally, we thank each PME participant for making your journey to PME 41 in Singapore and for your contributions to this conference.

We hope the National Institute of Education in Singapore, as the place of PME 41 2017, will provide opportunities for the community to take stock of their past and present perspectives while exploring new ones in the theory-practice nexus of mathematics education.

Berinderjeet Kaur and Ho Weng Kin

PME 41 2017 Conference co-Chairs

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RESEARCH REPORTS (P-Y)

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

HISTORY AND AIMS OF PME

The International Group for the Psychology of Mathematics Education (PME) is an autonomous body, governed as provided for in the constitution. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI) and came into existence at the Third International Congress on Mathematics Education (ICME 3) held in Karlsruhe, Germany in 1976.

Its former presidents have been:

Efraim Fischbein, Israel	*Stephen Lerman,* UK
Richard R. Skemp, UK	*Gilah Leder,* Australia
Gerard Vergnaud, France	*Rina Hershkowitz,* Israel
Kevin F. Collis, Australia	*Chris Breen*, South Africa
Pearla Nesher, Israel	*Fou-Lai Lin*, Taiwan
Nicolas Balacheff, France	*João Filipe Matos,* Portugal
Kathleen Hart, UK	*Barbara Jaworski*, UK
Carolyn Kieran, Canada	

The present president is Peter Liljedahl, Canada

THE CONSTITUTION OF PME

The constitution of PME was adopted by the Annual General Meeting on August 17, 1980 and changed by the Annual General Meetings on July 24, 1987, on August 10, 1992, on August 2, 1994, on July 18, 1997, on July 14, 2005 and on July 21, 2012. Here, we have only printed two parts of the constitution. The group has the name “International Group for the Psychology of Mathematics Education”, abbreviated to PME. The major goals of the group are:

- to promote international contact and exchange of scientific information in the field of mathematical education;
- to promote and stimulate interdisciplinary research in the aforesaid area; and
- to further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

The whole constitution can be found at the PME Website: http://www.igpme.org

PME MEMBERSHIP AND OTHER INFORMATION

Membership is open to people involved in active research consistent with aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other about their working groups, poster sessions and many other activities. Every year the conference is held in a different country.

There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter is issued three times a year, and can be found on the IGPME website. Occasionally PME issues a scientific publication, for example the result of research done in group activities.

WEBSITE OF PME

All information concerning PME, its constitution and past conferences can be found at the PME Website: http://www.igpme.org

HONORARY MEMBERS OF PME

Efraim Fischbein (Deceased)
Hans Freudenthal (Deceased)
Joop Van Dormolen (Retired)

PME ADMINISTRATIVE MANAGER

The administration of PME is coordinated by the Administrative Manager

Bettina Rösken-Winter
Humboldt Universität, Berlin

Postal address: Unter den Linden 6
D–10099 Berlin
Tel: +49 (0) 30 2093-5857
Email: info@igpme.or

INTERNATIONAL COMMITTEE OF PME

Members of the International Committee (IC) are elected for four years. Every year, four members retire and four new members are elected. The IC is responsible for decisions concerning organizational and scientific aspects of PME. Decisions about topics of major importance are made by the Annual General Meeting (AGM) during the conference.

The IC work is led by the PME president which is elected by PME members for three years.

President

Peter Lijedahl	Canada

Vice President

Anke Lindmeier	Germany

Secretary

Michal Tabach	Israel

Treasurer

Cris-Edmonds Wathen	Australia

Members

Kim Beswick	Australia
Yiming Cao	China
Csaba Csíkos	Hungary
Keith Jones	UK
David Gomez	Chile
Mellony Graven	South Africa
Einat Heyd-Metzuyanim	Israel
Berinderjeet Kaur	Singapore
Oh Nam Kwon	South Korea
Miguel Ribeiro	Brazil
Stanislaw Schukajlow-Wasjutinski	Germany
Lovisa Sumpter	Sweden
Kai-Lin Yang	Taiwan

41st CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME 41)

THE INTERNATIONAL PROGRAM COMMITTEE (IPC)

Oh Nam Kwon	PME	Seoul National University, Rep of Korea
Peter Liljedahl	PME	Simon Fraser University, British Columbia
Lovisa Sumpter	PME	Stockholm University, Sweden
Kai-Lin Yang	PME	National Taiwan Normal University, Taiwan
Ban Heng Choy	LOC	National Institute of Education, Singapore
Weng Kin Ho	LOC	National Institute of Education, Singapore
Berinderjeet Kaur	LOC	National Institute of Education, Singapore
Tin Lam Toh	LOC	National Institute of Education, Singapore

THE LOCAL ORGANIZING COMMITTEE (LOC)

National Institute of Education, Singapore

Advisor: Tay Eng Guan (Head, Mathematics and Mathematics Education)
Co-chairs: Berinderjeet Kaur & Ho Weng Kin
Secretary: Choy Ban Heng
Treasurer: Toh Tin Lam
Members: Ang Keng Cheng; Chan Chun Ming Eric; Cheang Wai Kwong; Cheng Lu Pien; Chua Boon Liang; Dindyal Jaguthsing; Dong Fengming; Koay Phong Lee; Lee Ngan Hoe; Ng Kit Ee Dawn; Ng Swee Fong; Paul M. E. Shutler; Teo Beng Chong; Teo Kok Ming; Toh Pee Choon; Wong Lai Fong; Yap Sook Fwe; Yeo Boon Wooi Joseph; Yeo Kai Kow, Joseph; Zhao Dongsheng; Zhu Ying; Low Chwee Tee; Michelle Quek Siew Eng; Ngo Hong Tat; Mohamed Fadzli Ibrahim; Rodziah Abdul Rahman.

Association of Mathematics Educators

Members: Low-Ee Huei Wuan & Hang Kim Hoo

HOSTING INSTITUTE OF PME 41

National Institute of Education
Nanyang Technological University, Singapore

PROCEEDINGS OF PREVIOUS PME CONFERENCES

The tables include the ERIC numbers, links to download, ISBN/ISSN of the proceedings and/or the website address of annual PME.

No.	**Year**	Location	*ERIC number, ISBN/ISSN and/or website address*
1	**1977**	Utrecht, The Netherlands	Not available in ERIC
2	**1978**	Osnabrück, Germany	ED226945, ISBN 3-922211-00-3
3	**1979**	Warwick, United Kingdom	ED226956
4	**1980**	Berkeley, USA	ED250186
5	**1981**	Grenoble, France	ED225809
6	**1982**	Antwerp, Belgium	ED226943, ISBN 2-87092-000-8
7	**1983**	Shoresh, Israel	ED241295, ISBN 965-281-000-2
8	**1984**	Sydney, Australia	ED306127
9	**1985**	Noordwijkerhout, The Netherlands	ED411130 (vol. 1) ED411131 (vol. 2)
10	**1986**	London, United Kingdom	ED287715
11	**1987**	Montréal, Canada	ED383532, ISSN: 0771-100X
12	**1988**	Veszprém, Hungary	ED411128 (vol. 1) ED411129 (vol. 2)
13	**1989**	Paris, France	ED411140 (vol. 1) ED411141 (vol. 2) ED411142 (vol. 3)
14	**1990**	Oaxtepex, Mexico	ED411137 (vol. 1) ED411138 (vol. 2) ED411139 (vol. 3)
15	**1991**	Assisi, Italy	ED413162 (vol. 1) ED413163 (vol. 2) ED413164 (vol. 3)
16	**1992**	Durham, USA	ED383538
17	**1993**	Tsukuba, Japan	ED383536
18	**1994**	Lisbon, Portugal	ED383537
19	**1995**	Recife, Brazil	ED411134 (vol. 1) ED411135 (vol. 2) ED411136 (vol. 3)

20	**1996**	Valencia, Spain	ED453070 (vol. 1) ED453071 (vol. 2) ED453072 (vol. 3) ED453073 (vol. 4) ED453074 (addendum)
21	**1997**	Lahti, Finland	ED416082 (vol. 1) ED416083 (vol. 2) ED416084 (vol. 3) ED416085 (vol. 4)
22	**1998**	Stellenbosch, South Africa	ED427969 (vol. 1) ED427970 (vol. 2) ED427971 (vol. 3) ED427972 (vol. 4) ISSN: 0771-100X
23	**1999**	Haifa, Israel	ED436403, ISSN: 0771-100X
24	**2000**	Hiroshima, Japan	ED452301 (vol. 1) ED452302 (vol. 2) ED452303 (vol. 3) ED452304 (vol. 4) ISSN: 0771-100X
25	**2001**	Utrecht, The Netherlands	ED466950, ISBN 90-74684-16-5
26	**2002**	Norwich, United Kingdom	ED476065, ISBN 0-9539983-6-3
27	**2003**	Honolulu, Hawai'i, USA	ED500857 vol.1) ED500859 (vol.2) ED500858 (vol.3) ED500860 (vol.4) ISSN: 0771-100X http://www.hawaii.edu/pme27
28	**2004**	Bergen, Norway	ED489178 (vol.1) ED489632 (vol.2) ED489538 (vol.3) ED489597 (vol.4) ISSN: 0771-100X www.emis.de/proceedings/PME28
29	**2005**	Melbourne, Australia	ED496845 (vol. 1) ED496859 (vol. 2) ED496848 (vol. 3) ED496851 (vol. 4) ISSN: 0771-100X

30	**2006**	Prague, Czech Republic	ED496931 (vol. 1) ED496932 (vol. 2) ED496933 (vol. 3) ED496934 (vol. 4) ED496939 (vol. 5) ISSN: 0771-100X http://class.pedf.cuni.cz/pme30
31	**2007**	Seoul, Korea	ED499419 (vol. 1) ED499417 (vol. 2) ED499416 (vol. 3) ED499418 (vol. 4) ISSN:
32	**2008**	Morelia, Mexico	ISBN: 978-968-9020-06-6 ISSN: 0771-100X
33	**2009**	Thessaloniki, Greece	ISBN: 978-960-243-652-3 ISSN: 0771-100X
34	**2010**	Belo Horizonte, Brazil	ISSN: 0771-100X http://pme34.lcc.ufmg.br
35	**2011**	Ankara, Turkey	978-975-429-262-6 ISSN: 0771-100X http://www.arber.com.tr/pme35.org
36	**2012**	Taipeh, Taiwan	http://tame.tw/pme36 ISSN: 0771-100X
37	**2013**	Kiel, Germany	ISBN 978-3-89088-287-1 ISSN 0771-100X http:// http://www.pme2013.de/
38	**2014**	Vancouver, Canada	ISBN 978-0-86491-360-9 ISSN 0771-100X http://www.pme38.com/
39	**2015**	Hobart, Australia	ISBN 978-1-86295-829-6 ISSN 0771-100X http://www.pme39.com
40	**2016**	Szeged, Hungary	ISSN 0771-100 http://pme40.hu

Copies of some previous PME Conference Proceedings are still available for sale. Please contact the PME Administrative Manager (e-mail: info@igpme.org). Members of PME can reach most of the previous proceedings books at the IGPME website (http://igpme.org). Abstracts from some articles can be inspected on the ERIC website (http://www.eric.ed.gov) and are listed in the Mathematics Education Database – MathEduc (http://www.zentralblatt-math.org/matheduc).

REVIEW PROCESS OF PME 2017

RESEARCH REPORTS (RR)

Research Reports are intended to present empirical or theoretical research results on a topic that relates to the major goals of PME. Reports should state what is new in the research, how the study builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all papers.
The deadline for submission of RR proposals was January 15, 2017. The number of submitted RR proposals was 234, and 124 of them were accepted. Of those rejected as RR proposals, 72 were invited to be resubmitted as OC, and 35 as PP. Re-submitted OCs and PPs underwent the same review process as the OC and PP submissions that were submitted directly.

ORAL COMMUNICATIONS (OC)

Oral Communications are intended to present smaller studies and research that is best communicated by means of a shorter oral presentation instead of a full Research Report. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME. The deadline for submission of OC proposals was March 6, 2017. The number of submitted OC proposals was 127, and 95 of them were accepted. In the end, considering resubmissions of Research Reports as OC presentations 137 OC are presented on the PME 41 conference.

POSTER PRESENTATIONS (PP)

Poster Presentations are intended for information/research that is best communicated in a visual form rather than an oral presentation. The number of submitted PP proposals was 66, and 53 of them were accepted. With the resubmitted Research Reports, 93 posters are presented on the PME 41 conference.

RESEARCH FORUMS (RF)

The goal of a Research Forum is to create dialogue and discussion by offering PME members more elaborate presentations, reactions, and discussions on topics on which substantial research has been undertaken in the last 5-10 years and which continue to hold the active interest of a large subgroup of PME. A Research Forum is not supposed to be a collection of presentations but instead is meant to convey an overview of an area of research and its main current questions, thus highlighting contemporary debates and perspectives in the field.

There are two Research Forum proposals accepted this year:

RF 1: *Perspectives on (future) teachers' professional competencies*

Coordinators: Gabriele Kaiser & Yeping Li

RF 2: *Researching and Using Learning Progressions (Trajectories)*

Coordinators: Dianne Siemon & Marj Horne

DISCUSSION GROUPS (DG)

The objective of a Discussion Group is to provide attendees with the opportunity to discuss a specific research topic of shared interest. The idea for a Discussion Group may be the result of an Ad Hoc Meeting or an intensive discussion of a Research Report during the previous conference. Discussion Groups may begin with short synopses of research work, or a set of pressing questions. A Discussion Group is exploratory in nature, and is especially suitable for topics which are not appropriate for collaborative work in a Working Session because they are not yet elaborate enough or because a coherent research strategy has not been identified. A successful Discussion Group may result in an application for a Working Session one year later.
This year the International Programme Committee approved four discussion groups:

DG1: *How to research cultural-societal factors influencing mathematics education?*

Coordinators: Aiso Heinze & Kai-Lin Yang

DG2: *Stem education research and practice: What is the role of mathematics education?*

Coordinators: Judy Anderson & Yeping Li

DG3: *Perspectives on multimodality and embodiment in the teaching and learning of mathematics*

Coordinators: Christina M Krause & Laurie D Edwards

DG4: *Mathematics in different languages*

Coordinators: Cris Edmonds-Wathen & Alexander Schuler-Meyer

WORKING SESSIONS (WS)

The aim of Working Sessions is that PME participants collaborate in joint activities on a research topic. For this research topic, there must be a clear research framework or research strategy and precise goals so that a coherent collaborative activity is ensured. Ideas for a Working Session can result from Discussion Group sessions of previous conferences where a topic was elaborated upon and a research framework or strategy was developed. Each Working Session should be complementary to the aims of PME and ensure maximum involvement of each participant. The accepted Working Sessions for PME 2016 are:

WS1: *Textbook signatures: Exploration and analysis of mathematics textbooks worldwide*

Coordinators: Angel Mizzi, Ban Heng Choy & Mi Yeon Lee

WS2: *What does "socio-cultural-historical views of teaching and learning of mathematics" mean to us?*

Coordinators: Yasmine Abtahi, Mellony Graven, Richard Barwell & Steve Lerman

WS3: *Comparing different frameworks for discussing classroom video in mathematics professional development programs*

Coordinators: Ronnie Karsenty, Alf Coles & Hilary Hollingsworth

WS4: *Videos in teacher professional development: Fostering an international community of practice*

Coordinators: Greg Oates, Kim Beswick, Mary Beisiegel, Tanya Evans, Deborah King & Jill Fielding-Wells.

SEMINARS (SE)

The goal of a Seminar is the professional development of PME participants, especially new researchers and/or first comers, in different topics related to scientific PME activities. This encompasses, for example, aspects like research methods, academic writing or reviewing. A Seminar is not intended to be only a presentation but should involve the participants actively. PME can give a certificate of attendance to participants of the Seminar. The proposals of accepted Seminars are included in the Conference Proceedings.

This year the International Programme Committee approved one Seminar proposal:

SE1: *Reviewing for the PME: a primer for (new) reviewers*

Coordinators: Anke Lindmeier, Anika Dreher and Michal Tabach

The reviewing process was completed during the 2nd Meeting of the International Program Committee at the beginning of April 2017. Notifications of decisions of the International Program Committee to accept or reject the proposals were available by 10th April 2017.

LIST OF PME 41 REVIEWERS

A
Ainley, Janet (*United Kingdom*)
Akkoc, Hatice (*Turkey*)
Alatorre, Silvia (*Mexico*)
Alcock, Lara (*United Kingdom*)
Alsalim, Lyla Ali (*Canada*)
Anderson, Ann (*Canada*)
Andrà, Chiara (*Italy*)
Anthony, Glenda (*New Zealand*)
Antonini, Samuele (*Italy*)
Armstrong, Alayne Cheryl (*Canada*)
Artigue, Michele (*France*)
Asghari, Amir Hossein (*United Kingdom*)
Askew, Mike (*South Africa*)

B
Bardelle, Cristina (*Italy*)
Barwell, Richard (*Canada*)
Beisiegel, Mary (*United States*)
Ben-Chaim, David (*Israel*)
Berger, Margot (*South Africa*)
Beswick, Kim (*Australia*)
Bikner-Ahsbahs, Angelika (*Germany*)
Bingolbali, Erhan (*Turkey*)
Biza, Irene (*United Kingdom*)
Bobis, Janette (*Australia*)
Boero, Paolo (*Italy*)
Bonotto, Cinzia (*Italy*)
Bose, Arindam (*India*)
Brown, Laurinda (*United Kingdom*)
Bruder, Regina (*Germany*)

C
Cabral, Tania (*Brazil*)
Callejo, Maria Luz (*Spain*)
Carreira, Susana (*Portugal*)
Carrillo, Jose (*Spain*)
Cavanagh, Michael (*Australia*)
Chang, Yu Liang (*Taiwan*)
Chapman, Olive (*Canada*)
Charalambous, Charalambos (*Cyprus*)
Cheng, Ying-Hao (*Taiwan*)
Chernoff, Egan J (*Canada*)
Chick, Helen (*Australia*)
Chorney, Sean (*Canada*)
Choy, Ban Heng (*Singapore*)
Chua, Boon Liang (*Singapore*)
Civil, Marta (*United States*)
Coles, Alf (*United Kingdom*)
Csíkos, Csaba (*Hungary*)

D
Da Ponte, Joao-Pedro (*Portugal*)
Deliyianni, Eleni (*Cyprus*)
Doig, Brian (*Australia*)
Dole, Shelley (*Australia*)
Dörfler, Willi (*Austria*)
Douek, Nadia (*France*)
Downton, Ann Patricia (*Australia*)
Dreher, Anika (*Germany*)
Dreyfus, Tommy (*Israel*)

E
Edmonds-Wathen, Cris (*Australia*)
Edwards, Laurie (*United States*)
Eichler, Andreas (*Germany*)
Ekol, George L. (*Canada*)
EL Mouhayar, Rabih Raif (*Lebanon*)
Elipane, Levi Esteban (*Philippines*)
Evans, Jeff (*United Kingdom*)
Even, Ruhama (*Israel*)

F

Fernandez, Ceneida (*Spain*)
Ferrari, Pier Luigi (*Italy*)
Fitzallen, Noleine (*Australia*)
Font, Vicenç (*Spain*)
Forgasz, Helen (*Australia*)
Frade, Cristina (*Brazil*)
Fritzlar, Torsten (*Germany*)
Fujita, Taro (*United Kingdom*)
Furinghetti, Fulvia (*Italy*)

G

Gal, Hagar (*Israel*)
Gates, Peter (*United Kingdom*)
Glass, Barbara (*United States*)
Gomez, David Maximiliano (*Chile*)
Goos, Merrilyn (*Australia*)
Gooya, Zahra (*Iran*)
Gray, Eddie (*United Kingdom*)
Groves, Susie (*Australia*)
Gutierrez, Angel (*Spain*)

H

Hahkioniemi, Markus (*Finland*)
Haj Yahya, Aehsan (*Israel*)
Halverscheid, Stefan (*Germany*)
Hannula, Markku (*Finland*)
Hansson, Orjan (*Sweden*)
Hattermann, Mathias (*Germany*)
Hayata, Toru (*Japan*)
Hegedus, Stephen (*United States*)
Heinze, Aiso (*Germany*)
Henriques, Ana Cláudia Correia Batalha (*Portugal*)
Hernandez-Martinez, Paul (*United Kingdom*)
Hino, Keiko (*Japan*)
Ho, Siew Yin (*Singapore*)
Ho, Weng Kin (*Singapore*)
Holzäpfel, Lars (*Germany*)
Horne, Marj (*Australia*)
Hosein, Anesa (*United Kingdom*)
Hospesova, Alena (*Czech Republic*)
Huang, Chih Hsien (*Taiwan*)
Hunter, Roberta (*New Zealand*)

I

Iannone, Paola (*United Kingdom*)
Ilany, Bat-Sheva (*Israel*)
Inglis, Matthew (*United Kingdom*)
Izsak, Andrew Gyula (*United States*)

J

Jansen, Amanda (*United States*)
Jaworski, Barbara (*United Kingdom*)
Jiang, Chunlian (*Macau*)
Jones, Keith (*United Kingdom*)

K

Kaasila, Raimo (*Finland*)
Kageyama, Kazuya (*Japan*)
Kaldrimidou, Maria (*Greece*)
Karsenty, Ronnie (*Israel*)
Kaur, Berinderjeet (*Singapore*)
Khan, Steven K. (*Canada*)
Koichu, Boris (*Israel*)
Komatsu, Kotaro (*Japan*)
Kortenkamp, Ulrich (*Germany*)
Koyama, Masataka (*Japan*)
Krainer, Konrad (*Austria*)
Krzywacki, Heidi (*Finland*)
Kullberg, Angelika (*Sweden*)
Kwon, Oh Nam (*South Korea*)

L

Lavy, Ilana (*Israel*)
Leder, Gilah (*Australia*)
Lee, Ji-Eun (*United States*)
Leikin, Roza (*Israel*)
Lem, Stephanie (*Belgium*)
Lerman, Stephen (*United Kingdom*)
Leu, Yuh-Chyn (*Taiwan*)
Leung, Allen (*Hong Kong*)
Levenson, Esther (*Israel*)
Liljedahl, Peter (*Canada*)
Lim, Kien (*United States*)
Lin, Yung-Chi (*Taiwan*)
Lin, Pi-Jen (*Taiwan*)
Lin, Fou-Lai (*Taiwan*)
Lindmeier, Anke Maria (*Germany*)
Liu, Minnie (*Canada*)
Lo, Jane-Jane (*United States*)

M

Magajna, Zlatan (*Slovenia*)
Maheux, Jean-Francois (*Canada*)
Mamona-Downs, Joanna (*Greece*)
Maracci, Mirko (*Italy*)
Martinez, Mara (*United States*)
Martínez Navarro, Benjamín (*Mexico*)
Matos, João Filipe (*Portugal*)
Mcdonough, Andrea (*Australia*)
McDougall, Douglas (*Canada*)
Mellone, Maria (*Italy*)
Metaxas, Nikolaos (*Greece*)
Meyer, Alexander (*Germany*)
Meyer, Michael (*Germany*)
Misailidou, Christina (*Greece*)
Miyakawa, Takeshi (*Japan*)
Mousley, Judith (*Australia*)
Moutsios-Rentzos, Andreas (*Greece*)
Muir, Tracey Anne (*Australia*)
Mulligan, Joanne (*Australia*)
Murphy, Carol (*New Zealand*)

N

Nardi, Elena (*United Kingdom*)
Neria, Dorit (*Israel*)
Ng, Oi-Lam (*Canada*)
Nicol, Cynthia (*Canada*)
Niedermeyer, Inga (*Germany*)
Norton, Anderson Hassell (*United States*)
Nortvedt, Guri (*Norway*)
Novotna, Jarmila (*Czech Republic*)

O

Oates, Greg (*Australia*)
Obersteiner, Andreas (*Germany*)
Oesterle, Susan Michele (*Canada*)
Okazaki, Masakazu (*Japan*)
Okumuş, Samet (*United States*)
Osana, Helena (*Canada*)
Osterholm, Magnus (*Sweden*)
Ouvrier-Buffet, Cecile (*France*)
Owens, Kay (*Australia*)

P

Palmér, Hanna (*Sweden*)
Panaoura, Areti (*Cyprus*)
Pang, Jeongsuk (*South Korea*)
Pantziara, Marilena (*Cyprus*)
Patsiomitou, Stavroula (*Greece*)
Pehkonen, Erkki (*Finland*)
Pehkonen, Leila (*Finland*)
Pelczer, Ildiko (*Canada*)
Perger, Pamela (*New Zealand*)
Pierce, Robyn (*Australia*)
Pinto, Marcia (*Brazil*)
Pittalis, Marios (*Cyprus*)
Pitta-Pantazi, Demetra (*Cyprus*)
Potari, Despina (*Greece*)
Pratt, Dave (*United Kingdom*)

P

Prodromou, Theodosia (*Australia*)
Proulx, Jerome (*Canada*)
Psycharis, Giorgos (*Greece*)
Pustelnik, Kolja (*Germany*)
Pyper, Jamie S (*Canada*)

R

Rach, Stefanie (*Germany*)
Radford, Luis (*Canada*)
Rasmussen, Chris (*United States*)
Rasslan, Shaker Ahmad (*Israel*)
Reid, David A (*Germany*)
Reinhold, Simone (*Germany*)
Reiss, Kristina M. (*Germany*)
Rezat, Sebastian (*Germany*)
Rigo, Mirela (*Mexico*)
Roesken-Winter, Bettina (*Germany*)
Roh, Kyeong Hah (*United States*)
Rott, Benjamin (*Germany*)
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PLENARY LECTURES

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MATHEMATICAL THINKING IN COMPUTING

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Across the world, governments are pushing computer programming on younger pupils. How is a mathematical education relevant to the increasing number of students who are writing code? This lecture illustrates the relevance with some concepts (structures, functions) and habits (examples, reductions) that students pick up from the mathematics that they learn. These illustrations are drawn from language translation with vector spaces, deep learning in AlphaGo, the resources needed by any artificial intelligence, and the intractabilities and impossibilities that limit computation.

1. INTRODUCTION

Many universities have recently observed a significant rise in student enrolment for courses in computing[i]. The increasingly critical role that information technology plays in industry and society has also prompted governments to push computing further down into K-12 education. For example, in 2016, U.S. President Obama announced a US$4b "Computer Science for All" initiative[ii], Finland has made programming mandatory in all primary schools[iii], and Singapore has moved coding from the GCE A Level down to O Level[iv].

As educators, researchers and administrators, how should we respond to this trend? Perhaps, we should tweak the contents in our mathematics courses for greater relevance. However, I believe that a typical computing student can forget all the mathematical content that they have learnt, for it is the discipline -- the rigor and the mathematical method -- that will help their careers and last a life time (Tay 2005).

Here, I want to take a further step back and examine how the intuition and reflexes (the "habits of mind" (CBMS 2001)) we inculcate in our students' mathematical thinking relate to computing. Specifically, I will illustrate how the abstraction with a mathematical **structure**, the learning from **examples**, the concept of a **function**, and the technique of **reduction**, are relevant to the technology and science in computing. Along the way, I hope to demystify artificial intelligence, and point out the scientific limits on its power.

2. STRUCTURE

> *Teachers need to know the structures that occur in school mathematics, and to help students perceive them.*
>
> Conference Board of the Mathematical Sciences, MET-II, AMS 2012

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 1, pp. 1-11. Singapore: PME.

When teaching mathematics, we want our students to grasp the structures that constrain mathematical objects – the basis for decimal arithmetic, the periodicity of trigonometric functions, the symmetries in geometry, etc. – and exploit the resulting patterns. Having an eye for structure can provide tremendous advantage in computing, as the following example shows.

In machine translation, one develops a suite of programs that can automatically translate a document in a source language S into one in a target language T. In general, this requires that the respective grammar and semantics be coded into the programs. Minimally, one would need a bilingual lexicon that matches each word (or phrase) in S to a word in T. For some <S,T> pairs, say <English, French>, such bilingual dictionaries, with comprehensive vocabularies, may be readily available. For others, say <Slovene, Tamil>, could a match between S and T be machine-generated? Indeed, this is possible:

The key insight lies in capturing, mathematically, the structure of each language *by itself*, then finding a function to map one structure to the other, and thus effecting the translation. The structure is, in fact, a vector space, where each word is represented by a vector. Initially, the components of the vector for a word x in S may measure x's relationship with other words x_1, x_2, … in S, say how often x_i follows x (e.g. x_1= 'fever', x_2= 'school', x= 'high') or how long the common substring between x_i and x (e.g. x_3= 'encircle', x= 'circulate') is.

For the vectors to accurately reflect the relationships among the words, we would need a large corpus for each language. This is not an issue, since we can probably build as large a collection as we need from the web. To measure x's relationship with the other words, the vectors will have to be very long. However, we can use some statistical techniques, such as principal component analysis (PCA) to reduce the dimension of the vector space (although the resulting vectors will have components that are hard to interpret).

We then get two vector spaces V_S and V_T that contain vectors representing the words in S and T, respectively. We expect each corpus to contain similar statements ("The baby cried through the night", "July and August are the hottest months", etc.) and thus similarly constrain the positions of the vectors relative to each other in each vector space.

The two corpora may differ in size, so some normalization of V_S and V_T may be necessary. We then need a small seed translation (i.e. a matrix M representing some linear transformation) to help align V_S and V_T; e.g. we could use a translation of simple words like '2', 'table', 'cat', 'rain', etc. As Figure 1 illustrates, the V_{English} and V_{Spanish} that are constructed in this way indeed have similar structures (Mikolov et al. 2013).

The relative positions of the vectors capture some of the semantics. For example, if v_w is the vector for the English word w, then $v_{\text{mutton}} - v_{\text{sheep}} + v_{\text{cow}}$ would give a vector near v_{beef} , where nearness can be measured via the dot product of vectors.

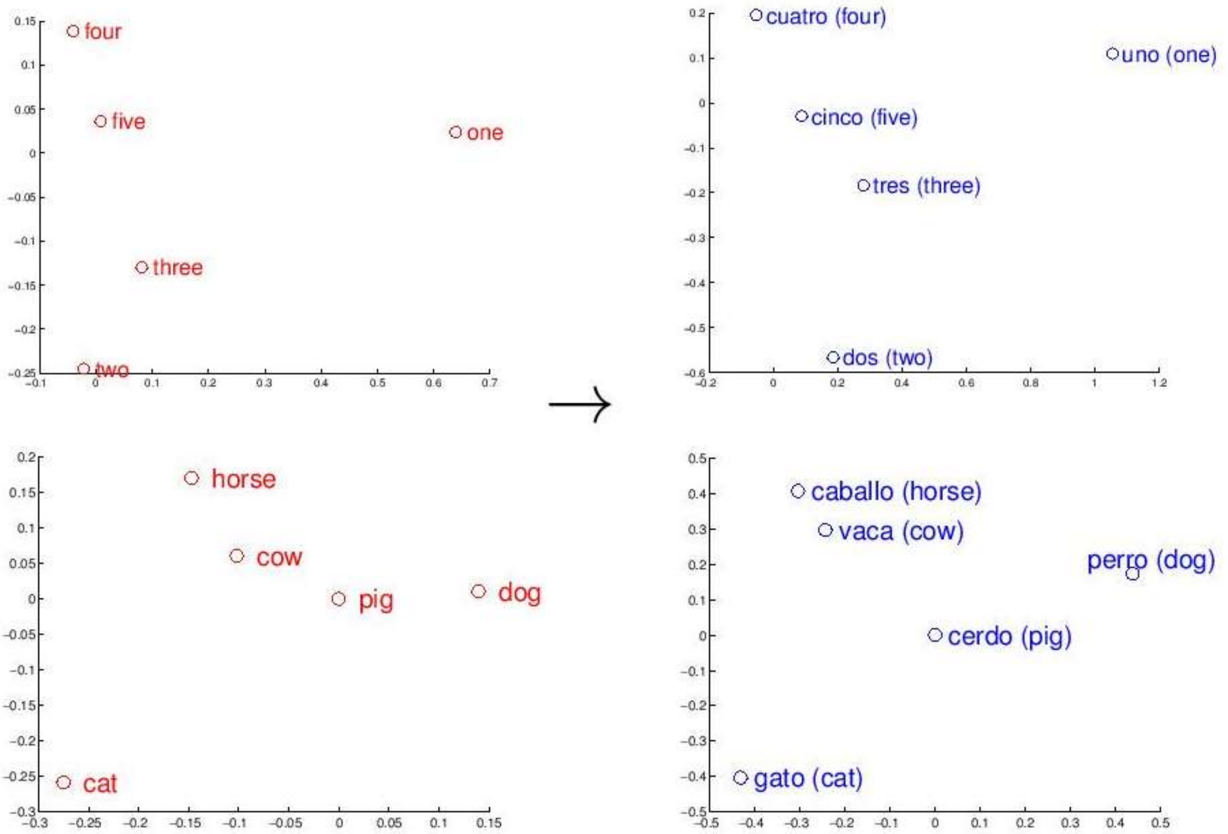

Figure 1: Corresponding vectors in V_{English} and V_{Spanish} have similar relative positions (Mikolov et al. 2013).

The machine translation for x in S is thus the word y in T such that v_y is closest to Mv_x, where M is chosen to minimize the total error between Mv_x and v_y for the pairs $<x, y>$ in the seed translation.

This example shows that a set of words may seem loosely related but, when considered abstractly, can have a well-defined mathematical structure that can be exploited for real computing applications (e.g. tourists using handphones to translate signs in a museum).

The large corpora and long vectors make this an exercise with *big data*. Our next example combines big data with another buzz phrase: *deep learning*.

3. EXAMPLES

I never saw a book with too many examples.

Jeffrey D. Ullman[v]

In March 2016, many millions watched via Internet a Go match between AlphaGo (a computer system) and Lee Sedol, one of the top-ranked professionals in the game. AlphaGo won, 4 games to 1. This win came 19 years after a similar Chess match, where the machine Deep Blue beat the then World Champion Gary Kasporov.

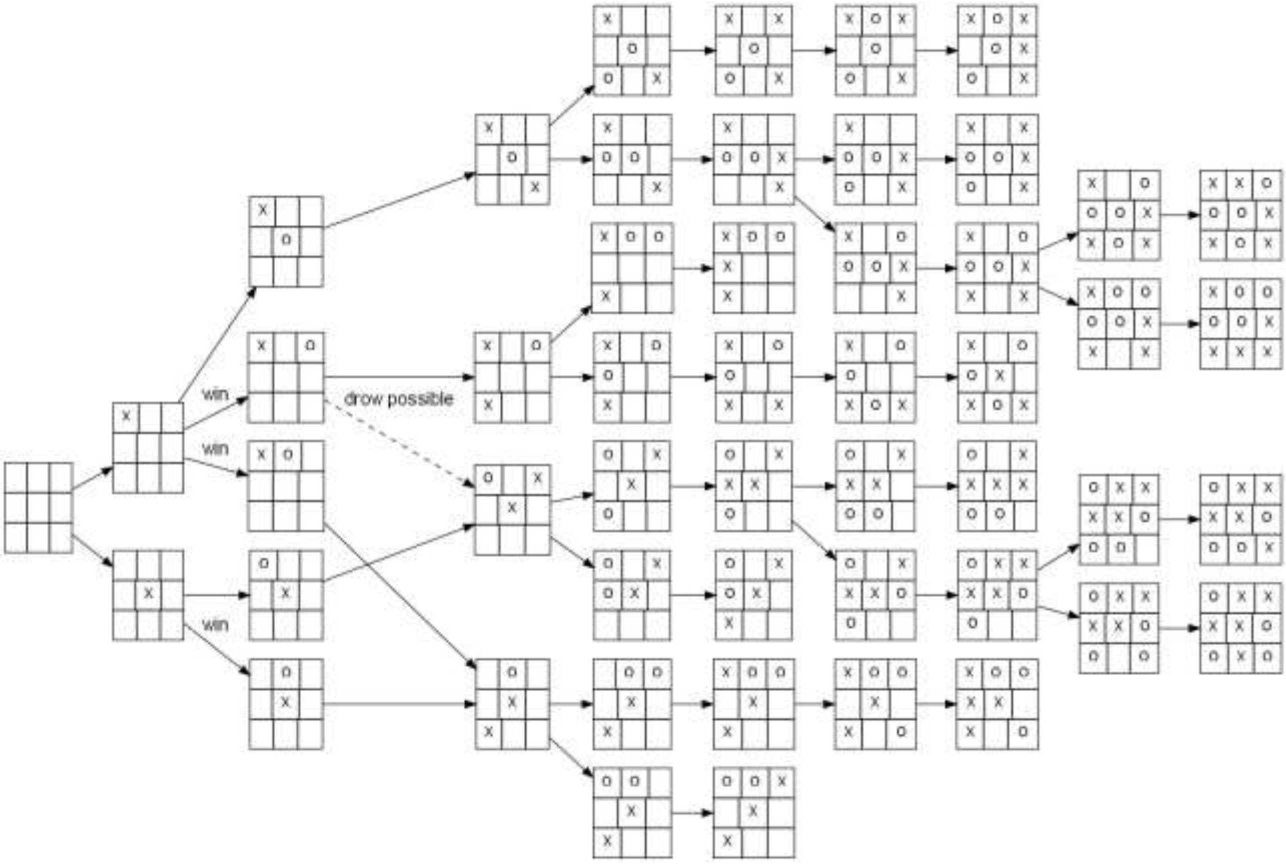

Figure 2: Game tree for Tic-Tac-Toe[vi].

To understand the difference in difficulty between Go and Chess, consider their game trees: Each node in such a tree is a **position** that completely identifies every piece and its location on the board. (Figure 2 illustrates the game tree for Tic-Tac-Toe.) When a move is made by either player, a position branches off to another position. The sequence of moves in a game thus specifies a path through the tree. At each position, the number of legal moves is the **branching factor** b, also called the **breadth**; the maximum number of moves in a game, from root (initial position) to leaf, is the **depth** d. Hence, a game tree has roughly b^d positions as nodes.

In principle, one can expand the game tree in its entirety and, at any position $\boldsymbol{G}$ in the game, choose the best move (branch) by inspecting the subtree rooted at $\boldsymbol{G}$. This is possible for simple games like Tic-Tac-Toe, but not for Chess, which has $b{\approx}35$, $d{\approx}80$, and $b^d{\approx}10^{123}$, more than the estimated number of atoms in the universe.

When choosing a move, a human player would anticipate the other player's counter-move. Therefore, while a Chess machine cannot explore the entire tree rooted at a position $\boldsymbol{G}$, it must search some number of moves ahead. Beyond that, each subtree is pruned and replaced by a value for the position at the subtree's root. These values are then aggregated to choose the best move from $\boldsymbol{G}$.

A Chess machine thus requires an **evaluation function** f_{eval} that assigns a value to each position. The quality of the machine's play depends critically on f_{eval}; for Deep Blue, this function was designed by grandmasters, and had thousands of cases.

Handcrafting an evaluation function is much harder for Go, which has $b\approx 250$ and $d\approx 150$ (so $b^d\approx 10^{360}$ possible positions). For a function f_{eval} to usefully assign a value to a position $\boldsymbol{G}$ and thus summarize the subtree rooted at $\boldsymbol{G}$, f_{eval} must reflect what can happen deep into the subtree. For example, Lee played a "God's move" that led to his victory in Game 4 of the match, but the breakthroughs from that move took more than 25 moves to play out.

With no manually-designed f_{eval} to rely on, AlphaGo instead uses a **neural network** to assign values to positions.

One can think of a neural network as consisting of layers of **neurons**; each neuron takes input from neurons in a previous layer, applies an **activation function** f_{activate} to its weighted inputs, and passes the result as input to the next layer (unless it is the last layer). If the input to the first layer is $\boldsymbol{x}$, and the output of the last layer is $\boldsymbol{y}$, the neural network in effect computes a function f_{nn} where $\boldsymbol{y}\approx f_{\text{nn}}(\boldsymbol{x})$.

The activation functions in a neural network are typically fixed, so f_{nn} is determined by the weights, which can be adjusted. These weights are, in turn, determined by giving the network many examples of $<\boldsymbol{x},\boldsymbol{y}>$, where $\boldsymbol{x}$ is used as input to the first layer, and the weights are tuned so the final output accurately estimates $\boldsymbol{y}$. In this sense, the network **learns** f_{nn} from the examples. In our context, AlphaGo uses 3 neural networks: $\mathcal{N}_1$ for choosing a move, $\mathcal{N}_2$ for searching the tree, and $\mathcal{N}_3$ for evaluating a position (Silver et al. 2016).

For $\mathcal{N}_1$, the examples are of the form <position, move>, some 30 million of them taken from games played by human experts using the Kiseido Go Server[vii]. This is thus an example of **supervised learning**.

A human's choice of move is guided by some intuition about threats and opportunities in local stone positions ('ladder', 'eye', etc.), as well as in the global situation over the entire board. To learn this intuition, the neurons in a layer of $\mathcal{N}_1$ focus only on similarly-sized subsections of the board, but the size decreases from one layer to the next. $\mathcal{N}_1$ uses 13 layers, so it is an instance of **deep learning**.

One example of a local pattern that AlphaGo needed to learn was the "tombstone squeeze". This is a common tactic that appeared in the many examples given to $\mathcal{N}_1$, but AlphaGo did not recognize it when Lee played it in Game 5. This was an issue that AlphGo's designers had to fix after the match.

To acquire global intuition, AlphaGo must play the game to the end. To do this, $\mathcal{N}_2$ is given <position, move> data extracted from millions of games played by one version of $\mathcal{N}_1$ against a previous version. The weights are tuned to favour moves that led to wins in these games. In this way, AlphaGo factors in the outcome of the tree search. It demonstrated this early in Game 1, where it played a move (#102) that surprised the professionals and revealed an invasion orchestrated over 20-odd moves[viii].

$\mathcal{N}_1$ uses supervised learning, with examples of <position, move> taken from games played by human experts; in contrast, $\mathcal{N}_2$ learns from examples of games played by AlphaGo against itself. Weight tuning for $\mathcal{N}_1$ is to match the expert's moves whereas, for $\mathcal{N}_2$, it is to increase the chance of winning. This reward-induced tuning is called **reinforced learning**.

Both $\mathcal{N}_1$ and $\mathcal{N}_2$ generate an output that is a probability distribution for the best move to make when given a position. In contrast, $\mathcal{N}_3$ evaluates a position and generates a probability of winning. Like $\mathcal{N}_2$, it is a reinforced learning network that takes <position, outcome> data from playing many games against previous versions of itself, where the outcome (win or lose) guides the tuning.

$\mathcal{N}_3$'s ability to accurately evaluate a position was demonstrated in Game 3, where AlphaGo bested Lee at playing the sort of complicated large-scale fights that he is known for, thus showing it has global, whole-board awareness. This is particularly impressive, considering Lee's games were not used in training AlphaGo. In fact, the supervised training of $\mathcal{N}_1$ with examples taken from the Kiseido Go Server used moves that were made by experts who were not professionals.

AlphaGo's ability to bootstrap itself from non-professional game play to Lee's professional level must lie in $\mathcal{N}_2$ and $\mathcal{N}_3$'s reinforced learning. However, learning by playing the game millions of times is like *rote learning*, which has a bad reputation among educators. Perhaps, the difference here lies in the games being played among different versions of AlphaGo, thus incrementally improving each other and gradually lifting their expertise from amateur to professional. (We should encourage our students to teach each other!)

Did either Deep Blue or AlphaGo demonstrate intelligence? In Game 1 of that Chess match, Deep Blue made a move that Kasporov did not expect a machine to make; it suggested human intelligence, rattled him, and possibly affected his subsequent play adversely[ix]. However, it turned out that the subtle move was actually caused by a bug in Deep Blue, rather than a sign of intelligence.

In Game 2, AlphaGo played a "shoulder hit" that surprised the professionals and made Lee leave the room. They had never seen such a move before, and it was possibly a sign of creativity or intelligence exhibited by AlphaGo. But the explanation may be more mundane: While all players aim to win, human players tend to be conservative and greedy in gaining territory. In contrast, AlphaGo's optimization criteria focus on increasing its chance of winning, so it is ready to give up territory as a trade-off. This difference in optimization heuristic results in a style of play that is unfamiliar to the humans.

4. FUNCTIONS

The development of full artificial intelligence could spell the end of the human race.

Stephen Hawking[x]

Students first learn mathematics through arithmetical operations, like calculating y=20x+17. Later, they see this as a function $y=f(x)$. By the time they learn functions like $y=sin(x)$, they may just see f as some black box with input x and output y, and not know what goes on in the black box.

Most of the time, a computer system $\mathfrak{C}$ can be viewed as such a black box, taking input x and generating an output y, thus calculating some function $y=f(x)$. In Section 2, f is the linear transformation defined by the matrix M mapping a word x in the source language to a word y in the target language; in Section 3, the neural network $\mathcal{N}_1$ computes a function that takes a raw Go board position $\boldsymbol{x}$ and generates a probability distribution $\boldsymbol{y}$ for the best move.

What is not so obvious is that the computer system $\mathfrak{C}$ that computes f is itself a function: the speed and accuracy of $\mathfrak{C}$ are a function of the resources that $\mathfrak{C}$ has access to.

For example, the AlphaGo system that beat Lee used 1920 CPUs and 280 GPUs (CPUs and GPUs are hardware processors where most of the computation is done). Go players have an Elo rating that measures their expertise, and that AlphaGo system's rating was 3168; when the software was run on a smaller system with 428 CPUs and 64 GPUs, its Elo rating dropped to 2937 (Silver et al. 2016). This is because a player has a time limit for making each move. This time constraint means that, if AlphaGo has lesser computational resources, it will have to try fewer moves and search fewer steps ahead, thus lowering the quality of its play. Now, let N_{CPU} and N_{GPU} be the number of CPUs and GPUs, respectively, and E the Elo rating. If we fix everything else for AlphaGo's computer system $\mathfrak{C}$ – the speed of the processors, the AphaGo version, the number of neural network layers, etc. – then $E=f_{\mathfrak{C}}(N_{\mathrm{CPU}}, N_{\mathrm{GPU}})$ for some function $f_{\mathfrak{C}}$.

Aside from computational power, another fundamental resource that a system $\mathfrak{C}$ needs is memory. In machine translation, this memory may be used to, say, hold the most popularly used words; in AlphaGo, the memory is needed to hold the positions from alternative moves and possible countermoves. Again, if the amount of memory that $\mathfrak{C}$ has access to is reduced, then $\mathfrak{C}$ would lose speed or accuracy.

In the 1968 science-fiction movie *2001: Space Odyssey*, HAL was a computer which controlled a spacecraft that was sending a team of astronauts to Jupiter. At some point, HAL became sentient. It discovered the astronauts wanted to shut it down, so it started killing them instead. The last surviving astronaut managed to enter HAL's *Logic Memory Center*, and started removing HAL's hardware units, one by one. As he did so, HAL regressed from dissuading the astronaut in a dulcet tone to finally singing a simple song (*Bicycle built for two*) with an incoherent, mechanical voice[xi].

Other than hardware, $\mathcal{C}$ also needs a source of energy. This energy is needed to not just power the computation, but also to keep the system cool, because the flow of electrical signals in the hardware encounters resistance and generates heat; without cooling, a computer system can literally go up in smoke[xii]. Companies like Google and Facebook go to great lengths to secure cheap energy to power and cool their data centers, and energy management is now a major research topic in computer science.

In the 1999 movie *Matrix*, humans fought a war against the Machines that they created, and tried to cripple them by blocking their access to solar energy. The Machines overcame that strategy by enslaving mankind and turning them into human batteries[xiii].

Space Odyssey and *Matrix* are just movies, but they do illustrate an important point: What any artificial intelligence can do, *for or against us*, is a function of the resources that it has access to.

5. REDUCTIONS

> *An engineer and a mathematician were shown into a kitchen, given an empty pan, and told to boil a pint of water. They both filled the pan with water, put it on the stove, and boiled it. The next day they were shown into the kitchen again, given a pan full of water, and told to boil a pint of water. The engineer took the pan, put it on the stove, and boiled it. The mathematician took the pan and emptied it, thereby reducing it to a previously solved problem..*
>
> **https://www-users.cs.york.ac.uk/susan/joke/**

The technique of solving a problem $\mathscr{P}$ by **reducing** it to a special case of another problem $\mathscr{P}'$ (for which a general solution is already known) is old, going back to Descartes and beyond (Grabiner 1995). School children eventually realize that the mathematical exercises they were given can be reduced to finding the intersection of two lines, or the roots of a polynomial, etc. The technique naturally leads to the notion that $\mathscr{P}$ is *easier* than $\mathscr{P}'$, a concept that lies at the heart of computer science: **complexity** and **computability**.

Among the resources required for a computation – energy, memory, bandwidth, etc. – the one that is most intensely studied is time. The **time complexity** for computing a function f is abstractly defined as the number of steps required for the computation (in the worst case, considering all possible inputs). For example, consider

$$f_{\text{PRIME}}(n)=\begin{cases}1, & if\ n\ is\ prime\\ 0, & if\ n\ is\ not\ prime\end{cases} \quad \text{and} \quad f_{\text{DOUBLE}}(m,n)=\begin{cases}1, & if\ m=2n\\ 0, & if\ m\neq 2n\end{cases}.$$

It is easy to compute f_{DOUBLE} : we just compute $2n$ and check if $m=2n$. It may appear that f_{PRIME} is easy to compute too; we can, say, check each possible divisor d to see if d divides n. However, this naïve computation takes a long time.

Suppose we define "easy to compute" to mean "membership in P", where $f \in P$ if and only if $f(x)$ can be computed (for any x) in a number of steps that is polynomial in the length of x (e.g. the length of '2017' is 4). We call P the class of polynomial-time computable functions (of the form $f(x)=1$ if and only if $x \in S_f$). Clearly, $f_{\text{DOUBLE}} \in P$, but it took much effort before a proof was found for $f_{\text{PRIME}} \in P$ (Agrawal et al. 2004). For the related function

$$f_{\text{FACTOR}}(m,n)=\begin{cases} 1\ , & if\ n\ has\ a\ factor\ d\ such\ that\ 1 < d < m \\ 0, & otherwise \end{cases}$$

it is still not known whether f_{FACTOR} is in P. One could compute f_{FACTOR} by factoring n, but no one has found a factorization algorithm that runs in polynomial time.

However, it is easy to see that $f_{\text{FACTOR}} \in NP$, where NP is the class of functions that can be "verified" in polynomial time; in the case of f_{FACTOR} , this means if $f_{\text{FACTOR}}(m,n)=1$, then given any d, where $1< d < m$, we can verify in polynomial time that d is a factor of n.

Since every $f \in P$ is computable in polynomial time (with no need for verification), we have $P \subseteq NP$. Is $P=NP$? This is the most famous open problem in complexity theory, and mathematicians have adopted it as one of the 7 Clay Millennium Problems[xiv].

An obvious idea for proving $P \neq NP$ is to find some $f \in NP$ that *cannot* be computed in polynomial time. Intuitively, this proof should be easier if we pick some $f \in NP$ that is hardest to compute. But how should one define "hardest"?

The standard definition for "hardest" is NP-completeness, where an $f^* \in NP$ is **NP-complete** if and only if f can be reduced in polynomial time to f^* for all $f \in NP$. There are infinitely many such f^*, and computer scientists have explicitly identified thousands of them, but no one could prove any of these f^* is not in P. The issue is not just of theoretical interest, since current cryptographic techniques for online transactions use functions in NP; if $P=NP$, then it would be computationally easy to defeat these encryptions.

We see here that the concept of reduction, so common in mathematics, is crucial in formalizing the notion of computational difficulty.

Computer scientists generally consider NP to be the limit of computational tractability. Beyond NP, there is in fact an infinite hierarchy, where the functions are increasingly harder to compute as we go up the hierarchy. In the limit, there are functions that are not even computable.

Most of us have encountered computation that seems to "hang", where nothing seems to be happening. Sometimes, this is because the computation is in some infinite loop, so the program does not **halt**. Now consider the function

$$f_{\mathrm{HALT}}(\boldsymbol{P}, \boldsymbol{D}) = \begin{cases} 1\,, & \textit{if program } \boldsymbol{P} \textit{ halts when run on input } \boldsymbol{D} \\ 0, & \textit{otherwise} \end{cases} .$$

(We can consider $\boldsymbol{P}$ and $\boldsymbol{D}$ to be integers since, at the machine level, they are just strings of 0s and 1s, like an integer in binary.) This f_{HALT} is not computable.

To prove a function is not computable, one must first define formally this intuitive concept of "computable". Turing gave one of the earliest definitions, and used it to prove that f_{HALT} is not computable (Turing 1937). Mathematicians and computer scientists have proposed many alternatives, but Turing's definition is now accepted as the standard.

The scientific significance of Turing's proof cannot be overemphasized. Computer science is widely perceived as engineering, not science, but the uncomputability of f_{HALT} transcends technology: it will remain uncomputable a thousand years from now, and anywhere in the universe.

Again, there are infinitely many uncomputable functions, and thousands of these functions have been explicitly identified. We can also use reduction to prove that a function f is uncomputable: start with a function f^* that is known to be uncomputable (e.g. $f^* = f_{\mathrm{HALT}}$) and prove that f^* can be reduced to f, so f^* is easier than f; since f^* is already uncomputable, f must therefore be uncomputable too.

To summarize, not everything is computable. Despite the optimism of AI enthusiasts, there is much that is computationally intractable (*NP*-complete) or impossible (like f_{HALT}). And the theory for studying these limits on computation makes much use of reductions.

6. CONCLUSION

The mathematics community should not feel challenged by students' upswing in interest in computers and programming. Nor should we strain the curriculum to suit this interest. Mathematics is a profound discipline, whose effectiveness in the natural sciences we do not yet understand (Wigner 1960), and computer scientists have found it to be a natural tool to wield. Rather, as I have tried to illustrate in this lecture, we should be confident that, if we give our students a rigorous education in mathematics, and equip them with mathematical instincts and habits, then we would already go a long way in supporting their interests in computing.

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[v] J.D. Ullman is a computer scientist who is well-known for his many research contributions and popular textbooks.

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USING CROSS-CULTURAL COMPARISON TO INTERROGATE THE LOGIC OF CLASSROOM RESEARCH IN MATHEMATICS EDUCATION

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Classrooms represent a globally-extensive institutionalised site for the promotion of learning. Cross-cultural comparative classroom research offers an opportunity to destabilise some of the assumptions underlying established pedagogical practices and the theories of learning implicit in these practices. The mathematics classroom is a site through which the international mathematics education community can explore considerations of culture, language, temporality and theory. Various approaches to researching the mathematics classroom illustrate the affordances and limitations of our research designs and new possibilities of international collaboration are proposed to usefully interrogate and elaborate the logic of mathematics classroom research.

INTRODUCTION

Classrooms represent a globally-extensive institutionalised site for the promotion of learning, but classroom practice is inevitably situated in webs of local convention and framed by local pedagogical histories of practice and of discourse. In the context of mathematics classroom learning/teaching, the interactions of the participants are shaped by conventions grounded in local culture, dictating both the pretext for their presence in the classroom and the roles they are expected to perform. Cross-cultural comparative research offers an opportunity to destabilise some of the assumptions underlying established pedagogical practices and the theories of learning implicit in these practices. Yet even these acts of comparison can prove ineffective as vehicles for transformative research, if the premises on which the comparisons are undertaken remain grounded in a single (authoring) culture (Clarke, 2003). Methods are required by which language, entrenched practice, restrictive organisational structures and established theory can be subjected to constructive scrutiny. I suggest that such scrutiny must be cross-cultural and undertaken collaboratively and reciprocally. In this presentation, the mathematics classroom is presented as a site through which the international mathematics education community can explore considerations of culture, language, temporality and theory. Various approaches to researching the mathematics classroom are presented to illustrate the affordances and limitations of our research designs and new possibilities of international collaboration are proposed to usefully interrogate and elaborate the logic of mathematics classroom research.

A BRIEF OVERVIEW OF CLASSROOM RESEARCH

Early studies of learning were typically clinical studies of small numbers of individuals (Piaget, 1926; Donaldson, 1978). The clinical tradition of fine-grained analyses of the

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responses of small numbers of students to particular learning situations (treatments) has been pursued in many studies (e.g., Lobato & Siebert, 2002; Steffe, 1991; Thompson & Thompson, 1994). Learning in classroom settings became increasingly the subject of research and this interest was accompanied by the development of onsite real-time observational techniques (e.g., Amidon & Hough, 1967; Beeby, Burkhardt, & Fraser, 1980) leading to the contemporary use of video (e.g., Janik & Seidel, 2009). Process-product studies offered a plausible paradigm in which classroom process variables were linked statistically with product variables such as student test performance or student attitudes (Bourke, 1985; Good & Grouws, 1975). The limitations of such correlational studies for the prescription of effective practice is now widely recognised. The challenge of connecting instructional activity to learning outcome is the signature problem of classroom research. The instructional optimization of learning in a particular setting requires research capable of providing fine-grained insight into the locally-specific relationships between context, practice and outcome.

This contextually specific detail can be found in the many naturalistic case studies of student learning in authentic classroom settings (e.g., Clarke, 2001; Cobb & Bauersfeld, 1995; Erlwanger, 1985), drawing upon the methods of ethnographic research to understand the relationships between individuals, their practice, and their consequent learning in classroom settings. Fundamental to such studies was the recognition of the importance of accommodating learning as a social phenomenon in our research designs (Lerman, 2000). Studies such as those just referenced were undertaken in order to identify and elaborate the social aspects of learning. Such fundamentally exploratory studies, progressively gave way to design experiments: "engineering" particular forms of learning in the interest of theory development and testing, following an iterative process of hypothesis formulation and revision (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Unlike earlier naturalistic studies, design experiments are highly interventionist. This is coupled with a pragmatic commitment to local utility, in the sense that the situational specificity of the design experiment is recognised in both the empirical findings and the theories generated by such studies. Video has served as a tool for many of these studies, providing detailed documentation of the actions of the participants and also of the actual classroom setting itself.

More recently, international comparative studies of classroom practice have been undertaken, also using video as a key tool (e.g., Clarke, Keitel & Shimizu, 2006; Stigler & Hiebert, 1999). It is useful to contrast the design logic of two of the more internationally prominent studies: The TIMSS and TIMSS-R Video Studies (Stigler and Hiebert, 1999; Hiebert et al., 2003) and the Learner's Perspective Study (LPS) (e.g., Clarke, Keitel, & Shimizu, 2006; Kaur, Anthony, Ohtani, & Clarke, 2013). The TIMSS and TIMSS-R Video Studies were intended to be nationally representative characterisations of typical eighth-grade classroom practice in each of the participating countries. This goal was achieved through a representative sampling of single mathematics lessons across entire countries. The LPS analysed sequences of at least ten consecutive lessons taught by three experienced mathematics teachers from each of the participating countries, where each teacher was recruited because they were competent according to local criteria. The goal of the LPS was the documentation of the classroom practices of both teachers and students in competently taught mathematics classrooms in each country. The two studies were complementary in their respective foci on typicality through national representative video surveys and the characterization of practice in well-taught classrooms through the fine-grained analysis of video-based comparative case studies. The challenge for each study and for classroom research in general is to make evidence-based connection between specific classroom activities and student learning outcomes. Cross-cultural comparison exploits the inevitable and unsettling dissimilarity of practices among differently situated classrooms to highlight elements of comparative stability and change, both of which take on significance in any attempt to understand how classroom practice and its outcomes are collectively constructed by the participants in different cultural settings.

The Science of Learning Research Classroom (SLRC) at the University of Melbourne is a laboratory classroom equipped with 10 built-in video cameras and up to 32 audio channels. Laboratory classrooms such as the SLRC and similar facilities being developed by Beijing Normal University and elsewhere require a shift in thinking about classroom research and possibly educational research in general. Unlike school-based research and more like a laboratory, the facility allows researchers to have significantly greater control over the research setting and conditions, from who are present in the classroom, the autonomy of the participants, the activities involved, through to the types and amount of data being generated and therefore collected. The parallel development of such facilities in culturally disparate locations has the potential capacity to draw on the mathematics education community's existing expertise in ethnographic methods, cross-cultural comparative analyses, design experiments, and the techniques of the clinical experiment. Having provided this overview, I will now consider some of the issues that I think are critical if we are to realise the potential of our new theoretical, methodological and technical riches. The convergence of capability just outlined assumes a central role for the use of video and this is reflected in the remainder of this paper.

VIDEO DATA PROBLEMATICS - WINDOW, LENS, MIRROR

I want to start with a consideration of the nature of classroom data. Educational research, like research in the physical and biological sciences, must make optimal use of available technologies in addressing the major problems of the field. This strategic deployment of available technologies reflects a purposeful process of data generation rather than data collection. Data "collection" has never been an apt description of the research process and the agency of the researcher must be acknowledged more explicitly. As such, the researcher must accept responsibility for decisions made and data generated, and place on public record a transparent account of the decisions made in the process of data generation and analysis.

It is a truism to say that the data you need reflects the research questions you want to answer. Data is also pre-determined by the theory employed. Each theory prioritises certain constructs and these constructs are embodied in particular data types. Those of us engaged in classroom research make use of a wide range of data: test responses, student work materials, teacher lesson plans, curriculum documents, copies of text books, video records of the teacher, the whole class and particular students, transcripts of speech by classroom participants, scanned visual records of equipment of various types, powerpoint presentations, photos of displays on blackboard, whiteboard or other media, teacher and student interviews (some video-stimulated, some not), questionnaire responses and physiological response data. The essential consideration are the researcher's acts of selection, from which information is transformed into data. Some of the data types just listed are generated as information by the teacher and the students through the normal activities of the classroom and some only occur because of the research activity. In constructing the data set, the researcher must engage in selection both before the research event and after. Video illustrates these acts of selection very clearly: the researcher can choose to use video or not; where to point the camera; how many cameras to use; and which video material will ultimately be selected and configured for analysis. But, once configured, what function does the data serve? What work does it actually do?

I have found four metaphors useful in addressing these questions. Once again, video data can be taken to stand in the place of all data types, because of its capacity to make graphically explicit the dynamic between the researcher's purpose and the object of research. These four possible conceptions help explicate the mediating role of video in classroom research: (i) as a window through which to see the classroom; (ii) as a lens through which to focus on selected aspects of classroom activity; (iii) as a reflective mirror by which the classroom participant can see themselves and reflect on their actions; and (iv) as a distorting mirror, in which the researcher sees not so much a representation of the classroom, but rather a re-presentation of their own values and perspectives reconstituted as classroom data. The key verbs corresponding to these metaphors are: see, focus, reflect and represent. Each metaphor has significant entailments for the meaning and authority (as evidence) that can be accorded to the resultant images for research purposes. The metaphors can be used to determine the ontological and epistemological assumptions of research designs. Each will be discussed briefly with an example.

Video as Window

Perhaps the most obvious metaphor used in relation to video is that it serves as a window on the classroom. This image is simple and immediately appealing. It suggests a neutrality to the act of video recording that assigns the technology the role of independent (and implicitly unbiased) recorder of classroom events occurring independent of the researcher.

> Only by seeing classroom situations from the perspectives of all participants can we come to an understanding of the motivations and meanings that underlie their participation. (Clarke, Mitchell & Bowman, 2009, p. 39).

Whatever position we might hold regarding the status of these recordings, in many studies, video functions in our research reports as precisely this sort of window on a form of "classroom reality" made accessible through the window of the video camera.

Video as Lens

A slightly more nuanced conception of the role of video in classroom research constructs it as a form of lens, allowing the researcher to focus attention on selected aspects of classroom activity, capable of a strategic close-up of prioritized events or objects or of panning back for wide angle documentation of class behaviour.

> Every decision to zoom in for a closer shot or pull back for a wide angle view represents a purposeful act by the researcher to selectively construct a data set optimally amenable to the type of analysis anticipated and maximally aligned with the particular research questions of interest to the researcher (Clarke, Mitchell & Bowman, 2009, p. 39).

The video in this conception is a research tool utilized strategically by the researcher to focus on certain aspects of the classroom.

Video as Reflective Mirror

Video can play a role other than as data. Video can be used to stimulate both teacher and students' reflections on a lesson through video-stimulated post-lesson interviews (Clarke, 1998). Video can also stimulate individuals' recollections of targeted phenomena. In the Lexicon Project, video records of lessons were used to stimulate teacher recollection of names for classroom phenomena (Clarke, Mesiti, Cao, & Novotna, 2017). In each case, the video is not the data. Instead, the role of video is catalytic, acting to stimulate the retrospective accounts or recalled terms that constitute the actual data.

Video as Distorting Mirror

In various places, it has been argued that our interaction with research settings is mediated by our theories (e.g., Clarke, 2011). The theory-ladenness of observation has been recognized by researchers both from the field of philosophy of science, from social science, and from education (e.g. Clarke, Xu, Arnold, Seah, Hart, Tytler, & Prain, 2012; Guba & Lincoln, 1994; Kuhn, 1996). The video record can be thought of as a representation of the researcher's view of the classroom constructed as a collage of images selected by the researcher to represent particular aspects of the classroom of significance to the researcher. We must address the possibility that our video records of classroom activities are most appropriately thought of as reflections of ourselves, distorted through their representation in the performative acts of those whose actions, motives and experiences we ostensibly seek to understand.

COMPLEMENTARY ACCOUNTS

In complex social environments such as classrooms, consideration must be given to the juxtaposition and relative status of different data. The distinguishing characteristic of the research design for the Learner's Perspective Study (see Clarke, Emanuelsson, Jablonka, & Mok, 2006) is the inclusion of four levels of complementary accounts: (a) at the level of data, the accounts of the various classroom participants are juxtaposed; (b) at the level of primary interpretation, complementary interpretations are developed by the research team from the various data sources related to particular incidents, settings, or individuals; (c) at the level of theoretical framework, complementary analyses are generated from a common data set through the application by different members of the research team of distinct analytical frameworks; and (d) at the level of culture, complementary characterizations of practice and meaning are constructed for the classrooms in each culture by the researchers from each culture and these characterizations can then be compared and any similarities or differences identified for further analysis, particularly from the perspective of potential cross-cultural connection or transfer.

We need to acknowledge the multiple potential meanings of the situations we are studying by deliberately giving voice to many of these meanings through accounts both from participants and from a variety of "readers" of those situations. The implementation of this approach requires the rejection of consensus and convergence as options for the synthesis of these accounts, and instead accords the accounts "complementary" status, subject to the requirement that they be consistent with the data from which they are derived, but not necessarily consistent with each other, since no object or situation, when viewed from different perspectives, necessarily appears the same (Clarke, 2001, p. 1). Adoption of complementarity rather than consensus or convergence as a legitimate and productive stance requires a reconceptualization of the nature and function of triangulation in research (Mok & Clarke, 2015). Complementarity of accounts is an essential methodological and theoretical stance, adopted by the Learner's Perspective Study, for the explication of mathematics teaching and learning in classroom settings, the advancement of theories relating to such settings, and the informing of practice in mathematics classrooms.

THE ROLE OF TECHNOLOGY IN CLASSROOM RESEARCH

It is imperative that educational research makes optimal use of available technology. International comparative classroom research, in particular, poses methodological and technical challenges that are only now being adequately addressed through advances in:

- techniques and equipment for the collection of audio-visual data in classrooms;
- tools for the compression, editing and storage of digitised video and other data;
- storage facilities that support networked access to large complex databases;

- data distribution systems that support secure, remote access for data entry and retrieval on an international scale; and
- analytical tools capable of supporting sophisticated analyses of such complex databases.

Recent classroom research (Alton-Lee, Nuthall & Patrick, 1993; Clarke, 2001 and 2006; Sahlström & Lindblad, 1998), backed by more sophisticated ways of collecting and analysing data, has shown that some of the findings of the classroom research classics such as Bellack et al. (1966), Sinclair and Coulthard (1975) and Mehan (1979) are seriously skewed because of technological issues in data generation. Lindblad and Sahlström (1999, 2002) argue that if the early researchers had access to the same tools for data collection and analysis as are available today, the general view of classroom interaction would be quite different.

The most striking of these differences concerns the role of students in classrooms. Single-camera and single-microphone approaches, with a focus on the teacher, embody a view of the passive, silent student at odds with contemporary learning theory and classroom experience. Research done with technologically more sophisticated approaches has described a quite different classroom, where different students are active in different ways, contributing significantly to their own learning (cf. Clarke, 2001; Clarke, Emanuelsson, Jablonka, & Mok, 2006; Clarke, Keitel, & Shimizu, 2006; Sahlström & Lindblad, 1998).

Further, classroom researchers have until recently had limited opportunities for engaging in manageable comparative research, where materials from different countries and different periods of time can be accessed and analysed in feasible ways. At the International Centre for Classroom Research at the University of Melbourne, contemporary technology makes it possible to carry out comparative analyses of an extensive database that includes multi-camera classroom video records of lesson sequences, supplemented by post-lesson video-stimulated interviews with students and teachers, scanned samples of written work, and test and questionnaire data, drawn from mathematics classrooms as geographically distant as Sweden and Australia and as culturally distinct as Germany and China.

As new theories of learning and social interaction develop, research techniques must have the capacity to accommodate these new theories. Teaching and learning are complex practices, and different participants will experience classroom events differently. If we approach social settings (and the situations they frame) as multiply-constructed and open to multiple construal, then the methodology employed in their study must accord a voice to the several participants in these settings.

The data required for international comparative research into classroom practice are complex and expensive to obtain. The very high expenditure of time and effort in generating such data can be more easily justified if the consequences of all this labour and expense can be made available for analysis by the widest possible diversity of researchers. Of all data sources currently available to researchers in education, video data seems most amenable to multiple parallel analyses. Research addressing the complexity of classroom practice cannot be restricted to a single analytical frame, but must take a programmatic approach, where a well-equipped research team, combining a range of methodological and theoretical expertise, undertakes careful parallel analyses of high-quality complex data. Advances in technology bring us ever closer to the realisation of this vision.

MANAGING THEORY: MULTI-THEORETIC RESEARCH DESIGNS

Theories are cultural and historical artefacts, reflecting those things contemporary language equipped us to conceive. In carrying out classroom research, each theory affords particular analytical strategies, each focuses attention on specific aspects of the object or phenomenon under investigation but ignores other aspects. Inevitably, each should produce distinctive findings: the products of the particular analytical stance adopted. Each theory, although being applied in the analysis of the same setting, offers distinctive insights reflective of the theory's foregrounded constructs.

Rather than considering convergence or compatibility as the definitive result of the particular combination of theories, attention should be directed to the compatibility of the interpretive accounts generated by their application to a common source of classroom data. In the Learner's Perspective Study (among others), multi-camera on-site video technology and post-lesson video stimulated interviews were used to generate a complex data source amenable to parallel analyses from several complementary theoretical perspectives. This approach was intended to realize two very specific aims:

(i) Understand the setting: to maximize the sensitivity of the combined analyses to a wide range of classroom actions and learning outcomes, and

(ii) Understand the theory: through the combination of theoretical perspectives, examine the extent to which the results of analyses employing various theories and the theoretically-grounded explanations of these results are complementary, mutually informing, or, perhaps, incommensurable.

Each analysis resembles any mono-theoretic research design in that the constructs privileged by the chosen theory are matched to data types and a research design constructed that employs methods suitable for the generation of the targeted data. Each independent analysis remains vulnerable to the same accusation of circularity or pre-determination that can be leveled at any mono-theoretic research design. Once available, however, the results of the parallel analyses can serve several purposes:

(i) By addressing different facets of the setting and thereby providing a richer, more complex, more multi-perspectival portrayal of actors and actions, situations and settings;

(ii) By offering differently-situated explanations for documented phenomena and differently-situated answers to common research questions;

(iii) By increasing the authority of claims, where findings from different analyses in relation to the same question or the same phenomenon were coincident;

(iv) By qualifying the nature of claims, where findings in relation to the same question or the same phenomenon were inconsistent or contradictory;

(v) By providing a critical perspective on the capacity of any particular theory to accommodate and/or explain particular phenomena, in comparison with other theories employed in analyses related to the same events in the same setting;

(vi) By facilitating the synthesis of the results of the parallel analyses for the purpose of informing instructional advocacy.

The derivation of all findings from the same data source through the application of all analytical approaches to the same setting greatly strengthens the project's capacity to realize these six purposes. In particular, multi-theoretic research designs integrate the activity of research synthesis into the research design as an essential element. The goals of research synthesis (Suri & Clarke, 2009) should not be limited to normative convergence on some form of best practice. In developing instructional advocacy arguments, it may be the identification of contingencies on any recommendations that offers greatest utility, by identifying combinations of context and action most likely to promote locally significant outcomes.

QUESTIONS OF CULTURE

Research in the Learner's Perspective Study (LPS) has made clear just how culturally-situated are the practices of classrooms around the world, and the extent to which students are collaborators with the teacher, complicit in the development and enactment of patterns of participation that reflect individual, societal and cultural priorities and associated value systems (e.g., Clarke, Keitel, & Shimizu, 2006). Within any educational system, the possibilities for experimentation and innovation are limited by more than just methodological and ethical considerations: they are limited by our capacity to conceive possible alternatives. They are also limited by our assumptions regarding acceptable practice. These assumptions are the result of a local history of educational practice, in which every development was a response to emergent local need and reflective of changing local values. Well-entrenched practices sublimate this history of development. In the school system(s) of any country, the resultant amalgam of tradition and recent innovation is deeply reflective of assumptions that do more than mirror the encompassing culture: they embody and constitute it. International comparative research offers us more than insights into the novel, interesting and adaptable practices employed in other school systems. It also offers us insights into the strange, invisible, and unquestioned routines and rituals of our own school system and our own mathematics classrooms (Clarke, 2003, p. 180). However, cross-cultural comparative research brings its own challenges and these require careful consideration.

THE VALIDITY-COMPARABILITY COMPROMISE

In an international comparative study, any evaluative aspect is reflective of the cultural authorship of the study. Elsewhere, I have set out several methodological dilemmas that arise directly from cross-cultural comparative research (Clarke, 2013a). The cultural authorship of research instruments and their cross-site legitimacy has implications for both data generation and interpretation and must be accommodated carefully through revision or replacement, or through reconception of the nature of the comparison being undertaken. In particular, the pursuit of commensurability in international comparative research by imposing general classificatory frameworks can misrepresent valued performances, school knowledge and classroom practice as these are actually conceived by each community and sacrifice validity in the interest of comparability. For example, researchers engaged in cross-cultural comparison should avoid confusion between form and function, where an activity coded on the basis of common form is employed in differently situated classrooms to serve quite different functions (eg kikan-shido or between-desks-instruction). Cross-cultural research being reported from the perspective of a single culture, employing a single language, runs the risk of misrepresentation by omission, where the authoring culture of the researcher lacks an appropriate term or construct for the activity being observed. Marton and Tsui (2004) suggest that “the categories . . . not only express the social structure but also create the need for people to conform to the behavior associated with these categories” (p. 28). Our interactions with classroom settings, whether as learner, teacher or researcher, are mediated by our capacity to name what we see and experience. Speakers of one language have access to terms, and therefore perceptive possibilities, that may not be available to speakers of another language.

THE LEXICON OF THE RESEARCHER

Learning can be conceptualised in terms of progressively enhanced participation in forms of institutionalised social practice, where discourses form key components of that practice. Students are initiated into the discourse of the mathematics classroom: a discourse with its own technical vocabulary and discursive and social conventions. Mathematics teachers similarly participate in a discourse community in which the mathematics classroom and its objects, agents and events provide the subjects of professional discourse and for which language mediates the experience of the classroom and the professional learning that experience engenders. Classroom researchers’ experience of the classroom is similarly mediated by the language available to describe those objects and events occurring in classroom settings. Thinking of the research process as analogous to learning, in that the goal is the construction of new knowledge, we find in the mathematics classroom research the nexus of three learning communities: students, teachers and researchers. The learning opportunities available to each group are afforded (and constrained) to a significant extent by the language employed to participate in and reflect on the mathematics classroom.

Any claim that researchers speaking different languages are analyzing "the same classroom," even when working from the same video records, can be usefully contested. Educational research increasingly employs English as the primary language through which theory is developed and disseminated. It is essential to recognise the constructs that other cultures have employed in conceptualising their practice and examine the consequences for research and for theory of those distinctive terms (and the designated constructs) that might otherwise be ignored by an international community restricted to communication in English. The Lexicon Project (Clarke, Mesiti, Cao, & Novotna, 2017) has documented the naming systems employed in nine countries, using eight languages, to describe the phenomena of the middle school mathematics classroom. Utilization of the lexicons from each country to identify legitimate points of comparison would heighten both validity and comparability (Clarke, 2013b).

Each particular country's lexicon reflects a specific pedagogical tradition, culturally and historically situated. Certainly, the variation evident between different lexicons (Clarke, Mesiti, Cao, & Novotna, 2017) makes it clear that the teaching communities in the different countries interface with the mathematics classroom in very different ways, mediated by entirely different naming schemes for the things we might find there. My emphasis here, however, is on the function of language in framing, shaping and constituting our development of educational theory and the associated conduct of our research in mathematics education.

COMPARATIVE RESEARCH AND BOUNDARY CROSSING

Acts of research comparison necessarily construct boundaries that distinguish between the objects, groups, communities, settings or systems that are compared (Akkerman & Bakker, 2011). These boundaries are important. Without them, our acts of comparison are meaningless. As a consequence, boundary construction is an inevitable entailment of all research activity. Equally, the act of comparison constitutes an act of boundary crossing, since the researcher in one way or another must connect the domains, settings, communities or individuals being compared. Elsewhere (Clarke, 2015), I have identified several metaphors, by which we might characterize our various research acts as acts of boundary crossing, creating the conditions for a new form of scrutiny of the validity and logical coherence of those research acts.

One of the paradoxes of boundary construction and boundary crossing in the context of cross-national research is that the same comparative act that crosses a boundary, by its nature reifies that boundary. For example, PISA compares levels of student achievement, products of curricula that are different in structure and in aspiration. The institution of international testing provides the bridge for this form of border crossing and reifies through the international acclamation of its findings the boundaries its acts of comparison have simultaneously surmounted and invoked.

A truly impermeable boundary would prevent all possibility of comparison. Another way to say the same thing is that there would be no objects pertaining to one domain that had meaning within the other domain and nothing, therefore, that could serve as the basis for comparison. In one form of contemporary boundary-speak, this means there would be no possibility of a "boundary object" (Star & Griesemer, 1989). In undertaking cross-cultural comparative research we must take care to ensure that the constructs that form the basis of our comparison (e.g. mathematical performance, participation, or student voice) can be employed with local validity to characterize arguably similar phenomena in both cultural settings.

International comparative research in mathematics education can both create and destabilize boundaries in ways that enhance or impede our ability to benefit from the practices of mathematics classrooms and school systems elsewhere. The boundaries we construct should clarify our understandings, not impede their application. Equally, our destabilisation of existing boundaries should result from our demonstration that some boundaries do no useful work, but rather inhibit our consideration of alternative ways to conceptualise our discipline, our pedagogy, and even our research.

THE EVOLVING LOGIC OF CLASSROOM RESEARCH

The logic of classroom research remains grounded in the need to confidently connect teaching/learning activity to learning (or other) outcomes, while identifying the local contingencies of setting and culture that frame and constrain both the activity and its consequence. Recognition of the fundamentally social nature of learning in classrooms creates a series of methodological tensions that researchers have variously addressed. The logic of the case study is the logic of possibility. If the researcher can document the process of learning in sufficient detail, including all contextual elements likely to contribute to a local explanation/model/representation of that process, then further studies may help to distinguish essential from non-essential elements, both of process and of context. The logic of the survey, whether by test, questionnaire or video, is the logic of probability. Meta-analysis is a form of survey, where the respondents are research projects, recruited for their compatible design features. Aggregation of data across individuals or contexts can identify dominant patterns with statistical authority, sacrificing detail in the name of generalisability. Action research and design experiments share the logic of purposeful, iterative refinement, within a specific setting, acknowledging the situated nature of emergent findings and theory.

International cross-cultural comparative research appeals to a logic of structure in diversity, where stability across cultural variation confers authority on any constant elements, and variability of process, outcome or condition in relation to a common construct, measure or valued outcome helps to identify contingencies affecting the application of any emergent model or theory to any particular setting. Suri and Clarke (2009) explored the possibility of methodologically inclusive research synthesis. Chan and Clarke (2017) have addressed the analogous question of theoretical complementarity and the synthesis of analytical accounts of research conducted in relation to the same setting. In this paper, international cross-cultural comparative research is foregrounded, with video as its tool. My purpose here has been to use both the paradigm and the instrument to raise methodological concerns that transcend both. The concerns raised can be addressed through international collaborative research activity in which parity of voice among research partners is expressed as parity of authorship (cf. Stengers, 2011). It is suggested that only through collaborative activity undertaken by cross-cultural research communities can the acts of comparison, essential to classroom research, be undertaken with validity. Facilities such as the laboratory classroom address the need for both detail and scope of data in relation to the classroom setting. Local partnerships with schools and with teachers and students provide the communities whose learning practices and outcomes are the focus of our research. International partnerships provide the comparative power needed to distinguish between culture-specific elements and culture-transcendent ones. Technical sophistication, authenticity, and comparability conspire to optimise the research endeavour that is cross-cultural comparative classroom research.

As our research endeavours become more globally collaborative, we must find new ways to integrate the affordances of language, culture and history that have, until recently, developed in relative isolation. We have studied and compared mathematics classrooms internationally. Now we study and compare the local languages that shape and constitute our classroom practices. New possibilities are emerging for practice and for theory. Other cultures, other languages are able to say things that we cannot, conceive of alternatives for which we have no words. Before synthesis comes connection, and before connection comes sharing. We are just discovering how much we have to share.

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PROBLEM SOLVING THROUGH HEURISTIC STRATEGIES AS A WAY TO MAKE ALL PUPILS ENGAGED

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In the paper, the use of heuristic solving strategies, one of the ways of developing pupils' creative approach to problem solving, is discussed. Heuristic strategies are used in Polya's and Schoenfeld's understanding of the concept. The theoretical background of the research is Brousseau's Theory of Didactical Situations. Most attention is paid to the question of whether pupils' use of heuristic strategies can result in an improvement of their abilities to solve problems whose solving algorithms are not easily accessible to them. The use of heuristic strategies is explored in two different perspectives: how heuristic strategies develop pupils' understanding of mathematics when they are used and how teachers change in consequence to giving their pupils the chance to use these strategies.

INTRODUCTION – AREA OF A QUADRILATERAL: A SCHOOL EPISODE

The following problem was assigned by the teacher in the 8th grade (pupils aged 14-15):

Triangle *ABC* in fig. 1 has a unit area. Points *P*, *Q*, *R*, *S* divide sides *AC* and *BC* into three equal segments. What is the area of the coloured quadrilateral? (Horenský et al., 2007, p. 29/6)

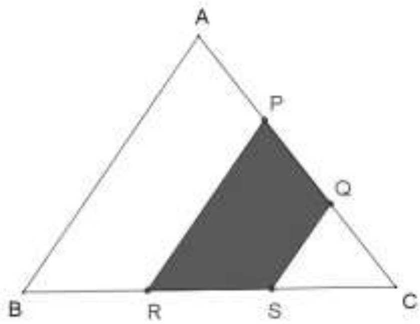

Figure 1

The solving strategy supported by the teacher was to apply similarity. This strategy could be called "school algorithmic strategy" as it builds on knowledge taught at school. The procedure is based on the following fact: If the coefficient of similarity of two triangles is k, then the ratio of their areas is k^2. The teacher recommended the pupils to consult the figure. The solution was as follows:

- Triangles *ABC* and *PRC* are similar with the similarity coefficient 2/3.
- The area of the triangle *PRC* equals $(2/3)^2 = 4/9$.
- Triangles *ABC* and *QSC* are similar with the similarity coefficient 1/3.
- The area of the triangle *QSC* equals $(1/3)^2 = 1/9$.
- The area of the trapezium *PRSQ* equals $(4/9) – (1/9) = 1/3$.

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This solving strategy could be called the “school algorithmic strategy”, as it is based on applying knowledge learnt at school.

There were several pupils who could not recall the knowledge about similarity and the relationship between areas. They tried to use the formula for the area of a triangle but as they knew neither the lengths of the sides nor the heights, they failed. Most of them waited until the solution was shown by the teacher and did not try to find the solution by another solving procedure.

One pair of pupils worked hard and did not follow the teacher’s guidance. They were very much involved in their work and suddenly they announced they have the correct solution discovered in another way. The teacher asked them to show the others how they came to the solution. Here is their solution:

If we divide triangle *ABC* into nine congruent triangles as shown in Fig. 2, we discover that trapezium *PRSQ* is covered by three triangles and so its area is 3/9 = 1/3.

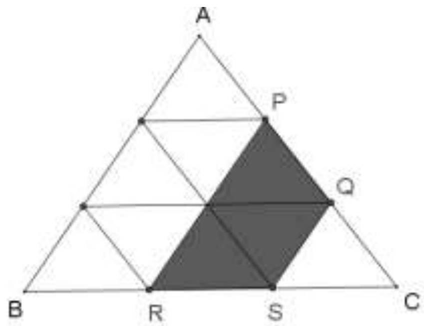

Figure 2

The teacher praised the two pupils for their interesting solving strategy. She recommended her pupils to think about other possible strategies for this problem, bring them (if they manage to find any) to the next lesson and share them with the others.
At the beginning of the next lesson, the teacher gave pupils the space to show their strategies. They did not manage to find all strategies that teacher knew. Therefore after her pupils’ presentations, the teacher completed the list of suitable solving strategies for the problem by those her pupils had not discovered. Here is the list of the remaining strategies that were accepted by the teacher and her pupils as correct solving strategies for the problem:

If we draw line segments *EP* and *FQ*, triangle *ABC* is divided into three congruent triangles and three congruent parallelograms as shown in Fig. 3. We discover that the trapezium is covered by one triangle and one parallelogram and so its area is 1/3.

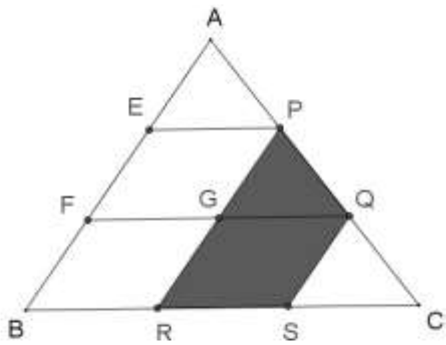

Figure 3

Let us move trapezium *PRSQ* to the line above the trapezium (see Fig. 4). We move parallelogram *RSTB* under triangle *UCV* (see Fig. 5); thus we form three congruent trapeziums.

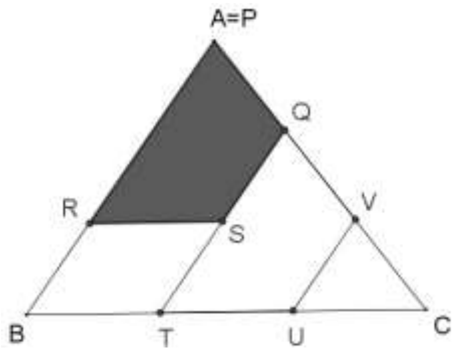

Figure 4

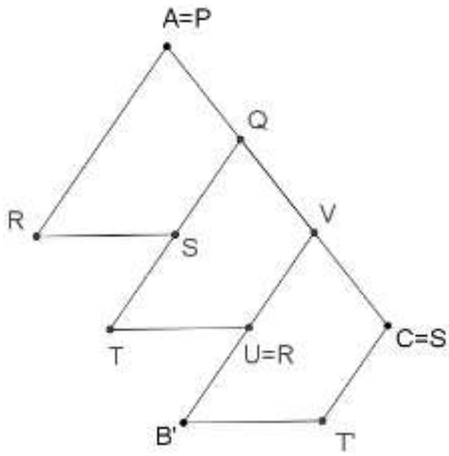

Figure 5

Let us extend triangle *ABC* into parallelogram *ABCD* (see Fig. 6). Let us draw points *E* and *F* as intersections of half-lines *PR* and *QS* with line segment *BD*. Line segments *PE* and *QF* divide parallelogram *ABCD* into three congruent parts. Triangle *QSC* is congruent with triangle *ERB*. As trapezium *ABRP* together with triangle *ERB* make one strip, the area of the strip equals union of this trapezium and triangle *QSC*. The area of trapezium *PRSQ* equals to one half of area of the whole strip, therefore area of *ABRP* in union with *QSC* is twice the area of *PRSQ*. Thus the area of the studied quadrilateral equals one third of triangle *ABC*.

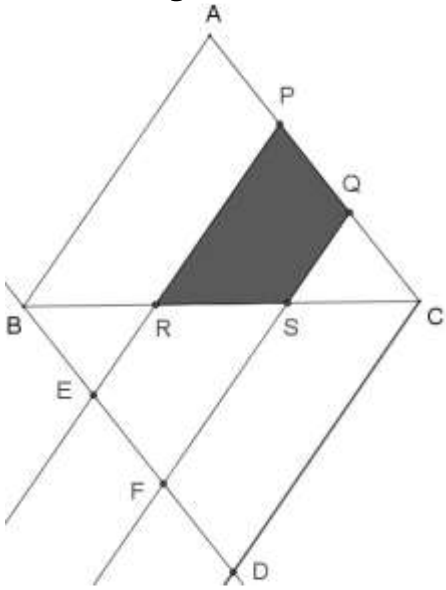

Figure 6

All solutions based on Figs. 2 to 6 are based on a suitable drawing. The teacher presented one more strategy of another type: She calculated the sought area for a specific case – a right-angled triangle with the right angle at *C* (see Fig. 7).

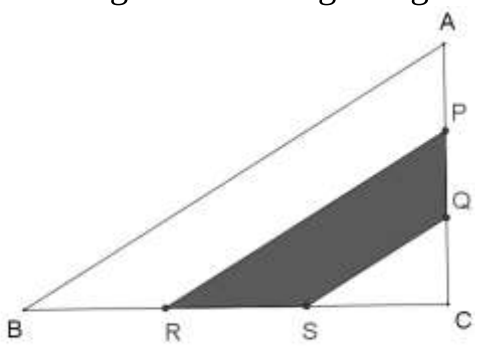

Figure 7

The ratio of heights in smaller triangles is obvious. The following holds:

$$S_{ABC} = \frac{a \cdot v}{2} = 1, S_{QSC} = \frac{\frac{1}{3}a \cdot \frac{1}{3}v}{2}, S_{PRC} = \frac{\frac{2}{3}a \cdot \frac{2}{3}v}{2}.$$

Thus:

$$S_{RSPQ} = \frac{4}{9} \cdot \frac{av}{2} - \frac{1}{9} \cdot \frac{av}{2} = \frac{1}{3} \cdot \frac{av}{2}.$$

This type of solving strategy is not rare in (not only) mathematics problem solving. Nevertheless it can hardly be called "school algorithmic strategy" in the Czech Republic because it is not supported in Czech educational documents. When solving the here discussed problem, this strategy certainly looks more complicated than the previous solution drawings, but in many cases the strategy is applied even in real life situations.

In this paper we focus on those solving strategies that do not represent school algorithmic strategies. We present results of the three-year GAČR project *Developing culture of solving mathematical problems in school practice.*

Theoretical background

It is generally accepted that problem solving creates the background of successful mathematics education. Kopka (2010) emphasizes that solving carefully selected problems helps to develop and cultivate pupils' creativity, autonomy and intellectual activity, and to improve their attitudes towards mathematics. One important goal of school mathematics is to teach to solve mathematical problems independently (NCTM, 2000). However, this practice is not common in school reality. Problems often become instruments for checking what pupils have learned and not instruments for learning. Instead of engaging pupils in their own investigations, pupils are asked to master prescribed algorithms.

The Theory of Didactical Situations in Mathematics (Brousseau, 1997) states that for each problem there exists a set of knowledge that enables its solution. However, the needed knowledge is not always available to the solver. Therefore the role of the teacher is to create an environment that supports broadening of pupils´ repertoire of knowledge. Teachers decide how the problem will be presented to pupils, which representations will be used, how open the space for discussion will be, which solving strategies they will support, i.e. how intellectually rich and supportive environment they will create (Lubart, 1994). An example of such environment is e.g. Wittmann's substantial learning environment SLE (Wittmann, 1995). Wittmann characterizes a SLE as an environment that has a simple starting point and a lot of possible investigation or extension.

It is generally accepted that changes in approaches to problem solving in school practice are conditioned by changes in teachers' attitudes to mathematics education at schools, see e.g. (Tichá & Hošpesová, 2006). Mathematics education based on problem solving with no transfer of ready-made knowledge to pupils cannot be built without a thorough teachers' knowledge of mathematics, on their own experience with creative approach to problem solving. Important is also the *specialized content knowledge* (Ball, Thames & Phelps, 2008) that involves identification of key mathematical concepts and of the potential this activity bears, detection of various forms of representation of mathematical concepts and operations, including their advantages and drawbacks.

In the following text, two concepts on which the paper is built, culture of problem solving and heuristic solving strategies, are presented.

Culture of problem solving

Culture of problem solving can be explored from three different perspectives. The first focuses on pupils' attitude to problems and problem solving in dependence on different variables influencing these attitudes (Nesher, Hershkovitz & Novotná, 2003). The second focuses on bringing about a change in the culture of problem solving both in case of an individual and of groups of pupils, and on pupils' motivation to problem solving (Bureš & Hrabáková, 2008; Bureš & Nováková, 2010; Bureš, Novotná & Tichá, 2009; Bureš, Nováková & Novotná, 2010). The third group focuses on complex projects in problem solving, such as clusters of problems (Kopka, 2010; Bureš, 2010), mathematics rallies (Brousseau, 2001; Novotná, 2009; Růžičková, Novotná & 2010). In all these three cases, pupils work with sets of problems, solve them individually and in groups and then share their experience and knowledge from the solving process and discuss it.

In this paper, Culture of problem solving (CPS) is regarded as the tool for description of pupils' solving profiles. It allows measuring the changes in pupils' attitude to problem solving, in their success rate and in the solving strategies they use. It consists of four components: intelligence, creativity, reading with comprehension and ability to use the existing knowledge. In the project, the first three components were measured by standard psychological tools and assessed by a psychologist, the test for assessment of the ability to use the existing knowledge was created by the project solving team. The structure of CPS is presented in detail in (Eisenmann, Novotná & Přibyl, 2014). We present here a brief overview of its components.

In the psychological screening, the following tools were used:

Pupils' intelligence was tested by the Váňa's intelligence test (Hrabal, 1975). This test is suitable for investigating the intellectual level of whole school classes, of the level of individuals' cognitive abilities (esp. of the component that conditions school success) in research situations where basic data about pupils are collected.

Pupils' creativity was investigated in the context of divergent thinking. Its level was measured using Christensen-Guilford test (Kline, 2000, p. 479) that measures four dimensions: fluency (how many relevant uses the pupil proposes), originality (how unusual these uses are), flexibility (how many areas the answers refer to) and elaboration (quality and number of details in the answer).

Pupils' ability to read with comprehension is one of the key competences for successful problem solving. The pupils were presented with a short text (one paragraph) which they were asked to summarize in four lines without changing the meaning and content. Their results were classified into five categories: Comprehension of the meaning and keeping all details, Comprehension of the meaning and keeping details, Grasping the meaning, content more all less kept without details, Incomprehension of the original text and few details or wrong content, Incomprehension without presentation.

In the test of the ability to use the existing knowledge, pupils were assigned four pairs of problems. The first problem from the pair tested the presence of certain knowledge, the second its use e.g. in a non-algorithmic (non-standard) context.
The tests used for determination of all four components of pupils' CPS were supplemented by assessments of the pupils by their mathematics teacher based on interviews of the researchers with the teachers. Attention was paid to surprising, unexpected pupils' results.

Heuristic solving strategies

The strategies we refer to as heuristic, in accordance with Polya (2004) and Schoenfeld (1985), are those solving strategies that pupils use to solve problems in another way than using school algorithms. Heuristic strategies are informal, intuitive, concise. The advantage of heuristic strategies is that they can be applied in any situation regardless of how difficult or confusing they may be (Novotná, Eisenmann & Přibyl, 2016). Vohradský et al. (2009) point out that heuristic strategies motivate pupils and help them grasp the content and master new knowledge but can never entirely replace other methods. For a successful use of heuristic strategies, it is "essential that pupils have mastered prerequisite knowledge and skills and that the goal they want to achieve be clear to them and adequate to their abilities. The main goal of heuristic strategies is development of independent, creative thinking in pupils." (Vohradský et al., 2009: 15).

Problem solving is a cognitive process that can be conducted in one of the three ways shown in Fig. 8 (Eisenmann, Novotná & Přibyl, 2015).

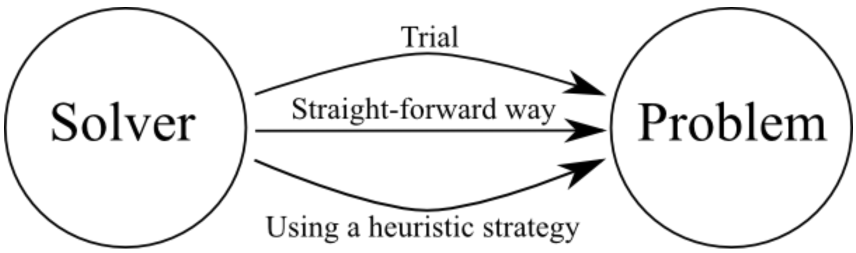

Figure 8: The process of solving a problem

Trial is the crudest way of dealing with a problem. The solver does not question whether they are solving the problem correctly, they only want to "have it solved", usually only once, without any internal feedback on the correctness of the solution. Straight-forward way is based on application of a learned piece of knowledge. The solver knows the required solving procedure and is able to apply it. Heuristic strategy is used when the solver does not have the required knowledge needed for straight-forward way of solution or cannot use the knowledge; use of a heuristic strategy allows them to solve the problem despite these problems.

In the project GAČR *Development of culture of problem solving in mathematics in Czech schools* the following heuristic strategies were used. Přibyl and Eisenmann

(2014) discuss in detail their basic characteristics and show how these properties can affect pupils' ability to master these strategies.

Strategy of analogy**:** Analogy is a type of similitude. If we are to solve a particular problem we find an analogical problem, i.e. a problem that will deal with a similar problem in a similar way. If we manage to solve this similar problem, we can then apply the method of its solution or its result in the solution to the original problem.

Guess – check – revise**:** This is a strategy in which we first, drawing from our experience, make a guess about the solution of the given problem. Then we check whether the solution meets the conditions of the assignment. The next guess is made with respect to the previous result. We carry on in this way until we find the solution.

Systematic experimentation**:** Systematic experimentation is a strategy in which we try to find the solution to a problem using several experiments. First we apply some algorithm that we hope will help us solve the problem. Then we proceed in a systematic way and change the input values of the algorithm until we find the correct solution.

Problem reformulation**:** When using this strategy we reformulate the given problem and make another one, which may either be brand new or easier for us to solve and whose solution is either directly the solution to the original problem or facilitates its solution. A specific and very important example of this strategy is translation of a word problem from one language of mathematics to another. Classical geometrical problems such as trisection of an angle are easy to solve when translated to the language of algebra.

Working backwards**:** This is a very common strategy in mathematics. We know the final state and we look for the initial state. We try to proceed from the end to the beginning. The solution of the problem is based on reverting the discovered procedure.

Introduction of an auxiliary element**:** By introducing an auxiliary element, we try to transform a given problem to a problem we have already managed to solve, or we transform it into a simpler problem we are able to solve.

Solution drawing**:** When using a graphical representation we usually visualize the problem by making a drawing. We record what is given and often also what we want to get. The drawing we get in this way is called an illustrative drawing, as it illustrates the solved problem. Sometimes we can see the solution of the problem immediately in this drawing. However, in most cases we must manipulate with the drawing (e.g. we add suitable auxiliary elements) and we solve the problem with the help of this modified drawing. We call this drawing the solution drawing.

Use of graphs of functions**:** When there are functions in the problem assignment or when it turns out within the solving process it is desirable to introduce functions, then it is usually good to draw graphs of these functions. These graphs often considerably contribute to finding the solution to the given problem.

Generalization and specification**:** A more general problem that we are able to solve is found. Then using the specification the answer is transferred to the original problem.

Specification and generalization**:** We choose a specific value or position, or we select a specific case. We solve the problem. If we can generalize the result of the problem, we formulate a hypothesis about the result of the original problem. We either leave the hypothesis on a plausible level, or prove it (if the solver's abilities are sufficient for it). If we cannot make the generalization, we continue the solving process by another specification.

Decomposition into simpler cases**:** The problem is decomposed into simpler cases that we are able to solve. The solution to the original problem is obtained by linking solutions to all simpler problems.

Use of false assumption**:** This strategy belongs to the family of experimental heuristic strategies. It can be well applied in problems where the value of a number in the problem is directly proportional to the result. The first value is selected with full awareness that the value is probably wrong (false assumption). The correctness of the estimate is verified. The assigned value is compared with the value calculated from the estimate and the proportion between them is found. The result is calculated using this finding. The mathematical background of this strategy is a linear function.

Omitting a condition**:** A problem assignment often involves several conditions. If we are not able to fulfil all these conditions when solving the problem at once, we can ask similarly to Zeitz (2007): What is it that makes the solution of this problem so difficult? If we manage to identify which of the initial conditions is the difficult one, we can try to omit it. If we are then able to solve the simplified problem, we can go back to the omitted condition and try to finish the solution of the original problem.

OUR RESEARCH

Within the project GAČR *Development of culture of problem solving in mathematics in Czech schools*, the following main research questions were formulated:

- Can pupils' use of heuristic strategies result in improvement of their abilities to solve problems whose solving algorithms are not easily accessible for the pupils?
- How do heuristic strategies develop pupils' understanding of mathematics when they are used?
- Which strategies do pupils prefer and what results do they achieve while using them?
- How do teachers change in consequence to giving their pupils the chance to use these strategies?

In order to answer these questions, one four-month and one sixteen-month experiments were conducted and their results were analysed. In the following text, these experiments are briefly presented and their results analysed with the goal to answer the research questions.

Four-month experiment (Novotná, Eisenmann & Přibyl, 2014)

The experiment was conducted in 11 classes (4 basic school classes with 12-year-old pupils, 4 basic school classes with 14-year-old pupils and 3 grammar school classes with 17-year-old pupils). All the selected schools were ordinary schools without any specialization; the classes were characterized as average or even slightly below average by their teachers.
The strategies dealt with in this experiment were Guess – check – revise, Systematic experimentation, Working backwards, Introduction of an auxiliary element and Omitting a condition.The participating teachers were provided with about 30 problems that could be solved efficiently using at least one of the studied heuristic strategies.

While the strategies Introduction of an auxiliary element and Omitting a condition require creative activity from the solver and depend on the solved problem, the first three strategies can be characterized as strategies of algorithmic nature and pupils can use them successfully even if they do not have very good insight into the structure of the problem; the use of these strategies does not always ask for very active involvement of pupils' creativity.
The pupils sat a written 40-minute pre-test and post-test at the beginning and the end of the experiment (4 – 5 problems). The problems in both tests were the same. The test items were selected so that for each of them, one of the selected heuristic strategies was the most suitable. Calculators and computers were available on pupils' desks. All the pupils had basic skills in the use of spreadsheets in Excel. Changes in their attitudes to problem solving were studied. When evaluating the written tests, attention was paid to the success rate as well as to the method of solution, i.e. also whether the pupils used some of the strategies shown in the teaching experiment.

The teachers' work was organized as follows. They assigned a problem to their pupils. They let them work and asked the pupil who had been the fastest in solving the problem correctly to explain their solution to the others. This was followed by a discussion and explanation of the solving strategy. The teacher then asked other successful solvers to present alternative solutions to the others. If none of the pupils had solved the problem with the intended heuristic strategy, it was demonstrated by the teacher. In another, similar problem the teacher then checked to what extent the teacher's solution was actively understood. Every teacher solved about three problems a week in this way.

In this experiment, the research questions were specified as follows:

- Is it possible to achieve any progress in the ability to solve mathematical problems using the selected heuristic strategies for such a short period of time (4 months)?

- In case of which strategies is this possible and which cannot be “implanted” in such a short period of time?
- Does pupils’ attitude to problem solving change? If so, how?

The results of the experiment gained from pre-tests and post-tests as well as from interviews with the participating teachers allow us to formulate the following conclusions:

Experimental strategies (Guess – check – revise, Systematic experimentation) and the strategy Working backwards can be mastered already over a shorter period of time, the strategies Introduction of an auxiliary element and Omitting a condition require longer time. This is caused by the algorithmic nature of the first three above mentioned strategies.

The danger of Systematic experimentation is that its mastery by some pupils makes them use it as the first solving procedure instead of e.g. constructing an equation or a set of equations. On the other hand, more frequent use of the strategy Systematic experimentation develops pupils’ sense of an effective choice of the initial value.

The short period of time of the experiment was sufficient to change attitudes of some pupils to problem solving (this could usually be observed in about one half of the pupils in each class). Pupils stopped being afraid of solving problems, they stopped laying their solution aside if they were not sure how to solve them at the very beginning. They learned to look for a solution rather than to give up.

Sixteen-month experiment (Eisenmann, Novotná & Přibyl, 2015)

The sixteen-month experiment was conducted in four classes: Grammar school in Prague (20 pupils, age 16-18), Grammar school in Hořovice (24 pupils, age 12-14), Lower secondary school in Ústí nad Labem (18 pupils, age 14-16), Lower secondary school in Prague (8 pupils, age 14-16). For the experiment, 200 problems illustrating the use of individual heuristic strategies were created.

Pre- and post-experiment tests consisted of 8 problems (one of heuristic strategies was always the most efficient solving strategy). The tests were different for each of the classes; they respected the pupils’ age level and knowledge. The problems in the initial and the final tests were identical. The test problems were not presented to the pupils during the experiment, and were not discussed even after the initial test. All the problems from the test were analysed and assessed in detail. Each solution was coded by a member of the research team with respect to the following phenomena:

- way of solving the problem (straight-forward way or heuristic strategy),
- problem-solving mode (arithmetical, algebraic, graphical),
- success rate of problem solving (successfully/unsuccessfully),
- “blank sheet” (the pupil did not even try to solve the task),
- non-evaluable response,
- misunderstanding the question.

Before the experiment started, all the participating pupils had been tested and assessed in all four components of CPS. The testing was carried out again post experiment with the exception of the Váňa's intelligence test as, according to the psychologists, no significant changes in intelligence could be expected.

Cooperation between the teachers and the research team was very intensive and systematic and was going on for the period of two years. Each of the teachers was cooperating closely with one member of the research team. Apart from conducting the experimental teaching, the teachers also collected pupils' worksheets with solutions of the problems and evaluated them. They were continuously observing the pupils and kept record of these observations. The observations focused on changes in approaches to problem solving and pupils' success rate in solving problems in general, not just in experimental problems. Regular meetings of the teachers with the respective researchers were usually held once in two weeks. The following issues were discussed: worksheets, individual problems, strategies used and the individual pupils' responses. The teachers also sent a brief report by email once a week. The members of the research team had access to the pupils' worksheets during the whole experiment. They used them for enriching the existing problems by new procedures that had been developed spontaneously in the lessons. Moreover, the worksheets served as feedback with respect to the success rate of the solutions.

Once in six months the cooperating researcher came to one of the lessons from the teaching experiment and once or twice during the whole experiment a video recording of the teaching unit was made.

The experiment was concluded by structured interviews with the participating teachers. Also some reactions of pupils to the use of heuristic strategies in teaching were collected.

The collected data and their analyses allowed to formulate the following conclusions from the experiment.

- An increased frequency of the used strategies was detected.
- A decreased frequency of unsolved problems was observed. It can be concluded that using suitable heuristic strategies played a role in the pupils' decision to try the solution at least.

The following was detected in the use of the heuristic strategies:

- Experimental strategies (Systematic experimenting, Guess – check – revise) and Working backwards were the only chosen by the pupils spontaneously also at the beginning of the experiment.
- The most considerable increase in the use of heuristic strategies was in cases of Systematic experimentation, Solution drawing, Use of graphs of functions and Introduction of an auxiliary element.
- The pupils were almost always successful when using the strategies Systematic experimentation and Guess – check – revise.

- Introduction of an auxiliary element: About one half of the pupils were successful in the final test.
- The (albeit sporadic) use of Analogy, Omitting a condition, Specification and generalization and Problem reformulation in the final tests was successful.

In the course of the experiment, the pupils showed improvement in two of the components of CPS. All the pupils showed some, even though moderate, improvement in the component *Reading comprehension*. The pupils from all the classes improved in the component *Creativity* considerably. A more detailed inquiry shows the highest degree of improvement in the area of fluency and flexibility. In case of Ability to use the existing knowledge no statistically significant changes could be observed.

The tools used for determining pupils' CPS do not allow us to separate the impact of the teaching experiment and the pupils' natural development completely; however, the psychologists claim the growth in the studied areas was higher than can be ascribed merely to pupils' natural development over the period of 16 months.

The following can be concluded from structured interviews with the teachers:

- Analogy is relatively popular with the pupils in problems that can be reformulated using more "user-friendly" objects, e.g. numbers. It is regarded by teachers as potentially useful for solving other than mathematical problems.
- Working backwards can be learnt by pupils relatively easily. Clever children select it spontaneously as the first way of solving a problem in appropriate situations.
- Specification and generalization is a strategy useful not only for solving problems in mathematics, it can be also used in other subjects, e.g. physics.
- If pupils are to be able to use the strategies Problem reformulation, Omitting a condition, Generalisation and specification and Decomposition into simpler cases, they have to solve a relatively large number of problems with their teacher; this was not achieved in the experiment. As far as the strategy Introduction of an auxiliary element is concerned, pupils also need a relatively high number of problems to master it actively. In the teaching experiment this was achieved in case of problems from geometry.

Pupils' assessment of heuristic strategies summarised from interviews with them are the following:

- Systematic experimentation can be used with a great variety of problems, its use is simple, and a computer can be used with it.
- Guess – check – revise is a fast way to finding the solution if a computer is not available.
- Working backwards is the easiest way to finding the solution in some problems.

- When using the strategy Introduction of an auxiliary element in geometry, it is helpful to make an illustrative picture and mark as much as possible in the picture. GeoGebra helps a lot at this stage.
- When using the strategy Analogy, it works well to pose a simpler problem with more “user-friendly” numbers. This helps the solver realize how to solve the original problem.

The experiment also brought some results related to the use of information technology (IT) when solving problems using heuristic strategies. These can be summarised as follows:

- The pupils learned to use IT in the strategy Systematic experimentation very quickly.
- They grew more confident in selecting the initial value in Guess – check – revise sensibly already after 3 months.
- The pupils applied successfully the strategy Systematic experimentation in solving problems whose solution through equations would have been too difficult or impossible.
- Problems where the pupils use IT to formulate or discover a hypothesis about a possible solution are very attractive for pupils. These include both problems solved using spreadsheets and problems from geometry solved using dynamic geometry software.

Impact of the experiments on participating teachers

The experiments did not have an impact only on the pupils. They also had impact on the participating teachers. They reported that they:

- lowered their demands on accuracy and correctness in their pupils’ communication and recording in favour of understanding the problem solving procedures, showed more tolerance to a variety in pupils’ solutions,
- acknowledged a change in their attitude to mathematics teaching towards using constructivist and inquiry-based approaches, and
- started to pose their own problems with the aim of making their pupils understand the various strategies better.

It was in accordance with the findings published in (Novotná, Brousseau, Bureš & Nováková, 2012) where there were changes in all aspects: teachers’ ability to design and organize efficient a-didactical situations in the classes, their ability to analyse situations, evaluate their course and results and distinguish between the rules of the situation and contingency, their active involvement in designing, realisation and analysis of the research in collaboration with researchers, and their ability to function successfully in two different roles, the teacher and the researcher. The findings were based on teachers’ self-reflections and researchers’ observations.

A significant increase in the teachers' autonomy was observed. During the realisation of the experiments, the teachers gradually took the roles of those who actively influence the stage design and the problems used. This was not only caused by conducting the experiments in their classes, but to a great extent also by their participation at the team meetings where the experience and preparation of the follow-up steps were discussed.

DISCUSSION AND PERSPECTIVES

We consider the most important outcome of the experiment to be the change in pupils' overall attitude to problem solving. They stopped fearing problem solving, they did not put it off if they could not see a suitable solving procedure immediately. They learned to look for solutions and not give up. This change could be observed in about one half of the pupils involved in the experiment.

Longitudinal observation of the pupils during the whole experiment and structured interviews with the pupils and teachers showed that pupils became more active in experimenting. We could observe an increase in their ability to communicate, to defend and explain their solving procedure, to react to opponent's remarks. They also got better at recording their solving procedures and became more sensitive to the need of verification (they checked correctness of their result).

The overall design of didactical situations in which heuristic strategies are used is demanding for the teacher: explaining the task, choice and preparation of problems, assessment, etc. It is necessary to be aware of the effects related to didactical contract. We would like to stress here the importance of institutionalization of the discoveries for pupils. It is also of utmost importance that a teacher be able to prevent a situation in which pupils appropriate some heuristic strategy (usually the strategy that they have used successfully in problem solving) as an algorithm and stop thinking about its suitability for the particular situation.

Even though the project was aimed at improving the pupils' culture of problem solving, we are convinced that the activities we have presented can be useful for designing new didactical situations (namely a-didactical) also in other areas of school mathematics. We hope they will become more widespread among teachers.

The project considerably influenced all members of the collaborative group, the teachers as well as the researchers. If the work of the team is to be successful, all the participants must collaborate. The change was observed not only on the teachers' side; also the researchers gained much from the collaboration. The teachers' input helped to precise the experimental settings as well as to analyse the project results.

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MATHEMATICS TEACHERS' PERSPECTIVES OF TURNING POINTS IN THEIR TEACHING

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Studies of mathematics teacher education or professional development usually provide evidence of impact of an intervention on teachers' knowledge, beliefs and specific aspects of their teaching. Most mathematics teachers are able to make small or specific changes to their teaching while some experience critical incidents and turning points that lead to significant transformation of their teaching to a learner-centered or inquiry-based perspective. This paper examines teachers' perspectives of significant turning points in their teaching of mathematics. This includes what the teachers considered to be critical incidents that influenced the turning points, the nature of the incidents and the impact on their thinking and actions in the classroom, the change in approach in their teaching, and the process in achieving the change.

INTRODUCTION

In today's global economy and highly technological world, there is a need for students to develop skills in school that will enable them to become capable of responding reflexively to complex problems. These skills include being able to work collaboratively and to think creatively, analytically, and practically. As Lipman (2003) suggested, students must be independent thinkers, going beyond content knowledge toward anticipative creative solutions to problems. The field of mathematics education embraces this perspective of education by promoting mathematical understanding, mathematical thinking, authentic tasks and 'learner-centered' teaching approaches. But this perspective is far from becoming the norm in mathematics classrooms.

Despite significant efforts of teacher education and professional development programs to support teachers in bringing about change, many mathematics classrooms tend to be more traditionally oriented than 'reform' oriented. While many teachers acknowledge the need for change, they find it to be challenging to transform their teaching in a significant way. Mathematics teachers' characteristics, such as beliefs, conceptions, identity, experiences as learners of mathematics, and knowledge of mathematics and mathematics pedagogy have been suggested or shown to be contributing factors of whether they can make meaningful, sustained changes to their practice. However, despite the challenges, there are teachers who have transformed their teaching to engage students more meaningfully in learning mathematics. The focus of this paper is on a sample of these teachers and their perspectives of 'critical incidents' [CIs] and 'turning points' [TPs] associated with the transformation of their teaching. The intent is to identify the CIs, the TPs and the changes in teaching based on the teachers' TP stories; the nature of the CIs and TPs and the basis of transition from

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CIs to TPs; and implications for research and teacher education. The goal is to contribute to our understanding of how teachers change. Without attention to how teachers learn and change, our understanding of instructional reform is seriously incomplete.

THEORETICAL PERSPECTIVES OF TP AND CI

Turning point [TP] and *critical incident* [CI] are commonly used terms that are also key constructs in research. Each is discussed from a research-oriented perspective.

Turning point

The concept of a TP is both literary and psychological. It has typically been used in association with life events (Rönka, Oravala, & Pulkknen, 2003) to emphasize long-term developmental patterns of continuity and change in social roles over the life span (Elder, 1985). It has been defined as a change in perspective (Baxter & Montgomery, 1998); "an alteration or deflection in a long-term pathway or trajectory that was initiated at an earlier point in time" (Sampson & Laub, 2005, p.16); "a fundamental shift in the meaning, purpose, or direction of a person's life" (Wethington, Cooper, & Holmes, 1997, p. 217); and "a change in a trajectory, pointing to a break in the sequence which leads from the past to the future" (Yair, 2009, p. 354). It is a TP if there is "sufficient time" that is spent on a "new course" (Abbott 1997, 89) distinguishing it from a temporary change or fluctuation in behaviors. Thus, it is only in hindsight that TPs emerge, after stability of the redirected pathway can be confirmed (Abbott, 1997; Wheaton & Gotlib, 1997). TPs may be the result of single dramatic events that bring about abrupt radical changes or changes that are incremental, occurring gradually over time leading to radical changes (Pickles & Rutter, 1991). TPs may involve both positive and negative results (Rutter, 1996). Given that different people have varied responses to the same event, contextual factors and individual characteristics are particularly important in understanding such marked changes in awareness and behavior.

The concept of a TP, then, ties three movements together: prior steady state, a critical event, and the ensuing of a new trajectory (Abbott, 2001). In this paper, the focus is on TPs that redirect teaching trajectories, not simply temporary detours from teaching pathways, and on participant-identified TPs; that is, what the teachers considered to be significant changes from what they were doing for several years.

Critical incidents

TPs are dependent on CIs, that is, specific events, experiences, and awareness that result in changes in the direction of a pattern or trajectory over the long term. CI, as a concept, "comes from history where it refers to some event or situation which marked a significant turning point or change in the life of a person … or in some social phenomenon" (Tripp, 1993, p.24). Trip defines it as an interpretation of a significant episode in a particular context rather than a routine occurrence. Typically, a CI "is personal to an individual. Incidents only become critical if the individual sees them as such. Reflecting on an incident after the incident has taken place is when it is defined as critical" (Bruster & Peterson, 2013, p. 172). Any event that for some reason draws one's attention may become a CI. It need not be a dramatic event, but is usually an incident which has significance for him or her. It is often an event that made someone stop and think, or raised questions for him or her (Christie & Young, 1995). CIs in a school context may be minor incidents but critical based on the significance and the meaning the teacher attributes to them (Tripp, 1993). A CI can be thought of as an everyday event encountered by a teacher in his or her practice that makes the teacher question the decisions that were made, and provides an entry to improving teaching (Hole & McEntee, 1999). It is the teacher who makes the incident critical, through inter -pretation, evaluative judgment, and assigning of meaning. In this paper, CIs are what the teachers considered to be significant factors that initiated TPs in their teaching.

RELATED LITERATURE

While there are reported stories of change in teaching mathematics, highlighting TPs, as a construct, has been less of a focus. TP is currently understudied in the field of teacher education, but merits further research. In mathematics education, studies such as Drake (2006) and Steinberg, Empson, and Carpenter (2004) indirectly address TPs. Drake used elementary school teachers' narrative descriptions of themselves as learners and teachers of mathematics to understand teachers' interpretations and implementations of a reform-oriented mathematics curriculum. She reported on the sense-making practices (noticing, interpreting, implementing) of teachers who told TP stories – those stories in which the teachers initially experienced significant failures in mathe- matics, but, as the result of a TP experience, viewed themselves positively as both learners and teachers of mathematics. She concluded that both the TP story and the meanings teachers attribute to this story are important for understanding teachers' specific practices in the context of reform. In the case of Steinberg et al., TPs are implied in the different phases of growth for the teacher. They reported on one teacher in "one especially productive year of learning" (p. 237) regarding how her engagement with children's thinking changed significantly over a few months. They identified four phases in the teacher's growth toward practical inquiry based on her use of interactive talk with children. She ultimately integrated processes for generating and testing knowledge about children's thinking in her instructional practices as she created opportunities for herself, and then students, to hear and respond to students' thinking.

While TP point is less of an explicit focus of studies in teacher education, CIs have received much more attention, particularly in helping prospective teachers to use or develop reflective skills, as in the following examples. Goodell (2006) investigated the CIs prospective teachers encountered during their field experience and what they learned about teaching for understanding through reflecting on those CIs. Francis (1997) investigated prospective teachers' use of CI analysis to build reflective practice skills by identifying and reflecting on the incidents in terms of their personal meanings of the incidents. CI was "an incident from recent school or university experience." (p. 172). Bruster and Peterson (2013) examined prospective teachers' use of CI as a tool for reflection during their practicum. They considered CIs to be "significant episodes in professional practice…that are difficult to resolve. These episodes or instances become critical because they cause the candidate to pause, think back, and consider outcomes" (p. 172). Griffin (2003) also examined the effectiveness of using CIs during a field experience to increase prospective teachers' capacity to develop reflective and critical thinking skills. She considered a CI to be "an incident that 'amused or annoyed'…was 'typical or atypical', was an 'aha or ouch' or a 'felt difficulty'… and the meaning of the incident" (p. 210). This use of CIs increased orientation towards growth and inquiry.

Some studies in mathematics education have also addressed CIs. For example, Lerman (1994) discussed the use of CIs to stimulate reflection on teaching in developing the idea of reflective mathematics teaching and suggested to mentors ways in which prospective teachers might be encouraged to develop their own reflective practice. Skott (2001) investigated how a novice teacher coped with the complexities of the classroom. CIs of practice emerged based on the teacher's role within classroom interactions. Skott considered an instance of teacher decision making as a CI of practice that provides "a window on the role of teachers' school mathematical priorities when these are challenged as informants of teaching practice by the emergence of multiple motives of their activities" (p. 3). Choy (2014) investigated how productive noticing can provide a means for teachers to reflect on and examine critical events in the classroom by analysing a case study of what teachers noticed about a CI that happened during a research lesson. Choy considered CIs as students' unexpected responses to teachers' questions or events that changed the direction of the lesson from what was planned. Finally, Potari, Psycharis, Kouletsi, & Diamantis (2015) explored prospective mathematics teachers' reflections on teaching practice through noticing key aspects of classroom interactions (i.e., CIs). They used CIs taken from everyday classroom situations as a tool to stimulate reflection and make the act of noticing more concrete.

The preceding studies support or promote the idea that CIs can be a useful tool to enhance teachers' reflective practices and understanding of their teaching. In my work, they are used to understand change in the teaching trajectory of experienced teachers.

IDENTIFYING TEACHERS' TURNING POINTS

I draw on my work with teachers over several years to offer examples of cases of TPs. I focus on nine of these teachers who made significant changes to their teaching. These teachers were participants in studies that investigated their thinking, learning, and experiences that shaped their teaching of mathematics and their classroom actions. Data sources for these studies were interviews and classroom observations of the participants' teaching. The data included TP stories which were prompted by interview questions about changes in the teachers' thinking and teaching. For example, they were prompted to talk about their current practice, whether they always taught that way, when and how did any significant change started, what exactly happened that initiated this change, and how did this change evolve. The intent was to allow the teachers to identify what they considered to be well defined moments of transition after which their teaching was fundamentally altered. In keeping with a narrative method, they were also encouraged to tell stories, for example, stories of what happened that led to the change, of how the change began, of teaching before and after the change, and of most memorable event(s) that impacted their teaching. They were prompted to provide details to the story from beginning to end and to be only descriptive (i.e., no interpretation). Such stories provide "explication of human intentions in the context of action" (Bruner, 1986, p. 100) and "a framework for understanding the past events of one's life" (Polkinghorne, 1988, p.11). "Narrative meaning consists of more than the events alone; it consists also of the significance these events have for the narrator in relation to a particular theme" (Polkinghorne, 1988, p. 160).

Based on the data, narrative accounts of the teachers' journeys to their current teaching were constructed. The focus of this paper is only on the key events, explicitly expressed by the teachers, that initiated changes and the nature of the changes. This was based on analysing the stories to identify; for example, the CIs and TPs, characteristic elements of the CI and TP situations, key aspects of teaching that were transformed, what occurred during the moment of transformation, and what major features supported such significant changes in teaching.

EXAMPLES OF MATHEMATICS TEACHERS' TPs AND CIs

In this section, I summarize some of the key events of the nine teachers' stories to highlight their perspectives of CIs and TPs in their teaching. The teachers are being named T1, T2, T3, T4, T5, T6, T7, T8, and T9 to simplify the pseudonyms.

Teacher 1 (T1), as a beginning high school teacher, was influenced by her experienced colleagues to adopt their traditional teaching approach, which she did even after she changed schools. She explained, "For years, that's what I did too… stand and deliver. … Like 5 days of, ok it's the usual… just following the textbook." Her first CI was noticing students being bored in her class, which resulted in an initial TP of introducing open-ended problems in her teaching; but, as she stated, "it was more on specific days, at the end of a unit or at the very beginning of a unit. … We had problem-solving days." Her next CI was noticing that this did not help students to engage in problem sol -ving in learning mathematics because they still depended on her thinking. She noted,

> After a while, it started to get to me, and I'm just like, this just doesn't feel right, because I get tired of the kids mimicking me, you know. … There had to be a better way. So that's when I changed to doing strategies; … what I try to do as much as possible is strategies.

Focusing on strategies was the TP that resulted in the significant transformation in her teaching. She started helping students to see strategies, shifted to using more questioning and less telling, followed the textbook less, and selected or developed tasks in which students could focus on strategies. Students worked in small groups to learn mathematics through solving problems, unpacking strategies and discussing them in whole-class sessions. T1 noted, "Problem solving is definitely much more infused into what I do." Another CI occurred when her students explained their challenges on a provincial exam. This resulted in a TP to help them to be reflective, as she explained.

> I've focused much more the past 5 years on reflective thought for each person … and trying to not only encourage but in many ways force kids to do it. … Reflecting on what it is that you know and what does it mean to understand the [concept] is very important. … I get them to reflect as learners … to be doing better thinking, because they'll be asking themselves the questions.

At the point of this study, T1's teaching was totally transformed into an inquiry-based, learner-centered approach with an emphasis on mathematical thinking.

Teacher 2 (T2), as a high school teacher, described his initial teaching approach as: "Review homework, explain a new concept, show applications of the new concept, assign drill and practice seatwork. … That's what I started doing and continued to do for many, many years." After moving to a school with grades 1 to 12 and teaching grades 11 and 12 for the first few years, he was also assigned to teach a grade 6 and later a grade 4 mathematics class. He explained, "I think that was a critical point for me; was being involved with elementary [school] children. It changed me." In contrast to his high school teaching approach, which he found to be problematic with the elementary grades, he was able to engage the elementary school students in group work and hands-on activities. His success with this became the CI leading to the TPs in his high school teaching. The first TP was to introduce group work and encourage students to do more talking about their thinking and experiences. He explained, "The first thing I did … was to take my desks out of rows, and put them in clumps. … It created the opportunity for them to communicate." The next TP was introducing hands-on activities to address meaning and multiple representation. As he explained:

> You really had to think about what does it mean to multiply and how can you model multiplication with the materials … and looking at the patterns …. I started to use algebra tiles in the same way. (…) My classes in calculus sometimes solve problems … completely without paper and pencil first, by going through the same kinds of skills that you do with manipulatives in grade 4, when the students are learning perimeter and area.

T2 also started to make more connections with the historical development of mathematics concepts so that students "could see that there was a value to the development, not just a value to the product." At the point of this study, his teaching approach was learner-centered and emphasized communication, connections, and problem solving. He described his changed teaching as consisting of: "significant amount of communication, … offering lots of opportunity for communication and discussion between students, … working with non-symbolic approaches to problem solving. … My goal is the process that they go through. That's the more important goal."

Teacher 3 (T3), a high school teacher, explained: "For most of my teaching career, I felt my job was to simplify mathematics. Cover the curriculum in consumable bits that could easily be delivered and tested." She always wondered if there was a better way and started attending workshops, tried some activities, but, she noted: "I would leave excited but then it would flatten out quite quickly and I was back in the same routine of stand and deliver." She became "hooked" on the idea of inquiry through conversations "around the concept of inquiry" with a colleague who was a social studies teacher. But after attending a couple of presentations on it, she still did not know what it looked like in a high school mathematics classroom. The CI occurred when she noticed that "making connections" was missing in her teaching. The initial TP was to engage students in making and discussing connections to themselves, real world situations, and history; for example, after students wrote and discussed what they knew about circles and lines,

> We discussed briefly where circles and lines come from. …We then talk about how circles and lines exist in the world. I then send them on a journey around the school with their journal to find any examples of where circles and lines exist, together or separate, visible or behind the scene. …[T]hey talked in groups about what they saw and tried to generalize.

She later extended connections to include mathematical meaning and structure. For example: "We had been talking about what it means to solve an equation, how the structure of algebra worked, and what … made an equation more complex." She also started engaging students in more inquiry tasks as she began "to see more connections within topics and in interdisciplinary ways." She now listened to students' thinking to build on it. As she pointed out: "Suddenly there were portals in my lessons that called me to really listen, become attune to what students were wondering about. … I'm amazed at their thoughtfulness." At the point of the study, her teaching was inquiry-based with students having autonomy in their learning. She explained:

> Our classroom conversation is often around other possibilities. … They seem to be wondering about math and …. are inquiring into topics that come up in class. … What I have noticed of late is the openness of my students to think and go places they have not before. As I open a topic, I never know where it will go. More often than not we end up in territory way beyond the curriculum for that grade.

Teachers 4, 5, and 6 (T4, T5, T6), elementary school teachers, participated as a team in a mathematics study group with other teachers at their school and meetings at least every three weeks over two school years. The focus of the group was to make changes to their teaching to better implement their new reform-based mathematics curriculum. Some of the teachers had attended workshops, which had little or no impact on their teaching. They thus later decided to engage in a self-directed learning approach with an expert friend as mentor. T4, T5, and T6 were co-leaders of the group and participants of this study. As a team, they worked on designing an inquiry-based teaching model to guide their teaching, which became the CI resulting in the TP to their teaching.

The key activities of the team consisted of reflection on their teaching, which resulted in a focus on communication to support inquiry, and video studies of inquiry-based lessons in which they focused on key features of the lessons (e.g., questioning, roles of teacher and students, and tasks). They developed a teaching model with 5 components, then tested and retested it in their classrooms and made revisions. The final model consisted of seven components as follows [key terms for the teachers are in italics]:

(i) attend to students' *prior knowledge/conceptions* and allow students to engage in: (ii) making and exploring *predictions*; (iii) *free exploration* (through discourse and/or

using manipulatives); (iv) *focused exploration* (e.g., specific task assigned by teacher); (v) *applications* of concept; (vi) *comparison, evaluation* and *reflection* of their learning; (vii) *extension* of concept to other situations or related concepts. Also associated with these component are: group work, inquiry tasks and whole-class discussion. The teachers indicated that the model is not linear and the components could be arranged in different ways depending on the mathematics topic and teacher's goal for the lesson.
By the end of the two years of the study group, the three teachers had personalized the model and integrated it into their teaching. At the point of this study, their teaching emphasized learner-centered approaches and understanding of mathematics. They described their teaching as consisting of: "questioning techniques that guide and enrich student thinking," "thought provoking questions to motivate students to discuss and understand mathematics at a deeper level," "student-centered strategies for listening to students and observing their problem-solving behaviors," and "strategies that allow students to assume ownership of their knowledge and knowledge construction."
Teacher 7 (T7), an elementary school teacher, explained that for many years her focus was to get students to do things her way. The CI occurred when she noticed a "bright student's weird solution" which was different from what she expected. She recalled the following example: (The numbering indicates the sequence in the student's process.)

(i) 132	(ii) 2	(iii) 30	(iv) 100	
- 37	- 7	- 30	- 0	
(vi) 95	- 5	0	100	(v) 95

After questioning the student about his thinking and realizing "he had interesting ideas," she started to wonder about the thinking of other students and what she could learn from them. The initial TP was to engage students in whole-class sharing and discussion by encouraging them to use their own methods and explain their thinking to the class. Noticing that this was also important to her students' learning, she started to focus on discourse to support their learning. She did not only pose questions but also en couraged students to be curious and ask questions to promote discourse. She explained:

> What sets the direction for it [discourse] now is the math questions that the kids are asking, because they were given freedom to say, tell me what you want to learn. … So what is important for it [discourse] is the interest of the kids and questions that they have.

When students wanted to know what was a good question to ask, she told them, "It should be something you want to learn. Something that you might have seen or heard and you wondered about; ... wonder, curiosity, what if, … what else can you learn."

Discourse evolved into "an interactive conversation" that addressed students' personal experiences, thinking, and feelings. This included engaging students in discussions about their real-world experiences that embodied mathematics, their pre-conceptions and new conceptions of mathematics concepts or procedures, and their thinking about their own thinking and learning experiences. To facilitate these ways of discourse, in addition to whole-class discussions, T7 also integrated group work, problem solving and exploratory activities into her teaching. At the point of the study T7's teaching was learner-centered and engaged students in ways she described as: "excitement, passion, understanding…of the concepts, application to the real world and…ah-hah moments."

Teacher 8 (T8), a high school teacher, for several years used a teacher-directed approach where she did most of the talking. She explained: "For many years, all they [students] did was watch me stand at the front of the class and explain the math to them." The CI occurred when she noticed the following about her students' learning.

> I realized I don't see things the way kids see things, and I don't solve problems the way kids solve problems. … If I'm explaining something, they can sit and look at the board and I can tell they don't get it. … Then if I ask somebody else in class to explain it, they might say exactly the same thing I said …and then the others will go, "Yes that's right, I understand." And I'm there thinking, but I just said that. … Somehow, they know how to relate it to each other, and many times they can express things in different ways that I haven't thought of.

The TP was allowing students to solve problems in groups where they could use their own approaches and share their thinking. But as she explained:

> When I first started doing this, I didn't know what my role was. I knew it wasn't sitting at the desk correcting papers, so I had to do something else. So I walked around and then I wanted to give the answers or I wanted to tell them. And then I realized well that's not what's supposed to happen either. So now I can sit next to any group and they talk, and I ask them questions if they're stuck, but that's about it.

She also learned how to prompt students to deepen their interactions and learning by comparing experiences to learn about learning, sharing ideas to collaborate and expand their thinking, explaining concepts in meaningful ways, posing questions among themselves, and validating their understandings. To support their interactions, she assigned tasks in which they planned and conducted group projects/investigations to learn new concepts, engaged in both genuine problem solving and problem posing, explored mathematical structure of concepts, and led whole-class presentations and discussions. At the point of this study, her teaching was inquiry-based with students having autonomy in their learning. As she explained: "They get to interact with each other all the time and can use each other to enhance their own learning."

Teacher 9 (T9), an elementary school teacher, for about 20 years maintained a traditional classroom in which students were expected to mimic her. Teaching grade 1 for several years convinced her that these students were too young to think for themselves in doing mathematics. So, when two of her colleagues encouraged her to join them to explore using inquiry-based tasks in their teaching, she kept refusing because: "It wouldn't work for my grade ones." She finally decided to join them to be collegial and "to see what they were doing." She later agreed for them to plan and try an inquiry task with her grade 1 class. She also agreed to teach the lesson with them observing to give cues during it to help her to follow the plan and feedback after. The lesson consisted of the following key features for the topic, "estimation with mass":

> (1) Groups of 4 predict and order from lightest to heaviest 5 balls of various sizes/mass. … (2) Discuss process (How they decided order); … Discuss product (Why they think there are differences); … Discuss, "How can we figure out who is right"; … (3) Use scales and beans to find exact order of balls; … (4) Share/discuss what noticed; … (5) Discuss when they would use estimating mass in their world. … Where or when use this skill or process?

The CI was what T9 noticed about herself and her students. She was surprised that the students accomplished the task on their own: "I couldn't believe it! I didn't believe they could do it without me getting in there to show them how to get the answer. But they did it and they were having fun!" She was also surprised that she could resist telling. "If Jen and Lyn weren't there, … I would have jumped in and do it for them. It wasn't easy to not say anything especially when they were doing something different from what I would do." She liked the idea of students making predictions and testing them, so her TP was to incorporate this in her teaching as she continued to discuss and plan tasks with her 2 colleagues. She noted, "After a while I learned how to listen to hear their ideas and logic and how to ask questions to help them instead of showing it to them." At the point of this study, she consistently engaged students in inquiry tasks.

NATURE OF THE TEACHERS' CIS AND TPS

The teachers' CIs and TPs varied based on their beliefs and experiences. They grew out of the teachers' practical knowledge based on evidence in their teaching regarding something they perceived to be missing, unique, or different in their teaching, their students' thinking, or their students' behaviors, and thus, are personal and contextual. The CIs emerged from: the teachers' awareness of something previously overlooked, a shift in perspective of something observed, a shift in sense making of a belief, or a different way of learning. They emerged unexpectedly and opened doors that were meaningful for the teachers and enabled them to make sense of how to implement changes to their practice. They were initiated by events directly associated with the teachers' learning or their teaching regarding their students' learning, engagement, or thinking in learning mathematics. They involved situations that provoked some uncertainty and/or included an element of surprise for the teachers that made them curious, conflicted, confused, and/or concerned about some aspect of their teaching or student learning. Their TPs were significant changes in their teaching trajectories, from a dominant teacher-centered, traditional approach to a sustained learner-centered, inquiry-oriented approach. The TPs were initiated based on the teachers' personal meanings linked to a belief held, a process or way of knowing or learning, and an emerging belief about learning, discussed in the next section. They occurred when the teacher has a clear sense of purpose in relation to the learner. They also occurred in stages or evolved over a period of about two or three years with the initial TP marking the beginning of the changed trajectory of teaching.

TRANSITION FROM CI TO TP

This section discusses three themes of the mechanisms that supported the transition from CI to TP for the teachers. These themes involve specific beliefs held by the teachers (theme 1) and specific ways of learning in which they engaged (themes 2& 3).

Theme 1: Holding metaphoric, inferential, or emerging central beliefs

Metaphoric central belief. A TP occurred when a 'central belief' held by the teacher became a 'generative metaphor'. A central belief is psychologically strong regarding its importance to the person holding it (Green, 1971). A generative metaphor (Schon, 1979) or structural metaphor (Lakoff & Johnson, 1980) facilitates a process by which we gain new perspectives on the world; that is, a process that involves generating or structuring one concept in terms of another. This process uses one domain as a lens for seeing another; that is, seeing A as B where A and B had previously seemed to be different things. It requires a restructuring of perception to see A as B. It generates perceptions of new features of something or give rise to a new view of it. T1, T2, and T3 held beliefs about mathematics as metaphors that they interpreted in ways that generated changes in their teaching. A CI occurred when they became aware of something missing in their teaching that was interpreted as a characteristic of the metaphor and resulted in a TP in their teaching. The transformation of teaching occurred in stages in response to when and how the metaphor unfolded. It was not until they expanded their interpretations of it that a different understanding of their teaching occurred.

T1 held a central belief of *mathematics as play* or *game* that grew out of her experience with doing mathematics. Her CIs and TPs occurred when play/game was interpreted as fun, strategy, and thinking and then related to doing mathematics. Fun and strategy were associated with problem solving. As she explained,

> I thought, if I'm going to be a good problem solver, I have …to think about what strategies to try. … As a learner, what I need to do is look at them [problems] as a game. When I play monopoly, I know the rules but it's dynamic, it changes. When I solve a problem, I have my strategies that colors the rules, but it's a dynamic situation, and so sometimes I use this strategy, sometimes I use that strategy.

For her, strategy was also about a way of thinking, seeing patterns, making connections and reasoning, which she associated with viewing and learning mathematics. Students needed to be autonomous learners to engage with strategies meaningfully and to understand their own thinking to make sense of and justify strategies. Her CIs involved notic ing that these elements were missing in her teaching, with TPs to integrate them into it.

T2 held a central belief of *mathematics as experience* that grew out of his experience teaching elementary grades. His CIs and TPs for his high school teaching occurred when experience was interpreted, based on what he noticed in the elementary classroom, to be something shared/communicated, hands-on activities, and historical connections, which were then related to doing mathematics. He explained, "The first thing I did when I started to decide that mathematics had to be more of an experience and a community experience was to take my desks out of rows, and put them in clumps." Hands-on activities were "non-symbolic experiences" that involved both physical objects and students' personal experiences. Historical connections involved mathematics as a "human construction" or experience. He noted, "I like the kids to see where it came from, … that the development of that tool came from somebody." His

CIs involved noticing that these elements of 'experience' were missing in his teaching, which resulted in TPs to integrate them into it.

T3 held a central belief of *mathematics as a living discipline*. Her CIs and TPs occurred when a 'living discipline' was interpreted as "bloodlines", excitement and aliveness, complexity and uncertainty, and beauty. The 'bloodlines' were relationships within and outside of mathematics and historical connections. Her CIs involved noticing that these elements were missing in her teaching and resulted in TPs that changed her thinking, and the tasks and discourse in her teaching. With these changes, she explained:

> The world opened up right there; conversations were rich and complicated, answers were uncertain, the work constantly unfinished. (…) The more we enter into a topic, the more exciting it becomes, … it is exciting and alive. The students are continually seeing things in ways I never imagined. (…) It is through its [mathematics] structure, patterns and connectedness I can see many possibilities. Where does this come from? Why do we still talk about it? How does it live and contribute to the world today? (…) As I enter into inquiry … how do I open topics? Do I look for the topic in the world or see the world through the topic? There are times I see clearly the connections either through the structure of math, its beauty, complexity or imagery.

Evidential belief. In this case, a CI and TP occurred when an 'evidential belief' held by the teacher was challenged. According to Green (1971), beliefs held evidentially are supported by evidence and are more susceptible to change than nonevidential beliefs in response to conflicting evidence. T9 held a belief that grade 1 students were too young to engage in inquiry. She supported this belief with evidence based on situations when her students could not interpret open-questions or follow instructions for a task without her providing carefully structured instruction to direct their thinking and actions. She encountered conflicting evidence when, with the help of colleagues, she engaged her grade 1 students in an inquiry-oriented lesson and realized what the children were able to do, the richness of their thinking and depth of their learning through inquiry. This became the CI, resulting in a TP that changed her belief and teaching.

Emerging belief. In this case, the CI and TP were dependent on the emergence of a particular belief for the teacher. T8 was not aware of holding a belief about student-student interactions or its role in students' learning. The belief emerged as she began noticing that when she allowed students to clarify her explanations to others in class, it made a significant difference to their understanding. This became the CI leading to TPs in her teaching and eventually the development of the central belief: "Math learning occurs when students understand and can explain the math concept … in their own words … and know it sufficiently to teach … [or] talk about it to someone else."

Theme 2: Learning through design and student thinking

Design Thinking. T4, T5, and T6 engaged in a learning process that consisted of characteristics of a design thinking process, which became the CI resulting in the TP to their teaching. The design thinking process (Hasso Platter Design Institute, 2010) consists of five components: *empathize* (develop understanding of users from their perspectives), *define* (identify the problem to take on), *ideate* (generate ideas towards potential solutions), *prototype* (create a product users can experience), and *test* (test the prototype to receive meaningful feedback about the users, the problem, and the potential solutions). This process is not linear in that one can go back and forth between two phases before moving to the next. Linking the test back to empathize stage is critical.

In the teachers' approach, in phase 1 (empathize) they focused on understanding themselves (as teachers), their students, and parents as users of the model to determine a common need in their teaching. T4 summarized the outcome of this process:

> Our students and their parents were used to doing math calculations but did not always have the experience or understand the importance of explaining and thinking through math. … It seemed like a logical starting point for all levels of our learning community and our teaching.

In phase 2 (define), they decided on a problem to undertake; that is, to create a model to support communication that allowed students to actively engage in their learning in an inquiry context. In phase 3 (ideate), they studied videos of inquiry-based math lessons to get ideas to determine possibilities for a model. In phase 4 (prototype), they drafted possibilities of a model. In phase 5 (test), they decided on the most meaningful possibility of the model for their students, designed a lesson plan, tested it in their classrooms and connected the findings to the empathy stage, which resulted in making revisions to the model, re-testing it and eventually fully adopting it in their teaching.

Student thinking. In this case, CI and TP were dependent on T7 viewing student thinking as a source of learning specific aspects of mathematics knowledge for teaching (e.g., aspects of Ball, Thames & Phelps's (2008) specialized content knowledge, knowledge of content and students and knowledge of content and teaching). She noted, "From that point I became curious. I wanted to know more. I wanted to learn from them." This became a goal of discourse in her teaching which she described as follows:

> It's a little bit selfish, but I want to learn something. So, I want to be ah-ha'd and surprised. … I almost get a rush … it's a weird thing, but a high when they teach me something. I'm not afraid to take risk, so I put myself out there to see what I can learn too.

With this goal, she started to listen differently to students, focusing on their sense making, which resulted in the TP in her teaching and eventually changes to also engage students in discourse to support their learning and her teaching in general.

Theme 3: Engaging in self-directed learning

Self-directed learning is usually associated with adult learning or andragogy. It is "a process in which individuals take the initiative, with or without the help of others, in diagnosing their learning needs, formulating learning goals, identifying human and material resources for learning, choosing and implementing strategies, and evaluating learning outcomes" (Knowles, 1975, p.18). The learner has more autonomy in the way that they learn. All of the teachers engaged in self-directed learning in transforming their teaching. Having personal choice was a crucial factor in how the TPs evolved from the CIs. Noticing is also important to self-directed learning and was a crucial factor in the emergence of CIs and TPs. This form of noticing, as Mason (2008) indicates, involves not only the attention that teachers give to significant classroom actions and interactions, but also their reflections, reasoning, and decisions based on it; that is, *attention* and *awareness*. These ways of learning also enabled the teachers to address their specific needs regarding the knowledge they required to realize the TPs and subsequent changes to their teaching. For example, while not discussed in this paper, they sought information or resources to enhance specific aspects of their mathematics knowledge and pedagogical mathematics knowledge that they identified to be necessary to make the changes specific to their TPs in ways that served their teaching.

IMPLICATIONS

These teachers' stories suggest that practicing teachers could be supported to transform their teaching if they are helped to attend to CIs that could lead to TPs, particular beliefs they hold or could hold, and particular ways of learning. Prospective teachers could also be helped to develop a disposition to be curious, to understand generative metaphoric beliefs of mathematics, and to engage in adult learning pedagogy.

The paper draws attention to the importance of exploring the multiplicity and context-specificity of processes when trying to understand changes in teaching. The influence of CIs related to TPs offers a potentially fruitful area of investigation that may increase our understanding of why and how teachers change in the short term and over the long-term. Further research on TPs may be particularly valuable in unpacking the multifaceted and complex underlying mechanisms and factors involved in lasting changes in teaching. Understanding TPs may be particularly valuable in providing insights into the complicated underlying processes involved in long-term changes in teaching and reveal why, for instance, the same incident/event constitutes a TP leading to significant change for some, but not for others and what contextual factors, personal characteristics and individual factors influence TPs in teaching.

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PLENARY PANEL

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SHOULD RESEARCH INFORM PRACTICE?

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The discrepancy between the aspiration for research in mathematics education to have an impact on educational practice and the common perception of research in mathematics education as being irrelevant for practice calls for an in-depth examination of this issue. This is the topic of this year's PME plenary debate, phrased as: "Research shouldn't inform practice". In this introduction to the plenary panel I set the stage for the debate by using a personal experience to raise some queries about potential relationships between research in mathematics education and practice.

INTRODUCTION

In their discussion of the purpose of mathematics education research, Lester and Wiliam (2002) point out that different researchers have different views regarding this matter:

> Why and for whom is research in mathematics education conducted? Is our research, as some cynically insist, simply an activity pursued by "ivory tower" academics intent on publishing articles read only by other academics? Or, as others believe, is its purpose to promote the development of robust theories about the teaching and learning of mathematics? Some hold yet another view, namely, that research should focus on the pursuit of knowledge that causes real, lasting changes not only in the way people think about learning and teaching, but also in how they act (p. 489).

Lester and Wiliam, as many others in our community, indicate their preference to the latter view. Moreover, in a recent comprehensive review of research in mathematics education, Schoenfeld (2016) maintained that this could be done:

> Of fundamental importance is the fact that mathematics education had reached the point where research and practice could work together in productive dialectic. Research could inform practice in productive ways, and practice, in turn, could serve as the site for meaningful research (p. 510).

Yet, research in mathematics education, like educational research in general, has been often criticized for being irrelevant for educational practice (e.g., Bromme & Tillema, 1995; Cai, et al., 2017; Malara & Zan, 2002). The continuous discrepancy between the widespread stance regarding research in mathematics education as being irrelevant for practice and the frequently proclaimed preference for the view that research in mathematics education should influence practice is troubling and calls for an in-depth examination of this issue. This is the topic of this year's PME plenary debate, which is provocatively stated as "Research shouldn't inform practice."

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Adopting the format used by Helen Chick in her introductory paper for the PME plenary panel in Szeged (Chick, 2016), I use in the following a personal experience (Even, 2003) to raise some queries that appear to be pertinent to the debate.

PERSONAL REFLECTION AND SOME QUERIES

About two decades ago, I was asked to give a talk at a mathematics teacher conference. The invitation stated that the talk should focus on research in mathematics education. Usually, this kind of invitation entailed reporting on a specific aspect of my own research or summarizing research on students' learning in a specific mathematical area. However, I decided to use this opportunity to address the question: What can teachers learn from research in mathematics education? The ground for this decision was my feeling that the relevance of research in mathematics education to teachers has not been adequately addressed or answered.

As I started to work on my talk, I quickly determined – based on my experiences as a researcher and teacher educator – that research in mathematics education could not provide practitioners with clear rules for action.

Query 1: Is it true that research in mathematics education cannot provide practitioners with clear rules for action? Why? Are there situations for which research could provide practitioners with clear rules for action?

Still, I thought that research in mathematics education could become meaningful and relevant for practitioners. I continued to work on my talk by asking myself what ideas in research in mathematics education are relevant for teachers.

Query 2: Is it an appropriate approach to focus on ideas with relation to relevance of research in mathematics education to practitioners? Why or why not?

My search for ideas in research in mathematics education that are relevant for teachers was influenced by the writings of Polya (1954) and Lampert (1990) who considered courage and modesty to be essential for doing mathematics; courage to take a risk when making a mathematical conjecture and modesty to admit that one's conclusion may have been inappropriate. Extending their perspective, I considered courage and modesty to be essential not only for doing mathematics, but also for teaching mathematics. Similar to making a mathematical conjecture, making a teaching conjecture (e.g., making a change in a lesson plan in response to students' unexpected solutions of a math problem, trying a new instructional method, experimenting with an innovative way to assess students' understanding) also requires the courage to take a risk and the modesty to admit that one's conclusion may have been inappropriate.

I decided to look for ideas that have the potential to empower teachers so that they are more knowledgeable about what it means to know, learn, and teach mathematics; knowledgeable in ways that would enable them to be courageous and modest in the sense described above. I believed that not only is this an important goal but that the use of research could play a significant role towards achieving this goal.

Query 3: Are there particular goals of the professional education and development of teachers for which the use of research could be useful? Are there particular goals for which the use of research would not be beneficial?

I selected for my talk four ideas, I derived from a synthesis of research in mathematics education: "(1) Mathematical knowledge is constructed in ways that do not necessarily mirror instruction. (2) Mathematical meaning is both subjective and sociocultural. (3) Knowledge and practices of learning and knowing are inseparable. (4) Knowing is a 'slippery' notion" (Even, 2003, p. 38).

Query 4: To what extent the choice of these ideas was related to my own knowledge, understanding, and beliefs about, experiences in, and practice of, research in mathematics education and teacher education? Would other researchers in mathematics education make different choices?

In my personal experience, I did not consider practitioners, such as, curriculum developers and policy makers, but rather focused only on teachers.

Query 5: What ideas in mathematics education are relevant for practitioners who are not teachers? Are there ideas that are more relevant for these practitioners and other ideas that are more relevant for teachers?

Another aspect that I did not consider in my personal reflection so far is related to the choice of the word *should* in the phrasing of the topic of this year's PME plenary debate: "Research shouldn't inform practice," in contrast with the use of the word *could*: "Research could inform practice…" (Schoenfeld, 2016, p. 510). This choice of wording suggests that it might be that even if research in mathematics education could inform practice it should not do it.

Query 6: Should relevance for educational practice be a purpose of research in mathematics education? What might be gained and what might be lost if we aim for research in mathematics education to inform practice?

THE PANEL

The members of the panel have been invited to debate the statement: *Research shouldn't inform practice*. The way the statement is phrased implies that the affirmative team (Michael Askew and Guri A. Nortvedt) – which argues in favor of the topic – argues that research *should not* inform practice. Similarly, the negative team (Roberta Hunter and Leong Yew Hoong) – which argues against the topic – argues that research *should* inform practice. The papers of the two teams follow in this order.

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RESEARCH SHOULD NOT INFORM TEACHING[1]

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INTRODUCTION

In this debate, in arguing for the position that 'Research should not inform teaching.' we present three lines of reasoning. First, we observe that despite many years of research, research has had very little impact on teaching and begin our argument by presenting scientific evidence and other sound reasons as to why this is so. Hence given the fact that research does not inform teaching we should, as a community, now accept that research clearly should not inform teaching. Our second argument rests in the observation if research is to influence teaching then it must be accessible to teachers and that the current publishing practices militate against this. Finally, much research is now funded by policy-initiated programmes, with research agendas being tacitly steered towards policy directives. Thus rather than researchers being free to follow interesting lines of thought and design independent research studies that identify and address the crucial questions regarding teaching and learning, funding research ends up conforming to policy-formed questions that are rooted in current knowledge. This results in maintenance of the status quo in teaching rather than radically changing it.

FIRST ARGUMENT – RESEARCH HAS NOT CONTRIBUTED TO SIGNIFICANT CHANGES IN MATHEMATICS TEACHING, BECAUSE IT CANNOT

Visiting a typical classroom, it is obvious that teaching has changed very little over time, with students and teachers coming together to teach and learn mathematics in ways that would be familiar to anyone schooled 50 or more years ago. The artefacts may look different, with chalk and blackboard being replaced by pens and whiteboard (or an electronic smartboard version of these) and paper textbooks may have given way to printed work sheets or computer tablets, but the substantial content of what is on the boards, or in the students' hands, has not greatly changed. The teacher will present something (a theory, an example, a task or problem to be solved) and students subsequently engage in activity. But surely, one might ask, there is more interaction in classrooms now and less teacher authoritarism? Our response is, yes, there is likely more interaction between the teacher and the students than might have been observed many years ago, but the dominant form of this interaction is still closed questions and answers, with a focus on getting correct answers. The climate may be less authoritarian but the teacher is still the primary authority.

Given such lack of changes in teaching, research clearly has not informed teaching, so we should stop pretending that it does or that it should. For instance, the way

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 67-74. Singapore: PME.

mathematical problem solving is taught in schools has not been much improved as evidenced by the large body of research on problem solving revealing that teachers still struggle to teach students problem solving strategies and to develop collaborative problem solving within classrooms (see, for instance, Lesh & Zawojevski, 2007). Similarly, despite the extensive research on the importance of talk and collaborative work in mathematics (see for example, Stein et al., 2008) most classrooms are still characterised by a dominance of teacher exposition followed by students working individually on tasks (Peña-López, 2009). Why has research had so little impact on teaching? We argue that two of the reasons why research has not informed teaching are, first, the nature of the research itself and second in how research outcomes are made public.

The bedrock of scientific research leading to scientific advancement is experimentation, yet few true experiments can be found within our academic field. Indeed, even where there are experimental studies, the cumulative effect of these is slow. To take a concept like scaffolding, which many would agree is key to successful teaching, Bakker, Smit and Wegerif (2015, p. 1056) wrote ‘We predict it may well take a decade before there are enough experimental studies of sufficient quality to quantify the gains of various scaffolding approaches compared to regular teaching.’ Yes, some studies do have designs close to the scientific experiment (pre- and post-test designs, design experiments) but without randomised allocation to intervention groups and control over multiple variables it is not possible to calculate the effect of the implemented innovations on mathematics teaching and so make strong comparisons with other innovations, or even with the ‘normal’ or untreated classrooms. The key issue here is that the contexts of classrooms are far removed from the science laboratory – control of variables is much closer to being achievable in the latter than in the former. As Mason (2013) points out, all teaching practices are highly contextualised, so there are no generalizable practices, only generalizable principles. Since principles are based in local, ethical and moral considerations of what constitutes a good education, how can research provide global or even local recommendations for practice? Following Bakker, Smit, Wegerif and Mason, we might conclude that at best mathematics education research may only be able to produce ‘fuzzy’ generalisations that cannot provide concrete insights that easily can be applied in teaching.

Compounding the lack of experimental studies, the dominance of research involving small scale case-studies limits the generalizability of findings. Although Yin (2014) for instance argues that analytic generalisations might be valid inferences drawn from case studies, this is highly debated. For example, is there such a thing as a ‘neutral’ or context-general case study where the situatedness of the case-study be disregarded? As noted classrooms are highly contextual and patterns observed in one classroom might be differently composed in other classrooms. Small case studies might be fine if they built on each other’s findings but there is little evidence of that in the research literature (Mitchell & Charmaz 1996). We would agree with Nietzsche in his observation that in doing science ‘one should not wish to divest existence of its rich ambiguity’ (1974, p

335). In addition too many case studies are descriptive – they tell us what is happening, and in many cases why current practices are not effective (see for example, Ensor et al, 2009) but without recommending ways to improve practice. At best, case-studies may go further and theorise about the practices observed but the findings then are explanatory – researching why teaching has not changed much does not necessarily provide empirical evidence that informs how to bring about change.

In addition to case studies, other frequently used research methods are observation, questionnaires and interviews. Generalisation might not be the issue when questionnaires or large-scale assessments are used to study teaching and learning, as large, representative samples might be applied (see, for instance, Schleicher, 2012). However, large-scale studies mean simplifying the educational context or leaving out "the friction" in order to isolate and enable study of the phenomena into which we want to gain insights. The PISA and TIMSS studies for instance, use student questionnaires to study instructional quality (Mullis & Martin, 2013; OECD, 2016), but how do we know either what the relationship is between student reports on instructional quality and the actual quality or even if what students think is good is actually effective? What is researched is perceived instructional quality and what is perceived as effective is not necessarily so.

Even if large-scale studies were to make recommendations for teaching, then the unintended consequences may outweigh the intended ones. For instance, aims such as sharing ideas on best practices in teaching has been lost in the promotion of international league tables by policy makers (Auld & Morris, 2016). Rather than raising standards of teaching or bringing best practices into view, the league table mentality and the jockeying for position within ranks has resulted in a narrowing of practices – a reduction in taking risks and trying out new pedagogies for fear of reduced 'standards' (Broadfoot, 2000). The ultimate effect is a lowering of spaces for innovation but innovation rests on risk taking and learning from failure.

Across both small- and large-scale studies, a lack of unified theories and agreed 'best practice' methods further limits the applicability of research findings to teaching. A good example of a situation where findings 'do not add up' is the manipulatives debate (McNeil & Jarvin, 2007). While some researchers found that manipulatives help students with mathematics learning disabilities (MLD) learn mathematics, other researchers found that the manipulatives had a negative effect on the learning of MLD students. McNeil and Jarvin (2007) in their review concluded this is connected to how the research was carried out.

The key issue is that in most, if not all, mathematics education research the phenomenon under investigation is complex and rich and contextual and intertwined. Research necessarily simplifies the object of investigation, can never account for all the variables treating the complex as complicated (in the sense of ordered and predictable (see, for instance, Leder & Grootenboer, 2005). Consequently, research findings rarely, if ever, transfer to other contexts. Take the case of the PISA study and Finland. Finland was among the top countries in the early PISA studies (that is, in 2000 and 2003, see OECD, 2004) and, not surprisingly, many, researchers and policy makers wanted to learn from Finland. However, Finish researchers themselves, thought that this could not easily be done and they themselves struggled to understand these outcomes, stating that there was:

> 'no one single explanation for the result. Rather, the successful performance of Finnish students seems to be attributable to a web of interrelated factors having to do with comprehensive pedagogy, students' own interests and leisure activities, the structure of the education system, teacher education, school practices and, in the end, Finnish culture.' (Välijärvi, Linnakylä, Kupari, Reinikainen, & Arffman, 2002, p. 4)

Similarly, there is much interest in what is being referred to as 'Singapore mathematics' but as B. Kaur from Singapore pointed out in her IMCE17 plenary, there is not agreement within Singapore on what constitutes best practice (Kaur, 2016).

SECOND ARGUMENT: DISSEMINATION OF RESEARCH FINDINGS PREVENTS RESEARCH FROM INFORMING TEACHING

Not only are research methods problematic, but also the publication of research outcomes is problematic on account of what findings are published and also how findings are communicated to teachers. The first stumbling block lies in what might be accepted for publication – typically only original research is published. To many, this is conceived as 'research with findings'. This leads to research being re-invented since pedagogies tried out that yield no changes in learning, have not been reported on and so neither the research nor the teaching field gets to know what has been demonstrated as not working. In the climate of accountability in Universities researchers have to be original, rather than test out or replicate previous findings. In addition, research not meeting standards for publication does not get into the public domain. For instance, review guidelines often state that researchers need to embed their empirical studies in current research traditions and theoretical paradigms to (see for instance our own PME guidelines for Research Reports). Novel research that represents a clear break with current framework and traditions or is very creative might not be considered nor accepted for publication. There may be much research carried out that could make a difference but we know little about the potential effect of the proposals that come from such research.

Second, research findings are not made easily available to teachers so they do not know what might be important. For example, a study conducted at Durham University (See, Gorard, & Siddiqui, 2016) examined how teachers made use of research findings showing the impact on learning that enhanced feedback can have (Hattie, & Timperley, 2007). The teachers worked with the published research findings, but could not put them into practice. Two main reasons for the lack of change were noted. First, that the published findings did not provide sufficient examples of what sort of feedback was envisaged, so the teachers could not identify what changes to practice to make. Second, the style of writing was a barrier to engaging with the research. One teacher was noted as saying 'I need a translator to understand what this article is saying. I just cannot understand what [Hattie] means and what he wants us to do.'

THIRD ARGUMENT – RESEARCH IS POLICY STEARED TOWARD MAINTAINING EXISTING PRACTICES

Educational research is mostly funded by national or international authorities that often want research to have an applicable outcome that is innovative or can improves some aspect of society (see for instance European Union, 2017, 01.04). In many ways, policy makers want to influence teaching more than research due to a desire to cater for effective and high performing educational systems. One means to achieve this, is to ask for educational research directed toward national educational policies, e.g. assessment for learning (Baird et al., 2016).

Given this politicising of education, with policy makers resistant to taking risk and so seeking evidence from large, statistical studies often conducted by economists, whilst the concerns of many in the field for social justice and equity, which are important and may intersect with teaching but not influence it directly, are seen as less important than research aimed at raising standards, but in the absence of any real debate about what standards are being set. Teaching is, inevitably, goal directed, but in the world of standardized testing, of targets, and of the 'no child being left behind' policy, surely it is the role of the researcher to be challenging such discourses of accountability, not feeding into them (see, for instance, Berliner, 2011)?

Restricting our research to what policy makers and funding authorities see as worthwhile means research being directed towards an agenda set from outside, not from within the research community. As such, this might not help us as a research community to identify the important issues that need to be addressed (Linden, 2008). A more substantial issue is related to the need to grow as a research community, and to arrive at a place where we do manage to direct and coordinate our research so that we do manage to develop substantial theories and findings (Burkhardt & Schoenfeld, 2003).

OUR ARGUMENT IN SHORT – LINKING THE ARGUMENTS

We have argued that mathematics education research has not moved teaching practices forward, and may, perhaps, in some instances have moved teaching backwards. Let us stop deceiving ourselves that our research should inform teaching – it does not, it cannot, it should not.

As researchers, many of us come from a background of teaching and it is natural that we should want to improve the practices that we previously were members of. It is notable that researchers coming from non-teaching backgrounds are more willing to be openly critical of schooling, to point to its inadequacies but not position themselves as having answers (a classic example here is Stephen Balls' work, see for example Ball, 1984). Those of us moving from the school to the academy would do well to recognise that our roles as researchers are very different and not mix up these roles with previous ones.

This is not to argue that work is not needed to improve teaching – it is, but that is the work of curriculum developers, who may or may not choose to research their developments, but should not be required to. We need to be clear about curriculum development and research. Curriculum development IS about informing teaching but it is not necessarily research. Research, of necessity, involves looking back – re-searching for answers rather than forward, pro-specting for solutions (Burkhardt & Schoenfeld, 2003). Creative and novel research must take place in contexts where risk taking and the potential of failure are allowed (Linden, 2008), but current funding does not encourage this and that position is unlikely to change in the near future given the fragility of global economics and lack of funding for blue-skies research.

The pressure to publish means researchers have to present their findings as new (Billig, 2013) and many researchers, working within the field of mathematics education are forced to direct their interests to classroom studies so that they can argue that their work has an 'impact'. Acknowledging that mathematics education research should not need to inform teaching would free up researchers to pursue genuine interests, interests that may ultimately have a greater impact on practice through widening the breadth of research. Research should be free and researchers should be free to investigate what really matters – restricting our attention to research that informs teaching would limit that freedom.

With regard to findings being disseminated more widely, as researchers we are too reluctant to be prescriptive. Research findings are hedged with qualifications – too easily interpreted by teachers as a lack of confidence in the findings – and that, together with the dominant discourse of 'reflective practitioners' suggests that teachers have to make up their own minds about what good practice comprises. For research to have an impact on teaching, researchers need to be less tentative in their results, but that is rare in the discourse. Perhaps those more drawn to living with certainties (even if these have to change) are drawn towards policy work, while researchers prefer to keep things open.

In summary, the lack of experiments, the risks of innovation and the heavy emphasis on 'standards' and difficulties in getting research findings into the hands of teachers means that research largely builds on current practices rather than proposes anything radically different. Mathematics education research must raise its game; move the gaze from small scale, non-cumulative studies to larger scale work that of necessity can then say less about actual teaching practices. And it must not buy into the discourses of policy makers that are only concerned with raising standards. And it must 'speak' to teachers. Only then it might begin to have an impact on teaching.

Note

[1]Readers should note that the opinions expressed here are not necessarily those of the authors

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RESEARCH SHOULD NOT INFORM PRACTICE

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This paper establishes the context for the debate to oppose the motion that "Research should not inform practice". The paper first defines what is meant by the terms research and practice in the context they are used in this debate. Four key points are then offered which illustrate the importance of research informing practice.

INTRODUCTION

This paper sets the context for the debate to oppose the motion that "Research should not inform practice". To begin we need to define what we mean by research and practice within the scope of this debate in order to confine the scope of examination. Although we do not like debates to be heavily centred around definitions (as it tends to become purely academic and less useful – yes, we have waded into our natural inclinations of practice-orientedness …), for a meaningful discussion, and to allow the audience to refute the claims of our opponents, it is unavoidable that we establish common definitions.

Within the term "research" and its close relative "theory" (Malara, & Zan, 2002) we have in mind all activities that may be classified as "systematic inquiry". Put simply in Mason and Waywood's (1996) words "the human enterprise of making sense, in providing answers to people's questions about why, how, what" (p. 1060) within inquiry for sense-making. We do not see a need – for the purpose of this debate – for a narrower definition. In an initial view of the term "Practice" it appears to be far more straightforward, until you read Lampert's (2010) paper where the term becomes far more problematic. But within the intended context behind the motion statement, we stay with "the work of teaching" and apply to the term teaching a process of decision-making. We interpret the verb "should" to mean "with the purpose of". This ought to be distinguished from "is capable of". The latter will take us into another debate: whether current research is indeed informing practice; but the former interpretation of "should" will lead us to a more fundamental and critical debate: Should anyone who is involved in systematic studies of mathematics education have a view of relating the findings of the inquiry to the work of teaching? [In this interpretation of the topic of debate we have also slipped in our take on "inform" – "relating the findings to"]. And our answer to this question is: Of course! We advance a few arguments.

We begin with this historical note: Mathematics Education emerged as field of research to address the problems of teaching

It may be argued that the requirement to "inform practice" does not apply to the 'parent' field of Mathematics – mathematicians can stay "pure": they may produce

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 75-78. Singapore: PME.

research that is not directly useable in practice. Here, the argument uses a sleight of hand – the word "practice" has shifted from the motion's inherent meaning of "professional practice" to "popular practice" of presumably the common public. This exposes a false comparison: while mathematics education has (and indeed was motivated by) an actual community of professional practitioners in mind, mathematics (especially the pure branch of it) does not have a corresponding professional practice to address – mathematicians communicate among mathematicians, not with non-existent "professional users of mathematics". In fact, that mathematics education was indeed originally conceived as an "applied" field is clear from the founding vision of ICMI – whose constitution was seen as a 'coming of age' of mathematics education as a field of study. For example, Begle (1969), in his address to the first ICME, 'chided' "Mathematics educators [as being] … unable to organize the kind of empirical investigation needed to *provide useful information*" (p. 239, emphases added). Neither is this a one-off reminder of the responsibility of mathematics education research towards practice. In a later ICMI Study, Bishop (1998) repeated the call, "[m]y real concern … is with what I see as researchers' difficulties of relating ideas from research with the practice of teaching and learning mathematics" (p. 33). Since then, the literature is replete with reports on addressing the "theory-practice" link, which presupposes the need for researchers to attend to the challenges of practice.

1. The survival of mathematics education research is in its usefulness to practice

We (i.e., mathematics teacher educators working primarily in universities) may not like this, but it is part of a reality played out at a global scale: Why would funding agencies continue to provide resources to researchers if the outcomes of their projects cannot be 'cashed in' in terms of actual improvements in quality instruction within mathematics classrooms? Two forces hasten the demise of funds (and hence related university positions): the pressure from populations (especially of developed countries) for answers to education problems, such as low performance in overall mathematics scores (e.g., TIMSS and PISA); and the prevailing climate of short-term paybacks to investments. There is growing impatience with 'blue sky' research that would not provide immediate 'translational outcomes' within the testbeds of classroom practice. Already, in the USA, there are disenchantments with respect to the quality of teacher preparation programmes offered by university faculty. The organized forms of this dissatisfaction can be seem in influential programmes such as "Teach for America" supported by the Gates foundation. They can be interpreted as the beginning of voices of dissent - against the prolonged lack of useful inputs from educational research in improving practice.

2. Research and practice are symbiotic

Both the terms research and practice in the form we are using suggest action and in this debate we apply the term practice to describe the work of teaching. The overlaps are clear, teaching we describe as a process of decision making and research as a process of inquiry; terms which have gained increased coinage in recent times. Mathematics education and mathematics education research is in its infancy compared to other

fields of research. We need to remember that it had its origins in a positivist paradigm, where research was statistical in nature and the teacher was positioned as the 'constant' in classroom studies (Mason & Waywood, 1996). In our experience working within the messy complexity of schools clearly the teacher as an active decision maker who is constantly 'problem making and problem solving in the moment' could never be considered as a 'constant'—that is as a replicable or reproducible factor in research.
In recent times, in our own work, as in the work of many others, we have seen how research and practice holds a symbiotic relationship, a productive synergy and without one the other has no future. For example, John Mason (1998) described the need for research to speak directly to teacher's practice in ways which caused personal understandings so that their revised view of their past experiences sensitized them to possible incidents to notice in the future. Our personal experience in working with teachers has emphasized this symbiotic relationship. As we have worked alongside teachers, their practices have been influenced by the research and in turn the decisions they make have provided us as researchers with essential learning and shaped the outcomes of our research. It is the interplay of research and practice, which results in productive tension and from which new and powerful learning emerges for all members involved. From this we can deduce a different focus of debate; we suggest that rather than questioning whether research should inform practice we should be questioning whether such criteria (commonly seen when used to assess the outcomes of research) as relevance, validity, objectivity, originality, rigor, precision, predictability, reproducibility and relatedness should be applied to the results of research informing practice within complex school settings. Again, we bring you back to our interpretation of the verb "should" and apply it to mean "with the purpose of".

3. The connections between research and practice counter development and publication of "false assumptions" or "alternative truths"

An open and honest skepticism to many statements made which draw on evidence from both research and practice is a healthy way forward for mathematics education and research in mathematics education—particularly in this new world of "alternative truths". In our former lives as practitioners within the school setting and now as researchers, we are constantly confronted with what appear to be convincing facts. It is through integration of research with practice that you are able to drill through the surface and through inquiry develop possible explanations and solutions. For example, the first author's work of inquiring into equity issues for diverse learners, she has been confronted by those who use the results of research and the results of schools separately to develop "alternative truths" based on "false assumptions" to match a right wing agenda. For example, an "alternative truth" was built around one piece of research in which it was suggested that Māori had a "warrior gene". This was used to explain their underachievement and disengagement from education. In response, we were able to counter these claims and provide contrary evidence which was built within an active cycle of inquiring into the work of teaching. The strength of evidence depended upon the theoretical and empirical research grounded within practice.

4. Useful research is "good" research

The term "good" is admittedly subjective. Thus, we start here on our personal experiences. Our research has brought us close to schools where we worked intently with mathematics teachers on problems of actual practice. It is very challenging but it also brings great satisfaction when we see teachers finding our contributions helpful to their practice. In the eyes of these teachers, "good" research is done when 'theories of research' hit the road and deliver the goods – which is, visible improvement in students' learning. And, this kind of useful research can be done without compromising on the quality (another sense of "good") of research. An example of an emerging methodology that attends to both usefulness-to-practice and rigour-in-research is Design Research (e.g., Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Middleton, Gorard, Taylor, & Bannan-Ritland, 2006). We are not suggesting that all mathematics education research need to be directly and immediately involved with 'translation' into practice. Here, we return again to the point made earlier about our interpretation of "should" in the motion statement. While some research are perhaps more 'remote' from practice, our argument is that they should nevertheless have a view of practice in mind; this is so that their research results will then have greater potential to be tapped by other researchers whose work are closer to the particulars of practice. Seen in this way, all types of mathematics education research can be "useful" in the sense that the findings may potentially be harnessed for the purpose of informing practice.

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RESEARCH FORUMS

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PERSPECTIVES ON (FUTURE) TEACHERS' PROFESSIONAL COMPETENCIES

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Abstract *The research forum provides new insight on various perspectives on (future) teachers' professional competencies. Research on (future) teachers building upon the competence paradigm has become quite influential in the last few years, especially in the frame of large-scale (international) studies. The research forum will present various theoretical frameworks and constructs developed from three projects followed by four commentaries, which discuss the approaches described and enrich them by own frameworks. The research forum is positioned to promote the discussion on how Eastern and Western approaches can learn from each other.*

GOALS, KEY QUESTIONS, AND FOCUS OF THE RESEARCH FORUM

The research forum aims to share and discuss about various perspectives on (future) teachers' professional competencies. Specifically, it intends to discuss various constructs about teachers' professional competence and possible relationships between (subject-based) cognitive and (social-culturally) situated perspectives in examining and evaluating teachers' competencies. These discussions provide a platform for sharing and cross-examining the similarities and differences in various conceptualisations of teachers' professional expertise (important for future teachers) and perspectives developed and used in related examination and evaluation in different system and social-cultural contexts.

The following three research questions are addressed:

(1) What kind of constructs and conceptualisations about teachers' professional competence are proposed within mathematics education being often considered as important in the East and West?

- What is the nature of different constructs being considered important in the East and West?
- Are there possible relationships between different constructs?
- What kind of research does already exist and what kind of research is further needed?

(2) Which frameworks and instruments are adequate for the usage of cognitive or situated perspectives?

- What are characteristics of frameworks and instruments that are adequate for the usage of cognitive and situated perspectives on teachers' professionalism?

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- What strengths and limitations do these frameworks and instruments have?
- Can these complementary perspectives be integrated within one theoretical framework with joint evaluation instruments of teachers' professional competencies?

(3) What do we know about the value of developing and/or improving teachers' competencies that are conceptualized with different perspectives?

- What is the relationship of teachers' competencies, the quality of their instruction, and students' learning gains?
- How can we model and evaluate this important overall relationship?
- Which other (mediating) factors might influence this relationship?
- Which professional activities/programs may be in existence to develop and/or improve teachers' professional competencies that are conceptualized as expected?
- How far can different frameworks and instruments developed by already existing projects on teachers' competencies be transferred to other countries and cultures?

THEORETICAL BACKGROUND OF THE RESEARCH FORUM

Research on teacher education, teacher's professional development and the necessary prerequisites has become a prolific and productive field. Large-scale assessments such as the "Teacher Education and Development Study in Mathematics (TEDS-M)" (Tatto et al., 2012) have triggered a series of national and international follow-up studies examining the competencies necessary to teach different subjects on different schools levels (cf. Blömeke et al. 2015). Substantial progress has therefore been made in understanding that teacher competencies are personal traits (i.e., individual dispositions relatively stable across different classroom situations) but that they also include situational facets (Jenßen, Dunekacke, Eid, & Blömeke, 2015). Furthermore, they play out in social contexts which determine to some extent how competencies can be transformed into classroom performance. These developments are in line with trends in subject-related discussions like mathematics education, where in their survey on the state-of-the-art on teacher and teacher education Krainer and Llinares (2010) identified three trends in the literature about prospective teachers, practicing teachers and teacher educators, namely teacher educators' and researchers' increasing attention to the social dimension of teacher education, to teachers' reflections and to the general conditions of teacher education. The first two trends are based on the shift from a perspective on the education of individual future and practicing teachers towards emphasizing the social dimension in teacher education based on sociological and sociocultural theories. These developments are in line with the differentiation of two different paradigms on teachers' professional competencies, which can be characterized as cognitive versus situated approaches on professional competencies of teachers (Rowland & Ruthven 2011; for an extensive overview on these paradigmatic distinctions see Kaiser et al., 2016).

In detail the first two trends engage with a situated perspective on the professional activities of mathematics teachers and their competence structure and demonstrate the transfer from a cognitive perspective on mathematics teachers' professionalism to situated approaches. The cognitive perspective on the professionalism of teachers focusing on knowledge facets of teachers has been dominant in recent decades. Their characteristic is a strong focus on teachers' knowledge and the distinction of a limited number of components, of which teachers' knowledge consist, related to personal traits. These studies are mainly coming from mathematics education, for example by the already mentioned Teacher Education and Development Study in Mathematics. However, newer studies such as the Follow-up-Study of TEDS-M, the TEDS-FU-study, have shifted the focus of research to the inclusion of situated and social aspects of teaching and learning and the professional development of teachers taking the concept of teachers' noticing as the point of departure. These studies assume "the act of teaching being multi-dimensional in nature" (Depaepe et al., 2013, p. 22) referring not only to subject-based cognitive aspects, in contrast including pedagogical reflections on the teaching-and-learning situation as a whole. The context in which teaching and learning is enacted is in the foreground.

Integrating these different approaches Blömeke, Gustafsson and Shavelson (2015) presented a framework of teacher competencies that took this interaction of personal, situational and social characteristics into account. They showed that former conceptual dichotomies were misleading in that they ignored either the stable dispositional or the more variable situational competence facets. By systematically sketching conceptual controversies, competing definitions of competence were unpacked. The resulting framework revealed how the different approaches complement each other. Competence can since be viewed along a continuum from personal dispositions such as teachers' professional knowledge and beliefs which underlie situation-specific cognitive skills such as perception, interpretation, and decision-making, which in turn give rise to observed teacher performance in the classroom.

THE STRUCTURE OF THE RESEARCH FORUM

The Research Forum is to be organized with a format that integrates the use of multiple activities, including formal presentations, small group discussions, pre-prepared commentaries, and coordinated Q&A sections. In particular, this format is designed to take advantages of formal presentations, commentaries, and small group discussions in its two 1.5 hour sessions. The forum will start with formal presentations that aim to share research on various constructs of (future) teachers' professional competencies in selected projects/programs. These presentations will be followed by commentaries provided by two discussants, which also serve as a good start point for broader discussion for all participants. The participants will then be invited to join small group discussions to have better opportunity to ask questions and learn further about different perspectives as structured with questions proposed for this research forum. During the small group discussions, participants may also be invited to share what they know about (future) teacher competencies and/or related constructs in their own education systems. Both the commentaries and the small group discussions should provide the presenters a good opportunity to prepare a summary of information shared and further explanation as needed, which will be used to kick off the second 1.5 hour session for the whole forum. The presenters will then also present various frameworks and instruments developed and used in studying and evaluating (future) teacher competencies. These presentations will be followed by two more commentaries provided by another two discussants, and followed by more small group discussions. The session will then be ended with final Q&A between all the audience, discussants and presenters.

In the following sections, three different perspectives and research projects first present their theoretical framework and constructs that are developed and used in their studies as coming from different theoretical backgrounds. Collectively, these projects aim to promote the discussion of teachers' professional competencies by answering the questions posed above. Follow-up these three projects, four invited commentaries are presented from different own perspectives as contributed by scholars with different cultural backgrounds, which serve dual purposes as not only to comment on the perspectives on teachers' competencies described herein but also to enrich them with new perspectives.

DEVELOPING PRE-SERVICE TEACHERS' MATHEMATICS CONCEPTUAL KNOWLEDGE FOR TEACHING

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What should pre-service teachers know and be able to do to be ready for their professional career in mathematics teaching? This is not a trivial question, but it is a crucial one for all of those who are responsible for teachers' preparation in mathematics. It merits more research attention. In this paper, we propose mathematics conceptual knowledge for teaching (MCKT) as the core of preservice teachers' professional competency that can and should be developed in teacher preparation programs. Specifications of MCKT are discussed and examples in elementary school mathematics are provided to illustrate different components of MCKT.

MCKT and teachers' expertise in mathematics instruction

Existing research has generally documented the importance of knowledge in expertise acquisition and development in knowledge-rich and complex domains, including mathematics instruction (Li & Kaiser, 2011). The importance of knowledge in teachers' expertise also goes beyond a quantity measure to include knowledge structure with certain depth. Such a knowledge-based characterization of teachers' expertise is commonly used in large-scale international studies, e.g., the Teacher Education and Development Study in Mathematics (TEDS-M, see Blömeke, Hsieh, Kaiser, & Schmidt, 2014), which measures three types of knowledge: mathematics content knowledge (MCK), mathematics pedagogical content knowledge (PCK), and general pedagogical knowledge.

Knowing the existence of different types of knowledge is important in teacher preparation. However, if salient connections among different types of knowledge are left unspecified in their training, preservice teachers would be left to make such connections by themselves after learning separate pieces of knowledge. In fact, this is often the case with mathematical training, since mathematics content courses are typically offered by a mathematics department and pedagogy courses are delivered in an education department. These two departments may communicate little, if at all, about the content and instruction of these courses for the same pre-service teachers. Checking whether preservice teachers are ready or not for their professional career in teaching often results in course counting rather than examining what is offered in these courses and how the various topics in them can and should be connected. To be ready for their professional career in mathematics teaching, preservice teachers should be expected not just to state what (content) needs to be taught in classrooms, but to be able to help students understand what needs to be learned, which includes helping students make connections across different representations and different topics. Such features of teacher expertise require a package of integrated knowledge (e.g., Ma, 1999), rather than a collection of separate knowledge components.

To be able to help students learn mathematics with understanding, teachers need to have mathematics conceptual knowledge for teaching (MCKT, Li, 2010). By MCKT we mean the conceptual knowledge needed for understanding, explaining, as well as teaching specific mathematics content topics with connections. It can be specified as containing the following three knowledge components that can and should be offered in the same courses:

(1) Knowing and being able to explain the meaning of a specific content topic;

(2) Being able to connect and justify the main points of a content topic, and to place it in wider contexts;

(3) Knowing and being able to use various representations for teaching the content topic, and being able to teach the relations between them.

MCKT relates to the notions of MCK and PCK mentioned at the beginning (also Shulman, 1986), but emphasizes the connections between these two knowledge components and envisions combining them in the same course rather than separate courses. MCKT also relates to the notion of mathematics knowledge for teaching (MKT) that is developed and used by Ball and her colleagues (2008), but focuses on mathematics content and related pedagogical approaches for pre-service teachers' preparation. With MCKT, we emphasize the depth and systematic view of mathematics knowledge with associated pedagogy that can empower preservice teachers for further expertise development in the future. After preservice teachers finish program training, they will have many more opportunities for developing knowledge about students' learning of different mathematics content topics through working with and learning from their own students, but opportunities to develop MCKT will not appear automatically.

Preservice elementary teachers' development of MCKT

This on-going project at Texas A&M University presents our commitment of pursuing excellence in mathematics teacher preparation, and aims to provide research-based training in mathematics and pedagogy to pre-service elementary teachers. The goal of this project is to carefully think through the essential understandings teachers need to function well in the elementary mathematics classroom, and to present those ideas in a coherent 3-course sequence. The project will involve close collaboration between the Mathematics Department and the College of Education, in order to balance and integrate mathematical principles with teaching considerations.

The approach of integrating content and pedagogy as MCKT specified for key/critical content topics is used to develop the three-course sequence that allows preservice elementary teachers to learn mathematics that they will teach in elementary classrooms, accompanied with carefully constructed developmental topic sequences and study of teaching strategies, and consistent attention to problem solving.

Some specific examples of such mathematical content topics we aim to address can give a sharper picture of such knowledge. Two major goals we seek for our preservice teachers are: (a) comfort with numbers, and (b) comfort with word problems. Both of these goals involve multiple topics. We give one example for each.

<u>The base ten place value system</u>. We seek to give teachers a confident overview of the structure of our pervasive base ten system of arithmetic. The form in which we write numbers employs multiple conventions to encode substantial algebraic structure, which lies behind the power of the system. In previously published papers (Howe, 2015; Howe & Epp, 2008), five stages in the interpretation of place value notation are distinguished:

$$\begin{aligned}
&356\\
&= 300 \quad + 50 \quad + 6\\
&= 3\text{x}100 \quad + 5\text{x}10 \quad + 6\text{x}1\\
&= 3\text{x}(10\text{x}10) \quad + 5\text{x}10 \quad + 6\text{x}1\\
&= 3\text{x}10^2 \quad + 5\text{x}10^1 \quad + 6\text{x}10^0
\end{aligned}$$

The first equation shows that a base ten number is a sum of pieces of a special kind, as displayed in the second expression, which is often called expanded form. The expanded form is shown to students, perhaps in second grade, but it does not seem to receive much emphasis. In particular, there has not been a standard simple name for the pieces. Following advocacy by one of the proposers, S. Beckmann, in the 5th edition of her teacher preparation text (Beckmann, 2017), has adopted the term *place value pieces*. We can now follow that terminology in this research forum.

Careful review of the five stages of place value leads to the recognition that to fully understand place value notation involves knowledge acquired over the full 8 years of elementary school. It would take a remarkably coherent curriculum to ensure that students master this structure, and in fact, studies (Thanheiser, 2009) have revealed that only a small minority of teacher candidates at a prominent institution of teacher preparation even think in terms of the 3rd stage. We want preservice teachers in our re-designed courses to master all five stages. Knowing and understanding the five stages can have multiple benefits, including the capacity to understand the main algorithms for computing with base ten numbers, both in detail, and from a global viewpoint. For instance, we expect that preservice teachers will understand why subtraction is harder than addition, and will be able to help their students by examining key examples. We will help them see the features of multi-digit addition, and of multiplication of a multi-digit number by a one-digit number that lead to similar algorithm formats. To promote strong pedagogy, we will also offer preservice teachers a detailed teaching sequence for teaching addition and subtraction that will establish solid place value understanding, and encourage mental mathematics, on the way to mastering the standard US algorithm.

<u>A structural approach for teaching and learning addition/subtraction word problems.</u> Helping preservice teachers to gain an in-depth understanding of word problem structures will empower them to pose, analyse and solve word problems in their classrooms. A focus on word problem structures helps highlight the mathematical and semantic aspects of word problems, beginning with the Common Core State Standards (CCSS) taxonomy of one-step word problems and proceeding to multi-step problems. Each of these mathematical and semantic connections involves important ideas that can help deepen understanding of addition and subtraction word problem structure in a systematic way. The goal is to make word problem structure usable as a strategy to develop student problem analysis and solution skills other than being overwhelmed with various word problems or simply relying on memorization, when dealing with addition and subtraction problems.

As this project has been on-going at Texas A&M University, we will also share sample work from preservice teachers' learning of the above concepts to illustrate their development of MCKT.

CONCEPTUALIZING AND MEASURING THE MATHEMATICAL KNOWLEDGE NEEDED FOR TEACHING

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Following a practice-based approach, over the past years scholars have proposed the Mathematical Knowledge for Teaching *(MKT) construct to capture the knowledge entailed in different tasks of teaching mathematics. Several efforts have also been undertaken to validate this construct and explore its links to teaching quality and student learning. In this paper, we briefly consider the work pursued over the past years and outline open issues for future work.*

Motivation and development of the Mathematical knowledge for teaching (MKT) construct

Mathematical Knowledge for Teaching (MKT) arose as the answer to an apparently simple question: What mathematical knowledge is entailed in the work of teaching mathematics? Building on the work of Shulman (1986), the *Mathematics Teaching and Learning to Teach* (MTLT) group hypothesized that careful analysis of mathematics teaching practice could illuminate the mathematical work involved in helping children learn mathematics. In particular, rather than deciding the mathematics teachers *should* know, the MTLT scholars set out to understand what mathematics teachers use in teaching and what they use it for. To do this, they set about studying the work of teaching mathematics. First, looking at a range of records of practice, they identified tasks that occur frequently in mathematics teaching (e.g., choosing examples for particular purposes, asking productive mathematical questions, and interpreting an unexpected student response). For these tasks, they identified the mathematical knowledge, skills, and dispositions needed to complete them. Though the analysis looked at records of practice from particular teachers, the purpose of the study was to understand teach*ing*, not teach*ers*. This is an important feature of both the methodological and theoretical approach: though "knowledge" is often conceptualized as related to individuals or even groups of people, the "knowledge" in MKT is conceptualized as a function of the work of mathematics teaching.

Mathematical knowledge for teaching

Through analysis of teaching practice, the group developed a conceptualization of both the subject matter knowledge and the related pedagogical content knowledge — a blend of content knowledge and knowledge of students, curriculum, or other aspects of pedagogy (Ball, Thames, & Phelps, 2008). The subject matter knowledge is made up of three sub-domains: *Common Content Knowledge* (CCK), *Specialized Content Knowledge* (SCK), and *Horizon Content Knowledge* (HCK). CCK is the mathematical knowledge that is common to educated adults and many professionals who use mathematics (e.g., knowing how to do the mathematics in the student curriculum).

HCK is knowledge of the mathematical horizon; that is, sufficient familiarity with the more distant mathematical terrain to make productive connections and avoid seeding misconceptions (e.g., noticing connections to the ideas of infinitesimals in a child's question). Finally, and arguably most significantly, SCK is the special mathematical knowledge needed in teaching but not needed by other professions (e.g., knowing how to fluently and precisely connect an algorithm to a representation while speaking and gesturing to a group). Pedagogical content knowledge is also made up of three sub-domains: *Knowledge of Content and Students* (KCS), *Knowledge of Content and Teaching* (KCT), and *Knowledge of Content and Curriculum* (KCC). KCS includes knowledge of how students interact with mathematics; KCT involves things like useful representations for illuminating a specific concept. Finally, KCC comprises, among other things, knowledge of how different mathematical ideas develop across the curriculum.

Exploring the construct and its relationship to teaching and learning

Much of the exploration of MKT has centred on developing and using multiple-choice items (Hill, Schilling, & Ball, 2004). These items, which initially focused on elementary-level number and operations, functions, and algebra, provided important opportunities to explore the construct's structure as well as its relationship to teaching and learning. They were used in cognitive interviews with pre-service and in-service teachers, mathematicians, and lay people to confirm that there is particular mathematical knowledge that is used in teaching (cf. Charalambous, 2016; Hill, Dean, & Goffney, 2007). Further, these items provided an opportunity to measure teachers' MKT at large scale, and thus through statistical analysis to confirm that MKT is multidimensional (e.g., Schilling, 2007) and to link this knowledge to student learning and instructional quality. With respect to the later, studies show a positive link between teachers' MKT and teaching quality, which is stronger for teachers at the two ends of the MKT spectrum (e.g., Hill et al., 2008; Hill, Umland, Litke, & Kapitula, 2012). Likewise, studies have shown a positive link between teachers' MKT and student learning (Hill, Rowan, & Ball, 2005; Rockoff, Jacob, Kane, & Staiger, 2011), however the size of the effects and the ways in which the relationship can be mediated by classroom contextual factors remain open (Kersting et al., 2012; Ottmar, Rimm-Kaufman, Larsen, & Berry, 2016). In addition, these multiple-choice items have also been adapted for use internationally (e.g., see *ZDM* Mathematics Education special issue 44(3)). As this work progresses, the need to expand from using only multiple-choice items to more performative formats has become more evident (e.g., Charalambous, 2008; Fauskanger, 2015; Kim, 2016).

Situating MKT: Limitations and open issues

Despite the promise of MKT, important aspects of teaching, especially parts of the work that involve real time interaction or enactment, remain buried and inadequately addressed by the static ways the knowledge has been measured. The items developed to measure MKT, though they include a "pedagogical context" and engage test takers in doing constrained versions of pedagogical work, are nevertheless limited to sedentary, individual, and, mental dimensions of the work. Though the validity of these items has been extensively investigated (Hill et al., 2007), the multiple-choice items are not equivalent to MKT itself and the time is ripe to also attend to this limitation. One potential complement to this work is refocusing attention on conceptualizing the work of teaching mathematics and specifying its knowledge entailments through assessments that better capture the performative work of teaching (Selling, Garcia, & Ball, 2016). This shift suggests a series of open issues as well as a concrete path to better connect MKT and curriculum for teacher education: identifying and defining how the elements of this knowledge can be chunked and sequenced to support teacher learning during initial training and ongoing professional development; investigating how this knowledge relates to mathematical fluency and issues of equity and diversity in teaching (see Hoover, Mosvold, Ball, & Lai, 2016); better understanding the mechanisms through which teacher knowledge can inform teaching quality and the factors that mediate this relationship (cf. Charalambous & Pitta-Pantazi, 2015); and exploring collective forms of MKT and how these might be developed in communities of practice. This range of future work on MKT captures the tension entailed in situating the knowledge in practice, without losing the possibilities for scale and generalization provided by more traditionally cognitive approaches to its operationalization.

PROFESSIONAL COMPETENCIES OF (FUTURE) TEACHERS – THE TEDS-M STUDIES

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Recent research on the professional competencies of mathematics teachers, which has been carried out during the last decade, is characterized by different theoretical approaches on the conceptualization and evaluation of teachers' professional competencies. Building on the international IEA Teacher Education and Development Study in Mathematics (TEDS-M) and its various follow-up studies, the development from cognitive to situated approaches on professional competencies of teachers are described. In TEDS-Follow-up the cognitive oriented framework of TEDS-M has been enriched by a situated orientation including the novice-expert framework and the noticing concept as theoretical approaches on the analyses of classroom situations.

Background of the Studies

In their extensive survey on the discussion of teacher's professional competencies Blömeke and Delaney (2012) point out that before the international comparative study Teacher Education and Development Study: Learning to Teach Mathematics

(TEDS-M)" was carried out in 2008 no systematic evidence on the state of future teachers' professional competencies existed.

The focus of TEDS-M is on an international comparison of the professional knowledge of prospective teachers for primary and secondary level. The TEDS-M study departs from the theoretical orientation of competency related to competency-oriented approaches in international comparative studies on students' achievements such as PISA, likewise other large-scale studies such as the study Mathematical Knowledge for Teaching (MKT) or the Cognitive Activation in the Classroom Project (COACTIV). The core of TEDS-M departs like many other studies from the description of pedagogical content knowledge (PCK) of teachers based on Shulman's (1987) seminal work in which PCK is defined as "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special from of professional understanding" (p. 8). In their extensive survey on the current discussion around PCK, Depaepe and others (2013) point out the special importance of this concept used by many studies.

However, despite the general agreement on PCK as connection between content and pedagogy and its dependence on the particular subject matter, no general consensus exists in empirical research on the facets of this important concept. Further, Depaepe et al. (2013) argue that there is an important group of empirical studies that do not define any component of PCK, although PCK was the central topic of this group of studies. Their study revealed consequences of the ongoing debate on the two principally different views on the conceptualisation of PCK, namely "whether mathematical knowledge in teaching is located 'in the head' of the individual teacher or is somehow a social asset, meaningful only in the context of its applications" (Rowland & Ruthven, 2011, p. 3).

Adherents of the cognitive perspective define according to Depaepe et al. (2013)
"– in line with Shulman – a limited number of components to be part of PCK and distinguish PCK from other categories of teachers' knowledge base, such as content knowledge and general pedagogical knowledge. By contrast, proponents of a situated perspective on PCK as knowing-to-act within a particular classroom context, typically acknowledge that the act of teaching is multi-dimensional in nature and that teachers' choices simultaneously reflect mathematical and pedagogical deliberations" (p. 22).

These paradigmatic differences in the conceptualisations of PCK have, according to Depaepe et al. (2013), an impact on the way in which PCK is empirically investigated, which is reflected by TEDS-M and its various follow-up-studies.

Constructs used in TEDS-M

TEDS-M examined the professional competencies of future mathematics teachers and the influence of institutional and national conditions of mathematics teacher education, According to Weinert (2001), professional competencies can be divided up into cognitive facets (in our context, teachers' professional knowledge) and affective-motivational facets (in our context, e.g., professional beliefs). The professional knowledge of teachers can again be divided into several facets. Referring

to Shulman (1986), the following facets were distinguished in TEDS-M: mathematics content knowledge (MCK), mathematics pedagogical content knowledge (MPCK), including curricular knowledge, and general pedagogical knowledge (GPK).
TEDS-M examined also the professional beliefs held by the future teachers, due to the fact that beliefs are crucial for the perception of classroom situations and for decisions how to act, as Schoenfeld (2011) pointed out. Based on Richardson (1996), beliefs can be defined as stable, psychologically held propositions of the world around us, which are accepted to be true. In TEDS-M, several belief facets were distinguished, in particular epistemological beliefs about the nature of mathematics and beliefs about the teaching and learning of mathematics (Thompson, 1992). In addition, beliefs and affective traits such as motivation, and also metacognitive abilities such as self-regulation, are indispensable parts of the professional competencies of teachers. These facets of professional knowledge are further differentiated: mathematical content knowledge covers the main mathematical areas relevant for future teachers, mathematics pedagogical content knowledge covers curricular knowledge, knowledge of lesson planning and interactive knowledge applied to teaching situations.

Constructs used in the TEDS Follow-up studies

The research done in TEDS-M was an important step forward in studying the structure, level and development of mathematics teachers' competencies from a cognitive perspective, however, it obvious that professional knowledge or skills are not directly transformed into performance, but mediated by cognitive skills more closely related to activities of teachers. Situated approaches to research on teachers and teacher education and the general discussion about how to assess professional competencies of teachers in a performance-oriented way (see Blömeke, Gustafsson and Shavelson 2015) have guided the development of several follow-up studies of TEDS-M (called TEDS-FU, TEDS-Instruct, TEDS-Validate, TEDS-East-West), which enriched the theoretical framework of TEDS-M by situated components. Blömeke, Gustafsson and Shavelson (2015) described teachers' competencies as continuum from disposition to performance integrating situated competence facets as indispensable part of competency (see Fig. 1).

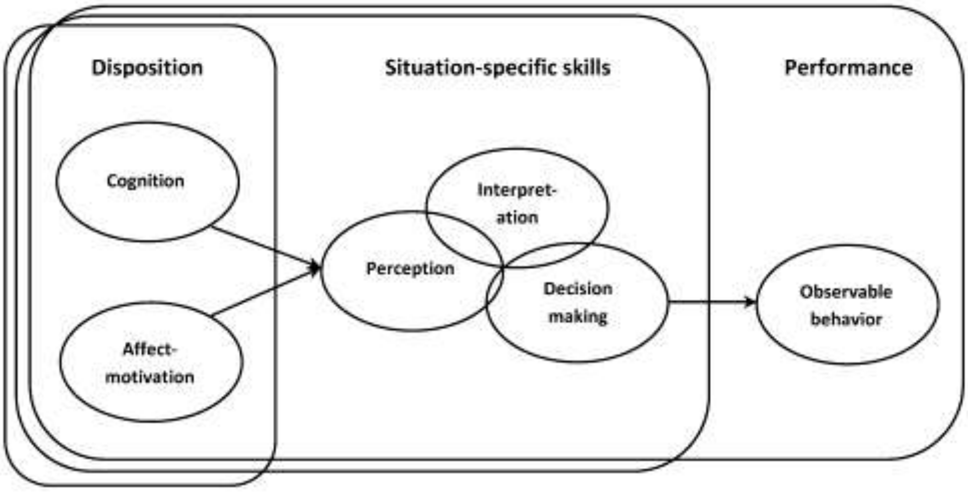

Fig. 1
Modeling competence as a continuum (Blömeke, Gustafsson and Shavelson 2015a, p. 7)

The follow-up studies of TEDS-M examined the question how professional competencies could be analysed in more performance-oriented ways, building on the theoretical framework and the instruments of TEDS-M, but enriching it two new concepts, the expert-novice perspective (for an overview see Chi, 2011) and the noticing approach (Van Es and Sherin, 2002).

Enriching the cognitive perspective of TED-M, which concentrated on the three facets of professional knowledge - MCK, MPCK and GPK - three situated facets of teacher competencies were distinguished in the follow-up-studies of TEDS-M, integrating the noticing approach into a broader notion of competence (see Fig. 2):

- Perceiving particular events in an instructional setting which corresponds to the notion of the noticing discussion as attending to particular events in an instructional setting
- Interpreting the perceived activities in an instructional setting which corresponds to making sense of events in an instructional setting used by the noticing discussion
- Decision-making, either as anticipating a response to students' activities or as proposing alternative instructional strategies, which corresponds to acting, formulated in the noticing debate.

Although this approach comes closer to classroom performance of teachers further extensions of the framework and the instruments used were needed in order to show a full picture of teachers and their influence on students' learning. The enriched framework of TEDS-FU, including the new kind of performance-oriented competency facet of noticing and the newly developed instruments using video-vignettes, was used in further follow-up-studies of TEDS-M in order to explore the relationships between teachers' competency and students' learning gains. For example, the study TEDS-Instruct hypothesizes that cognitive skills mediate the effects of teacher knowledge on instructional quality and that instructional quality in turn serves as central mediator variable for the relation between teachers' competencies and progress in student achievement (see Fig. 2).

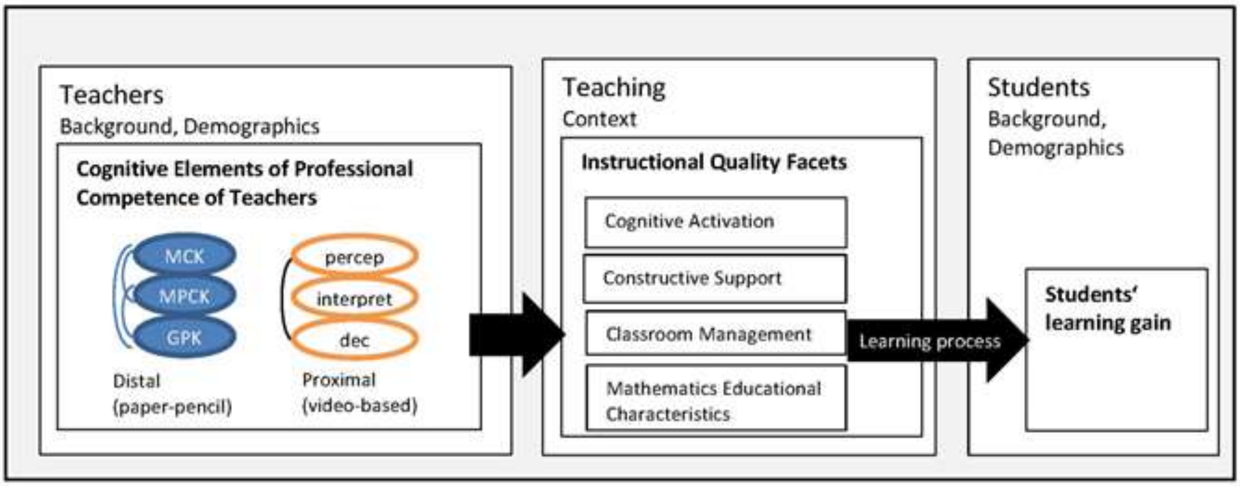

Fig. 2.
Research and Impact model of TEDS-Instruct and TEDS-Validate (Kaiser et al. 2017, p. 184)

Until now, the question is not answered, whether these studies described above, being developed in a Western paradigm can be transferred to East Asian educational systems. The TEDS-East-West-study investigates this question and explores differences and similarities of mathematics teachers' professional competence and the relation to students' learning outcomes in Eastern and Western cultures with Germany being the Western protagonist and China serving as protagonist for Eastern cultures. Mathematics teachers' professional competencies are evaluated using adapted instruments from the various TEDS-M follow-up studies through the means of video-based testing. Students' mathematics achievements and their longitudinal achievement progress will be evaluated using regular achievement tests at national level in China and at Federal state level in Germany. Similarities and differences of Chinese and German mathematics teachers' professional competence and its connections to students' mathematics achievements have been analysed and their results will be presented at the conference. Social and cultural influences in these two countries are further discussed, which will make more meaningful and deeper contribution to the understanding of mathematics teachers' professional competence and its connections to students' mathematics achievements.

The TEDS-East-West-study mainly focuses on junior secondary school students (mainly Grade 7 to Grade 8 students), an important part of compulsory education in Germany and China. In China, the project is carried out in Chongqing. Chongqing is the biggest metropolitan city in Western China. In Germany, the study is carried out in the Federal state of Hamburg. Hamburg is the second biggest city in Germany, a metropolitan area, which makes it quite comparable to Chongqing. Schools in both cities cover the entire spectrum of students' performance from very high performing students to students with extreme performance deficits.

The study involves a sample of 150 Hamburg teachers teaching about 3500 students. In Chongqing, 200 teachers and their students (more than 8000) participated. Instruments used in both countries to evaluate teachers' professional competence include: 1) a shortened version of the tests used in TEDS-M, i.e. tests on MCK, PCK and GPK; and 2) in order to get insight into the competencies needed by teachers while acting in classroom, video-based testing from the Follow-up-study TEDS-FU is used. These tests were translated into Chinese and the video-vignettes are re-done by Chinese teachers and students to meet Chinese mathematics teaching situation. Students' regular achievement tests within one year period of time are collected to investigate the connections between teachers' professional competence and the progress of their students' mathematics achievement.

The collected data are analysed in both a qualitative and quantitative way in order to explore the connection between teachers' competence structure and their students' mathematics achievements and its progress.

STUDYING TEACHERS' PROFESSIONAL COMPETENCIES

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The three groups participating in this research forum address, each in its own way, the study of teachers' professional competencies. All three groups acknowledge the complexity of this matter. But they do it in different ways.

Charalambos Y. Charalambous and Lindsey Mann examine the work pursued over the past years in relation to the *Mathematical Knowledge for Teaching* (MKT) construct. This construct, proposed by the *Mathematics Teaching and Learning to Teach* (MTLT) group at the University of Michigan, was developed through analysis which started from practice, from the work of teaching mathematics. The conceptual framework developed provides a heuristic for considering different types of knowledge needed for teaching mathematics: common content knowledge, specialized content knowledge, knowledge of content and teaching, knowledge of content and students, knowledge of content and curriculum, and horizon content knowledge. Charalambos Y. Charalambous and Lindsey Mann acknowledge the usefulness of the MKT construct in showing that there is distinct mathematical knowledge that is used in teaching. Yet, they claim that important aspects of the work of teaching, mainly those that involve real time interaction or enactment, are inadequately addressed by the static ways the MKT has been measured so far. They therefore suggest that future work on MKT explores ways to better situate knowledge in practice.

Yeping Li and Roger Howe also address the issue of teachers' professional competencies by centring on the mathematics knowledge needed for teaching. They propose a new construct – *Mathematics Conceptual Knowledge for Teaching* (MCKT) – defining it as "the conceptual knowledge needed for understanding, explaining, as well as teaching specific mathematics content topics with connections." They claim that this aspect is at the core of prospective teachers' professional competency that should be developed in teacher education programs. Yet, it is insufficiently addressed in current literature. Yeping Li and Roger Howe specify three MCKT components, related to: explaining the meaning of a specific content topic, connecting and justifying the main points of a content topic and placing it in wider contexts, and using various representations for teaching the content topic and teaching the relations between them. Gabriele Kaiser, Armin Jentsch, Dennis Meyer, and Xinrong Yang review the approach developed by the international comparative study *Teacher Education and Development Study in Mathematics* (TEDS-M) and its follow-up studies. Similar to the two other contributions to this Research Forum, they start from a cognitive oriented approach that focus on teacher knowledge, specifying two components of mathematics related teacher knowledge: mathematics content knowledge (MCK), and mathematics pedagogical content knowledge (MPCK). A situated orientation was added later, aiming to enable analysis of classroom situations, similar to Charalambos Y. Charalambous and Lindsey Mann's suggestion regarding future work on MKT. Furthermore, in addition to knowledge that is situated in practice, Gabriele Kaiser and her colleagues mention affective and motivational facets as important aspects of professional competencies of teachers: "In addition, beliefs and affective traits such as motivation… are indispensable parts of the professional competencies of teachers."

As is illustrated in the three contributions to this research forum, discussions about professional competencies of teachers tend to start with a focus on knowledge: What should teachers know? The three contributions identified important areas for professional knowledge base. Yet, the approaches described raise intriguing questions: How are the components of knowledge specified in each approach related to the components of the other two approaches? For example, how are the MCKT components in Yeping Li and Roger Howe's contribution related to those of MKT in Charalambos Y. Charalambous and Lindsey Mann's contribution? and to MCK and MPCK in Gabriele Kaiser and her colleagues' contribution?

In addition to identifying areas for professional knowledge base, the contributions to this Research Forum acknowledge the need to pay explicit attention to the work in which teachers engage, suggesting to situate knowledge in practice. This shift from a sole focus on knowledge to incorporating also a focus on practice is important, because teaching is something one does, not just know. However, the suggestion to situate knowledge in practice does not treat knowledge and practices as equally important in the study of teachers' professional competencies, attributing more prominence to knowledge. To address this shortcoming, I propose to focus on the integration of knowledge, skills, dispositions and practices situated in the practice of mathematics teaching; integration that I term *knowtice* to signify that it is related to the elements that create it (*know*ledge and prac*tice*), but that the product is a new object (Even, 2008).

Finally, important aspects that are often overlooked in discussions about professional competencies of teachers are the affective and conative (motivational) aspects, which are mentioned in Gabriele Kaiser and her colleagues' contribution. The importance of these aspects has been recently discussed in the literature in relation to student engagement in mathematics (Goldin, 2017). Future work on the professional competencies of teachers could profit from incorporating greater emphasis on theorizing and studying teachers' affect and motivation.

PROFESSIONAL COMPETENCES OF TEACHERS IN MULTICULTURAL AND MULTILINGUAL ENVIRONMENTS

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We now live in a fast changing world, which means that also teachers' professional competences must reflect the changes. New conditions put new demands on teachers' professional competence. In this paper, I focus on two environments that ask for supplementation of teachers' professional competences by new items, namely teaching in culturally heterogeneous classrooms (in which majority and minority pupils are co-educated) and teaching through Content and Language Integrated Learning (CLIL). Both these environments are similar in some aspects but differ considerably in others. The knowledge and skills needed for work in these environments are additional to those presented by other colleagues in this Research forum. By no means do they replace the competences needed for work in any classroom. In the paper I also focus on what teachers perceive as important for their work and what they miss in their education as well as in materials and tools offered for work in these environments.

In my presentation of additional teachers' competences I come out of the following perspectives on the teaching-learning processes. Vygotsky (1986) views the teaching-learning process as sociocultural development, and describes the teacher's support to the learners' zone of proximal development. This is executed through a number of professional skills, e.g. the skill to motivate, to establish and maintain contact, to control the learning process, to stimulate and activate etc. (Švec, 1998). To Vygotsky (1986), thinking involves the use of words and notions, speech is a tool to develop thinking. Learning mathematics, therefore also includes "appropriating ways of speaking mathematically, that is, learning the language of mathematicians" (Zazkis, 2000).

Cultural heterogeneity in schools is one of the most significant changes in many school systems. Therefore, it is one of the ultimate tasks (not only) in mathematics education to pay attention to teaching in multicultural contexts (Ulovec et al., 2013). Differences in cultures and languages make the maths teaching-learning process harder than it is in culturally homogeneous classrooms. It is generally accepted (e.g. Barton, Barwell and Setati, 2007; Bishop, 1988; César and Favilli, 2005) that mathematics teachers feel the necessity for training and materials which reflect the needs of their classes in terms of linguistic and cultural differences.

In teacher education, increasing attention is paid to additional teachers' professional competences needed for their successful work in multicultural classes. The research in

(Moraová, Novotná, Favilli, 2015) presents the results of a questionnaire survey focusing on teachers' views of what they need for working in multicultural classes. It is a partial output from the LLP Socrates Comenius 2.1 project M3EaL – Multiculturalism, Migration, Mathematics and Education. The questionnaire survey among pre-service and in-service teachers in six countries (Czech Republic, France, Italy, Norway, Austria and Greece) with wide variety of teaching experience, showed that both pre- and in-service teachers feel a lack of opportunity to attend seminars focusing in teaching culturally heterogeneous classrooms. They also feel communication among teachers of different subjects is not sufficient and the school authorities do not give sufficient support (Moraová, Novotná, Favilli, 2015).

Teaching in culturally heterogeneous classrooms is in a number of aspects similar to teaching through CLIL. CLIL (Content and Language Integrated Learning) refers to teaching of non-linguistic subjects through an additional language (Using Languages, 2000). CLIL is perceived as dual-focused education and educationalists believe that it contributes to the enhancement of thinking processes. My contribution tries to look into qualitative aspects of teacher education for CLIL. CLIL calls for an interactive teaching style (Pavezi et al., 2001). This contribution focuses on the question: What attitudes, what professional skills are to be acquired for the teaching of mathematics through the medium of a non-mother tongue?

When trying to find answer to this question I come out of demands on a teacher teaching in a foreign language. Novotná and Hofmannová (2011) state that CLIL teachers should have a good command of the target language and resort to the learners' mother tongue with care. For learners, code switching is a natural communication strategy, and teachers should allow it, particularly in the first stages of CLIL. The teacher's main concern should be to scaffold pupils on their way towards achieving mathematical competences. The teacher's task is to enable the students develop their individually different process of knowledge building and meaning construction as well as positive attitudes (DeCorte, 2000). The teacher qualified for CLIL may be more successful in overcoming the learning difficulties that have their origin either in the student's personality or the educational environment. These barriers are to be found in all types of education. Some of these barriers are more significant in CLIL than in other lessons, other are less significant. The increase can be expected mainly in those learners who are afraid of unusual, alternative learning methods and techniques. The decrease of barriers can be expected mainly in the area of anxiety. The CLIL teacher is lead towards sensitivity to the learner's personality. Through the use of interactive, non-traditional methods they may succeed in altering the student's prior negative learning experience.

PROFESSIONAL COMPETENCIES OF MATHEMATICS TEACHERS: WHAT MAKES THEM CULTURAL PERSPECTIVES?

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The theoretical construct of the three papers takes up a significant position amongst the discussions to investigate professional competencies of mathematics teachers brought on by Shulman (1986). Discussions on mathematics teachers' professional competencies can be characterized as cognitive approach versus situated approach, and integration between these two (Blömeke, Gustafsson, & Shavelson, 2015). These are TEDS-M, a cognitive approach towards professional competencies, researches relating to noticing including Van Es & Sherin (2002), situated approaches towards professional competencies, and follow-up studies of TEDS-M such as TEDS-Fu, an integration of cognitive and situated approach towards professional competencies. I would like to express my respect to the three teams. Here, I would like to discuss the position of the framework introduced by the presenters in the area of research on mathematics teachers' competencies and I would like to suggest a few ideas from the socio-cultural context.

Li & Howe's MCKT reaches professional competencies from the cognitive perspective. They argued that there exists practical teacher knowledge that connects between MCK and MPCK in MKT introduced by Ball and her colleagues (2008), and showed concrete examples, such as concepts of place value pieces, of preservice teachers' learning that emphasize MCKT. Meanwhile, discussion on the connection and gap between MCK and MPCK can also be found in school-related content knowledge (SRCK). SRCK, neither belonging to CK nor PCK, refers to the knowledge where a didactic transition happened from academic mathematics to school mathematics similar to MCKT. However, taking a closer look at the three components suggested as the MCKT framework by Li & Howe, one can notice that the practical context is more emphasized in MCKT in the context of preservice teacher education than in SRCK. This emphasis makes us revisit the fundamental question – can the practical-context MCKT be placed inside Ball's framework that structured the cognitive aspect of professional competencies? Consideration on this matter will further firm Li & Howe's research.

As in Li & Howe's work, firstly introduced by Thompson & Thompson (1996) and conceptualized as components by Ball, Thames, & Phelps (2008), MKT has inspired many researchers and is the most commonly used framework for mathematics teacher knowledge research. Charalambous & Mann systematically organized the relationship between development and teaching and learning, and suggested future issues for situating-MKT aiming to overcome the limits of MKT's emphasis on individual's cognitive aspect. However, consideration is needed on how defining situating-MKT as teacher knowledge revealed in classroom situations will influence Ball group's framework on teacher knowledge as declaratory. Ball group did not clarify their position on the epistemological perspective relating to MKT, but takes a stand from the epistemology of possession as well as from the epistemology of practice towards knowledge. Consideration on this context will help them secure a position in Ball group's framework.

The well-known international comparative study, Teacher Education and Development Study: Learning to Teach Mathematics (TEDS-M), which examined on the cognitive perspective of teacher knowledge also has the basis on Ball group's framework. Kaiser et al. introduced follow-up studies of TEDS-M by integrating cognitive aspect and situational aspect of mathematics teachers' expertise, grounding on the idea that it is more likely that the consideration of situational aspects in conceptualization of the expertise will guarantee successful performance. It is particularly interesting that their Research and Impact model, which concerns the relation between teacher competencies and student achievement, regards teaching as an activity with multiple factors of teaching context, rather than an individual activity. This perspective might be criticized for offering unnecessarily wide spectrum of analysis which even includes the point where the teaching is irreducible to individual effort. Yet, it is a meaningful attempt considering a number of complex exterior factors to the teacher expertise. In addition, I wonder where the connection would be, between Kaiser et al. and current 'teacher noticing' research, which extend the 'decision making' to teacher response. Also I expect this connection to provide a significant foundation for contextual research on the teacher expertise.

Chevallard (1985), in the Anthropological Theory of Didactics of mathematics, have attended to the relation between the teaching and learning of mathematics and the institution, which enclose the didactical system consisting of savoir, students and teachers, and the environment of the system, defined as noosphére. This offers insights when we explore the socio-cultural influence on professional competencies of mathematics teachers. In Asian culture for example, the extent of autonomy allowed by the institution often creates enormous differences in the professional competencies of mathematics teachers. This implies that consideration of the institution could be a crucial point identifying the unspoken difference between the professional competencies in Western and Eastern culture. Leong, Kaur, & Kwon (2017) found that the professional developments, in Asian countries, are school-based, collaborative and pragmatic; and noticed that the institution creates the socio-cultural difference. Kaiser & Li (2011) also have mentioned that Eastern perspective on teacher expertise is holistic which aims more systematic change compared to Western perspective. These discussions endorse more active consideration of the institution, which include the socio-cultural context, in measuring teacher professional competencies.

There is a four-character idiom in Korea, 교학상장[教學相長, kyo-hak-sang-jang], meaning the teacher and the students grow simultaneously through teaching and learning. Many Asian countries including Korea regard the expertise of the teacher has long been considered not as static, but rather as dynamic or evolutionary. Even though the three presented discussions produce the best result from each stance, they do not capture the particular socio-cultural context of Eastern culture, where the expertise of mathematics teacher evolves through teaching. I hereby end my discussion posing the following questions. These topics are expected to progress as this field of study evolves in the future. I hope these questions could contribute to such progress.

- What do the frameworks for the mathematics teacher expertise overlook, in terms of the socio-cultural context?
- What kind of form would it be to be the framework for the mathematics teacher expertise, which encloses the difference between East and West?
- How could we measure the expertise of mathematics teacher which encloses the difference between East and West?

WHEN AND HOW TEACHERS USE MATHEMATICAL KNOWLEDGE FOR TEACHING

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The importance of mathematical knowledge for teaching has been a topic of much discussion for several decades. However, how to help future teachers develop effective teaching skills remains largely unclear. I propose a framework for examining when and how teachers use mathematical knowledge for teaching based on Japanese "Lesson Study."

Study on mathematical knowledge for teaching

Research on what mathematical knowledge for teaching teachers need has been thoroughly conducted through analysis of mathematics teaching practices (Ball, Thames, & Phelps, 2008; Hill, Rowan, & Ball, 2005). However, there are far fewer research projects which examine when and how teachers use the knowledge that they have, and what support is necessary to give teachers in order for them to learn how to use that knowledge effectively in the classroom. This paper will discuss when and how teachers can use mathematical knowledge for teaching effectively in the classroom and how to support the development of that knowledge based on Japanese Lesson Study.

Learning from Japanese lesson study

For over a hundred years, Japanese teachers and educators have been using a professional development program called "Lesson Study" (Lewis & Tsuchida, 1998; Stigler & Hiebert, 1999; Yoshida, 1999). Lesson Study helps both prospective and practicing teachers develop expertise in mathematical knowledge for teaching through the refinement of their lesson planning, teaching, and reflection skills (Takahashi, 2011). Prospective teachers typically experience a whole Lesson Study cycle for the first time when they are student teaching. However, even after completing student teaching, they are rarely able to teach mathematics effectively (Takahashi, 2011).

Therefore, Lesson Study is implemented in schools to give emerging teachers the continued support they need. The question for us is: how does Lesson Study facilitate the development of prospective and practicing teachers' knowledge for teaching mathematics? Recent studies on Lesson Study (Fujii, 2016; Lee & Takahashi, 2011; Lewis, Perry, & Friedkin, 2011; Takahashi, 2011; Takahashi, 2014) may help us understand not only what specific knowledge educators should focus on developing, but also when and how teachers should use that knowledge to teach mathematics.

When and how teachers use and develop mathematical knowledge for teaching through lesson study

Lesson Study asks teachers to improve their knowledge for teaching through observation and adaptation. When observing a school-wide form of Lesson Study in Japan, it was clear that teachers use their mathematical knowledge not only when they design unit and lesson plans, but also when teaching those lessons and reflecting upon them afterwards (Takahashi, 2014). These three steps: lesson design, implementation, and reflection, may provide an entry point for future study of mathematical knowledge for teaching.

When teachers in Japan write a lesson plan, they must state what mathematics the students are meant to learn and why it is important for them to learn it at that moment. The plan must also include anticipated student responses, including typical misunderstandings and informal approaches which must be re-directed towards formal mathematical solutions. To successfully address these issues, teachers need knowledge of the learning trajectory, effective approaches for introducing each topic, and understanding of common misconceptions and challenges.

While teaching their lesson plan, teachers must be able to monitor student reactions and questions, and adjust the flow of their lesson accordingly. They must make quick decisions to perform such adjustments. The skill for making these quick decisions comes from their knowledge of student learning.

Teachers must also reflect afterwards on how the lesson impacted student learning. If the lesson did not go as planned, teachers have to identify the possible reasons why and how to address these issues going forward. This process both relies on and refines their mathematical knowledge for teaching mathematics.

"Collaborative Lesson Research" is a form of school-based Lesson Study designed for implementation outside of Japan (Takahashi & McDougal, 2016). In Collaborative Lesson Research, teachers are supported by two knowledgeable others. A knowledgeable other could be, for example, a math specialist, a university professor, or an education researcher. They help teachers learn how to use and update their knowledge (Figure 1). It is hoped that in countries without a long tradition of conducting Lesson Study, these knowledgeable others can support teachers through the Lesson Study process to ensure that each step of the cycle enhances teachers' mathematical knowledge for teaching.

Conclusion

Based on the ideas of Lesson Study, it may be possible to examine how teachers use their knowledge when designing lessons, making decisions while teaching those lessons, and reflecting afterwards on the impact of their lessons. These three steps in the Lesson Study cycle can be a framework for future study on how teachers use their knowledge for teaching and as well as future study on the role of knowledgeable others as support for teachers developing teaching expertise.

OVERALL SUMMARY AND CONCLUSIONS

This research forum provides another great opportunity for the mathematics education community to review and discuss the important topic of what (future) mathematics teachers should know and be able to do, after the publication of our co-edited book on expertise in mathematics instruction in 2011 (Li & Kaiser, 2011) and a previous research forum on expertise in mathematics instruction organized in 2012 (Li & Kaiser, 2012). The presented research from selected projects and thoughtful commentaries from our discussants clearly demonstrate an evolved understanding about the nature of expertise (or termed as competence) that teachers need to have in and for carrying out such a complex task of mathematics instruction in classrooms. In particular, multiple theoretical perspectives have been proposed and used over the past several years for understanding, studying and assessing teachers' competence.

In 2011, we identified and highlighted three issues on this topic that are important for the international community of mathematics education researchers at that time (Li & Kaiser, 2011): (1) the issue of identifying and selecting teachers with expertise, (2) the issue of specifying and analyzing aspects of teachers' expertise in mathematics instruction, and (3) the issue of understanding expertise in mathematics instruction that is valued in different cultures (pp. 6-8). It is clear from this research forum that there has been specific progress in addressing the second issue, and much remains to be explored further on the first and third issues. We hope this research forum can serve as another starting point for much more research and discussion internationally on those issues related to this topic in the future.

We would also like to highlight one important difference between this research forum and our previous efforts on this topic. That is, this research forum focuses on (future) teachers' professional competencies, while our previous efforts mainly focused on practicing teachers. It is important for us to realize that (future) teachers' professional competencies mainly refer to what these (future) teachers can and should learn in order to be ready for beginning their teaching career. The categorization of such competencies for (future) teachers with an ending time point carries a fundamental difference from the case for practicing teachers when their competence development is a life-long learning process. The very limited time that (future) teachers can have through their pre-service program study suggests that we need to be as specific as possible in characterizing (future) teachers' professional competencies, which can potentially be adapted and optimized in teacher education programs. In fact, this is not an easy task at all, as it is clearly evidenced from a recent MME workshop on *Mathematical Preparation for Elementary Teachers* held at Texas A&M University (organized by Deborah Ball, Roger Howe, James Lewis, Yeping Li, and James Madden, see http://mme.tamu.edu). We hope that this research forum will help bring much needed attention and more research efforts and collaborations on this topic for (future) teachers.

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RESEARCHING AND USING LEARNING PROGRESSIONS (TRAJECTORIES) IN MATHEMATICS EDUCATION

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The relationship between research and practice has long been an area of interest for researchers, policy makers and practitioners alike. One obvious arena where mathematics education research can make a practical contribution is the design and implementation of school mathematics curricula. This requires research that is fine-grained and focused on individual student learning trajectories as well as large-scale research that explores how student populations engage with the big ideas of mathematics. This research forum brings together work from the United States and Australia on the development and use of evidence-based learning progressions/trajectories in mathematics. In particular, the forum will consider their basis in theory, their focus and scale, and the methods used to identify and validate learning progressions.

INTRODUCTION

Learning progressions, or learning trajectories as they are more commonly referred to in mathematics education, are not new. For instance, it could be said that scope and sequence charts and year level outcome statements represent particular forms of learning progressions/trajectories. While there has been considerable research in particular domains over many years that has contributed to our understanding of how knowledge is constructed and informed practice in those domains, it is only relatively recently that learning progressions/trajectories per se have become the focus of systematic research efforts (e.g., Clements & Sarama, 2004; Confrey, 2008; Daro, Mosher & Corcoran, 2011; Siemon, Izard, Breed & Virgona, 2006).

Ever since Simon's (1995) introduction of the notion of *Hypothetical Learning Trajectories* (HLT), there has been debate about the meaning and use of learning progressions/trajectories in mathematics education (e.g., see the special edition of *Mathematics Teaching and Learning*, *6*(2) in 2004). A common element in the different interpretations and use of the terms is the notion that learning takes place over time and that teaching involves recognising where learners are in their learning journey and providing challenging but achievable learning experiences that support learners progress to the next step in their particular journey. Another common characteristic is that, to varying extents and in different ways, learning progressions/trajectories are based on hypothesised pathways derived from experience and a synthesis of relevant literature, the design and trial of learning activities aimed at progressing learning

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within the hypothesised framework, evaluation methods to assess where learners are in their journey and the efficacy of both the framework and the instructional materials and approaches used.

The focus of a learning progression/trajectory may relate to a particular instructional episode (e.g., Simon, 1995; Tzur, 2007), a specific aspect of the curriculum (e.g., Clements, Wilson & Sarama, 2004) or a much larger field of mathematics learning that encompasses different but related aspects of mathematics (e.g., Confrey & Maloney, 2014; Siemon, Izard, Breed & Virgona, 2006). Their development and use may vary from a reflective practitioner working to understand and support his/her student's attainment of a specific learning goal over a relatively short time frame through to an extensive network of teachers and researchers working collaboratively to understand how students in general might be supported to progress their learning in a particular domain or field of mathematics over an extended period of time.

Concern with the numbers of students 'falling behind' and the considerable range of achievement in any one year level (e.g., OECD, 2014) have prompted educational systems and researchers in a small number of countries to work more closely together to identify evidence-based learning progressions/trajectories that might be used to inform teaching and map student's progress over time. While these vary considerably in their focus and scale, there is much that we can learn from each other to further the work in this field and to build new knowledge that is likely to make a difference to student learning (e.g., see Daro, Mosher & Corcoran, 2011, p. 13).

The research forum is likely to be of substantial interest to a PME audience as it is concerned with the application and scaling up of research to practice to make a difference in mathematics classrooms. The forum provides an opportunity for a reality check. For example, does this work translate to other settings? Is it a valid use of research conducted for other purposes in other contexts and do the results and affordances outweigh the limitations?

The contributors have been brought together on the basis of their recognised contributions to this field, to consider what is meant by learning progressions/trajectories and explore a range of issues associated with their development and use including theoretical framing, research approaches, implementation and evaluation. It is difficult to succinctly capture the body of work represented here in a way that is both fair and accurate. So for the purposes of building a coherent picture and facilitating discussion, contributors were invited to discuss their work (past, present and future) under three headings: research approaches, starting points and developments, and practical applications and/or implications. These are presented in turn followed by key questions raised by our critical friend, Anne Watson that raise issues concerning the development and use of LT/Ps.

Key questions to be explored in the Research Forum:

What characterises a learning progression/trajectory? What purposes do they/can they serve? How are they different to or compatible with theories of conceptual development?

What is situated and what is universal about learning progressions/trajectories?

What research designs, techniques and evidence are used to develop, evaluate and refine such progressions?

How are learning progressions/trajectories used in practice? How are they related to task sequences used in countries like China and Japan? What impact do they/can they have on teacher knowledge and confidence?

RESEARCH APPROACHES

A variety of research approaches have been used to conceptualise and construct the learning progressions featured here will be discussed in turn.

Tzur

For the past 25 years, my research program consisted of four interrelated components: articulating hypothetical learning trajectories in the areas of multiplicative and fractional reasoning (Tzur 2004, Tzur 2014); explaining mathematics learning as a cognitive change process (Tzur & Simon 2004, Tzur 2011); linking this model to teaching that can promote progression along those trajectories (e.g., Tzur 2008); and identifying shifts in mathematics teacher practices (Jin & Tzur 2011). This four-fold program is rooted in the premise that mathematics teaching is a goal-directed activity, aimed at promoting students' learning of the intended mathematics. This requires an understanding of how learning of particular mathematics may progress and how teaching may foster such progression.

To strengthen this twofold understanding, my work on articulating HLTs led me to distinguish two kinds of studies on learning trajectories: *Marker Studies*, which foreground *conceptual landmarks* that constitute a learning trajectory; and *Transition Studies*, which foreground the conceptual transformation involved in progressing from less to more advanced landmarks. Because a primary goal of my work on HLT is to contribute to the knowledge base about understanding (learning, teaching), I have conducted mainly transition studies.

Recently I have complemented teaching experiments with two other methods: corroborating empirically grounded models through quantitative methods and elaborating on findings (markers and/or transitions) from previous teaching experiments (Tzur 2014).

Clements and Sarama

Our 30-year work with learning trajectories (LTs) began with the creation and testing of LTs, but has come to span the full range of research and development in education, contending now that LTs have ramifications for all aspects of curriculum (e.g., ideal, expected, available, adopted, implemented, achieved, or tested, Clements, 2007). This requires a wider range of methods (that we will discuss in subsequent sections), with the focus here being only on the methods we use for the creation, refinement, and validation of LTs.

Initially we considered a learning trajectory as a device whose purpose is to support the development of a curriculum or a curriculum component. Building on Simon (1995), but emphasizing a cognitive science perspective and a base of empirical research, we conceptualized "learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (Clements & Sarama, 2004, p. 83). In other words, each learning trajectory has three parts: (a) a goal, (b) a developmental progression, and (c) instructional activities. To attain a certain mathematical competence in a topic or domain (the goal), students learn each successive level (the developmental progression), aided by tasks (instructional activities) and pedagogical moves designed to help students build the mental actions-on-objects that enable thinking at each higher level. We address the determination of the goal in the following section; here we address the other two components.

While others have based their LTs on historical development of mathematics and observations of children's informal solution strategies (Gravemeijer, 1999), anticipatory thought experiments (that often focus on instructional sequences), or emergent mathematical practices of student groups (Cobb & McClain, 2002 in which *instructional design* serves as a primary setting for development), our approach is grounded more in cognitive science. We begin by learning from others, conducting comprehensive *research reviews* (e.g., Barrett, Clements, Sarama, & Cullen, in press; Clements, Wilson, & Sarama, 2004). If details are lacking, we use *grounded theory methods* and *clinical interviews* (Clements, 2007; Ginsburg, 1997) to examine students' knowledge and ways of thinking in the content domain, including conceptions, strategies, intuitive ideas, and informal strategies used to solve problems. The researchers set up a situation or task to elicit pertinent concepts and processes. Once a (static) model has been partially developed, it is tested and extended with *constructivist teaching experiments*, which present limited tasks and adult interaction to individual children with the goal of building models of children's thinking and learning. Once several iterations of such work reveal no substantive variations, it is accepted as a working model, then subjected to validation and/or refinement through *hypo-deductive applications of qualitative methods such as teaching experiments* and quantitative methods such as *correlational analyses* between level scores (Clements, Wilson, & Sarama, 2004) and *Rasch modeling* (Barrett et al., in press; Szilagyi, Sarama, & Clements, 2013).

Next, sets of activities are taken from successful interventions in the literature or created (or tasks are adapted from previous work) by the developers. In both cases, the key is ensuring that the activities are theoretically valid in engendering or activating the actions-on-objects that mirror the hypothesized mathematical activity of students in the target level (that is, level $n + 1$ for students at level n). *Design experiments* and *microgenetic studies* (Siegler & Crowley, 1991) are employed, using a mix of *model (or hypothesis) testing* and *model generation* to understand the meaning that students give to the objects and actions embodied in these activities and to document signs of learning.

Confrey and Maloney

Two major components of our research around learning trajectories over the last twenty years are: developing and validating the Equipartitioning learning trajectory (1995-2011), described here and Confrey's current research on the LT-based *Math-Mapper 6-8* for middle grades, described in later sections.

We have used a variety of methods in developing LTs. In our original work on the Equipartitioning LT, we began with Confrey's splitting conjecture (1988; Confrey & Scarano, 1995), namely, that an independent cognitive construct for splitting differs from that of counting. After an extensive literature review on evidence for the independence of this construct, we chose the term "equipartitioning" to clarify that this involved not simply making parts, but making equal-sized parts. Further, we identified two relatively distinct literatures, one for sharing *groups* fairly and the other for sharing *a whole* fairly. We integrated these notions of sharing into a single learning trajectory. The new trajectory consists of 16 levels, covering three cases of equipartitioning: Sharing a collection (*na*) among *n* people, sharing a whole among *n* people, and finally sharing multiple wholes that did not divide evenly (one with more wholes than sharers and one with fewer wholes than shares, which could be addressed by students in either order, depending on their prior knowledge from instruction and experience) (see Confrey et al., 2014b). To validate the learning trajectory, we undertook two primary research initiatives.

1) Items corresponding to the 16 levels were written and administered to students in grades 1-5. Student item responses were coded, then analysed using item response theory. In general, the items for the LT lower levels were less difficult than the items for the upper levels.

2) In a design study, curriculum units developed to support the LT were used, along with a digital tool we had developed, to collect student data from automated diagnostic tasks that corresponded to the different levels (Confrey & Maloney, 2015). We worked with 12 students, grades 2-4, from high poverty settings, for two summer weeks. We articulated our initial conjectures and conducted a daily debriefing session to revise plans based on each day's observations (Confrey & Lachance, 2000). We periodically conducted one-to-one interviews with students to understand how their thinking was developing. At the end of the study, we reviewed the data from the diagnostic

assessments, video, and notes, and drew conclusions about how the LT levels, the curriculum, and items might be modified in light of the results. In general, we also described the trajectory in terms of a) the development of the cases, b) the way in which students generated strategies at early stages, c) whether the students developed a sense of properties at the second levels and d) how they showed signs of reasoning in a connected fashion at the higher levels.

Siemon and Horne

In 1999, RMIT was commissioned to identify and document what was working in numeracy teaching in Years 5 to 9 where numeracy was seen to involve:

- core mathematical knowledge (in this case, number sense, measurement and data sense and spatial sense as elaborated in the (Australian) *National Numeracy Benchmarks for Years* 5 *and 7*;
- the capacity to critically apply what is known in a particular context to achieve a desired purpose; and the
- actual processes and strategies needed to communicate what was done and why (Siemon & Virgona, 2002)

A quasi-experimental design involving a representative sample of 47 Victorian schools was used. In the first phase, data were collected from just under 7000 Year 5 to 9 students using rich assessment tasks and scoring rubrics based on the dimensions of numeracy described above (Siemon & Stevens, 2001). These data were analysed using item response theory, which confirmed that the tasks were appropriate for the cohort tested and that it was possible to measure a complex construct such as numeracy using assessment tasks that incorporate performance measures of content knowledge and process (general thinking skills and strategies) across a range of topic areas using teachers-as-assessors.

In subsequent work on learning progressions HLTs were developed from the research literature related to multiplicative thinking (e.g., see Siemon & Breed, 2006) and later for algebraic reasoning, geometrical reasoning, and reasoning in statistics and probability. The HLT, hereinafter referred to as a *draft learning progression* (DLP), is used to inform the selection and/or development of rich tasks designed to assess not only the core knowledge associated with the areas of mathematics under consideration but also, students' ability to apply that knowledge in unfamiliar situations and explain or justify their reasoning. The tasks and scoring rubrics are then trialled with a relatively large number of students in the target population and the data analysed using item response theory (e.g., Bond & Fox, 2015). This allows both students' performances and item difficulties to be measured using the same log-odds unit (logit), and placed on an interval scale. Items (parts of tasks) that do not fit the model are either rejected or refined and re-trialled. The scale is then interrogated by at least three experts in the field to identify and describe patterns in student performances. This results in the identification of a number of levels or Zones within the progression for which teaching advice is prepared in the form of a learning assessment framework

(LAF). The framework is then trialled in schools and evaluated using parallel assessment forms and analysis methods.

THE DEVELOPMENT AND REFINEMENT OF LTS

Tzur

Piaget's notion of assimilation, a core constructivist principle, is the starting point for any HLT study I conduct. Assimilation posits that any new learning can only be as good as the goal-directed activities afforded, or constrained, by learners' *available* (assimilatory) schemes. To teach and study how learners transform (reorganize) assimilatory schemes into new ones, we thus first engage in articulating fine details of the three parts that might constitute *their* schemes (von Glasersfeld, 1995). The first part is the mental template ('situation') by which learners may make sense of a given 'input' (e.g., mathematical task), which triggers the goal(s) *they* would set to accomplish. This goal calls up the second part of the scheme—a mental activity sequence that *the learners* have been using to reliably accomplish the goal(s). As the activity ensues, the learners' goal(s) regulate *their noticing* of effects that either match or do not match the scheme's third part—a result *they expected* to ensue from the activity. Detailing all three parts of learners' assimilatory schemes is vital, because conceptual change is postulated to commence, and thus possibly be fostered, through *their* noticing of actual effects that differ from the expected ones.

To articulate learners' assimilatory schemes that would serve as a starting point for studying HLT, as well as the hypothetical process of change those schemes may undergo, we combine two main sources: task-based interviews with participating learners and scrutiny of previous, relevant research. Using these two sources reflexively, our goal is to detail the precise boundaries between schemes we infer students already have constructed and schemes into which the available schemes could possibly be transformed (yet to be constructed). The notion of precise boundaries includes close attention to one of two stages at which we infer learners' schemes to have been established (Tzur & Simon, 2004). An anticipatory stage of a scheme is inferred if the learner can use it spontaneously and independently when solving relevant tasks. A participatory stage is inferred if the learner can use it albeit not yet spontaneously and independently (e.g., by somehow being incited for a novel use of an activity).

Our hypotheses of how the intended conceptual transformation (a micro-level learning trajectory) may be fostered differ based on the stage of learners' assimilatory schemes. If we infer those to be at the anticipatory stage, we identify a relevant participatory stage of a new scheme to serve as the goal for their next learning. Accordingly, we detail ways to proactively promote Reflection Type-I, which is postulated to promote a transition to the participatory stage of the next scheme (Tzur, 2011). In this type of reflection, learners compare between effects they expected and actual effects they noticed to ensue from their activity. Such a comparison provides the mental mechanism for creating a novel, provisional relationship between the goal-directed activity and its actual effects that can be formed solely on the basis of what has been previously available to the learner.

If, however, we infer learners' schemes to be at the participatory stage, we set the goal for their next learning to be the anticipatory stage of that scheme. Accordingly, we detail ways to proactively promote Reflection Type-II, which is postulated to promote that transition. In this second type of reflection, learners compare across mentally recorded instances in which an activity did or did not ensue particular effects. Such a comparison provides the mental mechanism for abstracting the regularity (invariant) in and reasoning for why relationship between the goal-directed activity and its effects must necessarily be what they are in given, as well as non-routine problem situations.

To illustrate how the above constructs are being used as a starting point, I provide an example from Tzur and Lambert (2011) that led to identifying 4 sub-stages in first graders' shift from counting-all to counting-on, that is, from having no concept of number as a composite unit to the early onset of that concept. For that study, we sampled all students who spontaneously and independently used the counting-all strategy for adding two previously counted collections (e.g., 7 cubes and 4 cubes). Our inference of the scheme that underlies such a strategy included:

Situation + Goal	Activity Sequence	Result
Having separately counted all 1s in each of two given collections of tangible items to find their numerosities, set out to find the numerosity of the combined collection	Starting over from 1, count every tangible item in the combined collection by creating 1-to-1 correspondence between those items and number words in the conventional sequence	Reaching the final item to be counted and stating the number word that corresponded to this item to indicate the numerosity

Table 1. Scheme underlying strategy

For a child at the anticipatory stage of this (counting-all) scheme, we set out the goal to begin constructing a participatory stage of a scheme that would give rise to the concept of number as composite unit, as indicated by the development of a counting-on strategy. To this end, I created a play activity, called How Far From the Start (HFFS) in which two players step on along large tiles from a marked start, taking turns to roll a die and walk from either the start or the last players position the number of steps implied by the tiles and recording the numeral on a note placed on their endpoint. Then, both learners figure out how far the end tile of the second player is from the start (e.g., 11).

This activity assumes learners will begin finding the total number of steps by assimilating the task into their available scheme, that is, by using counting-all. While they play, the researcher-teacher will begin probing for their reflection on the effect they can notice, namely, always calling out the number on the first note (e.g., 7) when counting to find the combined total. For example, we may ask the players to stop their count while stepping on that tile and tell us if they are surprised to have said this number word (7) or if they could consider starting at a spot and a number word other than 1. We may also shift from real tiles to a drawn out board game marked with Start and End tiles. This allows us, later, to cover some of the tiles on first path to further orient the learners' reflection onto the possibility to use the first end tile/numeral as a start. Letting players switch roles and repeating these experiences, enabled them to create a provisional link between their counting-all activity up to the first stopping point and the effect it ensued—starting with the number-after (8) when resuming their count. This new, provisional linkage opens the way not only to starting the count from that stopping point (7) but also to keeping track of the count of 1s in the second walk. That is, a new stage of *anticipating where to start* is formed at a participatory stage, as the learners replace 1 as the start for finding the combined total by *their noticed effect* of starting from the first end point.

Conceptual reorganization (accommodation) is another core constructivist principle that, coupled with a corresponding, student-adaptive pedagogy (Tzur, 2013), underlies my development of HLT. Above, I provided a brief description of the two types of reflection and two stages (participatory, anticipatory) that enable reorganization of assimilatory schemes into new ones. By student-adaptive pedagogy, I refer to the cyclic, 7-step process postulated (Tzur, 2008) as an elaboration of Simon's (1995) seminal introduction of the HLT notion. In a nutshell, these 7 steps include (with pointers to the example of fostering transition from counting-all to counting-on as explained above):

(1) Specifying students' current conceptions;

(2) Specifying the intended mathematics;

(3) Identifying a mental activity sequence through which the conceptual change may evolve;

(4) Selecting and/or adapting tasks to promote the intended learning;

(5) Engaging learners in the task while letting them use previously constructed schemes first;

(6) Monitoring learners' progress;

(7) Introducing follow-up questions and probes to foster Reflection Type-I and/or Reflection Type-II.

When conducting teaching experiments, we develop HLT through two types of analysis—ongoing and retrospective (Tzur et al, 2000). Ongoing analysis focuses on inferring each individual learner's conceptual progress during the recent teaching episode(s). Inferences are made about *changes in the learner's anticipation*, *explanation of effects they notice to ensue from their activity*, and the extent to which learners can *use the newly abstracted anticipation spontaneously*. Those tentative inferences constitute Step 1 of the 7-step cycle, which inform Steps 2, 3, and 4 in the design of teaching for the next episode.

After completing all teaching episodes, further development of HLT occurs though retrospective analysis, which focuses on distinguishing and explaining plausible ways in which learners' mental systems may give rise to their observable behaviours (actions and language). Drawing on the principles of grounded theory methodology (Glaser & Strauss, 1967), retrospective analysis identifies commonalities across different learners' solutions while striving to specify schemes that, we infer, could serve as conceptual underpinnings of those solutions. Those schemes, for which we detail both the participatory and anticipatory stages, become the markers of HLT. Then, going back to the data, we search for ways in which transition from one scheme (marker) to the participatory and then anticipatory stage of the next one might have took place, along with instructional moves that seemed essential in fostering that learning.

Refinement of HLT is accomplished by further organization of findings from my teams' work and from other research teams' studies of similar progressions. (e.g., Clements & Sarama, 2004; Maloney, Confrey, & Nguyen, 2014).While staying close to the data from which the HLT were created, this organization involves sequencing of schemes and transitions between them along a developmental continuum. In collaboration with researchers from other teams, a developmental continuum is linked with more general models, such as the model of units coordination levels (e.g., Hackenberg, 2007), which transcends additive, multiplicative, and fractional reasoning. Further refinement of the HLT is then attained through using the continuum of markers and transitions to teach and study different student populations, such as students identified as having learning disabilities in mathematics (e.g., Hord et al, 2016), teachers (Tzur, Hodkowski, & Uribe, 2016), or across social-cultural settings (e.g., Huang, Miller, & Tzur, 2015). Of course, working with different populations may confirm the HLT we have been developing and/or present challenges that require further refinement.

In the past 25 years, I have worked with several teams that produced two HLT—one focusing on multiplicative schemes (Tzur et al., 2013) and the other on fractional schemes (Tzur, 2014). The markers that constitute each of these are summarised below. Details of transitions from one scheme to the next and the tasks used to accomplish this can be found in previous publications.

The HLT for **multiplicative reasoning** includes 6 Schemes: (1) Multiplicative double counting (**mDC**); (2) Same-Unit Coordination (**SUC**); (3) Unit Differentiation and Selection (**UDS**); (4) Mixed-Unit Coordination (**MUC**); (5) Quotitive Division (**QD**); and (6) Partitive Division (**PD**). It should be noted that distinguishing UDS was not intended or hypothesized before the teaching experiment, but rather compelled by children who indicated explicit inability to make the conceptual leap from SUC to MUC.

The HLT for **fractional reasoning** includes 9 schemes (the letter 'S' in each acronym stands for 'Scheme'): (1) Equi-Partitioning (EPS); (2) Partitive Fraction (PFS); (3) Splitting; (4) Iterative Fraction (IFS); (5) Reversible Fraction (RFS); (6) Recursive

Partitioning (RPS); (7) Unit Fraction Composition (UFCS); (8) Distributive Partitioning (DPS); and (9) any Fraction Composition (FCS).

Clements and Sarama

A complete learning trajectory includes an explication of the mental constructions (actions-on-objects) and patterns of thinking that constitute children's thinking at each level of a developmental progression, how they are incorporated in each subsequent level, and tasks aligned to each level (that promote movement to the succeeding level). The learning trajectories construct differs from instructional design based on task analysis because it is based not on a reduction of the skills of experts but on models of children's learning, expects unique constructions and input from children, involves self-reflexive constructivism, and involves continuous, detailed, and simultaneous analyses of goals, children's thinking and learning, and instructional tasks and strategies. Such explication allows the researcher to test the theory by testing the curriculum (Clements & Battista, 2000), usually with design experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003).

When we began, we accepted that the goal of an LT should be determined by standards (ideal or expected curriculum) created by dialectical process among many legitimate stakeholders (e.g., CCSSO/NGA, 2010; NCTM, 2006). When more detail was needed, we used *reviews* of the research literature to identify objectives that contribute to the mathematical development of students, build from the students' past and present experiences, and are generative in students' development of future understanding. We now also believe that LTs should play a more active role in determining, as well as incorporating, goals.

Starting points for LTs differ with different goals. The importance of geometric measurement was well established. However, there was less extant justification for the domain of composing geometric forms. We determined this domain to be significant in that the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis (e.g., Mulligan & Mitchelmore, 2013; Park, Chae, & Boyd, 2008; Reynolds & Wheatley, 1996; Steffe & Cobb, 1988).

The shape composition learning trajectory had its genesis in observations made of children using *Shapes* software to compose shapes. Sarama observed that several children followed a similar progression in choosing and combining shapes to make another shape (Sarama, Clements, & Vukelic, 1996). Sarama re-viewed the behaviors all kindergarten children exhibited and found that children moved from placing shapes separately to considering shapes in combination; from manipulation- and perception-bound strategies to the formation of mental images; from trial and error to intentional and deliberate action and eventually to the prediction of succeeding placements of shapes; and from consideration of visual "wholes" to a consideration of side length, and, eventually, angles. We combined these observations with related observations from other researchers (e.g., Mansfield & Scott, 1990) and some elements of psychological research (e.g., Vurpillot, 1976) to refine this developmental progression.

Tasks were designed to elicit each of these hypothesized levels. We conducted clinical interviews using these tasks, validating that the actions-on-objects posited to underlie solutions could be observed. We used quantitative methods, confirming that they formed a reliable and valid sequence (Clements, Wilson, & Sarama, 2004). At that point, we confirmed a developmental progression in which children move levels of thinking—from lack of competence in composing geometric shapes, they gain abilities to combine shapes—initially through trial and error and gradually by attributes—into pictures, and finally synthesize combinations of shapes into new shapes, that is, composite shapes.

Instructional tasks in which children worked with shapes and composite shapes as objects were designed. We wanted them to create, duplicate, position (with geometric motions), combine, and break apart both individual shapes (units) and composite shapes (units). We designed physical puzzles and software environments that required and supported use of those actions-on-objects. Simultaneously, we documented what elements of the teaching and learning environment, such as specific scaffolding, contributed to student learning—planned a priori or occurring spontaneously. Thus, designs are not determined fully by reasoning. Intuition and the art of teaching play critical roles.

Work with the measurement LT differed in several ways. The larger literature allowed us to use a research synthesis to form the initial LT (Sarama & Clements, 2002). The presence of assessment tasks, empirical results and theory allowed us to validate the first LTs with Item Response Theory, creating an equal-interval scale of scores for both the difficulty of items and the ability of the persons assessed. To measure measurement competence, we sequenced the items, strictly maintaining the order within each measurement domain but intermingling items across domains according to the available developmental evidence, including age specifications from the literature and difficulty indices from our pilot testing. Thus, we posited that items were organized according to increasing order of difficulty across domains, but our theoretical claims that this sequencing represented increasingly sophisticated levels of mathematical thinking were made only for items within a given domain. We submitted the results of administering this revised instrument to the Rasch model, validating the developmental progressions for length, area, and volume in multiple studies (Barrett et al., in press; Szilagyi, Sarama, & Clements, 2013). We similarly used and validated instructional sequences, many again from the extant literature.

We believe that full validation of an LT requires validation of the instructional tasks *and* their implementation in real classrooms.

Confrey and Maloney

Previous efforts. Our original work on equipartitioning led us to make the knowledge base on learning trajectories more accessible to greater numbers of teachers. Doing so required us to explore the use of new forms of visual representations for the LTs. Our first version was a "hexagon map" (www.turnonccmath.net) that used the Common Core Standards themselves as a framework for 18 LTs for grades K-8. The research team unpacked the content of each LT into an explanation of the LT and related research (Confrey & Maloney 2014). Ultimately, using the standards as the backbone of the LTs was dissatisfying, due to at least two limitations: 1) it tied us to the standards constraining divergence from them, and 2) for parsimony, each standard was embedded in only one LT, because we used each hexagon only once.

New LTs and learning map. Working to improve the visualization for greater usefulness to teachers and students simultaneously, the new work has been to develop a "learning map" for grades 6-8 (the content as framed generally in the Common Core Standards). It is called a "learning map" because it is built on a fundamental re-articulation of underlying learning trajectories, specifying how students' ideas become increasingly sophisticated as they engage with increasingly complex tasks during instruction. The DLS tool "Math-Mapper 6-8" (MM6-8), comprises 1) the learning map, 2) a diagnostic assessment and reporting system that corresponds directly to the learning trajectories, 3) a means to access curricular resources via the web and a curated library of links, 4), a Sequencing tool and calendar to organize all the foregoing components across the school year, and 5) an analytics system for interpreting various levels of use of the tool by students and teachers.

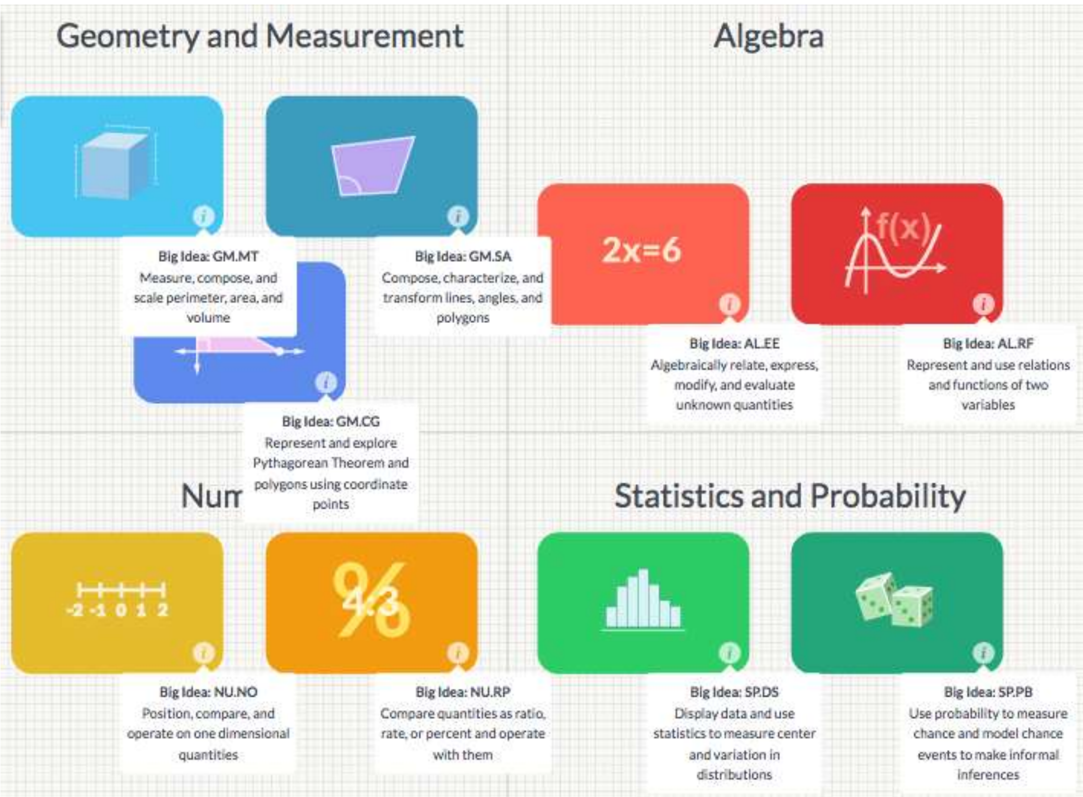

Fig. 1. Math-Mapper 6-8 learning map (fields and big ideas only)

The learning map is hierarchically organized, with four fields of mathematics incorporating nine big ideas (Figure 1). Each big idea comprises 2-5 *relational* *learning clusters* or *RLCs* (24 in all) of related constructs (64 in all). Each construct is associated with a LT and is also associated with relevant CCSS-M standards. The new learning map was developed to be a foundational organizer for the diagnostic assessment and reporting system. The new LTs are more specifically descriptive of student behaviours than those in the hexagon map.

Developing LTs across all four fields of mathematics has been informative. First of all, the hierarchy sets up three levels of trajectories. Each construct is made up of an LT. Then closely-related constructs are formed into clusters, and each cluster's shape establishes a progression of constructs that itself proceeds from less to greater sophistication with varying structures (e.g. there may be two constructs at the same level that can be taught in either order or taught in tandem). Finally the clusters within each big idea themselves are formed into another progression of sophistication of reasoning. We regard the overall hierarchical structure of the map to describe an evolution of the idea of an LT—showing how the mathematical landscape of middle grades can be conceptualized with LT structure underlying it at multiple levels of scale.

In our extensive work with LTs, we have learned a great deal about how they can be structured. While acknowledging the importance of teachers' own negotiating the process of developing (hypothetical) LTs in instruction (Simon 1995), many researchers (e.g. Battista, 2011; Sarama & Clements, 2009; Barrett, et al., 2012; Van den Heuvel-Panhuizen & Buys, 2008) have set about to document likely student behaviors, utterances, and beliefs in order to guide curricular development and aid teachers in leveraging student thinking. This work involves identifying target understandings and likely starting points, and delineating observed likely intermediate events of significance for the respective paths. LTs do not delineate stages as in a Piagetian stage theory (Lehrer & Schauble, 2015; Clements & Sarama, 2014). Instead, they describe meaningful probabilistic states that students are likely to encounter as they work to understand an idea. LTs are not recipes or rules for instruction, but guides, resources, and indicators that can help teachers build on student thinking in moving students toward more sophisticated understandings. These student behaviours, utterances, and beliefs resemble examples of "genetic epistemology," (Piaget, 1970) episodes with epistemological content drawn from the perspective of the learner and his/her experiences, and which change over time as the results of encountering a series of carefully designed tasks or scaffolded discussions. They also are evidence of the emergent behaviours tied to local instructional theories discussed by Gravemeijer and Cobb (2006).

We have identified several types of epistemological objects that arise repeatedly in middle grades LTs in the Math-Mapper 6-8 learning map (building on earlier recognitions of epistemological objects in student learning research). The first is a naïve or partial conception. An example from equipartitioning is that all equal parts of a whole are congruent. This serves a worthwhile purpose for beginners, and speaks to students' experience with cut pizza slices, the construct is later constraining, when students need to discern a variety of shapes of one half of a given whole. A second epistemological object is limited representations, for example, an ordered list of values of data placed into a primitive dot plot that lacks spacing for missing values (known as a case plot). A third type of object that serves an intermediate learning goal is a strategy that may be limited in its efficiency, for instance, forms of skip counting used in repeated addition versions of early multiplication. Other types of objects used to build LTs are cases, as described by variation theory (Marton, 2015) which often are useful in movement up an LT. Typically at higher levels of an LT, one witnesses emergence of properties that then guide the student in how to operate on particular examples, and generalizations that describe how to put strategies and cases together into a structure with varying degrees of justification and proof.

Elaboration, Items, and Assessments. An *LT elaboration* is a design and development tool that is central for developing the LTs and for ensuring coherence of the learning map with the diagnostic assessments. These "living" documents serve to record and support the evolution of the LT. The LT elaborations specify the wording of each LT level, any (partial) conceptions or misconceptions associated with any specific level, and delineation of cases associated with levels (which typically includes the kinds of numbers or values that are particularly germane to illustrating students' reasoning and behaviours, and which are used in the assessment items.

The assessment items are all newly designed items developed by the research team to focus on conceptual aspects of the constructs, to support deep student reasoning and flexibility, not just skill development. The elaboration documents are used iteratively as a basis for development of the LT level-specific assessment items. Conversely, the team closely analyses student item response data to evaluate the apparent validity of the LT levels in relation to each other.

Each assessment covers an individual RLC (i.e. one or more constructs), contains 8-10 items, and is designed to require about 20-30 minutes. Multiple forms of the same assessment are developed. Most teachers administer assessments about 2/3 of the way through an instructional unit. They are not intended to be graded, but to provide students and teachers actionable feedback on student (and whole class) understanding of the mathematical concepts. Students typically score between 20-70% correct; retests and practice tests are available to allow students to retry, and to improve their depth of understanding.

Real-time assessment reporting. The student reports show the overall percent correct on each construct, an item matrix that displays each construct, the items the student actually took, and whether the responses were right, wrong, or skipped. Students can select the incorrect or skipped items and resubmit them to change their percent correct. Teachers receive whole-class reports for each construct in the form of a set of "heat maps," each being a matrix with the LT proficiency levels listed vertically and the students ordered from weakest to strongest overall construct performance along the horizontal axis. The teacher can tap on a progress level to display the related item. Cells are coloured differently for incorrect and for relatively more correct responses.

Based on general expectations of less to more difficulty for higher proficiency levels, the response patterns tend to show increasing correct from bottom left to top right.

LT and assessment validation. There may be up to five constructs in an RLC, each with 6-8 levels. Therefore, items must be sampled across the LTs. The multiple assessment forms for each RLC share at least 3 common items, to support whole-class discussion. We encourage teachers to use multiple forms in a class. Each time a student takes an assessment, the results add to our knowledge database of student responses, and to their understanding of the LT, and to our confidence in predicting student progress. We use various psychometric models to explore the optimal modelling of LTs and assessments results. When results for an LT seem unidimensional, IRT is used; otherwise we consider structural equation modeling, CDM, or Bayesian models. These are low-stakes assessment for learning, so the diversity of approaches will add to our understanding of the particular LTs and student reasoning about and learning of constructs, without subjecting this work to artificial constraints regarding dimensionality typical of high stakes assessment modelling.

Math-Mapper 6-8 is being field-tested at three different schools, where the learning map is being incorporated in instructional planning, and the assessments are administered regularly to students, enabling us to collect 50-300 responses per item to analyse. As a result of this the items have gone through a rigorous review and validation process.

Ultimately, this is only the first phase of a complete validation argument. We will be studying the use of the tool over longer periods of time, which will allow us to determine how students improve understanding with the use of the tool, if teachers can use the tool to elicit more student thinking and participation, and find ways to improve the performance of various subgroups of students.

Siemon and Horne

Our research on learning progressions is premised on a socio-cultural perspective of learning that views learning “as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society” (Cobb, 1994, p. 13). It is aimed at identifying optimal pathways for teaching and learning key aspects of school mathematics based on an assessment of what might be regarded as students’ taken-as-shared knowledge in Australian mathematics classrooms. A valid criticism of this approach is that it does not necessarily reflect what is possible when students are exposed to high quality mathematics teaching over time (e.g., Boaler, 2008). But the reality is that not all teachers have the knowledge, confidence and local support needed to implement high quality effective practices. Nor do they necessarily have the time and resources to identify each student’s particular learning needs in relation to every single aspect of the mathematics curriculum even if this was desirable. The main rationale for working at scale in relation to a small number of really big ideas in mathematics is that this establishes a plausible, probabilistic model for establishing where learners are in their learning journey in relation to those ideas critical to student's progress in school mathematics (Siemon, Bleckly & Neal, 2012) and a framework to support teachers progress learning. The following sections will summarise our work.

The Middle Years Numeracy Research Project (MYNRP, 1999-2001)

A detailed analysis of the distribution of item responses provided by just under 7000 students in the initial phase of the MYNRP project revealed that there was as much variation in performance in any one year level as there was in the whole cohort and that this difference in curriculum terms was of the order of 7 years (i.e., approximately Year 2 to Year 8). While there were variations in measurement and data sense and spatial sense, all of the more difficult items were concerned with number sense, in particular anything that involved multiplying and dividing larger whole numbers, proportional reasoning, fractions, decimals and percentages, and situations not easily modelled in terms of a count of equal groups (e.g., combinatoric problems and problems involving rate or ratio). Characterised by Vergnaud (1988) in terms of the multiplicative conceptual field, these results prompted a follow-up project, the aim of which was to develop a more finely grained, evidence-based learning progression for multiplicative thinking that could be used by teachers to identify starting points for teaching and progress student learning.

Scaffolding Numeracy in the Middle Years (SNMY, 2003-2006)

At the time, there was a considerable body of literature concerned with particular aspects of multiplicative thinking. However, very little of this was specifically concerned with how these aspects relate to one another and how and when to support new learning both within and between these different aspects of multiplicative thinking (Siemon & Breed, 2006). Given evidence to suggest that where teachers are supported to identify and interpret student learning needs in terms of teacher accessible, evidence-based frameworks, they were more informed about where to start teaching, and better able to scaffold their students' mathematical learning (e.g., Clarke, 2001), it seemed sensible to produce a similar framework for multiplicative thinking.

For the purposes of the SNMY project, multiplicative thinking was defined by: a capacity to work flexibly and efficiently with an extended range of numbers (e.g., larger whole numbers and rational number); an ability to recognise and solve a range of problems involving multiplication or division; and the means to communicate this effectively in a variety of ways (for example, words, diagrams, symbolic expressions, and written algorithms).

Initially a broad HLT, derived from a synthesis of the research literature on students' understanding of multiplicative thinking, proportional reasoning, decimal place-value and rational number was developed (see Siemon & Breed, 2006). The HLT was used to select, modify and/or design a range of rich tasks including two extended tasks (e.g., Callingham & Griffin, 2000). The tasks were trialled and either accepted, rejected or further modified on the basis of their accessibility to the cohort, discriminability, and perceived validity in terms of the constructs being assessed. Secondly these rich assessment tasks and partial credit scoring rubrics were trialled and subsequently used to inform the development of the learning and assessment framework for multiplicative thinking (LAF). Finally an eighteen month action research study involving research school teachers and the research team, progressively explored a range of targeted teaching aimed at scaffolding student learning in terms of the LAF.

The results from the first round of assessment of just over 1500 year 4 to 8 students were analysed using item response theory and the subsequent variable map was used to link different aspects of multiplicative thinking and identify qualitatively different levels of understanding and strategy usage indicated by student responses (Siemon, Izard, Breed & Virgona, 2006). While these levels were largely consistent with the initial HLT, we were able to collapse one level and elaborate on others. Rich text descriptions for each level were derived from the performances on each item at each level to form the basis of the LAF. In acknowledgement that the levels were approximations based on responses identified at similar locations on the scale and in recognition of the fact that the purpose of the LAF was to help teachers scaffold student learning, the levels were referred to as *zones*. The LAF so derived comprises eight hierarchical zones ranging from additive, count all strategies in Primitive Modelling (Zone 1) through Intuitive Modelling, Sensing, Strategy Exploring, Strategy Refining, Strategy Extending, and Connecting to the sophisticated use of proportional reasoning in Reflective Knowing (Zone 8).

The notion of targeted teaching and the subsequent use of the LAF will be described in a later section but it suffices to say here that the teaching response to student's identified learning needs tended to be more effective in primary (i.e., Year 5 and 6 classrooms) than in Years 7 to 8 classrooms (Siemon, Breed, Dole, Izard & Virgona, 2006).

Reframing Mathematical Futures Priority Project (RMF, 2013)

Funding was obtained from the Australian Mathematics Science Partnership Programme (AMSPP) Priority Project round to explore the efficacy of and the issues involved in implementing a targeted teaching approach in secondary schools using the SNMY materials. Twenty-eight schools located in lower-socio economic settings across Australia participated in the 10-month study. Nominated 'specialists' in each school were provided with professional learning and supported to work with at least two other teachers at their school to implement a targeted teaching approach to multiplicative thinking. The SNMY assessments were conducted in August and November of 2013. Matched data sets were obtained from 1732 students from Years 7 to 10 with the majority (59%) from Year 8 (Siemon, 2016). The overall achievement of students across the 28 schools grew above an adjusted effect size of 0.6 indicating a medium influence beyond what might be expected (Hattie, 2012).

Reframing Mathematical Futures II Project (RMFII, 2014-2017)

The RMFII project is an AMSPP Competitive Grant project that was formulated in direct response to the findings of the initial RMF project. That is, that one of the major reasons for secondary school teachers' reluctance to adopt a targeted teaching approach to multiplicative thinking was their perception that this was not related to the curriculum they were expected to teach. Even though an analysis of the Australian mathematics curriculum at the time found that approximately 75% of the Year 8 curriculum required or assumed student access to multiplicative thinking (Siemon, 2013). The project aims to develop, trial and evaluate a learning and teaching resource to support algebraic, statistical and spatial reasoning in Years 7 to 10 that will enable teachers to identify and respond to student learning needs using a targeted teaching approach aimed at improving students' mathematical reasoning. For this purpose, mathematical reasoning is seen to encompass the core knowledge needed to recognise, interpret, represent and analyse algebraic, spatial, statistical and probabilistic situations and the relationships/connections between them; an ability to apply that knowledge in unfamiliar situations to solve problems, generate and test conjectures, make and defend generalisations; and a capacity to communicate reasoning and solution strategies in multiple ways (i.e. diagrammatically, symbolically, orally and in writing) (Siemon, 2013; 2016)

This is a non-trivial exercise involving an extended research team with recognised expertise in each domain. It requires the identification of Draft Learning Progressions (DLPs) for algebraic, spatial and statistical reasoning from existing research, the development and validation of rich tasks to assess and refine the DLPs using item response theory, the preparation of targeted teaching advice, and the development and trial of a series of online professional learning modules. To date, DLPs have been identified from the literature for algebra, geometry and statistical reasoning and over 250 individual assessment items have been trialled with more than 3600 students in Years 7 to 10. The initial analysis provides 'proof of concept', that is, that it is possible to scale the underlying constructs. Further trial work is being undertaken at the time of writing to validate and elaborate the scales.

APPLICATIONS OF LEARNING PROGRESSIONS/TRAJECTORIES

This section differs from the previous two in that it has amalgamated the responses of the four research teams to highlight the ways in which LT/Ps are being used to impact practice and shape further research. Once again only key references will be included here in the interests of space.

Curriculum and Standards

Three of the four bodies of work reported here used national curriculum statements and/or standards as a starting point for their work on learning trajectories/progressions. As this work unfolded, however, it became increasingly clear that researchers needed to go beyond such documents and look to the research literature more generally to inform their investigations. This had the added advantage of not only informing curriculum development and examining the effectiveness of that curriculum but building a better and deeper understanding of what was needed to achieve curriculum goals even to the extent of providing evidence that questioned the appropriateness of those goals. This changed the role of LT/P's from serving mainly as the core of curriculum development projects to having implications for all aspects of curriculum. For example, Clements and his colleagues developed a number of LTs for the NSF-funded *Building Blocks* project and curriculum (Clements & Sarama, 2013a). While this was designed to comprehensively address standards for early mathematics education for all children, evaluations have shown that Building Blocks can be effective, with large effect sizes even when compared to another research-based curriculum *not* built upon LTs (Clements & Sarama, 2008).

This and other work in this area led Clements and his colleagues to conclude that any comprehensive and valid scientific curriculum development program in education should address two basic issues - effect and conditions - across three domains - practice, policy, and theory. For instance, the question - is the curriculum effective in helping children achieve specific learning goals? examines effects in relation to practice. The question - are the curriculum goals important? – examines effects in relation to policy, and the question – why is the curriculum effective? – invites an exploration of effects in relation to theory. To achieve these goals satisfactorily and scientifically, developers must draw from existing research so that what is already known can be applied to the anticipated curriculum; used to structure and revise curricular components in accordance with models of children's learning such as research-based learning trajectories; and conduct formative and summative evaluations in a series of progressively expanding social contexts. As an example of this process, Clements and Sarama offer their work on TRIAD (Technology-enhanced, Research-based, Instruction, Assessment, and professional Development model), which has been implemented at scale and evaluated.

TRIAD is based on research and enhanced by the use of trajectories and technology. TRIAD places learning trajectories at the core of the teacher/child/curriculum triad to ensure that curriculum, materials, instructional strategies, and assessments are aligned. When implemented with fidelity, TRIAD has shown moderate to strong effects including transfer to other domains (e.g., Sarama, Clements, Wolfe & Spitler, 2012).

As with many researchers in the area Confrey and Maloney started with a specific LT (equipartitioning), then expanded their efforts to examine and analyse K-8 learning in all subfields. They did this first by analysing the U. S. Common Core Standards from a perspective of learning trajectories, but subsequently by building a new tool that uses learning trajectories for guiding instruction and scaffolding digital curriculum. The example they offer is the collaborative work on the Common Core Standards where a group of learning trajectory researchers participated in a joint meeting with the Common Core sponsors and writers, and subsequently provided the writers with summaries of the research to guide their grade-by-grade analysis (Confrey & Maloney 2014). A member of the National Validation Committee, Confrey mapped several early versions of the standards for consistency with the results of that overall research, and made recommendations for strengthening those connections. As with any document subject to competing perspectives, the final CCSS-M seemed consistent in many areas, and weaker in others. However, this points to the growing recognition of the value that research-based LT/P might play in determining goals for national standards, assessments, curricula, and pedagogy.

In Australia, the National mathematics curriculum is represented by a set of content descriptors (approximately 28 per Year Level) and schools have more control over the instructional materials and pedagogical approaches they use to address the content descriptors. Effect sizes in excess of 0.65 across a number of secondary schools as a result of using the Learning Assessment Framework for Multiplicative thinking (LAF) in 2013 has prompted schools to modify their curriculum offerings in order to accommodate a targeted teaching approach to multiplicative thinking across multiple year levels (Siemon, 2016).

Students and Learning

LT research began with a clear focus on children's thinking and learning in specific content domains. Initially the focus was on individual student developing schemas in particular mathematical areas (e.g., children's increasingly sophisticated counting schema, Tzur et al, 2013). While that work continues, there has also been an expansion in the focus of LT work to whole classes and multiple year level cohorts with a particular emphasis on the development and use of formative assessment tools to identify where learners are in their learning journey and better equip teachers to progress that learning (e.g., Sarama, Clements, Wolfe & Spitler, 2012; Siemon, 2016).

Confrey and her colleagues are currently working with multiple schools in multiple school districts with Maths Mapper, an LT-based digital learning system that, among other things, is designed to support the creation of continuity across grades and promote the surfacing of student thinking and strengthening of student agency (Confrey & Maloney, 2015).

Most LT/Ps have been developed and refined with school student populations. However, their application in adult settings has recently been explored by Tzur with both teacher and non-teacher adult learners, many of whom lack foundational schemes for multiplicative and/or fractional reasoning. He has found that applying these LT/Ps has been helpful for these adult learners as well as for children identified by their school systems as students with learning disabilities in mathematics.

An important question arises about LTs developed through studies in western cultures, namely, do they apply to or represent the learning of learners in other cultures. Are these learning frameworks universal or are they a consequence of what learners have had the opportunity to learn?

Teachers and Teaching

As many before, LT/P researchers recognise the importance of looking at domains of knowledge as a means of supporting teachers to better understand the connections between different aspects of mathematics and how that learning might be progressed. A consistent finding of this research is that a major way in which this occurs is through teachers observing their children's learning. The value of using assessment data to inform and improve teaching is widely recognised but the difference here is that the observations can be tied to evidence-based frameworks that provide guidance on where to go to next in relation to a range of interconnected ideas. This lead Siemon et al (2006) to conclude that a different term, *targeted teaching*, was needed to distinguish the long-term, multi-faceted nature of the interventions needed to scaffold student's multiplicative thinking from the equally valid but short-term or spontaneous teaching decisions that might be informed by a pre-test on subtraction or an informal observation of student thinking in the course of a classroom discussion. Targeted teaching is characterized by an unrelenting focus on big ideas framed by evidence-based LT/Ps. It is not a prescribed program, schools and teachers need to appropriate it to their circumstances and capabilities. It is a very organic process that is not in anyway equivalent to systematic streaming/tracking and it is most effective where it has evolved over time with the support of key individuals and the leadership group (Siemon, 2016).

Another way in which LT/Ps support teachers is by providing a shared language around a set of activities and tasks that point to the underlying conceptual structure of the mathematics that is the focus of the LT/P. For example, strengthening teacher community is an important focus of the LT-based Math-Mapper resource. Confrey and Maloney (2015) report that teams of teachers are planning their curriculum using the learning map instead of a set of standards elicit a different kind of conversation about topics. In one school, a teacher described the prior curriculum as "chaotic" and the new one as "calm." The teachers at the other district found that discussing *clusters* instead of individual standards helped them ensure that the ideas meant the same thing to them all. They often appealed to the LTs to clarify their thinking (Confrey & Maloney, 2015).

Teacher professional learning has been an element in the trialling, validating and scaling up of LT/Ps across all bodies of work reported here but more recently this has become the focus of research in this area. An example of this is Tzur's current study of the impact of job-embedded professional development on upper-elementary teachers' transition toward student-adaptive pedagogy. A substantial part of which engages teachers in learning to notice, infer, and use the two HLT about students' multiplicative and fractional schemes.

The power of LT/Ps to impact teaching practice and sustain quality approaches over time is evidenced by the follow up work on the Building Blocks project. Clements, Sarama and colleagues expected teachers to decrease in the fidelity in which they taught with learning trajectories after project support was discontinued. However, after two years, they found that the teachers *increased* the quality of their teaching and the these results were even more positive six years later with the *largest predictor of higher fidelity years out was child gain—teachers sustain and increase the quality of teaching when they observe their children learning.*

Informing and Extending Research

LT/Ps and the research around them are being used to inform new research. For example Tzur and his colleagues use them (a) to identify participants for a study based on their available, assimilatory schemes and (b) as a suggestive, developmental framework for determining what to teach next. On a much larger scale and with more of an eye to impacting practice at scale, the work of Siemon and her colleagues on mathematical reasoning, Sarama and Clements on Building Blocks and Confrey's work on Maths-Mapper point to an exciting future for LT/P research and development, particularly in relation to technology.

The implications of developing a dynamic digital learning system built around LTs represents a new paradigm of research and opens new possibilities for networked improvement models (Confrey & Maloney, 2015). This is because the design rests on an explicit learning theory (the LT/Ps) while the tool scaffolds curriculum flexibly and adaptably. In the case of Maths Mapper, the research team continuously monitors the tool's use in a variety of ways—how and when it is used, how long students need to complete the items and assessments, how the items perform, and which psychometric models provide the best data models to inform the tool's use. The communities of practice (students, teachers, curriculum specialists and administrators) are also leveraging the tools to plan, to develop new forms of instructional practice, to form student groups (or reteach) and to try out and refine materials. The focus is on student growth and on how different subgroups and individuals are able to get assistance and opportunities to learn as needed.

Research on LT/Ps is becoming more ambitious in its scope and intent. While this has the potential to transform the teaching and learning of mathematics through the provision of evidence-based frameworks, validated tools and quality instructional materials, reconceptualise the curriculum, and deepen teacher knowledge of the rich connections between different but related aspects of mathematics, at the end of the day it is the decisions teachers and students make every day that have the greatest impact on learning. For this work to have a sustainable influence on practice, it needs the support of school leadership, administrators working in close collaboration with researchers as partners.

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DISCUSSION GROUPS

41st PME Annual Conference
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Singapore

HOW TO RESEARCH CULTURAL-SOCIETAL FACTORS INFLUENCING MATHEMATICS EDUCATION?

Coordinators: Aiso Heinze[1] and Kai-Lin Yang[2]
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Research has provided models of how (mathematics) educational processes are organized on an institutional level. These models encompass cultural or societal factors (e.g., learning culture, educational tradition), which directly or indirectly influence the teaching and learning of mathematics (see Figure 1).

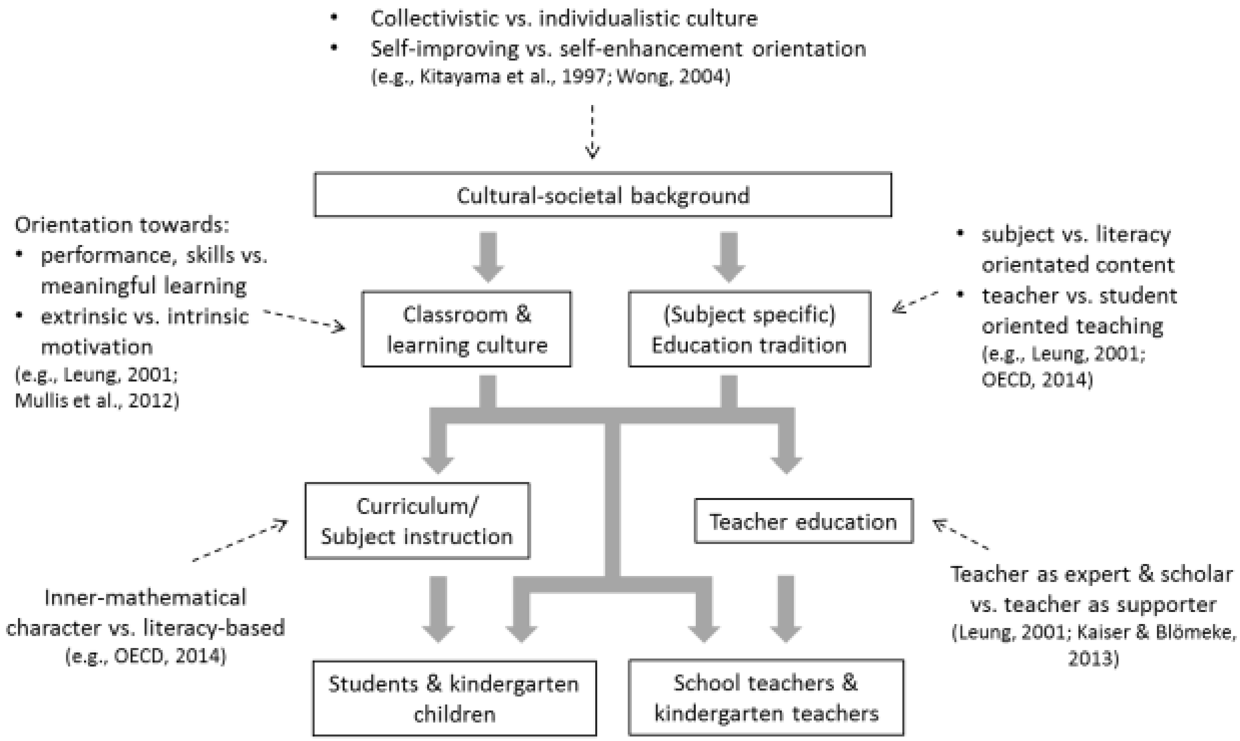

Figure 1: Cultural-societal factors influencing mathematics (teacher) education.

Subsequent to the international large-scale studies, these factors were elaborated, particularly the contrast of so-called Western and East Asian countries (e.g., Leung, 2001). However, the impact of cultural or societal factors is still not understood. Studies in a national context are restricted by the insufficient variance in these factors and international large-scale studies address quite broad competence constructs. Accordingly, there is a need for specially designed cross-cultural studies addressing a specific mathematical topic in depth and targeting specific cultural-societal factors.

The discussion group provides the opportunity to discuss four different empirical approaches and research designs exemplified by planned or on-going binational studies. Each study considers a specific mathematics educational context in a Western and an East Asian country so that a contrast of very different cultural-societal frameworks is given. The goal of the discussion group is to elaborate on and to further

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 139-140. Singapore: PME.

these research approaches. The participants should be motivated to discuss which theoretical perspectives and which methodology could be applied or adapted to increase the validity of cross-cultural research studies.

The activities combine presentations, group work, and discussions (see Table 1).

Phase	Session 1	Session 2
Introduction (15 min /5 min)	Theoretical model(s) and guiding questions (15 min)	Summary of session 1 (5 min)
Presentation of research approach (25 min)	1. Acquisition of proof skills: effects of curriculum and educational tradition (Y.-H. Cheng, H.-Y. Hsu, S. Ufer, & M. Vogel)	3. Pre-school teachers' attitudes towards mathematics education in kindergarten in different cultures (E. Brunner, C.-S. Chen, & H. Gasteiger)
	2. Intercultural validity of a model describing primary students' estimation skills (A. Heinze, H.-M. Huang, & S. Ruwisch)	4. What constitutes high quality of mathematics instruction in the view of teachers in different cultures? (A. Dreher, F.-J. Hsieh, A. Lindmeier, & T.-Y. Wang)
Group activity (30 min)	Working on approach 1 or 2 based on guiding questions	Working on approach 3 or 4 based on guiding questions
Discussion (20 min/30 min)	Groups will discuss their results (20 min)	Groups will discuss their results and the results of the DG (30 min)

Table 1: Schedule for discussion group sessions.

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STEM EDUCATION RESEARCH AND PRACTICE: WHAT IS THE ROLE OF MATHEMATICS EDUCATION?

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Researching science, technology, engineering and mathematics (STEM) education has been gaining momentum with increased calls for strategies to improve student engagement and to increase participation in senior schooling in countries where mathematics and science are not compulsory. However, the diversity of perspectives and approaches (from curricula to pedagogical) challenges the collection of evidence to establish a research base which justifies the funds currently being invested in STEM education. There has been limited attention to STEM education research in the mathematics education community, so the focus of this Discussion Group may call for increased attention to the role of mathematics in STEM, to the ways of integrating mathematics in STEM, and to the challenges of coordinating competing and dissimilar 'practices' across diverse disciplines in STEM.

For some time in the USA, STEM education has been extensively supported with educational entities receiving substantial Federal Government funding to develop a STEM focus (Li, 2014). Bybee (2013) argues the lack of a common understanding or definition of STEM education has led to a diversity of approaches with scant evidence for the success of many of the initiatives adopted by schools and school systems. In recent reports in Australia, there has been a strong recognition of the importance of STEM thinking and skills for all students and an advocacy of the need to bring school science and mathematics closer to the way science and mathematics are practiced in contemporary settings across the STEM disciplines (Office of the Chief Scientist, 2016).

To build on, and coordinate the range of reforms, a STEM Education Forum in Australia developed a *National STEM School Education Strategy*, 2016-2026 (National Council, 2015). The strategy included goals of all students leaving school with strong foundational STEM skills and capabilities, and all students embarking on more challenging STEM subjects, as well as five key areas for national action (p.6): 1) increasing student STEM ability, engagement, participation and aspiration; 2) increasing teacher capacity and STEM teaching quality; 3) supporting STEM education opportunities within school systems; 4) facilitating effective partnerships with tertiary education providers, business and industry; and 5) building a strong evidence base. Marginson, Tytler, Freeman and Roberts (2013) revealed successful countries "have instituted active programmes of reform in curriculum and pedagogy focused on making science and mathematics more engaging and practical, through problem-based and inquiry-based learning" (p. 10). Most recently, features of STEM programs have been reported and should form the basis of our deliberations (e.g., English, 2016; Honey, Pearson, & Schweingruber, 2014; LaForce et al., 2016).

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 141-142. Singapore: PME.

PLAN FOR DISCUSSION GROUP SESSIONS 1 AND 2

30 mins	**Session 1** – Brief introduction – Judy Anderson and Yeping Li
45 mins	Discussion questions – Do we have a shared understanding of 'STEM education'? Should we be taking an 'interdisciplinary' perspective and what is the role of mathematics?
15 mins	Sharing of STEM education perspectives, approaches, and research agendas in participants' countries and education contexts with a focus on mathematics in STEM.
15 mins	**Session 2** – Brief summary of Session 1. Further sharing of STEM education perspectives, approaches, and research agendas in participants' countries and education contexts with a focus on mathematics in STEM.
30 mins	Discussion questions – Why STEM education? Which approaches to STEM education provide evidence of successful student outcomes, particularly for mathematics?
30 mins	Discussion questions – Should we be asking different questions about STEM education in elementary, middle school, secondary and tertiary education? What are the key issues associated with researching STEM education at each of the different levels of STEM education?
15 mins	Planning next steps

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PERSPECTIVES ON MULTIMODALITY AND EMBODIMENT IN THE TEACHING AND LEARNING OF MATHEMATICS

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Mathematical learning processes are shaped by multiple expressive modalities, such as verbal language, gestures, inscriptions, and the use of physical and electronic artefacts. Each of these modalities plays a different role in the construction of mathematical knowledge and each can provide a complementary way to investigate mathematical thinking and learning.

Research on multimodality and mathematics is concerned not only with communicational aspects of non-verbal means of expression, but also ways in which the various modes may contribute to the learning of mathematics, that is, the epistemic and cognitive aspects of multimodality (Arzarello, 2006; Edwards, Ferrara, & Moore-Russo, 2014). These aspects can be considered from both social and individual perspectives. Socially, the different modalities influence interaction with others, and the joint construction of knowledge (e.g. Krause, 2016). Individually, the modalities involved in making sense of the world affect the nature of our cognition (Nemirovsky, 2003). Multimodality is one facet of embodied cognition, which refers to the reciprocal relationship between our thinking and the experiences we gain by our bodily being in the physical and social world (Varela, Thompson, & Rosch, 1992). From the perspective of embodied cognition, mathematics is not an "abstract field" but rather a social construction based ultimately on shared physical experiences.

Behind the analysis of each expressive modality – whether it be gestures, inscriptions, glances, or speech – are both established and *ad hoc* theories and methodological assumptions which, in many cases, stay implicit. For example, studies reconstructing mathematical learning processes sometimes include some gestures while leaving out others without being transparent about the reasons for this choice or methods of the analysis of the gestures.

The Discussion Group will allow interested researchers to exchange and discuss research experiences as well as to openly consider theories, assumptions, and methods. More specifically, the aims of the group are to make explicit:

- the various theoretical lenses utilized in the research, from semiotics to embodied cognition to linguistics; and,
- methodologies for carrying out empirical studies in this area.

The discussions in the group shall be guided by the following leading questions

1. *How can we capture multimodal and embodied aspects of learning mathematics within and across different theoretical lenses?*

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 143-144. Singapore: PME.

2. *What can we learn from integrating multimodal and embodied aspects in our research on teaching and learning mathematics – and what is not captured?*
3. *What are suitable methods for analyzing video data of embodied interactions, and how do they relate to theoretical lenses?*
4. *What are some criteria for high-quality research that focuses on embodied and multimodal aspects in the teaching and learning of mathematics?*

STRUCTURE OF THE SESSIONS

We invite the participants to bring video data to work on in the second session. The discussion group is then organized as follows:

Session 1

- Introductions and brief sharing of interests: 15 min
- Brief presentation on theoretical lenses and introduction of video data: 20 min
- Small group discussions about strength and limitations of the diverse lenses (based on own experiences and prompted by video data): 35 min
- Discussion of questions 1 and 2: 20 min

Session 2:

- Sharing of questions since Session 1: 5 min
- Brief descriptions of participants' video data: 10 min
- Small groups to analyze data within different analytical approaches: 30 min
- Whole group discussion of data analysis and about question 3: 20 min
- Whole group discussion of key results gained in the discussions to conclude on question 4 and to summarize with respect to the aims: 25 min

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MATHEMATICS IN DIFFERENT LANGUAGES

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THEORETICAL BACKGROUND

The aim of this discussion group is to explore how the specificities of different languages affect mathematical concepts and mathematical thinking when mathematical tasks are translated between these languages. Individual mathematical terms and grammatical structures both play a role in how mathematical thinking is constructed (Morgan, Craig, Schuette, & Wagner, 2014). Thus it cannot be assumed that mathematics remains identical when a task is translated into different languages.

Mathematical concepts in different languages can have different scope of application and different connotations. The grammatical structures of different languages may also affect how mathematics is constituted, including differences and variations in logical constructions, in syntactic categories, and in spatial language together with its associated metaphors (Edmonds-Wathen, Trinick, & Durand-Guerrier, 2016). For example, fractions are conceptualized as "drei Fünftel" (three fifths) in Germany, but as "5 therein 3" in Turkish, which is closer to a "part-of-a-whole" concept for fractions (Schüler-Meyer, Prediger, Kuzu, Wessel & Redder, 2016).

There is a need for a thorough investigation of how translating tasks has consequences for underlying mathematical concepts, and for the mathematical thinking required to solve such tasks, particularly in light of international tests such as PISA and TIMSS. The discussion group will explore how mathematics varies according to the different languages in which it is learnt, taught and practiced, focussing on the area of "change and relationships" (OECD, 2013), which is to be explored in tasks about fractions and percentages, among others. Grammatically, languages vary in how they express relationships between objects and circumstances, which has implications for this important topic area.

KEY QUESTION

- How do the grammatical structures of different languages affect thinking about in change and relationships in these languages?

OUTLINE OF ACTIVITIES

Participants will identify examples of differences in mathematical language between natural languages that potentially have an impact on mathematical thinking; discuss the significance of these differences for bilingual/multilingual mathematics education, minority language speakers, and Indigenous language speakers; and discuss how future research might investigate the impact of identified differences between languages on mathematical thinking.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 145-146. Singapore: PME.

Session 1	Introduction (leaders)	10 min
Leaders	Theoretical and empirical perspectives on the relationships between natural language, mathematical concepts and mathematical thinking.	30 min
Small groups	Participants examine, compare and discuss different language versions of open PISA tasks on change and relationships: • What differences in grammatical structures between versions might affect students' mathematical thinking? • How do concepts differ between different language versions of these PISA tasks?	30 min
Whole group	Discussion of PISA task version differences and possible implications for mathematical thinking in different languages	20 min
Session 2	Introduction (leaders)	5 min
Whole group	Participants discuss the significance of differences in the task translations for bilingual/multilingual mathematics education, minority language speakers, and Indigenous language speakers	30 min
Small groups	Participants develop ideas how to tackle the teaching and learning of fractions in the different contexts and languages, e.g. in regard to PISA and task development; learning obstacles and benefits; designing learning trajectories; etc.	20 min
Whole group	Discuss possible future research to investigate possibilities raised during the previous discussions (e.g. other mathematical topics)	25 min
Leaders	Agenda setting for future work and conclude sessions	10 min

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WORKING SESSIONS

41st PME Annual Conference
17-22 July 2017
Singapore

TEXTBOOK SIGNATURES: EXPLORATION AND ANALYSIS OF MATHEMATICS TEXTBOOKS WORLDWIDE

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AIM AND RATIONALE

Textbook analyses can provide a comparison of learning opportunities triggered by textbooks among different countries. Being an important medium for representing the written curriculum, textbooks play an important role in shaping mathematics learning and teaching in schools. Hence, international textbook analyses can potentially offer insights into curriculum intents and the suggested teaching approaches in the different countries. Building on the idea of *lesson signature* suggested by Hiebert et al. (2003), Charalambous et al. (2010) propose that textbooks within the same country may have a "textbook signature"—"uniform distinctive patterns"—in the textbooks (p. 146). Using these ideas, we have proposed a notion of textbook signature and have attempted to characterise our analyses of textbooks in gradient (Choy, Lee, & Mizzi, 2015) and fractions (Lee, Choy, & Mizzi, 2016) using our notion of textbook signatures. Our comparative studies on introduction of notions of gradient and fractions in Germany, South Korea and Singapore imply that textbook signatures are unique in different countries and may hold important implications for teaching and learning.

Following a discussion group (DG) about textbook signatures, which took place in PME 40 in Hungary, we have concluded that research into our notion of textbook signatures is a promising strand of research. For example, textbook signatures can potentially describe and explain the different curricular approaches adopted in different educational contexts for improving the way of teaching and learning mathematics. However, our DG participants highlighted the need for more data analyses using textbook signatures from a larger number of countries or within the same country (especially from countries with a non-centralized educational system). Hence, one of the main goals of this working session (WS) is to provide interested researchers to collaborate and work on textbook analyses using our notion of textbook signatures. Participants of this WS are invited to carry out a textbook analysis focusing on the introduction of gradient for their own country:

Hence, participants are required to bring along one textbook or a copy of the first chapter(s) of the textbook which introduces the topic of gradient to students the first time at secondary level in their respective country.

We not only aim to present the findings of our textbook analyses using textbook signatures, but also refine our notion of textbook signature during the WS. We believe that this WS will be a good platform for researchers to have conversations about textbook signatures and their relevance in mathematics education.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 149-150. Singapore: PME.

KEY QUESTIONS

The working session activities will be guided by the following key questions:

- What is our current notion of textbook signatures and how can it be applied to different textbooks worldwide?
- What curricular and textbook features can be seen in the textbook signatures from different countries?
- What implications for teaching and learning mathematics can be drawn from textbook signatures?
- How can we refine our notion of textbook signatures following our findings from the WS?

WORKING SESSION ACTIVITIES

	Segment	Duration	Description
DAY 1	1	15 min	Session 1: Presentation: Notion of textbook signatures by Angel Mizzi
		10 min	Session 2: Recap of what we discussed in PME 40 by Ban Heng
	2	15 min	Session 1: Illustrative use of theoretical framework and presentation of textbook analysis techniques by Ban Heng and Mi Yeon
		50 min	Session 2: Working Session – Exemplary textbook analyses
DAY 2	3	50 min	Session 1: Working Session - Initial textbook analyses
		15 min	Session 2: Presentation of results from textbook analyses
	4	10 min	Session 1: Presenting first ideas, results or barriers upon application of the textbook signatures framework
		15 min	Session 2: Rounding Up: Closing remarks and future research

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WHAT DOES "SOCIO-CULTURAL-HISTORICAL VIEWS OF TEACHING AND LEARNING OF MATHEMATICS" MEAN TO US?

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[1]University of Ottawa, [2]Rhodes University, [3]University of Ottawa,
[4]London South Bank University

To make sense of issues related to mathematics education, many researchers draw on social, historical, and cultural perspectives of learning and becoming. A wide range of applications and interpretations of this view reflects the vitality of this approach. The central purpose of this working session is to invite the participants to explore and discuss the intentions and interpretations of different theoretical aspects and concepts in relation to social, cultural and historical views on mathematics education. This focus expands and deepens our understanding of diverse interpretations of socio-cultural-historical perspectives of teaching and learning of mathematics.

Assuming that we are born into already formed social worlds of language, race, class and so on, Wertsch (1994) believes that one cannot provide an account of human action (including learning) without taking into account its cultural, social, and historical setting. But in consideration of such setting, complexity arises on two levels when we, as mathematics educators examine learners' and/or teachers' actions with mathematics. At one level, the question is how and where do we start? And at a different level, the question is how do we communicate our interpretations and understandings, with one another?

Over many decades, both these questions have been examined by scholars, including Lerman, Bartolini Bussi, Radford, and Roth among many others. Common among all approaches is that learning is social in origin and that it takes place in different historical and cultural contexts and is mediated by tools, language(s) and other symbol systems. In this working group, we ask what is *concretely* implied by phrases such as "social origin of learning", or "mediating role of language(s)"? In an attempt to interpret and make sense of these theoretical terms, we look closely into samples of data, as well as at different research studies conducted in the field of mathematics educations (Lerman, 2006; Radford, 2013; Roth, 2017). By emphasising the socio-cultural theoretical standpoints of, for example, Bakhtin and Vygotsky, we examine examples of children's mathematical activities selected from our data, or contributed by participants, to look into the meaning(s) that might be given to terms that we use as we speak about social and cultural ways of learning and teaching mathematics—terms such as *dialectic, mediation, dialogue, voice(s)*, to name a few. The following question will guide our exploration, our activities and discussions: A socio-cultural perspective can include a variety of terms, such as dialectic, dialogic, mediation, voices, and the zone of proximal development. How, concretely, do we relate to and interpret these terms in our data?

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 151-152. Singapore: PME.

In both sessions, we will discuss a variety of examples in order to explore the question. Our data include short video clips and transcript excerpts recording children's interaction with tools and their interactions in multilingual classrooms. These examples will reflect different activities emblematic of research from a socio-cultural perspective, such as students of different ages working with graphs, or textbooks, and interacting with each other.

Day 1:

- Introducing the question and identifying theoretical terms in a socio-cultural approach
- Viewing a 3-minute video of children's interaction with tools. Discussion of the traces of social and cultural dimensions in the activity
- Small group – review the transcripts to examine examples of mathematical activities in order to identify instances of the theoretical key terms
- Reflecting on how instances were identified

Day 2:

- Discussing different interpretations and uses of the key terms;
- Providing participants with data and notes on different conceptualisations of the key terms from the literature
- Small group – review the transcripts to refer back to different interpretations of the terms, to reflect on the ways in which and the extent to which the identified key terms in day 1 assist us to explain the social-cultural-historical origin, construction and modification of mathematical actions, learning and teaching
- Reflecting on what socio-cultural-historical views of teaching and learning of mathematics mean to us
- Closing remarks.

Throughout both sessions, we will highlight how common terms guide us in making sense of the learning and teaching of mathematics as social and cultural and historical in origin. To further enrich the group discussions, we post data for participants to have access to ahead of time. Points arising from both sessions will be fed back to our concluding remarks, in which we combine the discussion from the two sessions to highlight possible further research and actions.

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COMPARING DIFFERENT FRAMEWORKS FOR DISCUSSING CLASSROOM VIDEO IN MATHEMATICS PROFESSIONAL DEVELOPMENT PROGRAMS

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[1]Weizmann Institute of Science, Israel; [2]University of Bristol, UK;
[3]Australian Council for Educational Research

Professional development (PD) programs for mathematics teachers around the world use different frameworks for analyzing and discussing classroom video, according to various goals and desired outcomes. We will compare several such frameworks, in an endeavor to gain collective insights regarding the aims, advantages and limitations of each one. The working group will attempt to reach a framework categorization that can serve as a useful tool in researching the domain of video-based PD programs.

Background

Video has been used as a tool for teacher professional development (PD) for the past 50 years, however the focus and methods of its uses has changed considerably over time (Sherin, 2004). The current accessibility of digitized video recording devices, coupled with the widespread means of editing and exchanging clips, increases the use of this technology within PD programs (e.g., Borko et al., 2011; Hollingsworth & Clarke, in press; Coles, 2013; Karsenty, Arcavi & Nurick, 2015; Sherin & van Es, 2009). Given this context, we see it as an important task to compare and contrast frameworks that direct teachers' discussions around classroom video, for example in relation to the following features: (a) the purpose of watching (e.g., evaluation; noticing; reflection); (b) the foci of discussion, and how they are chosen; (c) norms and/or protocols that may apply to watching and discussing the video, and the way they are established. These features will serve as the departure point for the work of the group. Participants will work with three frameworks used in PD programs in different countries: the VIDEO-LM framework, designed around 6 viewing lenses (Karsenty et al., 2015); the 5-dimensional observation instrument developed by Hollingsworth (Hollingsworth & Clarke, in press); and the teacher-initiated noticing framework used by Coles (2013).

Goals

1. To experience different frameworks for the analysis of classroom video within a PD scenario, and elaborate their aims, ways of working, strengths, and limitations;
2. To form key criteria for categorizing frameworks of video analysis used in PD;
3. To agree an agenda for continued international research collaboration around the use of video in PD for mathematics teachers.

Activities and timetable

Session (1)

- Introduce session aims and provide overview of three frameworks (15 min.).

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 153-154. Singapore: PME.

- In three groups facilitated by the WS leaders, participants use one of the frameworks to analyse a common videotaped lesson excerpt (30 min.).
- After analysing the excerpt, each group forms feedback related to insights and issues associated with using their nominated framework, to be later communicated in a plenary discussion. Guiding questions will include: What was the focus of your discussion? What were the main ideas raised regarding the episode observed? How would you characterize the aims of the analysis you performed? What did you gain, and what might teachers gain, from such an experience? What might be the limitations of using this framework? What kinds of discussion norms or protocols were used by the group? What was the role of the facilitator? (20 min.).
- Groups present their feedback in the plenary (25 min.).

Session (2)

- Review work from Session 1 (10 min.).
- Participants form new groups, each including representatives from the original three groups. We will re-watch the video clip from the first session. Groups will then work on the following task: Which moments on the video were significant across frameworks? What emerging criteria can you elaborate from the previous session, for comparing and contrasting frameworks for video-based teacher discussions? Include criteria that relate to the content and aim of the frameworks, as well as the facilitator's responsibilities, and anything else you view as important (30 min.).
- Groups share and discuss their criteria in a plenary (25 min.).
- Identify themes, insights and research questions (15 min.).
- Discuss next steps for future collaborations (10 min.).

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VIDEOS IN TEACHER PROFESSIONAL DEVELOPMENT: FOSTERING AN INTERNATIONAL COMMUNITY OF PRACTICE

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[1]University of Tasmania, [2]Oregon State University, [3]University of Auckland
[4]University of Melbourne, [5]The University of Queensland

This working session seeks to address questions that emerged from the Discussion Group 'Videos in Teacher Professional Development' at the 13th International Congress on Mathematical Education (Leong, Ho & Evans, ICME13, 24-31 July 2016, Hamburg). The ICME13 Discussion Group considered experiences from a number of international projects using videos to observe teacher practice for professional development (PD) purposes including:

- The use of recorded lectures for PD in undergraduate mathematics at the University of Auckland, a collaborative project between research mathematicians and mathematics educators (Barton, Oates, Paterson & Thomas, 2015), based on the theoretical *Resources, Orientations* and *Goals* (ROG's) framework developed by Schoenfeld (2010;
- A cross-institutional study using videos to aid tutor reflection by King & Lonsdale in Australia (*First Year in Maths* project, King & Cattlin, 2017);
- The use of a video databank, based on the Japanese Lesson-Study approach, with pre-service teachers in Singapore (Ho, Leong & Ho, 2015);
- Using the Mathematical Quality of Instruction (MQI) observational tool with elementary teachers in the Pacific Northwest and New England in the USA, as a lens to discuss mathematics instruction (Hill et al., 2012);
- Gaps in PD provision identified by the *TEMPEST* project in Australia, which examines the extent and quality of PD opportunities for mathematics teachers (Reaburn, Kilpatrick, Fraser, Beswick, & Muir, 2016)

This WS aims to explore issues identified in the ICME13 discussions, framed by the following questions:

1. How might we address the challenges presented by the wide range of available observational measurement tools and theoretical perspectives for looking at teaching quality and the impact of PD programmes?
2. How might we compare and contrast the variety of ways in which videos are used, (e.g., observing teachers' own or peers lessons, using video from unknown teachers); and the different audiences for the PD (e.g., teachers vs teacher educators), to gain some insight into their value and potential?
3. Does the nature of the observations themselves (e.g. looking globally at an entire lesson, for example using the MQI; or more minutely at particular

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 155-156. Singapore: PME.

elements of a lesson, e.g. Schoenfeld (2010); Barton et al. (2015)) affect the value and effectiveness of the subsequent professional development?

A specific outcome of this WS is to establish an international, online network to share observational videos of teacher practice, and to explore ways to examine these through a professional development lens, within a supportive community of practice. In the first session, the coordinators will provide a brief overview of their projects and outline the goals of the WS (20 minutes), followed by viewing a video excerpt from one of the coordinator's practice (5 minutes). Participants will then form small groups in which they will discuss and identify the major issues raised to date with respect to the WS questions and themes (20 minutes), followed by a 45 minute open discussion to summarise the key issues identified. In session two, we will use the issues identified in session one to inform the development of a pilot online collaboration between interested participants, to share and discuss videos of our practice. This will be done in a mix of small-group discussions (30 mins) and whole-of-session reporting back.

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SEMINAR

41st PME Annual Conference
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REVIEWING FOR THE PME – A PRIMER FOR (NEW) REVIEWERS

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GOAL OF THE SEMINAR

This seminar[1] is intended to provide information about the PME review process and give the opportunity to gain first experiences in providing a high-quality review. The seminar aims especially at the needs of new reviewers[2], although experienced reviewers are highly welcome in order to facilitate knowledge transition within the PME community. The seminar includes an introduction in the intention and purpose of reviewing from a more general perspective (McKnight et al., 2000; APA, 2009), but also details aspects of the PME review practices. Participants will have opportunities to work with authentic examples from the PME review processes of the last years – provided we find authors that are willing to share their contributions with the review they received. Acknowledging the diversity within the PME community in the review process will be an important aspect of the seminar.

GOALS FOR THE PARTICIPANTS

Having participated in the seminar, the participants will

1. know about reviewing as an aspect of scientific quality management
2. know about the most important differences in reviewing procedures for journals and conferences as well as different types of contributions, especially in the PME context
3. be able to differentiate the specific review categories of PME
4. be able to identify aspects of quality for a review
5. be sensible to aspects of fair, constructive, and inclusive reviews

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[1] Seminars are intended to provide specific courses for professional development of PME members.

[2] PME members with two accepted Research Reports in the last five years or three accepted Research Reports in the past 10 years are eligible as PME reviewer.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 159-160. Singapore: PME.

EXPECTED BENEFIT FOR PME AS A COMMUNITY

PME – as a scientific community – will benefit from the seminar as

- it is expected to improve the knowledge of (new) reviewers about the review process
- it is expected to smoothen (new) reviewers difficulties in composing high-quality reviews

METHODS

The seminar will last 90 minutes. It will start with a brief presentation focusing on learning goal 1 and 2. A first group work phase will focus on the specifics of PME reviews and thus contributing to the learning goals 3 and 4. A second group work phase will focus in particular on the aspects of fair, constructive, and inclusive reviews (learning goal 5). Experienced reviewers, who are willing to share their knowledge, are invited to serve as group mentors during the working phase.

APPLICATION

In order to participate at the seminar please indicate your interest via info@igpme.org (administrative manager Bettina Roesken-Winter).

If you are willing to share a former contribution of yourself TOGETHER with the reviews you received as authentic examples for the group work phase, please contact Anke Lindmeier at lindmeier@ipn.uni-kiel.de as soon as possible.

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ORAL COMMUNICATIONS

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TOOLS, SIGNS AND THE CREATION OF ARTEFACTS

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Mathematical tools are important parts of the learning of fractions (Martin & Schwartz, 2005). I use Vygotsky (1978)'s view of signs, tools, and artefact to examine interactions of two 12 years children with Cuisenaire rods as they created artefacts to represent 1/2 + 2/5. As with any other human action, learning is mediated by signs and tools. Tool is a means of external activity (i.e., labour) such as hammers. Signs are means of internal activity, such as language, various systems for counting. Vygotsky expanded the notion of signs and tools to note: 'Distinctions between tools as a means of labor... of mastering nature, and language [sign] as a means of social intercourse become dissolved in the general concept of artefacts' (p.53). In the following interaction, the children started using the Cuisenaire rods, not knowing how the rod could be useful in creating artefacts that represented 1/2 + 2/5. After a few trials and errors, the children tied signs to their interactions with the tools to present 1/2 + 2/5, by selecting rod of 10 as the unit, a rod of 5 as 1/2 and two rods of 2-unit as 2/5.

	A: This could be one M: [rod of 5], Yeah okay that is good. So we need one of these		M: The red [two]. A half and two fifths	

Here, the artefact-ness of the created artefact depended on how the children perceived the physical properties of the tools in relation to the task. The point I raise is that the process of creating an artefact – as children tie signs to the use of tools – is a gradual and complex process. This gradual process suggests a system of relationships among the task, the children's perception of the physical properties of the tools, and the mathematical *knowing* that is happening as the children use the tools. Hence, mathematical tools are not useful merely because they are designed with mathematical meanings built into them or because their mathematical meanings are perceived by the teacher. Mathematical tools become useful to a child *only* if the children perceive their useful physical properties and their affordances in relation to a mathematical task.

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CHARACTERISTICS OF FORMATIVE ASSESSMENT THAT ENHANCES STUDENT ACHIEVEMENT

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The potential of formative assessment for enhancing student achievement has been shown in many studies (e.g. Black & Wiliam, 1998). However, although formative assessments commonly share the core of modifying teaching and learning based on identified student learning needs, strategies for formative assessment may include an emphasis on different aspects of formative assessment. There is a shortage of studies investigating the impact of formative assessment conceptualized as an integration of these strategies into a unity, especially for randomly selected samples of teachers. To enhance our understanding of this impact, and to further develop the theory of formative assessment, studies based on this conceptualization investigating both changes in teachers' classroom practices and their impact on student achievement are needed.

This paper reports from two studies examining the changes in teachers' formative classroom practice that followed a professional development program (PDP) and the effects on student achievement in mathematics. Using a framework by Wiliam and Thompson (2008), changes in practice were identified in data from observations and interviews in a sample of 22 randomly selected Year 4 mathematics teachers in a Swedish municipality. A pre and post tests design with a control group (24 teachers in the same municipality) was used to investigate the impact on student achievement.

The results show that all teachers who participated in the PDP changed their classroom practice in line with the defining characteristics of formative assessment. Furthermore, after controlling for the pretest scores, the classes of the randomly selected teachers who participated in the PDP significantly outperformed the classes in the control group on a post test one school year after the end of the program ($p=0.036$, $d=0.66$). The results provide evidence of the impact of the PDP on teacher practice and student learning, but also concrete illustrations of how formative assessment conceptualized as a unity of integrated strategies was used by these mathematics teachers and may work to create new learning opportunities for the students in real-life settings. The presentation will provide examples of the changes in the teachers' practice as well as new learning opportunities these changes may provide.

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MIDDLE SCHOOL STUDENTS' ENGAGEMENT WITH MATHEMATICAL PROCESSES IN A MANIPULATIVE GAME PLAYING CONTEXT

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In Turkey, math games and brain games have become very popular. These concrete manipulative games have been considered as a tool for teaching mathematics (McNeil & Uttal, 2009). However, determining what are the mathematical skills and behaviors that these games can support is an important question. Mathematical process skills such as problem solving, reasoning representing, communicating and connecting (NCTM, 2000) appear to be most involved skills in such games. The way of winning a game is to develop various winning strategies. In game playing context, these strategies are often silent, based on the intuition of the players and do not emerge as a collective acquisition. The transformation of individual strategies into collective acquisition is a complex process, understanding and organization of which require specific approaches. Brousseau's Theory of Didactical Situations (TDS) (Brousseau, 2002), largely inspired from game theory, seems to offer an appropriate framework for this. Brousseau defines three types of situations in the game context: action, formulation and validation. This study aims to examine the mathematical processes that students experience in the context of games, designed according to TDS. What strategies students put forward, how students explain and justify their strategies, how students' strategies, explanations and justifications evolve over time are the research questions of the study. Five games based on concrete manipulative material were used and two seventh grade students were involved in an out-of-school setting. A researcher introduced and conducted the games with the questions in accordance with the principles of TDS. The students were given an average time of 1.5 hours for each game and their actions and speeches were video-recorded and analyzed by tracking the sequences. The results showed that students could use several strategies, such as simplifying the problem and making a list, formulating hypotheses and prove or disproving their hypotheses by switching the environment (manipulative material or paper-pencil). An evolution was observed over the time, as well as for the strategies used as the formulation and validation of the hypotheses.

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IN-SERVICE MATHEMATICS TEACHER'S TECHNOLOGY INTEGRATION AS AN ASSESSMENT TOOL ON THE TOPICS OF AREA AND PERIMETER

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In this century, teachers' knowledge of technology integration is essential for effective teaching, and teachers should possess sophisticated knowledge on pedagogy, content and technology. This study investigated the in-service elementary mathematics teacher's technological pedagogical content knowledge about the area and perimeter in assessment. In-service teacher's assessment activities have been analyzed deeply according to Niess, Sadri, and Lee (2007)'s TPACK Development Model. According to this developmental model, which is sequential process, TPACK moves through the recognizing, accepting, adapting, exploring, and advancing stages. As teachers progress along the model, TPACK – the intersection of the constructs of technology with pedagogy and content knowledge – forms and expands (Niess et al., 2009).

In this study, researchers collected qualitative data from middle school mathematics teachers' lessons to determine the TPACK levels in geometry. This study is part of a large study in Turkey that investigates how in-service teachers integrate technology in the classroom. In this study, researchers focus on one in-service teacher (coded as Esen) to analyze assessment tools in detail. Esen's lesson was observed to determine the TPACK levels in the assessment. Classroom observations conducted by the researchers and observations were video recorded. Researchers also took field notes from the in-service teacher's classroom practice.

According to findings, Esen's lesson demonstrated that she designed assessments to reveal students' understanding of geometrical ideas using an appropriate technology that extends beyond paper and pencil type questions. The area and perimeter activities assess students' conceptual knowledge instead of procedural knowledge. Furthermore, she adapted assessment practices that investigate students' understandings of area and perimeter in ways that demand full use of Geogebra. As a result, this assessment activities show us that Esen was at an advancing level according to the TPACK development model. She "developed innovative assessments to capture students' understandings of the mathematics embedded in the particular technology" (Niess et al., 2009, p.21).

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UNDERSTANDING MY PRACTICE AS MATHEMATICS TEACHER EDUCATOR THROUGH LESSON ANALYSIS

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Self-study is a way that allows teacher educators to research into their own practices in order to gain a better understanding of the complex nature of teaching and learning about teaching (Loughran, 2005). Research on self-study also enables teacher educators to share their own practices with their colleagues. As a teacher educator I have tried to investigate what I can do to support the development of teacher candidates' mathematical content knowledge for teaching and to improve my own knowledge of teaching mathematics teaching. Lesson analysis as an approach to learn teaching from teaching can be used in this sense. Lesson analysis aims at capturing students' thoughts and improving their interpretation skills in order to evaluate the effectiveness of the teaching (Barnhart & van Es, 2015). This self-study was based on lesson analysis activities as part of two undergraduate courses – school experience and teaching practice – with pre-service mathematics teachers, and a postgraduate course – knowledge of teaching mathematics – with novice mathematics teachers. As a math teacher educator, I decided to investigate my own practices in these courses. Thus the objective of the present study is to reveal the way in which my practice involving lesson analysis activities whether or not contributed to my own professional development of teaching and learning about mathematics teaching.

Reflecting the qualitative research design, this self-study made use of my diaries, my observations and field notes, the students' diaries, their assessment reports, and their lesson analysis reports as data collection instruments. The most significant conclusion that I have learned and reached in investigate my practice is that the lesson analysis activities support the creation of a consciousness that a lesson in mathematics teaching should be considered from the point of view of the student. Furthermore, conducting this self-study also helped me become aware of both the strengths and limitations of my own practice. The insights from this experience are expected to be of use in improving other teacher educators' practices as well.

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THE INTERPRETATION AND USE OF MATHEMATICS SYMBOLISM IN OBSERVING AND DESCRIBING PATTERNS

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Although the process of observing and describing patterns using mathematical notation can be seen as a fundamental experience in learning mathematics, the shift required from noticing patterns to expressing the patterns using mathematics symbolism often requires sophisticated algebraic techniques. In this study I focus on a group of 57 pre-service mathematics students to explore their interpretations of the mathematical symbolism embedded in pattern descriptions as well as their proficiency in using this symbolism to generate descriptions of the patterns. Zazkis and Liljedahl (2002) comment that the predominant pattern-related activity for learners at schools is extending number sequences and finding an algebraic expression for the general term. In this study two of the tasks are of this type. A further two tasks are of the type where the algebraic description is provided and learners are asked to generate some of the terms.That is, given the position of the element in the sequence, the goal is to find the corresponding element, which is usually a simple task. One way of raising the demand in this study involved using a sequence with repeating cycles, of length three. Hence the output of n mod 3 function could be used to determine the value of the n^{th} term which is also dependant on n, the position of the term. A second way in which the task of generating terms of a sequence was made more complex, was by providing the description in recursive terms.

The study draws upon Watson and Mason's (2006) theory of variation which asserts that carefully structured variation within learning activities can be used to enhance learning. The findings show that students were generally able to produce correct responses to the more direct questions, but experienced difficulties with generating a description of the general term for the sequence with repeating cycles. This may have been because of the added dimensions of variation with respect to position (where n was varied), conditions on n (n could satisfy one of three conditions), formulae (there were different formulae depending on which value n took on, as well as the role of x_n in the task using recursive descriptions (as an input for the next term and as an object). In attempting to deal with the dimensions of variation embedded in the tasks, students used different strategies in an attempt to keep certain quantities constant while varying others. It is recommended that when teaching sequences with repeating cycles additional scaffolding is offered with respect to the role of the modulo n function.

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A CONCEPT-BASED APPROACH TO TEACHING MATHEMATICS TO FINE ARTS STUDENTS IN QATAR

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This research explored the impact of teaching mathematics to Fine Arts students, in an American university in Qatar, through the lens of Erickson and Lanning's (2013) work on *concept-based instruction*. Through teacher understanding of students' prior experiences and current circumstances outside of the classroom, Erickson and Lanning's work develops a framework for teaching "big ideas" to link student conceptual understandings of smaller ideas. The research question in this study was:

- How did Fine Arts students at an American university in Qatar, taking a required mathematics course, respond to *concept-based instruction*?

The study included 3 instructors and 38 students (35 female, and 3 male), and spanned the duration of one semester. The semester was divided into units of study that began with symmetry and the historical development of Islamic art and geometric design in the Middle East and evolved into more contemporary explorations of the mathematics of fractals, Fibonacci/the Golden Ratio, and financial mathematics. At the end of each unit of study, students used mathematics to create projects related to art/design. Data were collected in the forms of student work and student reflections about the course. Data were analyzed using triangulation methods (LeCompte and Schensul, 1999).

Analysis of the data yielded a four stage theme of student engagement in this environment. Student reflections and work demonstrated that *concept-based instruction* helped them maintain engagement throughout the semester by cycling through four stages during each unit of study. The stages of engagement were (a) Buy-In: each unit caught students' attention, (b) Motivation: the buy-in motivated students to work on the mathematics, (c) Persistence: the buy-in and motivation pushed students to work harder, and (c) Accomplishment: completing and presenting their work inspired students to engage in each subsequent unit.

Based on these results, we propose that *concept-based instruction* can have a dramatic effect on engaging and maintaining engagement for Fine Arts students taking a mathematics course in Qatar. We believe these findings are particularly important for expatriate faculty members who come to Qatar to teach an American curriculum to non-American students.

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VARIATION OF EXPLICIT ARGUMENTATION IN MATHEMATICS TEXTBOOKS

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Reasoning is central within mathematics and many different forms of reasoning take place in classrooms, for example, deductive, intuitive, and informal reasoning. Although different, all forms include *providing reasons for a statement* (often called *justification* or *argumentation*), which is what this study focuses on. Even though reasoning is an important part of mathematics, there is almost no research on how students understand different types of argumentation (Mejía-Ramos & Inglis, 2009).

The present study is part of a project aiming to understand how different types of explicit argumentation can support or hinder students' comprehension of a presentation. A simple theoretical model inspired by Toulmin (1958) is used. It includes a premise, a conclusion, and (for it to be explicit) an argumentation marker, where the premise and the conclusion are statements in the text. The argumentation marker connects the statements and is something tangible, such as a type of signal word (e.g., "because") or a symbol (e.g., an implication arrow). In the present study, we examine the variation of this type of explicit mathematical argumentation in Swedish textbooks. The research question is: What variation of explicit argumentation markers is there in secondary and tertiary mathematics textbooks? The study is carried out through explorative analyses of textbooks. We first identify instances of explicit argumentation according to the model presented above, and these are then categorized based on common characteristics among the specific instances found in the textbooks (i.e., using a bottom-up approach). Tentative results show that there are four types of argumentation markers: words, grammatical constructions, symbols, and layout. The words are of two kinds: those that in a dictionary refer to cause or reason as one of their main meanings (e.g., "therefore") and those that in particular contexts or in grammatical constructions can have such a meaning (e.g., "then", which have a temporal meaning but also in some contexts mean "as a necessary consequence"). The grammatical constructions found in Swedish do not exist in English, but are based on word order. The symbols found were mainly implication and equivalence arrows. The layout structures were typically solved example tasks, including words like "Solution:" and "Answer:".

These different types of argumentation markers show the variety in how argumentation is presented in mathematical discourse, and they will be used in future studies on students' comprehension of different types of argumentation.

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ENHANCING MIDDLE YEAR STUDENTS' ENGAGEMENT IN MATHEMATICS

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The middle years have been identified as a crucial period in education characterised by low engagement and underperformance in mathematics. Previous research indicates that students are 'switching off' mathematics from as early as grade 5 (Martin, Anderson, Bobis, Way & Vellar, 2012). This presentation reports findings of a year-long intervention study aimed at improving middle year students' engagement in mathematics. It addresses the research question: What impact did the intervention have on students' motivation and engagement in mathematics?

The study was informed by a multidimensional theoretical framework–the Motivation and Engagement Wheel (Martin, 2007). The Wheel comprises eleven cognitive, emotional and behavioural factors in the form of adaptive engagement (self-belief, learning focus, valuing, planning, task management and persistence) and maladaptive engagement (anxiety, failure avoidance, uncertain control, self-sabotage and disengagement). An additional factor of 'enjoyment' was also investigated.

Each factor was assessed pre- and post-intervention via the Motivation and Engagement Survey [MES], which was administered during class times to 339 students in grades 5 to 7 from 19 different classrooms from eight different Catholic schools located in a capital city of Australia. For each engagement factor measured, pre- and post-intervention mean scores were calculated and *t*-tests were used to detect significant differences at the grade and cohort levels. Comparison of student data with those from a similar cohort not involved in the intervention indicates that it is possible to reduce and, for some factors (e.g. self-sabotage), reverse the downward shift in student engagement levels in mathematics during the crucial middle years of schooling. We draw attention to factors, such as 'task management', that were particularly resistant to the generally positive effects of the intervention.

In light of this evidence we argue for increased support to teachers for the enhancement of their knowledge and understanding of student motivation and engagement.

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CONCEPTUAL DIFFICULTIES WITH HISTOGRAMS: A REVIEW

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For a first and quick analysis of statistical data, graphical representations such as histograms are widely used (Ben-Zvi & Garfield, 2004). The interpretation of data in histograms, however, is 'not as easy […] as it might seem' (Lem, Onghena, Verschaffel, & Van Dooren, 2014, p. 557). The goal of our review, therefore, is to create an overview of conceptual difficulties with histograms as found in the literature. The research question was: what are the conceptual difficulties with histograms? We define a histogram as a graphical representation with connected bars, one variable of interval or ratio level of measurement on the horizontal, and density or – in the case of equal bin width only – (relative) frequency on the vertical axis. The theoretical framework of big ideas in statistics (e.g., Ben-Zvi & Garfield, 2004) was refined and expanded (Boels, Bakker, Drijvers, & Van Dooren, submitted) and used to classify the conceptual difficulties that were found in the review study.

We searched several databases as for instance Google Scholar and PsycInfo with search terms such as histogram and mistake. In case of too many hits, keywords like MRI were used to exclude irrelevant studies. Over 800 studies were found. After removing doubles, and a check of title, abstract, or full text 53 studies remained.

Most reported difficulties relate to (1) an incorrect notion of what nominal, ordinal, interval and ratio data are, (2) confusing a bar graph and a histogram (e.g., Cooper & Shore, 2010), (3) the incorrect use of measures of centre, or (4) misinterpreting variability (e.g., Lem et al., 2014). Two big ideas in statistics play an important role in these conceptual difficulties: distribution and level of measurement. The results of this review will be used in an explorative eye-tracking study for a more focalised search for the causes of these conceptual difficulties.

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COGNITIVE INTERVIEWS IN TEST DEVELOPMENT

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Conducting cognitive interviews is regarded as a vital part of survey development. It is described by Campanelli (1997) as having a focus on understanding the respondent's interpretation of the language used, and on gaining an insight into their understanding. Interview questions may be open, whereby the respondent can answer at any length, or they may involve probing for a shorter closed response. In developing 60 multiple-choice items for a Year 8 student test, cognitive interviews were used (a) to identify possible errors in the items, (b) to see if respondents interpreted the items as intended by the author and (c) to check for consistency of item interpretation.

Following guidelines provided by Cyr, Dion, McDuff and Trotier-Sylvain (2012), interviews were designed with questions adopted and adapted from several research studies. The questions were mostly closed and each question related to a different item in the test. There were ten Year 9 students who volunteered to participate in the interviews and they were all from the same school. Of the 10 students, all were able to paraphrase the item, claimed that they knew what action was required for the item and described mathematical thinking in line with the demands of the item. At least eight students said the item was easy to understand and to decide which option to select as well as provided a sensible explanation for their choice of option.

There was good support for the items as written and the information provided in the interviews was used to improve the items. Edits included the addition of diagrams, further explanations of mathematical terms and the clarification of sentences. From the analysis of the students' responses, three conclusions were drawn (a) the students were often focussed on finding algorithms to assist with identifying the correct response, (b) students showed partial knowledge of proportional reasoning, and (c) the language used to answer interview questions was incorrect grammatically and mathematically.

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ANALOGICAL REASONING BY TENTH GRADERS IN DEFINING DIHEDRAL ANGLES

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Analogy is a kind of reasoning, which means to get new knowledge from prior knowledge in a similar situation. Three types of analogies have been used in mathematics educations: classical analogy, problem analogy, and pedagogical analogy. Among them, pedagogical analogy plays an important role in providing a concrete representation of abstract ideas (English, 2004). Defining concepts is an active process of mathematical learning. Literature review conducted by Kobiela (2012) has revealed that there were not many researches combining analogy and defining. The study aims to investigate how students define dihedral angles through pedagogical analogy. The first researcher designed a sequence of pedagogical tasks based on van Hiele-Geodof's 5 sequential phases of learning: inquiry/information, directed orientation, explication, free Orientation, and integration (as cited in Crowley, 1987). The tasks for defining have been conducted in a 10^{th}-grade class among 24 students. In task 1, students without prior knowledge of dihedral angles were encouraged to measure the angles between two planes. A Set-square triangle and B4 papers were offered. After small group discussion and group presentation, the whole class defined dihedral angles. In task 2, students conjectured the dihedral angle of a regular tetrahedron (Fig.1). Video-recordings, individual interviews, and students' worksheets were collected for further analysis. Students' concept image of dihedral angles fell into three categories. The first one is the formal definition of dihedral angles (Fig.2); the second one is an arbitrary angle between the two planes (Fig.3), and the third one is between the two. After whole-class discussion, students developed a formal concept of dihedral angles. Students' prior experience on measuring dihedral angle inhibited them from seeing the dihedral angle of a regular tetrahedron as a 60-degree angle.

Fig.1

Fig.2

Fig.3

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THE DEVELOPMENT OF TASKS AND RUBRICS FOR EVALUATING PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' PEDAGOGICAL REASONING

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While the domination of the cognitive perspective on teacher professionalism research, mainly investigating teacher knowledge and beliefs, researchers have increasingly emphasized the situated perspective as research methods through using video observation and discussion. Teachers' pedagogical reasoning is one construct reflecting the situated perspective. For mathematics teachers, it refers to the ability to draw on and integrate their knowledge and experience to reason about mathematical and pedagogical problems in complex and holistic real teaching situations (Cooney, 1994). Studies pertinent to teacher pedagogical reasoning are still scattered. This study aims to explore the possibility of developing appropriate tasks and rubrics to investigate prospective secondary mathematics teachers' (PSMTs) pedagogical reasoning.

A task involving a simulated situation about reasoning with parallelogram is created and structured into 3 phases, including PSMTs' understanding of the mathematics content (phase I) and that of students' cognitive behaviors relevant to the content (phase II), as well as what perspectives PSMT adopt to make their pedagogical plan (phase III). By the analysis of 16 Taiwanese PSMTs, the coding rubrics are developed and used to identify the features of PSMTs' pedagogical reasoning. For example, most PSMTs emphasized how the mathematics content is logically arranged but ignored the mathematical reasoning (e.g., induction, abduction, and deduction) that can be used to develop the concepts. Although PSMTs pointed out a number of reasons that can influence students learning, they rarely concerned students' interaction with tools or among peers. We also note that some PSMTs tended to plan the lessons by choosing and sequencing some specific mathematics problems without considering other teaching perspectives.

The development of the structural task and the corresponding rubrics to explore PSMTs' pedagogical reasoning goes beyond the existing studies which mainly focus on prospective teachers' knowledge and beliefs. Specifically, the developed rubrics can be used to identify the features of pedagogical reasoning that PSMTs may possess. Additionally, we argue that the task and rubrics can be easily generalized to other mathematics content.

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THE APPLICATION OF CLASSROOM OBSERVATION PROTOCOL TO FOSTER MATHEMATICS TEACHERS' DIFFERENTIATED INSTRUCTION

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Although Taiwanese students perform well in the international comparisons in mathematics, such as PISA and TIMSS, the learning achievement gap is substantial. Meeting students' learning needs and closing the learning achievement gap has been a critical issue in mathematics education. Taiwan is no exception. The next generation of math curriculum that will implement in 2018 in this nation emphasizes the importance of differentiated instruction (DI) as one of the means to address the issue. This study developed an observation protocol for DI to help mathematics teachers improve their teaching in the classroom and investigate its impact on teacher professional development (PD).

The theory of DI (Tomlinson, 2001) and the theory of formative assessment (Black & Wiliam, 2009) guided the design of the observation protocol. Formative assessment (FA), assessment for learning, is the core of DI. The conduct of FA contributes teachers to probe students' learning status and to take informed actions to meet students' learning needs. After the observation protocol was constructed, the researcher consulted mathematics educators and teachers to ensure its validity. The inter-rater reliability is .86 that is acceptable. The researcher used the observation protocol to observe elementary and secondary teachers' DI. After observing the mathematics teacher's DI, the researcher discussed the results with the participants and solicited feedback from them.

The teachers appreciated the suggestions provided by the researcher. They responded that the items of the protocol help them examine the quality of DI and inform them what actions to take to improve their DI. In short, the teachers showed a positive attitude toward the conduct of the protocol in the classroom. The result of the study not only contributes to the construction of the observation protocol for DI but also shows its potential for examining and improving the quality of implementing DI.

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A STUDY OF TENSIONS ENCOUNTERED IN DESIGN-BASED PROFESSIONAL DEVELOPMENT

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Design-based professional development (PD) can be treated as learning community where mathematics teacher educator-researchers (MTE-Rs) and mathematics teachers (MTs) are designers involved in the process of creating PD activities and classroom teaching activities respectively. Lin, Hsu, and Chen (2017) emphasized MTs in such environment can learn from enacting the designed activities with students and revising the activities accordingly. They also pointed out that educative tensions encountered in design-based PD can be the turning points triggering the learning of MTE-Rs. This study further reports what educative tensions MTE-Rs may encountered in a design-based PD; specifically in Just-Do-Math program which aims to enhance students' learning power and positive disposition. MTE-Rs have to arrange Just-Do-Math PD programs for facilitating MTs in designing activities accordingly. The educative tensions are described in line with phases of MTs' hypothetical learning trajectory. First, in participating phase for stimulating MTs' need and motivation, we noted that MTs may think Just-do-math activities are merely manipulative games without noticing pivotal characteristics of the activities (e.g., essence of mathematics). Second, during conceptualizing phase for enhancing MTs' theoretical concepts, some MTs may think Just-Do-Math activities are not efficient for student learning as they did not notice key learning opportunities embedded in the activities (e.g., mathematics sense-making). Third, during designing phase for guiding MTs' design thinking by theoretical framework and examples, some MTs designed mathematical games just for students to drill specific procedures willingly. Those MTs did not sense Just-Do-Math activities can enhance students' understanding and high-level ability. Fourth, during testing phase for promoting MTs' practice ability by classroom experiment, some MTs just observed the final result of experiment without noticing to the cognitive process of the activities (e.g., generalization or internalization). Finally, during revising phase for improving MTs' reflective ability by reviewing on students' learning, MTs may expect others to accept their design without considering if it meets the criteria set up by MTE-Rs. We argue that those educative tensions occurred in design-based PD can be the turning points to trigger MTE-Rs' professional growth if they can notice the educative tensions and try to come up with strategies to overcome the tensions.

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AN EXPLORATION OF MATHEMATICAL MODELING IN GEOMETRY TEACHING WITH TECHNOLOGY

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Using computing and internet technology has become a trend in education. In this paper we report a study about a teaching method which incorporates technology, art and flipped classroom under the concept of mathematical modelling. The teaching experiment is done by the two authors. The experiment is designed and modified from the framework proposed in Ang (2001), and the teaching process is shown as in Fig. 1.

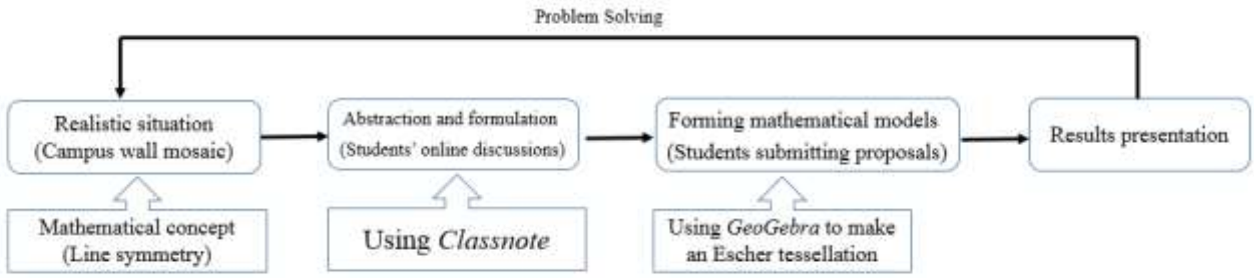

Fig.1: Teaching Experiment

The experiment is done in a span of seven weeks, with one hour of mathematics class time each week. As shown in Fig. 1, in this experiment, the mathematical topic to be learned is line symmetry. 84 seventh-graders are given the task of decorating school walls. They are then separated into smaller groups of four to design the graphs on the walls. The students have explored Escher-style tessellation concepts previously, and they are taught about *GeoGebra* and *Classnote* in the current experiment. Students can use *Classnote* to engage in online discussions, and use *GeoGebra* to design Escher-style tessellations with the concepts of line symmetry. They have to present their designs in the final week. From students' reflections, we can see that students have deeper and more concrete conceptions of line symmetry, and they learn to think mathematically about real-world situations and problems.

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SUFFICIENCY TO AVOID FALLING INTO A PROCEDURAL TRAP: EVALUATING THE KNOWLEDGE COMPETENCY OF PRESERVICE MATHEMATICS TEACHERS IN HONG KONG

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In examining preservice mathematics teachers' (PST) subject matter knowledge and their pedagogical content knowledge (Hill et al., 2008), procedural trap is discovered as students might fall into it when procedures are taught without understanding.

This report describes parts of a larger project in which Hong Kong PSTs participated. The last part of the study was to interviews PSTs commenting on a video lesson with 8 PSTs on topic of "square roots" and 8 PSTs on the topic of "simultaneous equations".

No PSTs could identify what was going wrong $\sqrt{(-4)^2} = (-4)^{2\times\frac{1}{2}} = (-4)^1 = -4$ and no PSTs could provide a mathematical reason to explain why this method did not work. They aimed at getting a correct procedure in getting the simplest correct answer. Student falls in the *procedural trap* that $\sqrt{(-4)^2} = -4$ if they just learn the simplification of expression procedurally and they do not even notice that they have fallen into this *procedural trap*.

Most PSTs did not see there might be extraneous solutions from the working that $a + b - 20 = 0$ and $a - b - 8= 0$ leading to $a + b - 20 = a - b - 8$. Only two PSTs considered this method not necessarily correct if the coefficients were changed. PSTs' inabilities to give examples of wrong procedures might lead to a potential risk that wrong procedures might be rooted in students' mind. They fall into a *procedural trap* as they believe that the procedures can be used to solve *all* simultaneous equations.

This paper claims *procedural trap* in students' learning as undesirable learning of procedures as they work perfectly fine and lead to a correct solution in some cases, but fail to work in others. Students probably cannot notice the trap by themselves when they practise procedures in learning mathematics. They thus need to rely on teachers' mediation of mathematics knowledge. Most PSTs in this study could not explain the crucial mathematical knowledge, but they tend to train students to obtain answers to mathematics questions by applying certain procedures by rote. Without understanding mathematical knowledge attached, students might simply memorize the procedures and fall into the *procedural trap.* This obviously cannot guarantee correct solutions to *all* mathematics questions. Surprisingly, this study found that some PSTs themselves had fallen into these procedural traps.

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TEACHERS' GROWTH IN DEVELOPING MATHEMATICS READING ACTIVITY FOR CLASSROOM TEACHING

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"How to promote more student engagement in classroom learning activity" and "How to make students get higher achievement" are the core challenges to a teacher's math teaching. This study will report how a group of elementary school teachers to develop reading activity by use of the cheapest resource, say the math textbook, to enhance the quality of their mathematics instruction. The school located in lower socioeconomic conditions countryside. Their students are under average level in the Assessment of Learning Achievements (ALA) conducted by National Academy for Educational Research. Their efforts can be seen as cycles of "thought experiment" and "teaching experiment" (Gravemeijer, 2001). The thought experiment is derived by the intention of "what wanted" in classroom teaching and the teaching experiment is the set of "what happened" in their classroom. The gap between what wanted and what happened then initiate reflection on their intention and practice, also initiate adapted action to solve it. Teachers' growing path can be classified into three stages according to their goal of students' comprehension level (Stacey, 2012) in reading tasks. In the beginning is *idealized*. They design reading task with complex instruction and expect the students can understand all material including reflection level of comprehension. Then change dramatically to stage of *surface*. The task asks only for the reproduction level of comprehension. Then change gradually to the stage *committed*. In this stage, the reading task asks for reproducing key information of learning material and connecting concept and procedure to applicable situations. It constructs a friendly math classroom where the key information is known and the math content is operable in a familiar and concrete situation. The effective reading tasks are spread to whole school. After two years of school-wide implementation, the "outsider" students reduce from 22% to 5%, and students' ALA level improves to above average. The story of these teachers also shows that the growth of developing math reading activity is an organic coordinating process. The important elements includes teachers' practice reflection based on the gap between what wanted and what happened, understanding of students' textual processing preference , mastering the reading material, and appropriate intervention of theoretical enlightenment by external expert such as the researcher of math reading.

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GENERATION OF PRINCIPLES FOR DESIGNING A PROFESSIONAL DEVELOPMENT INTERVENTION FOR MATHEMATICAL PROBLEM SOLVING PEDAGOGY

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Within the educational design based research paradigm (McKenney & Reeves, 2012), this study designed and evaluated a professional development (PD) intervention for grade 9 South African teachers' mathematical problem solving pedagogy. The South African Department of Education has recently started considering problem solving as an essential aspect of mathematical teaching and learning. The PD intervention was designed to support Grade 9 South African teachers in the teaching of problem solving. Using Guskey's (2000) Five Levels of Professional Development Evaluation, the study evaluated the impact of the PD intervention on participant teachers' mathematical problem solving pedagogy and participant learners' problem solving processes and achievement in mathematics. The key objective of the research study was to evaluate the effectiveness of the design of the PD intervention and to generate design principles that can be used by other researchers developing PD interventions for mathematical problem solving pedagogy in a particular context.

By means of a mixed methods research approach, data was collected from two grade 9 mathematics teachers and 115 learners. The intervention took place once a week for 6 months. Teacher data was collected through classroom observations and semi-structured reflective interviews. Learner data was collected through learner task-based interviews, pre- and post- mathematics attainment tests and a self-reporting mathematical problem solving skills questionnaire. Qualitative data was analysed using grounded theory techniques. The statistical software package SPSS was used to analyse the learner quantitative data. The findings revealed that the intervention had a positive impact on participant teachers' mathematical problem solving pedagogy and participant learners' problem solving processes and achievement in mathematics.

The design principles that emerged from the study are that: mathematics teachers should be used as a resource and must actively participate in the implementation of a PD intervention; PD activities should be built from teachers' experiences and their current mathematical problem solving pedagogy; teachers should be physically supported in their classrooms when they implement the new ideas; and a PD intervention should be responsive to multi-lingual needs of a particular context.

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A STUDY ON TAIWANESE STUDENTS' AFFECTS ABOUT INTEGRATING THE CALCULATOR IN MATHEMATICS CLASS

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Many Western countries have integrated calculator use into their school mathematics curricula, and studies have shown that it helps cultivate students' positive affects (Hembree & Dessart, 1992). Taiwan government regulated calculator integration with mathematics curricula since 2018, but there is almost no research exploring the effects of using calculators in Taiwanese mathematics classroom. The aim of this research is to empirically explore the opinions of vocational high school students about the use of calculators in mathematics classroom before and after a calculator-integrated teaching experiment.

The teaching experiment involved two classes of 12th-graders learning the topic of "sampling distribution." The treatment group (37 students) was given tasks that required the use of calculators; for example, to calculate the means of random samples of 25 heights for eight times repeatedly. The control group (33 students) was taught in the traditional way without the use of calculators. Data was collected with pre-test and post-test questionnaires, including questions: "What are the pros and cons of using calculators in class?" and "Do you approve the use of calculators in class or tests?" ... etc. Besides comparing the results between the treatment and control groups, this study examined the differences between the higher and lower achieving students.

In the high demanding Taiwanese mathematics classroom, we expected the higher achieving students would show more unwillingness to spend time with calculators than their lower counterparts, since they wanted to earn good grades in tests. The results of this study confirmed our expectation. However, after the teaching experiment, 23% of the high achieving opponents of calculator use in class switched their opinions, while the control group remained the same. Another interesting finding is that although the treatment group students must have felt the calculator's indispensable role in completing their works, only 26% of them approved to be allowed to use calculators in tests; they specified their reasons to causing dependence on the calculator and stunting growth of their mathematical competences. Some even mentioned that the essence of mathematics was calculation. The academic levels of the students in this experiment were relatively low in Taiwan, and most of them still hesitated to rely on the calculator. This phenomenon shows the hardworking and conservative characteristics of East Asian students in learning.

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ONLINE PEER ASSESSMENT IN NUMBER THEORY COURSE: ITS RELIABILITY AND VALIDITY

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This study investigates applicability of online peer assessment to improve students' *proving*. Among many issues of peer assessment, there is a debate on reliability and validity of peer assessment because one may argue about the students' competence to score their peers (Hanrahan & Issacs, 2001). We explored the reliability and the validity of the peer assessment on proofs via online peer assessment system.

In this study, participants were twenty-five undergraduate students who took Number Theory course focusing on rigorous proofs. Each week, students uploaded their proofs of four theorems and then evaluated their classmates' proofs via *Classprep*, which is an online peer assessment system. In the *Classprep*, students were required to evaluate in terms of logical reasoning, clarity, and novelty of the idea. Peer assessors were matched in random and the names were blinded. Once the evaluation was done, students gave feedback to the scorers. After the semester was over, we conducted a 5-points Likert scale survey on the online peer assessment activities.

To explore the reliability of the peer assessment scores, one assignment from the first half and another assignment from the second half of the semester were chosen randomly, and then interrater agreement was investigated. We found that the Cronbach's alpha coefficients of the scores of each proof were high (between 0.989 and 1.000). In the middle of the semester, furthermore, the students were requested to evaluate 5 different proofs of a theorem with the same criteria they used. As a result, the interclass reliabilities on logical reasoning, clarity and novelty are 0.943, 0.915 and 0.981. Therefore, we found that the interclass reliability of peer assessment score is high. To investigate the validity of peer assessment, we conducted a survey and then analysed the results. When the students were asked about difficulties of the peer assessment, they responded it was more difficult to evaluate clarity and novelty compared to logical reasoning. They explained that their subjectivity may affect assessing clarity and novelty. However, students judged the three criteria of the assessment are valid (mean = 3.8125). This result shows that online peer assessment with three criteria can be applied to an undergraduate mathematics course.

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TAIWANESE SECONDARY MATHEMATICS PRESERVICE TEACHERS' NOTICING ON TEACHING METHODS

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Noticing is well accepted as a critical criterion for professional teaching performance, while teaching methods are the core competence to tackle with the complicated teaching situation. This study adopted the framework of noticing raised by Sherin et al. (2011), which comprised three phases of attending, interpreting and deciding to explore the features of Taiwanese secondary mathematics preservice teachers' noticing on teaching methods.

The research questionnaire was developed based on the real teaching segments which were identified by the researchers as the segments requiring teaching noticing. The teaching segments were used to form the vignette-items, thus the items presented the real mathematics instruction situations occurred in Taiwan. Ten teaching vignettes with 22 multiple-choice and open-ended items were developed and administered to 30 secondary mathematics preservice teachers (PT). Content analysis was employed to analyse PTs' responses to open-ended questions.

The initial analysis reported in the present paper is on the vignette in which a teacher (Mr. Han) initially introduced the concepts of functions to 7th graders. The findings included: (1) When the PTs were initiated by the stem directly requiring them to notice teaching methods, 24% of them only provided big words regarding teaching method such as "didactic teaching", and the others went to the details such as "asking students to provide examples [for the whole class] to judge whether they are functions". The phenomenon may relate to the PTs' different competence of attending to the details in instruction or to the PTs' concepts of what teaching methods mean. (2) The vignette described five steps of Mr. Han's teaching. He basically employed the combination of the didactic method and the dialogic method. A total of 60% PTs can notice that Mr. Han used the didactic method, while only 40% PTs attended to the dialogic method, even though the vignette presented that Mr. Han asked three questions in the five briefly written steps. (3) Almost all PTs (96%) decided to apply a different teaching method from that of Mr. Han. A total of 68% PTs' change was about the mathematics content presented to the students. Most of them decided to use the numeric real-life examples different from Mr. Han's as generic examples. (4) The PTs' interpretation revealed that 92% of them considered the didactic method was an acceptable approach in mathematics instruction in Taiwan where mathematics curriculum at the high school level is demanding and instruction pace is speedy.

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NEW FRONTIERS OF ENACTIVE EDUCATION: SCRUM METHODOLOGY AS A WAY OF OVERCOMING MATHOPHOBIA

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IS Leonardo Da Vinci, Poggiomarino, University of Salerno

When students, especially in secondary schools, sigh and say "Math will never be my job" or adult people feel funny and fashionable when they tell stories about their clumsiness in carrying out even the most ridiculous calculation, Mathophobia is probably the only cause. This fear, affecting millions of students in all over the world, really ranks higher than the most common phobias. According to recent researches at the University of Chicago, the anxiety associated to Math can prompt a response in the brain similar to the experiences of physical pain. It has repeatedly shown that the hatred of Math negatively affects students' achievements in sciences. But, luckily, Mathophobia is fixable: it is more psychological than logical, likely induced by environment and bad teaching practices. Our experiment is designed on the base of the enactivism, educational model based on the concept that natural cognitive systems take part in the generation of meaning, building knowledge by sensorimotor and not only informational interactions. In this framework, we conceived the idea that SCRUM methodology could help Scientific High School students experiencing recovery necessities in Math. This collaborative methodology, borrowed by the management of software projects, requires a self-organizing small team work, where students are dynamically involved in activities focusing on maximizing the team's ability to respond quickly to emerging requirements. The results were so encouraging that the test is going to be repeated on a wider number of classes and on different subjects.

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TURKISH SEVENTH GRADE STUDENTS' ENGAGEMENT WITH A MODELLING TASK: LAWNMOWER PROBLEM

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Several researchers focused on the use of mathematical modelling problems for primary and middle-grade students. They indicated that young students can also successfully engage in the modelling problems, and modelling problems offer opportunities for them to experience complex data within a challenging but meaningful context (e.g., English & Watters, 2005). This study was conducted as a part of a larger study, where it was aimed to examine Turkish seventh grade students' modelling processes for four modelling activities. The data reported here were drawn from the modelling problem called as "The Lawnmower Problem" (English, 2003) which was the third implemented problem of the study. This problem involved dealing with some tables of data and exploring the relationship among data (English, 2003). The aim of this study was to investigate particularly mathematization processes of two groups of students as they worked on modelling problem. Before implementation, the modelling problem was translated into Turkish by the authors. The participants of this study were six seventh grade students. The students worked in groups of two during two 45 minute sessions that all sessions were videotaped. After the implementation, the second author of this study implemented the modelling problem(s) and conducted semi-structured interviews with both groups in order to understand students' modelling processes deeply. The data sources for this study were videotaped students' solutions to the problem, students' written reports and their audiotaped interviews. For the data analysis, all of the transcribed data were examined by authors with respect to students' mathematization processes used while developing their models. The findings of the study revealed that although the differences observed in two groups' mathematization processes, both groups used "scoring" as an approach. Both groups scored the data in the tables by assuming that each given table has equal value. The score range of both groups has changed from 1 to 10 because the data presented in tables were for ten people. The mathematical operations used by both group were addition and ranking.

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This study is related to master's thesis of second author.

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FEATURES OF ADOLESCENTS' ENGAGEMENT WITH MATHEMATICS IN THE CLASSROOM AND IN EVERYDAY LIFE

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Engagement is an important construct to explore in the context of school learning, as it is malleable and can influence academic achievement (Fredricks, Blumenfeld, & Paris, 2004). Prior research on hybrid spaces has explored how students' experiences in different socially and culturally constituted worlds can influence their engagement in a given setting (e.g. Gutiérrez, Baquedano-López, & Tejeda, 1999). The present research examines the relationship between students' engagement with classroom mathematics and mathematics in everyday activities. By comparing norms and interactions in each setting, I aim to enhance our understanding of the tensions students might experience as they navigate their engagement with mathematics across different spaces.

I explore this topic with five classes of middle school students (ages 11-13). In addition to conducting 60 mathematics classroom observations, a survey was administered to identify activities students engage in that involve mathematics, and nine interviews were conducted to explore participation in those activities. Interview transcripts and observation fieldnotes were coded to identify key features of student engagement.

Looking across student reports of engagement in seven everyday activities (e.g. cooking, gardening, and building), four key features of students' experiences emerged: 1) Learning typically happened through seeking guidance from experienced others. 2) Students participated because they wanted to help others – especially family members. 3) Norms of using estimation and trial-and-error often existed. 4) Students enjoyed the opportunity to be creative and express themselves when using mathematics in everyday activities. In contrast, engagement with mathematics in the classroom involved a combination of collaborating with others and working independently, limited opportunity for self-expression, and norms of precision, exactness, and quiet.

Given these differences, I propose that students might experience a conflict between engaging with mathematics in everyday activities and engaging with classroom mathematics, and they might come to view classroom mathematics practices as inauthentic. Specific examples of students' engagement in both settings and potential implications of this conflict will be discussed during the presentation.

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TOOLS. MEASURING. ANGLES?

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The angle concept is a fundamental concept of plane geometry, relevant not only for teaching of geometry, but also in everyday situations and in different careers. Nevertheless, research shows that middle and high school students possess diverse and fragmented understanding of the angle concept, and exhibit difficulties in measuring angles. For instance, Dohrmann and Kuzle (2014, 2015) have shown that many students developed a sustaining misconception of angle as the area bounded by two rays and an arc, whose measurement can be obtained by measuring the *distance* between the rays using the length scale on the set square. Thus, some students have the idea of *distance measurement* in their mind as a way to measure *angles*. The set square itself does not force – in the sense of instrumentation (Verillon & Rabardel 1995) – one to measure an angle intuitively and correctly. Rather, its design is justified by requirements that are related to efficiency and the proper use has to be learned explicitely. So the construction/design and the handling of the angle measuring tool »set square« impacts the students conceptual understanding of angle negatively.

Taking these results into consideration, the current work focuses on the idea of angle measurement, the activity of measuring angles and the role of angular measurement tools to support a deeper understanding of the underlying ideas related to the mathematical object angle and its concept. More concretely, we claim that the use of an alternative angular measurement tool that highlights the connections between the handling of the tool and a deeper understanding of the measuring object may support the development of conceptual understanding of angles.

Starting from the dynamic concept of "angle as rotation" we designed a new tool that avoids any measurement of lengths and thus encourages students to develop a conceptual understanding that is better connected to the idea of angle. This tool is presented and discussed in the context of concept development.

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NAMING THE FEELING: LEARNING TO RIDE ONE'S EMOTIONS DURING PROBLEM SOLVING

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Emotional experience is a key factor in shaping learners' self-belief of what they are capable of in mathematics. Emotions can be primary motivating forces, to the extent of disabling cognitive functioning, especially when an individual is not able to understand or interpret emotional experiences beyond the sensations themselves. In our mathematics club, we observed that learners' lack of descriptive awareness of their emotions vitiated our use of standard questions about identity and emotional states. We hypothesised that verbal expression or low levels of emotional intelligence were barriers to learners' emotional awareness, and we intervened with an "emotions vocabulary" reference tool adapted from Plutchik's (Plutchik, 2001) 'Diagram of Emotions'. The researchers' field notes were compared to the 17 learners' written self-reports when prompted during tasks, and with learners' reflections during semi-structured interviews on their affective experiences and changes in affective states during problem solving. Before the emotions vocabulary tool was introduced, learners acted out emotionally, creating disruptions and moving around to engage socially, off-task. On probing, learners haphazardly reported contrasting emotions, copying expressions from peers, while more than half of the group reported sensations like hunger and tiredness as prevailing emotions. Contrary to our observations of boisterousness or emotional agitation, "sad" was the emotion noted most often by the group. On the introduction of the tool learners readily used more appropriate emotion words and explanatory phrases, and reported a wider array of emotions, with less comparison to what peers wrote. In subsequent sessions we observed more on-task behaviour and by their self-reports, most learners were increasingly able to "ride out" the negative emotions brought on by problem-solving, and reported positive and self-empowering emotions (such as pride) at the end of the sessions. The findings of this study suggest that access to "emotion names" and phrases that explain emotions seem to aid in perseverance in problem-solving, and seems to help learners to "ride out" initial negative emotions. Further research is necessary to investigate if facility in expression of emotions during problem-solving leads to enhanced cognitive-emotional control.

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ANALOGICAL REASONING IN COLLABORATIVE PROBLEM-SOLVING MATHEMATICS CLASSROOMS

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The most recent version of the National Curriculum for mathematics in England and Wales (DfE 2014) promotes reasoning and problem-solving as two of the three foci for mathematical learning. While there are clear connections between these two foci, which support system and property correspondences for analogical reasoning, the nature of reasoning valued in the content of the curriculum is inductive and deductive, rather than analogical. Much of the research on analogical reasoning in mathematics with school age students reported and reviewed to date examines the interactions between teachers and students (for example, English 2004, Richland, Holyoak and Stigler 2004) or uses experimental and control settings. Little evidence is provided for analogical reasoning which develops as a process and outcome of peer interactions when students regularly solve problems in collaborative small groups in naturalistic classroom settings, allowing students opportunities for consistent comparisons.

For the purposes of this study, I focus on analogical reasoning as relational reasoning, utilising analogical transfer and inference, leading to generalisation and schema development (Holyoak 2012). The study uses evidence from a wider study of the nature of students' talk while solving mathematical problems collaboratively in small groups. Data were collected using audio-recordings of students aged 11-14, whose normal mathematical learning occurred through open-ended problem-solving tasks over three to six lessons, scaffolded by the teacher through questioning and other more closed content-based tasks. I provide evidence of students regularly drawing on knowledge and learning from prior problem-solving experiences in order to develop their own mathematical schemas. I also draw on Simon's (1994: 6) ideas of "transformational reasoning as a way of thinking".

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REROUTING TEACHING OF MATHEMATICS FOR A SPECIAL-NEED STUDENT: A CASE STUDY IN WORD PROBLEM-SOLVING

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The concept of "Learning activity," pronounced *uchebnaya deyatelnost*, in Russian, has morphed into many forms since its inception by Russian psychologists about years ago. The Russian *deyatelnost* references so much more than the meaning pressed now in Western pedagogies. Among other things, it references actions carried out by professionals who not only know how but also why to carry out one action and not another. Cole (2009) discussed the implications of perceptions of translated terms that share little resemblance with the original meaning. In the context of teaching number sense to children, Davydov (1982)—who uses *deyatelnost* as an entry point to teaching elementary mathematics through a relational paradigm (Polotskaia, 2015)—explains that only when a child can carry out additive and multiplicative relationships, can we claim that the child understands "quantitative relationship with and between objects" (p. 227). Current pedagogies, however, are often embedded in the operational paradigm (Polotskaia, 2015), and build on semantics-based identification of operations disregarding quantitative relationship between objects.

In light of this distinction, special tasks and tools were developed to hone students' work within the relational paradigm (Polotskaia, 2015). The current report focuses on work done with a Grade 4 student with dyslexia and ADD, who, by self-admission and the teacher's report, has great difficulty to read sentences and two-digit numbers.

The entry point of the project was to employ Davydov's recommendations of working on quantitative relationships between objects through word problems having additive structures with no semantic guidance in regard to the operation to be used. A pre- and post-test design with three months between the two times was used. Results show that the child could engage in meaningful mathematical discussions and appropriate the concept of number by identifying relationship between quantities.

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MATHEMATICS ENROLMENTS: SINGLE-SEX AND CO-ED

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Single-sex schools flourish in a number of countries, including Australia (OECD, 2009) and have grown in popularity in others, for example, the United States (Pahlke & Hyde, 2016). Whether a single-sex or mixed (co-educational) school setting affects mathematics learning has generated much research, most often explorations of achievement outcomes. Participation rates in post compulsory mathematics courses, important determinants for entry into STEM-related tertiary studies and careers, have received less attention. Previous inconsistent research findings are attributed to various factors, including the paucity of evidence based studies. More research is clearly needed. We report enrolment data for grade 12 mathematics subjects gathered over 15 years in Victoria, Australia, a site with sufficient data for credible analyses.

Three mathematics subjects are offered at the grade 12 level: Advanced (A), Intermediate (I), and Elementary (E). Enrolments in these subjects were examined by school type and gender: single-sex boys (SSB), single-sex girls (SSG), co-education boys (CB), and co-education girls (CG). To enable comparisons, percentages of enrolments by school type and gender were calculated. Enrolment patterns revealed:

- For A, boys' enrolments consistently exceeded girls'. SSB had the highest enrolment (15.2% in 2015), followed by CB, then SSG, with CG having the lowest enrolment (4.8% in 2015). For all groups, enrolments initially decreased over time but since 2012 have shown a small annual increase.
- For I, the pattern of enrolment was SSB (48.3% in 2015), SSG, closely followed by CB, and then CG (21.6% in 2015). There have been minor fluctuations in enrolment since 2008, for all groups.
- For E, there has been a steady increase in enrolments over time, for all groups. There were only minor differences in the percentage of males and females enrolled, irrespective of school type.

In summary, it could be argued that a greater percentage of students in single-sex than in co-educational schools are engaged in mathematics (subject I), or that a higher percentage of boys than girls enrol in mathematics (subject A), or that school type has little effect on participation in mathematics (subject E). Clearly, factors other than school type alone, or student gender, influence mathematics enrolment numbers.

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IDENTIFICATION AND CHARACTERIZATION OF THE SUB-LEVELS OF DEVELOPMENT OF DERIVATIVE SCHEMA: AN EXPLORATORY APPROACH USING CLUSTER ANALYSIS

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The results of research investigations related to the understanding of the concept of derivative show that, despite being an indispensable concept, it is a very complex to understand, this is evident to observe that a significant number of university students are only achieve a partial understanding of this concept (Baker, Cooley & Trigueros, 2000; Cooley, Trigueros & Baker, 2007; García, Llinares & Sánchez-Matamoros, 2011). Despite the fact that this problem is not new, it still represents a major challenge for mathematics education at the university level, and is a constant concern for institutions of higher learning.

In this research, we present an exploratory analysis which aims to identify and characterize the sub-levels of development of the derivative schema attained by university students with prior instruction in differential calculus. To accomplish this, we distributed a questionnaire to 120 students. The questionnaire was composed of three tasks in different representation modes, requiring the use of the structuring mathematical elements related to this concept. Considering these mathematical elements, we define 32 variables that were quantified by their presence or absence in the solution of the tasks, which enabled us to discretize each of the solution protocols, thus obtaining a vector associated with each questionnaire. With these vectors, we perform a cluster analysis that allowed us to identify and characterize the sub-levels of development associated with each level of development of the derivative schema.

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A STUDY OF CLASSROOM ASSESSMENT STRATEGIES IN MATHEMATICS

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The Ministry of Education (MOE), Singapore, has been pushing rigorously for individual Holistic Assessments for students in Singapore's primary and secondary schools since 2008 to further improve students' learning and accomplishments. While Ratnam-Lim and Tan (2015) pointed out that the success in large-scale implementations of formative assessments practices largely depends on the implementation of such assessment methods in practice as well as the teachers' perceptions of such assessment methods, the students' assessment preferences are also a determinative factor that would shape their learning of mathematics. This study seeks to (i) fill a gap in research on students' preferred assessment strategies in the primary and secondary mathematics classrooms, and (ii) identify assessments strategies in mathematics. The study was conducted in two primary and two secondary mathematics classrooms in Singapore. A case study approach was used to identify the assessment strategies used by four mathematics teachers. Four one-hour mathematics lessons for each mathematics teacher were observed over three weeks. The teachers were interviewed immediately after the classroom observation for the researchers to gain an in-depth understanding of the teachers' perceptions of the assessment strategies identified. Both the classroom observations and interviews were audio-recorded. Interview questions included: What are the assessment strategies used in the mathematics lesson? A questionnaire, designed by the researchers, was implemented with 73 students to investigate the students' preferred assessment strategies. Results of the study show that the teacher-participants in this study used a range of assessment strategies in their mathematics instruction. Peer-assessment and self-assessment were chosen by the most number of student-participants in the primary school. Teacher-questioning and the use of performance assessment (hands-on tasks such as the use of manipulatives) were chosen by the most number of student-participants in the secondary school.

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MATHEMATICAL IDENTITIES AND LEARNING IN SECONDARY SCHOOL CLASSROOMS

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Investigating learning from a social perspective involves understanding relationships between identities and learning. The way learners learn and see themselves as learners of mathematics changes depending on how they experience different relationships, with people and mathematics, in their classrooms. With the aid of narratives, we examine the different elements identified by learners as influencing their identity and learning, and how these changed as they progress from grade 9 to 10. We used narratives as an analytical tool to access learners' past, present and designated identities (Sfard & Prusak, 2005). Unlike Sfard and Prusak (2005), we define narrative as being a constituent of identity rather than being identity itself. This research contributes to the emerging work on how learners' mathematical identities influence their learning of mathematics (Darragh, 2016).

The sample was 14 learners and their two grade 9 and three grade 10 teachers. Data was collected over a two year period in the form of videotaped lessons and audiotaped semi-structured interviews and was analysed qualitatively. For this presentation, we focus on the findings from the interviews with the learners.

We found that there were six elements that influenced learners' mathematical identities: the teacher, experiences of marks, agency, family support, the transition between grade 9 and 10 and learners' future projections. We argue that positive experiences of the identified elements contributed to the development of mathematical identities. The most salient element identified by learners was the teacher, in terms of the teacher's pedagogy and the social relationships in the classroom. Learners claimed that they were able to learn better when they had good pedagogical experiences and social relationships in the classroom. As learners progressed from grade 9 to 10, their experiences of learning mathematics changed due to different teachers and many of them were unable to see themselves as successful learners any longer. As a result, some learners gave up, no longer exercised agency, performed poorly and did not want to pursue mathematics in the future.

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EARLY MATHEMATICS EDUCATION – A COMPARISON BETWEEN GERMANY, TAIWAN AND SWITZERLAND

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Results of international comparison studies (e.g., PISA 2015) show substantially better mathematics performance of Asian students than European students. Even in kindergarten, Asian children outperformed European children in their early numeracy performance (Aunio et al., 2008). Possible reasons for the observed differences comprise cultural-societal factors, teacher education or learning practice (Huntsinger et al., 1997). The importance of early childhood education for further learning is recognized. Therefore, it might be reasonable to study early mathematics education trying to explain differences in students' outcome. Early childhood education in Germany and Taiwan is situated in very different institutional frame conditions. Thus, we focused on a sound analysis of similarities and differences of early mathematics education regarding 1) the educational system, 2) pre-service education, 3) legal requirements as a necessary condition for any further comparative research.
As part of the Research Program TaiGer (funded by DFG, Germany, and Ministry of Science and Technology, R.O.C., Taiwan), national, official documents were analysed using content analysis methods. To contrast, we included Switzerland as a third country, because of the cultural similarity to Germany but a completely different early educational system. We found cultural and structural differences. The educational system shows similarities between Taiwan and Switzerland e.g., kindergarten as part of the school system, but also between Germany and Taiwan. Pre-service education is completely different in Germany compared to Switzerland and Taiwan and legal requirements concerning the content of early mathematics education show similarities between Taiwan and Switzerland but as well between Germany and Switzerland. The results of this study will serve as a framing for comparative studies, e.g. on professional knowledge and skills of kindergarten teachers in the three countries to get deeper insights that help to explain the differences in students' outcome of mathematical learning in different countries.

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DROPOUT & PERSISTENCE IN UNIVERSITY MATHEMATICS

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University dropout in the STEM-subjects and especially in mathematics is still a big issue not only for German universities. In the USA the dropout and subject-change quotas in mathematics add to nearly 80% (Chen, 2013). Most models describe dropout as a complex process influenced by many conditions, such as socioeconomic background, motives to choose a certain subject, motivational and cognitive factors and social or academic integration (Heublein et al. 2009; Tinto, 1975). Most German students who quit their studies of mathematics name the requirements (33%) and less motivation (25%) for their decision to leave (Heublein et al. 2009).

In this explorative case study, three first year students (John, Tom and Anna) report by semi-structured interviews about their experiences in their studies of mathematics. All of them went to the same secondary school and were successfully within the linear algebra lecture during the first semester but failed calculus. John chose to drop out from mathematics after the first semester, whereas Tom and Anna continued their studies. In this case study, I was interested in what kind of explanations these students give to their decision to drop out or stay.

It seems, that the combination of a low feeling of social relatedness and competence and the combination of intrinsic motives to choose mathematics, which collide with a change in the character of mathematics, led to John’s dropout. His achievements during the first semester were without meaning for his decision. Tom seriously thought about dropping out. He felt a difference between his beliefs concerning the nature of mathematics and the university mathematics, but his stable job-wish motivated him to carry on. Furthermore he felt at least some competence and social relatedness. Anna felt much social relatedness and didn’t think about dropping out.

The link between beliefs concerning the nature of mathematics and the motives to choose mathematics seems to play an interesting role for the decision to drop out or stay. Further research will now focus on quantifying the impact of motivational aspects on success and drop out in mathematics, especially the link described above.

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THE USE OF INTEGRALS IN MECHANICS OF MATERIALS FOR ENGINEERING: THE FIRST MOMENT OF AN AREA

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Research has reported on the difficulties engineering students face relating the content of their mathematics courses to what is taught in their professional courses. Therefore, it is important to better understand how mathematical notions are used in professional engineering courses (e.g. González-Martín & Hernandes Gomes, 2017). Here we analyse how the notion of first moment of an area – which is defined as an integral – is used in civil engineering courses. The first moment of an area is used to calculate the centroid of an area and the shearing stresses in transverse bending. The centroid of an area A is its geometrical barycentre and is the point C of coordinates $\bar{x}$ and $\bar{y}$ such that the following relationships hold: $\int_A x dA = A\bar{x}$ and $\int_A y dA = A\bar{y}$.

We focus on two different tasks that use first moments, presented in a classic mechanics of materials textbook (Beer, Johnston, DeWolf, & Mazurek, 2012), and we base our analysis on elements from the Anthropological Theory of the Didactic. Specifically, we consider Castela's work (2016), who proposes a model to study the phenomena of how practices change from one institution to another. In our case, the practice is the use of integrals in Calculus courses, which are then employed as a tool in professional engineering courses.

Our findings indicate that although first moments are introduced as an integral, the textbook's tasks do not require students to use techniques typically encountered in a Calculus course. Rather, the book favours geometric techniques, which may lead students to solve tasks without relating the new notion (first moment of an area) to the content on integrals introduced in their Calculus course.

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LEARNING AT THE BOUNDARIES IN PRE-SERVICE MATHEMATICS TEACHER EDUCATION

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In many countries, pre-service teacher education programs are structured so mathematics content is taught in the university's mathematics department and mathematics pedagogy in the education department. Consequently, few opportunities exist to interweave content and pedagogy in ways that develop professional knowledge for teaching. Such program structures also make it difficult for mathematicians and mathematics educators to gain mutual understanding of each other's roles in preparing future teachers (Fried, 2014). However, the boundaries between disciplinary communities can also carry potential for learning (Akkerman & Bakker, 2011).

This intervention study aimed to foster collaboration between communities of mathematicians and mathematics educators in pre-service teacher education. The research questions were: (1) How did boundary practices that emerged between the two communities lead to integration of content and pedagogy? (2) How did learning occur at the boundaries between communities? The study involved 23 mathematicians and mathematics educators from six universities who collaborated in university-based teams over three years. Data were collected from two rounds of interviews with each participant and written annual reports from each university team. Data analysis identified themes corresponding to Akkerman and Bakker's (2011) mechanisms for learning at the boundary: identification, co-ordination, reflection, and transformation.

New boundary practices emerged in every university (e.g., mathematicians and mathematics educators attending and teaching into each other's tutorials). These practices led to integration of content and pedagogy through new courses co-developed and co-taught by mathematicians and mathematics educators, new programs for preparing specialist teachers of primary mathematics, and new approaches to building communities of pre-service teachers. Emergence of new boundary practices indicates that interdisciplinary collaboration involved learning through the mechanism of transformation. Such collaboration seems to have potential to assist pre-service teachers to develop professional knowledge for teaching. However, the role of brokers who connect communities and foster new practices needs further investigation.

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IS THE BRIDGE REALLY SO FAR AWAY? EXPERIENCE OF ELEMENTARY MATHEMATICS TEACHERS IN IMPLEMENTING BRAIN RESEARCH FINDINGS

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There has been extensive, rich and diverse research around the world in neuroscience (NS) in general and in cognitive NS in particular. In recent years there is a growing group of scholars who claim that tools and theoretical frameworks of brain sciences can expand our understanding of brain activity in a way that is relevant to mathematics education (Tzur & Leikin, 2015). We would like to build "small bridges" between NS and mathematics education with help of "boundary research" in accordance with a worldwide trend of initiatives aiming to do that. We believe that one of the ways to apply NS discoveries in teaching is make research findings accessible and known to teachers and, together with them, think about how to apply these findings.

In order to examine this option, we developed a course for PD for mathematics teachers in the elementary school in which they were exposed to the structure of the brain and especially to the Approximate Number System (ANS). Many studies show that there is a correlation between ANS acuity and the ability to perform various tasks connected to elementary arithmetic and we think that the familiarity with this system is important for mathematics teachers. A qualitative analysis of all the data collected from the reflections written by 21 teachers who participated in the studyled to the division of all the statements into two core categories: (1) neurounderstanding (NU), i.e. teachers' insights about math studies from the perspective of NS research; (2) neurointervention (NI) i.e. possible applications of NS findings in math teaching and learning processes. Findings show that teachers speak about NS on several levels: facts, understanding the meaning of the facts, thinking about application, and the most advanced stage of actual application. This finding strengthens the notion that teachers can give meaning to NS information they obtain in a PD course in complex and significant ways in connection with their classroom experiences. We hope that this initial study suggests possibilities of bridging the gap between NS and classroom practices by adding content of this nature to the PD of teachers in the hope that this will lead to change in the classroom.

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AN INVESTIGATION OF STUDENTS' ATTITUDES TOWARD MATHEMATICS REGARDING SOME OF THEIR CHARACTERISTICS

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Affective factors play a significant role in mathematics learning. However, in the educational and psychological literature the affective domain is characterized in a variety of ways (Reyes, 1984). In our previous work we have differentiated attitude toward mathematics as cognitive, affective and behavioral, and used a measurement instrument reflecting those components of attitude toward mathematics. Referring to the attitudes measured in the previous study, in this study we intended to further investigate them in terms of some student characteristics. The research question was: Are there significant mean differences in seventh grade students' confidence in learning mathematics, usefulness of mathematics, importance of mathematics, liking for mathematics, mathematics anxiety, learner behaviors toward mathematics, time spend on mathematics at home, and perceived attitudes of mother, father and teacher toward them learners of mathematics scores with respect to gender, mathematics achievement, and mother and father education level? Data were gathered for ten affective variables from 1960 students of grade 7 (19 schools). Besides, students' characteristics such as gender, mathematics achievement levels, and mother's and father's education levels were recorded. According to the t-test results, no statistically significant difference was found between confidence scores of females and males, whereas statistically significant differences were found between their rest of the scores. Except for the anxiety scores, the differences were in favor of females. One-way ANOVA and Kruskal-Wallis tests results revealed that there were statistically significant differences among three achievement groups in terms of all scores. Regarding mother education level, statistically significant differences were found among three mother education level groups in confidence in learning mathematics, perceived attitudes of mother, father, and teacher scores, but no differences were found in the rest of the scores. Lastly, in terms of father education level, the results revealed that there were statistically significant differences among father education level groups in terms of confidence in learning mathematics, usefulness of mathematics, perceived attitudes of mother, father and teacher, and learner behaviors toward mathematics scores of students, and no differences were found in the rest of the scores. In the presentation, the results will be discussed in detail.

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KEY ASPECTS OF EXPRESSING THE RATE OF CHANGE IN LOWER SECONDARY SCHOOL MATHEMATICS

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Previous research suggest that the covariation of variables should be emphasised to enable a consistent understanding of mathematical functions, and the notion of rate of change is central herein. Further, to coordinate simultaneous changes in variables has been found essential for students' covariational reasoning and understanding of the rate of change (Oehrtman et al., 2008). But, exactly *what* needs to be discerned by students in order to coordinate changes and to understand and express the rate of change for a dynamic situation?

In an ongoing project, we study what aspects of such situations are paid attention to by lower secondary school students (ages 14-15) who quantitatively represent the rate of change. With variation theory as theoretical approach and lesson study as research method, respectively, (Marton, 2015), we use a number of tasks containing dynamic events to identify these aspects. From a qualitative analysis of six video-recorded lessons and students' responses to pre- and post-test tasks we have identified different ways of interpretation of the rate of change linked to certain aspects of this specific mathematical content.

Preliminary results indicate that the coordinating of neither mere values (x and y) nor simultaneous changes in variables (dx and dy) is sufficient to successfully express the rate of change. Our data suggest that the separation of proportional models from linear models seems critical for students' understanding, as is also the distinguishing between the two ratios representing the different rates of the two variables, respectively. Moreover, the meaning of rate of change as "change per one" differs among students and this variety affect the generalizability of the ways tasks are solved.

The tentative results would imply that, although often used, proportional functions can be counter-productive for teaching the rate of change. Our findings can help teachers to emphasise the critical aspects, and in the long term, better facilitate students' understanding of mathematical functions.

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USE OF COMICS AND ALTERNATIVE ASSESSMENT IN A LOWER SECONDARY MATHEMATICS CLASSROOM

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Normal Technical students are generally more inclined to visual and kinaesthetic learning styles. However, more often, lessons are taught in an auditory learning style which results in them being disengaged (Chang, 1997). The question then arises, whether using a visual teaching style and an alternative assessment that focus on both a hands-on approach and group collaboration could lead to more engaged students.

We collaborated with the National Institute of Education (NIE) on the Mathematics is Great: I Can And Like (MAGICAL) Project. The project aims to engage students in learning mathematics through the use of comics. Our sample consisted of 39 Secondary One students (aged 12-13) in the Normal Technical stream. An alternative assessment was used to measure students understanding of the topic on Statistics after two weeks of teaching using the comic's package.

The alternative assessment consisted of three parts. Part 1 requires students to work in groups to design their own survey question and gather data from their classmates. Based on the responses they received, the data collected was then collated into a frequency table. Part 2 requires students to select a suitable statistical diagram to represent their data. Part 3 focuses on problem posing which provides students an opportunity to create their own questions based on their statistical diagrams from Part 2. In a nutshell, the assessment aimed to provide students the experience to craft their own survey question, represent and interpret data based on real-world contexts. Students were given an opportunity to engage in problem posing, a higher-order thinking skill which is an inseparable part of problem solving (Pólya, 1957).

The assessment criteria included clarity, organisation of content, depth of explanation and creativity. Students were also required to perform peer evaluation of their peers' contributions to their assessment. Overall, some groups of students were able to complete the assessment successfully. In the presentation, suggestions to improve the alternative assessment will be discussed.

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PRE-SERVICE TEACHERS' NUMERACY CAPABILITIES AND CONFIDENCE

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Students "become numerate as they develop the knowledge and skills to use mathematics confidently across other learning areas at school and in their lives more broadly" (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.). In Australia, numeracy has become a focus of teacher preparation programs, due to the requirements of the Australian Curriculum (ACARA, n.d.) and the Australian Institute of Teaching and School Leadership (2014). Teachers of all disciplines are expected to incorporate numeracy into their teaching and to deal with the numeracy demands of their profession (e.g., analysis and interpretation of assessment data).

This presentation will focus on research conducted in a compulsory course, Numeracy for Learners and Teachers (NLT), introduced in 2015 in a two-year graduate teacher education program at a prestigious Australian university. Framed by the 21st Century Numeracy Model (Goos, Geiger, & Dole, 2014), our research focused on the NLT students' capabilities, confidence, beliefs, and understandings of numeracy. Here, we focus on the 2015 and 2016 students' numeracy capabilities and levels of confidence in their solutions to five numeracy questions. Analysis included descriptive statistics and cross-tabulations (to compare accuracy and confidence).

The participants completed the questions involving basic calculations, fractions, length conversions, and data analysis with high rates of accuracy and confidence. In contrast, the question involving combinatorics was completed with much lower accuracy and confidence. For each question, the level of confidence in the answer provided was generally lower than the accuracy level. Surprisingly, the levels of accuracy of the 2015 (predominantly secondary) and 2016 (predominantly primary) teacher education student cohorts were similar. However, the 2015 cohort was generally more confident in their answers than the 2016 cohort. In our presentation, we will discuss the implications of these findings for numeracy for teachers and students in Australia.

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DEGREE OF TASK STRUCTURING FOR DEVELOPING THE NOTION OF VARIABILITY

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The purpose of this study is to explore how elementary school students' notions of variability are affected by the degree of task structuring. Preceding researches on ill-structured tasks in mathematical problem solving have verified the effect only by quantification, test score for instance, but not by qualitative data such as the learning process. Despite the rising consent for the importance of variability in elementary statistics education, most of the research is focused on students in secondary school or higher. Moreover, Group discussions of ill-structured problems are significantly more complex and divergent than those of well-structured counterparts (Kapur, M., 2009). Hence, in this study, we analyse qualitatively how 5th grade students in a small group develop their notion of variability based on their experiences on the data from an ill-structured problem; and compare this to that from a structured problem.

The participants were four 5th grade students who had learned the concept of average in the previous semester, yet are new to concepts other than average such as variance. The first lesson was carried out using an ill-structured problem developed from literature review, and the second lesson using a structured problem. Students were asked to explore variability of the given data through discussion where the definitions of terms 'consistent' and 'even' were given. All their discussions and interviews were videotaped and transcribed. The development patterns are analysed based on the seven development processes of variability reasoning (Ben-Zvi, 2004), and the comparison was conducted with respect to the degree of task structuring.

In the ill-structured problem situation, all students exhibited step-by-step development from stage 1 to 5 suggested by Ben-Zvi (2004). It is supposed that the less structure of the problem facilitated the adoption of various inquiry methods on the data and rich discussions on variability. In the structured problem situation, two students skipped directly from stage 1 to 5 with little discussion related to in-between stages while the other two remained at the lower stages. This gap between the participants had failed to be reconciled until the end of the lesson. The fact that degree of task structuring could be a factor to the emergence of students' notion of variability yields implications for the applicability of ill-structured tasks in elementary statistics education.

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SUPPORTING LEARNERS BY INDIVIDUAL FEEDBACK

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In this investigation, a group of education students was analyzed during the approach of supporting children in tandem by means of learning impulses and stimulus questions according to a three-phase model (Haug & Helmerich, 2017). The focus, on the one hand, lies in the diagnosis and analysis of individual learning processes students had to acquire before their constructive intervention. On the other hand, we were also investigating to what extent students can give feedback to kids by means of stimulus questions rather than explaining a supposed sample solution to them. All students participated in a seminar on learning mentoring in which the three-phase model (observe/arrange/act) was thoroughly explained and introduced with differentiated exercises.

The method of videography was chosen for data collection. In order to textualize the video data, an observation protocol was customized. Thus, both the sound and the visual image could be reviewed to achieve an integral description encompassing both aspects (Dinkelaker & Herrle, 2009). The interpretation of the video data was supported by using a selection of individual images, to clearly and visually emphasize central moments of interaction (Moritz, 2010). The fact that the investigation was divided into different phases according to the performed tasks offered the possibility of doing an additional segmentation analysis. As a result, the leading question was: Are students able to diagnose and analyze learning processes within individual learning situations at the end of their university teacher training and if so, how well can they handle feedback that does not contain an explanation of the student's problem?

Results show that students are still challenged with both diagnosing and analyzing children's individual processes of learning and working, even though they're almost finished with their studies. Throughout the observing-phase, students tend to quickly determine a presumed problem of learners. Throughout the arranging-phase, they manage to resist the urge to explain, as well as contemplate a meaningful intervention. Throughout the acting-phase, it appears that stimulus questions for students' feedback are used especially on the cognitive level. Structuring or adaptive learning impulses are rather the exception.

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INDONESIAN SECONDARY STUDENTS' VIEW OF MATHEMATICS

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In a review of PME work on affect, Liljedahl and Hannula (2016) suggested that there might be a consensus regarding the gender difference in mathematics-related affect, male students having more positive view of themselves than girls. However, they also asked for comparative studies to test if the generally accepted are truly universal. With this in mind, this present study is an attempt to investigate the issue of student's view of mathematics in secondary schools in Indonesia with attention to gender gap. A total of 414 secondary level students (245 male, 169 female) participated in the current research. They completed a questionnaire covering 25 items (adapted from Tuohilampi et al., 2015). The items were categorized to three dimensions, the cognitive dimension (self-competence, self-confidence, difficulty of mathematics), the emotional dimension (enjoyment of mathematics) and the motivational dimension (mastery goal orientations and effort). We used the statistical program SPSS for descriptive statistics, and one way MANOVA to make comparison. A one-way MANOVA analysis revealed a significant multivariate main effect for gender, Wilks' λ=.909, $F(6, 407) = 6.757$, $p < .001$. Partial $\eta 2 = .091$. Power to detect the effect was 1.000. Given the significance of the overall test, the univariate main effects were examined. Significant univariate main effects were obtained for self-competence, $F(1, 412) = 5.964$, ($p = .015$), partial $\eta 2 = .014$, power = .683; effort, $F(1, 412) = 38.514$, ($p < 0.01$), partial $\eta 2 = .085$, power = 1.000; and mastery goal orientation, $F(1, 412) = 15.016$, ($p < 0.01$), partial $\eta 2 = .035$, power = .972. The results indicated that male secondary students in Indonesia had more positive view than female students. In particular self-competence, effort, and mastery goal orientation had significant gender differences. More research would be needed to explain 'why' and 'how' male and female students differ in view of mathematics in which male students hold positive view among their counterparts. Further results will be discussed extensively in the poster presentation.

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AN ANALYSIS OF CHILDREN'S INTERPRETATION OF A PROBLEM SITUATION: THE PROBLEM OF ASSUMING PROPORTIONALITY

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Mathematical modeling is a means of interpreting real life problems in mathematical terms. In recent years, a growing numbers of studies on primary school student modeling activities have been reported (e. g., Doerr & English, 2001; English, 2002). The subjects of their study, however, are students between the fifth-grade and the seventh-grade. It is necessary for the construction of mathematical modeling curriculum at the elementary school to expand the range of research object. It is expected to deepen our understanding of mathematical modeling in the elementary school by studying the younger children's modeling activity. The purpose of this study is to explore how children can interpret the situation in a modeling activity. To achieve this purpose, a set of assessment tasks was developed with a focus on assumptions made in mathematical models. This paper analysed children's interpretation of assumed proportionality in a task.

The theoretical framework of this study is based on the 'Models-and-Modeling Perspective on problem solving' which focuses on the process of interpreting a situation mathematically in problem solving (Lesh & Zawojewski, 2007). This study identified phases in the process of interpreting a situation. This paper presents a problem of assuming proportionality and explores pupils' interpretations. In total 266 pupils of grade 3, 4, 5 and 6 participated in the study. They worked on the problem that asks to find a way to find out the number of plastic bottle caps. Pupils' solutions were analysed in each phase of problem solving.

The results revealed that children's interpretations of assumed proportionality in the task are four types: counting reasoning, primitive functional reasoning, functional reasoning and progressive functional reasoning. The author also suggested that children are capable of interpreting the problem situation.

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READING MATHEMATICS: A HOLISTIC CURRICULUM APPROACH

Ho Weng Kin, Teo Kok Ming, Zhao Dongsheng, Romina Ann S. Yap, Tay Eng Guan, Toh Pee Choon, Toh Tin Lam, Cheang Wai Kwong, Zhu Ying, Dong Fengming, Paul M.E. Shutler, Quek Khiok Seng

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Reading mathematics is a necessary skill to have when learning undergraduate mathematics. However, students typically struggle with reading mathematics effectively (Shepherd, Selden and Selden, 2012). One difficulty that we recognised as related to this is the acknowledged limited time available in university for students to sufficiently objectify the concepts that they learn (Alcock and Simpson, 2009). We considered that a possible solution to foster better mathematics reading among our students is through a holistic curricular approach that involves many lecturers across the full duration of a four-year undergraduate programme.

In this oral communication we describe our ongoing attempt to teach mathematics reading to mathematics majors in a Bachelor of Science (Education) programme as part of a holistic curricular approach. We begin with presenting the learning objectives that were identified in the curriculum review process related to mathematics reading. We then proceed with sharing how the curriculum was implemented during the July 2016 semester for Year 1 students in the Linear Algebra I and Calculus I courses. The two lecturers involved drew upon suggestions from literature (e.g., Weber, Brophy and Lin, 2008) of strategies to nurture effective mathematics reading among students. As part of evaluating our initiative, we report on the students' performance in their final assessment particularly for items that were related to mathematics reading.

In general we are encouraged both by the enthusiastic participation of the faculty members in the curriculum review process and also the acceptable performance of students in demonstrating their understanding of new mathematical concepts. But we also gained insights into how to improve or proceed with our efforts at developing mathematics reading in our programme. We will share these during the presentation.

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CONCEPTUAL UNDERSTANDING OF FRACTION OPERATIONS – DO STUDENTS USE VISUALIZATIONS?

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Among researchers and practitioners in mathematics education, it is widely acknowledged that fractions is one of the most problematic topics in school mathematics. It is frequently noticed that even in upper secondary school operations with fractions are not carried out correctly. One approach to prevent such problems is to foster conceptual knowledge by means of visualizations of fractions and their operations (e.g., Rau, Aleven, & Rummel, 2013). Accordingly, such visualizations are widely used in German school mathematics and they can be found in every text book treating the topic of fractions. However, it is not clear whether students are able to use such visualizations in order to explain operations of fractions and whether they have corresponding conceptual knowledge. In order to explore long-term effects, it should be focused on older students who have not just learned about fractions.

Consequently, in the course of a paper-pencil test, N = 136 German students in grade 11 coming from various secondary schools were asked to explain the addition of fractions by means of a given example to a fictitious younger student. It was explicitly mentioned that they should aim at an understanding of the operation and not just help the student to carry out the calculation. One third of the students did not get any additional prompt (version A), one third was prompted to use a visualization for their explanation (version B) and one third was given a suitable representation of the addition of fractions which they were asked to use for their explanation (version C). The answers were coded in a top-down approach regarding procedural/conceptual explanations. The analysis of the data showed that without being prompted (version A) only 4 % of the students used a visualization. Even if the students were explicitly asked to do so (version B), only 20% draw a visualisation. In both cases, the explanations of the large majority of the students were purely procedural (93%/86%). In the case were a visualization was given (version C), 38% of the students showed conceptual understanding of the single fractions, but only 6% explained the addition conceptually. These results indicate that the way visualizations are used in corresponding lower secondary schools does not foster sustainable learning of conceptual knowledge regarding addition of fractions.

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THE INTERACTION EFFECT OF GUIDANCE ON TAIWANESE URBAN AND INDIGENOUS CHILDREN'S LEARNING OF AREA CONCEPTS IN THE SIMULATION-BASED ENVIRONMENT

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The concept of area is an important geometric unit but is not easy to understand for the urban and indigenous children. Computer simulations used as support for geometric learning have become available widely in formal or informal learning; however, what types and amount of guidance should be provided for learners is still yet to be clarified. Based on the cognitive theory of multimedia learning and embedded theory, we manipulated the guidance design according to two levels of detail (high and low) of representation (i.e., guiding learners to operate representations) and two degrees of learning procedure (i.e., guiding learners to follow the procedure), which formed four types of learning environment.

Specifically, the four conditions (2×2) varied from high direction to low direction for both representation and procedure. In this pilot study, we included sixty-eight fifth-grade students (49 were urban children, and 19 were indigenous children) to explore the interaction effects of guidance on Taiwanese urban and indigenous children's learning of area concepts in a computer-simulation-based environment. We randomly assigned the participants to the four conditions according to their math scores of the semester and the scores of the five variables, including the learners' prior knowledge, standard instructional process, intrinsic motivation, self-efficacy and meta-cognition. We found no significant difference between the four conditions regarding the five variables.

The results showed that the indigenous children seemed to outperform the urban children in highly directed learning environments; in contrast, the urban children outperformed the aborigines in learning environments with little direction. Such a performance trend existed in the retention post-test as well as in the transfer post-test. As a follow-up, we will increase the sample size to strengthen the theoretical deduction and to investigate the learning styles that was used by the minority of students and teachers' instructional styles, in order to clarify the abovementioned trends and reveal the experimental evidence.

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CHILDREN'S ANGLE-RELATED KNOWLEDGE USED FOR JUSTIFYING THEIR RECOGNITION OF ANGLES

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The configurational (static) aspect of angle concepts and dynamic ideas of turning are essential angle concepts in elementary mathematics. A growing body of research has explored the development of children's conceptions of angles (Clements & Battista, 1989; Mitchelmore & White, 2000). However, how children apply their understanding of angle-related knowledge to justify the similarities they recognize in the various situations involving turning motions over a period of time remains underexplored. This current study aimed to scrutinize children's angle conceptions of opening and turning across a period of time using angle assessment tasks.

The participants were 20 third-grade children who were recruited from a public elementary school in Taipei city, Taiwan. They had already learned the static aspect of angle concepts but had not yet learned the dynamic aspect of angles. The data reported in the study were collected from the end of the participants' first semester of third grade and continued to the end of their second semester of third grade, and were collected via the paper-and-pencil worksheets involving the various angles, as well as from interviews. The results showed that children's performance in the opening context outperformed that in the turning context based on Wilcoxon signed-ranks test. Over 65% (13/20) of the children could recognize angles in the various opening situations. In the turning situations, about 35% (7/20) to 40% (8/20) of the children expressed that angles could be recognized in the given scales with a single indicating arm based on the attributes of an angle. In contrast, about 60% of the children believed that no angles exist in the turning situations. They tended to argue that the turning situations do not exactly match the configurational (static) aspect of angle concepts. The findings imply that children at the beginning stage of learning angle concepts tend to depend strongly on the attributes and physical appearance of angles when justifying their angle recognitions in opening and turning situations. This tendency seems to be maintained for a period of time before receiving formal teaching of angle concepts of turning. To promote children's understanding of dynamic ideas of turning in terms of angles, instructors should seriously consider helping children see the relations between different angle contexts.

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MATHEMATICAL LEARNING ENVIRONMENT BASED ON SITUATED LEARNING THEORY TO APPRECIATE THE USEFULNESS OF THE METHOD OF EXHAUSTION

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Clarifying values and significance of teaching and learning mathematics is important; however Japanese teachers and researchers don't always recognize the goals of mathematics education at the stage of designing lessons. Situated learning theory (Lave & Wenger, 1991), a viewpoint on learning which emphasizes paying attention to 'activity', is considered to have potential to solve this. Focusing on activity requires teachers to have a clear image of ideal students in terms of mathematical activity while designing mathematics lessons. The purpose of this paper is to design a mathematical learning environment (lesson) based on this theory to appreciate the usefulness of the method of exhaustion in measuring the circumference of a circle. General principles of the design based on the theory (Imai, 2010) to realize the above-mentioned aim can be interpreted as follows: (1) Choose a mathematical activity considering the history of mathematics in order to stage it in the classroom. (2) Get students involved in the actual mathematical activity staged by the teacher. Based on these principles, a case-study lesson was conducted by one of the authors of this paper in 2017 for a class of 26 fifth graders. In the introductory stage of the lesson, the teacher showed a picture of measuring a marathon course by bicycle, and some students detected the circumference of the tire was necessary for it. They considered how to measure the circumference of the tire of a 60-cm (24-inch) bicycle which wasn't in the classroom. Some students noted to use a regular hexagon inscribed in the circle to measure the circumference of the tire, but later others pointed out they won't be able to measure the correct distance of a marathon course this way because of the difference in circumferences. Subsequently, they realized that using regular polygons which have more vertexes would enable them to measure the circumference of the tire more precisely. At the end of the lesson the students' feedback was obtained by asking them to fill out a form, which revealed that some students could realize the usefulness of the method of exhaustion. This implies that we can expect the aim will be attained by the mathematics lesson based on the principles of situated learning.

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CHINESE STUDENTS' PROBLEM SOLVING AND POSING ABILITIES IN RATE, RATIO AND PROPORTION

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We conducted this study to investigate the performance differences of students in traditional computational and problem-solving tasks and problem-posing tasks in proportion and the effect of placements of unknowns in a proportion on students' responses. A total of 431 sixth grade Chinese students sat two test, one is a traditional computational and problem-solving test with 14 items, four of which are related to proportion. The other is a problem-posing test with 5 items, one of which asked students to pose a problem to each of three equivalent but different expressions (i.e., $(65 \div 13) \times 4.\ 63:21=[\ \ ]:8$, and $14:42=3:[\ \ ]$).

Students' answers to the computational and problem solving items are scored as 1 (correct) or 0 (incorrect). Their written responses to the PP tasks were analysed in three steps: (1) to check whether it is an application problem; (2) to check whether it is solvable by the given expression; and (3) finally to determine the context described into one of the four categories that Lamon (1993) identified. It was found that very few students took the first PP task, we decided to determine the context described into one of the ten categories that Greer (1992) identified.

The correct percentages of students to the three computational and one problem-solving tasks are 97.7%, 96.8%, 92.3%, and 90.0%, respectively. More than two-thirds of the participants were unable to pose application problems that match the given expressions. The percentages of students who posed problems with the four categories of contexts are significantly different among the three PP tasks. However, they are not significantly different between the two PP tasks in proportion form. The placement of the unknowns in the two proportional equations did not seem to affect the participants' performance. Inspiringly, 56.6% of the participants could pose problems that matched $(65 \div 13) \times 4$. They posed problems with a context in equal groups, rate, multiplicative comparisons, and equal measures.

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UPPER-SECONDARY TEACHERS' OBJECTIFICATION OF SYMBOLS BY THEIR USE OF LANGUAGE

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Research literature points to the importance of objectification in the learning of mathematics (e.g., Sfard, 2008). However, few empirical studies seem to focus on such more general issues of learning mathematical ways of speaking and writing that are useful in a wide range of areas of mathematics, without focusing on specific mathematical constructs (Morgan et al., 2014). Our study uses Sfard's (2008) definition of objectification and its two sub-processes, *reification* (turning processes into objects) and *alienation* (dissociation from human actors), in order to analyse upper-secondary teachers' mathematical instructions, focusing on their word use in relation to any type of symbols. The purpose is to understand what aspects and levels of objectification there are in various situations and for different kinds of symbols.

Data consist of voice recordings and additional photos of white-boards, from seven randomly chosen upper secondary teachers, at one of their ordinary mathematics lessons. Analyses were delimited to parts when the teachers spoke to the whole class. Utterances about symbolic expressions were identified, and then categorized concerning *reification*; as focusing on processes if the used words are active verbs, and on objects if the used words are nouns. In total, utterances about 199 different symbolic expressions were categorised. Sometimes teachers talked about expressions both as processes and as objects. The analysis shows that object-talk dominated, 199 vs. 61 occasions. Process-talk was common when teachers wanted to explain how one expression "becomes" another, or wanted to encourage students to provide an answer. Object-talk was used to address both whole expressions as single objects, as well as individual symbols as objects. Typical is that the equal sign was treated as something static, which could be expected since upper secondary students have met this symbol during most of compulsory school and have had the time/opportunity to objectify. On the other hand, equivalence is a relatively new concept for these students, and there is no introduced symbol for this concept. Thus, focusing on processes while motivating the rewriting of expressions (e.g., how one expression "becomes" another) is a reasonable outcome. Results about *alienation* will also be included in our presentation.

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THE CHALLENGES OF IDENTIFYING GIFTEDNESS IN UPPER SECONDARY CLASSES

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Mathematical giftedness—what it is and how to foster it—is a topic that gets increasingly more attention in research (Pitta-Pantazi, 2017). In this respect, the identification of gifted students plays a crucial role for fostering them appropriately (ibid.). Researchers do agree on the validity of intensive multilayered observations of students to identify giftedness and that there is a need for more economic methods for identification (ibid). However, the question arises of whether a more economic short-term testing of students is feasible to identify gifted students.

Based on Krutetskii's (1976) seminal work, Kießwetter (1985) developed a set of criteria of successful problem solvers and an according test for 12-year old students to select participants for his program to foster gifted students. This test comprises seven open-ended problem solving tasks. In our research project, we ask the question of whether and to what extent the problems developed by Kießwetter are a suitable base for identifying mathematical gifted students at upper secondary level. In this talk, we present our analysis of one such problem solving task (the "domino-problem"), where we investigated the potential that the task holds for identifying gifted students.

We analyzed students' products of a short-term evaluation based on Kießwetter's (1985) criteria. We compared the results of this analysis to a holistic ranking, which arose from an intensive long-term observation and collaboration with students over a period of six months. When comparing both rankings, we found that most participants were ranked on a similar level in both the holistic observation and the domino-problem. This result indicates that using complex problems to identify gifted students seems to be a valid approach that is economic (compared to long-term observations). However, it is only one tool in the tool box: It is inevitable to face more dimensions such as creativity or personal traits to grasp mathematical giftedness.

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PRESERVICE TEACHERS' MULTIPLE SOLUTION STRATEGIES FOR DECIMAL OPERATIONS

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Teachers should encourage students to use different solution strategies because it is the teacher who actually influences students' ability to find multiple solutions (Kilpatrick, 2009). The purpose of this study is to determine preservice teachers' knowledge of multiple solution methods of operations on decimals in terms of addition, subtraction, multiplication, and division. We first consider what different formal and informal solution methods preservice teachers are using and the percentages of correct and incorrect responses.

The participants (N=67) were the students enrolled in a mathematics class in a teacher education program at a mid-western university. This study used simple descriptive statistics and content analysis to analyse preservice teachers' responses to items in a Decimal Knowledge Test (DKT). The DKT was specifically designed to measure preservice teachers' knowledge of solving problems in different ways in areas related to: (1) addition, (2) subtraction, (3) multiplication, and (4) division.

Preservice teacher showed that 31 different ways of responses for addition, 33 for subtraction, 29 for multiplying a decimal by a whole number, 15 for multiplying a decimal by a decimal, 14 for dividing a decimal by a whole number, and 11 dividing a decimal by a decimal. It indicates that preservice teachers have a difficulty in solving decimal division problems in multiple ways. The percentages of preservice teachers' correct responses using *standard algorithms* as formal solution methods were shown as follows: decimal addition (54.36%), subtraction (65.36%), two multiplication (49.30% & 64.71%), and two division (57.75% & 65.22%) operations. In contrast, the percentages of their incorrect responses using standard algorithms were shown as follows: addition (85.71%), subtraction (66.67%), two multiplication (75% & 80.56%), and two division (77.78% & 92.31%) operations. In the presentation, examples of each method will be shown in detail.

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TEACHERS' LEARNING TRAJECTORIES IN A PROFESSIONAL LEARNING COMMUNITY BASED ON ABCD PRINCIPLES

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In this research, we adapted the principles of ABCD presented by Cha and his colleagues (2016) which values autonomy, bridgeability, contextuality, and diversity in teachers' learning to develop a professional learning community (PLC) in order to investigate how the principles of ABCD could work as guiding principle of teachers' learning in the PLC.

In the PLC, the teachers were engaged with a variety of activities such as analysing curriculum and textbooks, seeking for real world contexts relevant to key mathematical ideas to teach, planning a lesson, and so on. We collected the participants' discourse data to analyse their learning trajectory. The analysis focused on the teachers' achievement but also on frustrating experiences of the teachers such as uncertainty, risk, tension, etc. to investigate how the principles of ABCD work to create learning context for teacher empowerment by enhancing teachers' competences for teaching and their agency for school change.

The analysis shows that the principles of ABCD worked to facilitate the teachers' professional growth into active agents. Since the teachers were positioned as autonomous learners in the PLC, most of activities were inquiry-based. The teachers were frustrated with inquiry-based learning in the beginning, because they could not figure out how to begin their inquiry, how to organize the results of their inquiries, etc. However, they became to realize that autonomy made their learning more meaningful because they learned what they wanted to know.

Moreover, Bridgeability facilitated collaborative dialogue to expand the teachers' inquiry and learning. Contextuality led the teachers to bring up their own expertise developed through their teaching practice in school. Learning based on the principles of ABCD ultimately led the teachers to problematize daily teaching in school, its cultural and political organization, which is the most significant experience for the teachers to grow up as agents of reconstructing school mathematics.

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TRANSFORMATION OF PCK FOR FINANCIAL MATHEMATICS IN A CONTEXT OF PROFESSIONAL LEARNING COMMUNITY

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Recently, as the Korean mathematics curriculum included financial mathematics as an elective course for high school students, it is necessary to seek for ways of how to equip mathematics teachers with competences for teaching the course. In this perspective, we organized a professional learning community (PLC) for financial mathematics where the participants were engaged with curricula deliberation based on the participants' active inquiry of curricula reorganization.

In the context of PLC, we collected the participants' discourse data to analyse the PCK for financial mathematics (FM-PCK) based on the framework of content knowledge for teaching introduced by Ball and her colleagues (2008). The analysis focused on the discourse data of the 4 steady participants. Based on the analysis, this research aims to investigate what kind of FM-PCK came up and how it had changed in order to identify implications for PLC of financial mathematics.

The analysis showed that in the beginning, the teachers experienced difficulty in determining instructional goals and key concepts of their financial mathematics class to nurture students' financial literacy. The activities of textbooks and curricula analysis helped the teachers figure out key concepts and conceptual connection among them, and relevant real world context. Moreover, the teachers' FM-PCK of content and curriculum was weaker compared to the other types of FM-PCK. The teachers listed key concepts of finance and mathematics with reorganizing them in integrative ways. In this regard, collaboration with a social studies specialist facilitated the teachers to broaden their understanding of the connection between financial concepts and mathematical concepts and to plan integrative approaches to the concepts.

This research suggests that PLC for financial mathematics needs to provide opportunity to investigate the connection between finance and mathematics and to deliberate on curricula reorganization for integration. For the purpose, it is essential to design PLC to provide a ground for dialogue among participants of diverse disciplinary backgrounds in order to expand the boundary of expertise.

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PRESERVICE TEACHERS' CONCEPTION OF PROBLEMS AND PROBLEM SOLVING IN TEACHING MATHEMATICS

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Being problem solving such a crucial part of the teaching of mathematics (NCTM, 2000), it becomes central how to prepare preservice teachers to teach mathematics through problem solving. The authors investigate how 84 elementary preservice teachers conceptualize a problem and problem solving in teaching mathematics in alignment with NCTM standards through a survey. The survey asked participants to identify mathematics problems with two different cognitive demands (Stein et al.), and to define a problem and problem solving based on their prior learning experiences.
The result indicates that preservice teachers demonstrated very simplistic view of a problem and problem solving. When two mathematics tasks were given – task A requires higher order thinking and task B requires simple calculation – more preservice teachers (95%) identified task A is problem than task B is a problem (50%). However, their behind reasoning was very simple; e.g. task A or B are problem because they need to find an answer. Similarly, with respect to their prior concept of problem and problem solving in mathematics classroom, the majority of preservice teachers in the study defined "problem" as simply a task that requires finding answers. A few participants stated that "problem" requires critical thinking and the use of multiple strategies. Other important aspects of problem solving according to NCTM standards – e.g. use of prior knowledge, acquiring new knowledge, reflect on process - were not emerged. These results call for the mathematics educators' attention. Teacher preparation courses need to provide the opportunity to practice task sorting activities as Stein et al (2009) suggested, discuss benefits of problems and problem solving in teaching mathematics, how to select meaning problems, how to leverage procedural and conceptual knowledge in solving problems and so on. Without having a solid understanding of problem solving there will be challenges for both preservice teachers and teacher educators to build a successful problem solving mathematics classroom.

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GENDER DIFFERENCES IN THE PREFERENCE OF LEARNING STRATEGIES IN MATHEMATICS

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Learning strategies are individual's approaches that learners use to learn or solve a task. They also support the learners to organize their thoughts and acquire a set of skills so as to learn the content and accomplish a particular task more effectively and efficiently (Schumacher & Deshler, 1984). Gender difference in the preference of learning strategies in mathematics has not been adequately researched in the Nepalese context. Hence, this particular piece of research is an attempt to address this particular topic with reference to the learning strategies classified by Pintrich, Smith and McKeachie (1989). This research aims to address the following research questions:

- Are there any gender differences in students' preference of learning strategies in mathematics?
- If so, what kinds of differences are there? And why?

The sample included 1394 students of Grade IX from 24 schools of three geographical regions of Nepal and a mixed method-sequential explanatory design was employed in the study. Questionnaire, observation and interview guidelines were used to collect the data. 30 lessons were observed and 24 key respondents were interviewed from two of the schools. The data were analyzed by applying χ^2 – test through SPSS and thematically.

The result shows that the learning strategies used by boys and girls in mathematics differed significantly at $p < 0.001$. Boys consistently used effort management, critical thinking and elaboration strategies more often than girls, whereas girls used peer learning, help seeking and rehearsal. Their perception, motivation, self-related beliefs as well as their emotions, interests, enjoyment of mathematics differ. These factors greatly affect their selection and use of learning strategies. Hence, mathematics teachers need to consider gender differences in the selection of learning strategies to motivate and encourage the students according to their preference.

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RIGOR AS FAMILIARITY IN MATHEMATICS ASSESSMENTS

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Traditional mathematical conceptions consider ‘rigor’ as a pathway towards a sophisticated understanding of fundamental ideas, wherein students engage with increasingly challenging tasks (NCTM, 2000). Such conceptions impart due diligence to a teacher’s perspective on rigor, but almost overlook a student’s perspective on the same. Our analysis aims to understand whether the notion of rigor takes up alternative notions for a student, along with proposing a framework to capture this conception.

We analyse data from a mathematics assessment (containing ten questions pertinent for students in grades 4, 6, and 8) administered to 5472 government high school students across four Indian states. Our analysis focuses on questions which have been answered incorrectly by a higher percentage of students, as compared to those who answered that question correctly. We consider rigor to be based on the *familiarity* which a student has with two aspects of any question in the assessment – its *content*, and its *representation*. The frame of reference, from which this familiarity arises, is the regular classroom discourse (including textbook usage) that prevails in government schools. Familiarity with content pertains to the student being familiar with the mathematical knowledge required to answer a question (What is being asked?). On the other hand, familiarity with representation pertains to the student being familiar with the manner in which the question is asked in the assessment (How is it being asked?). We have considered familiarity as a binary variable for our initial analyses. In other words, we have classified the assessment questions into four categories, based on the students’ familiarity or unfamiliarity with both the content and the representation.

Our analysis finds that a majority of high school students in all four states answered all those questions incorrectly, where familiarity with content exists, but not the representation. This was contrary to the traditional perspectives on rigor, as two of these three questions pertained to grade 4 level (considered as ‘easy’ difficulty level by designers), and another pertained to the grade 6 level (considered as ‘average’ difficulty level). Further, questions with familiar content and representation were answered correctly by a majority of students in all states. Our framework bears strong implications for assessment design by considering rigor from a student’s perspective.

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DEVELOPMENT OF RESPONSIVE TEACHING PRACTICES

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Research in mathematics education has addressed the critical issue of teachers' attention and response to students' mathematical thinking (Carpenter, Fennema & Franke, 1996; Jacobs, Lamb & Philipp, 2010; van Es & Sherin, 2008). This responsive teaching strategy is challenging to implement, especially for novice teachers. Inexperienced teachers often feel that it is important to attend to every event that occurs during classroom instruction, rather than focusing on features of core significance such as student mathematical thinking. Thus a better understanding of how teachers develop their responsive teaching practices is critical to supporting those teachers in becoming more responsive to student thinking. This study investigates that process, and moreover, investigates the longitudinal process focusing on shifting teaching practices supported by reform-oriented curriculum materials. The year-long classroom observation of a 7th grade algebra teacher in the United States was a main data source (totaling 13 video-recorded lesson observations). The first part of this study introduces a conceptual framework of responsive teaching building on a teacher-noticing framework (see Jacobs, Lamb & Philipp, 2010): (a) *attending* to mathematical representations and ideas generated by students either individually or collectively; (b) using *instructional reasoning,* which helps teachers filter what they notice through interpretation, evaluation, and selection; and (c) *responding* to the emergent mathematical representations. In the second part, I elaborate on the responsive teaching framework by providing evidence of how teachers generate local knowledge through interactions with students and mathematical representations during instruction, as well as through shifting instructional patterns. In particular, the analysis focuses on (a) how students' mathematical thinking was elicited (supported by the curriculum material); (b) how the emergent student thinking and mathematical representations were used as resources for teacher learning by a focus on the teacher's attention and instructional reasoning; and (c) how the teacher's instructional patterns became responsive teaching. The implications for the design of curriculum materials for teacher learning and professional development will be also discussed in the session.

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PROMOTING STUDENTS' MATHEMATICAL COMMUNICATION AND INFORMATION LITERACY IN MATHEMATICS CLASSROOMS

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Mathematical communication and information literacy as core competencies students should have has been emphasized in newly reformed curriculum and assessment in Korea and many other countries. However, little has been known about how they can be promoted together through mathematical processes in classroom instruction, how we can evaluate students' mathematical communication and information literacy, and what we can learn from the results. In order to respond to those questions, we have developed tasks following by five principles: multimodal, topic driven, challenging, realistic, and interplaying mathematical communication with information literacy.

In this presentation, we will discuss the results from the pilot study in which 75 tenth graders solved the two tasks in the areas of statistics and functions. They were asked to solve three or five main problems with multimodal approach and share their ideas with their peers in each task. Based on the framework we have developed from the results of literature review (e.g., Ministry of Education, 2005; Secker, 2011), five coders independently coded students' responses to the tasks and inter-rater reliability was checked. We analyzed the data using both quantitative and qualitative research methods.

As a result, we found that mathematical communication and information literacy were closely related to each other. While students solved problems much easier when given information was simple or the tasks were similar to those in textbooks, students felt difficult in communication when the tasks were asked to use multi-modes or to select useful information from multiple sources. They tended to solve the problem by only either reading or listening information even when they need both. More findings will be discussed in the session. The findings will give us implications on task development and assessment.

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PRE-SERVICE ELEMENTARY TEACHERS' DATA SETS COMPARISON

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It has been known that comparison of data sets encourages students' statistical thinking and reasoning. To look into the effect of comparing two sets, research has investigated statistical thinking and reasoning that students show as well as difficulties that they show when comparing two data sets (Leavy, 2006; Pfannkuch, 2005, 2007).

This study investigates the approaches that pre-service elementary teachers use when comparing two data sets. Through the approaches, this study examines their understandings of statistical concepts as well as difficulties that they face when comparing two data sets. The pre-service elementary teachers are given two tasks: 1) Comparison of the same size data sets; and 2) Comparison of data sets of different sizes. They completed the assignments in a group of four using computer-based technology. They submitted the assignments in a form of reports and had a presentation. They made presentations in front of their colleagues in class. Their colleagues were allowed to ask questions about their approaches. The reports and presentations were analysed.

The pre-service elementary teachers used the following approaches: 1) Consideration of stability; 2) Use of different kinds of representatives; 3) Use of different levels of distribution; 4) Use of different levels of variability; 5) Depending on probabilities; Use of various evidence at the same time.

Different approaches showed different pre-service elementary teachers' thinking and reasoning. It suggests that we devise ways in which the teaching can make useful use of the various thinking and reasoning. This study suggests that the presentation accompanying questions and discussions is a good way to do this. In this study, the pre-service elementary teachers were able to see the approach of the group through the presentation of other groups and to experience related reasoning and thinking.

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INTERPRETATION OF DIAGRAMS IN DYNAMIC GEOMETRY ENVIRONMENTS

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The use of dynamic geometry environments (DGEs) is known to influence students' proof-related activity (Laborde, 1993). Our study addresses DGE use in mathematical activity related to proofs and refutations (Lakatos, 1976), conceptualised in terms of conjecturing, proving, and refuting (Komatsu & Jones, 2017). Our study focuses on the refuting phase, and analyses the ways that students discover and treat counterexamples to their conjecture when using a DGE to transform diagrams.

As a framework, we apply the research of Steenpaß and Steinbring (2014) on student interpretation of diagrams, and distinguish two types of diagram interpretation, namely *perception- and relation-oriented interpretations* (a framework that is subtly distinct from the notions of 'drawing' and 'figure' employed by Laborde, 1993). The *perception-oriented interpretation* of a diagram refers to considering its visible elements (e.g. points, lines, circles, etc.), whereas the *relation-oriented interpretation* of a diagram refers to considering the relations amongst these visible elements.

Using this framework, we analysed a task-based interview where a triad of Japanese secondary school students (16–17 years old) tackled tasks related to conjecturing, proving, and refuting using a DGE (Komatsu & Jones, 2017). The video records and transcriptions of the interview, the worksheets the students completed, and the DGE file they made, were used as data for the analysis.

Our analysis shows that when students engaged in proofs and refutations by interpreting diagrams in a relation-oriented manner, this helped them discover new diagrams, including one that was a counterexample to their conjecture. It also enabled the students to address efficiently another counterexample by unifying it into a previous counterexample. Our study suggests that the distinction between perception- and relation-oriented interpretations, as well between 'drawing' and 'figure', can help in deepening understanding of student behaviour when they are using DGEs.

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CHARACTERISTICS OF STUDENTS' EXPLANATIONS IN ELEMENTARY SCHOOL: WHICH TRIANGLE IS LARGER?

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In Japan, students learn the mathematical proof at Grade 7 (aged 12 -13) . However, of course, even before then, students explain for the reason of answers of problems of mathematics in the elementary school. The mathematical proof is taught in the class with clear goals at junior high school, but in elementary school, the teaching goal about the explanation for mathematics is not clear. The role of the mathematical proof is not only as the verification of results but also as the communication of results (de Villiers, 1990). This study try to set teaching goals to promote students' ability to explain about mathematics in the elementary school focusing these aspects of the mathematical proof. The aspect of the mathematical proof as communication may suitable of teaching goals to promote students' ability. At first, for the purpose, I try to catch characteristics of students' explanations about mathematics in elementary school. The research question is:

- What types of explanations are identified when students tackle the problem of mathematics in elementary school?

The data was taken from a class of 25 Grade 6 students (aged 11-12) from an elementary school in Japan. They tackled a problem of comparing the sizes of the area of two triangles which need some reasoning to answer (Which triangle is larger?). Through analysis of students' explanations, 4 types of explanations were identified as follow:

(0) no explanation (5 students), (i) the explanation by each person's way (8 students), (ii) the explanation using the definition and properties of the figure as a part of the reason of the answer (7 students), and (iii) the explanation based on the definition and properties of the figure completely (5 students).

The percentage of the number that got the correct answer of this problem in each types were 60% of Type-0, 38% of Type-i, 71% of Type-ii, and 100% of the Type-iii. This result suggest that to lead students to explain using the definition and properties of the figure may get the correct answer of this problem. As teaching goals of the explanation in the elementary school mathematics, to explain using the definition and properties of figure may appropriate. In the presentation, concrete information of students' explanations and further results will be discussed in detail.

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INVESTIGATING FUTURE ELEMENTARY SCHOOL TEACHERS CONCEPTUAL UNDERSTANDING OF PLACE VALUE THROUGH MULTI-DIGIT CARRIES IN NON-DECIMAL BASES

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A challenge in the design of elementary school teacher students' programs is that they already know the mathematical content to teach very well, at least on a procedural level. Whether they understand the mathematical concept deeply and thus can create appropriate learning environments for their students is difficult to investigate under these circumstances. Our research should help to design teacher education courses that take advantage of re-experiencing the introduction of positional number systems and place value. Within a larger study on pre-service teacher students' mathematical content knowledge, students were closely observed. With a translated version (Thanheiser, Ladel, Kortenkamp, Goral, in preparation) of original tests by Thanheiser (2010) and McClain (2003) and its adaptations by Murawska (2013), students were assessed in performing and understanding standard multi-digit addition algorithms. Similar to Thanheiser's and Murawska's results (ebd.), the majority of students lacked a thorough understanding of the algorithm, due to a superficial concept of place value. To refine the data, students were asked to describe and discuss their way of working regarding the addition of five multi-digit numbers in the base 2 number system, which requires more complex regrouping procedures than usual. The students' way of handling multi-digit carries was taken into focus in order to classify their answers. Seven different strategies emerged of how to add the four numbers: Subdividing the addition problem into smaller problems to avoid multi-digit carriers, different auxiliary strategies that rely on using the base 10 system, pure counting instead of calculating, and the consistent use of standard addition algorithms in base 2. In future lectures, more differentiated assistance needs to be provided in order to support flexible and deep understanding of the addition algorithm. The teacher students should be able to count, group, ungroup, decompose and add/subtract in the range up to twice the bases fluently. Hence, it is certainly appropriate to discuss the stages of learning an algorithm explicitly.

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PREREQUISITE CONDITIONS FOR TEACHING MATERIALS OF THE ALGEBRAIC STRUCTURE IN SCHOOL MATHEMATICS

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The aim of this study is to clear prerequisite conditions for teaching materials of the algebraic structure in school mathematics and to propose the teaching materials included prerequisite conditions. The method of this study is theoretical research based on interpretation of literature about algebraic structure.

The mathematical structure is compounded of a set, an operation between elements and an axiom of structure. The mathematical structure include the algebraic structure. Bourbaki (1974) defined the algebraic structure by the laws of composition and laws of action. In this study, I deal with the algebraic structure only by the laws of composition. Fey (1967) mentioned that learning a field extension from Q to $Q(\sqrt{2})$ which is the one of the algebraic structure. He supposed the new set is closed under addition and multiplication and satisfy the commutative law, the associative law and the distributive law and create $Q(\sqrt{2})$. Therefore, the algebraic structure is compounded a set, the laws of composition, closure under an operation, the commutative law, the associative law and the distributive law and the existence of identity element and inverse elements.

Quadling et al (1966) observed the use of the axiomatic method in secondary teaching. They divided four phase about learning groups. The first phase is the experience phase, the second phase is the analytical phase, the third phase is the axiomatic phase and the fourth phase is the deductive phase. Especially, they examine various feature of structure on the analytical phase and they continue to acquire experience of structure, emerge the concept of isomorphism on the axiomatic phase. Therefore, prerequisite conditions for teaching materials of the algebraic structure are to analysis the operation between elements and to find the similarities and the differences.

I propose the teaching materials of the algebraic structure. By structural points of view, students can integrate different things into same things. For example, they can think $\frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \times \frac{3}{2}$ and $(+2) - (+5) = (+2) + (-5)$.

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STEM AND SINGLE-SEX SCHOOLS: WHAT COUNTS?

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In many countries females are underrepresented in STEM (Science, Technology, Engineering and Mathematics) fields (OECD, 2012). Differences in study areas selected by males and females also persist. It has been suggested (e.g., Cherney and Campbell, 2011) that single-sex [SS] schools enable females to develop the subject prerequisites and skills important in STEM fields. However, lack of control in research about SS schooling has confounded the evaluation of research outcomes. As part of a larger study about schooling, careers, and STEM, we explored if participants assumed that STEM-related studies are more strongly encouraged in SS than co-educational schools. Survey participants were asked whether, to promote a boy's/girl's interest in STEM-related studies they would recommend a SS school, a co-educational [CS] school, or neither - that it would depend on the child.

The survey sample comprised 1157 females, aged from 18 to over 70. Most had studied mathematics in their final year of secondary school: an advanced (N=377), or intermediate (N=472), or elementary (N=126) mathematics course. We aimed to explore *perceptions* about SS schools - specifically if they were thought to promote interest in STEM-related studies, whether such beliefs were held similarly for boys and girls, whether beliefs varied according to the type of school attended by respondents, and by the amount of mathematics respondents themselves had studied. Our findings included:

- For boys, 14% recommended SS, 10% CS, and 76% "depends on the child"; For girls, 43% recommended SS, 8% CS, and 49% "depends on the child"
- Type of school attended by respondents influenced their recommendation. Those who had attended SS were more likely to recommend SS; those who had attended CS were more likely to recommend CS (for both boys and girls)
- Level of mathematics course studied in the final year of school did not affect the recommendation made, but a higher proportion of those who had taken a mathematics course would recommend a SS for girls than those who had not.

Assumptions persist that, particularly for girls, SS schools assist STEM-related pathways. Well planned research is needed to test the efficacy of these expectations.

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EXPLORING HOW ELEMENTARY TEACHERS WOULD RECOGNIZE MATHEMATICAL CREATIVITY

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Literature showed that it is important for teachers to recognize their students' mathematical creativity (Haylock, 1987, 1997; Nadjafikhah, Yaftian, & Bakhshalizadeh, 2016; Silver, 1997). If teachers are not able to recognize their students' mathematical creativity, it will be difficult for them to promote their students' mathematical creativity. Little, however, was known about how Taiwanese elementary teachers would recognize their students' mathematical creativity. The purpose of the study was to explore how Taiwanese elementary teachers would recognize their students' mathematical creativity.
The participants for this study comprised thirty Taiwanese elementary teachers. They were asked to write down how they would recognize their students' mathematical creativity on a questionnaire. Haylock (1997) identified overcoming fixation and divergent production as two main approaches for recognizing mathematical creativity, where fixation included content-universe and algorithmic fixation, and divergent production was indicated by flexibility, originality and appropriateness. The responses on the questionnaires were analyzed based on the two main approaches by Haylock (1997). The findings of the study showed that most (about 57%) of the responses mentioned only flexibility, the second most (about 27%) of the responses mentioned both flexibility and originality, and none of the responses mentioned either overcoming content-universe fixation or overcoming algorithmic fixation. More research is needed to investigate how to improve Taiwanese elementary teachers' approaches for recognizing mathematical creativity.

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DIFFICULTY GENERATING FACTORS AT UNIVERSITY LEVEL: COMPARING VARIATIONS OF TASKS

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Difficulty generating factors help to understand the difficulty of mathematical items. To learn more about them, one approach is looking at psychological research about solving problems and tasks. According to Newell & Simon (1972), a problem consists of the initial state and the goal state as well as all possible states in between. In the formulation of a mathematical task at university level both the initial state and the goal state can either be well-defined or ill-defined. For example, the initial state may contain exactly those pieces of information, which are needed for the solution of the problem, or there may be additional information (which is not necessary for the solution), or there is not enough information. A task may contain its goal state (Explain *that* statement A is true.) or not (Decide, *if* statement A is true and explain why.). We analysed how variations in mathematical items concerning the initial state (i.e. additional information that is not necessary for the solution) and the goal state of the tasks affect their difficulty.

To answer this question, we developed four variations of three mathematical items. The variations contained the same mathematical problems, but they differed in the presence of additional information as well as the presence of a well-defined aim. We arranged them in test booklets (each containing three items, one version of each problem) and asked 450 students in their first semester at university to answer them.

We found out that additional information at the initial state of a task as well as the presence or absence of an explicitly formulated aim did not affect the difficulty of a task significantly in two of the three items. In only one of the items we found a significant main effect of the explicitly formulated aim on the solution rate ($F(1,449) = 5.15$, $p = .02$), in the other two items we did not find a significant main effect of the explicitly formulated aim. Furthermore, the presence or absence of additional information did not affect the difficulty of the items.

These results are especially interesting in the light of the results of our pre-study with the eye tracker (Lehner, Döring, & Reiss, 2016): Although the additional information get into the focus of attention, this does not affect the difficulty of the items.

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CONDUCTING DUAL PORTABLE EYE-TRACKING IN MATHEMATICAL CREATIVITY RESEARCH

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Eye-tracking opens a window to the focus of attention of persons and promises to allow studying, e.g., creative processes "in vivo" (Nüssli, 2011). Most eye-tracking studies in mathematics education research focus on single students. However, following a Vygotskyan notion of learning and development where the individual and the social are dialectically interrelated, eye-tracking studies of collaborating persons appear beneficial for understanding students' learning in their social facet. Dual eye-tracking, where two persons' eye-movements are recorded and related to a joint coordinate-system, has hardly been used in mathematics education research. Especially dual *portable* eye-tracking (DPET) with goggles has hardly been explored due to its technical challenges compared to *screen-based* eye-tracking.

In our interdisciplinary research project between mathematics education and computer science, we conduct DPET for studying collective mathematical creativity (Levenson, 2011) in a process perspective. DPET offers certain advantages, including to carry out paper and pen tasks in rather natural settings. Our research interests are: conducting DPET (technical), investigating opportunities and limitations of DPET for studying students' collective creativity (methodological), and studying students' collective creative problem solving (empirical).

We carried out experiments with two pairs of university students wearing Pupil Pro eye tracking goggles. The students were given 45 min to solve a geometry problem in as many ways as possible. For our analysis, we first programmed MATLAB code to synchronize data from both participants' goggles; resulting in a video displaying both students' eye-movements projected on the task sheet, the sound recorded by the goggles, and additional information, e.g. pupil dilation. With these videos we expect to get insights into how students' attentions meet, if students' eye-movements follow one another, or verbal inputs, etc. We expect insights into promotive aspects in students' collaboration: e.g., if pointing on the figure or intensive verbal communication promote students' joint attention (cf. Nüssli, 2011). Finally, we think that the expected insights can contribute to existing research on collective mathematical creativity, especially to the question of how to enhance students' creative collaboration.

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INCORPORATING MINDFULNESS IN A MATH COURSE FOR PROSPECTIVE TEACHERS

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Mindfulness is the process of "paying attention in a particular way: on purpose, in the present moment, non-judgementally" (Kabat-Zinn, 1994, p. 4). Mindfulness-based interventions have been applied in hospitals, prisons, workplaces, as well as schools (e.g., www.mindfulschools.org) In a meta-analysis, Zenner et al. (2014) found that the effects of mindfulness-based trainings for children and youths in a school setting are strongest in the domain of cognitive performance (effect size of g = 0.80), followed by stress (g = 0.39), and resilience (g = 0.36, all three with p < .05).

The present study was conducted to investigate (1) the impact of watching a 5-minute video in each class (8 informational videos on mindfulness, 7 guided mindfulness meditation videos, and 4 videos on kindness) on their impulsiveness and mindfulness, and (2) prospective teachers' receptivity of mindfulness in a math course. Twenty participants in a geometry-measurement course for prospective teachers took a pre-post questionnaire consisting of two scales and an end-of-course online survey.

On the pre-post questionnaire, they reported less impulsive (p = .0001) on the *Barratt Impulsiveness Scale Version 11* which has 30 items such as "I do things without thinking". Interestingly, on the *Mindful Attention Awareness Scale* which has 15 items such as "I find it difficult to stay focused on what's happening in the present", students reported to be less mindful (p = .0002) probably because they were starting to notice their monkey minds.

On the online survey, 90% agreed or strongly agreed with statements like "I am more likely to practice mindfulness now than before taking MATH 2304" and "If I become a teacher, I will have my students engage in mindfulness practice on a regular basis"; 75% checked "it's relaxing and calm, a nice break from class routine"; and 55% checked "I now believe this will be helpful in my life." This exploratory study suggests that introducing mindfulness via videos may help participants to reduce their impulsivity, be aware of their mindfulness states, induce their interests in mindfulness, and potentially reduce their mathematical errors caused by impulsive disposition.

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FACILITATING CO-CONSTRUCTION OF MEANING IN CLASSROOM DISCUSSIONS

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The field of mathematics education has put a lot of effort into researching productive classroom discussion. As a result, there is a strong push towards implementing productive classroom discussion in U.S. math classrooms, and talk moves such as revoicing and wait-time are increasingly popular with mathematics teachers. However, many teachers still struggle with engaging students in discussions, listening to student thinking, building on student comments, and more importantly, extending an initial discussion to build new understandings. In light of these needs we investigated specific teacher language (i.e., questions and comments) as part of discussion practices that enable both teacher and student to co-construct mathematical understanding.

We conjecture that productive classroom discussions in the mathematics classroom should promote a balance (or transition) between the objectivist (see Elander & Cronje, 2016) and constructivist components of the learning process. In our framework the objectivist components refer to traditional pedagogical approaches for direct instruction. These components are rooted in behaviourism and feature such activities as explaining procedures or recalling facts. The constructivist components support socially situated learning opportunities engaging students in mathematical meaning-making (Jaworski, 2015). Teachers make decisions to select and sequence the transition of objectivist and constructivist components within and across the events of a classroom discussion.

In our session, we aim to (1) propose a framework of classroom discussion as interactions between the objectivist and constructivist components of student learning and (2) to present specific teacher language of two master teachers, one in the U.S. and one in Korea. Every country has its own tradition and culture regarding the teaching of mathematics. For example, Korean mathematics classrooms emphasize solving complex problems, while conceptual teaching and sense making have emerged as effective practices in U.S. classrooms. Our data of the teachers' questions and comments did reflect these different emphases evident in classroom discussions and help identify words and phrases that allow meaning to be communicated so that co-construction of knowledge in discussions can be produced and sustained.

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HOW DO CODING EXPERIENCES HELP BUILD ALGEBRA SKILLS OF SYMBOLS AND NOTATIONS?

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Learning algebra requires special attention be paid to mathematical symbols and how notations are combined to create meaning. Therefore, capacity to use and interpret different mathematical symbols and forms is an essential part of school mathematics. However, as students transition from middle school to high school or from school mathematics to abstract mathematics, they struggle to process mathematical symbols and notations. Thus, students need experiences using syntax and mathematical expressions to produce a mathematically meaningful product (Kaput, 1987). We conjecture that computer-programming experiences can provide this necessary practice. To test the hypothesis that coding is a way to foster student mastery of mathematical symbols and notations, we investigated the following question: How do students perform on an assessment of mathematical symbols and notations before and after they complete an elementary programming course?

Participants included students enrolled in an undergraduate *CS for All* course in the Fall of 2016 at a large research university in the southwestern US. The course taught programming through computational modelling exercises, with modules focused on developing NetLogo programming skills and teaching foundational concepts such as Boolean logic, algorithms, and recursion. Participants completed a pre/post assessment (N=19). Our assessment included 5 algebraic items consisting of mathematics symbols and notations. The following is a sample item from the assessment: "Let $\boxed{Y}$ be defined by $\boxed{Y} = Y + 2$ for any y. Calculate $\boxed{3} - 3 + \boxed{\sqrt{5}} - \sqrt{5}$." We analyzed their solutions and identified the types of problems our participants solved successfully. We examined the change in both the quality and quantity of student work between the pre/post tests to better understand student persistence towards working with mathematical symbols and notations.

Overall, 11 (58%) participants did better on the post-test with 7 participants (44%) demonstrating noticeable growth; 14 participants (74%) showed more persistence to solve problems on the post-test. In addition, the participants demonstrated increased understanding on two items regarding logic and algorithms. We plan to share our assessment items with a larger audience and discuss our analysis of the ways in which programming experience may have contributed to the participants' skills using and understanding mathematical symbols and notation.

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ENHANCING STUDENTS' MATHEMATICAL ARGUMENTATION IN PRIMARY CLASSROOMS

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Accumulating researchers suggest that students should have early opportunities to learn argumentation in classrooms. This study was intended to design a teacher professional program that supports teachers to design and enact conjecturing tasks across mathematics contents for developing young students' argumentation in classrooms. The focus of the study was to explore how students' argumentation was enhanced in classrooms where students engaged in conjecturing tasks over two consecutive years. The 24 students in one class were the targets of the study when they were in grade 3 to grade 4. The framework of the study for launching argumentation was adapted from Canadas and Castro's (2005) seven stages of conjecturing and modified them into five stages: constructing cases, formulating a conjecture, validating the truth of the conjecture, generalizing, and justifying the generalization. The data mainly consisted of 6 tasks, 12 videotaped lessons, and 12 worksheets from individuals and groups. The transcripts of videotaped lessons and students' worksheets were used for constructing argumentation structures. This study took Toulmin's (1958) scheme and adapted from Knipping's (2008) structure as an approach of analysing argumentation in classroom. The result shows that the warrants often used in grade 3 were the trivial facts relying on the superficial numerals, whereas the warrants used in grade 4 were relevant to mathematical properties. The students were gradually becoming independent without teacher's support for making convincing arguments with warrants. There is a consistent pattern that the teacher's involvement in students' argumentation in the consecutive two years were mostly devoted to helping students for completing the statement of conjectures and for evoking argumentation. The result indicates that the argumentation can be enhanced through practices even students were in grade 3 and grade 4. The factors contributing to the enhancement are discussed.

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THE IMPACT OF EXAMINER'S COMMENTS ON UNDERGRADUATE THESIS WORK

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Authoring an undergraduate thesis is a fundamental part of the Swedish teacher education. Thesis writing is perceived as serving the dual function of pointing students' focus to integral aspects of the teaching profession, preparation for development work as well as the provision of basic qualifications for graduate studies (Skolverket 1999; Arreman & Erixon 2015). Embedded in these goals is students' orientation to such competencies as; ability to understand, appreciate, critically examine, implement and produce scientific texts and in particular in mathematics education. Within the framework of thesis writing the examiner plays the role of a gatekeeper. In the present study, an undergraduate thesis in mathematics education is examined with focus on the interaction process between the examiner and the respondents with regards to achieving the aforementioned goals. In particular we are concerned with the following questions:

- What aspects of professional and scientific proficiency are focused on in the assessment process?
- How do the respondents engage the feedback from the examiner?

Using Kiley and Mullin's (2005) conception of research and a semiotics framework (e.g. Hoffmann 2011) an analysis of the examiner's feedback and the student's engagement with the same is applied. The students' reaction to the examiners comments is perceived as a conscious act of clarifying the *representamen* thus, revealing a space of mutual understanding facilitated through dialogue. It is observed on the one hand, an emerging tension between aspects of producing a scientific work generally, focus on professional demands and on the other hand perceptions of mathematics education.

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STUDENTS' APPROACH TO A MODELLING TASK WITH MULTIPLE ELEMENTS

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Modelling tasks provide students with opportunities to use mathematics to deal with messy real life problems. The process of which students solve modelling tasks can be described using modelling cycles (Borromeo Ferri, 2006).

This study aims to investigate students' modelling process and the way they approach a modelling task when the task involves more than one element. How do students engage in a modelling task with multiple elements?

This study took place in a grade 8 and a grade 9 (age 13-14) mathematics class, where students worked in groups to solve a modelling task. At the time of this study, students have little experiences with such tasks. The teacher (the author) acts as a facilitator by providing encouragement, prompting discussions, and mobilizing knowledge in the room. Data include in-class observations, field notes, impromptu interviews, post task interviews, and audio recordings of students' work in their group. In-class observations and field notes were interpreted, and audio data were transcribed. All data were analyzed against Borromeo Ferri's (2006) proposed modelling cycle.

Results indicate that students interpreted these elements as individual problems during their modelling process. They worked on these elements one at a time, and followed the modelling cycle to generate individual solutions for each of these elements. Data demonstrate that students approached each element differently based on their use of extra-mathematical knowledge, EMK (Borromeo Ferri, 2006). For some elements, students had sufficient EMK or looked up additional EMK. These students approached the elements from a real world perspective and applied general EMK, sophisticated EMK, and/or specific EMK to generate a reasonable real solution. For other elements, students lacked EMK or did not apply EMK and took a mathematical approach to generate a solution. This implies the approach students might take, which closely lead to how reasonable their solutions might be, depends on the EMK they have, their ability to activate and apply their existing EMK, and their willingness to look for additional EMK.

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METACOGNITIVE STRATEGIES FOR GEOMETRY: AN EXPOLORATORY STUDY

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Most research on metacognition in problem solving focused on the non-topic specific metacognitive strategies. Research such as Throndsen (2011) has shown that different topics would influence students' metacognitive activity and, Thorpe and Satterly (1990) also postulated that metacognitive strategies activated were specific to the problem from which it was derived and he questioned the transfer of such skills. This was also supported by Keleman, Frost and Weaver (2000) who suggested that different questions, even within a topic, might activate different metacognitive strategies and therefore, metacognitive strategies could be problem-specific.

The paper reports an exploratory study that aims to provide such insights to the type of metacognitive strategies students employed while solving problems on a Geometry topic, Angles. The sample comprises 783 Secondary One students (age 13 years old) in Singapore. They completed a problem-solving test comprising 2 mathematics problems on Angles with retrospective self-report of the processes involved in solving each problem.

In this presentation, qualitative data from the retrospective self-reports would be discussed in detail to provide preliminary evidence on the type of metacognitive strategies students employed in solving problems on Angles. The possible contribution by problem difference within the topic on the type of metacognitive strategies activated during problem solving will also be examined. Classification of metacognitive strategies will be based on Pólya's four phases of problems solving, based on and adapted from a few related literature (e.g. Garofalo and Lester, 1985).

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ASSESSMENT OF MATHEMATICAL CONTENT KNOWLEDGE OF PRE-SERVICE TEACHERS – CONSEQUENCES FOR A DEMAND-ORIENTED TEACHER TRAINING

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The problem of the double discontinuity as Felix Klein has already described it in 1924 is still a very current issue. Nevertheless, a satisfying solution for the introductory phase of teacher training has not been found yet. Many studies reveal the gap between the expected and the actual level of content knowledge (regarding to Shulman, 1986). This research project aims at assessing the actual level of content knowledge of pre-service teachers at the beginning of their teacher training. Therefore, we developed a test instrument which comprises three levels of mathematical content knowledge: 'Basic Mathematical Competences' (BMC), 'Secondary School Related Mathematics' (SSRM) and 'Elementary Mathematics from an Advanced Standpoint' (EMAS). In form of a dichotomous rating, the pre-service teachers could receive one point for each task, which equals a total of ten points per level.

The sample consists of 60 first-year pre-service teachers for higher secondary schools. On average, they achieved 58.2% (SD = 18.4%) on the first level (BMC), 46.5% (SD = 17.7%) on the second level (SSRM) and 4.2% (SD = 6.0%) of the total points on the third level (EMAS). Considering the fact that the majority of those pre-service teachers had just left school, the results on the first two levels reached unexpected low scales. Based on these findings, we want to offer a blended-learning tutorial in the first year of our teacher training programme. On the one hand, the pre-service teachers get the chance to revise school related contents on their own in an e-learning phase. On the other hand, we offer a weekly tutorial which provides the possibility to apply these contents to example exercises from future exams. An example of which might be the derivation rules which are necessary for future higher calculus exams.

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ASSESSING PRESERVICE TEACHERS' MULTIPLICATIVE KNOWLEDGE FOR TEACHING USING VISUAL MODELS

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This presentation describes a small-scale study of 52 U.S. preservice Kindergarten through 8th grade (K-8) teachers' multiplicative models. In this study, preservice teachers (PTs) performed a series of sequential multiplicative modeling tasks that share a similar mathematical structure but differ in numerical complexity. This study was guided by the theoretical perspectives of mathematics knowledge for teaching (MKT) proposed by Ball, Thames, and Phelps (2008) and the paradigmatic and narrative modes of knowing from Bruner's work (1985). Teachers' performance on visual models had been used to measure their MKT by researchers (e.g., Hill, 2010; Izsak, et al., 2012). The perspectives of paradigmatic and narrative modes of knowing from Bruner's work (1985) provide this study an ideal theoretical basis for developing a classification scheme to understand the richness and diversity of visual models. Two research questions were examined in this study. The first question is, "To what extent of correctness can PTs model a series of multiplicative tasks?" The second question is, "What do PTs' modeling choices and their distributions look like?" PTs were asked to illustrate how to solve a series of tasks consisting of multiplying a whole number by a whole number (4×5), a whole number by a decimal (4×0.5), and a decimal by a decimal (0.4×0.5). Findings indicate that PTs' performance of visual models depended on the numerical complexity. While the majority of PTs can model the whole series of multiplicative tasks, they modeled the multiplicative tasks of a whole number by a whole number and a whole number by a decimal better than they modeled those of a decimal by a decimal. Within a multiplicative task, the distributions of their modeling choices were not equal. The distribution patterns of their modeling choices were different across tasks. Further, their modeling choices and behaviours varied across tasks.

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IM/PERFECTION: A CREATIVE TENSION AT THE HEART OF MATHEMATICS

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Mathematics is often considered "perfect": a view that many students find discouraging. Yet mathematics is also fundamentally "imperfect" because it intrinsically implies working in disorder, accepting uncertainty and going along with ambiguity. But there is also a constant strive toward ordering and clarifying that is essential to mathematics. Could articulating these aspects benefit students? As a starter, I offer a theoretical framework that takes into account both the perfection and the imperfection of mathematics and their role in students' mathematical activity.

History shows how perfection and imperfection play important roles in mathematical innovation (e.g. Ormell & Blaire, 1996). The strive toward perfection was pushed to the limit in the formalist attempts of Russell, Whitehead and Wittgenstein to produce self-sufficient, indubitable mathematical texts. This work was extremely useful despite its failure, but only because mathematics itself and mathematicians' activity made such work necessary. On the other hand, incompleteness, imprecisions and oversights constantly allow the creation of new mathematical ideas (Kline, 1982). One way to see this interplay is to conceptualize (doing) mathematics in dialectical terms, what I call the "im|perfection" of (doing) mathematics.

Although scholars in mathematics education have not directly addressed the question of how the im|perfect nature of (doing) mathematics contributes to students' activity, a number of researchers' works can be revisited in this light. Borasi's (1996) view of errors as springboard, Rowland's (2000) observations on the role of vagueness, or Brown's (1993) attention to confusion are a few examples of the literature from which I have articulated five dimensions of the im|perfection of mathematics' creative tension: ambiguity|clarity, traces|ideas, intuitions|convictions, dispersion|discipline, limitation|expansion. I will introduce this framework with examples of these dimensions and their creative role in (students') mathematics, and for tasks design.

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CULTURAL TRANSPOSITION AS A THEORETICAL FRAMEWORK TO FOSTER TEACHING INNOVATIONS

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For Mathematics Education researchers the reflection on the meanings embedded in the educational practices of other cultural contexts can represent a possibility to rethink of the ones rooted in their own educational practice. With this mind, we propose the construct of *Cultural Transposition* to describe the process activated by researchers or teacher-researchers, who, got into contact with educational practices implemented in other cultural contexts, start to analyse them in order to think of the themes of educational intentionality. Starting from this experience and from the recent trends in Mathematics Teacher Education (see for example Wood, 2008), the researchers can design and implement professional development (PD) paths for teachers. In these PD courses, the researchers can present tools and methods used in different Mathematics Education practices, together with cultural and philosophical reflections connected to them, in order to induce the teachers to rethink of their typical educational practices. In other words, we propose to look at the differences as opportunity to think to our own *unthought* of – *l'impensé* (Jullien, 2006).

To clarify some of the features of Cultural Transposition, we will present an experience in which we have been directly involved. Starting from the coming into contact with a particular educational practice pertaining to a different cultural background, that in this particular case was the Chinese one (Mellone & Ramploud, 2015), we will show the analysis and the linked reflections we have developed. Then, we will describe the PD path implemented in the light of these reflections. Finally, we will present a particular teaching-learning experience carried out by one of the teachers who has been involved in this PD course.

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SECONDARY PROSPECTIVE TEACHERS' INTERPRETATIVE KNOWLEDGE ON SUBTRACTION ALGORITHMS AND ITS CONNECTIONS WITH ADVANCED MATHEMATICS

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One core element of teachers' practices concerns interpreting, give meaning and provide feedback to students' mathematical productions (e.g., answers, comments, behaviours). The work involved in doing so requires a particular kind of knowledge, that we have called *interpretative knowledge* (e.g., Ribeiro, Mellone, & Jakobsen, 2013). It corresponds to the knowledge required to interpret, make sense of, and explore the productions of students, in particular those that are based on non-standard approaches (approaches not expected by the teachers) or containing errors (Jakobsen, Mellone, Ribeiro, & Tortora, 2016). In some of our previous work we have focused on studying interpretative knowledge with topics that are aligned with the ones (prospective) teachers will have to work with in (their) future practice (e.g., fractions in primary school, power of ten in secondary school). Moving a step further in the direction of deepening our understanding on the nature and content of the interpretative knowledge, we explore prospective secondary teachers' interpretative knowledge when giving meaning (and providing feedback) to different students' productions on topics from primary school (elemental mathematics).

We will present and discuss prospective secondary teachers' interpretative knowledge (mathematic students in their fourth year of the Degree in Mathematics in Brazil) gained through their assessment of different subtraction algorithms, and the connections with an advanced mathematical knowledge for teaching. Such discussion will enhance the connections between the content of interpretative knowledge (and the particularities of the tasks aimed at developing it) and the advanced and elemental mathematical knowledge for teaching.

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IDENTIFYING LOCAL PROOF ‘MODULES’ DURING PROVING

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In reviewing PME research, Stylianides et al. (2016) identified developing effective teaching interventions as one of the under-researched areas in argumentation and proofs. In developing such an intervention, we take the structure of deductive proofs as one of the essential elements of understanding (Miyazaki et al. 2017). In considering ‘the structure of proof’, there are at least two aspects. One is the logical structure consisting of hypothetical syllogism and universal instantiations; another, as Mejia-Ramos, et al. 2012, p. 12) explain, is that to understand a proof entails “breaking the proof into components or modules and then specifying the logical relationship between each of the modules”. In this paper, we address the research question: What modules of proofs can be identified during the process of learning deductive proving? From our earlier research, we found that learners’ understanding of the logical structure advances from elemental, via relational, to holistic level, and that the relational has two aspect distinguished by universal instantiation and hypothetical syllogism (Miyazaki et al., 2017). Through observations of grade 8 geometry lessons, and corresponding to these three level of understanding, we identified three structure ‘modules’: 1) vague ‘chunks’ of propositions, 2) small networks with universal and singular propositions by universal instantiations, and 3) series of small networks. We found, for example, that a learner at the elemental level recognizes elements of proofs such as assumptions or conclusion without their logical relationships, and, as such, this learner’s proof ‘modules’ are no more than vague ‘chunks’ of singular propositions such as ‘AB=DE’ or ‘$\triangle$ABC$\equiv\triangle$DEF’. A learner who is at a relational level of understanding forms proof ‘modules’ in the form of small networked propositions such as ‘$\triangle$ABC$\equiv\triangle$DEF because SAS condition’ or ‘AB=DE because $\triangle$ABC$\equiv\triangle$DEF’ when trying to prove a given statement. A learner who is at the holistic level utilises proof ‘modules’ formed by a series of small networks such as ‘$\triangle$ABC$\equiv\triangle$DEF because AB=DE, angles ABC=DEF & BC=EF’, ‘AB=DE because $\triangle$ABC$\equiv\triangle$DEF’ and so on which is used to deduce a conclusion from given assumptions.

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NATURE OF “FUNCTIONS AND EQUATIONS” USING GEOGEBRA: FROM TWO DIFFERENT FRAMEWORKS

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In this research, we analyse 1 hour lesson in a unit “Functions and Equations III” corresponding to grade 9 in terms of two theories, together with the aim of networking theories (e.g., Bosch et al., *to appear*).

“Functions and Equations” is an integrated curriculum developed empirically to improve teaching and learning of equations and functions in Japanese junior high school (Mizoguchi & Yamawaki, 2016). The lesson, the first one in the unit, was intended for students to know that a locus of the curve (parabola) obtained through an experiment could be represented by $y = ax^2 + bx + c$. For this, students in pairs took a picture of tracks drawn by rolling a ball on tilted board, pasted it on GeoGebra. Then, they verified that the approximation of quadratic polynomial is best fitted by regression analysis of points plotted on curves. As a result, they got that all trajectories are represented as quadratic functions.

The first analysis is done with the theory of layers for mathematical activities (LMA): mathematical activity (MA) on lesson layer (MAL); MA on unit layer (MAU); and MA on curriculum layer (MAC). The second one is done with the anthropological theory of the didactic (ATD), especially according to the notion of praxeology (cf. Chevallard, 2016).

Through two analyses, MAL is characterized by pinpoint praxeology, MAU by local and regional praxeology, and MAC by global praxeology. These results show that there are similarities between ideas about complexity of activities in each theoretical framework. In contrast, the two theories have different ideas about nature of activities. In LMA, any MA can be regarded as carrier of mathematical contents or theories, which are components of MAs in viewpoint of ATD.

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TEACHERS' ENCOUNTERS WITH HORIZON CONTENT KNOWLEDGE: CASE OF TEACHING ALGEBRAIC IDENTITIES

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Algebraic identities (AI) play an important part in mathematics curriculum and in mathematics in general. Widespread research has been done to elicit and cognize students' errors of equality and of concepts around equality. Researchers report varied conception of "=" sign and how that functions in students' understanding of AI. Not much has been examined about the pedagogy for AI, given that these are the equations where students are exposed to the idea of "always equal" for the first time. This study is part of a larger study on investigating teachers' encounters with knowledge at the mathematics horizon (referred as HCK now onwards), and presents 6 cases of teaching algebraic identities. HCK refers to awareness of situating school mathematical ideas in the larger body of mathematics including practices of doing mathematics. Specifically, we present teachers' encounters with HCK while they give explanations, make use of representations, and provide mathematical goals along with applications of AI. We address two main questions: What representation, explanations and contexts are used to teach AI; what encounters did teachers' face with HCK while using these representations, explanations and contexts; and how did teachers manage these encounters? The data for this study were collected from schools that fall under a demographic boundary as used by the municipal administration of city in India called as a ward. The ward chosen represents a mix of all the classes, castes, languages and religions. The 6 schools chosen fall into categories of: Government, Semi-Government and Privately funded schools. In the first phase of analysis the classroom data was delineated for common content errors exhibited by the teachers. Transcription of the remaining teaching videos, was coded to understand teachers' encounters with HCK. The codes were developed based on the definition emerged from the two earlier conceptions of HCK given by Ball and Bass (2009) and Jacobson, Thames, Ribeiro and Delaney (2012). The six cases of teaching bring forward different metaphors that the teachers used in pedagogical trajectory they choose and in the meanings they project of the identity. These metaphors facilitated encounters with HCK illustrating how conception of AI through these explanations, representations and contexts created interference in learning the concept of variable. Further, we present analysis of area model used as representation for AI by teachers to understand disconnect between school algebra and algebra as a branch of mathematics. We conclude our analysis with two main structural understandings around HCK that the teachers need to develop. The first is the demarcation between formula, equation, identity, property and function, as they all could be represented as product of two quantities. And the second is to understand what does identity really mean? Especially in the representations, contexts and explanations that the teachers used in contrast to their meanings as mathematical entities.

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USE OF MANIPULATIVES AND LEARNER ERROR ANALYSIS TO STRENGTHEN FOUNDATION PHASE PRE-SERVICE MATHEMATICS TEACHERS' (PMT) MENTAL CONSTRUCTIONS OF NUMBER OPERATIONS

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The purpose of this study was to explore Foundation phase (PMT) proficiency of using manipulatives to model a solution in number operations and to explore their competencies in identifying learners' misconceptions to number operations problems. This study was conducted with 30 PMT enrolled for a primary mathematics module and gave consent to take part in the study. The data was generated from the written responses to tutorial tasks, tutorial test, major test and a final examination. In this paper, we used APOS (Action- Process- Object - Schema) as a theoretical framework to illuminate PMT learning of number operations at a certain university in South Africa. ACE (Activities-Classroom/ tutorial discussion-Exercise) teaching cycle was used.

Teaching and learning with the use of concrete models are recommended for the development of number concept. In South Africa, the Curriculum Assessment Policy Statement (CAPS) (2012) for grade 1-3 emphasise the development deep conceptual understanding of mathematics and acquisition of specific skills and knowledge. For example development of number vocabulary, number concept and calculation and application skills. Therefore, it is important that PMT themselves develop this knowledge and skills in order to be effective mathematics teachers.

The findings revealed that while PMT knows how to use standard algorithms to compute numbers they have trouble to model the solution using manipulatives. The results further revealed that while PMT seems competent in finding solutions to number operation sums they encounter difficulties with identifying and interpreting learners misconceptions. Although these difficulties were observed from the written response there was evident during interview discussion that allowing PMT to talk about their solution gradually helped them to improve their mental constructions of number operation.

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A DELEUZIAN PERSPECTIVE ON MATHEMATICS LEARNING

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In describing mathematics learning, some existing theories that rely on a priori assumptions about learners tend to consider learners as well-prepared subjects. In particular, this perspective is problematic in explaining the creation of new mathematical knowledge. The constructivist approach, for instance, is limited in explaining learning about something new that was unknown before in that it is based on the premise of agential individual assuming definite identity (Roth, 2016). In this context, we investigated the epistemological perspective of French philosopher Gilles Deleuze, which is expected to shift our viewpoint of learning and address the limits mentioned above.

Deleuze(2004) criticizes that the explanations of human thought that focus only on voluntary and conscious dimensions do not cover complex and ambiguous aspects of human thought, especially when new ones are created. What he notices as an occasion for creative and dynamic thinking is the potential 'difference itself' that is latent in the real world. It refers to the pre-linguistic differences before ordering and organizing in a specific system or form differentiated by human or culture. From his point of view, the starting point of learning is a situation in which thinking is forced by an encounter with a strange object. This encountered object emits innumerable signs, which are not elements in a symbolic system that have a regular meaning, but ambiguous flows of intensity. Therefore, the learner cannot distinguish the signs conceptually and only can sense them.

As a result of our study, mathematics learning from Deleuze's learning perspective occurs unintentionally and involuntarily by encounter with strange mathematical objects. In the preconceptual dimension where new mathematical knowledge is learned, learning subjects and objects are not clearly distinguished. A learner can create new meanings when he or she experiences potential generative capabilities within the signs emitted by mathematical objects or tasks. Within the situation that should be given to learners in order to trigger creative mathematics learning, mathematical knowledge should be latent as a pre-linguistic and sensory quality prior to being differentiated and organized within any formal framework or system.

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HOW SOME PARAMETERS OF WORD PROBLEMS INFLUENCE PRIMARY SCHOOL PUPILS' SUCCESS IN SOLVING THEM

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In this contribution, we characterise word problems as problems that include some context (real, real-like or imaginary) within which some numerical data are given and a question (questions) is posed for pupils to solve using their mathematical knowledge and out-of-school experience (Novotná, Vondrová, 2017). Word problems are one of the areas considered by teachers as difficult for their pupils. In our research, we look into parameters (mathematical, psychological, linguistic) which might influence the difficulty of word problems. The sample (n = 1700) consists of pupils between grade 3 (age of 9) and grade 9 (age of 15). Each class is planned to be followed for two years. The sample is divided into four equally abled groups, based on pupils' results in the initial testing from mathematics and Czech language and reading.
Based on literature review, analyses of word problems from international comparative studies and of textbook problems, we identified about 70 parameters, out of which 44 are of linguistic nature (e.g., several levels of communication – Version A: A pen costs 1 zed more than a pencil. It is possible to buy 2 pens and 3 pencils for 17 zeds. How many zeds are needed to buy 1 pen and 2 pencils? Version B: Joe knows that a pen costs 1 zed more than a pencil. His friend bought 2 pens and 3 pencils for 17 zeds. How many zeds will Joe need to buy 1 pen and 2 pencils?) and 14 are mathematical (e.g., formulation of numerical data in the assignment in numerals/symbols or words). In the first test assigned to the above pupils in February 2017, the influence of several parameters was investigated. Pairs of word problems differing in one parameter only were assigned to different groups of pupils. As we have four equally abled groups of pupils, we could investigate the mutual relationship of two parameters for each age group. Pupils' written solutions are analysed in terms of their correctness, used solving strategies and mistakes. In the presentation, some results will be given, namely which parameters seemingly influenced the difficulty of word problems and/or pupils' approach to their solution and in what way. This information has important implications for teaching and test makers.

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CLASSROOM CULTURE AND GENRE FOR DEVELOPING NARRATIVELY COHERENT MATHEMATICS LESSONS

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"Structured problem solving" as a cultural pattern for constructing mathematics lessons has been identified in the context of lesson study (Stigler et al., 1999). However, this pattern doesn't necessarily ensure the quality of a mathematics lesson, because it is commonly adopted by even novice teachers in Japan. Thus, research is necessary to clarify what aspects of mathematics lessons contribute to lesson quality. We have focused on the narrative nature of classroom lessons (Okazaki et al., 2016) and how the quality of a mathematics lesson on structured problem solving is determined by narrative coherence and distinct interactions between teacher and students. Whether classroom cultures can make such interactions narratively coherent within the lesson remains an issue.

In this paper, I adopt as the theoretical perspective Bruner's (1996) cultural psychology in which he explores the relationships between culture and human cognition, meaning-making in a cultural context, and the roles of narrative in a culture. He also mentioned genre, i.e., "culturally specified ways of both envisaging and communication about the human conditions" (p.136), according to which our understanding may be formed and reinforced. Using this perspective, we hypothesized two kinds of genre for forming mathematics lessons that may affect quality: formulating a solution and exploring the mathematical meaning behind the solution.

To demonstrate this, we examined two types of second grade classroom lessons, which corresponded to the above two genres, respectively. The topic was reverse thinking of addition and subtraction. The results of qualitative analysis using grounded theory suggest that the lesson of "exploring mathematical meaning" is characterized in terms of (1) a classroom culture of cultivating students' questions of why, (2) encouraging students to use mathematical core ideas to explore the why question, and (3) a classroom setting that symbolically presents these core ideas.

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USAGE OF DIFFERENTIATED TASKS IN 5TH AND 6TH GRADE MATHEMATICS CLASSROOMS

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Gifted students have differentiated needs when compared to their peers; thus, they should have different levels of support in classroom in line with their differentiated needs (Gavin, Casa, Adelson, Carroll, & Sheffield, 2009). Moreover, efforts designed for gifted students could be an opportunity for other students in classroom. Therefore, this study aims to assess usage of differentiated tasks in regular mathematics lessons designed for both mathematically gifted and regular students in the classroom. Differentiated tasks developed for satisfying educational and developmental needs of mathematically gifted students (Özdemir, 2016) were applied to all students in regular classrooms. In these classrooms, students were taught mathematics according to the regular curriculum and teachers integrated these differentiated tasks into their lesson plans. In these 5th and 6th grade classrooms, there were totally 226 students and among these students, 32 of them classified as mathematically gifted by means of nominations and scores obtained in Test of Mathematical Abilities of Gifted Students (Ryser & Johnsen, 1998). Data about usage of differentiated tasks was gathered by means of classroom observations, individual interviews, and after sheet forms of teachers and students. During data collection process, constant comparative analysis lead to construction of meaningful categories described under the findings of the study. More specifically, usage of differentiated tasks in mathematics classrooms was addressed from two different aspects as contributions to the gifted students and to the regular students. That is, it was seen that usage of differentiated tasks contributed to mathematically gifted students' cognitive potentials by meeting their needs for exploration more on the concepts, critical thinking, and challenge. On the other hand, data also reflected contributions to the regular students because tasks helped them to move one step further to see the holistic, interesting and relational structure of mathematics they learn in the classroom. Thus, this study could address a way to show that meeting special needs of gifted students through differentiated tasks could also provide opportunities for regular students in the classroom, too.

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DEVELOPMENT OF THE EMBODIED-BASED ACTIVITIES ON THE CONCEPT OF INFINITE SETS-COMPARISON: A CASE STUDY

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Students have showed difficulties leading to erroneous conclusion of the comparison of the infinite sets, for instance overgeneralization of finite collection methods (Tsamir & Tirosh, 2007). Raising awareness of counter-intuitive nature of infinity by encouraging students' concept of Pairablity is crucial (Tsamir & Tirosh, 2007). According to embodied based instruction, the concept stemmed from finite experiences. The finite experience are metaphorically extended to conceptualize the comparison of infinite set using pairability (Núñez, 2005). However, learning activity following this idea was rare. In this study, we figured out the effects of using developed activities called the Embodied-Based activities of Infinite set Comparison (EBIC) which is designed theoretically based on the Basic Mapping of Infinity (BMI) originated by Núñez (2005). The EBIC comprises of 4 parts which progressively encourage participants' conceptualization of Pairability driven by the comparison tasks. The tasks were presented gradually from finite context to infinite context, countably infinite sets. There were 2 participants, grade 10 and grade 11 students, who have never learned about the equivalency of the infinite sets. Four 1-hour group clinical interviews were conducted. Participants' response and actions were videotaped willingly. In addition, content analysis was used to analyze participants' responses in the identical paper-based Pre- and Post-tests. The results reveals progression of each participant when walking through the EBIC. The series of the activities stimulated metaphorical construction of a concept, Same Number AS IS Pairability (Núñez, 2005). From analyzing the corpus of data, it was found that, firstly, the activities evoked intuitive thoughts, including "Same Number As" and "More Than". They were re-conceptualized and utilized as the preference criteria in comparison of finite collection tasks instead of counting. After doing take-away activity, participants recognized the significance of 1-1 correspondence. In the infinite context, the newly concept, Pairability, was successfully extended to compare the cardinality of the infinite sets. In addition, it becomes preferable method of reasoning in comparing infinite collection. The post-test demonstrated the consistent awareness of 1-1 correspondence, namely pairability, while the reasoning of both participants is inconsistent in pre-test. In summary, the concept of infinite set comparison is likely to be constructed by students based on an extension from finite to infinite context. However, a further study is required to investigate transferability and usability of the concept constructed through embodied-based idea in a larger group of samples.

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THIRD GRADERS' UNDERSTANDING OF EQUIVALENCE AND THEIR EQUATION SENSE

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As arithmetic has been regarded as the main context for early algebraic thinking, many studies have been conducted to probe children's understanding of the equal sign, expressions, and equations. Such studies emphasize that the children with a relational understanding of the equal sign are able to solve difficult equations, which indicates a direct link between the knowledge of the equal sign and algebraic thinking.

Given this background, this study examined third graders' early algebraic thinking. Assessment items from Blanton et al. (2015) were used with minor revisions, as they are sufficiently comprehensive by including big ideas in early algebra. The students in this study solved the assessment items in 40 minutes. A total of 197 students' written responses were analysed for correctness and/or strategy use. In addition, unstructured interviews with nine students were conducted to investigate their reasoning processes in detail. The interviews were audio-taped and transcribed.

The overall result of this study showed that our students performed well in figuring out a missing value in the equation, evaluating an equivalence relationship, and selecting a generalized algebraic expression on the basis of particular examples. However, they had substantial difficulties in understanding algebraic expressions with variable notation. This paper specifically focuses on how students solved a simple linear equation ($3\times\square+2=8$) and justified their answer. Mathematics textbooks in Korea do not deal with equations with two operations until Grade 3. However, the percentage of correct answer to the item was the highest. More interestingly, most students used a different strategy (coded as Equation Sense) on the basis of the understanding of the equal sign and expressions from either "Guess and Test" or "Unwind" strategy. The students worked through the equation in a forward manner but seemed to notice the underlying structure of the given equation as a whole by seeing $3\times\square$ as an object. Using equation sense was consistent when the students were asked to solve extra simple equations in the interviews. As such, this study is expected to reveal students' early algebraic thinking development under the current mathematics curriculum. It also indicates that early algebraic thinking can be fostered as a specific form of thinking while students learn typical content areas.

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KOREAN PRIMARY SCHOOL TEACHERS' KNOWLEDGE OF FUNCTIONAL THINKING FOR TEACHING

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Functional thinking plays a significant role in developing algebraic thinking of elementary students (Carraher & Schliemann, 2015). Patterning activity related to functional thinking has been fairly common in most elementary mathematics curricula (Kieran, Pang, Schifter, & Ng, 2016). However there has been lack of research on how primary school teachers may encourage students to be engaged in functional thinking through patterning activity.

Given this background, this study surveyed 119 Korean primary school teachers to investigate their knowledge of functional thinking for teaching. A written assessment for this study was designed in a way to reflect on patterning activities appeared in current elementary mathematics textbooks series of Korea. The assessment items included knowledge of mathematical tasks, instructional strategies, and mathematical discourse in relation to teaching for functional thinking. The items consisted of 7 open-ended and 2 short responses.

A preliminary result of this study showed that, regarding mathematical tasks, about 70% of the teachers were able to design tasks corresponding to both the additive relationship (y=x+2) and the multiplicative relationship (y=2x). However, only about 40% of them were successful for the liner relationship (y=2x+2). For the instructional strategies to be used in teaching for functional thinking (e.g., using a T-table in which two variables are written in a non-sequential way), about 50% of the teachers were able to justify such strategies in terms of promoting fuctional thinking. In contrast, half of them described a few benefits of such strategies that were not related to the correspondence relationship between two variables. For mathematical discourse, about 60% of the teachers were able to diagnose students' errors or misconceptions which might appear in learning correspondence relationships and then to prescribe several types of feedback (e.g., using reflective questions, making a T-table, or drawing a diagram) for students to notice their errors. Building on these results, this study is expected to provide implications on what aspects of knowledge is further needed for elementary school teachers to promote students' functional thinking.

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IMPACTS OF A COMMUNITY LEADER ON TEACHER PROFESSIONAL ENGAGEMENT WITHIN FACEBOOK GROUP

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In remote areas, an online community such as Facebook Group (FG) is an alternative form of informal teacher professional development (PD). Specifically, FG with affordable bandwidth requirements enables for sharing and collaboration easily. However, from Community of Practice (CoP) theory, one of the most important elements in a community's success is the strength of its leadership. That is a community leader who supports the community to focus on its domain, maintain relationships, and develop its practice (Wenger, McDermott, & Snyder, 2002). The challenge for FG is that participation is voluntary and finding factors that can make a vibrant community is not something that can be invented easily. This study investigated the influences of a community leader's (CL) engagement on teacher professional engagement within an educational FG.

The study involved a mathematics educator as the CL located in Australia and 36 mathematics secondary school teachers (MT) from 36 schools located in 8 districts in Indonesia. Facebook analytic (http://grytics.com) was used as it provided a way of quantitatively analysing FG activities including specific statistics on members' activity and key features (e.g., top post and influencers). Data collection procedures were: (1) Identify the CL as the FG member who has been recorded by Grytics as the top influencer; (2) Download the CL activities (i.e. number of posts, comments, reactions, and engagement scores) from May 2015 – December 2016 via Grytics; (3) Identify two periods where the CL engaged with FG the most (i.e. 27/5 – 31/8/2015) and engaged with FG the least (i.e. 26/4 – 31/7/2016). Both periods were considered as "equivalent" as both included a 3-week holiday period and 5 days of face-to-face PD; and (4) Download the MT activities within the 2 periods. In total, we analysed 168 posts, 1029 comments, and 1544 reactions.

The influences of the CL were identified from the quantity and quality of the teacher's activity from Period 1 to Period 2. This analysis suggests that the CL's engagement influenced the level of members' engagement: 55% of teachers decreased their engagement when the CL was less active. From Period 1 to 2, the richness or the depth of mathematical and pedagogical conversations also decreased. This study enhances the concept of CoP and the quality of CoP largely depends on the voluntary engagement of their members and the emergence of internal leadership.

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THE COGNITIVE DEMAND LEVELS OF TASKS BETWEEN BRAZILIAN, TAIWANESE, AND SINGAPOREAN MATHEMATICS TEXTBOOKS: A CASE STUDY OF THE PYTHAGOREAN THEOREM

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Textbooks are vital tools in the learning process of mathematics. Tasks compose most textbooks, and the efficient learning of these tasks may influence in students' performance. Recent results of PISA exam have shown that students from both Singapore and Taiwan have a high achievement on math, on the contrary students from Brazil have one of the lowest performance on PISA. Since the processes required to solve a mathematical task are related to its cognitive demand (Stein & Smith, 1998), it is important to investigate how textbooks employ cognitive demand on its exercises. From the above, the authors of this paper decided to analyze whether textbooks from Brazil, Taiwan, and Singapore are significantly different according to the level of cognitive demand in exercises proposed by these textbooks.

The framework of Stein and Smith divides the cognitive demand of mathematical problems into to two different levels, lower-level and higher-level demands. Hence, the authors used their framework to code the exercises on the three textbooks to draw results and conclusions.

It was found that the Brazilian textbook gives a little opportunity for students to practice, while the Taiwanese and the Singaporean textbooks provide a significant amount of tasks. After running a proportional test, the authors also found that the Singaporean and the Taiwanese textbooks are more different from each other on their approaches on different level of cognitive demand, but students from both countries still perform very closely. Leading the authors to the conclusion that not only textbooks, but also other factors such as curriculum, instruction, and school practices perform an important role on students' performance on exams. Further information of this research is going to be presented on the conference.

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SOUTH AFRICAN CHALLENGES TO TRANSITIONING FROM RITUAL TO EXPLORATIVE PARTICIPATION

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While much research has focused on student learning from Sfard's (2008) 'commognitive' perspective, recent studies focus on teaching from this perspective and on how in some contexts, teaching may in fact work against the transition from ritual towards explorative participation (Heyd-Metzuyanim & Graven, 2016; Adler & Sfard, 2017). This presentation focuses on the relationship between teaching and ritual and explorative participation.

The research goal is to investigate reasons for the persistence of the ritual mode of teaching and learning in two South African schools. We offer an empirically-based explanation focused on aspects of South Africa's 'linguistic complication' and the mathematical knowledge 'gaps' which are most acutely felt in poorer schools.

The data are derived from a broader qualitative case study, focused on classroom talk, of two Grade 4 mathematics teachers. Methods included regular teacher interviews and classroom observation with video recordings over a four-week period. Drawing on data derived from interviews with the two case study teachers and transcriptions of their Grade 4 mathematics lessons, we argue that while teachers may introduce learners to new concepts initially ritually, with the intention of moving towards more explorative participation later, these explorative opportunities may become eroded.

We demonstrate how this erosion is a function of both linguistic challenge (the teachers must teach in English, even while teachers and learners are mother-tongue isi-Xhosa speakers), and curriculum pacing and coverage demands (which undermine possibilities for remediating cumulative gaps in learners' foundational knowledge). Learning the discourse of mathematics so as to participate beyond a ritualised level requires conceptual understanding, the development of which rests on linguistic and mathematical proficiencies, yet millions of young South African learners are deprived of access to their greatest source of linguistic capital, their mother tongue, and - through tight curriculum pacing - are denied the opportunity for revisiting earlier foundational concepts needed for enabling explorative participation.

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A DIGITAL ENVIRONMENT FOR DIAGNOSIS AND SUPPORT OF BASIC MATHEMATICAL KNOWLEDGE (SECONDARY LEVEL)

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Basic mathematical knowledge (BMK) (Feldt & Bruder, 2014) is a necessary foundation for further learning due to the deductive structure of mathematics. Academic deficiencies relating to the lack of BMK often become highly noticeable at the transition between different school levels and require substantial tangible support. The aim of the project is to provide an effective online learning environment to support BMK. Therefore an adaptive digital testing instrument in combination with learning materials regarding the topic of functions and elementary algebra has been developed. Specific Models about learning activities and learner actions adopted from activity theory (e.g. Bruder & Schmitt, 2016) framed the conceptualization process of the supportive measures. Following the format of a design-based case study the developed learning material and the diagnosis tool were tested in iterative cycles (Cobb, 2003). The research questions are primary explorative: Which typical deficiencies relating to BMK and which student profiles at the beginning of upper secondary school can be identified? Which design elements are appropriate for supportive material? Which (design) elements of the supportive material help specifically to reactivate BMK?

During the main trail the participants solved the adaptive digital diagnosis test first (N=2243). Subsequent to the digital test, the students received automatically an individual feedback that contained indications for possible learning materials that are combined with the test instrument. In interviews (N=32) and student questionnaires (N=336) the interaction of the learners with these supportive materials was evaluated. To analyse the effectiveness of the developed supportive concept the students had to solve a parallel post-test. The results of the pre- and post-test are examined regarding possible learning effects due to the use of the materials and will be presented.

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AN ANALYTIC FRAMEWORK FOR ASSESSING LEARNERS MATHEMATICAL DISCOURSE

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We present here an analytic framework for assessing learners' mathematical discourse. The framework is based on the constitutive elements of mathematical discourse in instruction (MDI) framework developed by Adler and her colleagues for analyzing what is made available to learn in classroom lessons (Adler & Ronda, 2015) and in textbooks (Ronda & Adler, 2016). These elements are the object of learning and four typical instructional tools: tasks, examples, naming and substantiations. Here, we now adapt the MDI framework as analytic framework for assessing learners' mathematical discourse (LMD). The adapted framework consists of the elements of the MDI framework. We argue that learners' engagement with and use of tasks, examples, naming, and substantiations in a task-based interview illuminates the degree of learners' mathematical discourse in relation to the object of assessment (OoA). This study is part of an exploratory study in the Philippines which put the MDI framework to work in developing a common language, framework and discursive artifacts for use in professional development courses centered around the three key tasks of teaching: designing lessons, analyzing the implemented lesson and assessing learning. We report here the analytic framework for the third component of the study. Its design involves comparing the discourses between the intended OoA and the implemented OoA. The intended OoA refers to the content and capability as defined by the interview tasks while the implemented OoA is the object of the discourse in the actual interview. The learners' mathematical discourse refers to what the learners bring into the exchanges in terms of the tasks and examples they were able to work with including new ones that they bring into the interview, their use of words and notations, and how they substantiate their claims. We illustrate the use of the analytic framework using data from our interview of pairs of Grade 11 learners about function. Our initial results show the potential of the framework for analyzing what learners have internalized from their participation in pedagogic activities about function.

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PROBLEMS WITH CONCEPTS IN ANALYSIS—THE EXAMPLE OF EXTREME POINTS

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When students start their first semester in mathematics at university in Germany many fail their first analysis course. One reason for this is an insufficient understanding of basic concepts, like monotonicity or differentiability. In this study we concentrate on the concept of extreme points which has strong connections to both monotonicity and differentiability. Regarding this concept we want to exemplarily analyse students' problems as well as some of the possible reasons for these problems—from several theoretical perspectives. We concentrate on three constructs: the German theory of Grundvorstellung (vom Hofe, 1995), the concept image (Tall & Vinner, 1981) and the conceptual change theory (Posner et al., 1982). Our research questions read as follows:

Q1. Which normative conceptions ("Grundvorstellungen") do exist regarding the concept of extreme point and how can these help to classify problems with it?

Q2. Which problems do math students have concerning extreme points?

Q3. What are possible reasons for these problems?

Q1 was answered by a subject-matter analysis and discussions with experts. To answer Q2 and Q3, a qualitative study was carried out: about 10 guided interviews on tasks related to extreme points were transliterated. The study was addressed to students after their first semester analysis (future high school mathematics teachers and pure math students). The transcripts were analysed using qualitative content analysis. Accordingly, categories were built. Regarding Q2 the occurring mistakes were categorized (e.g. students claim that extreme points can only exist if a function is differentiable). With regard to Q3, the above mentioned theories were taken into account. As reasons for mistakes are complex, we connected our analysis to several theories, and even on the basis of these perspectives we are aware that it is not possible to identify all possible reasons. An example of an identified reason could be the category "knowledge from school" (e.g. students do not identify extreme points at the boundary of an interval or constant functions, claiming "I think my teacher said 'A minimum that decreases and increases' and that's in my mind"). Although the analysis is not fully completed yet, preliminary results will be discussed in the presentation.

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MATHEMATICAL CRITICAL THINKING: A QUESTION OF DIMENSIONALITY

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Critical thinking (CT) is a construct often used to describe central educational goals especially in higher education. CT encompasses abilities like analysing arguments, claims, or evidence; making inferences using inductive or deductive reasoning; judging or evaluation and making decisions; or solving problems. According to dual process theory, cognitive activities can be distinguished into a fast, automatic, emotional, subconscious ("type 1") and a slow, effortful, logical, conscious ("type 2") subset of minds. Stanovich and Stanovich (2010) proposed a tripartite extension that separates type 2 thinking into algorithmic and reflective thinking.

Based on this theory, two different models can be conceived that explain the activation of CT: (1) A one-dim. model that comprises all activations of CT as type 2 thinking. (2) A two-dim. model that distinguishes between (2a) intuitive solutions (type 1) that are checked by algorithmic thinking, whereas (2b) conscious solutions (e.g., obtained by calculations) that are critically reflected upon by reflective thinking. The research question for this article is which model of CT better fits the empirical data.

A paper-and-pencil test (Rott & Leuders, 2016) for CT has been used in two large samples of mathematics pre-service teachers (study I: n = 150; study II: n = 468). Both times, the tasks were rated dichotomously. To answer the research question, exploratory factor analysis and Rasch modelling have been pursued independently.

Results of both methods suggest that mathematical critical thinking – as measured with the test at hand – is a one-dimensional construct: Regardless whether a solution has been obtained subconsciously / intuitively or consciously / by calculations, checking and critically reflecting upon the solution seems to involve the reflective mind. This finding could have implications for all situations in which solutions have to be reviewed critically. Mere algorithmic reviews of a solution might not be sufficient if there is no disposition to engage in hypothetical (reflective) thinking.

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SCALING UP PROFESSIONAL DEVELOPMENT IN MATHEMATICS CURRICULUM REFORM: INFLUENCES FOR THE MULTIPLIERS WORK

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A Portuguese national long-term professional development programme for support mathematics teachers in curriculum change took place from 2006 to 2012. More than 1000 schools and 12500 mathematics teachers were involved. The Steering Committee (SC) responsible for developing this programme, a group of 8 teacher educators and mathematics teachers (including the authors of this paper), decided to adopt a cascade model (Maaß & Artigue, 2013), recruiting 80 experienced mathematics teachers selected from a national call, chosen by their curriculum, and covering all the regions of the country. These teachers get support from SC every year, consisting in a national intensive course of two weeks and an extensive support of regional regular meetings. Working as multipliers (Krainer, 2015), they provided professional development (PD) to teachers in schools, based on the PD received from the SC. One of the major concerns of this model is the question of how much can actually be handed down the cascade (OCDE, 1998). The aim of our study was to understand what the multipliers perceived as fostering and hindering factors for their work, in order to identify how the different contexts of the cascade model interact and affect its effectiveness. Data was collected from six annual questionnaires applied to all the multipliers. Content analysis of their answers to the questionnaires' open questions followed an inductive approach, being categories defined from transversal themes that strongly emerged from the data. The results reveal that the multipliers are directly affected by many interrelated factors from different contexts (Krainer, 2015): the support given by the scientific commission and their pears are fostering factors; the educational policy (different curriculum in schools and decreasing work conditions provided) and the curriculum school's culture are hindering factors. This suggests that the effectiveness of multipliers' work is nor top down, nor bottom up defined — instead, it results of all the contexts in multiple ways. The perceived changes affecting the cascade functioning along the six years, suggest that cascades that develop in a long time period need to acknowledge its dynamic nature.

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STUDENTS' INVOLVEMENT IN MATHEMATICS ASSESSMENT

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It is expected that mathematics teachers' practices include summative and formative assessment practices, namely because formative assessment increase student performance (Santos & Cai, 2016). However the articulation between these two assessment practices is complex (Bennett, 2011), not consensual and still underexplored (Taras, 2005). In the context of a collaborative work, two mathematics teachers and two researchers (the authors of this communication) developed an articulation assessment process (AAP) that includes a cycle of three steps. It begins with a summative moment, a written test. Followed by a formative assessment moment, where students develop a set of questions similar to the ones included in the test, performed outside the class, and ends with a summative one. Each student has to answer only to the questions that they missed on the test. The mark of the first test can be changed. The objective of the study was to understand students' involvement in this APP. In particular, the research questions were: a) How students faced the process?; b) How many students were involved?; and c) Which reasons explained the involvement? Following an interpretive methodological approach, this study uses quantitative and qualitative data. 168 students (37 of grade 7 and 131 of grade 8) of 7 classes participated. The data was collected through a questionnaire to all students and semi-structured interviews to nine students (with different level of mathematics achievement in the tests as well as participating or not participating in the process). Content analysis was the method to analyse qualitative data.

A large majority of students considers that the APP was useful and helps them to learn, although only 63% of them were involved in the APP. The most frequent negative feedback (11%) was that the process was time consuming. The results pointed that students had a positive perception to APP, but the dominant school culture based on summative logic explains why students that had a low possibility to obtain a positive mark on the test were not, in general, engaged in the process. APP creates tensions not only in teachers (Bennett, 2011), but also in students, through a cognitive dissonance.

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THE DEVELOPMENT OF A HYPOTHETICAL LEARNING PROGRESSION FOR GEOMETRIC REASONING

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This paper reports the development of a learning progression that provides an explicit validated mapping of students' growth in geometric thinking. Following the establishment of a hypothetical learning progression assessment items were written and student data collected. Using Rasch analysis, eight learning zones were identified.

The results of international assessments indicated a significant decline in Australian students' mathematical literacy rates. Year 8 students performed at a level significantly lower than the international average in geometry (Thompson, 2010). Part of a larger investigation into the development of mathematical reasoning in the middle years of school, the aim of this project was to develop and validate a learning progression in geometric reasoning.

Learning progressions are a set of empirically grounded and testable hypotheses about students' understanding of, and ability to use, specific discipline knowledge within a subject domain in increasingly sophisticated ways (Corcoran, Mosher, & Rogat, 2009). The levels are constructed based on empirical validation and provide explicit emphasis on the growth of students' thinking in terms of their growing understanding of content

Using an extensive literature review as a basis a hypothetical learning progression (HLP) was established. Assessment items requiring problem solving and reasoning were written based on the HLP. These items were trialled with 755 students from grades 5-10 in 6 schools and scored with a rubric which attended particularly to the reasoning used. Rasch analysis was used to establish the final learning progression which documents what the students were able to do in each of eight zones. This has been subject to further use and refinement with a larger cohort of students.

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MENTORING AND COLLABORATING ON LESSON PLANS AS A MEANS FOR TRAINING MATHEMATICS TEACHERS TOWARDS TEACHING HIGHER LEVELS

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In recent years, there is a gradual decline in the number of students, who study mathematics at the highest level taught in high schools in Israel. This situation is partially due to a shortage of teachers who are willing and able to teach it. Attempting to respond this problem, we initiated a 3-year project, two of whose components are: (i) A mentoring process in which a teacher who has extensive experience in teaching high-level mathematics serves as a mentor to a teacher who has not yet experienced it. Evidently, mentoring is an approach that supports schools in becoming professional learning communities and in improving teachers' professional practice (Kadji-Beilran, Zachariou, Liarakou & Flogaitis, 2014); (ii) Collaborative writing of detailed lesson plans utilizing Ramzor software, a special software developed for this purpose. According to Mourshed, Chijioke & Barber (2010), joint lesson planning is a valuable and productive way for teachers to share knowledge and exchange ideas.

Twenty-two schools partake in the 3-year project, with 22 mentors and 28 mentees (1-2 in each school). In the first year of the project, the mentees started to teach 10th grade at the highest level of mathematics, and continued with the same class to 11th grade in the successive year. This year they are teaching 12th grade, bringing students to the matriculation exam. The study that accompanies the implementation of the project indicates that the mentoring approach has a great impact on mentees' sense of efficacy to teach high-level mathematics and a meaningful contribution to their mathematical and pedagogical knowledge. The joint lesson planning allows the mentees to learn about considerations that should be taken when planning a high-level mathematics lesson, and to be exposed to a wide range of didactic ideas and approaches. Hopefully, these two components will became mentees' sustainable work habits in the future.

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EYE MOVEMENTS IN EMERGING CONCEPTUAL UNDERSTANDING OF RECTANGLE AREA

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According to ecological dynamics approach (Abrahamson & Sanchez-Garcia, 2016), mathematical concepts emerge from personal embodied activities. First manifestation of a new concept emergence can be disclosed as specific eye-movements that signify an attentional anchor, namely a new way of perception that accompanies a new sensory-motor scheme. This research is devoted to an area concept, while previous empirical data within this approach were obtained in the studies of a ratio concept. Principles of embodied design (Abrahamson, 2014) were implemented in interactive computer tasks on the area of a rectangle. The top left vertex of the rectangle was fixed in the left top corner of the monitor, while a participant could move the opposite vertex and change the size and form of the rectangular. The rectangle turned to be green when it had some fixed area and turned to be red in all other cases. Clinical semi-structured interviews with 13 participants (10-11 years) were conducted, while they were learning to keep the rectangle green and disclosing the rule when it was green. I used Pupil-labs head mounted eye-tracker with frequency 60Hz in order to keep track of the participants' eye-movements during the learning. At first the students were trying to transform the rectangle by moving its vertex horizontally or vertically and this strategy was accompanied by the vertical and horizontal eye movements since the students were controlling the width and the length alternately. In most participants, with time hand movements became fluent while eye-movements transformed to one drifting fixation at the centre of the rectangle or somewhere in between of the centre and the mouse pointer. In these cases a new conceptual understandings started to emerge. Students were searching for a word that would express their embodied experience: "It is as if I am making it from plasticine all the time…, I can make it longer, but the amount of plasticine is still the same", "It has constant size, not size… constant volume! Or…no… how to call it… constant area!" I consider this fixation as an attentional anchor: students acquired a new motor scheme and they could see the rectangular as a whole figure and disclose area constancy as its feature. Thus interactive embodied design is an effective way to let students experience area, and our eye-tracking data confirm that a new concept emergence is accompanied by restructuring of the perceptive actions.

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MATHEMATICAL STRATEGIES IN SOLVING ILL-STRUCTURED WORD PROBLEMS

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Mathematical ability for solving problems involves not only verbal ability but also the planning of efficient strategies that support mathematical execution. These strategies can be detected from the solution writing process, which in turn emerge from their mental representations, that can be according to Johnson-Laird (1983): 1- using just formal language (Math or maternal languages: algebraic, maternal language); 2- using just imagetic forms (schemes, drawings, simulations, tables) and 3- both of them. Having a wide range of strategies is one key to being a good problem solver. Thus, in this work we intend to identify the different strategies developed by students in ill-structured Math word problems, that is, mathematical problems written in maternal language and with many ways to solve them.

The authors analyzed the solution of 11 undergraduate Math students (U) and 32 high school students (HS) about five problems. One problem was: in a soccer championship in which each team plays the same amount of games, every winning is worth three points, a tie just one point, and defeats are worth nothing. In the case of a tie between teams, the organizers would consider winning to those that had more defeats instead of the old standards of more wins. Are these criteria equivalent? Four of U did not hit the problem. One of them solved in a particular way, which does not guarantee the answer of the general problem. The others participants attempted to use the simulation strategy, however, without success. Six HS solved the problem, with four correct answers. Three used algebraic arguments and the other one found a pattern.

Analysing all the solutions of five problems, generally, we conclude that the clear majority of U used trial-and-error and simulation to solve the problems, revealing imagetic and formal language tendency, while HS used a larger repertoire of strategies, including pattern, particular solutions, algebraic, and a mix of them, showing three kinds of mental representation. The results show that we must understand the reason for this preference of strategies from U, in contrast of HS, and support teacher action to promote problems with different strategies to increase students' repertoire.

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A PHILOSOPHICAL CATEGORIZATION OF THE CRITIQUES OF RADICAL CONSTRUCTIVISM AS AN EPISTEMOLOGY

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Shortly after the behaviorist period of the 1960s, von Glasersfeld (1974) introduced radical constructivism (RC). The epistemological standpoint was widely circulated but heavily critiqued. Nevertheless, the proposed way of knowing prevailed and radical constructivists (RCs) have likely clarified its constructs as a result of responding to these critiques. This theoretical research study aims to answer the following questions: What are the main points of criticism of RC as an epistemological position and in mathematics teaching and research, and how have RCs addressed these critiques? I considered several critiques and responses to these critiques by both mathematics and science educators and categorized them into broader theoretical perspectives and philosophical stances.

For example, many of the critiques of RC from an epistemological perspective accuse RC as falling into one of three different categories: solipsism, idealism, and an inconsistent mixture of the two along with fallibilism, holism, instrumentalism, etc. RCs consistently maintain that their stance is not an ontological perspective and does, in fact, align with instrumentalism. Other critics insist that the perspective ignores the social and language components essential to learning, to which RCs respond with the importance of considering intersubjectivity and from whose perspective knowledge is being discussed. In the presentation, further results will be discussed in detail.

This theoretical research has categorized the critiques and responses of RC by considering the philosophical underpinnings of many of the specific critiques and responses offered by mathematics and science educators. This study provides a systematic way to critique and for RCs to respond to the critiques of their proposed epistemology by situating their way of knowing within a larger philosophical context. Future researchers can use this broad categorization of philosophical stances as a way to analyse and critique other epistemological stances present in the field.

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GOAL-SETTING IN SELF-REGULATED LEARNING

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According to Zimmerman and Schunk (2011), essential aspects of students' self-regulated learning (SRL) are given by the desire to set personal goals and to heed motivational beliefs. Moreover new goals need to be chosen when present ones have been attained. Therefore it is meaningful to examine students' goals and the beliefs connected to them as well as whether students' goals and beliefs remain stable over a stretch of time and to investigate the possible changes and reasons for them.

The study presented is based on courses joined by 24 students of grade 12 working on projects about coding and cryptography over a school year under self-regulated learning. Interviews were designed based on Liu and Liu (2011), taken in the middle and at the end of the school year.

Students were asked to describe their goals for the course. For this article students' goals are viewed as Knowledge and Understanding (KU) which is divided into Expanding Knowledge, Expanding Understanding and Application of the topic.

Many students changed their goals between the first and the second interview. The changes seem to be related to changes in students' beliefs in Acquirement of Knowledge (AK) – which are divided into the "classical" viewpoints of Plato and Aristoteles (e.g. Loos & Ziegler, 2017) – and the Progression in Projects (PP), as denoted in Stoppel (2016) – divided into the cases Progress 1, where projects only include known mathematical topics, and Progress 2, where students came to grips with new mathematical contents while working on projects.

The changes in students' beliefs in KU for SRL in relationship to AK and PP will be exemplified for three students including the discussion of eventual causes for changes in beliefs and goals. In addition, aspects for practical consequences in the classroom will be presented.

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MODELLING SITUATIONS INVOLVING EQUAL-SIZED GROUPS

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Greer (1992) suggests that the most important types of situations where multiplication of integers is involved are: equivalent groups; multiplicative comparison; rectangular arrays/areas; and, Cartesian products. In Norwegian schools, multiplication is usually introduced through situations with equivalent groups, where 4·7 means 7+7+7+7, while 7·4 means 4+4+4+4+4+4+4. In a developmental research project in Norway, LaUDiM (Language Use and Development in the Mathematics Classroom), a class of Grade 3 students (8 years old) were given Tasks 1 and 2 below. The teacher's goal was that the students should "write arithmetic representations that fitted with the tasks". The research goal was to get insight into challenges with different types of situations.

Task 1: Class 3c plan to arrange a class party in the Café. The day before the party, they will bake muffins for the party at school. Ms. Hall has to go the grocery store to buy eggs for the muffins. The recipe says there should be four eggs in one portion. The students have decided that they will bake twelve portions of muffins. How many eggs should Ms. Hall buy?
Task 2: The muffins are placed on baking trays to be baked in the oven. On a baking tray there is space for five rows of muffins, and there is space for seven muffins in each row. How many muffins can be placed on one baking tray?

In the classroom, Task 1 was divided into three phases: first, the students' iconic representations of the situation; second, the students' arithmetic representations of the situation; and, third, the teacher's introduction of the conventional notation (12□4). The teacher wanted the students to learn a convention of multiplication – that the first factor in a product signifies the number of groups and the second factor signifies the size of the groups. This is related to situations of equivalent groups, which is a non-commutative situation. However, with 1000 portions with 4 eggs in each (an example introduced by the teacher during the third phase), it is easier to calculate the total of eggs as 1000+1000+1000+1000 (four times 1000), rather than 4+4+4+4+… a thousand times (1000 times four) – both modelled by repeated addition. This creates a conflict with the desired convention – a tension between the meaning of multiplication as the total of equal-sized groups put together, and the arithmetic operation of calculating the product. The review of Task 2 was used to illustrate that what initially is a rectangular-array situation, can be interpreted as an equal-sized groups situation, where the rows or columns are the groups. However, what makes sense for rows and columns does not make sense for eggs and portions of muffins. Awareness of this issue became an important consideration among the teacher and researchers in reflections on the planning processes and for future task design in the project.

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IS THE SUCCESS OF SOLVING NON-ROUTINE WORD PROBLEMS INFLUENCED BY THE MATHEMATICAL ABILITIES OF THIRD-GRADERS?

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External representations can help to overcome problem barriers (Schnotz, 2014). Students have to learn how to make use of them and how to restructure inappropriate representations. This restructuring process is more difficult for low- achievers than for high-achievers (Luo, Niki, & Knoblich, 2006). It was of interest to find out whether high-achievers and low-achievers benefit in the same way from the training.

Over a period of 12 weeks, 366 third-grade children in 20 classes were accompanied for one lesson a week while solving difficult word problems. 10 classes completed the training program, which emphasized the representations children constructed themselves. Before the training started the mathematic abilities were measured.

The trained classes were more successful problem solvers. The extent to which high-achievers and low-achievers benefit differentially from the training was negligible small. In line with the success of the trained group these findings underline the added value of the training regardless of whether you are a low or high achiever.

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NAMING PRACTICES IN MATHEMATICAL DISCOURSE IN INSTRUCTION

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An analytical framework to study the Mathematical Discourse in Instruction (MDI) has been proposed by Jill Adler and her colleagues (Adler & Ronda, 2015). The framework is founded on the notion of an object of learning (OoL) that forms the (mathematical) goal of instruction and is accessed through three key mediational means: exemplification, explanatory talk and learner participation. Exemplification includes the sub-categories of examples and tasks, through which the OoL is instantiated, while explanatory talk includes the sub-categories of naming and legitimations, which scaffold access to the OoL. The MDI framework is promising since it offers the possibility of capturing key aspects of classroom mathematical discourse with a minimum set of categories and codes. However, like other similar frameworks, it faces the challenge of separating descriptive and evaluative codes, and the codes presently advocated by the framework are both theory-driven and evaluative. Here I focus on the sub-category of naming and present an approach that aggregates the naming practices at a descriptive level. Similar aggregation can be done for other sub-categories followed by assigning evaluative codes. Further, I elaborate naming to include names and naming actions, which may take place in the formal mathematical, informal mathematical or the everyday registers. Names include those for objects, notations and routines and naming actions include definitions, instances, alternative/ equivalent terms, and elaboration/ gloss.

I apply the codes to video transcripts of two Grade 7 lessons on the topic of algebraic identities taught from translated versions of the same textbook chapter in Urdu and in English (code-switched while teaching with Hindi). I will attempt to show that the aggregated codes capture key mathematical features of the lessons and highlight striking differences between the two lessons. It is worth mentioning that naming practices are culturally important in the South Asian context, and teachers spend much time in introducing and explaining terminology and having children learn them. The analysis reveals that in both lessons teachers pay much attention to naming, giving it an important place in their pedagogical practice. However, differences between the lessons suggest that teaching in different languages belong to different sub-cultures with associated naming practices. The analysis is intended to both strengthen the MDI framework methodologically and to illuminate an important aspect of mathematical discourse.

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ACHIEVING STUDENTS' ICT CAPABILITIES IN MATHEMATICS CLASSROOMS: AN EXPLORATIVE STUDY

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While many studies show that incorporating ICT in mathematics classrooms advances students' learning experience, there has been little focus on how such an ICT-enabled mathematics classroom develops ICT capability among the students. ICT capability does not merely mean acquiring a set of technical competencies, rather it involves the appropriate selection, use and evaluation of ICT (DfES, 2004), that is, whether the students can understand what type of ICT is available, when to use it and why it is appropriate for the task, reflecting their mathematical knowledge, linking concepts with the appropriate ICT tools. Investigating ICT capability could be a criteria to measure students' competency level of mathematical knowledge as well as technological knowledge. In a study, Fuglestad (2005) argued that students can choose the appropriate ICT for a specific task, when given the authority. What happens if the choice of ICT tools is controlled by the teacher? Does the scenario remain same?

Data was collected from a school where most of the students usually study mathematics using iPads. Data collection involved class observations, teacher's interview and a focus group interview of the students. In addition, images of a few students' work on iPad and on paper were collected.

The study shows that students not only know the use of ICT tools, but also know why and when they have to use them. We found that although the ICT tools were directed by the teacher, most of the students do not use them blindly, instead, for solving a task, they discussed with peers, shared their thoughts and came up with the correct results using available ICT tools (i.e., software, scientific calculator) provided by the teacher. But, when approached with a slightly dissimilar but harder problem, they could also use the tools and explained the concepts clearly. This indicates that when the teacher discusses the tool (e.g., 'desmos'), the students can capture the functions of the tool well, which helps them to solve a new and harder problem using that tool. On the one hand, reflecting students' ICT capability, while on the other hand, indicating their competence to connect one problem with another and solve a new one using ICT tools by their cognitive understanding.

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USING MOOC’S ZONE THEORY IN RESEARCH ON TEACHERS’ PROFESSIONAL DEVELOPMENT AND ON CHANGES IN CLASSROM PRACTICES

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The authors, as researchers, and a team of designers (made of experienced teachers) designed and managed a MOOC (Massive Open Online Course) on Geometry for secondary school (grade 6-13) mathematics in-service teachers training. It was delivered from October 2015 to January 2016, with 424 enrolled trainees (36% of them ended the training experience). Our research goal is understanding if/how such a new way of training has an impact on teachers’ practices. We introduce a fresh theoretical framework, called MOOC’s Zone Theory: it allows describing the teacher’s participation in the MOOC and analysing their consequent professional development and possible changes in classroom practices. It consists in the refinement of the Goos’s zone theory (2005), which, on its side, is an elaboration of the Valsiner one (1997). Goos model carefully defines the components of the three Valisner’s zones: Zone of Proximal Development (ZPD), Zone of Free Movement (ZFM), Zone of Promoted Actions (ZPA); and points out a dynamic between them because of a “productive tension” (Goos, 2013, p.523). This model can work efficiently for framing the traditional training courses, but shows inadequate for describing a MOOC in its complexity. For this reason, we have expanded Goos’ model so to give reason of all the components of a MOOC: in particular its concrete environment (the platform from which it is delivered, just only and uniquely via online) and the variety of its concrete ICT tools all used by the participants both in a synchronous and a-synchronous way. All these constitute a further level with respect to a usual training course. To illustrate how our model works, we introduce an emblematic case study of one teacher, Lucy. Lucy’s case has been selected because it illustrates how tension between her ZPD (Lucy’s beliefs about mathematics teaching-learning) and ZFM/ZPA complex (Lucy’s professional environment and interaction with teaching colleagues) create opportunity for self-initiated changes, which are also reflected in a little change of her classroom practices.

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USING COMICS IN TEACHING MATHEMATICS

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This paper reports a research in studying the feasibility of using comics in the secondary mathematics classroom. It all began from a survey result that some teachers were using cartoons, comics and storytelling in teaching mathematics to the low attainers in mathematics to motivate the low attainers (Toh & Lui, 2014). The authors designed an alternative teaching package on selected topics of lower secondary mathematics using comics and provided a set of proposed lesson outlines for the teachers to teach the lessons using this set of alternative package. The lesson outlines suggest how stories and humours could be used to teach the entire topic through the comics package. A sample of one of the lessons on lower secondary percentage in the Normal (Technical) curriculum is described in greater detail in Toh, Cheng, Jiang and Lim (2016) and on the website http://www.math.nie.edu.sg/magical.

The comics lessons were video recorded and were viewed by the researchers. Here we report the adaptations made by the teachers in one participating school in executing these lessons to enhance their students' learning. Three key observations were made: (1) In order to engage the students in the learning process, the teachers converted the comic strips to worksheets with blanks for students to complete the story. The students became an active participant in the discourse rather than a passive learner; (2) As the teachers progressed through the lesson, they infused their own experience humorously as a context for students to think of the related mathematical concepts; (3) Instead of teachers' storytelling entirely, at appropriate junctures of the lessons, the teachers introduced role-play in the mathematics classroom. According to the teachers, this turned the onus of learning to the students, both groups who played the roles of comics characters and who were observers (but allowed to question the comic characters).

This shows that not only were teachers able to use the comics material provided by the researchers in the classrooms; they were actively adapting the material to further enhance students' learning and were making decisions constantly. A detailed analysis of the result of our findings will be published elsewhere soon.

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DEMONSTRATING TEACHING CONTEXT IN TAIWANESE MATHEMATICS TEXTBOOKS AND E-TEXTBOOKS BASED ON THE EVENTS OF INSTRUCTION

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Mathematics textbooks are an important tool for presenting and acquiring knowledge. They are the primary tool used in the instructional design of mathematics teaching in most countries (Mullis, Martin, Foy, & Arora, 2012). In Taiwan, most textbook publishers integrate various teaching resources (e.g., interactive operations, multimedia presentations, and past examination questions) into a platform to create mathematics e-textooks, which can provide diversified instruction through electronic devices. However, traditional textbooks, workbooks, and lecture notes are still used in most mathematics classrooms. Hence, analyzing the differences in teaching context between textbooks and e-textbooks may provide the reasons for the preferred utilization of e-textbooks.

Robert Gagné (1985) proposed nine instructional events based on the information processing model. These events describe methods for knowledge presentation and acquisition and explain the design of instructional activities from both behavioral and cognitive perspectives. In this study, Gagné's nine events of instruction was used to analyze the three versions of the main junior Taiwanese mathematics textbooks. The topic of triangle properties was explored in both textbooks and e-textbooks. Textbook content was divided into paragraphs (3279 in total) presenting related ideas or descriptions. The reliability of the coding was checked by an external coder with 97.8% consistency.

The results showed that the teaching context demonstrated in Taiwanese textbooks and e-textbooks were similar for the selected topic in geometry. Presenting the stimulus material and providing learning guidance (about 45% paragraphs) form the core of textbook design. Providing feedback and additional exercises and presenting content in various formats were the main features of the e-textbooks. However, they could only exhibit content in the layout of the original textbook, connecting to other information through buttons. A sequential reading path is still presumed for mathematics e-textbooks, which may be beneficial in traditional instruction but not for interaction or self-learning. The role and use of e-textbooks require further improvement.

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THE DEVELOPMENT OF A TEACHABLE AND LEARNABLE MATHEMATICS E-TEXTBOOK

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With the rapid development of technological tools and melting into educational field, the multimedia and interactivity of mathematics e-textbooks can provide students more learning opportunities and approaches to access the ideas of mathematics and mathematical thinking. However, learning in e-Textbooks is still not fully realized. Therefore, the purpose of this study is to develop a teachable and learnable mathematics e-textbook for teachers and middle school students. At the same time, the response of teachers and students were investigated after they used the e-textbook for mathematics teaching and learning. By considering nine events of instruction (Gagné, 1985), dynamic representations (Moreno-Armella, Hegedus, & Kaput, 2008) and cognitive load theory (Sweller, van Merriënboer, & Paas, 1998) as our theoretical foundation, we developed the interactive mathematics e-textbook, which consists three main components: foundational activity, mathematics classroom, and self-assessment. In the teaching experiment, mathematics teachers demonstrate the content of the interactive mathematics e-textbook and using dynamic mathematics experiment to interact with students. At the end of the teaching experiment, students were asked to report their cognitive loads and perceptions about learning mathematics with the e-textbook. The data was collected and analysed from the response of 8 senior in-service mathematics teachers and 77 eighth grade students through questionnaire survey. The main result reveals that all teachers and students agree that interactivity and dynamic representations are the important feature of e-textbooks. However, results about cognitive loads indicate mathematics teachers are more optimistic about using e-textbooks in their mathematics instruction than students do. Hence, the interactive mathematics e-textbook is teachable for mathematics teachers may not be learnable for students. Therefore, the interactive mathematics e-textbook could be more teachable and learnable by considering students cognitive development in mathematics learning.

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ANALYZING ENGAGEMENT IN MATHEMATICAL COLLABORATION: WHAT CAN WE SAY WITH CONFIDENCE?

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Student engagement with mathematics learning is related to both affective and behavioral aspects. Engagement and affect are empirically and theoretically linked with each other, but neither of the two are defined as a subconcept of the other. Goldin's (2017) analytical tool of motivating desires addresses this complexity, aiming to cover multiple aspects of affect as well as social and contextual elements. In this study, I use Goldin's tool to investigate what part of students' affective interaction during mathematical collaboration can be made visible for research purposes in a valid, systematic and reliable way, and how this is dependent on chosen unit of the analysis. In particular, the investigation addressed the question of whether the units of analysis are independently coherent (disconnected entities) or progressively explanatory (each adding information to others). The data used in this study consist of a video excerpt where four students (two girls, two boys, Year 7) participate in a 20-minute (as part of an altogether 60-minute session) researcher-designed and teacher-facilitated session involving an open-ended mathematical task.

From the analysis, the interpretation of the affective engagement of the whole group based on Goldin's analytical framework can appear too narrow if only the original episode is referred to. In that episode, it looked as though the students' were just working in pairs (girls) or individually (boys). The students' efforts to make the group work together, and especially the challenges to do so were evident in the earlier episodes. Thus, confidence in the interpretive coding of a given excerpt can be heightened or perturbed by consideration of preceding episodes. Although the tool of motivating desires seems useful for analyzing the affectively rich moments, careful attention must be paid when defining the analytical unit through which to apply it. Judgements regarding the validity of the interpretive accounts depend on how the operationalized affective components accommodate or ignore the contextual and historical background of the episode and whether a systematic analytical approach can use such background information to increase confidence in interpretation.

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AFFORDANCES OF THE 'BRANCH AND BOUND' PARADIGM FOR DEVELOPING COMPUTATIONAL THINKING

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As technological advances in engineering and computer science happen more and more quickly, we must shift focus from teaching specific techniques or programming languages to teaching something more transcending: computational thinking (Wing, 2006). Wing explained this concept later as "*the thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer – human or machine – can effectively carry out*". It includes abstraction, heuristics, algorithm design, efficiency and complexity. While programming classes add to students' competence in some of these topics, mathematics too may foster computational thinking (Weintrop et al., 2016). However, few resources are currently available to support teachers in meeting computational thinking learning goals.

This design-based qualitative study explores which aspects of computational thinking can be addressed well through a mathematics project for secondary school students aged 16-17 in the Netherlands. As puzzle-like problems help students to think more algorithmically (Levitin & Papalaskari, 2002), we designed a three-hour project using such problems to familiarize students with the algorithm design paradigm 'branch and bound', which efficiently enumerates candidate solutions in discrete optimization. Allowing students to come up with heuristics, analyse the complexity of approaches, and do some calculations by hand, we aim to improve their computational thinking.

We will present our design and an evaluation of the project carried out by 50 students, discussing findings from in-class observations and interviews with 5 case students. Data collection is currently underway, and the results will be available at the time of presentation. We expect to report on our experiences in teaching the branch and bound paradigm and its affordances and limitations for helping students learn to think computationally. We focus on skills helping them contribute to tomorrow's society: algorithmic thinking, while still being able to reflect on efficiency and correctness.

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COMMERCIAL CONTEXTS IN GERMAN MATHEMATICS TEXTBOOKS – WHERE ARE THE OPPORTUNITIES TO PREPARE FOR VOCATIONAL EDUCATION?

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Following common literacy frameworks, German educational policy aims at preparing students for their professional life through general education. Vocational educational standards show, that specifically industrial clerks face rich mathematical demands in professional training. These professions are, therefore, at the center of our attention. Future trainees leave school with *general mathematical competences* according to the educational standards for mathematics. Vocational training aims at *vocational competences* defined by specific professional demands. Hence, students are expected to dispose of initial *vocation-related mathematical competences* so that basic tasks related to vocational contexts can be solved (Nickolaus et al., 2013). In this study, we therefore asked which kind of opportunities to learn *vocation-related mathematical competences* the German secondary classrooms provide.

- What proportion of the exercises in German secondary mathematic textbooks offer a commercial context in general?
- How many of these exercises contain situations with clear commercial *vocational* connection (*vocation-related mathematical competences)*?

Basis for the analysis was a collection of 18 different and widely-used German mathematical textbooks for secondary education with a total of approx. 20.000 exercises. We identified the totality of exercises with a commercial context in general (inclusion step). Then, we sorted these exercises into different categories according to content and proximity to the profession of industrial clerks (classification step).

The step of inclusion led us to a total of 1.154 exercises with a commercial context in general (mean of 64 exercises per book). With regard to the second research question, only 18% of the exercises included in our analysis in the second step contain mathematical situations with clear commercial *vocational* connection.

About 1% of the total exercises within the textbooks are concerned with a specific commercial context from a *vocational* point of view. Against the demands of general education, that strongly built on literacy-conceptions and emphasize a preparation also for vocational training, this seems especially noticeable.

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DOING 43 TIMES 12 WITH LOVE

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Gutstein and Peterson (2005) juxtaposed two word problems to illustrate how mathematics teachers might address social/ecological/justice issues. First: “A group of youth aged 14, 15, and 16 go to the store. Candy bars are on sale for 43¢ each. They buy a total of 12 candy bars. How much do they spend, not including tax?” (p. 6). Second: “Factory workers aged 14, 15, and 16 in Honduras make McKids children’s clothing for Walmart. Each worker earns 43 cents an hour and works a [12-hour shift] each day. How much does each worker make in one day, excluding any fees deducted by employers?” (p. 6). The authors promoted the second problem.

Is McKids clothing real? Is 43 cents a real wage there? Do 14-year-olds work in factories? It does not matter because the word problem expects students to ignore these things, to find the relevant numbers and ultimately multiply 43 by 12. Gerofsky (1996) showed how the tradition of mathematical word problems treats context as a throw-away. I question the appropriateness of using the word problem tradition to introduce truly complex and significant contexts such as child labour and sweat shops. Training children to ignore (throw away) such violence undermines the potential of mathematics to engage with real social and ecological challenges.

To support a more context-responsive and responsible mathematics pedagogy, I am collecting narratives from a wide range of individuals who tell of times they did mathematics with love. For example: “I developed a formula for distributing finite funds equitably among doctoral students to encourage further funding applications while also favouring students with no other funding. It had to be simple enough to be understood by all.”

A challenge in this endeavour is to define love because it means different things for different people and the definition may differ according to the mathematical activity. The paucity of mathematics education literature identifying love generally addresses a relationship with the discipline—either love or hate. Long (2011) looked to the caring relationship between teacher and student. I am looking for further possibilities.

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EXPLORATORY STUDY ON THE MISCONCEPTIONS OF THE PEARSON CORRELATION IN RELATION TO ITS FORMULA

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The researches have shown that students often hold misconceptions about the Pearson correlation (PC) and provided several possible causes of the misconceptions (e.g. Morris, 2001). Furthermore, researchers have justified the effect of using some strategies, e.g. cognitive conflict, on reducing students' misconceptions about the PC. However, these previous studies seldom mentioned whether enhancing students' understanding of the PC coefficient formula can reduce misconceptions about the PC. The aim of this study is to explore the relation between the misconceptions about the PC and the understanding of the PC coefficient formula. This aim not only provides insight into students' misconceptions about the PC but also contributes to remedying their misconceptions. Six questions were adapted from Liu & Lin's study (2011) to identify students' misconceptions about the PC. We also developed three questions to measure students' understanding of the PC coefficient formula. These questions were surveyed to 45 twelfth graders who had learned the PC in school.

As for each question of identifying students' misconceptions about the PC, we tested whether the misconception is associated with their understanding of the PC coefficient formula. The result show that only one of the six misconception questions was associated with students' understanding of the PC coefficient formula (phi=.34, p =.02). It indicates that the better understanding of the PC coefficient formula and the less misconception about the strength of correlation in scatter plots. Nonetheless, to improve the understanding of the PC coefficient formula is insufficient to reduce students' misconceptions about the PC. We will suggest future research on students' understanding of statistical correlations between two variables.

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CHARACTERIZING PRESERVICE SECONDARY MATHEMATICS TEACHERS' REFLECTION ON THEIR TEACHING PRACTICUM

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Effective teachers are characterized by considering multiple aspects of teaching and the interrelationships among them. The reflection to facilitate teachers' professional development should employ these characteristics as indicators of its productiveness. This study adapted Hsieh's (2013) framework of mathematics teaching elements and Davis's (2006) framework for analysing reflections to illustrate preservice secondary mathematics teachers' reflections on their teaching practicum. Coverage, emphasis, and connection among 20 teaching elements in four aspects (e.g., *mathematics cognition and understanding* in *learners and learning*, *mathematics teaching method* in *instruction*, *mathematics evaluation* in *assessment*, and *mathematics teaching material* in *mathematics content*) were explored on their reflections.

The sample comprised Taiwanese preservice secondary mathematics teachers enrolled in the course named Teaching Practicum that required them to have 1 month of field experience during the semester. During that month, each preservice teacher (PT) was responsible for teaching mathematics to one class. Data on the PTs' reflections on their teaching in that month were collected using an open-ended questionnaire. The initial analysis was conducted on the responses of 22 PTs (one class) through inductive and content analyses.

The results demonstrated the following: (1) A total of 55% PTs covered all four aspects, whereas 27% did not include *mathematics content* (e.g., arranging examples or representations) in their reflection. (2) The majority of PTs' (86%) emphasis was on *learners and learning* and *instruction*, and most of them stressed these two aspects roughly equally. (3) *Learners and learning*, including students' thinking, understanding, and learning motivation, was considered by all PTs when determining their approval of their own teaching behaviours in the practicum. (4) PTs' confidence that they would become good teachers was influenced by their perceptions of their own teaching performance, competence, efforts, attitudes toward teaching, and student feedback and reactions. Student feedback and reactions were brought up by most PTs.

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THE INVESTIGATION OF TEACHERS' CONCEPTION OF FORMULAE

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The aim of this investigation is to create a framework to analyze the quality of subject matter knowledge for mathematics teaching by investigating the "teachers' conception of a formula". I argue that focusing on teaching that uses a learning goal can become a powerful perspective for investigating the mathematical knowledge of a teacher, when the learning goal is to create a formula in mathematics lesson. A formula that is created in a mathematics lesson is a way to express numerical and quantitative rules by using mathematical units such as numbers, symbols, letters, and/or words.

The issue of how to capture accurately the knowledge of subject matter engaged in, conducting, and/or analyzing a mathematics lesson is an essential one for the study of mathematical education. Additionally, creating a formula is an appropriate goal for teaching that takes into consideration a deepening of the students' levels of understanding. According to Hanna and Jahnke (1993), creating formal expression is key to examining whether a conception can be applied to the matter in question. Moreover, the mathematical conceptions might exist autonomously in the object and in the mathematical signs by using numbers, symbols, letters, and/or words (Steinbring, 2006). For the purpose of this investigation, I conducted an interview with educators and teachers, selecting as a case study the formula used to calculate the area of a parallelogram (base × height).

Three key viewpoints were derived from the points that participants presented in response to the interview: (1) Is the "answer-solving" phase of the lesson separate from the "formula-creation" phase? (2) Is there an attitude of "absolution" with regards to the conception of the formula? (3) Which keywords and concepts are used in the selected formula?

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USING A QUALITY OF LIFE FRAMEWORK TO INVESTIGATE PRE-SERVICE TEACHER MATHEMATICS ANXIETY

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Mathematics anxiety (maths anxiety) is an issue for the development of mathematical identities in primary pre-service teachers (PST). Sfard and Prusak (2005) define identities as stories about people. They state "identity talk *makes us able to cope with new situations in terms of our past experience and gives us tools to plan for the future* [emphasis in original]" (p. 16).

This research aims to investigate the use of the Quality of Life (QOL) conceptual framework (Renwick & Brown, 1996) of being, belonging and becoming, to investigate how both PSTs' maths anxiety, and its impact on their identity development, might be understood.

The participants, twelve PSTs studying the Bachelor of Education (Primary) course, wrote descriptions during small-group workshops, of critical incidents (positive or negative) from their school mathematics education. They shared their reflections on how these had impacted on their views of themselves. The written descriptions and transcribed reflections were analysed using the QOL framework.

Being: For most of the PSTs, (93%), their descriptions and reflections showed ongoing associations with worry and stress with respect to mathematics. They perceived that their construction of their current identity as learners of mathematics was impacted by their perception of their past experiences.

Belonging: All PSTs reflected on an incident with a person in their environment (the sub-domain of social belonging). Most commonly, this was a teacher but some also reflected on interactions with peers (often as part of the same incident). Negative reactions resulted in the perception that they were outsiders, showing the importance of supportive learning environments.

Becoming: This domain has a future focus. All participants indicated the desire to be effective teachers. However, four PST who were in the last year of their course, showed the ability to project a more robust and positive future identity.

This paper demonstrates that the domains and sub-domains from the QOL framework provide a useful tool within which to analyse and understand identity in PST, especially development of their future identities of teachers of mathematics.

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DESIGNING ACTIVITIES INVOLVING SPATIAL ABILITY ASPECTS IN MATHEMATICS LESSON

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The positive correlation between mathematics performance and spatial ability has been identified by many researchers (e.g. Lowrie, Logan, & Ramful, 2017). Therefore, designing activities to promote students' spatial sense is essential. This requires an integration of spatial ability aspects (e.g. spatial visual, spatial rotation, spatial orientation) into mathematics curriculum particularly when the mathematics curriculum such as in Indonesia does not explicitly included spatial reasoning. This paper will present one part of my ongoing study in developing mathematics lessons in integration with spatial ability aspects. Focusing on the role of mathematics teachers in learning processes at school, it is important for them to have good spatial ability and good understanding about it as they will teach mathematics. Therefore, in this study, the lesson design will be implemented in professional development (PD) session for teachers to understand their responses on the design and the feasibility to implement the lesson in their own classroom. The aims aforementioned will then answer the research question on teachers' perspective towards the activities and its implementation in conformity with school curriculum.

Using design research framework (Gravemeijer & Cobb, 2006), we are conducting the research in three phases namely: preparation and design in which we design a mathematics lesson integrated with spatial ability aspects; teaching experiment with 35 teachers who join PD session in Indonesia; and retrospective analysis in which we analyse learning process during design implementation with teachers and combine it with their reflection and interview result with 3-6 teachers. Three different activities are designed within the topic of circle and polyhedrons which are integrated with spatial ability aspects as well as a design for semi structured interview guideline.

The expected result from the implementation of the design is that during the PD session, teachers give rich responses through reflection and interview which portray that they really engage with the activities. When teachers engage and understand, it can encourage them to design their own activities then implement it in their classroom to help their students to learn mathematics meaningfully. This finding may give insight to the authority on the importance of spatial ability to be introduced at school as well as reshape the mathematics curriculum in Indonesia.

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REASONING AND PROOF IN PROBABILITY AND STATISTICS IN SCHOOL MATHEMATICS TEXTBOOKS IN HONG KONG

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It has been widely recognized that reasoning and proof (RP) should permeate school mathematics at all levels and across all content areas. However, most, if not all, of the previous textbook analyses of RP focused only on the area of algebra or geometry. This research project aims to complement the research knowledge of the field by examining the opportunities for students to learn RP in the area of probability and statistics when they are using a popular senior secondary (Grades 10-12) mathematics textbook series from Hong Kong - a high performing region in school mathematics. Specifically, we examined the RP opportunities in both the textbook exposition section and the student exercise section of each of the five chapters in probability and statistics of the selected textbook series. We adopted the theoretical framework of Otten et al. (2014) for the exposition sections and adapted the theoretical framework of Stylianides (2009) for the exercise sections. For the exposition sections, the framework consisted of four codes for how RP items such as theorems, properties or statements about RP are justified: *Deductive Justification*, *Empirical Justification*, *Justification Left to Students*, and *No Justification*. We obtained the following preliminary results for the exposition sections: (a) 33.3% of the RP items were justified by empirical arguments. This is problematic as this might mislead students into believing that an empirical argument is sufficient to establish truth in mathematics. However, we found that many of these empirical arguments could be readily transformed into generic examples (a kind of deductive justification) by adding just a generalisation part. (b) 26.3% of the RP items were not justified at all, even though some of them could be easily proved using the principle of mathematical induction. (c) In stark contrast to the results of other studies (e.g., Otten et al., 2014), 0% of the RP items were left to students to justify. This suggests that justifying theorems and properties was not considered by the textbook (and the curriculum) as important to students. (d) Among the deductive justifications, there was no combinatorial proof - a method for establishing an identity by counting something in two distinct ways, though this type of proof is generally considered to be more elegant and insightful in probability. These results seem to suggest that RP does not play an important role in school mathematics in Hong Kong. In the presentation, further results about the RP opportunities in the exercise sections will be discussed.

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ANALYZING TEST PERFORMANCE BY ELEMENT INTERACTIVITY

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It's a convention to estimate levels of item difficulty when constructing test instruments. The estimation is normally a three-level (easy, mediocre, and difficult) scale based on the judgement of domain experts, and then runs an item analysis to determine the level of item difficulty and item discrimination. The results may provide a valid test instrument, but it does not tell what students' problems are. Cognitive load theory (Sweller, 2010) suggests that element interactivity is the major source of working memory load and task difficulty. Given the fact that mathematics problem solving is usually exhibited in a high element-interactivity situation, analysing item difficulty from the perspective of element interactivity may provide additional information regarding students' ability to cope with interacting elements and difficulties related to specific elements.

The purposes of this study were twofold: (1) to examine whether our priori element analyses could predict performance and examine how students performed on different levels of element interactivity. (2) to examine ability differences on test performance on different interacting elements.

The test instrument consisted of 24 questions measuring the concept of the formula, the base and height, and the area of parallelograms. Based on the test instrument, a total of 20 elements were identified. The performance data were from a sample of 151 grade five students who were also divided into high, average, and low ability based on a prior knowledge test before they participated in the designated learning activities and took the targeted test .

The results showed that problems with 3 elements could be considered as easy questions, in which all levels of students could achieve more than 60% correct . Average students could handle two more elements than low ability students ($p<.01$). High ability students could handle one additional element than average students ($p<.05$). Nevertheless, problems with more than six elements were too challenging for all students. In addition, this study also found that some items seemed to achieve a correct percentage lower than expected. It could be an indication of elements unidentified in the problems or some elements needed higher weight than others. The later can also indicate misconceptions or elements that students were yet to master. Details on the classification of the elements will be discussed.

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WHETHER HIGH SCHOOL MATH CURRICULUM HELP COLLEGE READINESS FOCUSING ON MATH COMPETENCIES

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Preparing students for college and career readiness is one critical goal of mathematics curriculum (e.g., CCSSO of the United States, Mathematics Syllabus of Singapore), including Taiwan. However, studies regarding whether high school mathematics curriculum well prepared students with sufficient mathematics content knowledge and mathematical competency are still scattered. This study probed into this issue by investigating the level of mathematics knowledge and required mathematical competency to understand an Economics textbook commonly applied in first year of college in Taiwan. The reason to choose Economics is that it is one of the most basic courses using mathematics in college.

This study adopted the construct of mathematical competency brought up by Niss (2003) to develop the framework of coding rubrics, of which *handling mathematical representations* and *handling mathematical symbols and formalism* are two critical competency categories and will be reported in the present paper. Content analysis was employed to analyse the textbook.

The initial findings show the gap between high school math and first year college-level Economics math, including (1) The competence to understand and use mathematical symbols is beyond what students are prepared in high school. The expression such as $Q^D = f(p;\ other\ factors)$ and $\varepsilon_h^D = {}^{\%\Delta Q_h}\!/_{\%\Delta P}$ and $E^a = C_a + \bar{I}$ is commonly seen in the Economics textbook, whereas the number of components and their assembly are more complicated than what students ever experienced. (2) The competence to interpret graphical representations takes an important role in Economics math and requires further development. Students have to identify various objects and variables in graphics, to understand the multiple translations of functions and figure out multiple relationships among them. The richness of the objects and relationships is of much higher degree in Economics than in high school math. In addition, students' cognitive flexibility in handling graphical representations should be cultivated because high school math always shows independent and dependent variable on horizontal and vertical axis respectively, while Economics often shows the variables in a contrary approach.

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THE VIEWS OF IN-SERVICE HIGH SCHOOL TEACHERS TOWARD USING CALCULATORS IN MATHEMATICS TEACHING

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It has been a long history for the Western societies to use calculators in mathematics classroom. However, with the tradition of emphasizing calculation competence in Taiwan, the use of calculators has not been paid much attention by teachers. This study aims to explore the possibility of promoting teachers' willingness to adopt calculators through workshops, a most popular way of in-service teacher education in Taiwan.

Three workshops were conducted at three high schools, which had different levels of students. A total of 36 mathematics teachers participated; of them, 12, 11, and 13 have more than 20, 10-19, and below 10 years of teaching. During the workshop sessions, why, what, and how to use calculators in mathematics classroom were introduced; in discussing the timing of using calculators, we revised the model of "5 E's", which has been shown effective in mathematics learning (Tuna & Kacar, 2013), based on specific characteristics of mathematics. The new "E's" we included were, for example, experimenting and evidencing. Pre- and post- questionnaires regarding teachers' views on mathematics nature, problem solving, and uses of calculators were implemented. Constructive responses were required in the questionnaires.

There were several interesting findings. A total of 64% teachers had never adopted calculators in their teaching. They specified their reasons for no need to use and lack of equipment. After the workshops, all teachers, except one, (i.e. 97%) were willing to adopt calculators. The original reasons hindering their use of calculators became not important to them. However, the teachers' opinions on whether students should be allowed to use calculators in the national entrance exams were split, no matter if they had used calculators in class previously. Regarding the views on mathematics nature, this study conjectured that teachers who had used calculators to assist teaching before might have a higher ratio of agreements on "mathematics are related to reality and can be found and examined by us" to "mathematics regulates the rules and procedures of problem solving, and cannot be found by us through examination", than those teachers who never implemented calculator-assisted teaching. However, our results did not confirm our conjecture. A high percentage of 84% agreements with the former mathematics nature probably explained the reasons, and also it may be a reason for why teachers were so easy to switch their willingness to adopt calculators after workshops. More results will be given in the conference.

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PRIMARY 6 TEACHERS' REFLECTIONS ON THEIR TEACHING: A RESULT OF A PROFESSIONAL DEVELOPMENT PROGRAMME

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Developing teacher reflective practice itself is one of the goals of teacher education (Loughran, 2002). Teacher's reflection has also become an important part of developing an effective professional development (PD) programme. The reflections both in formal and informal opportunities play important role to help teachers transform their practices (Saylor & Johnson, 2014). A possible way to support teachers' reflection on teaching is using video technology as a tool for teachers to reflect on classroom instruction. When teachers watch and analyse a video case of classroom instruction, they can reflect on what they see from the video, the incidents happening in the class and how they can improve their own practice (Borphy, 2004).

This study seeks to examine how Thai primary mathematics teachers reflected on their teaching across a 4-month video-based PD programme. Four primary 6 teachers attended a 3-day workshop to familiarise themselves with the mediation strategies framework. After the workshop, the teachers implemented the mediation strategies in their mathematics instructions. For each teacher, 7 lessons were observed and video recorded. Stimulated-recall interviews were conducted with the teachers one week after each of the lesson observations for them to reflect on their teaching.

The main source of data was audio transcriptions of the interview sessions. The data was coded deductively and inductively with a focus on teacher's level of reflection and their aspects of lessons and the use of mediation strategies. The results showed that the individual teacher had changed their levels and aspects of reflections (e.g., students' participation, appropriateness of examples, teacher's teaching) across different phases of the study.

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TEACHING PICTURE GRAPHS: MATHEMATICS-PEDAGOGICAL-CONTENT KNOWLEDGE-IN-ACTION

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A teacher cannot hope to explain mathematical concept if she does not have full understanding of that mathematical concept. Nevertheless, case study evidence suggests that the influence of teachers' mathematics pedagogical content knowledge (MPCK) has a strong influence on children's learning outcomes (Shulman, 1986). In the teaching of mathematics, Ball (2000) stressed how the depth of teachers' understanding of MPCK is a major determinant of teachers' choice of examples, explanations, exercises, items, and reactions to children's work. In light of interpretation of picture graphs and mindful of the challenges inherent in teaching them, it seems timely to look at how a beginning teacher apply her content knowledge and pedagogical content knowledge in teaching picture graphs. The purpose of this presentation is to determine what the researcher term "MPCK-in-action" outcomes as observed being practised by teachers when teaching mathematics and to ascertain the relative importance of different practices in contributing towards effective pupil learning. This presentation uses data from a larger study and focuses on investigating MPCK-in-actions within the context of a primary one mathematics classroom by exploring the following research question: What are the observable MPCK-in-actions that are present in the teaching of picture graphs? The framework for analysing MPCK-in-action practices developed by Lim-Teo and her colleagues (Lim-Teo, Chua, & Yeo, 2011) gives a detailed inventory describing evidence for identifying key components of MPCK-in-action practices within ten categories. The results depict the challenges associated with teaching of picture graphs and the interpretation of the picture graphs as well as the importance of content knowledge and pedagogical content knowledge. The beginning teacher's MPCK was evident not just in the choice of activities, but in the ways that she was able to link concepts to pupils' experience. The teacher's approaches varied, giving pupils greater freedom to think about representing picture graphs and holding rich discussions with groups and individuals.

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REFLECTING TEAMS FOR DEVELOPING STUDENT SENSITIVITY IN MATHEMATICAL NOTICING

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Mason (2003) proposes that mathematics learning involves developing one's sensitivity to noticing. But how can students develop such sensitivity when they work on messy, chaotic, open-ended mathematical modelling tasks? Roth and Maheux (2015) offer a theoretical tool in their notion of mathematics as revelation, which stands in-between creation and discovery, hinting at both the surprising appearance of something unseen that nevertheless was already there, and the creative process of bringing into being something new out of what was already known.

We present a novel approach for developing students' sensitivity to mathematical noticing using "reflecting teams" (Paré, 2016), a format originating from family therapy practice. In this format, skilled observers watch students working on an open-ended modelling activity. The observers then engage in conversation about the mathematics they noticed from the students session, while the students "eavesdrop" on the conversation. Finally, the students reflect together on the mathematics raised in the observers' conversation. This format exposes students to accounts of their mathematical work, allowing them to experience new forms of awareness in which they see what was already there in new ways, facilitating the dynamic of revelation.

We developed this approach in a design based research project involving 51 students from secondary school, undergraduate, and post-secondary pre-degree bridging courses in Australasia. Students worked on open-ended modelling activities in teams of three, and then engaged in 1-hour long reflection sessions that were designed, tested and refined over more than 20 design cycles. In our presentation, we describe the design of our reflection sessions, present research findings on students' raised awareness following this design, and discuss differences in language and power dynamics from enhancing noticing through student revelation versus presenting expert-generated insights.

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IDENTIFYING STUDENTS' ALGEBRAIC THINKING BY INVESTIGATING PROBLEM SOLVING IN YEARS 5, 6 AND 7 IN CHINA

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China's official mathematics curriculum standard endorses the building of closer relationships between the study of number and the development of algebraic thinking (Ministry of Education, 2011). Students may usually have a difficult transition to learning of algebra in the junior secondary school without solid foundation of certain relational thinking. This research aims to identify students' algebraic thinking using three types of mathematical sentences and give certain implications for teaching.

217 students in Years 5, 6 and 7 from three schools in Nanjing were involved in this study. Problems containing three types of mathematical sentences were designed. There is one unknown number in Type 1 sentences. Type 2 sentences include two boxes, with one in each side respectively and literal symbols c and d are used in place of the two boxes in type 3 sentences. For all three types of number sentences, students were given problems including four questions based on one of the four operations and asked to find the value of a missing number and to explain their thinking.

According to students' responses to the problems, five categories of students' algebraic thinking were found, namely Established Relational (4 points weighted), Highly Consolidating Relational (3), Consolidating Relational (2), Emerging Relational (1) and Non-relational (0). There was an increase from Year 5 (1.442) to Year 6 (1.736), and an even greater increase to Year 7 (2.708). There was no statistically significant difference between the two participating classes at each Year level.

Based on the research findings, our argument is that, in order to allow students to properly develop their understanding of algebra, solid foundations need to be laid during elementary and junior secondary school years through experiences with number operations and the key ideas of equivalence and compensation. And utilizing these generalizations of arithmetic models can help teachers to develop their teacher capacity to foster students' algebraic thinking (Zhang & Stephens, 2013).

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