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POSTER PRESENTATIONS

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ENHANCING STUDENTS' VISUAL SPATIAL SKILLS (VSS) AND GEOMETRY THINKING (GTL) USING 3D GEOMETRY TEACHING STRATEGY THROUGH SKETCHUP MAKE (SPPD-SUM)

Abdul Halim Abdullah, Rohani Abd Wahab, Mohd Salleh Abu, Mahani Mokhtar and Noor Azean Atan

Faculty of Education, Universiti Teknologi Malaysia

Difficulty in learning geometry is generally associated with students' weakness in visual spatial skills (VSS) and low level of geometry thinking (GTL). This study was conducted with the aim of achieving two objectives: (a) to develop a 3D Geometry teaching strategy through SketchUp Make (SPPD-SUM), and (b) to study the effects of SPPD-SUM in assisting students to elevate their VSS and GTL. This study was conducted in two phases; Phase I involved the design and development of SPPD-SUM to elevate students' VSS and GTL. Meanwhile, Phase II involved studying the effects of SPPD-SUM towards students' VSS and GTL. The development in Phase I was conducted based on ADDIE model of instructional design consisting of a five-phase cycle. The Analysis phase studied the fundamental information relating to students' VSS and GTL. In addition, suitability and selection of selected Geometry topic contents were also being investigated in this phase. The Design phase involved establishing the structure, arrangement and design of the activities which integrated the components of VSS into the GTL according to van Hiele Model of Geometry Thinking. Meanwhile, the Development phase involved constructing learning activities in accordance to each van Hiele's GTL and learning phase, as well as, the components of VSS. The Implementation phase comprised of two series of pilot study. It involved the implementation of SPPD-SUM upon 12 students for a span of three weeks. Data analysis obtained from the Evaluation phase was evaluated by seven experts in the mathematics field. They agreed that SPPD-SUM was expected to function well pedagogically. The study in Phase II involved quantitative data collection whereby descriptive and inferential statistical analysis was conducted by using single group quasi-experimental time series design. It was carried out for six weeks upon 34 Form Five students. Inferential statistics computed from the mean score of VSS and GTL suggested that the use of SPPD-SUM assisted the students to elevate their VSS and GTL with a significant difference of $t = 35.5$ and $Z = -5.21$, $p = 0.05$ respectively, by comparing before and after the intervention. These findings showed that SPPD-SUM can be used to enhance students' abilities in rotating, viewing, transforming, and cutting 3D objects mentally and hence concurrently elevating GTL in the aspect of recognising, analysing, making relationship and making formal deductions of geometry series and characteristics.

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CURRICULUM EVALUATION FROM THE VIEWPOINT OF MATHEMATICAL LITERACY: A CASE OF 'FUNCTIONS AND EQUATIONS III'

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[1]Niigata University, [2]Tottori University, [3]Hokkaido University of Education

In mathematised society, literacy to recognize implicit mathematics is required (cf. Chevallard, 1989/2007). For this, we demand to reform our curriculum towards competence-based one, which is required knowledge, competencies and affects through more comprehensive activities rather than multiple works as a means to acquire mathematical content as in the past.

In this research, we analyse a curriculum of "Functions and Equations III" corresponding to grade 9 from the viewpoint of the mathematical literacy. "Functions and Equations (F&E)" is an integrated curriculum developed empirically to improve teaching and learning of equations and functions in Japanese junior high school (Mizoguchi & Yamawaki, 2016). The first mathematical model in an unit is the graph, and this model is refined and modified while the student operates by using it. In a sequence of activities, students repeat mathematical modelling and argumentations for solving problems. So, it is described that the feature of this curriculum is "ability to use functions and equations as problem solving tools".

In this unit, F&EIII, the titles of sub-units are described by not names of concepts but of activities. These were labelled following developmental direction that the learning of mathematics is attained through mathematical activities. Indeed, in each section, different mathematical concepts could be constructed by students together with repeating mathematical modelling and argumentation as "core" of mathematical activities. So, each concept can be organized locally throughout the activity (Shinno *et al*, 2015). Such curriculum construction is expanding in the change of the prospective role and function for modern school education.

Acknowledgement

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DESIGNING QUADRATIC EQUATIONS TASKS SATISFYING STUDENTS' BASIC PSYCHOLOGICAL NEEDS

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According to Self Determination Theory (Deci & Ryan, 2000), individuals have three psychological needs: autonomy, competence & relatedness. If these needs are satisfied, then, individuals are more intrinsically motivated and their well-being increases. To make students engaged and motivated in the mathematical tasks, is it possible to redesign tasks is such a way, they satisfy students' basic psychological needs and therefore, their autonomous motivation is expected to increase? Available research points at a clear potential of this approach. For instance, Wæge (2009) showed that when students are given choices in tasks and to develop their solutions to mathematical problems, or to collaborate with peers and being presented with step-by-step approaches, they develop a better understanding and reflect higher mastery. Therefore, the main aim of the present study is to test the BSPN in quadratic equation (QE) tasks in classroom settings. Currently we redesign typical tasks to satisfy the BPN.

Let be $x^2 - 6x + 8= 0$ the quadratic equation. Traditional QE Task would be *"Find the roots of the quadratic equation." "Draw the parabola equivalent to this equation." "Find the roots by using the discriminant formula."* However, Need Supportive QE Task would be *"Find the roots of the equation by using the questions below:" "What will you do in this task?" "What could be an easier quadratic equation to start with?" "You can choose either parabola, factorization or discriminant method." "Create your own quadratic equation which has 2 different real roots." "Discuss with your peer about your own equation by considering the following: Does your peer's equation has real roots or not? You can compare your equations in terms of representative parabola, the discriminant and the roots"*. By conducting a 6-week intervention study, we expect these tasks to make a significant difference on students' autonomous motivation, their self-efficacy and hence their learning performance.

The study is expected helping teachers to adopt this new approach to foster students' learning performance. This study introduces a new perspective of mathematics education integrating self-determination theory to mathematical tasks.

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STRUCTURING OF WITHIN AND BETWEEN RATIOS WITH THE HELP OF HORIZONTAL AND VERTICAL RATIO TABLES

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A considerable amount of research is focused on students' conceptions and misconceptions of ratio and proportion concepts and informal strategies for related problems. In addition, structural features of the ratio and proportion tasks have been the topic of study. However, the implementation and testing of this accumulated knowledge in real classroom contexts remains an unaddressed issue. In this study, we aim to propose an instructional sequence for improving the instruction of ratio and proportion by referring to the results of a classroom teaching experiment conducted with a 7th grade classroom. Theory of Realistic Mathematics Education was used to develop a hypothetical learning trajectory and horizontal ratio tables (HRT) and vertical ratio tables (VRT) were models that were used to create meaning with a context. For instance, for the problem "How many food fishes can be fed with 9 food bars if 1 food bar can feed 3 fishes?" the following HRT and VRT were used:

Food bars	1 food bar	9 food bars
Fishes	3 fishes	*x* fishes

Table 1. Horizontal ratio table

Food bars	Fishes
1 food bar	3 fishes
9 food bars	*x* fishes

Table 2. Vertical ratio table

Within ratio is a ratio of two quantities in the same setting and between ratio is a ratio of two corresponding quantities in different settings. Throughout this teaching experiment, our goal was to support students' reasoning with HRTs and VRTs and to explore whether their conceptual understandings of within and between ratios were enhanced as they engaged in instruction. Class sessions were both observed and videotaped, and field notes related to researcher's reflections were taken. The transcriptions of the videotapes and field notes were analysed in terms of the cognitive processes of students (claiming, warranting, asking and answering questions, agreeing, disagreeing etc.) by an adaptation of Toulmin's (1958) model of argumentation. The findings of our analysis suggest that using HRTs and VRTs for representing the proportional situations has a big potential for enhancing students' understanding and structuring of within and between ratios.

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FACILITATING GEOMETRY LEARNING THROUGH BLENDED CURRICULUM

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The quality of mathematics learning has been affected by many systemic issues in state-run secondary schools in India. These include a lack of resources, paucity of teachers, pedagogical practices that promote rote learning (obfuscating the intended curriculum), and low student motivation. This poster illustrates a curriculum for geometry learning developed by the Connected Learning Initiative (CLIx) project at the Tata Institute of Social Sciences, Mumbai. CLIx aims to address some of these by designing solutions that factor in the complexity and scale of the challenges, selectively using technology to do so. The curriculum is based on the van Hiele theory of geometric learning, and takes a blended learning approach by providing opportunity for learners to engage in digital and hands-on activities. A digital game 'Police Quad' forms a central part of the module. It encourages students to begin by looking at the properties of shapes and hence leveraging students' insights from the initial level of geometric reasoning (visualization), further helping them to move towards the next level (analysis). The last level in the game facilitates an understanding of class inclusion among students. For every level, there are hands-on activities designed for students which involve classifying and describing geometric shapes. Pertaining to the third level (formal deduction), the game intends to help students understand conjectures, generalizations and value of proofs in mathematics.

The core of the CLIx geometry module is a learning game, but other vital components include – classroom discussions that complement the gameplay, and activities (both hands-on and digital) where students construct shapes, reason with them, make and prove conjectures. The CLIx-mathematics curriculum provides an opportunity for teachers and students to engage actively with the subject using interactive tools. It has been found that the students collaborate extensively while engaging with the digital component, and learn while trying and correcting their own mistakes. While the curriculum will be implemented in schools, pre and post assessments and formative assessments will be conducted in order to understand the impact of the curriculum on teaching-learning.

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DIGITAL DIAGNOSTIC TESTING TASKS (DDTA) – THEORETICAL DESIGN AND INTERACTIVE EXAMPLE

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Digital Diagnostic Testing Tasks (DDTA) can be characterised as adaptive digital sequences of items with a real time analysis of the entered answers. Based on this analysis the DDTA-tool is able to provide a content-related adaptive reaction. The goal of the DDTA-tool is an automatic supportive feedback for each learner by an individual clarification of mistakes.

The diagnosis of Basic Mathematical Knowledge (BMK) aims on diagnostic information as detailed as possible. Both the learners and the teachers obtain this information as a basis to initiate suitable supporting measures. The automatic analysis of answers is based on diagnostic item-distractors in closed items (Winter, 2013), and works with a computer algebra system (Sangwin, 2013) to analyse open answer formats. If it is not possible by these means to identify the mistake behind a given answer, the learner is getting adaptive test-items with a content-related elementarisation of the diagnostic task (Feldt-Caesar, 2017, Bruder & Schmitt, 2016).

Elements of DDTA are the diagnostic task, the algorithm for automatic analysis of the answers, the adaptive sequence control inside of the DDTA, the decision tree to generate feedback, elementarisations of the diagnostic task (if needed), and a parallel diagnostic task in case of an intended control of learning effects.

DDTA allow one to present and analyse tasks with a high complexity of content and activities. The result of using DDTA is an individual analysis of mistakes and supportive feedback (Winter, 2013) for learners and teachers.

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PLACE-VALUE CONCEPT PREDICTS CHILDREN'S MATHEMATICAL LEARNING

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Place-value concept (e.g., 2 in "24" means 20) is challenging to many children (Chan, Au, & Tang, 2014). How important is it to early mathematical learning? Previous studies have already found various domain-general (e.g., working memory) and domain-specific abilities (e.g., subitization) underlying mathematical acquisition. On top of these abilities, does place-value concept still play a role in children's mathematical learning? This study set out to fill this gap. We hypothesized that place-value concept – which is the syntactic knowledge in the symbolic number system – plays a unique part in children's mathematical learning.

Five hundred fifty-three children completed a battery of tasks in their first grade (phase 1), and 510 of them completed them again in their second grade (phase 2). Tasks in phase 1 included an IQ test, word reading test, a place-value task and a series of domain-general (i.e., working memory, spatial working memory, and processing speed) and domain-specific tasks (i.e., approximate number representation, number line estimation, subitization, and number fact retrieval). In phase 2, children completed the place-value task and the domain-general and -specific tasks again. They also completed an arithmetic task. In both phases, children completed a mathematical achievement test.

Using two-step hierarchical regressions, we found that first graders' performance in the place-value task explained a significant, additional 4% variance in their concurrent mathematical achievement, over and above the control variables (i.e., IQ, word reading, all the domain-general and -specific tasks), and contributed a significant, additional 2% variance in their second year's mathematical achievement, on top of the control variables. In Grade 2, children's performance in the place-value task explained a significant, additional 9% variance in their concurrent mathematical achievement, over and above the control variables including the arithmetic task. Hence, place-value concept is a unique predictor of children's early mathematical achievement. Such findings support our hypothesis that place-value concept taps into unique, syntactic knowledge of the number system – which is separable from the early cognitive abilities – thus highlighting its important role in early mathematical learning.

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A PROSPECTIVE TEACHER'S KNOWLEDGES IN FRACTION TEACHING

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Fraction is a difficult concept for most of the students, and teachers' knowledges are important factors to affect students' mathematics learning. Thus, the issue of teachers' knowledges in fraction teaching becomes more critical. It is also found that some prospective teachers are not sure how to appropriately interpret the mathematics textbook and design plan in teacher education programs. Therefore, the purpose of this study is to explore a prospective teacher's knowledges in fraction teaching. Depaepe, Verschaffel, and Kelchtermans (2013) made a systematic review of the way PCK is conceptualized in mathematics educational research and found that even though most authors refer to Shulman (1986) when they define PCK, there is no consensus in the literature about the components covered by it. Even so, grasping teaching materials and promoting students' learning are two main aspects in this study to explore a prospective teacher's knowledges in fraction teaching.

The data of this study was collected from a prospective teacher, Priscilla; the method used is in-depth interview, and the interview problems include posing problems and interpreting textbook about some fraction units.

Here are some discoveries of this study. First, Priscilla's design of main problems doesn't always meet the teaching goals of textbook. For example, the textbook uses hands-on activities to teach equal-sharing concept, but Priscilla uses graphic representation first. The textbook offers many problems with different equal-partition graphics and hopes that students can find equivalent fractions through observing those graphics, but Priscilla's problem posing uses purely mathematical symbols, like $\frac{1}{3}=\frac{()}{6}=\frac{()}{9}=\frac{()}{12}=\frac{()}{15}$, after one word problem. Second, when students can't understand the meaning of the problems in textbook, Priscilla may help them by direct explanation and questioning, assisting after listening to their explanation of the problem meaning, and providing real life examples in relation to the problem. Third, when students meet difficulties in solving problems in textbook, she may help them by asking them to do actual operation, assisting the identification of fraction terms via living language and teaching tools, reviewing prior knowledge, and using graphic representation.

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USING DIFFERENTIATED INSTRUCTION TO PROMOTE TEACHERS' AND STUDENTS' MATHEMATICS SELF-EFFICACY

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Within the "multiple and heterogeneous" classrooms that have students with "academic and neurodiversity diversity", the implementation of "differentiated instruction (DI)" is truly valuable and helpful in reaching the goals of considering both "individual differences" and "learning in the whole class environment" (Tomlinson, 2001). Applying DI in elementary mathematics, it will be beneficial to the development of mathematics teachers' professional knowledge, capabilities and efficacy beliefs, which in turn enhance the development of their students' mathematics self-efficacy and mathematical learning outcomes (Banrura, 1997; Chang, 2015). Based on the developing trend on the co-learning structure of "students-teachers-teacher educators", we established and developed a mathematics teacher professional development (PD) program, which applies the differentiated mathematical instruction in enhancing targeted elementary students' mathematics self-efficacy. Accordingly, we tried to reach these objectives: assisting targeted in-service teachers to design "differentiated mathematical instruction (DMI)" activities and implement the pilot studies within the support of the mathematics teacher PD program. A single-case holistic design was employed in this two-year qualitative and "explanatory and descriptive" case study. Elementary classrooms of higher grade levels of a public elementary school in Taiwan were selected. Mathematics teachers were chosen to participate in the teacher PD program. Data were gathered through semi-structured observations, individual summative and follow-up interviews, and various kinds of documents, and then analyzed qualitatively by template and editing analytic strategies and narrative self-study approaches.

Based on the research objectives and data gathered, the findings were as follows: First, at the beginning of this study (and before the implementation of the teacher PD), targeted teachers' current teaching performance in mathematics and their mathematics teacher efficacy were analyzed. Based on the research objectives and these teachers' current status and needs, a series of teacher PD programs on the design and implementation of DMI, where the developmental process of the targeted teachers' professional knowledge and capability about differentiated mathematics instruction and their efficacy beliefs is analyzed. Thirdly, we analyzed the beginning status of targeted students' mathematics learning profiles (focusing on students' mathematics self-efficacy) in order to select special cases and further analyses while implementing differentiated mathematics instruction in the future. Here we provide examples of how the targeted teachers employed the DMI to promote their students' mathematical learning outcome and self-efficacy.

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THE INVESTIGATION OF MATHEMATICS TEACHERS' PERSPECTIVES ON LEARNING TASKS AND ACTIVITIES FOR GROUNDING STUDENTS' MATHEMATICAL CONCEPTS

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Enhancing mathematics teachers' awareness of students' mathematical thinking through learning is one major mission in teacher professional development (TPD), however, research on teachers' learning focuses mainly on the investigation of teachers' process reflection and collaboration of the community, rarely on the learning instruments used by teacher educators in the existing studies (Llinares & Krainer, 2006). Teachers experience their activity in students' activities with the mathematical tasks can promote the connection between theoretical knowledge and their teaching practices. This study aims to inspect the quality of one nationwide TPD in Taiwan by investigating mathematics teachers' perspectives on tasks and activities they practice in the workshops via the developed questionnaires.

The contexts of workshops in this study are within a nationwide project, Just Do Math (JDM), in Taiwan. This JDM project sets a goal to improve students' motivation and cognition in learning mathematics, via several stages of TPD. The tasks for students learning and teachers utilizing in this project are all designed by mathematics teachers and reviewed by mathematics educators. Moreover, in utilizing the tasks, mathematics teachers need to attend workshops to learn the designed tasks and simulate students' activities in those corresponding tasks in order to be a qualified 'spreader activity teacher'. Since not all participated mathematics teachers thoroughly apprehend the missions born in the workshop but treat it as one opportunity for TPD, therefore, it is necessary to investigate their perceptions of the workshop to keep the quality of this type of TPD.

The 5-point Likert scale questionnaires were developed for teachers' self-evaluation on their perceived usefulness, ease of use, and acceptance on the tasks and activities of the TPD workshops. The preliminary questionnaires included 3 items for each corresponding task in three levels, and 22 items for teachers' activities were devised with the support of 6 mathematics educators' review and comments. After a pilot study with 66 mathematics teachers, the questionnaires were adapted to 6 items for each task, and 26 items for teachers' activities. We therefore found that teachers value those designed tasks at their teaching and students' mathematics learning, and such TPD can stimulate them to reflect on their present teaching (KMO = .913; N = 703).

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PRESCHOOL TEACHERS' KNOWLEDGE OF COUNTING STRATEGY

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Childhood mathematics education is an important part of the curriculum for preschool teachers worldwide. The foundation for understanding mathematical concepts related to counting begins early in life, and early childhood classrooms can provide the roots for mathematical skills needed later in life. Regarding counting knowledge, Gelman and Gallistel (1978) outlined five principles of counting objects: one-to-one correspondence, stable order, cardinality, abstraction and order irrelevance. Several studies found that 6-year old students could use subgroups as a counting strategy (Newman, Friedman, & Gockley, 1987). Fennema & Frank (1992) addressed the important of teachers' knowledge of student that influence students' learning outcome. This study is to explore teachers' knowledge of counting and counting strategy with subgroup on large number object-counting teaching children. In particular, after accepting base-ten lessons did teachers change their method of teaching counting? The research combined qualitative and quantitative methods. A total of 70 subjects came from different preschools. Four lessons related to base-ten strategy were implemented to help teaching counting. Questionaries about counting concept (five items haves five scores) and two tasks assess teachers' mapping ability of counting strategy. The tasks consist of different numerical magnitudes: 48 chick, and 51 candies pictures which are randomly displayed on the size of A4 arrays; tasks include scenarios related to children's real life experience for teachers to generate instructional strategies. Data analyses with statistical test and interview scripts coding were performed. The result indicated before invention, participants gained the score of counting concepts *M*(70)= 2.54. It was fewer for those using subgroup-strategy to count; there were 10.4% of participants who used base-ten strategy to count the objects. Most of them used one-to-one method to teach children counting, and some subjects preferred base-2 and base-5 as strategies for counting large number objects before intervention. When interviewed, participants interpreted that they liked the strategy of base-2 because they thought many children had been taught odd and even, and they liked the base-5 strategy because applying "正" represents as a unit for five times counting. So, they preferred to use base-2 and bese-5 as subgroups to teach students count large number objects. However, after intervention the score of counting concept was *M*(70) = 2.70, but 74% of the participants applied base-ten counting strategy to solve problems. So, intervention promoted teachers' concept and strategy of counting to teach students.

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PROBABILITY-BASED JUDGMENT AMONG PRESCHOOLERS

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Probabilistic thinking is vital for daily life, especially when faced with uncertain situations that require decision-making. Probability has been a topic of research for many years; nevertheless, various opinions still exist regarding the ability of young children to make probability-based judgments. Previous research such as Piaget concluded that young children are unaware of the concept of chance, and thus cannot effectively estimate probability before the age of 7. However, many researchers have obtained different findings showing that children can perceive probability. Neo- Piagetian suggested that teachers should offer opportunities for students to discuss and predict outcomes for uncertain situations by using operations with various materials to develop efficient probabilistic intuition. Čadež and Škrbec (2011) also found that half of a group of children aged 4−5 years could make accurate predictions after teaching. The present study examined children aged 4−5 years to determine whether their probability-based judgment performance could be improved through innovative teaching methods. A total of 28 children in a kindergarten class participated in the experiment, which involved 6 weekly 1-hour classes based on teacher-formulated questions related to life experiences, as well as group discussions, for practicing probability judgment. Relevant data were collected, organized, and displayed for analysis. To understand the efficient development of probabilistic thinking, six tasks were conducted for evaluation consisting of the following: two bags, one containing five red balls and one yellow ball, the other containing five yellow balls and three red balls; two boxes, one containing eight red candies and four blue candies, the other containing five red candies and five blue candies; and two bowls, one containing five red fish and two blue fish, the other containing four blue fish and two red fish. In the evaluation, the children were asked to predict the most probable outcomes for the six tasks. For a correctly predicted outcome on one task, a child would obtain one point. The results indicate that the childrens' probability judgment performance improved over the course of the experiment. Performance levels varied significantly from pretest to post-test (pretest M = 3.28 and post-test M = 4.13; $t(27)$ = 2.465; P = .001). The findings of this study are consistent with the conclusions of past researches that probabilistic thinking can be improved through instruction, a claim that was subsequently reinforced by Čadež and Škrbec (2011), who showed that children aged 4−5 years can make accurate probability judgments. In the present study, more than 70% of the participants made correct predictions on the probability judgment tasks. In summary, studying probability during early school years is necessary.

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DECODING ORTHOGONAL VIEWS OF CUBES: ANALYZING AND ELABORATING STUDENTS' BEHAVIORS

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Representation of 3D objects and spatial structuring are two important parts in 3D geometry thinking (Pittalis, & Christou, 2010). Building and drawing solids made of cubes are beneficial for spatial visualization (Ben-Chaim, Lappan, & Huang, 1988). However, students face difficulties in coding and decoding orthogonal views of cubes because lack of coordination (Battista, & Clements, 1996).This raises the research questions: What do different kind of tasks influence students decoding behaviors? How do students construct their mental models of cubes?

An assessment was developed by hypothetical learning trajectory (HLT) of orthogonal views of cubes, which comprised of 3 tasks: finding 3D corner views (given 2D orthogonal views), enumerating by the base and 2D orthogonal views, and finding compatible 2D orthogonal views. The sample consisted of 263 6^{th} students. In order to investigate how manipulatives influence students' performance. Students were separated into two groups. The first group decoded orthogonal views of cubes with concrete cubes; the second without. Semi-structured interviews were conducted to analyze students' decoding behaviors.

The findings were, the two groups had no significance difference in decoding. But the tasks were with different difficulties. The compatible task was the most difficult, while the enumerating task was of moderate difficulty. Among the compatible tasks, given top view was the easiest. Students' behaviors in decoding were influenced by two modes of thinking: analytic and intuitive. Given the base, students of analytic thinking examined the number of cubes in each position; while students of intuitive thinking combined two views to construct a mental model. In conclusion, mental model of cubes was more likely constructed given the base or the top view. Students' construction of mental models was influenced by their interpretations of one 2D orthogonal view.

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WHAT MALAYSIAN PRIMARY SCHOOL PUPILS FIND IMPORTANT IN MATHEMATICS LEARNING: A PRELIMINARY ANALYSIS

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Pupils from different ethnic groups might value different aspects of mathematics education in different ways. To date, however, limited research has been conducted to explore the respective area (see Jerrim, 2014). This research aimed to find out what Malaysian primary school pupils have valued as important in learning mathematics and set up a suitable framework to analyse the open-ended responses in one section of the validated questionnaire used.

Questionnaire data were collected from 370 Malaysian primary Five pupils, which involved three main ethnic groups in Malaysia: Chinese (158); Indian (14) and Malay (198) pupils. Analysis was conducted based on the value framework proposed by Bishop (1988, 1996) namely, mathematical values (relating to the discipline), mathematics educational values (relating to mathematics pedagogy), and general educational values (relating to the ethical and moral principles).

The analysis shows that pupils thought of “food” as the most important element in learning mathematics, perhaps due to the nature of the questions. Furthermore, 15.06% of Chinese pupils had perceived “ability”, 13.33% of Indian pupils thought of “spiritual elements” and “knowledge and skill” while 17.36% of Malay pupils valued “effort” as important for learning mathematics. This research provides evidence that pupils from different ethnicities valued mathematics and mathematics learning differently. This has implications towards catering to the learning needs of pupils representing different ethnic groups in the society, and indeed, within a class too. Further research can be conducted to gather more insights about the cultural aspect that is hidden beneath each ethnics group. In addition, the coding categories for this section can be refined and used to analyse similar data of different grade levels.

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SOLUTION OF WORD PROBLEMS BY MALAYSIAN STUDENTS: INSIGHTS FROM ANALYSIS OF REPRESENTATIONS

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The solution of word problems is an essential component of primary and early-childhood mathematics curriculum. Despite recent instructional advances, this area of learning continues to present considerable challenges to many children because multiple steps are involved in the solution process (Verschaffel et al., 2007). Children have to understand the text, identify key parts of the text that are relevant to decoding the problem and apply appropriate computational strategies. Research in the field showed that students' use of computational strategies during the solution process vary according to the problem context and even within the same problem context, their choice of strategy could change on the basis of their prior earning experiences (Torbeyns et al., 2009). However such findings are not examined in Malaysian classrooms. Neither have we tackled the question of why children choose a particular strategy when it can be shown that they can access more than one. In the present study, we draw on the framework of representation to track how students negotiate selected word problems in Malaysian classrooms. Representations provide a powerful theoretical lens into not only children's conceptual knowledge but also procedural knowledge. Computational strategies, we suggest are components of children's procedural knowledge and that the nature of representation will affect the type of computational strategy used by children. The aim of our larger study is to document the range of and interplay between representations that Malaysian children construct thus permitting us to identify reasons for their choice of computational strategies. In this report we provide preliminary data about the representational range as a cohort of children attempted to solve two problems that involved 2-digit numbers. Data provide evidence that children in this study tend to work within a limited range of representations and computational strategies. We speculate on current Malaysian instructional practices as reasons underpinning this relatively narrow representational range.

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THE DIFFERENCES AMONG MATHEMATICS ANXIETY GROUPS BY THE EEG MEASUREMENT IN RELATION TO THE MIDDLE SCHOOL STUDENTS' FUNCTIONAL THINKING

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There has been a lot of research focused on reducing the Mathematics Anxiety (MA) in Korea in accordance with the characteristics of Korean students (e.g. Jung & Whang, 2013). However, most studies in the brain science were mainly dealt with arithmetical addition and subtraction of mathematics. This study was to investigate how mathematic anxiety (MA) of the students in the eighth grade could be reduced by comparing analytically their cognitive neuroscience as well as the test result after they completed a three-hour Complex Treatment Program (CTP) about the quadratic function of mathematics.

In the summer of 2016, we collected data from the pre and post MA tests (Ko & Yi, 2011) and from the percent of correct answers (PCA) and reaction time (RT) through event related potentials (ERP) of Electroencephalograph (EEG) based on computer-based functional tasks with one class of a regular middle school.

The results indicated the program to be effective according to the test results. In specific, the results in RT demonstrated that group- higher mathematics anxiety (HMA) and group- low achievement (LA) took longer in all functional tasks than group- lower mathematics anxiety (LMA) and group- higher achievement (HA). The results in PCA, demonstrated that group LMA and group HA scored higher than group HMA and group LA in translating equation-to-graph (task F), but group HMA and group HA scored higher than group LMA and group LA in translating graph-to-equation (task G) by a help of CTP. The fact that CTP reinforced on task G was effective onto group HMA indicated that MA could be reduced through a well-designed treatment of this kind.

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SINGAPORE PRIMARY FOUR PUPILS FIGURING OUT PATTERN POSITION: HOW WELL? WHAT STRATEGIES?

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Understanding and recognition of patterns, relations and functions is emphasised through the use of pattern generalisation (PG) tasks throughout a Singapore pupil's primary mathematics education. Lower primary pupils learn to identify and extend not only figural repeating patterns that vary in a number of attributes but also number sequences that involve common differences between consecutive terms in the sequences. Upper primary pupils encounter linear figural PG tasks. The importance of the teaching and learning of PG in mathematics is often stressed by many researchers including Kaput (2008) and Stacey (1989), as PG tasks are powerful tools in examining pupils' ability to generalise and in determining their algebraic skills (Becker & Rivera, 2004). A typical linear figural PG task for primary pupils displays the first few terms of a pattern and asks for a particular term and the position of a particular term. In some tasks, pupils are even asked to establish the general term. This poster presentation reports on how 57 Singapore Primary Four pupils figured out the position of a given term in three linear patterns. Data were collected through administering a 75-minute written test and one-to-one interviews of nine selected pupils. Pupils' responses and strategies used were obtained, coded and analysed. *Undoing* emerged as the top successful strategy for finding the position of a given term in all three linear patterns (56%, 20% and 14%). Strategies used by Singapore pupils and comparison with strategies used by pupils in other studies will be presented.

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LESSON STUDY AS A CONTEXT FOR THE DEVELOPMEMT OF MATHEMATICS TEACHERS' SPECIALIZED KNOWLEDGE

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The development of a Lesson Study (LS) is perceived as one of the contexts for promoting teachers' professional development. In the context of an already stablished collaborative working group (Grupo de Sábado [Saturday Group]), a LS is being devised. It involves teachers from primary, lower and upper secondary, prospective teachers and researchers – in some moments working in three subgroups accordingly with the school level. The participants chose to focus on improving their practice by addressing the nature, kind and focus of the tasks they prepare and implement in their classrooms. Such choice allowed us to introduce the discussion around the development of the LS and the notions of Mathematics Teachers' Specialized Knowledge (Carrillo et al., 2013) and Interpretative Knowledge (Ribeiro, Mellone, & Jakobsen, 2013) as a ground for promoting the participants' professional development. Goals of two different natures are pursuit: studying the development of a LS as teacher education strategy in the Brazilian context and understand the critical elements promoting the participants' professional development processes. Data collection concerns audio and video recordings of the working meetings (in the subgroups and large group); classroom practices; interviews to teachers; interviews to students after the implementation of the tasks; students' productions when solving the tasks, and teachers' narratives grounded on their own experience.

We will present preliminary results pinpointing some critical elements for mathematics teachers' professional development. Such elements emerged on the group's process of conceptualizing tasks and discussing on students' answers and productions. Such critical elements sustain some discussion on the role and development of the participants' specialized and interpretative knowledge.

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AHA! MOMENTS IN MATHEMATICS CLASSROOMS

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The aim of the work is to investigate facilitation of Aha! Moments on the basis of Koestler (1964) bisociation theory and their detailed descriptions found in literature, which are modest in number. In agreement with the suggestions of reviewers the poster will start with the diagrammatic concept map presentation of theories of creativity used in Math Ed leading to the theory of bisociation, that is the theory of Aha! Moments. In words of Koestler, "*bisociation is a spontaneous leap of inside which…connects previously unconnected frames of reference and makes us experience reality at several planes at once*"(p.45). These leaps of insight are further described as the process of unearthing "*hidden analogies*" between two or more previously unconnected frames of reference. The definition of bisociation suggests that the cognitive content of Aha! Moments is the construction of the schema of thinking in relation to the problem in question.

Following this presentation, we will present and discuss two characteristic descriptions of Aha! Moments found in literature, which show two different methods of facilitation, through collaborative complex problems solved by students with the minimal help of the instructor and through student – teacher dialogues with the maximal involvement of the teacher. The discussion will address the significance of the circumstances leading to the insight of Aha! Moments, the role of the teacher and the assessment of the depth of knowledge (DoK) reached during the insight. DoK assessment will be conducted with the help of the Piaget and Garcia (1987) PG triad of conceptual development and the concept of "restructuring" of Gestalt theory (1994). We will show the existence of two separate frames of reference which are connected during the insight by the "hidden analogy" as an application of Koestler theory.

Finally, we will draw conclusions concerning methods of facilitation and present several new descriptions of Aha! Moments obtained during the pilot teaching experiment in the Spring 2017 conducted on the basis of the results discussed above.

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PRE-SERVICE MATHEMATICS TEACHERS' PREDICTIONS FOR STUDENTS' ALGEBRAIC WAYS OF THINKING

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Numerous studies investigating students' conceptions and interpretation of letters in algebra have reported students' common misconceptions and difficulties by paying attention students' inadequate understanding of letters (e.g., Stacey & Macgregor, 1997). Recognition and anticipation of students' common challenges/difficulties were an important pedagogical competency that teachers need to have. This study investigated to what extent pre-service middle school mathematics teachers (grades 5-8) were able to predict students' conceptions of letters in algebra. This study was conducted as a part of a comprehensive study where the investigation of PSTs' predictions was the first step of this study. The data was collected in a methods course offered in a mathematics education program of a public university, and 44 third years pre-service mathematics teachers (PSTs) enrolled in the course participated to study. During the study, PSTs worked in a group of 4-5. For this study, the authors designed a test including 25 items by using Küchemann's (1978) test items. Before this study was conducted, the student data were collected from almost sixty seventh grade students attending a middle school. For PSTs, the test items were purposively divided into three groups by authors, and then, each question group was presented to PSTs during three weeks of the course. Each week, before analysing students' actual responses, PSTs were asked to predict about students' possible solutions to the given questions, but particularly students' possible common incorrect solutions/thinking ways. They were also required to write down their predictions in detail to the given documents. The data were collected through PSTs' written documents. For the data analyses, initially, students' common incorrect solutions for each item and students' common mistake for the same items reported in research literature were identified. Then, PSTs' predictions of students' incorrect solutions on questions with students' possible incorrect solutions were compared, and the data was coded accordingly. Initial analyses showed that in the first week of the study, the predictions of all groups mostly could not correspond with the student common incorrect responses. However, the data presented evidence that all groups' predictions became diverse and consistent with students' common incorrect thinking ways reported in research and in students' actual responses. Implications for PSTs' professional development will be discussed.

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EDUCATIVE CURRICULUM MATERIALS: MODEL OF ANALYSIS OF CONTROL OVER THE SELECTION OF STRATEGIES TO SOLVE MATHEMATICAL TASKS

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In this poster I present results of a research whose objective was to elaborate a model of analysis of control over the selection of strategies to solve mathematical tasks in the classroom, from the reading of four educative curriculum materials (ECM), elaborated by a collaborative group, in Brazil.

Educative Curriculum Materials (ECMs) are designed to promote student and teacher learning (REMILLARD et al., 2014). The four ECMs referred to were analyzed using the framing concept of Basil Bernstein's theory.

In solving a task, it may occur that: (case 1) only the pre-defined strategies in the task (PdE) have been used; Or (case 2) the PdE and classroom emerging strategies suggested by the teacher (EmEP) have been used; Or (case 3) has been used the PdE, the EmEP and emerging strategies suggested by the student (EmES); and (case 4) only the PdE and EmES strategies have been used.

With this assumption, I constructed a model of analysis of the selection of task resolution strategies, with the following components: the ECM_i (i = 1,2,3, ..., n) in column 1 of a table; in column 2, there is the mathematic task of each ECM; in the columns 3, 4 and 5, there are the strategies (PdE, EmEP and EmES, respectively), with the indication of the words or phrases that reveal the strategies which can be read in the different links of the ECMs and, in column 6, I added the type of framing, determined by a certain decision criterion.

Applying this model, one of the main conclusions is that the framing in the case of control over the selecting of task resolution strategies is strong (case 2) for two ECMs and weak (case 3) for the other two ECMs. Therefore, this research communicates that there is a gap between what ECM can suggest before its implementation in the classroom and what actually happens in the classroom, allowing ECM designers to reconsider the need to improve these materials.

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A COMPARISON OF STUDENTS' AND TEACHERS' CONCEPTIONS OF THE USEFULNESS OF MATHEMATICS

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Teachers often strive to help their students see mathematics as useful, as perceived usefulness can have positive benefits including improved academic achievement and enhanced interest (e.g. Hulleman, Godes, Hendricks, & Harackiewicz, 2010). However, we know little about the ways in which teachers and students think about what it means for mathematics to be useful. This research explores this issue by asking, How do middle school students conceptualize the usefulness of mathematics, and what is the relationship between students' and teachers' conceptions?

Twelve seventh-grade students (ages 12-13) participated in interviews in which they viewed six images of students doing mathematics in different ways and with different people. The participants identified the images in which students were engaged in "useful" mathematics and "not useful" mathematics and explained why. Responses were coded using open coding to gain insight into the criteria students used to judge whether mathematics was useful. A comparable activity with teachers is underway.

Students spoke about the usefulness of mathematics in two ways. First, they sometimes discussed whether mathematics content could be applied in various settings. Here they primarily considered whether the images portrayed students doing mathematics that might be used in jobs or in everyday activities. Second, they considered whether features of the learning experience were useful. In these cases, students tended to view math as useful when it involved collaboration or when students were actively engaged and able to show their thinking. For example, José viewed an image of students sitting at their desks with the teacher at the front of the room and said, "I believe this is not really useful 'cause kids…want to…get up on the board and show ideas, but the kids are just sitting there and watching the teacher do the math."

Existing research focuses primarily on the usefulness of mathematics in terms of the applicability of content; however, students also considered usefulness in terms of the ways in which they learn mathematics – a new perspective that might be leveraged to enhance students' perceptions of usefulness. Data from teachers will be examined to explore whether the same themes arise in teachers' conceptions of usefulness.

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MUTUAL CONSTITUTION OF MATHEMATICS, LANGUAGE, AND CULTURE: A MODEL

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University of Melbourne

A theoretical model was developed to describe the relationships between (first) language, (home) culture and mathematics (Figure 1). The purpose of the model is to help describe how cultural mathematics programs can utilise the affordances of diverse local languages and cultures. Drawing on both theoretical and empirical research, each pair of concepts is described as mutually constitutive, with each intersection being an affordance for developing cultural mathematics programs.

Mathematics and culture: Mathematics is developed to fulfil cultural needs and also enables cultural development (Bishop, 1988). By drawing on cultural activities, a mathematics program can utilise the interaction between culture and mathematics.

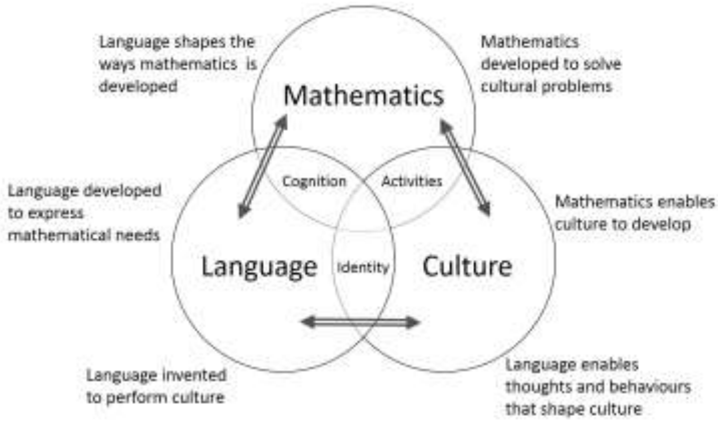

Figure 1.

Language and culture: Language both develops to enable and express the needs of a culture and influences cultural development (Vygotsky, 1963). Utilising both first languages and home cultures in a mathematics program helps create and affirm cultural identity.

Language and mathematics: Language is developed to express mathematical needs and also shapes the way that mathematics is developed (Barton, 2009). The interaction between language and mathematics is observed in the cognitive benefits of using first languages in mathematics teaching and learning.

Language and mathematics are thus both cultural expressions and drivers of cultural development, but are also both drivers and expressers of each other in a mutually constitutive manner. A cultural mathematics program that includes students' first languages can extract the embedded mathematics from cultural activities, and draw on the affordances provided by these intersections to benefit cognition and identity.

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TECHNOLOGY BELIEFS OF NOVICE SECONDARY TEACHERS

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The appearance of portable technological tools has given rise to a growing body of research at various levels of mathematics education. It remains interesting how and why secondary teachers employ technological devices (e.g. graphing tools or CAS) in the teaching and learning of calculus at upper secondary level. According to the framework of Hannula (2012), belief systems and goals are parts of mathematics-related affect that consist of cognitive, motivational and affective aspects. As research has shown (Zbiek et al. 2007), the role of technology in mathematics teaching requires an assiduous distinction between technical and conceptual mathematical activity. The function of technology in learning mathematics effectively is thus a central question for the teaching practice. Results from a qualitative study with 20 preservice and teacher trainees will be discussed centred on how their beliefs on technology correlate with pedagogical and mathematical beliefs on calculus teaching. Data were collected by two semi-structured interviews within a time span of 15 months and analysed by qualitative coding close to grounded theory.

After finishing their university education beginning teachers mention the use of technology as a visualization instrument in several parts of the interview. The majority of teachers see the key advantage of using technology in the possibility of visualizing mathematical objects so that students can form a mental picture of functions. With respect to the teaching of calculus, using technology as a learning tool is a belief that serves as a means to an end for many teachers. Beyond the advantage of visualizing mathematical objects, many teachers see the possibility to incorporate more complex modelling tasks as well as to further their students´ heuristic competence. Taking into account the variation in our results, our data yields the emergence of two antithetical belief clusters on the integration of technology in upper secondary calculus courses: a rejectionist and a supportive stance. Assuming that technological tools and mathematical tasks do not automatically lead to learning, teachers´ beliefs about the meaningful integration of technology are determining factors for the *what*, the *how* and the *why* in understanding teachers´ objectives and goals. However, the challenge of a meaningful integration of technology into calculus teaching needs further studies and more insight into relationship between teachers´ beliefs and actual classroom practice.

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HARNESSING COMPLEXITY: A FRAMEWORK FOR TEENAGERS' IDENTITY AS LEARNERS OF MATHEMATICS

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Learners' identity in the context of mathematics education is not a simple concept to explore especially given the dynamic and ever-changing notions of—inter alia—autobiographical identity, socio-culturally available selves, discoursal identity, authorial identity (Ivanič, 1998), and actual and designated identities (Sfard & Prusak, 2005). While Ivanič's (1998) work is situated within the field of academic writing, her framework provides a helpful tool in illuminating the complex nature of learners' identity in mathematics education. Both frameworks were thus used in a study on newcomer teenagers' identity in mathematics. However, questions relating to the interconnectedness, and system-like emergent nature of identity-associated constructs that collectively shape learners' identity remained open.

These questions emerged during the analysis phase of data that were collected through a three-part-interview design—comprising a family interview (FI), a one-on-one interview (ID), and an all-parent and all-teenager focus groups (FG). Seven newcomer families who moved to Canada with their teenage children reached out to participate in the study. Twenty-seven interviews—seven FIs, 16 IDs, and four FGs—which lasted 90 minutes each, yielded about 40 hours of research data.

While each of the analytical frameworks mentioned above highlights different areas of identity work, we were looking for a framework that will push forth and bring forward the idea that these notions may be distinct but they are not at all separate. Treating these notions separately not only unduly demotes the role of teenagers' identity as learners of mathematics to the backstage but also distracts our attention from the bigger picture that shows how other—sometimes overlooked—notions are strongly linked to shaping a teenager's identity as a learner of mathematics. Ergo, we use a set of vocabulary from complexity theory that includes, but is not limited to, agent, strategy, population, interaction, and copying to provide a networked perception of learners' identity in the context of mathematics education.

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A COMPARATIVE ANALYSIS OF STATISTICAL PROBLEMS BETWEEN JAPAN AND NEW ZEALAND

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The purpose of this paper is to elucidate the unique characteristics of Japan (JPN) and New Zealand (NZ) through a comparative analysis of statistics education textbooks. The theoretical framework to analyse is based on Bishop's six universal mathematical activities('counting', 'locating', 'measuring', 'designing', 'playing', and 'explaining')(Bishop, 1991, p.23). Additionally, the author transforms them into five mathematical activities to do for students: 'numerical calculations and formula operations', 'approximation and measuring', 'designing and locating', 'making reasonable decisions', and 'explaining'. The subjects of the analysis are questions and practice problems in the first year of high school. It is five textbooks for JPN and one for NZ that the author analyses.

84 problems from JPN and 322 problems from NZ are classified by the five activities.

	JPN (84 problems)	NZ (322 problems)
Numerical calculations and formula operations	59.1 (70.3%)	120 (37.2%)
Approximating and measuring	0 (0%)	16 (5%)
Designing and locating	14.7 (17.5%)	73 (22.5%)
Making reasonable decisions	0.2 (0.2%)	16 (5%)
Explaining	10.0 (11.9%)	97 (30.2%)
Total number of activities	92.4	489
Number of activities per problem	1.1	1.52

(The proportion among all questions is indicated in parentheses. To facilitate a comparison with NZ, the average for one textbook is used for JPN.)

Table 1: Classification result of problems

As a result, the unique characteristics in JPN include the fact that it aims to form the statistical concepts. On the other hand, statistics education in NZ is characterised as the handling of statistical inquiry through the sorting of the necessary data, the transformation from this data into the necessary information, and the decision-making using explanations of the information.

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A CASE STUDY ON STUDENTS' CONSTRUCTION OF THE AESTHETIC QUALITIES OF MATHEMATICAL OBJECTS

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BACKGORUND AND FRAMEWORK

In mathematics, the aesthetic sensibility plays an important role (Sinclair, 2006). Nevertheless, it is hard to say that the school mathematics reflects the role of the aesthetic qualities. One of the main reasons for this situation, there is a consensus that the general students cannot identify the aesthetic qualities of mathematical objects. Dreyfus and Eisenberg (1986) found non–mathematician did not pursue and could not identify the aesthetic qualities which defined by authors.

On the other hand, Sinclair (2006) used computer–based tasks and demonstrated that students focused on symmetry. The purpose of this study is to extend and complement the findings of Sinclair. In particular, this paper will show that students may focus on the aesthetic qualities of mathematical objects in non–computer environment. For this purpose, author analyzed the mathematical problem–solving process of high school students in Japan.

This study defines the aesthetic qualities as feeling about unity from the "form" of mathematical objects. This "form" includes symmetry, proportion, etc.

This study used the following problem related to the parabola. That is, to investigate the mechanism of parabolic antenna. The survey was conducted on 43 high school students belonging to 2nd grade (Grade 11).

RESULTS AND CONCLUSION

The students thought about the cross–section of the parabolic antenna. Then, they considered the figure that was expected to parabola on the coordinate plane.

Students' attention to the symmetry could be classified into the following 2 categories. First is symmetry with respect to the axis of the parabola. Second is symmetry with respect to the axis of the parabola and symmetry between the directrix and the focus. They obtained the standard form of the equation of the parabola using the symmetry.

The results of analysis, this study clarified that computer environment is not necessary condition for construction of aesthetic qualities of mathematical objects by students. In addition, the importance of learning about the function of symmetry was suggested.

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TRACKING VISUAL ATTENTION DURING COLLABORATIVE PROBLEM SOLVING

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Information in learning context is often visual. Gestures and diagrams are part of multimodal communication. The student cannot look at all possible sources of visual information at the same time, but one must choose what to attend to. In this poster we will present first results of the MathTrack project that uses mobile gaze-trackers to study students' collaborative problem solving. The methodology is ground breaking, as we study visual attention of several persons in a natural learning context.

The design of the gaze tracking device and the corresponding software were developed at the Finnish Institute of Occupation Health and released as open source (see, e.g., Toivanen & Lukander, 2015). We also used a video camera, which recorded students' actions and utterances, recorded students' work using smart-pens and screen capture video, and interviewed the students using the video recorded material as stimulus.

We present a case study of how three mathematics education students visual attentions shifted as they step by step generated a solution for a geometry problem. The task was to find the shortest path connecting four cities located at the corners of a square. A group of four students (one was not wearing a gaze tracker) had first generated ideas individually, then worked in pairs, and we will describe part of the process as all four work together and find a better solution. We begin from the moment, when one student first sketched a solution of the right kind. We follow how the students' visual attentions shifted as they first refuted the solution, later returned back to it, then moved to the smartboard to empirically test and refine the solution.

In the poster we present our cutting-edge methodology and demonstrate the potential of this new type of data for research of classroom interactions. We will present the visual attentional processes of the three focus students and show the similarities and differences between the individuals. The results will give us insight on thinking of those students that appear not to participate actively in creating the solution but are still part of the group and engaged with the task.

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IMPLICIT AND EXPLICIT VALUES OF A MATHEMATICAL CAMP FOR GIFTED STUDENTS

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In this presentation we examine the implicit and explicit values in a summer camp for mathematically gifted high-school students. This challenges the cognitivist framework of giftedness by studying processes in which the social identity of the gifted, as well as participation in a community, are related to the development of mathematical skills. The conceptual framework is based on Schein’s theory (2010), which characterizes organizational culture by layers, including hidden values and espoused values. Connecting Schein's theory with Sfard's (2008) commognitive approach, we searched for the ways in which the organizational culture of the camp supports certain mathematical identities and certain forms of participation in mathematical activity. This was done using discourse analysis to analyze and classify instructors’ stories about students as told in private staff meetings, problem solving episodes in the classrooms, and ideals of mathematical success as reflected in camp-leaders' talk during ceremonies. Three rounds of the two-weeks camp were studied, taking place during the years 2013-2015. All rounds were very similar in content and structure, focusing on number theory at a B.Sc. university level, and organized around groups of 4-5 students led by an instructor who usually had a B.Sc. degree or higher (the first author being one of the instructors). The staff was mostly uneducated in pedagogy of mathematics, thus the camp provided an opportunity to study teaching-learning interactions in a relatively "natural" or apprentice-oriented environment. Activities centered on individual problem solving, discussed in group "peer review" sessions led by the instructors. We found several explicit values prominent in the camp, including the importance of using accurate mathematical terminology both in written and in oral communications, and valuing persistence in the face of obstacles during problem solving efforts. Explicit values thus highlighted malleable aspects of mathematical activity. In contrast, implicit values had more to do with stable characteristics of students such as their enjoyment of doing mathematics during free time, or their talent and creativity. Talent, specifically, was often judged based on implicit assumptions about background knowledge that students were supposed to have, despite this knowledge not necessarily being part of the school curriculum.

A case of one student will be used to exemplify this tension between explicit and implicit values. This student made every attempt to align herself with the explicit values (by taking care to be precise in her writing, for example) but failed to exhibit the background knowledge and creativity expected according to the implicit values. As a result, she was both identified by the staff and by herself as relatively less successful than other students.

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INVESTIGATING SECONDARY STUDENTS' CONCEPT IMAGES ON DIVISION ALGORITHM AND RELATION BETWEEN THESE CONCEPT IMAGES TO EARLY LEARNING EXPERIENCE

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It is observed that students in senior secondary levels (Grades 10-12) often have significant difficulties in utilizing topics that are closely related to the algebraic division (e.g. remainder theorem, factor theorem, partial fractions, etc.). Such topics are based on understanding of the division algorithm, which is covered implicitly, in junior levels. This paper investigates possible sources of these difficulties, in particular, secondary school students' concept images (Tall & Vinner, 1981) of the division algorithm and how these concept images play a role in their thinking about more advanced topics.

Selected students from Grade 7 and Grade 10 were asked to complete certain tasks related to multiplication, division and division algorithm for the purpose of analyzing their concept images towards the topics. In the first part, they completed a set of questions and engaged in post-test interviews, where they were asked to explain their solution. In the second part, Grade 10 students are asked to try some questions on algebraic division, a topic they have not yet learnt in school. This part was intended to see if they could handle these unlearnt topics using their existing concept images.

Results showed that although students are competent in the operation of multiplication and division, most of them had an incomplete or inaccurate concept image on division algorithm. In particular, these students' conceptions were built with a lack of relations between multiplication and division, as evident in their written tasks and in the interviews. In the second part, when students were asked to solve questions on unlearnt mathematical topic (algebraic division), students who had previously shown a more complete concept image of division algorithm had a better performance in handling such advanced topics than those who are otherwise.

This poster discusses possible explanations of students' developed concept images from a pedagogical perspective. Moreover, we suggest that the structure and rationale of the mathematics syllabus may account for how these concept images evolve over time. This study sheds light on the teaching and learning of topics related to algebraic fractions in Hong Kong; in particular, it is argued that an emphasis on procedural understanding rather than a conceptual understanding on the topics of multiplication and division in early mathematics education, and the lack of discussion on the relation between them, play a role in the phenomenon observed.

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INVESTIGATING BACKWARD TRANSFER EFFECTS

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University of Delaware

I define *backward transfer* as "the influence that constructing and subsequently generalizing new knowledge has on one's ways of reasoning about related mathematical concepts that one has encountered previously" (Hohensee, 2014, p. 136). My research on backward transfer extends Lobato's (2006) *actor-oriented transfer* perspective on the transfer of learning. While there exists a long history of transfer of learning (or generalization of knowledge) research, *backward transfer* research has rarely been conducted in the context of mathematics education.

In a new study, I examine backward transfer in real classrooms. My research questions are: (1) What changes in students' previously-established ways of reasoning about linear functions are observed after students complete a quadratic functions unit, and (2) What classroom processes during the quadratics unit play a role in those influences? An NSF Early Career Grant supports this project.

The participants for my study will be students and teachers from four Algebra 1 classes at two schools (i.e., two classes per school). All four classes will use a linear functions-quadratic functions instructional sequence. One school will use a traditional Algebra 1 curriculum and the other will use a reform Algebra 1 curriculum. Data collection will begin after the linear functions unit and before the quadratic functions unit. First, students will complete five linear function problems. Next, all four classrooms will be video recorded during their quadratic functions unit. Finally, students will complete another five linear functions problems, similar to the first set of problems. The pre- and post-test results will be used to answer the first research question. The video recordings will be used to answer the second research question.

The following hypotheses guide this study: (a) students' linear function covariational reasoning changed (became more or less productive) after the quadratics instruction (i.e., a backward transfer effect); and (b) *focusing interactions* (Lobato et al., 2013) during the quadratics unit play a role in changes in students' covariational reasoning (i.e., a process underlying backward transfer). This research may reveal insights into important new instructional and curricular principles for minimizing unproductive backward transfer effects and promoting productive backward transfer effects.

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A STUDY OF COGNITIVE DEMAND EMBEDDED IN RIGID TRANSFORMATION OF GEOMETRIC DIAGRAMS

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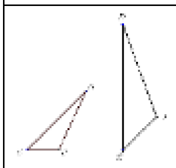
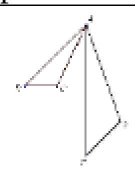
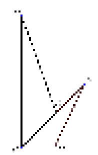
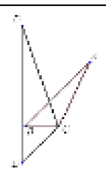
Researchers in psychology and mathematics education have stated that rigid transformations (flip, rotation, and slide) of diagrams require different level of cognitive demand. In this study, we created a survey instrument and used it to further investigate student performance on rigid transformation of geometric diagram. Specifically, we attempt to explore how rigid transformation influences students in recognizing the diagram properties and constructing mental images accordingly. In this regard, we first show students two similar triangles for 20 seconds and take out one of the triangles. Then we ask students to construct the disappeared triangle themselves.

We designed the survey not only to consider different kinds of rigid transformation, but also another two important factors: the degrees of rotation of the triangle (e.g., rotating 45 degrees to right hand side), and the location relation of the two triangles.

Separated diagrams	Diagrams shared a point	Diagrams shared a segment	Overlapped diagrams
			

The four kinds of location relation of the two similar triangles are shown in the above table. A total of 38 items involving different rigid transformation, degrees of rotation, and location relations were included in the survey. Each participating student had to answer the survey individually but with different item sequences. Totally fifty 8th grade students answered the survey. Regarding the three kinds of rigid transformation, the data analysis shows that students performed the best on recognizing the triangles and constructing the disappeared one when the two triangles involve slide transformation. Then is flip transformation. Students performed the worse on rotation transformation. Analysis also reveals the discrepancy of student performance on one kind of rigid transformation with different degrees of diagram rotation. For example, students performed better on recognizing and constructing the disappeared triangle when it is rotated to 45 degrees than that of 90 degrees. For location relation of the two similar triangles, data analysis also reveals that it significantly influences student performance. Student performance from high to low sequentially is the case that the two similar triangles are located by sharing a segment, then the case by sharing a point, then the case of the two triangles overlapped, and finally the case of being located separately. Those findings provide insights into the level of cognitive demand embedded in different kinds of rigid transformation as well as rotation degrees involved in the transformation, and the location relation.

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THE DEVELOPMENT OF A GEOMETRIC REASONING BOARD GAME: A PILOT STUDY WITH SIXTH GRADERS IN TAIWAN

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Ting-Hua Chu and Pei-Chin Tseng
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Board games can be enticing for children and can promote discussion and collaboration among students. The effect of board games on mathematics learning, however, is less certain, given the lack of research in this area. The purpose of this study is to evaluate a geometric reasoning board game entitled Shape Matchmaker. The study seeks to understand: (a) how this game is received by elementary school students and (b) how students articulate their thoughts while playing the game.

The van Hiele model of geometric thinking (Clements & Battista, 1992) and de Villiers' (1994) work on classification of geometric shapes were used to identify learning objectives and to guide the design of the game content.

Shape Matchmaker is designed for four players to play simultaneously. The game contains a game board, 32 shape cards, and 12 statement cards. Each shape card has a polygon figure printed on a grid (the lengths of the sides are provided when necessary). Statement cards are descriptions of geometric properties. For example, a statement card may show the message "find shapes with at least one pair of parallel sides." Players are required to match statement cards with shape cards individually in stage one. In stage two, players work together to figure out common properties among as many shapes as possible. Players in stage three are divided into two groups of two. The task in this stage is to categorize six shape cards into two groups and explain the underlying principle for the partition.

Twelve sixth graders were organized into three heterogeneous groups of four based on mathematics ability. Interview data, field notes, and reflective essays were analyzed. Students felt that the game was fun because it encouraged discussions and stimulated thinking. Most indicated that they preferred a cooperative game mode (i.e., stages two and three) over a competitive one (i.e., stage one). A few students were able to offer coherent arguments for making meaningful connections among shapes. Most statements made by students regarding common properties, however, were superficial. In addition, it was found that property statements from students were more likely to be based on ideas they were most familiar with but which might be irrelevant to a shape from a conventional mathematical viewpoint.

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THE COMPREHENSION OF RELATIONAL CONCEPTS (<, >, =) BY PRE-SERVICE AND PRESCHOOL TEACHERS

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Mathematical language is a language of symbols, concepts, definitions, and theorems that does not develop naturally like a child's natural language, but needs to be taught (Ilany & Margolin, 2010). Preschool teachers often use the knowledge and experience they bring from daily life, meaning that they might not always give the correct mathematical importance to the symbol. If the teachers incorrectly understand the use of the symbol, they will subsequently pass this on to the children, leading to their incorrect use in the future. They don't see a problem if the child writes: **5**>5. They say: "We teach the child to use the sign > between two objects, in this case the size is important, in another case the length is important. It depends on the context." (Ilany & Hassidov, 2012). This research presents a quantitative and qualitative study comparing how pre-service and preschool teachers perceive the relational symbols (<, > and =). The study population comprised 71 pre-service teachers participating in a course dedicated to teaching and learning early childhood mathematics and 149 in-service preschool teachers.

The data were collected through questionnaires and semi-structured interviews. The 25-item questionnaire was designed by the authors as part of a larger study examining the perceptions of mathematical symbols. Four questions that address mathematical symbols between different types of numbers (fractions, identical numbers, different numbers, and mathematical expressions) were analyzed. In each question, there were differences in the sizes and thicknesses of the numbers. Respondents had to add one of the relational symbols between the two numbers or indicate "X" if they believed there was no appropriate answer. They were asked to give the reasons for their answers. A large proportion of the participants did not answer the questions correctly or give suitable reasons for their answers. There was a significant difference between the two groups, with the pre-service teachers giving a significantly greater number of correct answers and explanations. The conclusions arising from this study are that preschool teachers do not correctly comprehend the true significance of <, >, and =, and therefore do not teach them correctly.

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RESEARCH ON PROBLEM IN EARLY OF PROBABILITY

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Among the countries of East Asia, the students learn early probability at the elementary school. The purpose of this research is to clarify the conditions with which the problem should be equipped. At the 1st step, we arranged about the worth of teaching materials. Sriraman & English (2004) mentioned that there was hardly any topic which could improve the skills of pupils so extensively as combinatorics. As a result, we pointed out worth of permutation and combination. At the 2nd step, we surveyed about the method of the present instruction. As a result, we paid our attention to the relation between asking for the number of permutation and combination, and calculation. At the 3rd step, we considered the difficulty of the instruction. Szitanyi and Csikos (2015) concluded that college students often chose one of the algebraic expressions from their high-school repertory, and use it without sense-making of the problem. As a result, we pointed out that calculation was interfering with achievement of the purpose of the lesson. At the 4th step, we examined the method for the instruction to improve. As a result, we proposed the instruction which does not ask the number of permutation and combination (Ishii, 2016). At the 5th step, we carried out the lesson by the improved teaching problem. As a result, we recognized the importance of the problem. At the 6th step, we analysed the lesson qualitatively. As a result, we specified the effective problem.

By all the steps, we clarified the following two points. The 1st point is not asking the number of permutation or combination. That is, it is important to ask how to put permutation and combination in order. The 2nd point is that the student searches for neither permutation nor combination by calculation. That is, it is important that the student sets up the problem which is not called for by calculation. When carrying out problem in early stages of probability, we need to fill these two points.

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11th GRADE STUDENTS' BELIEFS ABOUT THE SELF IN MATHEMATICS CLASSROOM

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Students' beliefs have an important influence on learning process (Kloosterman, 2002). The study of students' belief by Op't, Corte & Verschaffel (2002) found that belief about the self is another kind of students' mathematics-related belief systems. Belief about the self is show the belief in the knowledge and ability to predict the result in solving mathematical problems.

This study aimed to explore students' beliefs about the self in mathematics classroom. Data collected from 223 eleventh grade students who are studying in the second semester of academic year 2016. These classroom focused on teachers describe the content and explain an example. Then the students will do the exercises to understand the content. The instruments of research consist of mathematics-related beliefs questionnaire (MRBQ) containing 20 items that are scored on a 5 point likert-scale. Data were analyzed by using basic statistics.

The study result revealed that students' beliefs about the self in mathematics classroom had average as following; 1) beliefs which have the average at partially agreement level containing; a belief that I would be to solve mathematics problem if I would learn all rules has the average at 3.56 and a belief that If I do more exercises enough, I can understand in more mathematics has the average at 3.84. 2) beliefs which have the average at uncertain level containing; a belief that I can understand the difficult topics in mathematics has the average at 3.13 and a belief that I can solve mathematics problem with my attempt has the average at 3.33. And 3) beliefs which have the average at partially disagreement level containing; a belief that I think, studying mathematics is a waste of time has the average at 2.23 and a belief that mathematics has no relevance to my life has the average at 2.15.

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A STUDY ON CHINESE PRE-SERVICE TEACHERS' SUBJECT KNOWLEDGE OF MATHEMATICS

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With the deepening of the reform and development of higher education in China, some normal universities have adopted the joint-cultivation mode of teacher education. That is, pre-service teachers of different majors are implemented by the unified enrolment and management of the college of teacher education, but they need to take courses for credit at both the college of teacher education and the relevant professional colleges. For example, the pre-service mathematics teachers should take educational courses at the college of teacher education and mathematical courses at the school of mathematics. Against this background, there is a debate that the level of pre-service teachers' subject knowledge decreases since they can hardly see the use of the subject knowledge learnt in university for their teaching in primary or middle schools. In the present study, we investigated a class of pre-service mathematics teachers' (n = 75) subject knowledge by asking them to construct concept maps for certain topics.

RESEARCH METHODS

The participants were the student teachers enrolled in one of the first author's courses. Before the survey, the first author had introduced in class what a concept map is and how to construct a concept map to ensure that they acquired the basic concept mapping skills. At the end of the course, the participants were asked to self-select a topic in the middle school mathematics, list the relevant core concepts, and construct a concept map with the core concepts by referring to the knowledge system of the textbooks. The concept maps were then analysed qualitatively and a holistic score was assigned for each concept maps.

FINDINGS

First, the student teachers had a good grasp of the knowledge of mathematics in middle schools, but their subject knowledge was not improved with the increase of schooling stages. For example, few of them were able to relate the matrix transformation learnt in university to the solution of simultaneous equations learnt in middle schools. Second, the concept maps reflected individual difference in processing knowledge and a gap was detected generally between the student teachers' knowledge structure, as reflected by their concept maps, and that of mathematics textbooks, which suggests that the student teachers were unfamiliar with the mathematics textbooks of middle schools. Third, the concept maps also showed that some of the student teachers lacked deep understanding or even held misconceptions of the mathematical concepts. For example, many preservice teachers indicated that a function can always be expressed by a general algebraic expression. The findings suggest that teacher educators should pay attention to the advancement of student teachers' subject knowledge.

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THE STUDENTS' CREATIVE THINKING ABILITY THROUGH ELPSA FRAMEWORK

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Creative thinking as one of the competences needed to solve non-routine problems set in various real-world contexts at the Program for International Students Assessment (PISA). Meanwhile, Indonesian students' mathematics performance at PISA has been at the lowest five countries in the last five assessment periods. Indonesia's pedagogical practices restrict opportunities for creative problem solving (Johar, Patahuddin, & Widjaja, 2017). The ELPSA framework was introduced into Indonesian schools to encourage classroom teachers to focus on mathematics processes rather than direct instruction only. ELPSA contains five components within a learning cycle, such as Experience, Language, Pictorial, Symbol, and Application (Lowrie & Patahuddin, 2015). ELPSA framework sees learning as an active process in which learners construct their own way in understanding a problem through the process of creative thinking leading to students learn mathematics meaningfully. This article investigated the development of students' creative thinking abilities through the ELPSA framework in teaching transformation geometry.

Data was collected over five lessons, two lessons for teaching the topic of reflection and one lesson for each topic of translation, rotation, and dilation. The teacher designed non-routine tasks that required divergent thinking for component A (application) of ELPSA framework. Fifteen students from Grade 8 (14 years old) in a junior high school in Aceh participated in the study, with data from five students were analysed to ascertain their creative thinking abilities by test and followed by the interview. The participants were of varying mathematics ability. Creativity assessments were given to the students at the end of each topic, its called quiz 1, 2, 3, and 4. Students' creative thinking ability was scored by using a rubric, level 1, 2, 3, 4, and 5. Results revealed the progression of students' creative thinking ability, except for quiz 4. However, there is no student who fulfils the originality level (level 5 of creativity). Indonesian students are unfamiliar with these kinds of expectations, so more sustained periods of engagement are required in order to foster students' creative thinking ability through ELPSA framework.

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TEACHERS' EFFORT IN IMPLEMENTING DEMOCRATIC CLASSROOM IN MATHEMATICS LEARNING

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The mathematics learning in Indonesia tends to be started by presenting abstract information such as definition, formula or standard algorithm. Teachers rarely give students opportunities to solve problems using their own way (Fauzan, Plomp, & Gravemeijer, 2013). Consequently, students do not have opportunities to give their opinion in solving problems, the learning is not interactive and students feel 'being forced' to accept the knowledge that is not meaningful for them. In order to solve this problem, teachers need to implement democratic classroom in the mathematics learning. The research problem is 'what are teachers' efforts in implementing democratic classroom in mathematics learning?'

Observations were conducted in one of the junior high school class in Aceh, Indonesia, one teacher was involved. The observation was conducted for two meetings. The teacher gave three open ended problems in the first meeting and two open ended problems in the second meeting. The lesson was recorded. A questionnaire about teacher's perception of her efforts in implementing democratic classroom during the lesson was administered to the teachers after the lesson, followed by a short interview. The observation indicator of democratic classroom is adapted from Daher (2012). Data was analysed by video watching for several times, transcribing and re-reading the transcripts. Furthermore, the transcripts were compared to the field notes to gain deeper insights of teacher's effort.

The results show that teacher's effort in implementing democratic classroom include: 1) giving students freedom to present their opinions, ask questions, choose resources and discuss how to solve problems using their own way, 2) agreement on the conclusion, 3) being fair in treating the students and providing feedback and, 4) not interrupting students when they are presenting. The results indicate that a democratic class enable students to learn mathematics in a more engaging environment.

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LIFELONG EDUCATION FOR ADULT NUMERACY: IMPLICATIONS OF PIAAC BY OECD

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As Korean society is being transformed into an aging society, social interest in lifelong education is increasing. In particular, since numeracy is emphasized as one of the core competence for the 21st century, it is necessary to prepare educational interventions to enhance adults' numeracy to their contribution to social development sustainable. When considering that only limited groups of people are benefited by quality of higher education, educational intervention for adult numeracy is concerned with equity and justice. In the perspective, we analysed the data from PIAAC to identify implications for the development of lifelong education program for adult numeracy. PIAAC is a Programme for the International Assessment of Adult Competencies conducted by OECD. It assesses the proficiency of adults from age 16 onwards in literacy, numeracy and problem solving in technology-rich environments that are relevant to adults in many social contexts and work situations, and necessary for fully integrating and participating in the labour market, education and training, and social and civic life.

In this paper, we analysed descriptive statistics for numeracy by diverse backgrounds in order to assess Korean adults' numeracy level and compare it with other countries. The analysis shows that the mean in the numeracy for adults of 16-64 years old is lower than the OECD mean. In addition, the age 16-19 group is the highest mean score 281. On the contrary, the mean of the age 60-64 group is 221, which ranks the 23rd among the 26 participating countries and the achievement gap between the two age groups is the biggest among the participating countries. Also, we found out the gap between groups of diverse social backgrounds within Korea. The groups of born in country(264), male(269), full-time employed(265), employee(267) show significantly higher mean score than the groups of born in regions other than country (231), female(258), part-time employed/unemployed(255), self-employed(256). This research implies that it is of essence to develop a lifelong education program to enhance numeracy for adults who are in a marginalized group in ages or social economic status in Korea.

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A COMPARATIVE ANALYSIS OF MATHEMATICAL TASKS FOR GIFTED STUDENTS IN KOREA AND THE UNITED STATES

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Korea and the United States have implemented different gifted education systems with different histories and culture. There are many previous researches on gifted education and it is still going on. However, in both countries, research topics on gifted education are concentrated on characteristics of students and curriculum (Min et al., 2011; Jolly&Kettler, 2008). In other words, there are relatively few comparative studies among countries. Therefore, the purpose of this study is to analyze the tasks of each country, see what and how they differ and find implications.

There are two research questions; a) How much the cognitive demands are required of gifted students in Korea and the United states when they learn?, b) What are the characteristics of the two countries in gifted education tasks?

The data are the tasks developed by one of the gifted education institutes in Korea and the Center for Gifted Students in USA, respectively.

Although Korea and the United States have different gifted educational systems, a similar pattern is observed by Bloom's Taxonomy revised. According to Davis and Rimm(2004), gifted learners should be given more time for high order thinking; analyzing, evaluating, creating. As a result of analysis, both Korea and the United States are experiencing a considerable shortage of evaluating and creating. In addition, there were more geometric lessons with various manipulative materials than other areas. And the contents of geometry were weigh are weighted towards some.

In conclusion, there are common aspects of task although they have different history and educational systems. Gifted learners should have opportunity to evaluate and create something with various experiences, improve their cognitive level, and to develop gifted education. More specific field studies are needed.

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PRE-SERVICE TEACHER REFLECTIONS ON PRACTICES OF ASSISTANT TEACHER

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RESEARCH PURPOSE

This study aims to investigate reflections of pre-service teachers through practicing as assistant teachers in mathematics classrooms. For the purpose of this study, two research questions on reflections are addressed; 1) What do pre-service teachers reflect on their assistant teacher practices? and 2) As times go by, how do the reflections of pre-service teachers on the practices shift?

METHODS

We designed a case study as a qualitative method. During 2015 spring semester, secondary pre-service mathematics teachers took a course designed for working as assistant teachers in regular mathematics lessons in a middle school. Eleven pre-service teachers took this course and participated as assistant teachers in mathematics lessons in the middle school. One or two pre-service teachers went to the lessons once a week and ten times in total. The pre-service teachers talked with students when a teacher gave time to his students for solving problem and waited sometimes when the teacher had time to explain things such as mathematical concepts and principles to his students. Whenever they participated in a lesson, the pre-service teachers made self-report of reflections on their practices. As a case study, two pre-service teachers were selected in this study. During the spring semester, data such as self-reports on their practices and videotapes of five meetings were collected. The self-reports as main data were analysed by four aspects of reflections.

RESULTS

As results, the reflections on assistant teacher practices of two pre-service teachers were examined in the four aspects such as teacher, students, mathematics contents, and themselves. We will show the different percentages of four aspects between the preservice teachers in a bar graph and show changes over time in a table. The assistant teacher practices were intended to extend the chance of field experiences for secondary pre-service mathematics teachers. In previous studies, pre-service teachers during their field experiences reflected more on themselves than other aspects in general, while this study found that most reflections on assistant teacher practices included reflections on teacher and students. This difference implied that preservice teachers could have different reflections according to the types of field experiences that they participated in.

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STUDENTS' VALUES ABOUT MATHEMATICAL STORIES AND CONNECTIONS IN TERM OF LEARNING MATHEMATICS IN THE CONTEXT OF CLASSROOM USING OPEN APPROACH

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This study aimed to explore Students' Values about Mathematical stories and connections in term of Learning Mathematics in the Context of Classroom using Open Approach. The target groups were consisted of 640 students- fourth grade students in the second semester of 2015 academic year. The target groups were separated into 2 groups; 1) students who were taught by internship students, and 2) students who were taught by in-service teacher. Both were used Open Approach as a teaching approach based on Inprasitha, 2011 as following; 1) posing open-ended problem 2) students' self-learning 3) whole class discussion and comparison and 4) summing up by connecting students' emergent mathematical ideas. Data were collected by using questionnaire students' values about mathematics (Seah, 2013) that divides opinion into five levels. The research findings found that two groups of students agree with all item as the following table.

Students' Values	means ($\bar{X}$)	
	teach by internship students	teach by in-service teacher
connecting maths to real life	4.42	4.25
hands-on activities	4.17	3.92
using concrete materials to understand mathematics	4.13	3.90
students posing maths problems	4.12	3.90

Table 1: shown the means of students' values.

The table above shows that the means of students' values between the 2 groups who were taught by internship students and in-service teacher.

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CONSTRUCTING THE HIGHER-ORDER MATHEMATICS COMPETING MODELS FOR JUNIOR HIGH SCHOOL STUDENTS IN TAIWAN

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The focus of mathematics education has gradually transferred from content knowledge to the factors of affect, beliefs, motivation, and disposition. De Corte et al. (2004) proposed a theory of self-regulated mathematical learning and problem processes that incorporate affect, volition and cognition, and included five factors of domain specific knowledge, heuristics strategies, meta-knowledge, self-regulatory skills and mathematics beliefs. Although they provided the possible factors in this higher-order mathematics model, the relations among these factors remains unclear. The study aimed to construct three higher-order mathematics competing models and examine the model fit of these competing models. According to different perspectives, the first model which is the “integrated relation model” indicates that heuristics strategies, meta-knowledge, self-regulatory skills and mathematics beliefs correlate to one other and all jointly predict domain specific knowledge. The second model which is the “affect－metacognition/meta-knowledge－cognitive-constructivist model” suggests that affect/belief is the primary factor to predict other self-regulatory factors and then influence domain specific knowledge. The third model which is the “affect－volition－strategy－achievement model” conducts that meta-knowledge, self-regulatory skills, and strategies are the mediators between beliefs and domain specific knowledge. The research question is: Among these three competing models, what is the best model to represent the higher-order mathematics model for junior high school students in Taiwan? Questionnaires were administered to 1,093 seventh grade students enrolled in middle schools of Central Taiwan. Structural equation modeling was utilized to examine three competing models. Based on model evaluation criteria of χ^2, CFI, TLI, RMSEA, critical N, and ECVI, the results indicated that the goodness-of-fit for the third model “affect－volition－strategy－achievement model” fits the data well among three competing models. The results also suggested that there were several mediating effects, including the first mediating effects of meta-knowledge and self-regulatory skills, from mathematics beliefs to heuristics strategies, and the second mediating effects of heuristics strategies, from meta-knowledge and self-regulatory skills to domain specific knowledge. According to the results, the study suggests that an effective mathematical learning should value the influence of different factors on learning process. The mathematics belief factor is the primary factor, the meta-knowledge, self-regulatory skills, and heuristics strategies are the mediating factors and these factors predict achievement directly and indirectly. The study offers a reason for why a higher-order mathematics model is so important in explaining Taiwanese students’ mathematics learning; an intervening effect from the perspective of cultivating students’ affection, metacognition, and strategies should be taken into account.

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MATERIAL BASED DEVELOPMENT OF MATHEMATICS TEACHERS' KNOWLEDGE FOR TEACHING PROBLEM SOLVING

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The (inter-)national educational standards have recognized the importance of problem solving in mathematics education and for that reason have strongly endorsed its inclusion in school mathematics. Nevertheless, mathematics problem solving continues to remain a challenge for many teachers for different reasons, such as limited knowledge for teaching problem solving (Chapman, 2016) and lack of practical teaching materials with didactical comments (Kuzle, 2016). Efforts to design, use, and do research on problem solving material for teachers in a real setting through collaborative work between researchers and practitioners may promote development of teachers' knowledge for teaching problem solving and with it implementation of problem solving in school mathematics.

In this poster presentation I demonstrate the possibilities for developing materials using design-based research (DBR) focusing on the development of mathematics teachers' knowledge for teaching problem solving. In accordance with DBR, the design process was informed by theoretical basis with respect to models of teachers' knowledge for teaching problem solving (e.g., Chapman, 2015, 2016). More concretely, I focus on the question: *How does material informed by models of teachers' knowledge for teaching problem solving foster or hinder the implementation of problem solving in mathematics?* The results are based on eight DBR-cycles with four teachers through which I demonstrate how the theoretical ideas got implemented and what design elements fostered or hindered the implementation. On this basis I offer suggestions for how material related to development of teachers' knowledge for teaching problem solving might be designed to support their diverse needs.

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A STUDY OF DIFFICULTIES WITH SIMPLE ARITHMETIC WORD PROBLEMS

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The study was to investigate why a group of 27 students were having much difficulty in handling simple word problems. The students in the study were aged 7 to 9, studying in the same school. Students were presented with three particular kinds of problems designed to reveal how they would approach the problems, what difficulties they would encounter and why. The first kind of problems was relatively simple, requiring the students to combine two known quantities to produce a total. The second kind was slightly more difficult and required the children to find a missing addend – the first addend and the total being known quantities. These two problems were based on the work of Riley, Greeno & Heller (1983) who had categorised *change problems* according to the unknowns to be found. The third kind of problem was a guessing game based on the work of Neuman (1987) where the total was known and the children had to work out different possible combinations of the two parts of the whole.

The conceptual framework of the study has its central concern with the part-whole relationship of quantities as the core mathematical concept. The framework involves the interrelationship among word problems, mathematical tools and world experience – and their impact on the conceptual understanding of the part-whole relationship.

Difficulties were identified in both procedural knowledge (inability to subitise, insufficient flexibility in the choice and use of arithmetic procedures, inability to use counting on, inability to count by grouping or regrouping) and conceptual knowledge (failure to understand the part-whole relationship). It also appeared that the students had little real-life experience that was directly relevant and applicable to the arithmetic problems encountered in the school context. They thus found it difficult to make sense of the arithmetic word problems.

For teachers struggling to understand why their students are not succeeding as they would like, the study will help them identify particular difficulties on the part of the students and design appropriate remedial instruction.

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SOLVING AND UNDERSTANDING MULTIPLE PROPORTIONALITY PROBLEMS

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Vergnaud's (1983) analysis of multiplicative structures and of scalar and functional strategies for solving missing value proportionality problems led to a wealth of work on how different populations solve simple proportionality problems favoring one or the other strategy. These strategies, which focus on the relationships between the quantities in a problem, may constitute a sound basis for the meaningful learning of the rule-of-three algorithm to compute the fourth proportional.

We analyzed how 16 high school students, who solved the following multiple proportion problem using the rule-of-three, considered scalar and functional relations among the quantities in the problem:

> "Marina is preparing a chocolate cake. The recipe states that, for each cup of milk, one should use 2 eggs. For each egg, one uses 3 cups of flour. How many cups of flour does Marina need to make a cake with 3 cups of milk?"

Textbooks adopted by the students' school emphasized scalar and functional relationships among quantities in verbal problems, before introducing the rule-of-three as a sequence of computational steps to find a solution. The students solved the problem in writing and, in individual interviews, explained how they did so.

We found that the rule-of-three representation was set up by the students after considerations about the scalar and/or functional relations among quantities. These were depicted as data tables or as pairs of values, usually with referents and with arrows to indicate the scalar or the functional relationships between two elements in a pair. They also explained their solutions in terms of these relations. Only a few students represented the rule of three as an equality of two ratios and only a few showed the computational steps to reach the solution to the problem, perhaps because the numbers in the problem were small and results could be easily found by mental computation.

Our data suggest that students' understanding of relationships among quantities in verbal problems can in fact constitute a basis for their appropriation of algorithmic procedures. In this process, it is helpful for students to consider pairs of physical quantities or pairs of numbers with their referents, rather than pairs of pure numbers, as they usually appear in the rule-of-three representation.

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THE EFFECT OF 'JUST DO MATH' ACTIVITIES ON CHILDREN FROM A REMOTE AREA IN TAIWAN

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The results of TIMSS (2015) showed that more than half of the 4th grade children possessed negative attitude toward math (56% disliked math and 60% had low confidence in math) although Taiwan ranked number 4 in mathematics. Additionally, the polarization of Taiwanese children's math performance was extreme among the top 7 Asian economies. A new curriculum project called 'JUST DO MATH' (JDM) has thus developed to fix such problems (Lin & Chang, 2016). The theories for JDM are meaningful learning with game-like and hands-on activities tackling various mathematical concepts which are considered by teachers difficult for most children to learn. The activities are played before the concepts are officially introduced in class. The results shown in this paper were part of the JMD project of which children were from a remote area in Taiwan called Taitung. The 5th grade children were given math attitude questionnaires before and after playing the JDM activities before the geometry topics officially taught in class. Also, a math test was given to them before the test and at the end of the semester. The results of T-test (Table 1) showed that the differences of the mathematical scores was significant for all children at time 2 and between two ethnic groups (indigenous vs. nonindigenous). Although boys scored higher than girls, the difference was not significant. The improvement was not significant for the nonindigenous children; however, the improvement was evident for the indigenous group. On the other hand, the change of math attitude was not significant. The reason might be it needs more time for the attitude to change.

Experiment	Group 1	Group 2	p-value
Test	Time 1: 58.86	Time 2: 80	0.004*
Ethnicity	Indigenous: 63.67	Nonindigenous: 79.82	0.021*
Gender	Male: 71.74	Female: 66.10	0.492
Cross-analysis 1	The 1st test of nonindigenous: 73.17	The 2nd test of nonindigenous: 87.80	0.072
Cross-analysis 2	The 1st test of indigenous: 48.13	The 2nd test of indigenous: 76.10	0.003*

Table 1: The results of t-test for math test of difference groups

"*" denotes significant at α = 0.05

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A COMPARATIVE ANALYSIS OF VISUAL REPRESENTATIONS IN ELEMENTARY MATHEMATICS TEXTBOOKS OF KOREA AND SINGAPORE: FOCUSED ON ADDITION OF FRACTIONS

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Addition and Subtraction of fractions with different denominators are central to mathematical topics that are directly linked to quantitative reasoning with three levels of units (Steffe & Olive, 2010). As quantitative reasoning with units is not automatically developed, a variety of visual representations which help students be engaged in such reasoning process are employed in curricular materials. Given this background, this study compared and contrasted in what ways the visual representations of fraction addition are used in the textbooks of Korea and Singapore with a focus on reasoning with units. The subjects for this study were current mathematics textbooks in Korea and *My Pals are Here! Maths* in Singapore. Analytic foci were three big ideas related to reasoning with units by Lee and Pang (2016): (a) fixed whole unit, (b) necessity of common measure, and (c) recursive partitioning linked to algorithms. The topics of analysis included equivalent fractions, comparison of fractions, and addition of fractions. Although Korean and Singaporean textbooks had commonality in using various visual representations, there were differences in when to deal with the topics. More interesting differences occurred in why, how, and how often the textbooks used visual representations in relation to the big ideas. For instance, regarding recursive partitioning linked to algorithms, visual representations in Korean textbooks are used to show that the sizes of two fractions are the same by equi-partitioning each of two equal units. In contrast, Singaporean textbooks tend to subdivide the partitioned unit when making equivalent fractions or adding fractions with different denominators. However, it is necessary for both textbooks series to reconsider the use of visual representations which help students reason with three levels of units through recursive partitioning and link them with algorithms. This poster will include detailed examples of visual representations used in both Korean and Singaporean textbooks, followed by strengths and weaknesses of such representations. As such, this poster is expected to elicit implications of constructing textbooks on fraction addition with different denominators.

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EXPLORING HOW PROBLEM SOLVING STRATEGIES WERE TAUGHT AT ELEMENTARY SCHOOLS

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As mathematics is used in more and more technology and workplaces, mathematics teaching and learning has become a crucial part in today's classrooms (National Governors Association, 2007). Problem solving plays an important role in mathematics teaching and learning (National Council of Teachers of Mathematics, 2000; Schoenfeld, 1992, 2002). Literature indicated that problem solving strategies can be taught and need to be taught (Krulik & Rudnick, 1995). Little, however, was known about how problem solving strategies were taught at elementary schools in Taiwan. The purpose of the study was to explore how problem solving strategies were taught at elementary schools in Taiwan.

Ten Taiwanese elementary-school teachers participated in the study. They received interviews on how they taught problem solving strategies in mathematics classrooms. The findings of the study indicated that the teachers didn't specifically teach problem solving strategies such as working backwards in their problem solving instruction; they, instead, tended to merely use the strategies to solve problems when teaching problem solving. More studies need to be conducted with a larger group of participants to understand better how problem solving strategies were taught at elementary schools in Taiwan. It also would be worthwhile to investigate how to enhance teaching in problem solving strategies at elementary schools in Taiwan.

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IMPROVING THIRD TO SIXTH GRADERS' LEARNING EFFICIENCY BY MAKING MATHEMATICAL SENSE

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The researcher had proposed making mathematical sense and diagnostic teaching to facilitate teachers' professional development when PHD research. After 15 years' research, the researcher (Lee, 2016) proposed content theory of mathematical sense as the knowledge system, applying the five core contexts (to give examples, to simplify, to draw picture, to ask why, and to recall) as the teaching strategies in the teacher education program and optimizing multiple teaching method. This research was to investigate the learning effectiveness (including cognition and attitude) for third to sixth graders after one-year participation in the teacher education program.

The research method was experimental teaching. A total of 616 participating students were served as the experimental group, while the rest of 1485 non-participating students were served as the control group. The data collection included the four mathematics exams from two semesters (T1-Oct. 2015, T2-Jan. 2016, T3-Apr. 2016, and T4-Jun. 2016) and three questionnaires on attitudes (A1-Sept. 2015, A2-Jan. 2016, and A3-Jun. 2016). The questionnaire on attitude was mainly adopted from TIMSS 2011, which was a four-point scale and consisted of 18 questions.

In this research, using T1 as the covariate and T2~T4 as the dependent variables, the exam scores revealed significant differences between the experimental and control groups. The results showed that the experimental group was superior to the control group. Meanwhile, the mean differences between these two groups were 2.56, 3.98, and 5.09, respectively; this indicated the differences in score for the two groups began to widen. The Mean (SD) of three questionnaires on attitudes are 2.03(0.60), 2.11(0.65), and 2.11(0.63). The paired-samples t-test showed significant differences between A1 and A2, and between A1 and A3; which indicated student didn't perform very positive attitudes during the program. Nevertheless, no significant differences between A2 and A3 also indicated students' attitude performance did not worsen.

Making mathematical sense needed students to actively show the five core contexts; these were all novel learning experiences for them. When we asked students to do these at the very beginning of the research; they often felt pressured. That's why students' learning achievement improved significantly but their attitude performance did not. According to the above results, the researcher believed that once students could be aware of mathematical sense, their attitudes would change positively as well.

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EXPLORING THE MATH CREATIVITY PERFORMANCE OF GIFTED ELEMENTARY SCHOOL STUDENTS

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Deal and Wismer (2010) believed that math creativity can be developed through open-ended problems, while Stillman, Cheung, Mason, Sheffield Sriraman and Ueno (2009) believed that challenging tasks can help develop math creativity. The purpose of this study was to investigate gifted elementary school students' math creativity performance through teaching both open-ended problems and challenging tasks.

The researchers defined the math creativity of this study with the math creativity at the K-12 level proposed by Liljedahl & Sriraman (2006). The theoretical framework of math creativity curriculum design was based on Sriraman (2005) and Stillman et al. (2009). The research was conducted through class observation. The subjects were 20 gifted fifth-graders. 11 two-hour courses were provides. The teaching material included: (1) open-ended problems, such as: among 2/3, 3/8 and 7/15, please explain how each number differs from the others; (2) challenging tasks, such as: please find the four graphs that five connected equilateral triangle constitute, choose three, and arrange them to form a line symmetric graph.

With open-ended problems and challenging tasks as materials, the subjects were taught through group discussion, dialectics, the generalization of solutions, and affective support. This study found that gifted elementary school students' math creativity performance was as follows: (1) they were able to produce various solutions; (2) they were able to think about and classify their solutions from different perspectives; (3) they were produce new solutions on the basis of those others had provided; (4) they were willing to try to produce unusual solutions.

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A CASE TEACHER’S TEACHING OF MATHEMATICAL THINKING IN CHINA

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Mathematical thinking (MT) is essential for learning and teaching mathematics. It is revealed that Asian mathematics teachers used to place emphasis on thinking and understanding in classroom teaching (Stigler & Perry, 1988). Especially in China, MT has been one of the “Four Basics” (basic knowledge, basic skills, basic thinking and basic activity experience) for Chinese mathematics education (Ministry of Education, 2012). This study aims to examine the teacher’s teaching of MT by case study. An experienced female teacher was selected from a Beijing common school. Two questions are considered: a) how is the teacher’s lesson planning for MT, and b) what about her classroom teaching from the perspectives of teacher questions and oral statements? Relevant data (textbooks, lesson plans and videotaped lessons) were collected and analysed. Based on western and Chinese literature, a framework for major aspects (specializing, generalizing, conjecturing, convincing, categorizing, analogizing, thinking of symbolic-graphic combination, thinking of transformation (*Huagui* e.g. transfer a complex problem to one or several simple)) of MT was developed for analysing the selected tasks. For the first question, tasks selected from the textbooks and then analysed them in the teacher’s lesson plans to see the thinking demanding level. For the second question, the types and their frequencies of MT involved in the questions and oral statements were considered. In this case, the teacher seemed like to raise the level of demand for MT in her lesson plans but lower it in classroom teaching for students. MT was easily involved in teacher questioning. For instance, the teacher explained a task of equations in a very detailed way to guide the students to generalize the common characteristics. “*Let’s look into the four group equations, what characteristics can you find? Did you notice them? We can observe the characteristics of the given expressions. Can we know how many items of the polynomial multiply the polynomial respectively of the left equations?*” However, this possibly limited the opportunity for the students to express their own ideas and thoughts during the lessons. Results indicated the teacher had the intention to make MT in a higher demanding level than MT but failed in classroom teaching.

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EMBEDDING QUESTIONS IN VIDEOS FOR A HYBRID COURSE

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In flipped classrooms, content is delivered through online videos whereas "homework" and activities are done in class. Benefits include: (a) freeing up class time for meaningful activities and formative assessment; (b) increasing student engagement; and (c) allowing learners to re-watch any part of a video (Bergmann & Sams, 2012). Jensen et al. (2015) found that learning gains are due to active learning rather than flipped classroom. Active learning in class requires students to first learn the content at home. One way to engage students is to require students to answer embedded questions as they watch math videos. Questions can be embedded to (a) introduce a concept, (b) demonstrate a procedure, (c) explain why, (d) pose a challenging problem, and (e) illustrate using real-life scenarios (Lim & Wilson, in press).

We conducted a study to investigate the use of embedded questions in math videos and their effect on student learning in a hybrid math course. For the 50% on-line portion, students watched videos and answered questions that require them to explain what they had learned from the videos. Students still had to turn in weekly homework which consists of math problems. A treatment group (T) of 19 prospective teachers had to answer embedded multiple-choice question as they watch videos watched videos whereas the comparison group (C) of 22 prospective teachers watched the same videos but without any embedded questions. Data were collected across 6 weeks (out of 15) covering angles, conversion of units, area-perimeter, Pythagorean Theorem. The pre-to-post improvement for group T was higher than group C (effect size of 1.60 vs. 1.35) but not significant; the test items are substantially different from embedded questions. Group C re-watched parts of the video more often than group T (1.4 times vs. 1.2 times) but not significant; T students took more time to watch the videos because of embedded questions. In pre-post surveys, opinions of the course improved for C but not T students. The embedded questions were designed to elicit student errors so that students can learn from their mistakes; however, students' confidence and video-watching experience might be negatively affected by wrong answers. This study suggests embedding questions must consider both cognitive and affective aspects.

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CAN ACTION SUPPORT THOUGHT AND PROMOTE SPATIAL REASONING?

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There are a growing number of research studies based on embodied cognition theories that support the idea that the physical manipulation of objects and the sense of touch affect thinking and learning (Barsalou, 2008). However, when teaching mathematics, often knowledge is still ideally expressed in verbal and written form and coming from the mind. Thinking has also been facilitated by direct actions on objects such as manipulatives and touchscreens showing actions that are directly connected to the thinking on a task will improve performance (Segal, Tversky, & Black, 2014). In designing such tasks, the challenge is to prescribe bodily actions that directly support the targeted learning outcomes of the activity. This study will explore the effect of gestures and gestural interfaces on children's performance in math – in particular, spatial reasoning. The research questions are: What types of gestures (iconic, deictic, metaphoric) emerge when children are solving spatial reasoning problems? Do gestures and gestural interfaces help promote a child's understanding of spatial reasoning?

Fifteen four- to six-year-old children were asked to perform spatial reasoning tasks involving the composition of shapes using different interfaces. Firstly, children were given tasks with concrete shapes to form pictures. Children then used the developed iPad app to complete similar spatial tasks. Each child completed two culminating tasks, first combining pattern blocks in different ways to form triangles, followed by an open-ended spatial question. The children's hands were videotaped and an analysis of the types of gestures produced, time to finish the task, and accuracy was completed for each child and each task.

The results showed how various types of gestures revealed embodied knowledge. Representational and metaphorical gestures used, could be interpreted as physically linking mental processes to the physical environment to help give the child meaning. Children who used gestures such as slides, rotations, and reflections on the touchscreen to complete the task performed better on the culminating tasks. Combining the use of gestures with concrete materials and the use of gestural interfaces resulted in performing the best on the culminating tasks, supporting the claim that cognition is based on the body and that gestures promote spatial thinking.

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PRESERVICE TEACHERS FOMAL AND INFORMAL SOLUTIONS IN DECIMAL OPERATIONS

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Formal solution methods refer to strategies which are deeply based on symbolic knowledge or complicated algorithms that are explicitly taught in mathematics lessons so students acquire mathematics knowledge through formal classroom instruction such as precedence rules for order of operations. Informal solution methods refer to the strategies that are not acquired through formal classroom mathematics lessons (Baroody & Coslick, 1998; GroBe, 2014). We investigate what types of formal and informal solution methods they have been using, what solution methods produces higher responses on the decimal operation test. Then we identify what types of correct and incorrect solution responses between formal and informal responses they have been most frequently used.

The participants were the students enrolled in a mathematics class in a teacher education program at a mid-western university. The DKT was specifically designed to measure preservice teachers' knowledge of solving problems in different ways using decimal operations in areas related to: (1) addition (one item), (2) subtraction (one item), (3) multiplication (two items), and (4) division (two items).

The result indicates that preservice teachers substantially produced higher formal solution responses on the decimal operation test. This implies that the majority of preservice teachers were heavily influenced by their previous lessons that are explicitly taught in mathematics classrooms. The findings also revealed that the majority of the preservice teachers demonstrated the most correct responses between formal and informal solution methods when using the *standard algorithms* method. Interestingly, they produced most incorrect responses between formal and informal solution methods using *standard algorithms* as well. Throughout this study, we found that preservice teachers demonstrated higher formal solution methods when computing decimal operations.

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THE INVESTIGATION OF COGNITIVE COMPONENTS OF GEOMETRY ITEM DIFFICULTIES

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The quality issues of an assessment for the test developers are not only concern the precise measure of student abilities, but also to help teachers achieve effective instruction by regulating teaching strategies and materials based on the feedback of the test information. Through item analysis, task difficulty is one of the significant issues, with cognitive component analysis can help test developers construct tests from the cognitive perspective to make the test constructing process more efficient and systematic (e.g. Embretson & Daniel, 2008). This study aims to develop an encoding framework of the cognitive component related to geometric achievement test. Item difficulty is concurrently calibrated under One-Parameter-Logistic Item Response Theory model of the response of 4800 participants from the sixth to eighth grade. In this research, geometry items (from the sixth to eighth grade) are used for coding, exploring how well it works for explaining the variety of item difficulty and describing the cognitive characteristics of different levels. 50 items are analysed; all items are coded with cognitive components, e.g. formula information, visualization, problem-solving steps needed, distractions, and novelty of context. Multiple regression analysis is conducted to calibrate the correlation between the components and item difficulty. The proportion of item difficulty variance explained is 54%. Dividing 50 items into 3 item difficulty levels: based on the difficulty of item is less than - .5, between -.5 to .5, and greater than .5. We find items in different levels show a little difference on cognitive components. The Researcher hopes that the result is in expectation to make the task constructing procedure more simplify, to make the task difficulty under controlled within the desired extent; another, the cognitive components proposed in the present research will provide references for constructing geometry test items.

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THE LATENT CLASS GROWTH ANALYSIS OF GOAL ORIENTATION AND MATHEMATICS ACHIEVEMENT

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De Corte (1995) indicates that one of the characteristics of effective learning is goal-oriented learning, that is, learning goal which can lead students' study behaviour is an important factor affecting learning performance. However, few studies have explored the relationship between changes in student goal orientation and learning performance changes. Therefore, this study aims to investigate the patterns of change of goal orientation and mathematical performance of students from sixth grade to eighth grade, as well as the association between this two patterns of changes.

There are 1518 students participating in this three-year study. This study uses the Rasch model to estimate the student's math abilities and goal orientation scores in grades 6 through 8, respectively. This study further conducts the latent class growth analysis (LCGA) by using Mplus software to examine the students' optimal growth models of goal orientation and mathematics achievement, as well as conducts the corresponding analysis to examine the relationship between this two growth models finally. The results show that the optimal model of LCGA of goal orientation is a four - class of models, that is, the changes of students' learning goal orientation can be divided into four groups. The change in the orientation of most students' math learning goals is the higher the grade, the more negative the goal orientation of mathematics learning. Only those students who initially have positive learning goal orientation will develop their learning goals positively. The optimal model of LCGA of students' mathematics achievement is five-class model. Based on the intercepts and slopes of the five groups, it shows that there is half of the students' math achievement is progressing year by year, however, the math achievement of the low achiever, their mathematics achievement is not increasing year by year. Finally, the relationship between the latent classes growth of mathematics achievement and goal orientation is analyzed by using the corresponding analysis. The result shows that there is an association between this two growth models, it suggests that there is something close to a large effect size, χ^2 (12, n = 1518) = 313.469, $p < .05$, and Cramer's V = .262. Based on the above research results, it is shown that the change of mathematics learning goal orientation is related to the change of mathematics achievement. If the student's learning goal orientation is positively developed, the mathematics achievement would be improved. It suggests that teachers should not only focus on the instruction of the math knowledge and skills, but also pay attention to the development and cultivation of students' goal orientation.

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DEVELOPMENT OF A THEORY-BASED AND EMPIRICALLY SUPPORTED COMPETENCY LEVEL MODEL

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Based on the implementation of the standardised final written examination in mathematics in Austria ("Matura") a theory-based and empirically supported competency level model was developed. This model distinguishes three process oriented mathematical aspects, namely operating, modelling, and reasoning (O-M-A). The competency levels are described based on activity theory (Bruder & Schmitt 2016) and are formulated separately for every process oriented aspect. Basically, the first two levels represent different stages of complexity regarding pattern orientation. The third level adresses field orientation, i.e. to solve a task the student has to recognise a mathematical problem in a different context or to conduct an independent analysis. The fourth level includes field orientation as well as a high degree of autonomy and creativity (c.f. Siller et al. 2015 for the full model).

The empirical evaluation of the Austrian Matura 2015 (n=17450) and 2016 (n=16919) is currently in progress. With regard to difficulty the analyses of the 2016 data show that out of the 24 tasks 9 were assigned to Level 1, 10 were assigned to level 2, and 5 were assigned to level 3, which corresponds well with item difficulty.

Exploratory models of the data so far do not suggest the three-factor solution O-M-A but a one-factor model. We assume that a focus on more homogeneous tasks will change the outcomes. Interestingly, we found that the distribution of the tasks varies from test to test: in the 2014 test run for the standardised Matura, the share of tasks concerning O-M-A was 62%, 20% and 18%. In the first standardised Matura 2015 the share was more balanced into 40% O, 44% M and 17% A. In 2016 this changed again to 42% O, 33% M and 25% A. Thus, the trend goes towards a more even distribution of the process oriented aspects. With the O-M-A-model, a tool is provided to check whether an examination addresses an array of mathematical processes or not.

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RELATIONSHIPS BETWEEN TRAIT EMOTIONAL INTELLIGENCE AND BEHAVIORAL PROBLEMS OF CHINESE STUDENTS WITH MATHEMATICS LEARNING DISABILITY

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Trait emotional intelligence (trait EI) has received much attention in the literature and generated intense demand for applications in educational settings (Petrides, Pita, & Kokkinaki, 2007). Trait EI concerns emotion related dispositions and self-perceptions measured via self-report. Despite all this research, there has been few research on the relationship between trait EI and internalizing and externalizing behaviours in mathematics learning disabilities (MLD).The present study represents some further development in the study of assessment of trait EI and its relationship with problem behaviours of Chinese students with MLD. More specifically, it was hypothesized (H1) that there will be significant differences in global trait EI scores between two groups; (H2) that there will be different components of trait EI relate to problem behaviour in Chinese students with MLD.

In the entire sample, 63 children met criteria for MLD (21 males), 75 children were classified as control group (44 males) (Table 1). They ranged in age from 8 to 12 years. The mean age in the MLD group was 9.60 years (SD = 0.97), in the control group was 9.59 years (SD = 0.96). We received the permission of all the students. The Trait Emotional Intelligence Questionnaire-Child Form and The Child Behaviour Checklist-Teacher's Report Form were used.

Analysis revealed significant individual differences in global trait EI, with non-MLD participants scoring higher in comparison to their MLD peers. While there were no gender differences in global trait EI scores and nine factors, further analysis revealed significant group differences on Adaptability, Emotion perception, Self-esteem and Self-motivation, with MLD participants scoring lower than non-MLD participants on all four factors. The result revealed a significant main effect of group on global trait EI scores. For MLD group, Emotional Intelligence was significantly correlated negatively with Attention Problems and Thought Problems. The regression found that Social Problems and Thought Problems can be significantly predicted by Peer Relations. Attention Problems can be significantly predicted by Adaptability and Self-Esteem. Hyperactivity/ impulsivity can be significantly predicted by Emotion Expression. Based on these results, we noted that personality traits are useful as long as they can help explain and predict relevant behaviours. If we can teach MLD students positive strategies for expressing emotion and the generation of new problem-solving ideas, then their socio-emotional well-being may be enhanced.

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THE INFLUENCE OF LESSON STUDY ON THE DEVELOPMENT OF A TAIWANESE NOVICE MATHEMATICS TEACHER'S KNOWLEDGE AND STUDENTS' LEARNING ACHIEVEMENTS OF INQUIRY TEACHING

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Inquiry-oriented learning is one of the mainstreams of education nowadays, as science, technology, and inquiry are crucial skills in the 21st century (NRC, 2010). Although inquiry teaching has been mostly heard in Taiwan for some time, the teacher-centred didactic teaching still dominates the teaching practice in Taiwan (Jang, Lee & Hsieh, 2013). As lesson study has been proven to be an effective model for promoting teacher's professional development (Suh & Seshaiyer, 2015), we try to investigate how lesson study influences a novice teacher who participates in a school-based professional learning community to learn inquiry teaching, and the effect on his students' learning of mathematics. The professional learning community [PLC] consists of four in-service, three pre-service teachers, and a teacher educator. Based on Yoshida's model (2008), five cycles of lesson study are implemented as each cycle includes teaching activity preparation, classroom teaching and observation, and post-teaching discussion. One of the PLC members, John, who is a novice teacher, is elected as the research subject. Besides the five cycles of lesson study, John also invites other PLC members to observe his classroom teaching on every Friday. By means of the related qualitative data, the progress of John's mathematical knowledge for teaching [MKT] (Ball et al., 2008, 2009) is analysed. In addition, John's students' term exam scores are compared with the other classes of the same grade. The results are as follows. At the beginning, we noticed that John could not grasp students' prior knowledge and was not familiar with the sequence of the content in the textbook. Hence John often provided improper inquiry tasks for the students. So we integrated more inquiry theories in the stage of teaching activity preparation. By following with the classroom teaching observations and the post-teaching discussions, John's knowledge of content and student [KCS] and knowledge of content and teaching [KCT] had been clearly improved after a year of lesson study. In addition, we also find that John's students' learning achievements appeared to be closely related with John's growth. At first, students' exam scores dropped a bit as John was struggling with inquiry teaching. However, they started to make progress as John could gradually grasp the strategies of inquiry teaching.

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EXPERIENCING MATHEMATICS THROUGH ALGORITHMS

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Calculation by hand is often seen as a dull, mindless activity. Thus, we tend to forget how powerful these algorithms are, and their importance in the history and present day of mathematics. Moreover, algorithms seem to offer a very special kind of mathematical experience, and provide a unique ways to conceive the discipline. How could we give students the opportunity to truly encounter algorithmic mathematics?

In the 1990, many mathematics educators explored the possibilities of the teaching and learning of algorithms in school mathematics (see in Morrow & Kenney, 1998). An algorithm may be defined as "a set of rules to follow in order to obtain a certain result". Unpacking this definition reveals the existence of different kind of algorithms. We can think of multiplication and "long division" techniques (of which there are many variations), but also need to consider formulas for example (like the one we use to solve quadratics), and more complexes procedures such as Newton's method.

One of the aspects surprisingly left behind in the study of algorithms is the epistemological dimension. As Morley (1982) once observed, mathematical ideas are "transformed through algorithmization, come to signify something different" (p.51). Finding the square root of a number means and demands something quite different if one uses a digit-by-digit algorithm, Heron's method, a Taylor series, continued fractions or the reciprocal method. One important revelation from the study of the history of these algorithms is that developing techniques and concepts goes hand in hand (e.g. Chabert, 2012). Another is the constant interplay between precision, rapidity and simplicity. These are fundamental aspects of nature, the history and the practice of algorithms, but they are hardly appreciable without special attention (Berlinski, 2001).

In this presentation, I will analyse the epistemological dimensions of algorithms in the case of root calculations, and discuss how offering students the occasion to play with such a variety of algorithms could present them (algorithmic) mathematics in a different light. I will share observations made with a group of in-service teachers, and questions relating to task design will ensue.

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PROFESSIONAL ROLE REFLECTION

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Moments of reflection within teacher training at university are regarded as being important parts of a pre-service teachers' professional development. An instrument of professional role reflection encourages pre-service teachers to consider their beliefs, motivations and self-regulation as well as five central roles of a mathematics teacher. Those five roles referring to the model of teachers' professional competence by COACTIV (cf. Kunter et al., 2013) are being an expert scientist in mathematics, an expert of subject teaching principles, a pedagogue, an organizer and a counselor. Focusing on the role regarding the principles of teaching mathematics, our aim is to answer the question, how pre-service teachers reflect those principles through categorizing the participants into different reflection groups. Reflection, therefore, is defined as a process of framing and reframing (Schön, 1987).

A sample of pre-service teachers, who have not finished their bachelor's degree, filled in an online questionnaire which consists of closed and open questions and covers the participants' general perception of mathematics teaching principles. While some answers will be evaluated quantitatively, categories, based on the answers to the open questions, will be constructed through the method of inductive qualitative content analysis. The examined cohort has only had one lecture regarding the methodology of mathematics teaching before, so we expect that they have a vague concept of mathematical education and keep a student-centered point of view. Also, we expect that they take over a mathematical, pedagogical, psychological and/or educational perspective when talking about the principles of teaching mathematics. In the presentation, those expectations will be discussed by including the results and conclusions of the survey. The analysis of the obtained data can help to improve teacher training courses at university on the basis of pre-service teachers' needs.

Acknowledgement

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PROMOTING ARGUMENTATION IN THE PRIMARY SCHOOL CLASSROOM

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The study reported here, suggests a process for designing successfully, argumentation-based mathematics teaching sessions. Such sessions, aimed specifically for very young pupils, are an area that has not, so far, been given the appropriate attention in Mathematics Education research. At the heart of such a process is the creation of a research-practice nexus which provides the combination of a research-based background and a practice-based design and implementation.

For the research-based part of the study, groups of pupils were formed and involved in guided discussions. The discussion data were then analysed by drawing on a method designed for an earlier study (Misailidou and Williams, 2004a and 2004b). The analysis produced tabular representations of pupils' discourse which were labelled as 'Generalised Patterns of Changing Arguments' (GPCA). These patterns indicated specific elements that aided the production of mathematical arguments. More importantly, they were designed with the intention of being a practical tool that could be easily communicated and then used by practitioners.

The practice-based part of the study involved the communication of the research data (including the GPCAs) to a 'Teachers' Inquiry Group' ('TIG'). A TIG was a group consisting of teachers and researchers who worked together with the aim of developing effective teaching practice. The TIG members, having the research data as a guide, designed argumentation-based teaching, appropriate for a whole classroom.

The teaching plans and tools developed by the TIG were firmly based upon the research data but aimed to address the practicalities of a school classroom as well. Consequently, the creation of the appropriate research-practice nexus is proposed as the necessary condition for the effective integration of mathematical argumentation in the usual teaching practice.

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A THEORETICAL ANALYSIS OF THE MATHEMATICS CURRICULUM FROM THE PERSPECTIVE OF METADOMAIN: A CASE OF THE PYTHAGOREAN THEOREM

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INTORODUCTION

The Pythagorean Theorem is one of the most important theorems in the mathematics curriculum of middle school. Previous curriculum have viewed this theorem as to learn and to apply. However, I think that the Pythagorean Theorem has the educational values more than to apply the real world. The purpose of this paper is to explore the place of the Pythagorean Theorem in the mathematics curriculum. To achieve this purpose, I analyse the mathematics curriculum of Japan and the United States.

METHOD

The theoretical framework of this study is based on the 'three domains of mathematical activity' which consists of (i) the domain of concrete experience; (ii) the domain of formal procedures; (iii) the metadomain (Noddings, 1985). The first domain is to consider situations of the real world or concrete things. The second domain is to deal with formal procedures such as algorithms and deductive proofs. The third domain is to consider the second domain as the target in order to discuss formal procedures themselves. This paper uses this framework and presents the place of the Pythagorean Theorem in the 'Course of Study' and the 'Common Core State Standards'; the standards of curriculum in these countries.

CONCLUSION

The results show that the curriculum of the United States is emphasized to learn the Pythagorean Theorem itself and to apply the theorem rather than the Japanese curriculum. It means that the curriculum focus on activities in the domain of concrete experience and in the domain of formal procedures. On the other hand, it is difficult to identify the activity in the metadomain, for example to organize theorems and to construct a system. Genuine mathematical activity should relate to all three domains. It is necessary to reconstruct curriculum from the perspective of metadomain.

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EXPLORING MATHEMATICS EDUCATION IN TIMOR-LESTE

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This presentation reports the issues of mathematics education in the top university in Timor-Leste. During our fieldwork in March 2016, the authors conducted interviews based on the life history method for university staff and students in the department of engineering and administered questionnaires in order to reveal their thoughts about mathematics education and the relevant cognitive and affective aspects and the extent of their influence on mathematics.

Firstly, targeting faculty members, the qualitative analysis for the university faculty's life stories (N = 8) reveals differences between young and old generations in terms of the language usage in mathematics education due to political and historical influences. The interview also showed that the faculty members, particularly those who studied abroad, also recognised the low level of mathematics. They thought that the quality of primary and secondary mathematics education should be improved in terms of linguistic aspects and teachers' professional development. In the quantitative analysis by using questionnaire for the faculty members, the results revealed that younger generation have a higher personal belief on professional development and teaching and learning than the older generation by using t-test (N = 24).

Secondly, targeting the university students, in order to clarify the relationship between students' cognitive and affective aspect of mathematics, the authors conducted a four items mathematics test and questionnaire that have four affective aspects. The correct answer rate of test items was 0.74, 0.11, 0.19 and 0.15 respectively (N = 34). Grasping which aspect is more related to cognitive aspect to get correct answers, the logistic regression analysis was used as depended variable was the item which have 0.74 correct answer rate and independent variables were four affective aspects. Applying backward stepwise selection, two variables such as "Interest in and enjoyment of mathematics" and "Anxiety in mathematics" were selected as well McFadden R^2 was 0.230. The result of logistic regression analysis shows that the students who not only have a similar degree of interest in and enjoyment in mathematics but also have less anxiety in mathematics are about 3.6 times more likely to get the correct answer than those who do not on the item. This implies that easing their anxieties is a pedagogical challenge.

Consequently, the authors found out the followings: firstly, linguistic characteristics were identified among different age groups of faculty members. Secondly, young faculty members have a higher degree of personal beliefs about both professional development and teaching and learning than experienced members do. Thirdly, relieving anxiety in mathematics could be effective for university students to improve their cognitive aspects.

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VISUALIZING 3D SOLIDS WITH 3D PRINTING TECHNOLOGY

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Research shows that young children have significant difficulties visualizing in 3D. While digital technology and manipulatives have been shown to support children's visualization, the effect of 3D printing technology has yet to be examined in research. This exploratory study asks: what does 3D printing, i.e. "drawing in space", afford in the visualization of 3D solids in the primary grades?

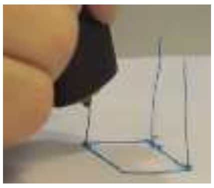

Figure 1.

A 3D Drawing Pen is a handheld 3D printing device that enables one to draw in the 3D space. As the pen moves along with hand holding it, a "3D drawing" is created at once (Figure 1). This enables children to explore 3D solids in ways that they could not traditionally with paper-and-pencil, the computer screen, nor physical manipulatives, as they could construct the 3D solids through their very hand movements. Adopting Chatelet's (2000) theory which addresses the role of diagramming in mathematical thinking, we consider these hand movements, much like diagramming, as important processes that capture mathematical thought and gives rise to new ones.

A teaching experiment involving two lessons was undertaken in a Primary 5 mathematics classroom in Hong Kong. The lessons were designed with the aim to incorporate students' 3D drawing processes into their learning of prisms and pyramids (we report here only the lesson on prisms). Upon drawing the solids with 3D Drawing Pens, the teacher led a class discussion on their various properties. Our results show that, interestingly, all students drew rectangular prisms in the same way, by drawing four edges vertically upward after drawing a rectangular base (square or rectangle). They finished off with drawing the opposite parallel rectangular base "in the air". Most students employed a similar strategy for drawing triangular prisms. When asked to describe their drawing process, they expressed that they "pulled" the edge "up" from each of the vertices of the triangular base. In contrast, two students drew a lateral rectangular face rather than the triangular base first. Without the aid of their 3D drawings, the students determined the number of edges and vertices of the prisms by making gestures that imitate exactly their drawing processes.

We conclude that, in contrast to the use of pre-made manipulatives such as sticks and nets, 3D drawings enabled students to construct 3D solids of varying sizes and in different ways. In addition, the construction process calls upon drawing a series "1D" lines (edges) and 2D shapes (such as rectangles and triangle), through which the decomposition from 3D solids into its 2D and 1D parts were facilitated.

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A SURVEY ON PRESERVICE-TEACHERS' PROBABILISTIC EQUIPMENT

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In this presentation, we investigate the nature of preservice teachers' probabilistic capacity. For this, we conducted a survey with a didactic and mathematical problem about probability as following:

> A student "A" answered 1/3 to the following problem: find the probability two coins coming up heads when you toss two fair coins at the same time. Conjecture this student's way of thinking and explain problematic points of the way.

This question was answered by 40 preservice teachers who are undergraduate students from at first to at fourth grade in mathematics teacher training course in Japan, and many of them may be going to become elementary school teachers or junior secondary school mathematics teachers. The preservice teachers' answers are analyzed in terms of the notions of *praxeology* and *ostensive* which are constructs within the Anthropological Theory of the Didactic (cf. Arsarello et. al., 2008). The praxeology is a general model of human activity which consist of two parts: *know-how* and *knowledge*. The ostensive is "material" entity in praxeology: symbol, gesture, voice and so on. The ostensive has a twofold function for praxeologies. First is the *instrumental value*, that is to say, ostensives are useful as tools of constructing something. Second is the *semiotic value*, that is to say, ostensives are useful as tools of representing something. All preservice teachers could correct explicitly or implicitly the answer of the student A to "1/4" and explain reasons why the student A made a mistake. From viewpoints of praxeology and its ostensive dimension, their explanations have two characters. First, they focused on know-how part of the fictional answer by the student A, that is, how to construct all outcomes. In other words, they ignored reasons why construction of all outcomes by the student A is irrelevant, which could be explained in terms of "equally possible" and symmetric structure of coins. This character of the preservice teachers' answers is interrelated to one more character of the answers. The second character is that they do not use terminology within stochastics even what they had already been taught "officially": event, equally possible and so on. This fact shows that stochastic terminology has less instrumental value for the preservice teachers in this situation.

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ANALYZING AND CHARACTERIZING JAPANESE SEVENTH-GRADE STUDENTS' IDEA OF CENTRAL TENDENCY

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Statistical educators should encourage children to develop the idea associating the two interrelated but complementary concepts of center and variation, which is called the idea of central tendency in this paper, in teaching statistics. However, in Japanese course of study, the two ideas are supposed to be taught separately, and measurement of center is highly emphasized. In the sixth grade, where quantitative data are handled for the first time, the concepts of average (center) and variation are taught, but their association is not explicitly. To what extent can children associate center with variation without explicit teaching? Clarifying the hidden outcomes can contribute to designing the teaching and curriculum which make it easier for children to develop such idea. The purpose of this paper is to explore the extent to which the Japanese seventh-grade students can associate center with variation.

The performance task was developed to do so. In brief, in the situation where flying distance data (n = 20) of two ski jumping players were given in the histogram, students were asked to select one of the two players likely to jump farther while comparing data characteristics. It was implemented with total 71 seventh-grade students (13-year-old) in two classrooms of a lower secondary school attached to a national university in Japan in February 2016. They all had learned the concept of average (center) and variation, but their association had not been explicitly learned yet. Arguments which they made and described were analyzed and characterized by the following three levels with reference to Reading & Shaughnessy (2004): Level 1, neglecting variation and over-dependent on center; Level 2, ignoring center and focusing on variation; Level 3, balancing center and variation.

As the result, 47.9% of the students were classified as Level 1, 39.4% as Level 2, and only 12.7% as Level 3. The findings suggest that about half of the students are over-dependent on center alone and that most students were not able to possess the desirable idea of central tendency to associate center with variation without explicit teaching. It seems to be the reason that they have not recognized the necessity of considering variation and the inappropriateness which only considers either center or variation. It is required to develop some didactic approach that encourages children to develop the idea of central tendency.

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CLASSIFYING THE VARIETY OF STUDENT'S INTERPRETATIONS OF MATHEMATICAL STATEMENTS

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To be a great mathematical problem solver or a better mathematical thinker, it needs to see mathematical statements from various perspectives, in which it needs appropriate interpretations for different purposes. However, not many students can select appropriate interpretations from various interpretations (e.g. Durand-Guerrier, 2008). Therefore, this research aims to examine how to interpret a mathematical statement for one's purpose.

In order to assess students' interpretations, we analysed the result of national assessment on specific issues in logical thinking (NIER, 2013) that was intended for high school students in Japan. The survey items consist of two frameworks that are expression forms of items (mathematical or non-mathematical) and six practices of logical thought. In this analysis, we applied mathematical logic for describing students' interpretations, and distinguishes three contexts as students' sense of purpose; understanding, proving and utilising.

Results focus on two different interpretations of a given equation "$(x - m) + x + (x + m) = 3x$" in the problem which is whether the equation can be applied in other cases. If some students regard the equation with availability, they should focus on integer "m". If others do not regard it with non-availability, they should focus on integer "3", and this interpretation produces a new equation "$(x - n) + (x - m) + x + (x + m) + (x + n) = 5x$".

The findings indicate that mathematical statements which are interpreted in the context of understanding can be utilised within a broader range than an intended range. Another result is that mathematical statements which are the evidence can be proved within a narrower range than an intended range. The reasons seem to be common. That is, it seems that a mathematical statement which are interpret in the different contexts affects interpretations in the other contexts.

Furthermore, a survey for undergraduate students is planed with the same test items. In the presentation, further results will be discussed in detail.

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CONSTRUCTING SIMILARITY CONECTIONS BETWEEN MATHEMATICS PROBLEMS: THE CASE OF "WEAK" HIGH SCHOOL STUDENTS

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A significant difficulty among high school mathematics students is their inability to solve new mathematics problems. One reason for this difficulty stems from their lack of effective strategies for organizing knowledge to make it available for implementation in new learning situations. Moreover, they have difficulty storing and organizing new knowledge over the long term to be used in the relevant context. Learning that incorporates mapping and constructing connections between problem concepts and between problem-solving strategies allows students to borrow and identify old knowledge and connect it to new knowledge (Lobato, 2014), thus prompting students to apply what they learned previously about problem solving to the new problem. Such a learning structure promotes the preservation and organization of information in long-term memory (Sweller, 2015). The research questions are: (1) What types of connections between mathematical problems did the students identify at the beginning, middle and end of the intervention period? (2) How did the students' independent mathematical problem-solving skills develop over the course of the intervention? The data consisted of 150 videotaped mathematics lessons, 36 one-on-one teaching experiments with twelve high school students (grades 11 and 12) at three different times during the intervention year and a researcher diary that documented the intervention period. Analysis of the 36 interviews revealed three developmental profiles describing the shift in the students' strategies over the course of the three interviews, from constructing one simple type of connections (formulation level) to constructing a different and more advanced type of connections (heuristic level). The three profiles are as follows: (a) Tri-stage development of all connections: development in three stages, ranging from constructing similarities at the formulation level to the algorithm level to the heuristic level. (b) Bi-stage development in two stages, from algorithmic connections to heuristic connections. (c) Bi-stage development in two stages, from formulation connections to heuristic connections. Analysis of the 150 videotaped mathematics lessons and 36 one-on-one teaching experiments revealed three stages in the development of independent mathematical problem-solving. These stages describe the students' perceptions about the act of constructing similarity connections, their motivations for choosing the problem to be solved and the relationship between their approach and autonomy in problem solving.

One practical contribution of this study is that it enables the formulation of intervention principles that can serve as a means for teaching classes of “weak” high school students. Another practical contribution is that it defines two sets of stages, one describing the development of heuristic literacy for problem solving and the other clarifying processes for developing self-sufficiency in solving problems.

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THE PROGRAM DESIGNED FOR THE PROFESSIONALISM OF CHARACTER EDUCATION BY MATHEMATICS TEACHERS

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This study is to foster practical competences of mathematics teachers who devote themselves to the education responsive to the social changes such as globalization, diversification, informationization, and conversion of knowledge through the cultivation of democratic character – the core competence of the future society (Partnership for 21st Century Learning, 2015). In that sense, this study aims to equip mathematics preservice teachers with competences to practice character education which is emphasized in the 2015 Korean national mathematics curriculum.

For the purpose, this study is conducted as part of development research of a mathematics teacher education course 'Mathematics Teaching and Learning'. Based on a comprehensive review of literature concerning the recent trends of educational reform, key competencies, and character education, this study identified the guiding principles and the methods of how to develop mathematics preservice teachers' competences of embodied level to practice character education .

Specifically, this study presents how to introduce flipped learning to expedite active reflective learning activities of preservice teachers Moreover, this study will present learning activities for mathematics preservice teachers such as inquiry into curriculum based on the core competencies of character education, planning mathematics instructions for character education, developing mathematical tasks and materials based on real world context related to issues about democratic world citizenship, implementing teaching strategies, setting up educational environments, and developing leadership to achieve the goals of character education in mathematics class.

Character education is an emerging issue of world-wide reform discourse. In this context, this study will contribute to the development of guiding principles and methods of how to prepare mathematics preservice teachers for character education. Specifically, this study will provide mathematics teachers' competences for practicing character education by respecting the diversity of students. In addition, this study will offer instructional approaches in which students are encouraged to be active producers of mathematics by thinking about mathematics based on related issues of life. In the presentation, further results will be discussed in detail.

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DUAL MODE PROFESSIONAL DEVELOPMENT RESOURCES FOR DISADVANTAGED COMMUNITY

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The challenges of conducting professional development (PD) on a large scale and targeting sustainability are global issues. One common practice is to undertake ToT (Training of Trainers) so more teachers have an opportunity to attend PD sessions. However, in addition to demographic challenges, the enacted PD sessions are often mismatched with the intended sessions after the ToT. This necessitates an alternative form of delivering PD, particularly in "disadvantaged" or remote areas.

The Internet allows for the flexible of delivery of PD sessions from anywhere and at any time. However, learning resources must be designed carefully to engage users and promote learning. This poster will describe the process of transforming face-to-face PD resources into a dual mode form. The term dual mode indicates resources, which can be utilised to conduct face-to-face PD as well as online PD. For the online PD, the teachers use the Internet to access learning materials, interact with the content, facilitators, and other teachers and seek support in the learning process.

This development will be exemplified through the transformation of one face-to-face PD module called "Productive Questioning" into a dual mode design. The PD module aimed to develop teachers' understanding of productive questions in mathematics lessons (Martino & Maher, 1999). Within this module, teachers are scaffolded to develop pre-planned questions, which promote mathematical thinking, facilitate classroom discussions about their students' mathematical reasoning and to help make connections between and within mathematical concepts. The module includes text and videos of classroom practices with/without productive questions.

To achieve quality learning from a dual mode system, the design of the PD resources is central. Our design is directed to focus on teacher learning and promote collaboration. It is designed to create a high interactivity between content, teachers and facilitators. The design is also directed to link theories and practices, cater for individual differences, provide feedback and encourage teacher reflections to acquire knowledge and build personal meaning. The dual mode design also provides opportunities for teachers to apply what they have learned into daily teaching practices. This poster presentation will also describe the challenges faced to develop the dual mode system and approaches used to resolve these challenges.

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IPADS IN GRADE 6 CLASSROOMS: EFFECTS ON STUDENTS' CHOICE OF STRATEGY FOR COMPARING FRACTIONS

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Fraction comparison tasks are known to be difficult for students. A frequently found misconception is not comparing the fractions but their components as natural numbers, which leads to incorrect answers in specific tasks (e.g. *incongruent* tasks). Research suggests that even expert mathematicians cannot overcome this natural number bias completely, despite reaching very high solution rates by applying a variety of strategies (Obersteiner, Van Hoof, & Verschaffel, 2013). Therefore, it seems reasonable to support students' choice of appropriate strategies to compare incongruent fractions in classroom instruction. We studied whether this is possible using interactive teaching material on the iPad.

To this end, 242 sixth grade students split into three groups participated in a four-week intervention. Group 1 (n = 80) worked with an iPad-assisted learning environment, group 2 (n = 100) received the same material as a regular paper-based book. To control for effects of the specifically designed material, group 3 (n = 62) used conventional textbooks. In a posttest, the participants were asked to explain how the size of fractions can be compared, given two incongruent problems. In item 1 (8/9 versus 7/6) the fractions had no common components, but one fraction was greater than one. In item 2 (5/8 versus 5/10) the fractions had the same numerator and one fraction equalled one half. *Feature-based strategies* (i.e. drawing a picture, benchmarking to a third value, arguing using the size of the pieces) were coded with 1, *rule-based strategies* (i.e. expanding or reducing fractions to get the same denominator or numerator) were coded with 0, so that scores between 0 and 2 were possible.

When the use of feature-based strategies was counted in both items, students from the iPad group in fact reached the highest score (M = 1.64, 1.52 and 1.02, respectively). A Kruskal-Wallis test showed a significant main effect of the treatment on the choice of strategy, $H(2) = 23.79$, $p < .01$. Indeed, both the iPad and the book group differed significantly from the control group, $p < .01$. However, no significant difference between the two treatment groups was found, $p = .55$. Our findings suggest that a flexible use of strategies for fraction comparison tasks can be encouraged with appropriate teaching material, but using the iPad has no additional effect.

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CHILDREN´S ESTIMATING COMPETENCES IN LENGTH AND CAPACITY

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In recent years, measurement estimation gains more attention in German classrooms. At the same time, it became obvious that little is known about the abilities of children in this field, especially on primary level.

Since most research in measurement estimation is focused on lengths and mainly on older students and adults, we focused on younger children and included lengths as well as another–visual–measurement area. Our tasks were constructed with reference to Bright's (1976) typology of requests in estimating length. In each measurement area and four types of requests five tasks were constructed, so overall 40 estimation tasks were presented. 46 (27 ♀; 19 ♂) 4th-graders from different schools solved these tasks in individual interviews which lasted about 15-20 minutes and were videotaped.

The results concerning the strategies show that 4th-graders use all strategies known from literature (e.g. Hildreth 1983; Joram et al. 2005; Siegel et al. 1982), and that the strategies to estimate capacities are mainly the same as those to estimate lengths. A detailed categorical system was presented at PME 39 (Ruwisch et al. 2015).

Even if the strategies are the same, the frequency and the precision in estimating measurement differ. The poster will present data comparing the frequency and accuracy in estimating lengths and capacity as well as in different magnitudes of the to-be-estimated objects (Heid 2016).

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DIFFICULTIES OF FIRST-SEMESTER MATHEMATICS STUDENTS

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At the start of her studies a student of mathematics described that she had good skills in solving tasks and that she could remember all the materials of the last years. In the 7th week of lectures she answered on the same question (What is she good at in mathematics?): "At the moment actually nothing, maybe I'll be ready to tell something in the next survey…." Apparently, the perception of her own mathematical competence has changed.

High abandonment rates of university students in mathematics are nationally well known (Heublein et al, 2014). Theoretical studies indicate the differences between mathematics in school compared with mathematics at the university regarding to proving (Fischer et al, 2009.). In 2016 a digital questionnaire was developed, which intends to investigate the difficulties related with proving at the beginning of mathematics studies. The pilot of the questionnaire in autumn 2016 indicates a dwindling perception of mathematical competence (see the student above). Also student S described difficulties with a changed competence experience:

> "I have problems with not succeeding directly as it used to be in school. In mathematics and physics studies the moment of success usually remains absent and a feeling of failure occurs. Hence, you doubt about your abilities and finally on yourself. […]"

The pilot study indicates a group of students with good grades in school but less experience in proving inside the classroom (8 out of 23). The assessments change with time (1st &2nd questionnaire) and by the exchange of the terms "reasoning" and "proving". This points to a deliberately distinguish between the terms by the students and, furthermore, this might indicate different imaginations of "proving" in school and university. For this target group it seems promising to develop a concept to improve the proving competence in the period between school and university. So this is the long-term goal of the project.

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ENGAGEMENT IN MODELING ACTIVITIES PROMOTING A CHANGE IN BELIEFS ABOUT MATHEMATICS AMONG PRACTICING MATHEMATICS TEACHERS

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The aim of this study is to examine whether the engagement of mathematics teachers in modeling activities and subsequent changes in their views about these activities affect their beliefs about mathematics. The sample comprised 52 mathematics teachers working in small groups in four modeling activities. The data were collected from: (1) reports by the teachers on completion of each activity about whether or not the activity was mathematical and about both the mathematical and non-mathematical features; (2) interviews conducted after the intervention process; and (3) questionnaires on teachers' beliefs following Stipek, Givvin, Salmon and MacGyvers (2001) with 27 items used to examine traditional beliefs and 16 items to examine constructive beliefs. Internal consistency was calculated using Cronbach's α which was .71 and .77 respectively. The questionnaires was filled in both before and after the intervention. The main research findings indicated changes in teachers' views about the modeling activities. Most of the teachers referred to the first activity as a mathematical problem but emphasized only the mathematical notions that appeared in the elicited model or the mathematical operations in the modeling process; only 13.5% of the teachers related to features of the whole modeling process. Regarding the fourth activity, 70% of the teachers referred to the modeling activity as a mathematical problem and emphasized features of the whole modeling process. The results of the interviews indicated that changes in teachers' views can be attributed to four main themes: the structure of the activities, group discussions, the solution paths (modelling process), and the elicited models. These themes are considered the main features of the modeling activities and are not characteristic of the traditional problems in mathematics textbooks (English & Watters, 2004). In order to examine whether changes in the teachers' views about the modeling activity were reflected in their beliefs about mathematics, a paired simple t-test was conducted. The results regarding traditional beliefs were ($M = 3.34$, $SD = .41$) before the intervention and ($M = 3.24$, $SD = .40$) after the intervention; the change was not significant. The results regarding constructive beliefs were ($M = 4.50$, $SD = .50$) before the intervention and ($M = 4.78$, $SD = .74$) after the intervention; the change was significant $t(52) = -2.67$, $p < .05$.

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BIOLOGICAL BASES OF THE RELATIONSHIP BETWEEN ORAL VOCABULARY AND LATER MATHEMATICS ACHIEVEMENT

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INTRODUCTION

In order to predict mathematics achievements as early as possible, contributors before mathematics instruction were explored by researchers and educators. Since learning the symbolic number system of mathematics relied on verbal skills, children with better vocabulary knowledge should more easily understand concepts and solve words problems. Having a larger oral vocabulary has been proved to result in greater mathematics achievement (Morgan, Farkas, Hillemeier, Hammer, & Maczuga, 2015), but the biological bases underline the long-term relationship remained unknown.

METHOD

Totally 79 children in China were measured oral vocabulary at the age of 3. They attended a neuroimaging study to scan their brain structure at age 14 and forty-seven of them participated mathematics achievement test one year later. The association between early vocabulary and gray matter volume was explored and the significant region was used to predict mathematics achievement.

RESULTS

One cluster within the right anterior temporal lobe was found to significantly correlated with early vocabulary size. The averaged gray matter volume within this cluster can predict later mathematics achievement and mediate the effect of vocabulary in early childhood.

DISCUSSION

Right anterior temporal lobe was critical neural substrate for conceptual knowledge, and has been proved to specifically activated for a prodigy (Pesenti et al., 2001). Early vocabulary can influence mathematics achievement through shaping brain conceptual regions.

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A METAPHORIC APPROACH TO BAYESIAN PROBABILITY

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The relevance of metaphorising and enacting in mathematics education is increasingly acknowledged (Proulx, Simmt and Towers, 2009; Diaz-Rojas and Soto-Andrade, 2015, 2016). We address here from this perspective Bayesian problems involving causal probability and argue that iterated metaphorising can significantly help especially not mathematically inclined learners to fathom causal probabilities and the involved causal network. More precisely, we point out that a Bayesian problem may first be metaphorised as a random walk (a 2-step one in a typical false positives problem), then solved thanks to a pedestrian metaphor (Diaz-Rojas and Soto-Andrade, 2015) and also metaphorised as a *flow*, that turns out to be stationary. An illustrative paradigmatic example, that we have worked out with first year university students majoring in social science and humanities, is the following: A schoolgirl bikes downhill to school in the south of Chile, where in this season it rains 2 days out of 3. On wet road, she falls from her bike 1 out of 4 times, instead of only 1 out of 10 on dry road. Bayesian question: *If you know that she fell today, how likely is that it had rained?* Bayes theorem may be dispensed with, by metaphorising as indicated above, to obtain the systemic flow in Fig. 1, that shows the causal relationships between Rain, Fall, No Rain, No Fall, both ways. Learners notice that the flow is stationary and that a few data determine the remaining data, while solving the problem metaphorically or algebraically. So they construct in an elementary and metaphoric way a Bayesian network (Pearl, 2000) that they can easily enact.

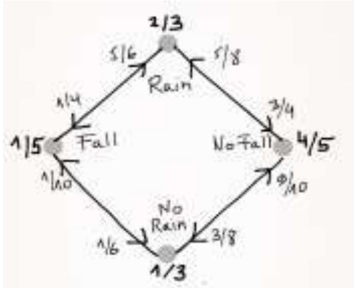

Figure 1.

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STUDENTS' PERCEPTION OF CREATIVITY IN CONNECTION WITH CONTINUED PROJECT WORK

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Creativity is at the heart of many mathematical breakthroughs. In order to enthuse students for this subject, experiences of independent project work, creativity, and their interconnections are worth exploring.

Over a year 24 students of grade 12 worked on several projects about coding and cryptography. After projects in the middle and at the end of the school year, data was collected in semi-structured interviews. Beside statements concerning perception of mathematics, the interviews also covered students' beliefs in *creativity* (Liu & Liu, 2011). Furthermore students were encouraged to write a learning diary beside their project notebook.

There are several definitions of creativity relevant for research (e.g. Wallas, 1926; Liljedahl & Sriraman, 2006). Students' perceptions of creativity might differ from these definitions. According to Stoppel (2016), creativity can be divided into the two cases: *Creativity 1*, where students understand creativity as an extension of knowledge to reach more comprehension of mathematics looking out from their projects, and *Creativity 2*, where students consider creativity in connection with the topics treated. Similarly, the progressions of projects can be divided into two types. In *Progress 1* the projects include applications of topics of the course processed before the beginning of projects. If students come to grips with new mathematical topics while editing their projects, the progression will be denoted *Progress 2*.

Students' perceptions of creativity and the changes therein in the course of a year seem to stand in relationship to the progression of their projects (Stoppel, 2016). The observations are underpinned by students' notebooks and learning diaries. The development of these coherences will be illustrated with a poster using typical examples.

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A STUDY ON DEVELOPING AND PRACTICE OF MATH CULTURAL MATERIALS FOR ELEMENTARY EDUCATION

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Reading ability and lifelong learning are closely related. In recent years, countries are increasingly focusing on students' reading ability. In order to understand students' reading performance, a number of international assessments have been organized, such as PISA (Programme for International Student Assessment) and PIRLS (Progress in International Reading Literacy Study). Taiwan students generally stand out in international competitions. For instance, the 2015 TIMSS (Trends in International Mathematics and Science Study) reported that Taiwanese Grade 8 students were ranked third, and Grade 4 students were ranked fourth. However, when it comes to proper learning attitude and confidence, Taiwanese students are low.

It is often said that mathematics is the "Mother of Science." However, how can we draw the veil of mathematics such that its secret charm may be discovered by the learner? Making students value and appreciate mathematics have become the overriding passion of educators. This article attempts discuss methods to add historical reading material in mathematics teaching, to help students have a cultural perspective on mathematics.

In the development of mathematics reading, we incorporate Jahnke's(1994) hermeneutics cycle (text-context-reader) in our study and reflection. The present study consists of a team of researchers and teachers, discussing and interpreting the history of literature on mathematics, the origin of mathematical symbols, like multiplication and division, the development of the decimal system, as well as other related materials, and how all this can be incorporated into empirical classroom instruction.

A total of 293 elementary students participated in this study. The result of this study and the use of cultural materials in actual teaching have been confirmed positively by both students and parents; thus, it is highly feasible that these materials can be used in teaching mathematics. Furthermore, we propose the following principles in the design of these cultural materials: (1) the topic of the material must conform to the mathematical content; (2) the materials must be interesting and educational; (3) the learning unit should be arranged in modules and must employ digital media. It is hoped that the above principles can be used by different educational study groups, and this study may contribute to the store of cultural mathematical materials and be made available for the use of elementary school teachers.

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DIFFICULTIES OF RECOGNIZING INDUCTION HYPOTHESIS IN THE PROOF BY MATHEMATICAL INDUCTION

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Mathematical Induction (MI) is a mathematical proof technique used to prove a statement for the set of all natural numbers. Due to the complex nature, proof by MI has been known to be difficult for high school students for many decades (e.g., Ernest, 1984). In the process of proof by MI, the validation of the first condition is called the induction basis and the validation of the second condition is called the induction step. Ron and Dreyfus (2004) identifies that the assumption used in the second condition, namely that the statement is true for k, is called the induction hypothesis. The goal of this study is to identify difficulties of recognizing the induction hypothesis.

There are two types of propositions for induction hypothesis which are treated objects of proof by MI in the high school curriculum in Japan. One is expressed as a generally relational expression and the other is expressed as a recurrence relation. For example, the story of the Hanoi towers produces the recurrence relation. Because of learning about progression ahead of learning about proof by MI, learners get the satisfaction of struggling with finding of the general term when they are offered a recurrence relation. Dogan (2016) recommends that learners should be guided toward the discovery of a non-recursive closed formula, in turn the questioning of the validity of the closed formula for larger quantities may bring out the need for MI proofs, and the role deductive processes play in the proof of its components. In addition, it is said that one needs not just a model that provides geometric means, but needs a learning environment that effectively put into view the crucial role recursion plays at the inductive step. In order to address the issue, we propose a teaching experiment of a set of questioning about an isoperimetric and equivalent problem for secondary school students.

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THE RELATIONSHIP BETWEEN MENTAL ARITHMETIC AND ACADEMIC SUCCESS IN UNIVERSITY MATHEMATICS COURSES

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Mental arithmetic is an early foundational mathematics skill that is taught during the elementary grades. Since early mathematics skills are one of the strongest predictors of later achievement (Duncan, 2007), it is crucial that students are capable in mental arithmetic. However, it is not uncommon for students to leave elementary school without being proficient in mental arithmetic due to the ubiquitous use of the calculator in the classroom. This leads one to question whether students are still able to be academically successful later on in life in their university mathematics courses. Thus, the research question is: What is the relationship between students' mental arithmetic abilities and academic success in their university mathematics courses?

The sample consists of 45 first-year and second-year female students (two classes) at a public university in the United Arab Emirates. The sample of students that participated in the study are essentially homogeneous, as the they are female, Emirati, 17 to 20 years old, wealthy, and English Language Learners (ELLs).

The data comes from two basic university statistics courses. Students were given 3 minutes to complete a worksheet of 60 mental arithmetic questions, (e.g. 5×9, $20 + 4$, $16 - 3$, $42 \div 3$). Their score was compared to their final grade which consists of the following components: a midterm (40%); a final exam (40%); a project (10%); and homework, classwork, and mini quizzes (10%).

Mental arithmetic scores ranged from 10 to 49 (out of 60) and final grades ranged from 60.66% to 101.7%. The mean mental arithmetic score was 30 and the mean final grade was 87.2%. The results of the study suggest that there is a slightly positive correlation between mental arithmetic capabilities and mathematics grades at the university level ($r = 0.542$ and $p = 0.00012$). The findings also indicate that for every additional correct answer on the mental arithmetic worksheet, there is a 0.52% increase in the final grade. Additional results will be included in the poster presentation, as current semester classes will be added, thus increasing the sample size.

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REPRESENTATION AND TRANSFER IN EARLY ALGEBRA (K3)

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RESEARCH CONTEXT

Mathematics education is a priority in Mexico, but gets very low levels in national and international assessments (*Program for International Student Assessment*). Counting appears to be dominant in children, over functional relationship of magnitudes.

THEORETICAL FRAMEWORK

It is use Vygotsky's (1980) mediation as representation in the development of the magnitude concept. Representations in arithmetic table, graphical and algebraic equations, are used as mediation to transfer understanding (Price and Fuchs, 2016).

STUDY

The study was conducted in a group of public school with 25 students (K3). To work collaboratively, students were distributed in 12 work teams. The teams manipulated a fixed magnitude of water (10 cm^3 = constant = C), transferring different amounts from one container to another (variables = A and B). Quantities were transferred to an arithmetic representation in a table. Then, the table data were transfer to its representation in the Cartesian plane, and later into its algebraic expression ($A + B = C$). A test with seven items was applied to evaluate the transfer in algebraic thinking.

RESULTS

A reliability of 0.948 was obtained between two evaluators. The results show that 80% percent of the students performed the numerical functional relationship. 96% built the numerical table. 64% elaborated the graphical representation and 28% expressed the functional relationship in an algebraic equation.

CONCLUSIONS

Most of the children can transfer the observed phenomenon to 3 forms of representation (arithmetic, graphical and algebraic). But the finding is that as water magnitude is a continuum were children cannot count and must understand abstract order of magnitudes, just 28% could express in their own words the functional relationship of magnitudes in the algebraic equation ($A + B = C$).

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A PRELIMINARY STUDY OF THE INFLUENCE OF MANIPULATION-BASED REMEDIAL TEACHING ON FIFTH GRADERS' MATHEMATICS LEARNING

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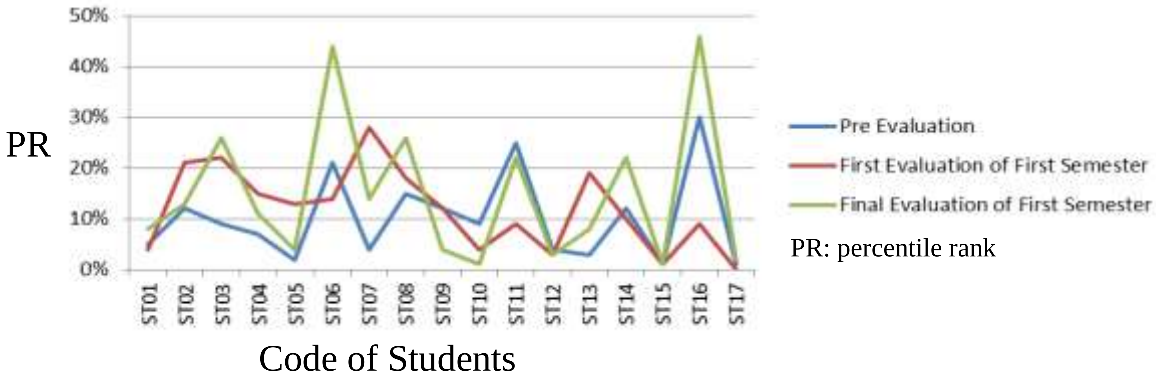
In Taiwan, because of the high student number in each class and teacher's pressure from the course progress, didactic teaching has been dominating classroom teaching practice for a long time. Even in the primary schools, teachers seldom teach mathematics by spending time for pupils' manipulation, but we should still cherish the special need of children (Chang, Wu & Yang, 2012). In this presentation, we present a preliminary study which provides sufficient opportunities of manipulation for some low-achievers in the remedial teaching and investigates whether the participants' learning improves.
The research subjects are seventeen fifth graders from different classes who are all volunteers and are within the last 20% in their fourth grade mathematics learning. Besides the four usual mathematics classes each week, these seventeen students take an extra remedial class every week, which is all about the "hands-on" activities by using different manipulatives to help them learn mathematics.
After a semester's effort, most participants do make progress in their mathematics learning. The following table shows their PR values for the two term exams compared with all the fifth graders. Therefore, the manipulation-based remedial teaching seems to be effective for improving some low-achievers' learning, since manipulating the concrete objects might be helpful for them to grasp some symbolic arithmetic concepts. However, they need time to adapt themselves to the new learning approach.



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THE USE OF GRAPHICS REPRESENTATIONS ON MUSIC COMPOSITION BASED ON MATHEMATIC IDEAS

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During two weeks of the month of January an interdisciplinary course of music and mathematics was conducted, directed to students with potential of academic talent coming from different schools of the Metropolitan Region in Chile. The aim of the course was to introduce students to one of the practices of musical composition, where musicians rely on mathematical models and notions to create their works. The course was mainly based on the methods of composition developed by authors such as Iannis Xenakis, Michael Winter and Schönberg's twelve-tone technique (Arbonés & Milrud, 2011; Xenakis, 1992).

Among the works of these composers is the use of graphic representations to describe the musical variations of the work. These are used both for the overall description of the piece – the temporary organization –, and as punctual, that is, in the form that the height, duration and intensity of each musical note is determined throughout the work. That is why it was decided to incorporate this type of representation to the course.

A positive reception was observed to the use of this type of representations. The main interest of the students was in the design of graphs, inspired on different geometric figures. Then, over these figures were added the musical parameters and the rules of variations. In addition, students also used visual representations to temporarily organize their compositions. Through these representations the groups could create rhythmic melodies and also serve as scores for their presentation in front of their peers.

In general, graphic representations seem to be a meeting point between music and mathematics. Graphics, schemes, shapes and figures were an important part of the process of creation and coordination, both for composers and students.

Acknowledgement

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NON-INTELLECTIVE CHARACTERISTICS OF HIGH-EFFICIENCY MATHEMATICS LEARNERS

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Studies have shown that Chinese speaking students enhance their mathematics learning mainly by prolonging their study time in and out school (Hu, 2015). However, it seems that students from Chinese speaking areas have low efficiency in learning mathematics though they spend a huge amount of time on their studying. To know more about students' efficient learning, this study aims to address the following questions: (1) What are the similarities and differences between students' high-efficiency mathematics learning and their average or low efficient peers in high school? (2)How do non-intellective factors relate to high school students' mathematics performance? (3) To what extent do non-intellective factors influence high school students' mathematics performance?

Participants

The sample included 690 high school students from 10 schools of 5 cities: Beijing, Tianjin, Jiangsu, Fujian and Guangxi.

Measures

The students were required to respond to a self-report questionnaire that included 82 items on student motivation, emotion, attitude, willpower and personality. The questionnaire was developed by Lin's non-intellective learning structure (Lin & Yu, 1994) and the Cronbach's coefficient, split-half reliability and test-retest was 0.95, 0.80 and 0.86 respectively. We used SPSS to describe the data, and analyse the correlation between one's non-intellective and his or her mathematics performance. AMOS was applied to show how non-intellective factor influences high school students' mathematics performance.

The results revealed high-efficiency mathematics learners of high schools were much more significantly than their average or low efficient peers. The first three have a direct effect on students' high-efficiency mathematics learning, while personality has an indirect effect and there is no sign showing high-efficiency mathematics learning students' willpower affects their performances. For high-efficiency mathematics learning students, their attitude towards studying has great influence on mathematics performance while emotion, motivation and personality show less influence.

High-efficiency high school students scored higher in many non-intellective factors, such as attitudes, emotions, motivations, and personality, but not apparently for willpower. They were doing better in cognitive motivation, achievement demand, learning anxiety, learning efficiency and skepticism than the other two groups and didn't show any difference in persistence.

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DEVELOPING COMPETENCE IN TASK DESIGN IN A MATH TEACHER PROFESSIONAL DEVELOPMENT PROGRAM

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The project Just Do Math was launched to address problems in mathematics education at the primary and lower secondary levels in Taiwan, including students' lack of interest in mathematics and the unacceptable percentage of low-achieving students revealed in international comparative studies such as TIMSS and PISA. The project involved multiple levels of mathematics education, including teacher professional development (TPD) involving task design, which is an effective approach for cultivating teaching competence (Zaslavsky, 2008). The TPD program comprised five stages: theory learning, task evaluation, task design, teaching experiments, and reflection and refinement. The tasks, named grounding activity modules, were required to reflect the goals of Just Do Math, namely to increase students' learning motivation and help them grasp the prerequisite fundamentals of a mathematics concept before learning it in regular class time.

This ongoing case study explores teachers' development of competence in task design during the TPD program, employing natural inquiry with intense fieldwork and interviews on five primary and six lower secondary mathematics teachers. Each version (ranging from initial to final) of the tasks they designed was collected. Content analysis was used to analyse the data in four aspects: identification of prerequisite fundamental mathematics ideas, arrangement of mathematics representations, use of sources of learning motivation, and sequences of learning activities.

The initial findings included: (1) The teachers knew which concepts were difficult for students to learn, but they could not identify the prerequisite fundamental ideas of these mathematics concepts without help from the teacher educators. (2) The teachers employed various concrete representations such as real objects and graphics in their design, but their arrangements were not always appropriate. (3) The teachers used various sources of learning motivation in their design. Some teachers stressed intrinsic motivations such as challenge, curiosity, and control, and others focused only on extrinsic motivations. (4) The sequences of learning activities varied in quality, and the teaching experiments and reflection and refinement stages were critical for teachers to develop their pertinent competence.

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IMPACTS OF PLANNING ON THE QUALITY OF A LESSON BASED ON MATHEMATICAL PROBLEM SOLVING

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Open up Math problems are known for having many ways to solve them. Classes based on such problems are recurrently recommended by mathematical educators around the world. However, the unpredictability of these classes inhibits teachers for causing insecurity about the large number of solutions students can have, mainly. The Japanese method - Lesson Study - was used to minimize or eliminate factors that constrain teachers in such classes, and can give them more confidence and mastery in teaching. Thus, the research sought to know the impacts of a shared and reflexively constructed planning, foundations of the Lesson Study, of 16 Math teachers in an 8th grade class on the following problem: You have a full cup of coffee and a full glass of milk, about 6 times the size of the cup. Take a spoon full of coffee from the cup and pour its content into the glass. Then, take the mixture with the same spoon and return it to the cup. Is there more coffee in the glass of milk than milk in the cup of coffee, the opposite, or the same amount?

The authors analyzed the quality of the class with the instrument "Quality Assessment of Instruction in Problem Solving - QAISP" - built mainly under the assumptions of Hill et al. (2011) and Fernandez & Yoshida (2004) - which seeks to examine 46 factors that should be a concern in the planning of an open up Math problem lesson.

Main results: there was concern with the familiarity of the context, level of difficulty of understanding and elaboration of resolution strategies; prediction of students' reactions, doubts and errors; pertinent questions that have promoted progress on student reasoning; sharing of resolutions enriching repertoires of strategies, among other advantages. The QAISP also pointed out factors that deserved to be reformulated by the teachers: execution and evaluation of the looking-back, connection between mathematical representations/strategies, verification of knowledge of similar problems by students, association of mathematical content with the problem and use of incorrect mathematical procedure. The application of Lesson Study showed the potential for such class by providing teachers with details that minimized the unpredictability and insecurity inherent in the management of classes based on solving mathematical problems.

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HOW STUDENTS WITH LOW AND HIGH ABILITY READ GEOMETRY WORKED EXAMPLES: EYE MOVEMENT STUDY

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Students commonly study worked examples while learning to solve geometric problems. Reading such worked examples is a complex process because they include reference text and figures to construct a mental model. Using eye movement data, Lee and Wu (2017) reported that adults' reading of geometry examples was text-guided and heavily dependent on included figures. This study investigated how ninth-grade students with low- and high-ability read worked examples of geometry problems and their comparative patterns of eye movement.

A geometric prior knowledge test (score ≤ 10) was administered to 226 ninth-grade students from two schools in Taiwan; 68 participants were selected and then separated into low- (score = 1–7, *n* = 34) and high-ability (score = 10, *n* = 34) groups. They studied three worked examples of geometric problem solutions and then completed comprehension tests and transfer tests. Their eye movements were recorded using the Eyelink 1000 eye tracker and categorized into five patterns. Most students in both groups read the text and examined the figures as they were mentioned (named text-directed). For the problem layout, low-ability students finished reading the text and then examined the figure significantly more frequently than the high-ability students (named text-first). For the solution layout, low-ability students were significantly less likely to show text-directed pattern and significantly more likely to show text-concentrated pattern (most time was spent focused on the text) than high-ability students. Thus, both groups of students used text and figures to understand the problems, but low prior knowledge students relied more heavily on text when reading solutions. The results imply that the literacy skills required to read a geometric figure are more difficult than read a geometric text for low prior knowledge students.

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WHAT STUDENTS VALUE TO MATHEMATICAL SOLUTIONS

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INTRODUCTION

Values are important factors in learning mathematics because they relate to an individual's motivation and will to act in particular ways. Students' values are analysed in questionaries, interviews, lesson observations and so on (e.g., Seah & Peng, 2012). But they are not be tied up with each student's mathematics learning. Therefore, I research students' values with their actions. In this time, I focus on students' reason for choosing a solution as representation of their values.

METHODOLOGY

I attempted to a questionary survey for 77 third junior high school students. First, they were proposed a 9/2-star polygon problem and 2 mathematical solutions and asked to choose the solution which they felt best. One was with a figure in static (type A) and the other was with a variable through some figures (type B). Next, they were proposed another solution with a table (type C) and asked how they thought about that.

SUMMARY OF STUDENTS' CHOICES AND REASON

25 out of 77 students chose type A and said type C was good. 28 out of 77 students chose type A and said type C was not good. 12 out of 77 Students chose type B and 7 out of 12 students said type C was not good. As the overall tendency, the students who chose type A reasoned from their own subjective feelings. The students who said type C was not good reasoned from mathematical contents in that solution.

WHAT STUDENTS VALUE TO MATHEMATICAL SOLUTIONS

When students choose a mathematical solution, they reason from their own subjective feelings and/or mathematical contents in the solution. In other words, those aspects of the solution are so valued that they have taken for a reason. It means they value subjective feelings and/or mathematical contents in choosing a mathematical solution.

CONCLUSION

Tendentiously, students valued type A more than type B according to subjective feelings and judged type C according to mathematical contents. Therefore, students' values on those aspects need to be payed attention to.

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A STUDY OF FIFTH GRADERS' PERFORMANCE ON THE THREE-TIER NUMBER SENSE TEST

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To assess the strength of conceptual understanding or lack of knowledge on number sense, a three-tier number sense test was developed for 5^{th}-grade students. The number sense three-tier test included a content-tier which assesses content knowledge of number sense; a reason-tier, which assesses a reason selected for the first-tier response; and the confidence-tier, which assesses how confident the students are in their answers to the first two-tiers. Earlier studies has showed that a two-tier test includes both quantitative benefit of collecting a lot of data without consuming much time and qualitative strength of assessing students' thinking in terms of an explanation for their answer choice and possible causes of misconceptions (Authors, 2010, 2016). However, the two-tier test does not allow us to measure their confidence as to why they select their answer to the first two-tiers. A low level of confidence with low performance on the number sense will be treated as a lack of knowledge and A high confidence with low performance on number sense will be treated as a sign of significant misconception (Caleon, & Subramaniam, 2010; Pesman & Eryilmaz, 2010; Stankov & Crawford, 1997). A total of 819 fifth graders in Taiwan joined the study. The results showed that this test has good reliability and validity. Results indicated that many sample students performed poor on number sense but with extreme high confidence indicating that many students have significant misconceptions and some students may lack number sense. This study also confirmed that a third-tier (with confidence rating) number sense test can be used to mitigate the weakness of a two-tier test. Educational implications of the findings are discussed.

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LEARNING GEOMETRY CAN BE FUN! – DEVELOPMENT OF STORY-BASED REMEDIAL INSTRUCTIONAL MODULES FOR DISADVANTAGED STUDENTS

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National Chia-Yi University

BACKGROUND AND PURPOSE

Educational equity is a core element and principle for school mathematics. In fact, mathematics for all has been internationally considered a key issue of mathematics education as noted by the Ministry of Education in Taiwan in 2010.

Geometry is fundamental to understand and to explain the physical environment. Enhancing geometric thinking is very important for high level mathematical thinking and in daily life. However, in Taiwan, many research findings indicated there was a lack of conceptual understanding of area for elementary students. They had an inadequate understanding of area and area measurement, and also commonly confused area and perimeter.

The main purpose of this research was to develop story-based remedial instructional modules for elementary school students to learn area and perimeter.

RESULTS

Three elementary schools were involved in this research. And three story-based remedial instructional modules were designed for the participants, including “The Legend of Long-Hair Elder” for grade 3, “Annual Purdue Festival” for grade 4, and “Mission of solving Problem for School” for grade 5. The researcher tried to help students understand geometry could be useful and beautiful, and learning geometry can be fun.

Through analysis of observation, interviews, tests, and related documents, the findings indicated that students enjoyed learning geometry through these "story-based remedial instructional modules", and their mathematics achievements were improved. Hopefully, this research could create appropriate and happy environments for disadvantaged students to learn area and perimeter.

Figure 1. Examples of the "story-based remedial instructional modules"

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RESEARCH REPORTS

(A - G)

41st PME Annual Conference
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HIGHER ORDER THINKING, ENGAGEMENT AND CONNECTEDNESS IN LESSONS BASED ON STEM CONTEXTS

Judy Anderson and Zenobia Katrak

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Retention rates of students enrolled in calculus-based mathematics courses is declining in Australian secondary schools. Pedagogical practices, engagement, perceived relevance and usefulness of mathematics have been identified as factors that exacerbate this issue. Informed by a quality teaching framework, and using responses to surveys, interviews and lesson observations, this interpretive study sought to investigate whether using STEM context-based tasks supported higher order thinking, engagement and connectedness in two mathematics lessons. Students identified challenge, collaboration, open-endedness and connections as factors which enhanced their interest and engagement. Teachers noted increased student engagement and greater connectedness with the intention to use more STEM context-based tasks.

INTRODUCTION

Using mathematics tasks embedded in a context (real or imagined) has been promoted and investigated by researchers for some time (Hodge, Visnovska, Zhao, & Cobb, 2007; Sullivan, Clarke, & Clarke, 2013) with findings suggesting their purpose is not always realised (Beswick, 2011). From promoting transfer of knowledge (Boaler, 1994) to enhancing the relevance of mathematics (Vasquez, Sneider, & Comer, 2013), the use of context-based tasks has been advocated in curriculum and other policy documents. Most recently, the STEM (science, technology, engineering, and mathematics) education agenda in Australia has endorsed the use of tasks which connect the four subjects with the suggestion that such connections will further promote the study of STEM, address falling student achievement and poor attitudes (Marginson, Tytler, Freeman, & Roberts, 2013; Office of the Chief Scientist, 2016). With declining engagement in mathematics, particularly in the middle years during the transition from primary to secondary school (Martin, Anderson, Bobis, Way, & Vellar, 2012), and with declining enrolments in senior school STEM subjects and in university STEM degrees (Kennedy, Lyons, & Quinn, 2014), there is a desire to implement approaches in secondary mathematics classrooms to address these declines.

Based on a survey of mathematics teachers and careers advisers to investigate reasons for the falling participation of students in senior mathematics, McPhan, Morony, Pegg, Cooksey and Lynch (2008) identified poor pedagogical practices and lack of understanding of the usefulness of mathematics as key issues impacting students' decisions. The study reported in this paper investigated teachers' use of STEM context-based tasks to enhance students' perceptions of the usefulness of mathematics

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by collecting data on students' and teachers' reactions to the use of such tasks in lower secondary mathematics lessons and to investigate whether the tasks promoted higher order thinking, engagement and connectedness. These three quality teaching elements form part of a *Quality Teaching Framework* (QTF) which is a pedagogical framework based on a synthesis of solid and reliable research that empirically links these general qualities of pedagogy to improved student learning (Ladwig, 2009).

LITERATURE REVIEW

With teachers typically using teacher-centred approaches such as explanation of skills and procedures followed by extensive practice, many students find mathematics boring and uninspiring (McPhan et al., 2008). Mathematics questions or problems tend to be lower order, repetitive and decontextualized. Based on the TIMSS Video study of Year 8 mathematics classrooms in Australia, Stacey (2003, p. 119) described the approach as a "cluster of features that together constitute a syndrome of shallow teaching, where students are asked to follow procedures without reasons". The approach reinforces a disconnection between school mathematics and real-world mathematics which heightens students' beliefs that mathematics is not useful and tends to reduce the potential for exploring and investigating rich, problem-solving tasks (Sullivan et al., 2013). However, using task types which "provide appropriate contexts and complexity; that stimulate construction of cognitive networks, thinking, creativity, and reflection; and that address significant mathematical topics explicitly" has the potential to promote learning and engagement (Sullivan et al., 2013, p. 14). While Boaler (1994) suggested context-based tasks provide students with a familiar metaphor, motivate and promote interest for students, and enhance the transfer of mathematical learning through links between school mathematics and real world problems, much of the earlier research examined the use of context-based word problems. Based on her examination of the research investigating such word problems, Beswick (2011) argued the identified purposes for using them do not always match outcomes.

A unique feature of this study is the tasks were not just context-based word problems but were more challenging inquiry questions which take longer to solve, have a STEM context, and are connected to students' interests as judged by their teachers. Given the complexity of student learning and catering for individual differences, designing inquiry tasks which support a learning focus, promote self-belief and persistence and reduce anxiety is challenging (Martin et al., 2012). However, finding ways to enhance mathematics learning and improve student affect is critical if we are to improve achievement, attitudes and participation in mathematics (Sullivan, et al., 2013). Previous studies suggest students need to be provided with choice (Boaler, 1994), the task should not be overly simplified so that it becomes unrealistic (Sullivan et al., 2013), the mathematics needs to be explicit and evident (Beswick, 2011), and the context needs to connect with students' lives (Hodge et al., 2007). To further focus the investigation, the study examined the use of tasks which promoted higher order thinking (or challenge) (Sullivan et al., 2013), behavioural, cognitive and affective engagement (Fredricks, Blumenfeld, & Paris, 2004), and connectedness (Ladwig, 2009), all of which are elements of the QTF; the framework is comprised of three dimensions – intellectual quality, quality learning environment and significance. Table 1 shows how each of the key elements under investigation is connected to the dimensions and provides a brief description of what each element might look to an observer in a classroom.

QTF Dimension	**Element (one of six)**	**What does it look like in classrooms?**
Intellectual Quality	Higher-order thinking	Students are regularly engaged in thinking, to organise, reorganise, apply, analyse, synthesise and evaluate knowledge and information.
Quality Learning Environment	Engagement	Most students, most of the time, are seriously engaged in the lesson rather than going through the motions. Students display sustained interest and attention.
Significance	Connectedness	Lesson activities rely on the application of school knowledge in real-life contexts or problems …

Table 1: QTF dimensions and elements investigated in this study (NSW DET, 2003)

The QTF is a model of pedagogy designed as an analytical framework for diagnostic development of classroom practice and assessment tasks (Ladwig, 2009); a QTF coding sheet has been developed to assist with the identification of the level of evidence for each element during classroom observations. Intellectual quality is described as the central to quality teaching but is not sufficient without attention to the other dimensions. Based on observations in more than 600 classrooms from Kindergarten to senior schooling and in all subject areas, Ladwig highlights the importance of connectedness which he suggests is rarely observed:

> … it is relatively rare to see links to other contexts or larger social purposes of what is being learned, such that the significance of a lesson or task is largely unarticulated. In short, the quality of pedagogy these studies have documented resembles much of what we all have probably experienced: reasonably warm, mundane schooling (p. 275)

METHODOLOGY

To investigate whether teachers' use of STEM context-based tasks enhances students' higher order thinking, engagement and connectedness, a small-scale study was conducted in two comprehensive secondary schools in a large metropolitan area. This paper includes data from an experienced mathematics teacher from each school who was teaching a year 7 and/or year 8 mathematics class. Both teachers had attended a STEM professional learning program and were keen to investigate their students' reactions to the use of STEM context-based tasks, particularly those which promoted challenge (Sullivan et al., 2013) and inquiry-based learning (Marginsen et al., 2013). Teacher A, a Deputy Principal at a small independent coeducational high school, was teaching mathematics to a composite year 7 and 8 class. Teacher B, an experienced teacher at a Catholic girls' high school, was teaching a top-stream year 7 mathematics class. For the study, each teacher designed a STEM task that connected with the curriculum requirements from the mathematics program, and that would engage and challenge their students. Both tasks were open-ended and relied on specific mathematics knowledge and understanding for completion (Sullivan et al., 2013).

Data were collected using researcher observations of each lesson, student surveys and teacher interviews after the lesson. A QTF coding sheet was used by the researcher to code the lesson for higher order thinking, engagement and connectedness with real-life contexts (Ladwig, 2009). Informed by a questionnaire used by Martin et al. (2012), the student questionnaire comprised 12 four point Likert scale items about students' interest ("I like doing mathematics"), the value of mathematics ("Maths is useful for solving everyday problems"), and their reactions to the lesson ("Maths makes more sense when it is linked to the other STEM subjects") to which they indicated their level of agreement from "Strongly Agree", "Agree", "Disagree" to "Strongly Disagree". Two open-ended questions asked students to describe the differences between the STEM focused lesson and "usual maths lessons", and how the lesson was more engaging than usual. An Excel spreadsheet was used to analyse responses to the Likert items and responses to the open-ended questions were coded to identify key themes. Teacher interviews sought their views on students' levels of engagement and interest in the lesson, and how that differed to typical mathematics lessons as well as whether they would continue to use STEM context-based tasks. Triangulating the data from the QTF coding sheets, student responses and teacher interviews enabled each lesson to be analysed for evidence of higher order thinking, engagement and connectedness.

DATA ANALYSIS AND DISCUSSION

This section presents a brief description of each lesson and then links data from the lesson observations, the teachers and their students to highlight similarities and differences in higher order thinking, engagement and connectedness. In teacher A's class, the lesson formed part of a STEM project entitled "My Kitchen Rules". Students had been provided with a 'challenge brief' for the project indicating they needed to develop and make an entrée or dessert that must include an egg as one ingredient. After the project students were to submit the following as evidence of their learning:

> Submit the recipe with ingredients, equipment and method for a party of 20 people. Identify the physical and chemical changes that take place and describe the types of heat transfer used in the process. Submit a nutritional label informing the consumer of the ingredients in descending order of proportion. Submit a detailed folio of the processes, experiments, mathematical calculations and graphics used to prepare your signature dish.

While mathematics and science formed key components of this project work, the mandatory technology curriculum was also evident as this task is classified as a "food design project" which "may result in food products, menus, food preparation systems, diets for special purposes, food presentation" (BOSTES NSW, 2003, p. 15). In the observed lesson, project-paired students were tasked with using an Excel spreadsheet to create a formula to show how the quantities of their ingredients would change, based on the number of serves. Students were required to ‘think big’ to investigate how their formula for serving size could be applied to real-life to enhance the relevance of the mathematical findings they had discovered. For example, one pair of students chose to look at the relation between serving size and *Recommended Daily Intake* as presented on a government health website.

In teacher B’s class, the lesson focused on the concept of optimisation, as students had to determine why beehives were hexagonal in shape. The lesson had three phases; during the first part of the lesson, students worked in small groups to calculate the maximum area of a rectangular enclosure that had a perimeter of 12 centimetres. This was followed by class discussion and teacher exposition on optimisation and its application in real-life contexts. The third phase of the lesson was framed by a video that provided stimulus for the students to further explore why beehives are hexagonal in shape. This lesson did not form part of a STEM project but the teacher indicated he was keen to use STEM contexts in as many of his mathematics lessons as possible.

Using the QTF coding system, the researcher determined that in both lessons there were high levels of higher order thinking since students were continually analysing, assessing, evaluating and deconstructing the problem-based contexts through engaging in experimental practices; one class used technology to facilitate the investigation while the other used concrete materials. Within both lessons, most students adopted a 'trial and error' approach to begin their task. In class A, there were multiple approaches to creating a formula which allowed them to interpret the problem in varying ways. Students having difficulty with their formula were prompted by the teacher to collaborate with their peers to uncover new ways of approaching their problem-solving task. For example, after discussing with two of their peers, one pair discovered that to find the relationship between ingredient quantity and serving size, they needed to start by converting the quantities of each ingredient into a percentage of the final product. In teacher B’s class, students were provided with paper cut-outs of hexagons to fit together to optimise the area covered within the rectangle (Figure 1).

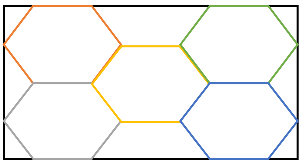

Figure 1: Sample of student solution to optimising the area covered with hexagons

Students were observed questioning the way their peers had reached their conclusions, and tried different ways of fitting the hexagons together. When interviewed, teacher B commented that he was “surprised by some of the solutions the girls came up with”, as they were unique and demonstrated creativity in spatial reasoning. Further evidence of higher order thinking occurred when one student announced to her group that they should draw a four-sided shape with a side length of three centimetres to maximise the area of the enclosure, and another student responded with “that’s not a rectangle” thus providing an opportunity for a deep mathematical discussion about the properties of quadrilaterals.

In both lessons, collaborative work to problem solve was an integral part of the lesson, as students were required to work in pairs or small groups for each activity. Students questioned each other and challenged each other's responses and mathematical justification was evident, particularly in teacher B's class. There were clear indications of student engagement (behavioural, cognitive and affective) with the content, with most students working on the tasks for most of the lessons, students made substantial contributions during class discussions, and all contributions included strategies to solve the task under investigation (Hodge et al., 2007).

However, there was evidence that some students in both classes were off task for a small proportion of the lesson. Both teachers were aware that some students were more dominant and inclined to 'take over', do most of the directing of activity, and ask most of the questions. Each teacher employed strategies to overcome this; B commented that: "there were a couple of girls as I went around that were letting the other girls do the work", so he used questioning when the class regrouped, and selected students to explain and justify their mathematical reasoning who appeared to be less engaged. Noticeably most students could justify their answers when required demonstrating they had engaged cognitively with the task. Teacher A commented that some pairs appeared to disengage if they both reached a blockage in their solutions. However, they were observed self-regulating their behaviour and re-engaging with the lesson by working with another pair. For example, teacher A commented

> I was surprised that the boys down the front [of the classroom] were asking for help from the girls at the back. That was good because the boys usually don't do that …

Importantly in teacher A's case, he acknowledged that establishing a problem with real world applications did not necessarily make mathematics more appealing, relevant and understandable for some students, and this may have been a contributing factor to one student's disengagement. He commented: "the boy at the back of the classroom, his partner was absent, and he relies on his partner to do the work and explain it to him".

Despite this, 13 out of 17 students in class A and 19 out of 20 students in class B agreed that they felt more engaged with mathematics, and that they preferred learning mathematics in a problem-solving context, especially when the contexts were relevant to their lives. In the open-ended responses, students reported that the lesson was "more engaging as we had experimented with the actual problem we were solving" and that they could see real world applications. Students also favoured a "hands-on" approach to mathematics compared to a textbook approach, as they could "do something ... we can actually see when we make mistakes and do things correctly". Collaborative work was also cited by multiple students as a contributing factor to increased engagement in the observed lesson, as students could learn from each other and "work out the answer by listening to each other's opinion", and that working with peers is "definitely an opportunity to find out more answers". The overall findings suggest that levels of engagement in mathematics classes are higher when open-ended problem-solving tasks are used to teach mathematical concepts to students, and highlight the importance and usefulness of mathematics in real world contexts.

Further evidence of connectedness occurred within teacher B's class as he described how optimisation is used in other real-life situations such as packaging and manufacturing cereal boxes. Links to engineering and construction using optimisation were also made. His exposition created background knowledge for the next phase of the lesson, as students explored the link between why bees form beehives in hexagonal shapes, as this increases the area in a beehive while minimising the amount of wax needed. In both cases, most students were making progress in the problem-solving process, and students were connecting the problem solving with mathematical concepts; in B's class, students could see the connection between perimeter and area.

From the student surveys, several items examined whether students felt that the lesson demonstrated the relevance mathematics had in the real world. From A's class, 16 out of 17 students agreed that mathematics is important and is used to solve practical problems in life, with all seeing the relevance mathematics had in the world from the lesson. B's class followed a similar trend, with 19 out of 20 students connecting mathematical knowledge to problem-solving in a real-world context. Student responses justified their choice by explaining that the lesson allowed them to "learn the purpose [of maths] … most of the time math class is boring and pointless", and that "maths lessons should be linked to how we can use it in real-life".

CONCLUSION AND IMPLICATIONS FOR FURTHER STUDIES

Although the data from only two mathematics lessons form the basis of this paper, observations as well as student questionnaire responses and teacher interviews suggest that teachers' use of STEM context-based tasks do enhance students' engagement and interest in mathematics. These tasks allowed teachers to create lessons that provided students with open-ended, challenging experiences to explore mathematical concepts in a creative way, and forge a deeper understanding of how mathematics is relevant to real-world contexts. Students were less reliant on the teacher to help them through the tasks as they interacted with each other. The evidence reported here indicates that the tasks promoted higher order thinking, enhanced engagement and enabled connectedness. The study supports recommendations from Sullivan et al. (2013) and Hodge et al. (2007) but builds on earlier work by incorporating a STEM focus. The implications of this study could have profound effects on the pedagogical practices employed in classrooms by teachers; there may be a greater push to forgo traditional pedagogical practices for open-ended investigations with cross-curriculum connections, particularly with the other STEM subjects (Vasquez et al., 2013). Both teachers and students agreed such experiences would enhance mathematics lessons and might encourage more students to participate in the post-compulsory study of STEM subjects (Kennedy et al., 2014; Marginson et al., 2013; McPhan et al., 2008).

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REFRAMING TEACHERS' VIEWS OF STUDENTS' MATHEMATICS CAPABILITIES

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Reforms interventions are mediated to a large extent by teacher's social construction of their students. Analysis of changes in teachers' perceptions of students' capabilities using a diagnostic and prognostic framework suggests that professional learning that includes opportunities for teachers to make sense of the reasons for student struggle through inquiry into practice can be effective in shifting teachers' perceptions.

INTRODUCTION

Inequitable opportunities to learn associated with ability grouping, teacher deficit thinking, and impoverished curriculum-based expectations are frequently cited as sources of disparity student mathematical outcomes (Anthony & Hunter, 2017). While acknowledging the many factors that impact on learning outcomes, this paper focuses on the influence of teacher perceptions of students' mathematical capabilities. Perceptions play out in expectations of achievement, assignment to 'ability' groups differential feedback, and opportunities to engage with challenging tasks, to participate in mathematical discourse, and the development of students' mathematical identity (Jorgensen et al., 2014). And importantly for any reform interventions, policy coherence as intended by reformers ultimately "is achieve or denied in the subjective response of teachers—in teachers' social constructions of students" (McLaughlin & Talbert, 1993, p. 248).

In this paper, we explore how ambitious forms of teaching advanced in the *Developing Mathematical Inquiry Communities* (DMIC) project impacts on teachers' perceptions of traditionally struggling students. Utilising complex instruction, DMIC focuses on the use of (a) groupworthy tasks connected to students' cultural experiences and the learning of big mathematical ideas; (b) instructional practices that support respectful group norms and the development of mathematical argumentation; and (c) status and accountability structures that raise the participation and intellectual expectations of each student (Alton-Lee et al., 2011). DMIC's ambitious teaching practices required that many teachers within the project school reorganise their vision of teaching and learning, including changes in their assumptions about how each student learns, the provision of mathematical activities, expectations for student participation, and norms of social interactions with and between students. As part of this learning journey we were interested to explore how teachers (re)framed their views of students' capabilities, and to develop a sense of what might be influencing these changes.

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CONCEPTUAL FRAMEWORK

Teachers' views of students' capabilities are important in efforts to supports instructional reforms, particularly so in settings that serve students from economically impoverished communities (Jackson et al., 2017). However, despite policy initiatives in New Zealand directed at improving mathematics achievement outcomes for diverse learners, achievement outcomes for Māori and Pasifika remain consistently low. Related, are concerns re teachers' deficit-oriented view of Maori and Pasifika students and their families and the communities in which they live (Turner et al., 2015) and the widespread practice of ability grouping in primary schools (Anthony & Hunter, 2017).

In this paper, we adopt a lens of *problem framing*—an approach proposed by Jackson, et al. (2017)—to provide insights into our teachers' views of their students' capabilities in the context of the DMIC instructional reform efforts. Drawing on Goffman's (1974) work on understanding how interactions are socially organised, frames are used to give meaning to an event or experience and enable people to answer the question "what is going on here"? (p. 8). Importantly, "the elements and processes assumed in reading of the activity often are ones that the activity itself manifests" (p. 26). For example, Horn (2007) working with two high-school mathematics departments noted differences in the ways the teachers framed the problem of differential success—in terms of inherent traits of the students or learning opportunities provided in the classroom—and consequential instructional actions. In the former, when students were not engaged or struggled the teachers placed the blame on the students, in the second group teachers were more likely to consider how they might alter instruction. Horn argued that a key aspect of teachers' substantial participation in ambitious instructional reform involved framing differential student success as a problem of instruction. Moreover, Horn conjectured that teachers' problem framing linked to their views of mathematics, with the latter group aligned to views of mathematics as a web of ideas rather than a sequential ordering of topics.

Bannister (2015) used two framing categories to provide insight into teachers' views of students' capabilities in relation to ambitious teaching reform efforts: *diagnostic framing*—identification of a problem and the attribution of blame; and *prognostic* framing—a proposed solution to the diagnosed problem that specifies what needs to be done. Linking frame analysis to the theoretical traditions of the communities of practice literature, Bannister argued that frames are co-constructed objects among group members that represent existing meanings in a group at a particular time. Hence, teacher framing of problems of practice has "the capacity to provide evidence of changes in teacher' participation and reification patterns over time, yielding empirical evidence for learning with a community of practice" (p. 350).

Jackson et al.'s (2017) study of 122 middle-grade mathematics teachers' perspectives on reform efforts categorised varying levels (unproductive, productive, or mixed) of teachers diagnostic and prognostic framing. They found that nearly 30% of the teachers coded for diagnostic framing (n = 100) "attributed students' difficulty solely to inherent traits of students, and/or deficits in their families and communities" (p. 7). The most common prognostic response (70% of the sample of 74 teachers) was to describe a lowering of the cognitive demands of activities for students perceived as struggling. They also noted for those teachers categorised as expressing a productive diagnostic framing related to instructional opportunities, half did not describe ways to respond to those difficulties "in ways that would enable students to participate substantially in rigorous mathematical activity" (p. 8). These researchers urged that professional learning opportunities need to simultaneously attend to supporting teachers to make sense of differential student success and ways in which teachers can learn to support students facing difficulty that align to ambitious teaching goals.

RESEARCH DESIGN

The data consists of a series of semi-structured interviews, each approximately 25 minutes in duration, with 17 teachers from one school. The school, located in an economically challenged community, comprised multiple ethnic groups and transient students. Questions related to struggling students: "Who are the strugglers in your class and why do you think they struggle?" and "How do you support students who experience difficulty in maths?" were woven into each interview about their ongoing experiences in the DMIC professional learning project. Analysis for evidence of changes in teachers' perceptions of students' capability and learning potential examined teachers' reported instructional strategies associated with student engagement and participation patterns within each teaching group (organised into New Entrants (NE), Years1/2, Years3/4, and Year 5/6). Interviews were coded with regard to references to managing diversity and changes in instructional and participation patterns for students in their classes. In seeking evidence for changes in framings of the struggling student problem (adapted from Jackson et al., 2017) we coded against the diagnostic frame (see Table 1 for example coding) to understand how teachers conceptualised the struggling student problem and the (b) prognostic frame (likewise see Table 2) to understand how teachers conceptualised interventions related to the struggling student problem.

Productive	Student performance is described as a relationship between student and instruction or opportunity to learn	I guess [now] I figured anybody could do maths. …I have got children with very little English and children from pretty traumatic backgrounds and they come in and they believe they can solve the problem as well as anybody else and they give it a go. (Y5/6)
Mixed	Some but not all students' performance is related to un/productive framing or that performance is a combination of factors.	Generally those children are low in all areas. They have difficult homes. Maybe they haven't had breakfast. Maybe they've been up late, they're tired, they don't like maths. Their experiences of maths aren't positive. Maths is boring. Maths is repetitive. Maths is pointless. (Y5/6)

Unproductive	Student performance is attributed to an inherent property of students or their home or community.	It's just home life. Pure and simple. It's those first five years and the moment we meet the families we can tell why the child is like they are (NE)

Table 1: Diagnostic framing coding scheme

Productive	Instructional support aimed at rigorous learning goals	I look at three different ways some kids struggle to explain things through words or pictures or materials. I don't care how they explain it as long as it's in some form. I know a lot of this is based around verbalising and talking. (Y1/2)
Mixed	Support aimed at supporting rigorous activity, but some actions lessen the cognitive demands.	The constraints around it is that a lot of those kids that are less capable sometimes sit back a lot more and take a bit more—and often they don't like maths because they're not good at it and because they don't like maths, they then don't participate in the way that we would like them to but then you just ask the right question. (Y5/6)
Unproductive	Instructional actions aimed at lessening the cognitive demands of a task.	I've usually been a teacher that's explained a lot and set it up for them just so they can figure it out a lot easier. (Y3/4)

Table 2: Prognostic framing coding scheme

FINDINGS

Table 3 provides a visual summary of how teachers from each syndicate shifted into a more positive framing space over the year, with upward movements represented in both diagnostic and prognostic framing of struggling students. At the beginning of the year the teachers across most of the year levels focused on fixed student attributes and gave advice about changes students needed to make in terms of references to "bottom" or "top" students. Several teachers appeared taken aback with probes about why some students struggle. For example, a Y1/2 teacher responded: "I'm not sure why they struggle. I don't have an answer for you" but later in the interview the teacher explained that prior to the project she was "a teacher that's explained a lot and set it up for them just so they can figure it out a lot easier". Mid-year, the teachers expressed greater awareness of how struggling students could participate—with many reporting examples of 'surprises' and increased expectations for their students. However, still concerned with filling gaps in basic knowledge, several teachers viewed the problem of managing diverse groups of students as being about getting the 'weaker' students to listen more to the 'more able' students. At the end of the year there was a noticeable decrease in teacher talk that labelled students by perceived ability.

Prognostic Framing				
	Productive			T5/6
			T1/2 T1/2 T1/2 T5/6 T5/6	T1/2 T3/4 T3/4 T5/6 T5/6
	Mixed	T1/2 T1/2 T1/2* T3/4* T3/4 T3/4	T1/2 T1/2 T3/4 T5/6 T5/6 T5/6	
		T1/2 T3/4	TNE TNE T1/2 T3/4	T3/4
	Unproductive	TNE TNE* T1/2 T3/4		
		Unproductive	Mixed	Productive
		Diagnostic Framing		

Table 3: Shifts in framing from start (unshaded) to end of Year 1 (shaded)

(Note: *assessed at mid-year interview)

In the next section we discuss three key features of DMIC that we conjecture contributed to the positive shifts evidenced in Table 3.

Making sense of the reasons behind differential success

Helping teacher understand the reasons behind their students' struggle involved challenging teachers' views about who and how students participate in mathematics. In seeking to engage more students DMIC included a focus on developing teachers' awareness of their students' cultural backgrounds and interests. For many teachers they developed this awareness through partnerships with their students:

> we ask the kids to give us the information about their culture rather than us just assuming that we know about their cultures. It's been a big learning curve; you have to change your mind-set. [Y5/6]

A second significant feature of DMIC involved the whole-school change from ability grouping practices to using mixed-achievement groups, combined with changing the nature of student collaboration in group tasks. These embedded changes in groupings, combined with the establishment of new group norms and inquiry discourse practices challenged teachers' perceptions of student agency:

> They are a lot more independent. You can give them a problem and obviously unpack it with them and then leave them to it. Last year working with the purple book and you would come across a question with a group in front of you they would sit there looking at you. Now they ask each other rather than asking us. This [inquiry group discourse] is starting to slowly transfer to the other learning areas now like in reading a boy nudged another boy and said "do you agree that this word is this"? [Y1/2]

Supporting students facing difficulty in ways that maintain challenging goals

More than challenging teachers' views about who can do mathematics, DMIC provided teachers with productive strategies for supporting students who have traditionally struggled. While towards the end of Year 1 only two teachers (compared with 10) reported deficit views of their students, all teachers reported using some productive strategies to support students who struggled. This shift was supported by changes in the opportunities to learn through the enactment of challenging groupworthy tasks, and by changes in the group participatory practices associated with mathematical inquiry. As a Y5/6 noted:

> So the whole structure of the way you do the groups making other people write and listening to other people feedback has just taken away the fear. If everybody in the room makes a mistake you quickly realise it's okay to make a mistake. ….The mixed ability groups promote the risk taking, questioning and classroom culture is so important.

As teachers persevered with the establishment of group tasks they noted that students were more confident and open minded, more independent of the teacher, and purposeful in their mathematics learning:

> The thinking and puzzling things through and discussing and justifying the thinking. And being able to say, "I'm going to try this way and show that to somebody else". Sometimes, like with patterns, I've got this way in my head, but kids have a different way. …Those conversations and discussions, you can see little light bulbs go on in their heads, listening in on conversations you hear "oh I get that know" because they've asked "can you tell me that again cause I still don't get it". [Y5/6]

Monitoring ongoing instructional improvement efforts

However, equally important in supporting shifts in framing problems of practice, is the development of teacher disposition to inquiry into practice. Only when a teacher weighs up the evidence of learning, thinking about the impact of specific instructional moves can the teacher develop adaptive expertise that is needed to response to diversity in his or her classroom (Anthony, Hunter, & Hunter, 2015). Inquiry was both individual, in terms of teachers noticing what children in their own classroom could do, and collaborative in terms of planning and co-teaching experiences.

Freed up from the role of explainer, teachers' close listening to their students informed their learning:

> I'm listening more, I hope I was listening to them before but it's definitely a lot more balanced out with them telling me, teaching me what they know. I'm learning more about their different thought processes, the conversation, and the reasoning behind the maths. [Y5/6]

Teachers freely reported examples of 'surprise' at what their students could achieve with the support of others in their group, and with the stimulus of greater challenge with the permission to take risks:

> They are just more open minded and more problem solvers than they ever were. I taught a couple of these kids a few years ago and they really struggled with maths but now they are problem solvers and there is not one way of doing things. They really branch out and are willing to give things a go. [Y3/4]

However, listening to and understanding children's thinking was challenging for many of our teachers. As a Y1/2 teacher noted at the end of Year 1:

> What I'm still finding a little bit hard is when I'm modelling something that a child has said and I don't always model it the correct way cause of a fixed mindset of what I want them to do—and they use a different strategy and so I need to get out of that habit and think "what are they trying to say when they say that and model it so it's representing what they said.

Looking at the shifts in framing, we were struck by the commonality of the directional shifts in each syndicate. Consistent reports of the value of collaboration—be it through shared planning, teaching, whole-school development, or in-class mentoring—in supporting their learning was a feature of interviews. With the requirement to design culturally responsive tasks, rather than reliance an 'ability group' text, shared team planning involved pooling of ideas about how students might respond to the group problem. Reflecting a more productive way to support students, a Y3/4 teacher noted:

> [Before DMIC] we would think about the different stages the children were at and you had to find the worksheet to follow up for each stage. Now we are thinking about what are their needs, what can we write a problem about that is going to get them to think about it in that way and to come up with that themselves. So pre-planning what the kids are going to think and give them problems to get them to where you want them to go.

Dynamic mentoring involving in-the-moment in-class discussions with the teacher and DMIC mentor (see Hunter et al., 2016) and co-teaching within shared teaching spaces also provided valuable opportunities for teachers to learn from and in. Working in a shared space with three teachers, a Y1/2 teacher noted:

> we have learned ideas off each other while we are teaching. If you are by yourself you don't get those moments of looking at what other people do. Sometimes we don't agree with the way others teach but then sometimes we learn other ways to do something, ways to teach.

DISCUSSION AND CONCLUSION

Consideration of the impact of reform programs needs to look wider than student learning outcomes, typically measured with some immediacy through standardised testing regimes. Indicators of how teachers frame the problem of student struggle in mathematics could and should be an important indicator of reform efforts. Indeed for New Zealand, a country that currently exhibits one of the greatest disparities in student outcomes (Caygill et al., 2016), understanding current framings should be a central to designing and monitoring the impact of policy directions around raising achievement of priority learners. With the DMIC project, in this school and others, we have observed significant whole-school shifts toward productive diagnostic and prognostic framing of students' capabilities, with a tendency for the nature of movement to be shared and localised within collaborative teacher syndicates. Exploratory analysis suggests that these shifts were occasioned by attending simultaneously to supporting teachers to critically understand students' source of struggle and how they as teachers are can engage students in more equitable and culturally appropriate ways. Given the differential trajectories in teachers' perceptions of the problem of struggle, more analysis is needed to explore how the impact of professional development opportunities is mediated by teachers' personal histories, and collective experiences and expectations within grade levels, and the impact of access to co-teaching and mentoring.

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COMPARING THE PROFESSIONAL LEXICONS OF CZECH AND FRENCH MATHEMATICS TEACHERS

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The research presented here is part of the Lexicon project whose aim is to identify and compare the naming systems available to and used by middle school teachers to describe mathematics sessions in classrooms. It compares the Czech and French national lexicons that have been produced in the first phase of the project, using the Anthropological Theory of the Didactic as main theoretical background. Through this case study are also developed and tested methodological tools that will contribute to the comparative dimension of the project.

INTRODUCTION

The research presented here is part of the Lexicon Project, a project involving nine countries (Australia, Chile, China, Czech Republic, Finland, France, Germany, Japan, USA), and in each country a team of mathematics education researchers and experienced mathematics teachers. In this project, we consider that our experiences of the world and reflection on those experiences are mediated and shaped by available language, and that the use of English as lingua franca for international communication substantially limits what can be expressed and shared. The goal of the project is thus to document and compare the naming systems employed in mathematics teacher communities in the nine countries to describe the objects and events in their classrooms, in order to expand on the variety of constructs available for the purpose of theorizing about classroom practice and for identifying the characteristics of accomplished practice (Clarke & Mesiti 2010), (Mesiti & al. 2016).

The first two years of the project have been devoted to the identification and validation of the nine naming systems called national lexicons. For this identification, it was agreed that the voice of experienced teachers would be given a predominant role, and that all teams would use as a spur the videos of grade 8 classrooms produced in each country in the first phase of the project along with their English transcription. However, it was also agreed that each team had to develop a strategy adapted to the institutional and cultural specificities of the country. After a week of collective work in November 2015, the first drafts were revised and complemented, then submitted to a collectively agreed process of local then national validation. The research presented here initiates, in some sense, the comparative phase of the project. It involves two

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countries: Czech Republic and France. Its aim is to identify similarities and differences between their national lexicons, and the possible origin of these. It is also to better understand the respective potential of the two lexicons, investigating how similarities and differences reflect in the way the Czech and French teams describe classroom sessions, in what they notice and are sensitive to. Finally, the research has also an important methodological aim: to develop comparative tools adapted to the specificity of the Lexicon project, to be used beyond this particular case study. In this paper, after presenting the theoretical background of the research, we detail the methodology we have developed; we then present and discuss the main results already obtained.

THEORETICAL BACKGROUND

As shown by the literature, comparative studies have exponentially increased in the last decade ((Gómez (2005) speaks about a “wave of comparative studies”), using a diversity of theoretical frameworks. For this particular research, we mainly rely on ATD, the Anthropological Theory of the Didactic (Chevallard & Sensevy 2014), and especially on two ATD constructs, the concept of *praxeology* used to model human practice and the *hierarchy of levels of didactic codetermination*, which potential for comparative research is well established today (see for instance the meta-study (Artigue & Winslow 2010)). In its most elementary form, a praxeology is made of two blocks: a praxis block consisting of a type of task and of a technique for solving it, and a logos block consisting of a technological discourse (technology) describing and justifying the technique, and a theoretical discourse justifying the technological discourse (theory). Elementary praxeologies aggregate into local praxeologies sharing a same technology, then into regional praxeologies sharing some theory, to form mathematical and didactical organizations, mutually influencing each other. ATD pays specific attention to the conditions and constraints shaping what is actually taught (mathematical praxeologies) and how it is taught (didactical praxeologies), and to the fact that these depend on the institutional and cultural contexts. The hierarchy of levels of codetermination distinguishes 9 connected levels of conditions and constraints (topic – theme – sector – domain – discipline – pedagogy – school – society – civilization). Due to the general character of the Lexicon project, which does not enter into the vocabulary specific to the teaching of particular mathematical domains, we hypothesize that the main levels of this hierarchy influencing the national lexicons, and helping to understand their specificities, are to be looked at the discipline (here mathematics), pedagogy, school and society levels, and that the civilization level, considering conditions and constraints transcending the limits of a particular society should pay a more important role in the comparison of the national lexicons.

METHODOLOGY

In line with the research “problématique” presented in the introduction, the methodology for this study is two-fold. It combines a formal comparison of the Czech and French lexicons (CL and FL in the following), and an operational comparison of their use. Each national lexicon is made of a list of terms or expressions with a description/definition complemented by some examples and non-examples. Terms are grouped into categories proper to each lexicon. Due to this format, the formal comparison of the two lexicons has been organized in three main phases: comparison of the structures of the two lexicons (categories and subcategories); comparison of the selection of terms and the terminology used; comparison of the term descriptions/definitions, of examples and non-examples. To facilitate this formal comparison, we used the English provisional version of CL and FL, but aware of the associated risks, we systematically checked the original meaning of the terms having close translations, and we also paid particular attention to the terms for which finding an English version had been difficult even through the use of a longer expression. The fact that one of the co-authors spoke the three languages was a substantial help.

The operational description was developed from the coding template of the videos of the Czech and French mathematics classrooms produced by each team. This coding template had been proposed in the first phase of the project by the Australian team directing the project. It obeyed the following structure, with three additional columns for the expression in local language, not reproduced here:

Tean mame: France	Lesson name: France	In English		Additional Comments
Time Stamp IN	Time Stamp OUT	Activity or Action	Description of Activity or Action	
00:00:00	00:02:40	**Session start**	Welcoming students and start of the session before engagement in the mathematical work, and associated rituals	Rituals for the entry in the classroom differ from one class to another. Here the teacher welcomes students at the door, request them to go to their place and sit down s's then to take their notebook and to write the date. In this phase, the teacher also

Table 1: Lexicon template

This structure was a priori appropriate to investigate how the respective lexicons oriented the views on mathematics' classrooms, but the coding process resulted in several very long spreadsheets decomposing the session into a succession of small episodes. We thus decided to create a new methodological artefact in the form of a narrative of the classroom session based on the template. With this new artefact, in line with the use of narratives in education research (Clandinin & Connelli 2000), our intention was to make it visible again that, for each team, each video tells a particular story generated through the selection, arrangement and interpretation of events which contain meaning for the teller, without forgetting that the stories we produce are socio-cultural products. Introducing these narratives also allowed us to include the additional information about the sessions and their context in the description, that each team considered necessary to generate the story. As is the case for the templates, the narratives are a collective production of each team, agreed by both researchers and teachers. The analysis of these narratives is still under process and its methodological tools, partly inspired by narrative research, are also in development.

RESULTS

We first present the main outcomes of the formal comparison of the two lexicons using the English provisional translation of categories, terms and definitions to save space, and some questions arising from this comparison. These have been particularly paid attention to in the analysis of the narratives that we evoke then more briefly.

The formal comparison of the two lexicons

A first obvious difference between CL and FL is their respective number of terms. CL is the smaller in size with currently 47 terms, while FL is among the biggest ones with 115 terms. However, CL is not at all a subset of FL, as will be shown below. The two lexicons are respectively structured into 5 and 6 categories, as shown in table 2, some of these also including sub-categories. For instance, the 50 terms of the category Pedagogical and didactic management of the classroom in FL are distributed into three sub-categories: Organization with 17 terms, Interactions with 20 terms, and Exploitation and assessment with 13 terms.

Czech lexicon categories	French lexicon categories
Stages of a lesson (9)	Phases of a session (13)
Organization forms of teaching (7)	Forms of pedagogical organization (10)
Type of problems (4)	Nature of tasks (activities) (17)
Teaching methods (23)	Pedagogical and didactic management of the classroom (50)
Use of didactic means (4)	General terms (9)
	Mathematical activities (16)

Table 2: Categories and associated number of terms in the two lexicons

The three first categories have very close titles, and one can expect also connections between the fourth categories. However, two categories in FL do not have their counterpart in CL, and reversely, one category in CL does not have its counterpart in FL. This does not mean of course that terms belonging to these categories are necessarily absent from the other lexicon, but these differences of structures are certainly meaningful. We will come back to this point later.

Entering into the categories shows that, even for categories with similar names, important differences are observed. The comparison of the terms of the categories Stages of a lesson and Session phases, for instance, makes clear that the two lexicons make a different interpretation of these categories. The 13 terms of this FL category are related to the progression of the mathematics classroom activity (Recall phase, Research phase, Kneading-up, Institutionalization…), while the 9 terms of the CL category are mainly related to organizational matters (Students' and Teachers' organizational questions, Maintaining the discipline, Written record on the board…).

In FL, such terms, when present, belong to the category Pedagogical and didactic management of the classroom. As a result, the two categories about stages or phases of a session have very few common terms. Even when there is no such difference in interpretation, the distance between the two lexicons is evident. For instance, the category Types of problems in CL only includes 4 terms while the category Nature of tasks in FL is especially large with 18 terms; however, two of the terms of CL - Determining problem (a problem asking for the determination of a number or magnitude), Proof problem - do not have equivalent in FL. The two lexicons have nearly the same number of terms associated with assessment (4 and 5); however, once again the terms do not exactly point out the same characteristics.

As mentioned above, two categories in FL do not have their counterpart in CL, Mathematical activities and General terms. We conjecture that this difference has deep cultural roots and shows the influence on the mathematics teachers' professional discourse of the highest levels of didactic codetermination. In the Czech educational culture, indeed, the main lenses are not mathematical lenses but pedagogical lenses, in line with a tradition of general didactics, which can be traced back to the *Didactica Magna* by Comenius in the XII^{e}. The French educational culture is different, with the historical investment of leading mathematicians in educational issues, and the development of the didactics of mathematics as a genuine research field closely connected to mathematics, since the early seventies. The two categories mentioned above reflect the resulting importance attached to mathematical processes and activities in the professional discourse of teachers and classroom descriptions. It also reflects the fact that some terms and distinctions used in French didactic research have reasonably disseminated through teacher education. De facto, the General terms category of FL mostly includes terms and distinctions used in French didactic research, such as the distinction between the tool and object status of mathematical concepts, and the notion of mathematical setting due to Douady (1984), or the notions of register of representation and conversion between such registers due to Duval (2000). Beyond the sole General terms category, the influence of French didactic research is also visible through for instance the inclusion of terms coming from the theory of didactical situations (Brousseau 1997), such as didactic contract, devolution, institutionalization, milieu, action situation, etc. in other categories. The existence of the specific structure of the IREM (Institute of Research on Mathematics Teaching (www.univ-irem.fr/)) in France, functioning in mixed thematic groups close to math departments and playing an interface role between didactic research and practice, through the publications they produce and the Professional development institutions that they represent, contributes without doubt to explain why the research didactic discourse seems to have more disseminated in France than in Czech Republic, and is thus more present in FL.

Other differences between CL and FL appear when considering the description/definition of terms. Some descriptions are quite close with differences just resulting from a change of perspective, reflecting the different categories the terms have been allocated. However, CL descriptions are generally short, in active form expressing some teacher's or student's action; they describe actions but do not enter into their possible function. FL descriptions are more in nominal form, and quite often they make explicit the purpose of described actions, through the use of words such as "aim", "goal" or "purpose". The case reproduced below is a typical example:

> Summarization (CL): Recapitulating steps of the solution of the problem.
> Summary, synthesis (FL): Phase whose purpose is to present and to discuss students' ideas and productions after an individual work phase, or to identify important points to remember at the end of a session.

Differences are also observed between the examples and non-examples of the two lexicons. Indeed, some terms in CL have many examples, the extreme case being Teacher's controlled solving of a problem, with 9 associated examples describing different possible configurations and distribution of roles, while in FL, with a few exceptions, all terms have at most 2 examples. In fact, examples in CL are partly used to compensate the limitation of the agreed professional vocabulary to speak about common practices.

In the limited space of this research report, we can neither enter into more details regarding the formal comparison of the two lexicons, nor fully develop the analysis initiated above of the conditions and constraints situated at different levels of the hierarchy of codetermination contributing to the differences observed. All the more that, beyond this formal comparison, our goal is to understand how these lexicons orient the vision and analysis of mathematics classrooms. For that purpose, as explained in the methodological part, we have developed a specific artefact in form of narratives of the Czech and French videos. We briefly present below the questions guiding their analysis and some first insights from this work in process.

The analysis of the narratives

For each video, the narrative artefact agreed by each team appears as a text based on the template's episodes and terms, with the occurrences of lexicon terms systematically highlighted. Each narrative is first the object of a separate analysis, trying to capture the density of lexicon terms, their distribution in the whole lexicon and its different categories. We also investigate what the narrative adds that could not be captured through the lexicon naming system due to its general specification, for instance regarding the mathematics specificity of the lesson or the use of specific tools, but also due to the characteristics of each lexicon, trying to answer questions that emerged from their formal analysis. Do the Czech narratives, for instance, add an inferential dimension to the strict descriptive dimension of CL? If so, how? Considering the fact that the students' perspective is under-represented in FL, is this limitation compensated in the narrative and if so, how? The second step is the comparative analysis of the two narratives produced for each video by the Czech and French teams, considering both the results of each local analysis and the two visions of the same lesson that emerge from these narratives.

The first analyses carried out show that the narratives make a substantial use of the respective lexicons, as shown by Table 3 below for the French case.

Categories / Number of terms (occurrences)	General terms	Nature of tasks	Phases of a session	Forms of pedagogical organization	Mathematical activities	Pedagogical and didactic management
Narrative of the Czech lesson	6 (8)	3 (3)	8 (19)	4 (7)	4 (5)	31 :12-14-5 (98 :34-52-12)
Narrative of the French lesson	2 (2)	5 (9)	11 (37)	4 (12)	8 (10)	33 :12-16-5 (108 :45-47-16)

Table 3: Number of terms and occurrences of FL by categories and sub-categories (last column) in the narratives of the Czech and French lessons

Combining the two narratives, 72 terms among the 115 of FL are mobilized in the description of these two lessons, as such or with some variation due to the inscription in a text. Moreover, despite the evident differences between the two lessons in terms of content and management, interesting regularities are observed. Roughly speaking, the two narratives describe the classroom story mainly through the progression of the mathematical activity, and the respective roles of teacher, students in this progression. This is confirmed by the proportion of words offering precise information about the mathematics at stake, which represent about 27% and 26% of the respective descriptions.

In the narratives produced by the Czech team, a substantial use of terms from CL can be also traced. For instance, 41 items from CL are mobilized in the narrative of the French lesson. However, 22 from them are not terms from CL but items used in CL as examples. In contrast with the French narratives and in line with CL characteristics, the narratives focus on pupils´ and teacher´s actions, without inferring about their purpose. We can see repeating sequences of CL items, which offer an opportunity to analyse the lesson from the perspective of didactic patterns. Also in contrast with the French case, mathematics items are very limited, representing less than 7% of the descriptions.

DISCUSSION AND PERSPECTIVES

From its start, the Lexicon project has been a very challenging project, notably because today still the teaching profession is not a full profession, with the socially agreed professional lexicons attached to these. During the two first years of the project, nine national lexicons have been produced. These are today reasonably stabilized and validated, opening the way to the comparative dimension of the project. The research presented here is just one form among many others that this comparative dimension can and will certainly take. Moreover, it only involves two countries. However, it has allowed us to identify dimensions that can contribute to the distance between national lexicons, and to link some of these to institutional and cultural characteristics; it has allowed us to design and test, in a case study, methodological tools, the combination of which seems promising to support the comparative enterprise.

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EXPLORING ISRAELI, ENGLISH, AND AUSTRALIAN STUDENTS' CONCEPTUALISATIONS OF FUNCTION

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This international comparative study sought insights into students' development of different conceptions of function from Years 9 to 12 using their written responses to a series of tasks. Forty high-achieving students from each country (ten per year level) were prompted to respond to several fictitious students' views on what a function is, and then to provide their own definition. The data analysis examined the students' responses for evidence of students' meanings for the word 'function' as well as dominant and contributing conceptualisations. This paper discusses some of the similarities and differences found between the three cohorts, and shares some initial conjectures about the influences of curriculum and teaching approaches in each country on students' concept image development.

The idea of 'function' evokes varied meanings, interpretations, and representations across different areas of mathematics and for different people – students, teachers, mathematicians, and mathematics educators. Students' mental concept images of functions may be different from mathematical definitions (Vinner & Dreyfus, 1989) and educators may hold different norms for what "students' understanding of function" looks like. "We avoid speaking as if there is a standard, generally accepted meaning of function against which others should be compared" (Thompson & Carlson, 2017, p. 421) but we can examine multiple meanings and conceptions to learn more about the different facets of function and how these might develop over time. Several studies of undergraduate students have shown that many bring limited meanings and concept images for function (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Since concept images are influenced by prototype examples (Schwarz & Hershkowitz, 1999) and examples to which students have been exposed at school (Vinner, 1983) it is reasonable to infer that curricular emphases and pedagogical approaches, used in a particular school context, influence the meanings students develop for function. This qualitative study sought to explore this influence with forty Years 9 to 12 students and their teachers from each of three different countries (n = 120) to gain insight into the nature and multiplicity of conceptualisations of functions the students had developed. This paper addresses the following research question: *How might students' conceptualisations and definitions of function relate to their experience of curricular and pedagogical emphases in their context?*

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BACKGROUND

Theoretical perspectives on the concept of function

The concept of function is of fundamental importance in the learning of mathematics and has been a major focus of research attention for several decades (e.g., Sfard, 1991; Vinner & Dreyfus, 1989). Various approaches have been offered to explore the concept in mathematics teaching and learning: action/process; point-wise/global; correspondence/covariation, with process, global and covariation being more closely associated with eventually understanding functions as objects. Two main approaches are correspondence and covariation (e.g., Confrey & Smith, 1995). The correspondence approach associates a unique y-value with an input x-value, thus building a connection between x and y. It is emphasised in teaching methods such as mappings and input-output models, and could be said to focus on a point-wise perception of functions as the emphasis for learners at first is on the relation between particular instances or subsets of the variables, either as numbers, objects or expressions. A covariation approach to functions involves understanding the manner in which a change in one variable is related to a change in another, or how those variables change together (Confrey & Smith, 1995; Thompson & Carlson, 2017). A dual view of functions as representing both correspondence and covariation allows the student to better understand actions performed on a function, such as a 'shift' translation (e.g., changing $f(x)$ to $f(x + 3)$) or taking a derivative.

Vinner and Dreyfus (1989) showed that students' mental images of functions may be different from mathematical definitions. They are frequently based on a concept image, which refers to "the set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them" (p. 356). Students have difficulty abandoning action and pointwise views of functions if that is how they first met them (e.g., Breidenbach et al., 1992). We are most interested in the effects of the formal introduction of the word 'function' at different stages in students' mathematics. The essential role of the word in concept development, as sign, has been conjectured and studied since Vygotsky (1986). The association of concept images with the mediated role of the word early in the learning of functions may well facilitate richer connections and meaning during students' development than an apparently fragmented set of activities building concept images that are associated by a unifying term later in the curriculum. Curricula are designed with a particular conceptual progression in mind, so it is not surprising if students display an order of learning similar to their curriculum. Thus curriculum, as well as teacher decisions, is of interest in any discussion about learning.

Curriculum and teaching approaches on function in the three countries

The following brief overview of the curricula on functions from each of the three countries were informed by content from curriculum documentation, typical textbooks used, and the perspectives shared by the teachers in the study on their pedagogic approaches for functions at different year levels. As a country with a centralised educational system, the Israeli school curriculum is regulated and textbooks need official approval. The national curriculum on functions (Ministry of Education, 2009) introduces the word ‘function’ in the context of numerical functions and a variety of representations. It is defined as the matching of a unique number to each number we choose. The idea of ‘mapping’ to connect ordered pairs is not used. The rate of change of one variable in relation to the other is calculated using quotients of change on an interval, and can be constant, zero or changing; the change can be increasing or decreasing. Linear functions are presented as a special class of functions. Students discuss graphs and rates of change explicitly in the context of realistic phenomena. Function notation is introduced for 12 to 14 year olds, either before or during a formal treatment of linear functions.

The curriculum in England (Department for Education, 2014) does not explicitly introduce the word ‘function’ and its notation until Years 11/12 with higher attaining students for transformations of linear and quadratic functions. Earlier introduction to functions is through generalising linear and quadratic sequences and studying mappings that connect domain values to range values. These mappings then give rise to data tables that can be used to plot graphs, starting with linear graphs. Mappings, sequences, and data tables might all be used to generalise an underlying sequential ‘position-to-term’ rule. The teachers from England explained that students meet input-output models in Year 7, for example, ‘function machines’. The idea of mapping between sets is developed in Year 12. ‘Rate of change’ is discussed by older students in the context of non-constant rates of change for quadratics, and finding derivatives from first principles.

As with the curriculum in England, the Australian curriculum also introduces functional concepts in a more informal way. In the upper primary years, students are to continue and create sequences, and describe the rule (Australian Curriculum Assessment and Reporting Authority [ACARA], 2014). Years 7/8 textbooks typically cover functional concepts within different topics, such number patterns, straight-line graphs, equations, and ratio and rates. The curriculum does not refer to the word ‘function’ until the advanced Year 10 level and function notation is introduced in the higher-level Year 11 mathematics units. The Australian teachers explained that they expected younger students not to have developed a clear idea of function yet, and for advanced students at Years 10 to 12 to describe mapping concepts, with Year 12 students holding multiple views of function.

These generalisations provide a framework for analysing differences among Israeli, Australian and English students’ responses to our task. We posed the same questions in each country, and prepared these with input from teachers from both countries in order to ensure cultural validity and curriculum fidelity. The study aims to learn about school students’ conceptualisations of function as briefly described above.

RESEARCH DESIGN

The task discussed in this paper was designed (as part of a larger survey; see Ayalon, Watson, & Lerman, 2015a,b) to prompt students' thinking about a variety of conceptions of functions. We chose five meanings for function to reflect the variety of conceptualisations they might have met and constructed, according to the teachers, curricula and research (Watson, 2013):

- Arthur said: I see functions as input-output machines, which receive some input and give an appropriate output.
- Ruth said: I see function as a mapping of each element of one set to exactly one element of a second set.
- Ian said: Functions for me represent relations between variables.
- Naomi said: A function shows how one variable changes in relation to another variable.
- Liz said: I see functions as expressions to calculate y-values from given x-values. For example, $y = 4x + 7$.

Students were asked to read each of the five ideas, and to write their response to it in the following way (illustrated with respect to Arthur's statement):

Which one of the following statements reflects your thinking about Arthur's description of functions? Mark your response and explain your choice.

- o All functions fit Arthur's description.
- o Some functions fit Arthur's description.
- o Arthur is wrong.

Explanation: ____

These ideas are not mathematically distinct, and the use of fictional characters to present them was intended to engage students in thinking about each description as they would if someone had said it in class. The second part of the task asked for the student's idea of function: Now, after you have responded to the students' ideas, write what is a function for you.

The task was given in each of the to two relatively high achieving classes from Years 9 to 12. Socio-economic backgrounds of all schools were similar relative to their national norms. The sample space was opportunistic, which is adequate for our exploratory research purposes. We use random anonymised samples of 10 scripts per class, which totalled 40 scripts from each country, 120 altogether. Our analysis was qualitative, designed to identify characteristics of student responses that might give insight into their understanding of functions. We analysed all their explanations for mathematical content and categorised the ideas they expressed. The analytical process was iterative and comparative, and required several passes through the whole data. Students' explanations for their choices were often more informative than the choices themselves. The second stage of the analysis, which we report here, included a holistic view of the set of responses from individual students, taking into account their five reactions to the given ideas and their response to the second part of the task (i.e., writing what is a function to her/him). This led to characterising an individual's 'dominant' idea, that is one that appeared most frequently in their response to the whole task, and other contributing ideas that occurred less frequently and either in tandem or separately from the dominant idea. Seven ideas emerged in our data set either as dominant or contributing ideas. Five of them were presented explicitly in the task ('input-output machine', 'mapping', 'relations between variables', 'covariation', and 'algebraic calculation'). Two additional ideas were 'patterns in outputs' and 'domain'. 'Covariation', 'patterns in output' and 'domain' were found to be contributing ideas only.

RESULTS AND DISCUSSION

Figure 1 presents the data analysis for the ten students' responses from each country and at each level (three tables side by side). Each table row of seven boxes represents one student with a dominant idea (if any) being shaded black and contributing ideas (if any) being shaded grey. Examples of students' response will be presented at the conference as space prohibits their inclusion here. A noticeable feature of the Israeli students' overall responses is that the students appeared to hold multiple views of function across the four year levels. The dominant idea was clearly 'relations between variables', although 'input-output machine' and 'covariation' were major contributing ideas. The English data showed a different pattern with most students holding one dominant idea and one contributing idea throughout the four year levels. For Years 9 to 11, 'input-output machine' was the dominant idea and 'algebraic calculation' the contributing idea with very few exceptions. This pattern shifted in Year 12, with 'mapping' becoming the dominant idea and 'input-output machine' the main contributing idea. Different again to the Israeli and English data, the Australian responses showed little noticeable development of any ideas about function in Year 9, and then a strong preference for the 'input-output machine' idea in Year 10. In Years 11 and 12, the students' responses then demonstrated multiple ideas about function, but no clear pattern of only one major dominant idea, and more similarity with the Israeli data than the English data. 'Mapping' at these two year levels was the most frequent dominant idea followed by 'relations between variables'.

It appears that the Israeli students have developed a more comprehensive conception of function at an earlier age than English and Israeli students. The English students seem to have developed input-output (dominant) and algebraic (contributing) conceptions side-by-side and hold to these with little variation until a shift to mapping (dominant) and input-output (contributing) conceptions in Year 12. This pattern matches the expectations of the English teachers based on their teaching approaches. The Australian responses showed that students' understanding of the concept of function appears to be delayed, compared to both the Israeli and English students. It is not until Years 11 and 12 that they hold multiple views of function. A number of these students' responses referred to 'one-to-one', 'many-to-one', and 'one-to-many' mappings and also the 'vertical line test'. These concepts are explicitly outlined in the content for the

Year 11 Australian curriculum for the Mathematical Methods unit (ACARA, 2014) studied by the student participants and were also mentioned by the teachers.

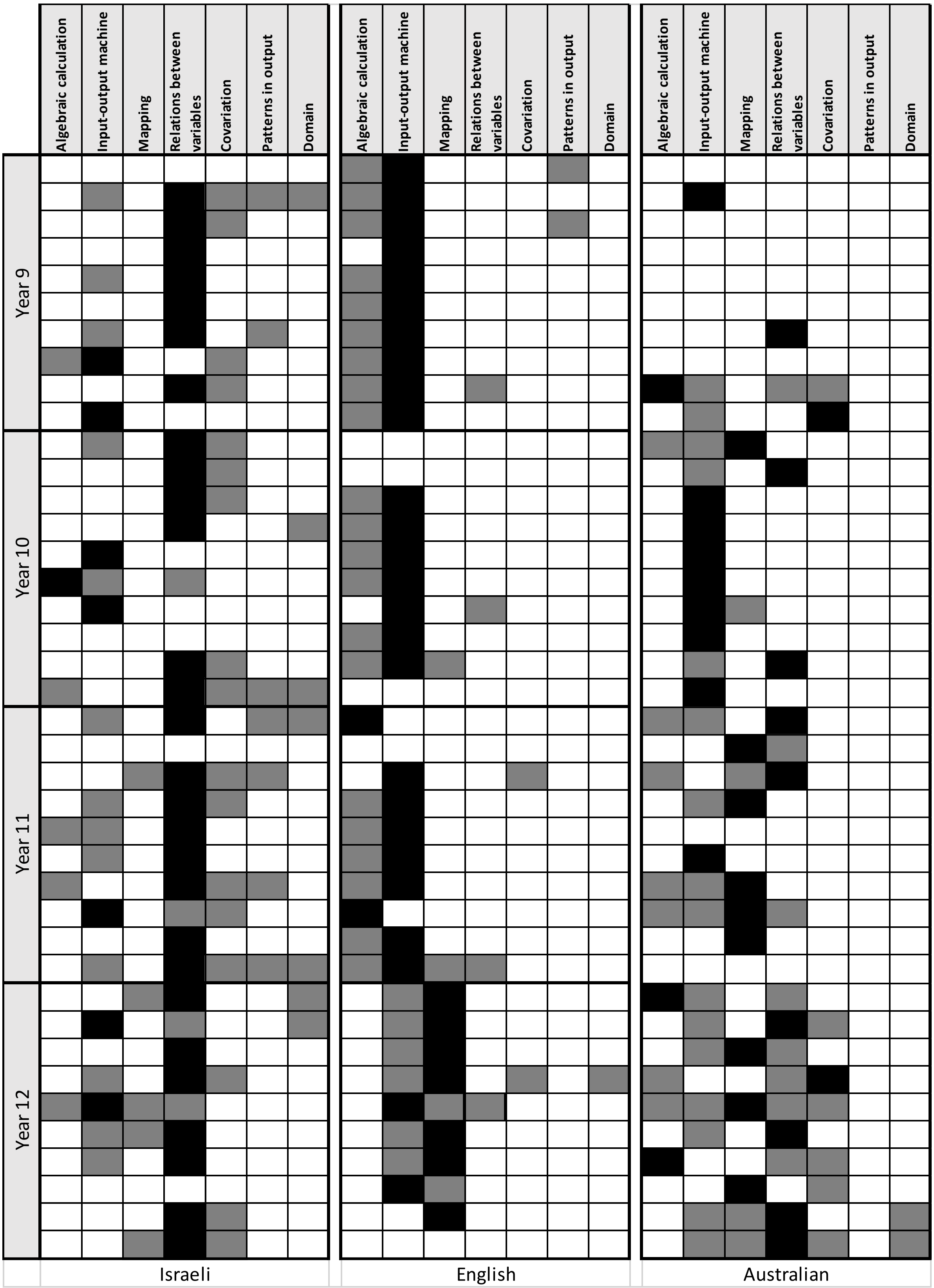

Figure 1: **S**tudents' dominant (black) and contributing (grey) conceptions of function (n = 120)

Overall these patterns demonstrate that not only that curriculum content and teaching approaches in a particular context can provide valuable insight into students' development of mathematical ideas at different year levels, but also that students are capable of holding multiple views of function at Year 7. Even though the English and Australian curricula are considered similar in structure, these results highlight the noticeable differences in students' development of conceptions based on the teaching approaches and language used in each context. It seems that the idea of an input-output *function* machine, which is more widely used in English schools than Australian schools, leads to students connecting this idea with 'function' even though they are not formally taught about functions yet. The Australian approach of introducing functions concepts across several topics but without reference to the word 'function' or the term 'function machine' seems to result in students' lack of any particular conception of function until much later. The Israeli curriculum is noticeably different, and from the students' responses, it seems apparent that it is possible to develop students' ideas about function more explicitly and connectively at an earlier stage.

Our findings are not a claim about these students' complete knowledge about functions, because all we have analysed is their responses and explanations to our tasks, and some of the context that has contributed to their concept images and responses. Nor can we generalise about countries on the basis of a small and opportunistic sample, probed qualitatively. Instead we believe that they provide one window into their understanding and suggest influences on their development. Vygotsky (1986) retained the idea that word is the beginning of cognition, in the same sense that pointing to or pointing out an object begins the development of sense and meaning. We might conjecture, then, that having a name, a label, in this case function, early in their learning, around which concept images are built over time, is a support for students in their learning and may well be critical in the early development of the more coherent structure of understanding as seen in general among the Israeli students than the English and the Australians. The word is the beginning of cognition, which then calls for all the work of the teacher and the resources to bring meaning, through concept images and activities, building towards sense. The findings have implications for any discussion of whether it makes more sense to name mathematical ideas formally after students have experienced them, or whether to name them before. The association of concept images with the mediated role of the word 'function' early in the learning of functions may well facilitate richer connections and meaning during students' development than an apparently fragmented set of activities about functions that are not associated by a unifying term until much later, and only for students who continue on to study calculus. This is not a question of whether formal treatment has to precede exploration and elaboration, but of whether and how students are supported to make connections, through language, between various mathematical ideas.

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SOCIALISATION IN FOUR SECOND-LANGUAGE MATHEMATICS CLASSROOMS

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There exist various situations in which mathematics is learned in a second language, including in the context of immigration, indigenous languages or immersion programs. Little research on this topic has examined mathematics learning across a range of such contexts. In this study, I collected ethnographic data in four different second-language mathematics classrooms. Drawing on a view of learning as socialisation and a Bakhtinian conceptualisation of mathematical discourse, I report on a number of socialisation situations that arise in these classrooms. The findings offer new insights into the nuances of learning mathematics in a second language in different contexts.

INTRODUCTION

The linguistic diversity of mathematics classrooms has been increasingly recognised in mathematics education research (e.g. Barwell et al., 2016). This work covers a wide range of contexts in which language diversity arises in mathematics classrooms. In this research report, I focus on second-language learners—students who are seen in the context of their school system as learning mathematics through a second or additional language. The objective of the research is to understand how learning in a second language influences the process of learning mathematics.

Research on second-language learners of mathematics shows that in the right conditions, they can outperform monolingual students in mathematics (e.g. Clarkson, 2007; Ní Ríordáin, & O'Donoghue, 2009). In other conditions, performance may be lower than expected (e.g. Clarkson, 2007). Research in a variety of contexts of language diversity, including second-language classrooms, shows that learners in such contexts make use of a wide variety of discursive resources to participate in and make meaning in mathematics. These resources include different languages (e.g., Setati, 2005; Planas & Setati, 2009), genres (e.g., Barwell, 2005), narratives (Barwell, 2005), gestures (e.g., Moschkovich, 2008), and diagrams (e.g., Moschkovich, 2008). Learning and teaching mathematics in such contexts creates challenges for students and teachers, relating to, among other things, the use of students' home languages (e.g., Adler, 2001; Parvanehnezhad and Clarkson, 2008; Setati, 2005), and the role of students' informal mathematical language (e.g., Adler, 2001; Khisty, 1995).

Existing studies have been largely situated within single linguistic contexts. Most classroom-based research has been conducted in single classrooms, or in a small number of similar classrooms, so that findings are limited to the particular linguistic

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 129-136. Singapore: PME.

context in which they were conducted. In the study described in this research report, I included four different kinds of second-language mathematics classroom (described below).

THEORETICAL FRAMEWORK: SOCIALISATION INTO MATHEMATICAL DISCOURSE

Learning mathematics can be understood in terms of students' socialisation into the discourse of mathematics, where this discourse involves various semiotic resources (words, symbols, diagrams, gestures, etc.) and forms of argumentation (posing problems, conjecturing, reasoning, etc.). Duff (2002), defines socialization as:

> the linguistic and interactional processes that mediate newcomers' participation in routine cultural practices [...] and facilitate their developing competence and membership in discourse communities. (p. 290)

Learning a second language can also be understood as a process of socialisation into the proficient use of the language practices of the classroom language. In addition to the teacher, students are also agents in the socialisation process, so that there is a reflexive relationship between the learning of individuals and the learning of the class as a whole (Duff, 2002).

Mathematical discourse, however, is not simply a neutral medium for the expression of students' ideas—discourse is itself a complex social phenomenon. To theorise the nature and role of discourse within the socialisation process, I draw on Bakhtin's (1981) work and, in particular, his concept of heteroglossia. According to Bakhtin, every utterance involves a unique combination of words, accents, style and so on, and so contributes to the constant diversification of discourse. Busch (2014) distinguishes between three dimensions of heteroglossia: multi-discursivity, multi-voicedness and language diversity. Mathematical discourse displays multi-discursivity, in the discourses of different subdomains of mathematics, or in the differences in school, university and academic mathematical discourses. Multi-voicedness arises in mathematics classrooms through students learning to use words introduced by their teacher, the textbook, or the curriculum, so that multiple voices are present in each utterance. Language diversity refers to the presence and interaction of multiple languages.

While heteroglossia is always present, so is the tendency of discourse to standardise, and, especially in the case of languages, to be seen as rule-governed. These two tendencies within all discourse creates a tension that shapes every utterance, as the utterance both conforms to the norms of the discourse and reflects the particularities of the context in which it is uttered (Bakhtin, 1981). This perspective on language is consistent with a view of learning as socialisation:

> Language is not a neutral medium that passes freely and easily into the private property of the speaker's intentions; it is populated—overpopulated—with the intentions of others.

> Expropriating it, forcing it to submit to one's own intentions and accents, is a difficult and complicated process. (Bakhtin, 1981, p. 294)

That is, we learn to use language through a process of coming to use the words of others. This view describes well the sometimes "difficult and complicated process" of learning mathematics in a second language.

RESEARCH DESIGN AND METHODS

The study involved the collection of ethnographic data in four different elementary school second-language mathematics classrooms in Canada (a country with two official languages, English and French): a *mainstream* class in which some second-language learners were present; a *sheltered* class for students considered to be learning English as a second language, who were mostly speakers of Cree, an indigenous Canadian language; a "*welcome*" class for new immigrants to Canada, from all over the world, where the main goal was for students to learn French; and a French-*immersion* class. All except the French immersion class were for students aged 10-12 years; the immersion class was for students aged 8-9 years.

Data collection drew on ethnographic methods, including classroom observation, including participant observation, as well as interviews, audio-recordings, collection of copies of students' work and photos of classroom artefacts. Fieldnotes were made during observations and a short summary was prepared after each visit to three of the four classes (I did not introduce this practice until after the completion of data collection in the mainstream class, which was the first to participate). Observation periods varied according to the constraints of school timetables, access protocols and other factors. I visited the mainstream and sheltered classes each for most of an academic year, and the welcome and immersion classes each for 2-3 months towards the end of the school year.

The main approach to data analysis consisted of reviewing summary reports and fieldnotes to identify situations in which the tension relating to heteroglossia and standardisation was particularly salient (see Barwell, 2014). These situations were important sites for the socialisation of students into the discourse of mathematics in the language of instruction (either English or French). In this report, I present the main socialisation situations that I identified across the four classes.

RESULTS: SOCIALISATION SITUATIONS IN FOUR CLASSES

I identified seven common socialisation situations. This set of situations is not exhaustive but represents the most common ones. I have organised them according to the three dimensions of heteroglossia, since for each situation, one of the dimensions was most significant, although there is overlap between the dimensions.

Language diversity

Multiple language use was observed in all classes and represented an important socialisation situation. In the mainstream class, other languages were rarely heard, and never in connection with mathematics. In the other three classes, however, at least two languages were regularly used. In the sheltered class, the students often interacted in Cree to talk about mathematics, such as to explain ideas or show each other what to do. In the welcome class, students often asked to use another language, such as Spanish, although the request was always declined by the teacher. In the immersion class, where French was the target language, students routinely also used English, often mixing the two languages, as in the following examples from some work on mass (translation is shown in brackets and in italics):

> "I'm gonna go like this (lifts like a hand weight). I'm gonna toss it up in the air."
>
> "Ça c'est deux grammes? Oooohhh!" [*That's two grams? Oooohhh!*]
>
> "Quatre pommes equals that?" (1kg) [*Four apples equals that?*]
>
> "Est-ce que c'est ça le plus heavy?" [*Is it that one, the most heavy*]
>
> (Visit report, 3 May 2012)

In the three classes in which more than one language was heard, the teachers explicitly requested students only to use the language of instruction.

A second socialisation situation was the *occurrence of non-standard accents, pronunciation or orthography*. This situation was not observed in the mainstream class. In the sheltered class, I noticed that the Cree-speaking students spoke English with an accent, but this was never commented on. In the welcome class, the teacher actively corrected students' pronunciation. In the following example, from a unit of work on geometric properties, a student answers that a particular shape is "convex", but with a Spanish accent. The teacher then rehearses the pronunciation with him:

Student 1:	conbexe [*conbex*]
Teacher N:	non: (.) convexe ok redis-le (.) non-convexe [*non (.) convex ok say it again (.) non-convex*]
Student 1:	non (.) con (.) bexe non-converse? [*non (.) con (.) bex non-converse?*]
Student 2:	non (.) con-vexe [*non (.) con-vex*]
Teacher N:	non-convexe con (.) v (.) v (.) vexe [*non-convex con (.) v (.) v (.) vex*]
Student 1:	non-convexe? [*non-convex?*]
Teacher N:	ouais c'est pas un B ah tu comprends? [*yes it's not a B ah you understand?*]

(Transcription from recording, 10 May 2010)

In the immersion class, non-standard pronunciation was often revoiced by the teacher in a more standard way.

These socialisation situations reflect the tension between the language diversity in each class and the expectation that only one language should be used. This tension was reflected in different ways, from the silencing of other languages in the mainstream class, to the mixing of French and English in the immersion class. Socialisation practices included explicitly asking students to use English or French, explicitly forbidding them from using other languages, ignoring the use of other languages, and revoicing non-standard pronunciation or utterances in a non-official language.

Multi-discursivity

There were several socialisation situations relating to various discourses of mathematics. One situation was *when students encountered genres particular to the mathematics classrooms*, such as word problems, textbook explanations, definitions and mathematical questions. In the mainstream class, for example, students discussed three different definitions of a prime number and, at the suggestion of a student, wrote them in their “math dictionaries”. In the sheltered class, I noticed that the students struggled with text-rich word problems, in some cases not recognising the cultural context of the problem (Barwell, 2014). In the welcome class, the teacher took the students through a series of activities through which a definition of a polygon was introduced, given meaning and rehearsed. These activities included:

- Informal sorting activities in which the class had to deduce the criterion used by the teacher.
- Students sorted a set of mixed shapes and explained their criteria.
- The teacher drew two groups of shapes on the blackboard—polygons and non-polygons—and asked students to deduce what criteria distinguished them.
- A worksheet that included a formal definition of a polygon and a classification task using the definition.
- Following review of the worksheet, the teacher asked a student to define polygon and another to define non-polygon.

In these situations, students had to interpret and respond to standardised genres in appropriate ways. Their own less formal forms of expression of mathematical thinking were regularly corrected or reformulated or students were prompted to reformulate them.

Additional socialisation situations relating to mathematical discourses involved *explicit attention to features of mathematical discourse*, such as vocabulary or morphology, and the *use of gestures* in mathematical interaction. In the mainstream class, during work on a complex problem about television schedules, the teacher initiated a class discussion about the meaning of consecutive, which was particularly crucial for the completion of the problem. The class and the teacher generated several examples and synonyms for this word. In the welcome class, the teacher devoted much attention to vocabulary during the geometry unit. Vocabulary items covered included polygon, convex, closed, curved, and the names of various shapes. The students often used informal alternatives, which the teacher revoiced using the standard terms. This teacher also responded to students’ frequent use of gestures by asking them to explain in words. At one point, she covered her eyes to encourage students to go beyond gestures and pointing. In the immersion class, the teacher also highlighted important vocabulary, including kilogram, numerator, denominator, the names of fractions, right angle, acute angle, obtuse angle, horizontal, vertical, reflection, and rotation. In this class, I observed attention to other linguistic features, such as the word endings used in the names of fractions in French:

> "cinquième...est-ce que c'est un sur cinq ou cinq morceaux?" [*fifth...is it one on five or five parts?*] Students were unsure, and P explained : "-ième est toujours un sur quelque chose" [-*th is always one on something*] (Visit report, 9 May 2012)

This teacher used gestures and physical movement on some occasions, such as during some work on geometric transformations. In these situations, standard vocabulary and students' informal alternatives, including informal names for things and gestures, were all present. Socialisation practices included revoicing informal vocabulary with standard mathematical terms, prompting students to use these terms, opportunities to use common mathematical genres, and the use of prompts to interpret and respond to word problems or other genres.

Multi-voicedness

Two socialisation situations were related to the presence of multiple voices: occasions when *students needed to explain their mathematical thinking*, and *moments of reduced participation*. In the mainstream class, second-language learners sometimes struggled to articulate their thinking:

Teacher L:	number two Darryl, where does it [the decimal point] go?
Darryl:	(indicates where)
Teacher L:	how do you know Darryl?
Darryl:	I just know.
Teacher L:	you know what if you write that on your exam, what do you get? (Fieldnotes, 10 December 2008)

The teacher's response represents a clear example of socialisation, invoking institutional requirements relating to assessment to underline the importance of elaborated explanations of mathematical thinking. In the sheltered class, I noticed that the students were generally able to solve word problems orally but struggled much more to complete written solutions to 'show their work'. In both classes, students were often silent or participated minimally with single-word responses to the teacher's questions. In the welcome class, students also expressed difficulty in explaining their thinking in a suitably mathematical manner in the language of instruction. For example, during the geometry unit, when the teacher asked another student 'what is a parallelogram?' she replied "I know but I don't know how to say it" (Visit report, 17 May 2010). In the immersion class, students were often called on to explain their thinking (in French); to do so, they often mixed English with their explanations, which the teacher would reformulate in French.

In these situations, the expectation that students should explain their thinking in a particular language and using particular aspects of mathematical discourse was in tension with their potentially wider range of informal forms of expression. In general where the requirements of a standard form of expression were stronger, the students were more constrained. Socialisation practices included requesting explanations, reformulating students' explanations in more formal mathematical discourse and/or in the language of instruction, and citing the requirements of assessment tasks to justify and encourage more elaborated explanations.

DISCUSSION AND CONCLUSION

My analysis has identified a number of socialisation situations in which second-language learners of mathematics are socialised into mathematical discourse (and hence into mathematics) in particular ways that are directly related to their status as second-language learners. These socialisation situations include: occasions when students used languages other than the language of instruction, or used non-standard pronunciations or accents; occasions when students encountered genres specific to school mathematics, such as word problems and definitions; occasions when specific features of mathematical discourse, such as vocabulary, became a focus of attention; the use of gesture; the production of mathematical explanations; and occasions in which students' participation was reduced. These situations arose at moments in which the tension between heteroglossia and standardisation was particularly salient. As a result, particular practices were deployed, often by the teacher, but sometimes by students, that socialised students into expected ways of talking about mathematics.

Some differences were apparent in the various second-language classrooms. In particular, in the mainstream class, there was little explicit recognition of second-language learners' other languages. In the immersion class, in contrast, students freely used both French and English, although the teacher was consistent in using French. In general, the welcome and immersion classes displayed more attention to language and to the discourse of mathematics than the mainstream or sheltered classes.

These findings suggest that the learning of mathematics is influenced by students' status as second-language learners, with a range of situations arising across different second-language contexts. These situations, however, played out differently in the different contexts, suggesting that the nature of the context has an impact on students' learning. Second-language mathematics classrooms, and more generally, language diverse mathematics classrooms, are all different, and learning mathematics in such classrooms varies from one context to another. More research is needed to better understand these contextual influences.

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A 2-DIMENSIONAL MATRIX FOR ANALYZING MATHEMATICAL TASKS

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This paper presents a 2-dimensional matrix for analyzing mathematical tasks. One dimension is the TIMSS' levels of cognitive domains and the other dimension is the type of mathematical tasks in terms of number of answers (Isoda and Katagiri, 2012). The proposed 2-dimensional matrix may be used as an analytic tool for analyzing the distribution of mathematical tasks in a textbook as well as to compare the distribution of mathematical tasks among textbooks. It can also be used as a pedagogic tool for designing mathematical tasks in standardized exams. This paper also illustrates an example how to use the framework.

INTRODUCTION

It is widely accepted that textbooks have major influence on classroom practice (Valverde, Bianchi & Wolfe, 2002) and TIMSS studies show that textbooks are present in almost every participating country, regularly used in instructions, and considered by teachers as primary source of information (Houang & Schmidt, 2008). Statistically significant relationships have also been found between textbooks and classroom instruction (Schmidt et al., 2001). For these reasons, textbooks have gained the attention of mathematics educators and researchers.

In a survey by Fan, Zhu, and Miao (2013), it was found that there have been a growing number of researches about mathematics textbooks in the past three decades. Most studies done involved the role of textbooks, textbook analysis and comparison, and textbook use. The textbook analysis and comparison was divided into the following categories: (1) mathematics content and topics; (2) cognition and pedagogy, gender, ethnicity, equity, culture and value; (3) comparison of different textbooks; and (4) conceptualization and methodological matters.

Earlier textbooks studies focused on comparison about content, structure, and performance expectations; e.g Stigler (1986). Some of the latter studies involved examining how a topic is taught (Mayer, Sims, & Tajika, 1995), comparison of a particular concept (Powell, 2012), placement of topics (Fuson, Stigler, & Bartsch, 1988), and discourse (Ronda, 2015). The most comprehensive comparative study about textbooks to date was done by TIMSS which compared the content, structure, and performance expectations of more than 400 textbooks from 40 countries which

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 137-144. Singapore: PME.

participated in TIMSS (Valverde et al., 2002). In the TIMSS study, textbooks were analyzed in terms of blocks. Blocks include narratives, graphics, activities, and worked examples. Although numerous studies have been done about content topics, less attention has been placed about the problems or tasks presented in textbooks (Li, 2000). Thus, this study presents a new analytic framework for analyzing mathematical tasks. This framework aims to answer the following questions:

(1) How can we analyze textbook tasks using the proposed framework?

(2) How can we use the framework to compare the tasks on two or more textbooks?

THEORETICAL UNDERPINNINGS

A mathematical task is a set of problems or a single complex problem that focuses students' attention on a particular mathematical idea (Stein, Grover, & Henningsen, 1996). In the TIMSS video study, students in all the participating countries spend at least 80% of their time working on mathematical tasks (Hiebert et al., 2003). Since mathematical tasks influence student learning (Doyle, 1988), and many mathematical tasks used in the classroom are taken from textbooks, it is important to pay attention to their quality. As Doyle (1988) argues, "the work students do, defined in large measure by the tasks teachers assign, determines how they think about a curricular domain and come to understand its meaning" (p.1 67).

Classifying Mathematical Tasks by Cognitive Demand

Several frameworks have been developed for analyzing mathematical tasks in terms of cognitive demand. Stein et al. (1998), for instance, developed a framework for analyzing mathematical tasks using four levels: memorization, procedures without connections, procedures with connections, and doing mathematics. Porter (2002) developed a very similar framework but categorized tasks into four levels: memorize, perform procedures, communicate understanding, solve non-routine problems, and conjecture/generalize/prove. Porter's framework, however, was not designed for individual tasks but rather for "descriptors of mathematical topics" (p.4).

A more simplified framework was developed by National Assessment of Educational Progress (NAEP, 2009). NAEP classified tasks into low, moderate, and high complexity items. Low complexity items expect students to recognize concepts or procedures. Items under this category typically specify instructions that can be followed mechanically. Items in the moderate complexity category involve more flexibility of thinking and choice among alternatives than to those in the low-complexity category. The students are expected to decide what to do and how to do it. High-complexity items expect students to use reasoning, planning, analysis, judgement and creative thought.

The TIMSS Cognitive Domain follows another classification of mathematical tasks which is very similar to that of NAEP's. Over a period of 20 years, it has been simplified into three: Knowing, Applying, and Reasoning (Mullis et. al, 2009).

Knowing involves recalling facts, concepts, procedures; Applying involves application of knowledge and conceptual understanding to solve problems; and Reasoning involves solving non-routine, complex contexts, and multi-step problems which expect students to reason and justify their answers.

Classifying Mathematical Tasks by Answers

Mathematical tasks can either be multiple-choice or constructed response. Each has its own advantages and disadvantages. Generally, multiple-choice questions are efficient to use for a large number of test takers and automatic scoring can be employed (Dufresne, Lenard, & Gerac, 2002). On the other hand, constructive response items eliminate random guessing, unintended corrective feedback, and working backward (Bridgeman, 1992). In TIMSS, both multiple-choice and constructive response tasks were employed, however, there was no classification of mathematical tasks in terms of answers.

Several frameworks were developed to compare mathematical tasks in terms of answers. Li (2000) created a framework with three dimensions. The dimensions are mathematical feature, contextual features, and performance requirements. In this framework, mathematical features and response type classified the tasks in terms of answers as single or multiple computation procedures. It also classified the response types as numerical answer only, numerical expression only, explanation or solution only.

Bennett, Morley and Quardt (2000) presented three-response types for a computerized-adaptive admission test. In their framework for constructed response items, response types are categorized into mathematical expressions, generating examples, and graphical modelling. Mathematical expressions requires a single best answer, generating examples are tasks that can have multiple correct answers, and graphical modelling asks for graphical representations of which can have a single or multiple answers.

Isoda and Katagiri (2012) classified mathematical tasks into three types according to number of solutions and answers. The tasks in Figure 1 illustrate this classification.

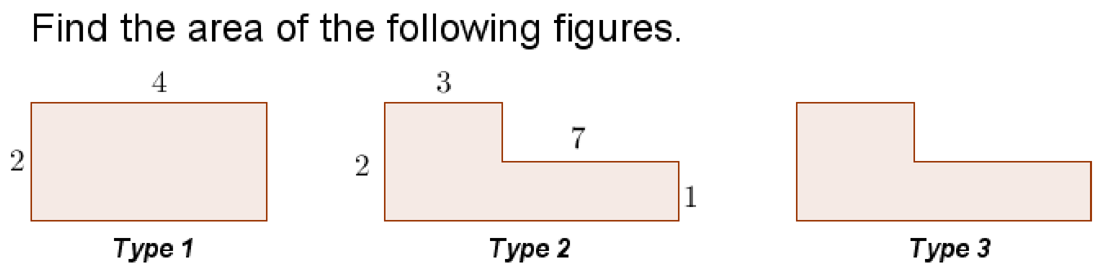


Figure 1: Types of Mathematical Tasks by Answers

A Type 1 task has one solution and one answer. Students who have already learned about area of a rectangle can calculate using the formula. A Type 2 task has various solutions but one correct answer. As shown in Figure 2, students can solve the problem using addition (Solution 1 and 2) and subtraction (Solution 3) of areas.

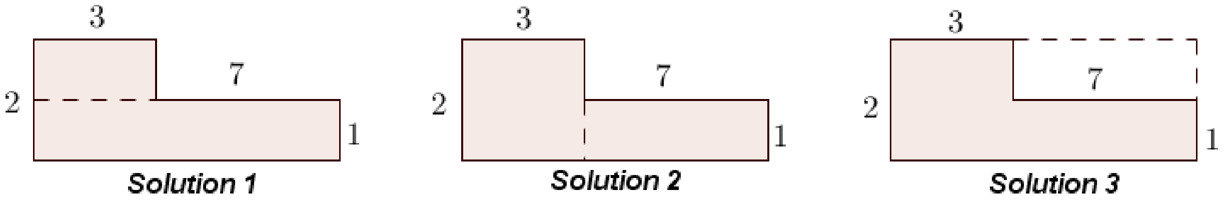


Figure 2: Multiple solutions for finding the area

A Type 3 task has many answers and solutions. Since there are no measurements in the figure, students have to use a measuring instrument. In so doing, their measurements may be different so their answers will depend on their measurements.

THE 2-DIMENSIONAL MATRIX

From the frameworks discussed above, the author has developed a 2-dimensional framework for analyzing mathematics tasks. One dimension is the level of cognitive demand and the other dimension is the number of solutions and answers. In the levels of cognitive demand, among all the frameworks stated above, the author has selected TIMSS' Cognitive Domain because of the following reasons: (1) it was "a consensus developed among many individuals and groups from around the world (TIMSS, 1995)"; (2) it has been improved over two decades of revision and implementation; and (3) the domains have clear demarcations. In terms of the number of solutions and answers, the author has chosen Isoda and Katagri's classification of mathematical tasks because of its clear demarcation.

Using the TIMSS Cognitive Domains and Isoda and Katagiri's classification of mathematical tasks, a 3 by 3 matrix can be created as shown in Figure 3. The columns of the matrix contain the cognitive domains namely Knowing (K), Applying (A), and Reasoning (R). The rows contain Isoda and Katagiri's classification of mathematical tasks namely Types 1, 2, and 3. In this matrix, K1 tasks are generally the easiest (Knowing with one answer and one solution), while the R3 tasks are generally the most difficult (Reasoning with various answers and/or various solutions).

	K	A	R
1	K1	A1	R1
2	K2	A2	R2
3	K3	A3	R3

Figure 3

In what follows, the author will demonstrate the use of the matrix. It will be used as a lens to give the reader a bird's eye view of the comparison of the distribution of mathematical tasks between two textbooks.

METHOD

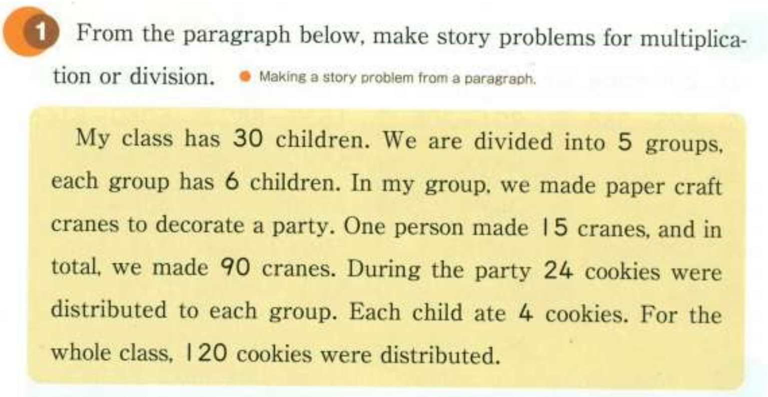
To illustrate the use of the framework, the author compared tasks from Japanese and Philippine textbooks in a lesson on division. It should be clear that the purpose of this comparison is not to make conclusions about the textbooks but rather to demonstrate the use of the framework. In addition, we cannot make any generalization from the findings here because we only selected one lesson from one textbook from each country.

The textbook used from the Philippines is the *Math Learning Materials* published by the Department of Education and used by Grade school students all over the country. The textbook used from Japan is the *Study with Your Friends Mathematics for Elementary School Mathematics* published by Gakkohtosho. This textbook is an English translation from a Japanese textbook. All the tasks in both books were analyzed including developmental tasks as well as exercises. All tasks in both books are constructed response tasks.

To facilitate the comparison, each task under division was coded independently by two coders. In terms of the TIMSS Framework, the tasks were coded as Knowing (K), Applying (A) or Reasoning (R). In terms of Isoda and Katagiri's (2012) Framework, they were also coded as Type 1, Type 2, or Type 3.

DISCUSSION AND RESULTS

In the task from the Japanese textbook shown in Figure 4, students should be able to make connections between the different sentences and link them to be able to create a problem. This is classified as Synthesize category under the TIMSS Reasoning domain. In addition, since multiple problems can be created from the text, it is also classified as a Type 3 problem. Therefore, this task is classified as an R3 task in the analytic matrix.

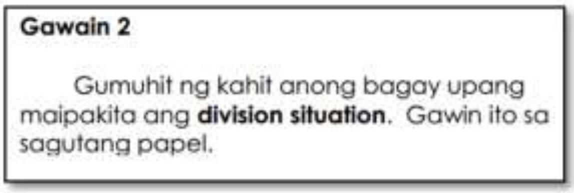


1 From the paragraph below, make story problems for multiplication or division. ● Making a story problem from a paragraph.

My class has 30 children. We are divided into 5 groups, each group has 6 children. In my group, we made paper craft cranes to decorate a party. One person made 15 cranes, and in total, we made 90 cranes. During the party 24 cookies were distributed to each group. Each child ate 4 cookies. For the whole class, 120 cookies were distributed.

Figure 4

Figure 5 is an example of task from the Philippine textbook. In this task, students are asked to draw things to show a division situation. This task is classified as Applying in the TIMSS Cognitive Domain. Since students can generate multiple models, it is classified as a Type 3 tasks. So, the task is classified as A3 task in the analytic matrix.


Gawain 2

Gumuhit ng kahit anong bagay upang maipakita ang **division situation**. Gawin ito sa sagutang papel.

Translation:

Activity 2. Draw things to show a division situation. Do this on your answer sheets

Figure 5: Sample question in Filipino textbook.

Shown in Table 1 is the spread of tasks in the 2 dimensional matrix from both countries. At first glance, it can be seen that in division, the Philippine textbook has relatively more tasks than the Japanese textbook. The bulk of the task in the Philippine text book is in cell K1 and A1 and there is no task under the Reasoning domain. In the Japanese textbook, the bulk of the items are in cell K1 and there are more Reasoning tasks than Applying tasks.

Type	Philippines (N = 112)				Japan (N = 41)			
	K	A	R	**Total**	K	A	R	**Total**
1	*62.5%*	*31.3%*	*0%*	***93.8%***	*56.1%*	*4.9%*	*2.4%*	***63.4%***
2	*0.9%*	*0.9%*	*0%*	***1.8%***	*12.2%*	*2.4%*	*2.4%*	***17.0%***
3	*0.0%*	*4.4%*	*0%*	***4.4%***	*9.8%*	*0.0%*	*9.8%*	***19.6%***
Total	***63.4%***	***36.6%***	***0.0%***	***100%***	***78.1%***	***7.3%***	***14.6%***	***100.0%***

Table 1

In terms of percentage, the tasks in the Japanese textbook are more "well-distributed" than the Philippine textbooks. In the Japanese textbook, 78.1% of the tasks are under Knowing, 7.3% are under the Applying and 14.6% are under Reasoning. In the Philippines, 63.4% of the tasks are under Knowing, 36.6% of tasks are under Applying and no tasks are under Reasoning. In terms of number of answers, 93.8% are Type 1 in the test while 63.4% are Type 1 test in the Japanese textbook.

CONCLUSION

The result above shows that the matrix can be used as an analytic tool to examine mathematical tasks in mathematics textbooks and give a birds' eye view of the composition of mathematical tasks. It can also compare the distribution of mathematical tasks between two textbooks. Of course, the comparison can be scaled up to include more topics and more textbooks.

Secondly, the matrix may be used in the future as a pedagogic tool for developing tasks in mathematics textbooks as well as standardized exams. Writers can use the matrix to determine the appropriate distribution of mathematical tasks in terms of cognitive demand and number of answers before writing the mathematical tasks in the textbooks or standardized exams.

Again, it should be noted that the result does not mean to generalize the quality of mathematics textbooks from Japan and the Philippines since the analysis only involved one topic from a pair of textbooks.

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UNDERSTANDING THE TRAJECTORY OF A TEACHER'S IDENTITY AS AN EMBEDDER-OF-NUMERACY

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Across the curriculum approaches to numeracy have shown promise but present challenges for teachers of subjects other than mathematics. This paper reports on an approach to understanding how teachers can be supported to promote numeracy learning in the subjects they teach. The findings illustrate a sociocultural approach to tracing the trajectory of a teacher's identity as an embedder-of-numeracy that may allow affordances and constraints to strengthening this identity to be revealed.

INTRODUCTION

Being able to cope with the mathematical demands of life is important for individuals and countries in an increasingly globalised world (OECD, 2013). Numeracy (or mathematical literacy), the capacity to do so, encompasses more than proficiency with mathematics and needs to be developed beyond the mathematics classroom (Steen, 2001). One way of promoting numeracy learning in schools that has shown promise is to take an across the curriculum approach (Geiger, Goos, & Forgasz, 2015) where numeracy is seen as the responsibility of *all* teachers. However, this approach requires teachers to identify numeracy learning opportunities in the subjects they teach, design appropriate tasks, and implement these tasks in their classrooms. Although research on professional development interventions that support an across the curriculum approach to numeracy is growing (e.g., Goos, Geiger, & Dole, 2014), research does not seem to have focussed on factors that influence how teachers interpret and translate learning from such interventions into their classroom practices. This issue was addressed in a study that sought to identify ways to support teachers to embed numeracy into subjects across the curriculum.

The study had two aims: (1) to identify how a teacher's identity influences his/her capacity to promote numeracy learning across the curriculum and (2) to investigate how a sociocultural approach could contribute to understanding how to support teachers in this endeavour. The second of these aims provides the focus for this paper. Specifically, the paper addresses the following research question: How can a sociocultural approach contribute to understanding the trajectory of a teacher's identity in the context of promoting numeracy learning across the curriculum?

BACKGROUND AND OVERVIEW

The study was conducted in Australia in the context of the introduction of a new national curriculum that identified numeracy as a general capability to be developed in all school subjects (ACARA, n.d.) and professional standards for teachers that set out

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what teachers need to know and be able to do to promote students' numeracy development (AITSL, 2012). Although some pre-service teacher education programs in Australia have included courses that specifically address numeracy for some time (e.g., Groves, 2001), these have not been widespread. In this context, there is a need to find ways to assist practicing teachers to develop the capacity to exploit numeracy learning opportunities in the subjects they teach, especially if this was not addressed during their pre-service teacher education.

Conducted in two interrelated phases (theoretical and empirical), the study developed and evaluated a sociocultural approach to understanding how teachers could be supported to develop the capacity to promote numeracy learning across the curriculum. Teacher identity, seen by many researchers as providing useful insights into the learning and practices of teachers, was employed as the analytic lens. However, identity is complex, situated, and changes over time (Wenger, 1998). Consequently, one of the challenges was to design an empirical study that captures both the complexity and dynamic nature of teacher identity. Using the situated nature of identity, it is possible to develop a framework that encompasses factors that seem particularly relevant in a given situation. Drawing on an extensive review of literature, this approach was used in the study to develop a framework for *identity as an embedder-of-numeracy* (Bennison, 2016). This framework is useful for guiding the design of empirical studies and provides a snapshot of a teacher's present identity. However, it does not capture how contributing factors shape this identity nor does it capture the temporal nature of identity. Valsiner's (1997) zone theory was employed to complement the identity framework thus overcoming these limitations. The way in which Valsiner's (1997) zone theory could be used to understand how factors that contribute to a teacher's identity as an embedder-of-numeracy interact to produce particular identities has been reported on previously (e.g., Bennison, 2015). This paper builds on this work by addressing the second limitation, that of capturing the temporal nature of identity.

THEORETICAL FRAMEWORK

The framework for identity as an embedder-of-numeracy (Bennison, 2016) developed in the study is organised by five Domains of Influence: Life History, Knowledge, Affective, Social, and Context Domains. These Domains include factors that seem to be particularly relevant for a teacher exploiting numeracy learning opportunities in subjects across the curriculum. The framework was developed through an extensive review of literature in the theoretical phase of the study then evaluated and refined iteratively during the empirical phase.

Valsiner's (1997) conceptualised development as the interactions between three zones: zone of proximal development (ZPD), zone of free movement (ZFM), and zone of promoted action (ZPA). The ZPD is the set of ways in which an individual could develop resulting from interactions with their environment and the people in it. The ZFM determines development allowed under existing conditions, whilst the ZPA includes actions that are promoted. Valsiner considered these two zones as a ZFM/ZPA complex, with development taking place under successive ZFM/ZPA complexes. Structuring this complex allows development to be directed. This theoretical framework lends itself to understanding the developing identities of teachers because it is consistent with situated learning theories in which learning contributes to identity development (Wenger, 1988). The practices in which an individual participates occur under the influence of a ZFM/ZPA complex, the cyclic process of development under successive ZFM/ZPA complexes may provide a way of capturing the dynamic nature of identity. Furthermore, it is possible to accommodate the agency that an individual has in identity development by the freedom an individual has to accept or reject actions that are promoted. Valsiner's zone theory allows insights into development through analysis of how an individual's ZPD maps onto their ZFM/ZPA complex. Extending the approach taken by Goos (2013) who used this theoretical framework to understand teacher learning, factors included in the framework for identity as an embedder-of-numeracy were mapped onto Valsiner's three zones.

RESEARCH DESIGN AND METHODS

Participants in the study were eight teachers from two secondary schools in Australia. The study was conducted over a two-year period (2014-2015) within the context of a larger project (*Numeracy Project*) involving more schools and teachers. Changes were observed in the way Kylie (pseudonym) promoted numeracy learning and what she said about numeracy over the course of the study. For this reason, her case illustrates how Valsiner's (1997) zone theory could be used to understand possible trajectories of a teacher's identity as an embedder-of-numeracy.

Teachers' involvement in the Numeracy Project included participation in a series of professional development workshops that promoted engagement with Goos et al.'s (2014) numeracy model (where numeracy is seen as involving *mathematical knowledge*, *context*, *tools*, and *dispositions* embedded in a *critical orientation*) and provided opportunities for teachers to plan and share numeracy rich tasks in a range of disciplines. Thus, conducting the study in the context of the Numeracy Project meant that there was at least one known contributing factor to each teacher's ZPA.

On each of the six occasions Kylie was visited, she was observed teaching one or more lessons and interviewed. Content analysis of interview transcripts was used to identify aspects of her ZPD, ZFM, and ZPA. For example, comments about confidence in dealing with numeracy were coded as part of her ZPD because lack of confidence in this area may limit the ways in which she might develop. Her personal conception of numeracy and the tasks she used were analysed in terms of Goos et al.'s (2014) numeracy model, as has been done previously by these researchers.

THE CASE OF KYLIE

Kylie was a qualified history and English teacher with a major in Ancient History. She had studied mathematics in her final two years of schooling and, although her university studies did not include any formal mathematics courses, she reported using mathematical knowledge, especially statistics, in some of her history courses. Her opportunities to learn about how to address numeracy in history had been limited. Kylie was in her second year of teaching when the study began. The findings presented here focus her identity as an embedder-of-numeracy when teaching history.

Professional context

Implementation of the Australian Curriculum (ACARA, n.d.) presented Kylie with some challenges throughout the two years of the study, particularly in relation to the amount of historical content to be covered. For example, she indicated that the amount of content to be covered would make it impossible for numeracy to be incorporated into a task where students wrote a newspaper article about a key aspect of Medieval life (e.g. by asking students to provide some data as evidence in their article): “We don’t have enough time at the moment and that’s what I am particularly concerned about (Year 1, September). Later in the study, Kylie noted how reduction in historical content would allow a greater focus on development of historical skills:

> I’ve really pushed, and a lot of other teachers have, to reduce the amount of content we teach and focus on the skills because at the end of the day a student can Google when Balboa found the Pacific Ocean but if they can’t read a timeline or read a map or construct a graph then, you know, they’ve lost significant skills. I would like to see the focus on skills more I think. (Year 2, October)

Perceptions about numeracy

Before her participation in the Numeracy Project, Kylie had seen numeracy as the responsibility of the mathematics teachers, perhaps indicating that she saw numeracy and mathematics as the same:

> I was probably one of those teachers who was like, “Numeracy, well I’m sure they’ll cover that in maths”. Um and like when I went to the [Numeracy] project if you do this and this and I thought well I do do this but it wasn’t explicit. (Year 1, September)

This comment suggests that her ideas about numeracy were changing and she noted that it was becoming more recognisable in her teaching: “It [numeracy] has become so much more, not prevalent, but obvious in what I do” (Year 1, September).

Kylie provided further evidence of changes in her personal conception of numeracy and her confidence of dealing with it in the subjects she was teaching when she described how she felt about the idea of numeracy across the curriculum:

> I’ve obviously seen an increase of just recognising how much numeracy there is. I think also though I noticed by coming into this, I felt much less like I would have two years ago. I would have freaked out at the thought of doing maths. I’m, like, I don’t know how to do maths or anything like that. I’m much more confident. (Year 2, June)

At the end of the study, Kylie noted how her views about the importance of numeracy had changed to the point where she saw numeracy as important as literacy:

> One thing I've realised over this time [the duration of the Numeracy Project], if you had asked me before the project, I would have said obviously numeracy is important but literacy is the most important … [now] I see it on a par with literacy … if you are innumerate, that's on a level with not being able to read. (October 2014)

Kylie's comments indicate a changing personal conception of numeracy, and increased awareness of, and confidence with attending explicitly to numeracy.

Numeracy and historical understanding

Kylie could see benefits for students' historical understanding in explicitly attending to numeracy learning opportunities. Throughout the study she described several instances of how numeracy could enhance learning in history. For example, she related the proportion of people who died to the same proportion of students in the class to help students understand the significance of the Black Plague:

> We looked at the Black Plague and how many were affected … we did how many people in the world would have been killed, one to two thirds, one to two thirds of the world, of Europe and then we looked at the school and then we looked at the classroom and decided who gets killed by the Black Plague … They just needed to understand how bad the Black Plague was. So it was a very easy concept to apply numeracy to … we said it was devastating and I think the problem was that they didn't understand, like, they have a lot of difficulty identifying the concepts in the Medieval world … it was trying to build their understanding. (Year 1, September 2013)

In one of the observed lessons in the second year of the study, Kylie implemented a timeline activity with more explicit attention to numeracy than she had in a lesson on the same topic approximately a year earlier. A teacher-led discussion about the important features of a timeline was used to introduce the task. Kylie asked questions that included: Why do you need to measure? How far apart in time were the events? A summary of the important features was provided and students were given time to *construct* their timeline. Kylie's rationale for employing the timeline was to address difficulties students were having with the concept of time:

> Students are having trouble with the concept of time … the fact that there were actually multiple Spanish people moving at once … It wasn't just like Columbus went and came back and then another one went and came back and then another one went and came back when really they were all just all over the place at once. So it's about understanding the concept of time. (Year 2, October 2014)

In this instance, she had used the timeline, an *historical skill* (ACARA, n.d.), to develop the time-related concepts of *duration* and *concurrency*.

Kylie also described how more extensive use of maps in lessons on the Spanish conquest of the Americas could help students understand the historical concept of *cause and effect* (ACARA, n.d.), acknowledging that available time had prevented her from taking such an approach:

> I think we could use some more mapping because there's so many questions that arise … Where did they live? Why do they speak English here? Which is a very obvious question. We keep talking about Spanish, Spanish, Spanish, Spanish. Why are the Americans English or speak English or look English? So that's a good cause and effects kind of question. If we had more time probably we could look into it. (Year 2, October)

Planning for numeracy in history

Kylie began the study reasonably confident that she could deal explicitly with the mathematical content in history lessons, even though some revision might be necessary: "Probably for me the mathematical knowledge needs a bit of a refresher for a lot of it. Once I look at it I can probably do it as long as it's not too complicated" (Year 1, September). She expressed similar views towards the end of the study about confidence in her knowledge of mathematics: "I looked through the curriculum, like I felt confident with all of the mathematics I was presented with" (Year 2, October); and the need to revise some mathematical content prior to incorporating it into her lessons:

> Timelines, obviously, are old hat but … I had to double check I knew how to do the types of graphs but I mean I do that with most of my content if I haven't taught it for like a year. (Year 2, October)

Confidence does not indicate competency, but Kylie's confidence with mathematics seems unlikely to prevent her from attending to numeracy in her history lessons.

KYLIE'S IDENTITY AS AN EMBEDDER-OF-NUMERACY

Drawing on data collected during the first year of the study, the analysis of Kylie's identity as an embedder-of-numeracy represents her initial identity with respect to the time frame of the study. Her ZPD could be seen as including the possibility of developing her capabilities to effectively promote students' numeracy learning through history. Her mathematical knowledge was probably adequate for the mathematics she was likely to encounter while teaching history and she expressed some confidence in her mathematical ability in this regard. Importantly, Kylie had a strong disciplinary background in history and was able to articulate how numeracy could support learning in history. However, her development could be constrained by limited opportunities for developing pedagogical content knowledge for numeracy. The ZFM/ZPA complex experienced by Kylie included a new history curriculum that promoted an across the curriculum approach to numeracy (ACARA, n.d.) but in an environment where there was pressure to content.

There was evidence of a change in Kylie's practice in the second year of the study (for example, her explicit attention to numeracy in the timeline activity described above), suggesting that her identity-as-an-embedder-of-numeracy had strengthened. This development appeared to be canalised by the zone of free movement/zone of promoted action (ZFM/ZPA) complex she experienced leading to an expansion in her ZPD. The Numeracy Project seemed to have been influential in raising her awareness of numeracy in history and may have led her to consider scale in addition to sequence when representing historical events on a timeline: only sequencing of historical events is identified in the Australian Curriculum (ACARA, n.d.). Thus, it could be argued that her beliefs had changed and her pedagogical content knowledge for numeracy had increased, with consequent changes to her classroom practice. Other changes evident in some of the factors contributing to Kylie's ZPD included more confidence in dealing with mathematical aspects in her lessons and increased recognition of the importance of numeracy, now considering it as important as literacy. While Kylie did not engage in any further formal mathematics learning, her increased confidence in her mathematical knowledge may be the result of being more familiar with the mathematics she was using in her history lessons and could indicate increased mathematical content knowledge. These changes could be seen as opening up more ways in which she could develop.

There did not appear to be any significant changes to elements of Kylie's ZFM/ZPA complex in the second year of the study that would impact on how she was able to promote numeracy learning through history. While she reported that she would like to see a reduction in the history content to allow greater emphasis on historical skills, she was able to manage the competing demands of explicitly addressing numeracy and covering content in the observed lessons. Overcoming this constraint in her ZFM could be a result of her changed beliefs and increased understanding of numeracy.

When Kylie's identity as an embedder-of-numeracy towards the end of the study is compared with that of the previous year it could be argued that this identity had strengthened. When analysed in terms of Valsiner's (1997) three zones, there appears to be increased overlap when her ZPD was mapped onto her ZFM/ZPA complex. Her pedagogical content knowledge seemed to have increased and her beliefs about numeracy had changed, with consequent changes to her classroom practice. Some of these changes could be attributed to her growing experience as a history teacher but it could be argued that the Numeracy Project (as part of her ZPA) influenced the direction of this development.

CONCLUDING REMARKS

Debate about the best way to promote numeracy learning in schools is occurring amidst increasing recognition of the importance of numeracy (Geiger, et al., 2015). There are connections that can be made between numeracy and learning in subjects other than mathematics (e.g., timelines). However, across the curriculum approaches to numeracy present challenges for teachers who may not have the appropriate knowledge base or see numeracy as something extra to be added to an already crowded curriculum. Professional development interventions alone are insufficient to address teachers' needs because of the many factors that influence whether and how they implement learning from such interventions into their classroom practices.

Valsiner's (1997) zone theory considers the influence of both cognitive and non-cognitive factors on development. The case of Kylie illustrates how teacher identity could be used in conjunction with this theoretical framework to follow the trajectory of a teacher's identity as an embedder-of-numeracy. This approach has potential to enable identification of affordances and constraints to strengthening this identity. One of the limitations of the study was that it was limited to a small number of teachers in two schools. Further testing of this approach is necessary and could be achieved by extending the study to more teachers and school subjects.

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EVALUATION OF A LARGE SCALE PROFESSIONAL DEVELOPMENT PROGRAM

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This paper reports on a par of an evaluation of the professional development program (PDP) Boost for Mathematics in Sweden. Around 200 mathematics lessons were observed, and the teachers were interviewed after each lesson. The findings indicate that the PDP has had a significant impact on the teachers' knowledge about the mathematical competencies as they are presented in the national curriculum documents, and that the teaching practice had improved and now gives the students better possibilities to develop the competencies. The results also show that these improvements are still present one year after the program had ended.

INTRODUCTION

There is an international trend towards competencies (reform mathematics), and several professional development programs (PDPs) have been performed (e.g., Boesen et al., 2014). In Sweden, Boost for Mathematics (Sw: Matematiklyftet) was a professional development program carried out in the years 2012-2016. It was a large scale program, and about 80 % of all Swedish mathematics teachers in compulsory and upper secondary school (school years 1-12) have participated, a total of 33,580 mathematics teachers. The two-folded aim was to develop the teaching culture and the professional development culture at schools. Four didactical perspectives guided the professional development program: 1) Teaching for the development of mathematical competencies, 2) Formative assessment, 3) Routines and interactions in the classroom, and 4) Classroom norms and socio-mathematical norms. The result of this PDP has been evaluated (Österholm et al., 2016). The evaluation was extensive, for example, including around 200 lesson observations, interviews with teachers and principals, observations of collaborative meetings, and document analyses. In this paper, we report on one part of this evaluation.

Research studies on large scale professional development programs mainly use teachers' self-reports as empirical data; "One limitation of the large-scale studies we reviewed is that they relied primarily on self-report data of teachers' changing classroom instruction." (Goldsmith et al., 2014, p. 24). In view of this large literature review, the evaluation presented here contributes to what is known in an important way, since the conclusions are based on observational data of changes in classroom instruction in a large scale professional development program.

In this paper, we focus on how the teaching culture developed regarding the first didactical perspective, that is, *teaching for the development of mathematical*

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competencies. The mathematical competencies here refer to the abilities the students are supposed to develop, according to the national curriculum, and include, among other aspects, problem solving and communication.

In this paper, we aim to answer the following research questions:

RQ1: To what extent has teachers' knowledge improved concerning mathematical competencies when they participated in the PDP Boost for Mathematics?

RQ2: To what extent has the teaching practice improved concerning teaching for the development of mathematical competencies, when teachers participated in the PDP Boost for Mathematics?

The paper is arranged as follows. First, we describe the professional development program and the theoretical framework underpinning the evaluation. Thereafter, we outline the method, followed by results from the analyses. Finally, we discuss the implications for large scale PDPs in relation to implementation of reforms.

DESCRIPTION OF THE PDP BOOST FOR MATHEMATICS

The professional development program *Boost for Mathematics* conforms to research findings concerning quality of large scale in-service programs (see Boesen et al., 2014). The main part of the program was supervised teacher collaboration and discussions, where web-based support material was used, which was developed by researchers and teacher educators at Swedish teacher colleges. As it is important that principals and school leaders are a part of a PDP (e.g., see Zehetmeier, 2015), professional development for them was also part of Boost for Mathematics.

The support material for teachers consists of different modules that for compulsory school cover a specific mathematical content area (e.g., algebra or geometry), while the modules for upper secondary school cover a specific educational area, such as problem solving or teaching in accordance with the mathematical competencies. The quality of the material was monitored by appointed mathematics education researchers. The material is hosted by the Swedish National Agency for Education, and is still available for anyone to utilize, also after the program has officially ended.

The modules contain didactical support material; scholarly texts, research articles in mathematics education, video, and audio, together with instructions for lesson activities and questions for collegial discussions. The modules framed, in a four-step model, how the PDP should be conducted at the schools; 1) individual preparation, 2) collegial preparation, 3) lesson activity, and 4) collegial follow-up discussion. The four-step model was developed in order to reach the program goals that the teachers to a higher degree should reflect on their decision-making in their mathematics teaching, as well as developing a wider range of teaching methods and teaching approaches, to be able to adapt to students' different needs.

There are also results from other evaluations of Boost for Mathematics. A large survey with participating schools and teachers show that the PDP mainly has been implemented as intended (Ramböll, 2016). This survey also shows that the teachers in general are satisfied with the program.

The evaluation of the professional development program *Boost for Mathematics* that will be reported on in this paper took place in the years 2014-2016. A project group of four researchers (the authors of this paper) developed tools for collection of empirical data (interviews, observations and questionnaires) and for analysis of development concerning the four didactical perspectives mentioned above. Eight specialists in relevant areas (mathematics education, assessment, statistical analyses, and evaluation) supported the project group in critical phases of the evaluation. Sixteen project assistants carried out the collection and preparation of data, together with some initial analyses. The main analyses and the reporting of the evaluation to the Swedish National Agency of Education (Österholm et al., 2016) was done by the project group.

ANALYTICAL FRAMEWORK

The framework guiding our analyses regarding teaching for the development of competencies takes its departure in the descriptions in the Swedish national curriculum documents for mathematics. Five competencies (Sw: *förmågor*) are common for school years 1-12: 1) problem solving, 2) conceptual understanding, 3) procedural competency, 4) mathematical reasoning, 5) mathematical communication. Two additional competencies exist only for upper secondary school (years 10-12): 1) mathematical modelling, and 2) the relevance for using mathematics.

For each of the competencies, we constructed a tool that could be used to analyze whether a teaching activity could give the students the opportunity to develop that specific competency. The tool, based on Lithner et al. (2010), looked at two aspects: A cognitive aspect (to identify, interpret and so on), and a productive aspect (to carry out, use, choose and so on). Here we describe this tool in detail for only one competency; problem solving, as an example.

Problem solving is defined as follows: "Mathematical problems are, in contrast to pure routine tasks, situations or tasks where the students don't directly know how the problem should be solved" (Swedish National Agency of Education, 2016).

For the competency of problem solving, we framed the cognitive aspect as 1) being able to identify different components of a problem, to see alternative solution possibilities, and to understand methods, tools and goals of problem solving, 2) being able to evaluate and assess solutions, strategies and methods, and 3) to judge the plausibility of the result in relation to the problem. We subsequently framed the productive aspect as 1) being able to use mathematics to solve problems arising in mathematics and other contexts, 2) being able to use and adapt problem solving strategies and methods, and 3) being able to formulate and specify different types of mathematical problems.

METHOD

This paper focuses on the development of the teaching culture in the participating schools. We analyze changes in the classroom practice and in teachers' knowledge, when it comes to teaching for the development of the mathematical competencies.

Sample

35 Swedish schools that participated in the professional development program were randomly chosen for the evaluation. At each school, three mathematics teachers were randomly selected, that is, a total of 105 teachers were selected. Half of the schools were visited *before* and *during* the PDP, and the other half were visited *during* and *after* the PDP. Since the same type of data was collected at all schools, and we had data from *before, during* and *after* the professional development program, we could examine *changes* in a direct way. Each school was visited twice, with a one-year interval, collecting data from the same teachers. If a teacher had left the school between the visits, a new teacher at the same school was randomly selected.

The structure with two visits offered the possibility to explore changes in a direct manner. We could not visit all schools in all three stages (before, during and after the PDP) due to time limitations. However, all schools did not start the PDP the same year. In spring 2015, we visited the first half of the schools *before* the PDP, and the second half *during* the PDP. At the second visit, in spring 2016, the first half was visited *during* the PDP and the second half *after* the PDP. This made it possible to perform two different types of analyses; the *same teachers* are analyzed two different years and *different teachers* are analyzed the same year.

Data collection and data processing

For each teacher, we observed a mathematics lesson with audio recording of the teacher's voice. The recordings were supplemented with copies or photographs of curriculum materials used during the lesson, and notes on what was written on the whiteboard and on what tasks the students were working with. After the lesson, a structured interview was conducted with the teacher. Each interview took about 75 minutes. The same interview guide was used at both visits.

The interview questions covered a range of issues for all four didactical perspectives. The following questions were used for the analysis presented in this paper:

1. What did you want the students to learn during the lesson?

2. The national curriculum documents describe different mathematical competencies. How do you think the competencies affect your planning of lessons in general?

3. For each competency, describe the core of the competency and give one example of how you have worked during a lesson to give the students the opportunity to develop this specific competency.

4. Sometimes learning goals are divided in content goals and competency goals. What is your view on this division?

5. Do you use learning goals directly from the national curriculum document? If so, from what part of the document?

All interviews were transcribed and the lessons were described and divided into activities based on the working methods we could observe (e.g., teacher presentation, whole class discussion, group discussion, or individual work).

Method of analysis

Two result variables were constructed concerning the didactical perspective *Teaching for the development of mathematical competencies*. The first variable describes to what extent each teacher plans and reflects in line with the didactical perspective. The second variable describes to what extent the teaching of each teacher is in line with the didactical perspective.

The analysis focused on three aspects: What knowledge do the teachers have regarding the competencies? What understanding do the teachers have for how classroom activities can give students opportunities to develop the competencies? What competencies are the students given the opportunity to develop?

Several assessments were made for each aspect. The two first aspects were analyzed using 19 assessments of the interviews to form the first result variable. The second aspect was analyzed using 10-14 assessments (two for each of the 5 competencies in grades 1-9 and 7 competencies in grades 10-12) of each activity in the lesson observations to form the second result variable. The final score for each of the two result variables was calculated as an average of all assessments.

All assessments were described very carefully to ensure the quality of the analysis. One assessment will here be presented in detail, but all assessments were done using similar tables and descriptions.

Example: Assess to what degree the teacher shows knowledge and will to let problem solving affect the planning of lessons.

Value	Definition	Example
0	The teacher never spontaneously talks about the competencies in relation to the planning of lessons, and gives only short answers to direct questions.	*The students should of course develop their problem solving ability during the lessons.*
0.5	The teacher stresses the importance of the impact of the competencies on the planning of lessons, but only in general terms.	*I give the students many problems to help the students develop their problem solving ability.*
1	The teacher exemplifies how the competencies affect the planning of the lessons. The teacher mentions both the cognitive and the productive aspect.	*When we work with problem solving in the class we always discuss both different ways of attacking a problem and ways to specify the problem at hand.*

Table 1. Assessment of teacher knowledge in relation to the planning of lessons.

This assessment is connected to the second aspect above, and was based on the answers to two different interview questions, one concerning how the competencies in general affect the planning of lessons and what the core of each competency is. The second question concerned learning goals, and to what extent the teacher used learning goals directly from the national curriculum documents, since the competencies are described in those documents. The value of the assessment was decided using Table 1.

We were able to analyze the change in both result variables, since we had data collected with a one-year interval from each teacher. Half of the schools were visited *before* the PDP and *during* the PDP, while the others were visited *during* the PDP and *after* the PDP. Statistical analyses through t-tests were used to identify significant differences, using $p<0.05$ as the limit for statistical significance. Two types of differences were analyzed, which increases the reliability of the results. In the first analysis, we compared the same group of teachers at different times, when they were at different stages of the PDP. In the second analysis, we compared two groups of teachers at the same time, but when the groups were at different stages of the PDP.

RESULTS

As a result of the professional development program *Boost for Mathematics*, the teachers plan and reflect to a higher degree in line with the competencies, see figure 1.

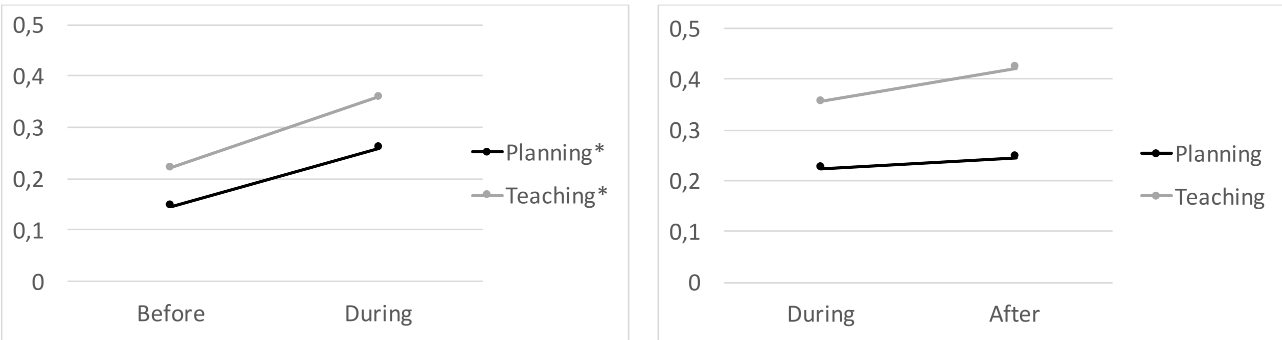
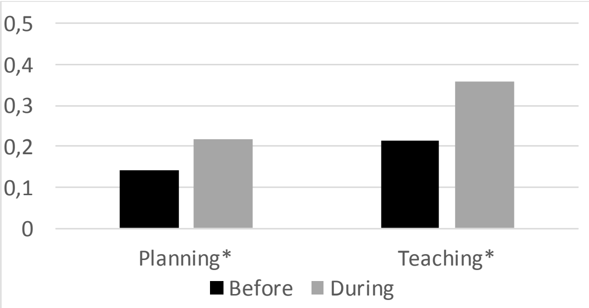


Figure 1. *Changes in teachers' planning and reflection on the one hand and teaching activities on the other hand, during different stages of the PDP (before, during, and after). * marks changes that are statistically significant (p<0.05).*

Figure 1 also shows that the teaching had changed during the PDP. The teachers work more in line with the didactical perspective *Teaching for the development of mathematical competences.* Furthermore, the right side diagram shows that there is no drop in how teachers work or plan one year after the PDP. This indicates that the effects of the program are stable.

Figure 2 shows a comparison between different groups of teachers the same year, but at different stages of the PDP. The patterns are the same as in figure 1, with significant differences between *before* and *during* the PDP, showing that the program has affected both the planning and the teaching. Figure 2 also shows that there are no significant differences between teachers one year *after* the PDP and teachers *during* the program.



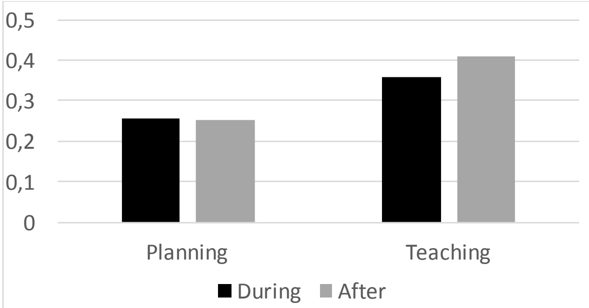


Figure 2. *Differences between groups of teachers' that are at different stages of the PDP (before, during, and after) concerning their planning and reflection on the one hand and teaching activities on the other hand. * marks differences that are statistically significant (p<0.05).*

In total, the results in figure 1 are the same as in figure 2, showing the reliability of these results. The effect sizes of the significant differences between *before* and *during* the PDP are medium to large, with Cohen's *d* value of 0.56 for planning and 0.81 for teaching.

What may the changes represent?

Here follow two examples of teacher changes, the first concerning the teacher's planning and reflections, and the second concerning activities in the classroom.

As one aspect of teachers' planning and reflections, we asked about the balance between content goals and competence goals. One teacher gave the following answer when interviewed before the PDP: "The competence goals are more of survival abilities. You practice them in all subjects. The content goals are the foundation the students should have." When the same question was asked to the same teacher during the PDP, the answer was: "The mathematical competencies are connected to the content areas I choose to focus on". In the first answer, the competence goals were clearly talked about as something outside the subject. In the second answer the teacher had changed to a balance between content goals and competence goals. The first answer was given the value 0 and the second answer was given the value 1.

At the lesson observed before the PDP, no competencies at all were discussed, and the students were not given the possibility to develop the reasoning competency. The lesson during the PDP could be seen as a contrast to the first. The teacher started by talking about the reasoning competency and what it could be. Later in the lesson, in a group assignment, the students were given the task to explain to a classmate what the equal sign stands for. The students were also to asked to present and argue for their explanations to the rest of the class. Discussions where the students have to argue for their descriptions, are seen as central for the reasoning competency.

CONCLUSION AND DISCUSSION

After the change of the Swedish national curriculum documents in 2011, all schools in Sweden were given the opportunity to participate in the large professional development program *Boost for Mathematics*. In this paper, we have shown that this large-scale PDP did in fact change the teaching culture concerning mathematical competencies. We argue that the reason for this change is that the teachers were given organized possibilities to develop their knowledge and abilities to teach in line with the new curriculum documents, as this result differs from previous change in curriculum documents in Sweden. For instance, Boesen et al. (2014) show in their study that the 1994 curriculum did not have the desired effect. The teaching did not change in any significant way, and was still focused on procedural knowledge. The competencies were present in the curriculum documents to a large extent, but they were not clearly conveyed (Bergqvist & Bergqvist, 2016). But, the large scale PD-program following the 2011 curriculum change made a difference (Österholm, et al. 2016).

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PARENTS' AND TEACHERS' VIEWS ON THE DISTINCT ROLE OF MATHEMATICS AS A SCHOOL SUBJECT

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In this research we aimed to investigate the mathematical views of two different groups that both may have enormous impact on students' mathematical achievement and attitudes, and on their future well-being as well. Elementary school teachers (N = 74) and parents (N = 955) filled in two analogous questionnaires concerning different aspects of learning mathematics. Data collection was done in a multilingual region of Europe. The results indicate that both in parents' and teachers' views Mathematics as a school subject has a distinct role in the system of school subjects. Differences between the two language-groups (Hungarian and Serbian) have been revealed in judging the pragmatic role of mathematics in several fields of future well-being.

INTRODUCTION

In the Western world, Mathematics as a school subject has always been an integral part of the curriculum since 1599. Baba, Iwasaki, Ueda and Date (2012) describe how the Western mathematical ideas changed even Japanese math education. During the 20th century, due to the IEA studies (FIMS, SISM and then the TIMSS-series), and especially with the advent of PISA-studies, its leading role has been reassured both by policy makers and educational researchers. The special role Mathematics fulfils in the system of school subjects can be described from several aspects. Kollosche (2014) claims that the mechanisms how mathematics teaching and learning take place in the classroom give power to those who are able to do mathematics. Similarly, Valero (2012) points to the role Mathematics and other STEM subjects may play when causing a gap between the two forms of subjectification (a Foucauldian term): in the mathematics classroom, the processes of subjectification are rather different that of other subjectification in other areas of social life.

A study with 3rd grade students' parents was conducted by Räty, Kasanen and Kärkkäinen (2006), and they revealed the special role Mathematics and Finnish (mother tongue) as two school subjects play in elementary schools. There is still much empirical research needed to reveal how Mathematics as a school subject is different from other subjects *in the system of school subjects*.

Parents' and teachers' views on mathematics

Why parents' views should be explored and taken into account by policy makers is justified by the need of winning parental support when introducing new approaches in mathematics teaching and learning. This is especially true in educational systems (like in the region where the current research has been conducted) where parents are free to

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 161-168. Singapore: PME.

choose the school where their children may receive mathematics education that fulfils parental expectations.

According to Albersman and Rolka (2013), the topic of investigating parental mathematical beliefs is rather neglected (e.g., the seminal work by Pehkonen and Torner, 1996, deals with students' and teachers' beliefs and not with that of parents'). Recently Csíkos and Dohány (2013) explored parental beliefs about secondary school parents' pragmatic values of mathematics and music, and Albersman (2015) conducted a research among the parents of 5th grade students about the pragmatic role of mathematics.

Teachers' mathematical beliefs have been more extensively explored in the last decades. Research ranges from case studies on philosophical values of mathematics education to large sample validation studies of questionnaires. E.g., FitzSimons, Bishop, Seah and Clarkson (2001) revealed that mathematics teachers are aware of rather different values mathematics teaching may explicitly or implicitly develop. Whereas Platas (2015) developed a questionnaire (MDBS, Mathematical Development Beliefs Survey) for pre-school teachers about early mathematics.

Our current research intends to investigate both parents' and teachers' mathematical views with the same questionnaire, and within the same sampling procedure. According to Dede (2013, p. 703.), "institutional values play an important role in mathematics teachers' decisions on classroom practices", consequently the institution-based sampling procedure seems to be an important novelty.

Cross-cultural considerations

The extent to which the results of the current investigations may be generalizable calls forth the question of cross-cultural (or cross-linguistic) differences. Although there may be relevant differences between teachers' explicit beliefs (see e.g., Andrews, 2007), studies on students' implicit mathematical beliefs (for a brief summary, see Csíkos, Kelemen, & Verschaffel, 2011) show a greater level of culture-independency. In the current study, parents and teachers from two ethnic groups within the same school system comprise our sample; consequently, our research design allows for exploring a level of cross-cultural differences or similarities.

Aims and hypotheses

In line with the literature review we proposed the following hypotheses.

(H1) In the system of school subjects, mathematics is considered as having a distinct role in both parents' and elementary teachers' views. This role may be indicated by the fact that parents often ask their children about school marks (especially in mathematics) and teachers frequently talk to parents about students' achievement in Mathematics.

(H2) Mathematics as a school subject is considered very important with regards to getting a job and earning high salary, but other aspects of well-being like sense of beauty and creativeness are less tightly associated with mathematics.

(H3) There are no relevant differences in mathematics-related views between the two language groups.

METHODS

Sample

The primary choice of sample units in our research were towns in Vojvodina (autonomous province of Serbia) where the language of instruction in primary schools is not only Serbian but Hungarian as well. Data on these schools is accessible in the database of the Vojvodina Methodology Center and can be found on the Hungarian Education Map of Vojvodina (http://vmoktatas.org.rs).There are 27 such towns in the province. The towns where the majority of the population is Hungarian are concentrated in the North Central regions of Vojvodina. We have chosen twelve of them, and nine school principals have permitted us to do the survey. We have asked Serbian and Hungarian teachers in grades one to four to hand out and to collect parental questionnaires, and also to fill out the teacher questionnaires. Participation was voluntary. This way, we received 1111 parental questionnaires. We left out 156 parental questionnaires from the research as we could not match them with any teacher questionnaires. Eventually, 955 parental and 74 teacher questionnaires were analyzed. 607 parents have filled out the form in Hungarian and 348 in Serbian. Out of the 47 teachers whose first language is Hungarian and the 27 whose first language is Serbian, 14 teach in grade one and 19-19 in grade two, three and four. One of them works in a composite class, teaching all grade levels. Out of the 74 teachers, only two are men, the rest of them are women.

Questionnaires

Two questionnaires were used in this investigation entitled “Parental questionnaire about learning” and “Teacher questionnaire about learning”. Both questionnaires have the same structure and items with the exception of the background items and some grammatical and syntactical adjustments. The questionnaires have the following sections:

(1) General beliefs on learning – five-point Likert-scale on the level of agreement.

(2) Importance of learning targets – five-level Likert-scale on the level of importance.

(3) Importance of school subjects – five-level Likert-scale on the level of importance.

(4) Frequency of discussion on school marks in different school subject – five-point scale on the frequency (not at all, monthly, weekly, several times per week, daily)

(5) Frequency of discussion on the content of learning in different school subjects – the same five-point scale as in part (4).

(6) The importance of mathematics with regard to students' future well-being – five-point Likert-scale on the level of agreement

Background questions covered demographic variables like age, level of schooling, type of settlement.

Both questionnaires have two versions: Hungarian and Serbian. Respondents could choose any of them, so another background variable of this investigation is the language.

Procedure and analysis

The data have been coded as quantitative data in the SPSS software. Except for part (4) of the questionnaire, the Likert-scale variables are considered as interval scale variables, whereas part (4) variables were treated as of ordinal scale level.

In the current phase of data analysis teachers' and parents' questionnaires are analysed separately. Nevertheless the data will enable for analysing connections between them, since parents' data can be connected to the teacher's data, and those parental questionnaires that cannot be matched to a teacher questionnaire have already been filtered out from the sample.

RESULTS

The place for mathematics in the system of school subjects (H1)

Teachers' and parents' judgments on the importance of "developing mathematical thinking" showed an average of 4.86 (SD=.45) on the five-point Likert-scale which clearly shows to what extent parents agree upon the utmost importance of mathematics. The highest average went to the development of first language skills (4.96), while other areas were all considered rather important (all mean values were above 4.24).

Part (3) of the questionnaire contained a list of all compulsory school subjects, and parents and teachers indicated how important each subject is in their opinion and in their child's opinion. The subjects were listed alphabetically. Table 1 shows the Mean and SD values for each item for both samples.

Comparing the mean values in each row, paired-samples t-tests indicated that teachers considered the school subjects significantly more important than – in their opinion – the students. There are only two exception: Information Technology and Physical Education. Parents, on the contrary, judge several school subjects less important than – in their opinion – their child does. Due to the large sample size, only the Technology and Lifestyles values are non-significantly different. As for mathematics, both parents and teachers considered it the second most important subject of the primary school, and according to their opinion, students may consider mathematics as even more important.

School subject	Parents		Teachers	
	Own	Child's	Own	Child's
Singing and Music	3.83 (1.03)	4.04 (1.00)	4.34 (.69)	3.62 (.94)
Religious Education	3.90 (1.17)	4.00 (1.11)	3.68 (1.27)	3.27 (1.06)
Second Language	4.84 (.51)	4.62 (.68)	4.66 (.58)	4.27 (.77)
Information Technology	4.65 (.72)	4.60 (.75)	4.54 (.74)	4.47 (.66)
Environmental Studies	4.75 (.54)	4.61 (.68)	4.78 (.45)	4.01 (.74)
Hungarian/Serbian Language	4.89 (.42)	4.74 (.58)	4.97 (.16)	4.48 (.67)
Mathematics	*4.87 (.42)*	*4.79 (.54)*	*4.96 (.20)*	*4.63 (.61)*
Drawing and Visual Education	4.05 (.93)	4.29 (.92)	4.41 (.66)	3.93 (.92)
Technology and Lifestyle	4.31 (.84)	4.30 (.85)	4.30 (.84)	3.76 (.95)
Physical Education	4.77 (.55)	4.72 (.63)	4.81 (.39)	4.66 (.69)

Table 1: Mean (and SD in parentheses) values on the importance of school subjects as judged by parents and teachers.

Interest in school marks

School subject	Parents		Teachers	
	Mark	Content	Mark	Content
Singing and Music	3	3	3	3
Religious Education	3	3	3	3
Second Language	4	4	3	3
Information Technology	4	4	3	3
Environmental Studies	4	4	4	3
Hungarian/Serbian Language	5	5	4	3
Mathematics	*5*	*5*	*4*	*3*
Drawing and Visual Education	3	3	3	3
Technology and Lifestyle	4	3	3	3
Physical Education	4	4	3	3

Table 2: Median values of the frequency of discussing school marks and content issues in each school subjects. (1 = not at all, 2 = monthly, 3 = weekly, 4 = several times per week, 5 = daily)

We hypothesized that parents discuss with their children their school marks and the content they learn in school frequently, and this might be especially true for Mathematics. The results are shown in Table 2. In the teacher questionnaire, the analogous items concerned the frequency of how often they discuss the school marks and the content issues with parents.

There is no tendency revealed that school marks are more often discussed than content issues (neither in parents-child nor in teacher-parents relations). Nevertheless, a clear picture is seen in Table 2, i.e. Mathematics as a school subject is among the top two subjects where both school achievement and the content to be learnt are fairly frequently discussed.

Why is mathematics important? (H2)

The questionnaire offered eight aspects from which Mathematics as a school subject might be judged important for well-being in adulthood. Table 3 shows the Mean (SD) values of each item in the two questionnaires.

Aspect of well-being	Parents	Teachers
Getting to work	4.23 (1.05)	3.59 (1.18)
Participation in social life	3.92 (1.15)	3.45 (1.09)
High salary	3.70 (1.28)	3.31 (1.13)
Balanced private life	3.21 (1.43)	2.92 (1.17)
Openness in social interactions	3.14 (1.38)	2.96 (1.07)
Creativity in work	3.97 (1.20)	3.84 (1.17)
Successful problem solving	4.10 (1.20)	4.39 (0.96)
Sense of beauty	2.29 (1.43)	2.41 (1.25)

Table 3: Mean (and SD in parentheses) values on the importance of different aspects from which Mathematics is important as judged by parents and teachers.

Table 3 indicates that with the exception of the aspect of problem solving and sense of beauty, parents generally overvalue mathematics as compared to elementary school teachers.

As for the third hypothesis (H3), the possible differences between the two language groups have been checked for the mathematics-related items of each preceding analyses. In the teachers' sample the only significant difference between the two language groups were found on the item about the role of mathematics in fostering creativity ($t(72) = 2.25$, $p = .03$). In the parents' sample, due to the large sample size, several differences proved to be significant. The pragmatic role of mathematics is seen differently in “high salary”, “creativity in work” and “successful problem solving” items ($p < .05$). Similarly to the teachers' questionnaire, Serbian questionees judged the role of mathematics in problem solving much higher; furthermore the Hungarian parents gave higher scores to the role of mathematics in creativity and in earning a high salary.

DISCUSSION AND IMPLICATIONS

Main points

Our research partly reassured our hypotheses. Parents and teachers seem to agree on the utmost importance of mathematics as a school subject. Mathematics is among the most frequently discussed school subjects both in parent-child and teacher-parent relations. Several aspects of well-being were unexpectedly undervalued (e.g. sense of beauty and balanced private life may and should be explicitly claimed as important aspects of making mathematics). In this respect it would be salutary to explore the opinion of another important group of stakeholders, namely, mathematicians.

Novelty

We would like to highlight two possible novelties of our research. First, we claim that investigating mathematical views in the framework of a systemic approach, i.e. mathematics is explored with an eye on the system of all school subjects, may bring new insights about the distinct role mathematics plays in the school. Second, the scarcity of research on parental views, and especially on a simultaneous enquiry on parents' and teachers' views indicates the need for such investigations.

Limitations

The usual limitations any similar study may face are to be mentioned here. One major point can be the issue of sampling and the selection of a multicultural region in Europe. We do not have enough information as to what extent our results might be generalizable concerning language and cultural boundaries. Another limitation factor is the typical indirect nature of the data collection when using questionnaires. In order to get the most possibly honest and objective answers, we aimed to use simple, straightforward items in closed-question format.

Practical considerations

Getting information on two key stakeholder groups concerning mathematics education provide a more comprehensible picture on the opportunity to meet with refusal when introducing new curriculum or instructional approaches in mathematics.

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"THE BEST EVER": MATHEMATICS TEACHERS' PERCEPTIONS OF QUALITY PROFESSIONAL LEARNING

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This paper reports on the responses of 109 Australian mathematics teachers to an online questionnaire designed to identify their perceptions of the characteristics of quality professional learning (PL). They were asked to consider the best PL that they had ever undertaken, rate possible contributors to its quality on 5-point Likert type scales, select five features of quality PL that most contributed to the PL being the best they had experienced, and nominate any other factors that they believed contributed to it being the best ever. Teachers most valued PL that they considered to contribute to improved teaching and improved student learning. Nevertheless, the PL nominated was not necessarily characterised by all of the widely accepted features of effective PL. Implications for the conceptualisation of quality PL are discussed.

The importance of ongoing professional learning (PL) (we use the term professional learning to mean the same as is meant elsewhere by professional development) for teachers is widely recognised. For example, in 2012, participation in PL was considered a professional duty of teachers and in some way linked to promotion in most European member states (European Commission, 2012). In Australia the Teacher Education Ministerial Advisory Group (2014) highlighted both the importance of ongoing PL and the dearth of research on effectiveness of PL for student outcomes.

The literature on PL typically refers to "effective" PL with improved mathematics learning by students the implied ultimate goal. Rigorous evaluation of PL, particularly at the level of impact on student learning is, however, a difficult undertaking and so claims of effectiveness are typically based on teacher self-reports of changed thinking, confidence or practice (e.g., Jacob & McConney, 2013); measures of teacher knowledge (e.g., Beswick, Callingham & Watson, 2012); or classroom observations of teacher practice (e.g., Scott, Clarkson & McDonough, 2012). Rarer attempts to measure PL effectiveness at the student level include that of Warren and Miller (2013) but the issue remains of linking positive results from such studies with the specific PL. In view of the difficulty of establishing effectiveness we use the term quality to describe PL that meets criteria typically associated with effectiveness in the literature. We contend that quality PL according to these criteria is likely to be effective but effectiveness cannot be claimed in the absence of rigorous evaluative evidence.

TEACHER LEARNING

The study was informed by a view of teacher learning in which reflective practice is central (Schön, 1983). Schön's theory is underpinned by a constructivist view of

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learning and incorporates notions of reflection-in-action and reflection-on-action. In considering PL, reflection-for-action (Killion & Todnem, 1991) by which teachers generate insights that will inform future actions and ongoing learning is also useful.

Research on teacher learning has focussed primarily on pre-service teachers and, where in-service teachers have been the focus, has predominantly reported on the effectiveness of particular programs. The role of informal learning - the ways in which teachers learn from their daily professional activity - is under-represented in the literature (Jones & Dexter, 2014). In addition, although consideration of formal PL programs has resulted in consensus about the broad characteristics of effective PL programs (as described below in relation to quality PL), minimal attention has been paid to why particular initiatives are successful (Beswick et al., 2016) or to the processes whereby learning occurs in these contexts (Vrikki, Warwick, Vermunt, Mercer, & Van Halem, 2017). Based on detailed study of dialogic interactions in small groups of teachers engaged in lesson study, Vrikki et al. (2017) attempted to go beyond studies of PL that examine inputs and outputs but not the intervening processes. They concluded that teachers learn through group conversation (not confined to the context of lesson study) that involves building on others' ideas and focussing on particular students, and that requires them to engage in reasoning to support their claims.

Collaboration and teacher learning

Consistent with the social aspects of constructivist views of learning, collaboration has been recognised as central to teacher learning (Jäppinen, Leclerc, & Tubin, 2016). Professional learning communities (PLCs) have been widely promoted as a means to facilitate collaboration and hence teacher learning (Jäppinen, et al., 2016). PLCs can be a context in which teachers have the kinds of conversations that stimulate reflection (Killion & Todnem, 1991; Schön, 1983) and contribute to learning in ways described by Vrikki et al. (2017). Jäppinen et al. (2016) identified five ways that apply across cultures and contexts, in which PLCs contribute to learning. Among these, was the role of supportive, not necessarily formal, leadership in providing a context and culture in which a PLC can flourish.

Quality professional Learning

The broad consensus about the characteristics of effective (or quality) PL has changed little over recent decades: Hunziker's (2010) list of characteristics of effective PL is markedly similar to that of Hawley and Valli (1999), and Beswick et al. (2016) acknowledged the currency of these characterisations of effective PL. Characteristics identified include having a shared purpose; being connected to the teachers' school contexts and sustained over time; attention to the explicit development of theoretical understandings and connections to practice that challenge and extend teacher knowledge through modelling; balancing individual learning needs within the development of a community of practice; and incorporating ongoing evaluation.

At least one study (Clarke, Roche, Cheeseman & van der Schans, 2014/15), although small and not evaluated in terms of teachers' actual use of strategies, described PL that did not conform to accepted tenets of effectiveness. The authors reported positive changes in teachers' awareness of teaching strategies likely to encourage students to persist with challenging mathematical tasks, as a result of observing a lesson designed to model such strategies and participating in a focus group on the same day. Clarke et al. noted that the teachers concerned were part of an existing active PL network. It could be that in such contexts, teachers can incorporate one-off PL experiences into their own ongoing 'program' of learning and hence be effective. Beswick et al. (2016) argued that more nuanced approaches to conceptualising the effectiveness of PL that take account of the particular context and aims of specific initiatives are needed. To inform such approaches the study reported here aimed to examine the views of recipients of mathematics PL regarding what characterises quality PL. It sought to answer the question, "What do mathematics teachers regard as the most important features contributing to the quality of the best PL that they have experienced?"

THE STUDY

Data were drawn from a larger study of quality PL for teachers of mathematics.

Participants

Participants were 109 teachers, of whom 78 (72%) were female. Just over half (56, 51%) were qualified to teach mathematics. Most (88, 85%) were secondary teachers (Years 7-12). The teachers were mainly experienced with 74% having taught for more than 10 years and 71% having taught mathematics for this length of time. Fifty-four percent taught in government schools with the remainder who specified a school system, divided between independent (24, 22%) and Catholic (23, 21%) schools.

Instrument and procedure

An online questionnaire was disseminated by way of a link on the website of the Australian Association of Mathematics Teachers. The link was promoted through the association's eNewsletter and the networks of individuals interviewed about the nature of quality PL as part of the larger project. Implementation Officers employed across the country for the larger project also promoted the survey through their networks.

Relevant questions asked participants to: name the best PL for teaching mathematics that they had ever undertaken. In relation to their nominated PL they were asked to select from options in drop down lists to indicate the focus of the nominated PL (mainly mathematics content, mainly pedagogy, or a combination of mathematics content and pedagogy); the format (e.g., one-off workshop, presentation); the year level focus; the participants (e.g., year level taught, leadership or other role); the frequency, length and number of sessions. They were also asked to name the facilitator(s) and location of the nominated PL and then to rate on 5-point Likert type scales the contribution of each of 16 PL features to the PL being the "best ever" and could add anything else that they believed made a contribution. A later question asked participants to select up to five of the 16 PL features that they considered essential for quality PL. A further Likert type item asked participants to rate the impact of the PL on students' learning outcomes. The 16 features of PL were derived from the literature on effective (quality) PL as well as interviews conducted with PL experts as part of the larger study. The questionnaire did not define PL although some questions about the PL nominated, implied that the PL was a formal session or program of activities.

RESULTS

Seventeen teachers described best ever PL that was clearly not a formal program. Informal activities included interacting with or preparing workshops for colleagues, participating in duties beyond their classrooms such as marking examination papers, participating in moderation processes, or working on curriculum materials.

The focus of the nominated PL was most commonly a combination of content and pedagogy (65% of the 106 who responded). Remaining responses were fairly evenly divided between Mainly pedagogy (19%) and Mainly mathematics content (15%). One respondent chose Other. In terms of format of the PL, 35% (n=98) indicated that it was a one-off workshop, 19% a presentation, 4% self-directed, and 3% online. The most frequently selected response was Other (39%). Table 1 shows the numbers and percentages of respondents selecting each option for the year level focus of, and participants in, the nominated PL. Respondents were able to select all options that applied, in relation to the year level of participants. A total of 295 options were selected by the 105 teachers who responded to this question. NA indicates that the option was not provided for that question.

	Year level focus of PL (n=96) Number (%)	Year level of participants (n=105) Number (%)
Early childhood	0 (0)	13 (12)
Primary	29 (30)	63 (60)
Middle school	9 (9)	NA
Secondary	36 (38)	69 (66)
Senior secondary	17 (18)	NA
School leaders	NA	29 (28)
Experienced teachers	NA	54 (51)
Beginning teachers	NA	37 (35)
Out-of-field teachers	NA	17 (16)
Teacher assistants	NA	10 (10)
Don't know/not sure	0 (0)	1 (1)
Other	5 (5)	2 (2)

Table 1: Year level focus of PL and year level of PL participants

The most commonly mentioned providers of the PL were professional associations and state education departments both nominated by 24% of teachers. Next most frequently named were private consultants (14%), university staff (9%) and school staff (8%).

The PL location was typically a provider venue (34%) or school (31%).

When asked about the frequency of sessions of their best ever PL, 48 (46%), of the 105 responses were that it was One-off, 14 (13%) selected Annually, 8 (8%) Monthly, 6 (6%) Weekly, and 4 (4%) Fortnightly. Twenty-four (23%) selected Other and 1 (1%) Unsure/don’t know. Regarding the numbers of sessions, 75 (72%) of the 104 who answered selected One, 16 (15%) More than 10, 9 (9%) Two, 1 (1%) Other, and 3 (3%) Don’t know/unsure. One hundred and five teachers indicated the length of each session. Most frequently selected was One day (32, 31%) followed by One hour (19, 18%), 2-3 days (17, 16%), Two hours (14, 13%), a Half day (11, 11%), Other (7, 7%), 4-5 days (4, 4%), and Don’t know/unsure (1, 1%).

Table 2 shows the means and standard deviations for each of the 16 possible contributors to quality PL, along with the number of times each was included in respondents’ five essential features of quality PL. The percentages relate to the 99 respondents who answered this question.

	No. of times in ‘top 5’ (%)	Mean	SD
It contributed to improved teaching	73 (74)	4.51	0.86
It focused on student learning	69 (70)	4.51	0.87
It contributed to improving student outcomes	52 (53)	4.46	0.87
It was immediately applicable	35 (35)	4.45	0.84
It encouraged reflection on practice	34 (34)	4.15	1.2
It was relevant to my context	33 (33)	4.59	0.82
The deliverer modelled effective pedagogies	32 (32)	4.14	1.37
It provided opportunities for collaboration	32 (32)	4.08	1.3
It provided opportunities to apply new practice	31 (31)	4.04	1.4
It was relevant to me	30 (20)	4.61	0.7
It was evidence based	27 (27)	3.99	1.33
It linked theory with practice	24 924)	4.08	1.29
It linked to the Australian Curriculum	13 (13)	3.37	1.73
It targeted the appropriate participants	10 (10)	4.26	1.09
The venue was appropriate	4 (4)	3.77	1.49
I found out where I could go to learn more	4 (4)	3.62	1.42

Table 2: Characteristics of quality PL

Table 3 shows the suggested additional features of the best ever PL that contributed to its quality that were mentioned at least twice by the 63 teachers who responded to this question. Qualities of the presenter included being engaging, enthusiastic, knowledgeable, and experienced. Suggestions mentioned once included such things as the mathematics content was rigorous, it was applicable to any year level, it provided personal and intellectual satisfaction, and the delivery was concise.

In relation to 'providing opportunities for collaboration', teachers mentioned collaboration with colleagues in similar (usually leadership) roles, from diverse geographic settings, teaching the same year level, from the same school, or being like-minded. One mentioned that they appreciated collaborating with a small group of colleagues who had participated in a series of separate PL events. Eighty-six (83%) of the103 teachers who responded to the relevant item indicated that the PL had a significant or very significant impact on their students' learning.

This was quality PL because …	Number
the presenter was expert, enthusiastic …	14
it provided opportunities to collaborate	13
it provided quality resources	6
it was immediately applicable in my context	5
it included tasks to complete between sessions	5
there was an expectation that I would share with colleagues	3
it was hands-on	3
took place in a positive atmosphere	3
examples were provided	2

Table 3: Additional characteristics contributing to quality PL

DISCUSSION

The fact that the majority of the PL described was one-off is consistent with the finding of Reaburn, Kilpatrick, Fraser, Beswick and Muir (in press) from an audit of Australian PL available for teachers of mathematics, that 61% of programs on offer were either one-off or annual events. Such programs are clearly not sustained over time and are unlikely to be designed collaboratively with participants in order to establish a shared purpose. The frequency with which such programs were nominated may be a related to the fact that they dominated the PL on offer (Reaburn et al., in press) and hence reflect limited experience of Pl that is sustained over time and/or collaboratively developed. Other aspects of effective PL apparent in the literature are encompassed in some of the characteristics that contributed to the quality of the PL as indicated in Table 2. For example, the frequencies with which being "immediately applicable", "relevant to my context", providing "opportunities for collaboration", "focused on student learning", and linking "theory and practice" were selected among the most important contributors to the quality of the PL, suggest that the teachers' views aligned with the literature in relation to these aspects of effective PL. Qualities of the presenter were also important and could be interpreted as ways in which presenters provided leadership, as described by Jäppinen et al. (2016) that allowed collaboration.

Several of the additional characteristics of quality PL shown in Table 3 duplicate aspects listed in Table 2 (e.g., opportunities to collaborate, immediate). The fact that teachers mentioned these as additional features may signify their importance or, in the case of collaboration at least, appeared to arise from a desire to specify the particular types of collaboration or the impacts that it had. Collaboration was valued as a part of both formal PL programs and informal PL. The importance of professional associations and state based education systems as PL providers is also noteworthy.

Teachers overwhelmingly valued PL that helped them to teach in ways that enhanced students' learning of mathematics and valued collaboration in achieving this end. Collaboration though, was not necessarily formally arranged nor part of the program nominated. Rather, PL can provide a site for teachers to meet and exchange ideas and a focus for conversation and reflection. Interestingly, given the high value placed on collaboration, no respondent described participation in a formal PLC or community of practice. This may mean that these are rare, that the teachers in this study who had engaged in a PLC did not rate it has highly as other PL, or that they did not recognise it as PL because it was not formally constituted. PLCs were similarly absent from the PL audit reported by Reaburn et al. (in press) possibly for similar reasons.

One-off or regular but infrequent PL events such as annual conferences may serve this purpose if teachers structure their own learning such that PL events, in conjunction with informal learning from experience, constitute a 'program' that meets their needs. The extent to which teachers are aware of their needs, however, is not clear. Reaburn et al. reported data from PL experts that suggested more mathematics focussed PL is needed for early childhood and out-of-field teachers as well as teaching assistants. While this is consistent with the absence of PL focussed on these audiences cited by participants in this study (See Table 1) it is also relevant that these groups were unrepresented or under-represented among participants in this study.

CONCLUSION

Beswick et al. (2016) argued that more nuanced approaches to conceptualising the effectiveness of PL that take account of the particular context and aims of specific initiatives are needed. The findings of this study suggest that part of the context that needs to be considered by PL providers is the range of learning experiences and activities, both formal and informal in which teachers are engaged. The results also highlight the commitment of teachers to student learning and their desire to collaborate and thereby learn from one another.

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LANGUAGE PRACTICES IN MULTILINGUAL MATHEMATICS CLASSROOMS: LESSONS FROM INDIA AND SOUTH AFRICA

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There is currently no comparative study that analyses language complexities in multilingual mathematics classrooms in developing countries with similar socio-cultural-economic milieu. This paper looks at similarities and differences of language practices in mathematics classrooms in India and South Africa and explores how such practices shape learners' mathematical communication. Analysis done using data transcripts of mathematics classrooms observed in both the countries indicates occurrence of complex factors that shape learners' mathematical communication and that the mere use of learners' home language may not be sufficient for building effective usage of mathematical language.

INTRODUCTION

Debates on which language(s) to use, why, how and when in multilingual mathematics classrooms are not new. Of note are studies that focus on how language can be used as a resource in multilingual mathematics classrooms (e.g. Moschkovich, 2002, 2007; Planas and Setati, 2009; Setati, Molefe and Langa, 2008). Language as a resource has been considered as an ideal orientation of language for multilingual mathematics classrooms over the two other orientations, namely, language as a right and as a problem (Planas & Phakeng, 2014). India and South Africa, emerging economies with a similar history and language infrastructure, offer valuable settings to provide insights on how language serves as a resource in different mathematics contexts. It is important to understand the complexities, commonalities and challenges emerging from such cross-country contexts.

Four decades since inception of the PME conferences and with the world being more multilingual than ever before, it is pertinent to explore complexities of language practices used to support mathematics teaching and learning in different contexts. The purpose of this paper is to explore similarities and differences in language practices of teachers in multilingual mathematics classrooms in India and South Africa. We draw on classroom observation data from two primary mathematics classrooms - one in South Africa and the other in India - and specifically ask the following questions:

- What are the similarities and differences in the language practices used in multilingual mathematics classrooms in India and South Africa?
- How do the language practices used in these classrooms shape the learners' communication of mathematics?

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Why South Africa and India and why now?

One of the aims of comparative studies in education is to improve the education systems of the respective countries by learning from other countries with similar educational situations and problems. In a world in which the research infrastructure is overwhelmingly concentrated in the North (see Altbach, 1982; Skovsmose, 2011), the vast majority of comparative studies in education are either between developing countries and developed countries or between developed countries. The potential of comparative studies between developing countries is a neglected field. Compounding the case for South Africa, in particular, is the current discourse in the country, which argues that given the similarities between the two countries South Africa can benefit from learning from India's successes in mathematics education. The two countries have similar challenges of poverty, inequality and unemployment, which can be alleviated through the provision of high quality mathematics education. The long standing friendship between the two governments as well as a shared colonial history, struggle for liberation and the existence of contemporary multilingual societies makes this exploration even more important.

Data shows that mathematics learners in India consistently perform better than those in South Africa in the International Mathematics Olympiad (IMO) and TIMSS. Thus India is ranked higher than South Africa on these international mathematics assessments. Unfortunately these rankings provide only a comparison of performance and not of what is done to produce the performance. Hence the importance of this exploration, which provides a comparative analysis of language practices in multilingual mathematics classrooms in both countries.

The language context: Case of India

India has adopted 22 languages (listed in the Constitution) as official languages collectively for its different states/provinces. Hindi is adopted as the official language of the Union. This list however does not include English, which functions as an "associate" official language. Soon after gaining independence from the British, there was a plan to gradually do away with English and replace it with Hindi. This plan however did not succeed. There are 100 other languages spoken in the country, classified as mother tongue spoken by 10,000 people or more, and 1635 dialects, which are referred to as rationalised mother tongue (Census of India, 2011). Different Indian states have one or two official languages, which may be entirely different from the regional languages of the neighbouring states. The language-in-education-policy (LiEP) for school education, prescribes a three-language formula, which comprises of the state's primary official language and two other languages. Multilingualism in India is visible at the official level where policy is formulated and also at the societal level where informal use of language occurs in addition to implementation of the policy. Sometimes languages used at the societal level are not even recognised as official even though an entire community speaks them. Being a populous country, the size of a population speaking a language is large compared to countries with lesser population.

Generally classroom interactions are much more formal than interactions that happen outside the formal learning contexts. Policy encourages acknowledging and drawing on learners' home languages during teaching and learning.

Language Context of South Africa

Linguistic diversity is an important feature of the South African nation, with eleven official languages and a Language-in-Education Policy that encourages multilingualism. According to this policy, not only can South African schools and learners choose their Language of Learning and Teaching (LoLT), but there is a policy environment supportive of the use of languages other than one favoured LoLT in school, and so too of language practices like code-switching. An additive model of multilingualism is encouraged. Under this model, learners are to add languages to their repertoire of linguistic resources for learning. An essential feature of the additive model is that the learners' main language is maintained, developed and used in the teaching and learning situation as a LoLT, alongside other language(s) of learning.

Language policies unfortunately have limitations - not all languages are equally 'powerful'. As Janks (2011) argues, 'access' is a double-edged sword (access paradox) – providing access to English, which in South Africa and India, is the language of power increases and entrenches its power. It is therefore not surprising that twenty-three years after the dawn of democracy, challenges with the implementation of the South African Language Policy remain as English continues to dominate and multilingualism is not as valued as was initially envisaged. Recent research suggests that most schools are not opting to use learners' home languages as LoLT as parents demand access to English from the first year of schooling. It is thus timely to explore how South Africa's experience in multilingual mathematics classrooms compares with that of India, which achieved its liberation almost 70 years ago.

Teacher practices – a glimpse from the lesson transcripts

In this section we present an analysis of carefully selected excerpts from our lesson observation data collected in multilingual mathematics classrooms in low socio-economic income areas in India and South Africa. Data in terms of teacher and student utterances were selected based on instances of code-switching and use of the mathematical register (Pimm, 1987) and social languages (Gee, 2005). These utterances were then analysed using Gee's framework to look for different "Discourses" and "intertextuality" where language is used to build "significance" or "connections" (Gee, 2005) that influenced learners' mathematical communication during classroom activities.

Excerpt 1 below is drawn from a Grade 7 vacation teaching camp following a teaching design experiment aimed at charting a pedagogical approach to connect learners' out-of-school mathematical knowledge and school mathematics. The school was located in an economically active low-income settlement in central Mumbai, India. The numbers on the left indicate line numbers in the original transcript while "T" and "S" stand for teacher and student respectively. Excerpt 2 is drawn from a low socio-economic class in a township school west of Johannesburg in South Africa formerly officially designated for black occupation by apartheid legislation.

In Excerpt 1 the teacher encouraged the students to come up with proper reasoning and justification and not to accept or believe any result without a justification. For example, the teacher used phrases like "*hum kyun maane*" [*why do I accept*] as a trigger to move towards a mathematically discursive practice. Such norms guided the classroom Discourse.

Excerpt 1.

152	T:	kyun teen batte chaar ka matlab pauna hai? hum kyun maane? aap keh rahe hain to sahi hai to aap prove kijieye ki teen batte chaar pauna hai/ [why is three-quarter three upon four? why do I accept? if you're saying it is correct then you prove that three by four is three-quarter/]
155	S_1:	sir pauna hota hai to teen batte chaar hota hai/ [sir if it's *pauna [three-quarter]*, it's three by four/]
156	T:	Kyun? [why?]
158	S_2:	teen batte chaar/ [three by four/]
160	T:	yahan aake koi explain ker sakt hai ki kyun … pauna ko teen batte chaar kehte hain? [can anyone come here and explain why … *pauna* is called three by four?]
163	S_1:	kyunki wo teen paav hai/ [because that's three quarters/]
167	S_1:	kyunki teen paav... do paav se aadha banta hai aur ek paav me pauna hai na isilieye teen... [Because three quarters … two quarters make a half and one quarter [more] gives *pauna*, isn't it therefore three...]
168	T:	teen paav/ woh to bilkul sahi hai lekin teen batte chaar kaise aaya teen paav se? [three quarters/ that's perfectly correct but how is three by four obtained from three *paav* [quarter]?]
169	S_2:	ek paav/ ek paav ho gaya aur ek paav/ [one *paav*/ one *paav* done one *paav* more/]
170	S_3:	ek batte chaar jama ek batte chaar jama ek batte/ [one by four plus one by four plus one by four/]
171	T:	Haan? [yes?]
172	S_1:	(writes on board) 1/4 + 1/4 + 1/4 = ¾

Students explained various strategies that they used to compute these multiples to their peers. This indicated their robust and confident awareness about decompositions of fractions (twice of 1½ [*dedh*] is the same as three; half of 1½ [*dedh*] is equal to ¾ [*pauna*], half of 2½ [*dhai* or *adhai*] is equal to 1¼ [*sawa*], etc.). The current curricular policy for mathematics teaching and learning in schools in India emphasises that learners' “home, communities, languages and cultures, are valuable as resources for experience to be analysed and enquired into at school” (NCERT, 2005: 14). However, in regular mathematics classroom practices, only formal mathematical language is used which are often alien to the learners who are usually more familiar with the informal registers as part of their "social language" (Gee, 2005) (for example, “pauna” in the above excerpt). The use of the Hindi/Urdu word “pauna” is interesting since unlike its closest English counterpart (which is three-quarters), this stand-alone term doesn't literally stand for a “three-quarters” but for “a quarter less than a whole” in a syntactic sense of the term. Interestingly, "pauna" is not part of the formal Hindi/Urdu or mathematical register. The learners had difficulty in moving from oral to written mathematics (formal representation) even though the classroom Discourse in Excerpt 1 encouraged use of informal and everyday mathematical words. Such intertextuality as part of the language practice enabled connecting with the everyday discourse with a possibility to unpack in a better way the abstraction embedded in the mathematical phrases such as “teen batte char” or “three by four” to connect with "teen paav" or "three quarters". In spite of that, such cross-referencing to other mathematical registers did not enable learners towards effective mathematical communication, especially while using non-binary fractions or following the language of mathematics, for example, arriving at the decimal representation for "teen paav".

Indian school mathematics curriculum places importance on moving from narrow goals to higher goals which necessarily includes communicating mathematically. The way in which learners in the excerpt above use the words “pauna” as “paav” added three times, once again indicates that the challenge they are facing is more about communicating mathematically. Our analysis shows that learners in South African classrooms share similar challenge as is emergent in Excerpt 2 below.

Excerpt 2.

130. T: We have to find out ukuthi [that] exactly how much is needed to add to this R5000 so that he can buy all his expenses. How are we going to find that out? We want to find out how much the farmer needs, how are we going to do that? Yes Thokozani.

131. L: Miss, nga khuluma ngesiZulu? [mam, can I speak in IsiZulu?]

132. T: Yes.

133. L: Miss, siya susa. [Miss, we subtract]

134. T: Good, siya susa. Siya minasa, but what is that, that you are minasing? [Good, we are subtracting. We are subtracting, but what is it that you are subtracting?]

135. L: Si minasa u five thousand no eight thousand five hundred and seven rand and one cent. [We subtract five thousand and eight thousand five hundred and seven rand and one cent]

136. T: Good, that's very good. We minus, in order for us to find the difference we say eight thousand five hundred and seven rands and one cent minus five thousand of the farmer. The farmer has five thousand rands. After minusing, we are going to find out exactly ifarmer idinga how much to add to his five thousand. [The farmer has five thousand rands. After subtracting we are going to find out exactly how much the farmer has to add to his five thousand.]

While multilingualism is encouraged by policy, learners in this classroom still ask for permission from the teacher to use their home languages, which suggests that there are limitations in terms of which language(s) can be used when and perhaps for what. In a context where multilingualism is valued and code switching is encouraged learners should not be asking for permission to use their home languages during mathematics teaching and learning especially when the teacher shares a home language with them. It is clear that language policies do not necessarily translate into unproblematic implementation in multilingual mathematics classrooms.

The manner in which the learner in the Excerpt 2 above is using the home language is instructive as he uses two different isiZulu words – 'susa' and 'minasa'- to indicate that the operation that they need to use to find the solution is subtraction. The word 'susa' when directly translated means remove and the word 'minasa' is a reformulation of the English mathematics word 'minus'. Utterance 133 above suggests that while the learner is given permission to speak in IsiZulu, his challenge is not just fluency in the English language but also in mathematical language as he only indicates that what needs to be done is subtract without indicating the values that must be included in the operation. When probed further by the teacher the learner uses the logical connective 'and' between the two values involved in the operation in his explanation (see utterance 135), which further indicates the challenge with mathematical language. Instead of saying we subtract "*a* from *b*", the learner says "we subtract *a* and *b*". What is at issue here is communicating mathematically and not just communicating in English. Communicating mathematically requires fluency in the LoLT as well as in the language of mathematics. Research shows that the challenge of communicating mathematically is faced by all learners including those who learn mathematics in their home languages. This is mainly because mathematical speech and writing have a variety of language types that learners need to understand in order to participate appropriately in any mathematical conversation. These are ordinary and mathematical language, or logical language and meta-language (Pimm, 1987; Rowland, 1995). One of the difficulties of learning to use mathematical language is that in its spoken (generally also in its written) form it is blended with ordinary language, and the distinction between the two languages is often blurred. In multilingual mathematics classrooms these difficulties are exacerbated by the fact that to communicate mathematically learners have to manage the constant interaction between home language, LOLT and language of mathematics.

Learning from similarities and differences

The above excerpts from Indian and South African multilingual mathematics lessons show a high prevalence of code-switching as a classroom norm, which is encouraged by the teacher and seem to have facilitated the learners' thinking and responses. It is also true that in both settings, teachers encouraged learners' formulation of sound mathematical justification. However, the use of learners' home language was not effective in improving their usage of the mathematical language. The learners' mathematical communication remained a challenge as learners struggled with the use of, for example, logical connectives. It is clear that while encouraging learners to use their home language is important, it is not sufficient to facilitate mathematical communication in these classrooms.

It is not just the use of learners' home language, LOLT and the language of mathematics or a mix of them that is critical for facilitating effective mathematical communication – necessary for developing sound conceptual understanding. Different languages function differently at the interplay with mathematical language depending upon the language's intonation, syntax and diction. Hence, uniform policy formulation may not be effective in such multilingual mathematics contexts as South Africa and India.

As emerging economies, South Africa and India have to deal with learners' identities in the classrooms that emerge from the language settings. The socio-economic statuses of these two countries are vastly different from the developed countries or other developing nations. The languages that learners use and how they use them during mathematics lessons often serve as an indicator of the social class they belong to or the social context they grew up in. This was visible in Excerpt 1 where learners' familiarity with everyday mathematical registers came from their exposure to the micro-enterprises around them. Their justification and reasoning revolved around their identities drawn from the work practices. On a similar note, in South African context learners' identities emerge from their racial identities. However, there is a dearth of research that explores links between identities and language use and how they influence learners' communication of mathematics.

Learners' attempts to communicate mathematically saw their efforts to come up with different mathematical representations for the mathematical objects under discussion. Language practices juxtaposed between home language, work context language, LOLT and language of mathematics create and shape their mathematical communication. Above analysis from these two emerging economies provide us with different insights about language practices in mathematics classrooms and calls for revisiting a few older terrains to cover newer terrains.

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STRUCTURE OF EARLY CHILDHOOD EDUACTORS MATH-RELATED COMPETENCE

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Previous studies stress the importance of math-related competence for early childhood educators. Currents studies therefore take a closer look at the structure of this competence with different results. As one study examines prospective and the other in-service educators it could be assumed that the differences can be related to the samples. However, also the instruments differ greatly. Hence, we take a first step to examine systematically if the differences are related to the samples by examining a sample of in-service early childhood educators with a set of measuring instruments used on a sample of prospective educators in previous studies. Our results reveal no systematic difference in the competence structure of in-service and prospective educators indicating that previous studies measured different competence facets.

MATH-RELATED COMPETENCE STRUCTURE OF EARLY CHILDHOOD EDUCATORS

Empirical results indicate that children's learning outcomes in mathematics depend essentially on support by early childhood educators (Bruns, 2014; Gasteiger, 2012; Klibanoff, Levine, Huttenlocher, Vasilyeva & Hedges, 2006; Peter-Koop & Grüßing, 2008) and pre-school math quality (Lehrl, Kluczniok & Rossbach, 2016). Therefore, math-related professional competence of early childhood educators is of great relevance to foster children's mathematical competence.

Models of different disciplines describe professional competence of early childhood educators theoretically in their structure and their development (in general: Fröhlich-Gildhoff, Weltzien, Kirstein, Pietsch & Rauh, 2014, mathematics: Gasteiger & Benz, 2016; Jenßen, Dunekacke, Eid & Blömeke, 2015). These models include mathematical content knowledge (MCK), mathematical pedagogical content knowledge (MPCK) and affective-motivational aspects (e.g. math-related beliefs).

At the moment, two research projects in Germany focus on early childhood educators' math-related competence. Both projects come to different results concerning the structure of the math-related competence of early childhood educators as shown in figure 1. Dunekacke, Jenßen, Blömeke and colleagues examined i.a. MCK, MPCK and beliefs towards mathematics in general as a characteristic of the affective-motivational competence facet of prospective early childhood educators (Blömeke et al., 2015; Dunekacke, Jenßen & Blömeke, 2015; Dunekacke, Jenßen, Eilerts & Blömeke, 2016). They found a close connection between the two knowledge facets (MCK, MPCK) and one belief facet (process-related orientation) as well as between MCK and the

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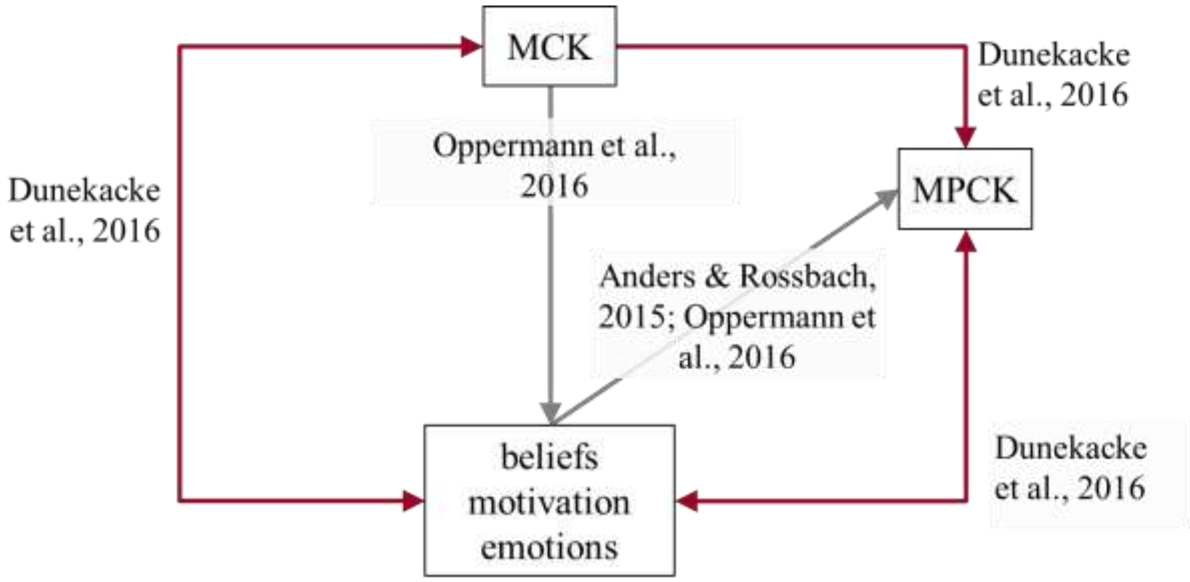
application-orientation. Furthermore, a direct effect of MCK on MPCK was found. The second project of Oppermann and colleagues (2016) examined MCK, MPCK and mathematical ability beliefs (self-efficacy, self-concept) as characteristics of the affective-motivational competence facet of in-service early childhood educators (Anders & Rossbach, 2015; Oppermann et al., 2016). They also found a relationship between MCK and MPCK in the linear regression model, however, the results of the mediation model showed that mathematical self-efficacy fully mediates the effect of



MCK on early childhood educators' MPCK (Oppermann et al., 2016).

Figure 1: Comparison of research results concerning competence structure of early childhood educators

As Dunekacke and colleagues (2016) examined prospective early childhood educators and Oppermann and colleagues (2016) early childhood educators in practice one might assume that the different results concerning the competence structure are related to the different samples. A closer look at both studies, however, also suggests differences in the measured competence facets as reason for the different results. Although both projects use the same terms to describe the examined competence facets (MCK, MPCK, beliefs), the underlying constructs and therefore, the measuring instruments differ greatly. Oppermann and colleagues (2016) define MPCK as the sensitivity to capture mathematical content in children's play. As measuring instrument, they use a play-based scenario task developed by McCray and Chen (2012) and half-standardized questionnaires. Dunekacke, Jenßen und Blömeke (2015) on the other hand consider MPCK as the knowledge of fostering mathematical literacy in informal and formal settings, the knowledge about how children's mathematical literacy develops as well as how to diagnose and support this development. They measure MPCK using a standardized test. Concerning the construct of MCK both research group describe MCK as mathematics of the primary school from a higher point of view. However, the study of Oppermann and colleagues (2016) measures MCK with four items that have been adapted from the TIMSS 2003 mathematical test while Dunekacke and colleagues (2016) use a standardized test with 24 items. Most obvious are the differences with regard to the affective-motivational aspects. Dunekacke et al. (2016) distinguished on the basis of Benz (2012) three facets concerning beliefs towards mathematics in general. As measuring instrument, they use a questionnaire. Oppermann and colleagues (2016) focus on math-related ability beliefs. Therefore, they measured early childhood teachers' mathematical self-concept and mathematical self-efficacy using established scales from teacher research.

In summary, it can be stated that it is not possible to reason based on the existing results that the competence structure of in-service early childhood educators differs from the competence structure of prospective early childhood educators. Even though both groups use the same terms it is also reasonable to assume that the different instruments actually measure different constructs. Though, to clarify if the different results regarding the competence structure are related to the samples or the measurements, a first step would be a study that uses comparable constructs and instruments to one of the presented studies but a sample of the opposite group. As research results concerning in-service early childhood educators have more relevance, since they provide information on the concrete situation in practice, this study should examine a sample of in-service early childhood educators using the same instruments as Dunekacke and colleagues (2016).

RESEARCH QUESTION

Following the presented argument this study focuses on the mathematics-related competence structure of in-service early childhood educators using the same instruments as Dunekacke and colleagues (2016). In particular, we pursue the following research question: How are MCK, MPCK and beliefs toward mathematics in general (process-related orientation, static orientation and application orientation) of in-service early childhood educators connected?

To answer this question two different models are compared. Model A reflects the structure found by Dunekacke and colleagues (2016), model B the structure found by Oppermann and colleagues (2016).

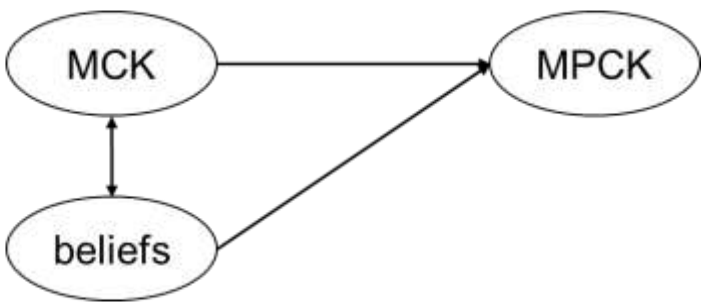


Figure 2: Model A (based on the results of Dunekacke et al., 2016)

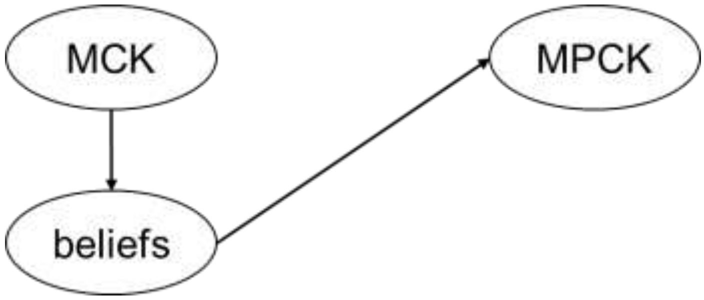


Figure 3: Model B (based on the results of Oppermann et al., 2016)

METHOD

We examine our research question with a convenience sample of the 10 largest pre-schools (*N* = 453 pre-schools) in Berlin, Germany. The sample consisted of *N* = 99 in-service early childhood educators. The average age of the study participants is 45 years and 2 months. Only 5 participants were males.

To measure mathematical content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK), Rasch-scaled tests (Blömeke et al., 2015; Dunekacke et al., 2016) were applied. The MCK test consists of 24 items which are mainly multiple choice and partly in open format. Topics of the tests are: number and operations; measurement, quantity and relation; geometry; data, combinatorics and chance. The test measures MCK at a secondary school level. The MPCK test consists of 35 items. MPCK is tested in the following areas: fostering mathematical literacy in informal and formal settings, knowledge about development of mathematical literacy, diagnosing and promoting early mathematical skills. Most of the items are in multiple choice and some in open format.

A questionnaire is used to assess beliefs towards mathematics in general (Dunekacke et al., 2016). It is based on a questionnaire developed by Grigutsch, Raatz und Törner (1998) and the study of Benz (2012) and distinguishes three beliefs facets: a static orientation (SO, e.g. “Hallmarks of mathematics are clarity, precision and unambiguousness”), a process-related orientation (PO, e.g. “Mathematics is an activity involving thinking about problems and gaining insight”) and an application-orientation (AO, e.g. “Mathematics helps solving everyday problems and tasks”). The 17 items (6 AO-items, 7 SO-items and 4 PO-items) are rated on a 6-point Likert scale from “strongly disagree” to “strongly agree” (Dunekacke et al., 2016).

To analyse the data, we used structural equation modelling with the R-package lavaan and compared the fit indices of model A and B. Due to our sample size we used item parcel rather than the individual items.

RESULTS

While the model fit values of model A indicate a good fit to our data ($\chi^2(44) = 50.620$, $p = .229$, RMSEA = 0.039 [0.00,0.081], CFI = .987) the fit values of model B indicate a misfit of the model to our data ($\chi^2(49) = 119.326$, $p = .000$, RMSEA = 0.121 [0.094,0.149], CFI = .861). Therefore, only the results of model A are presented in figure 4.

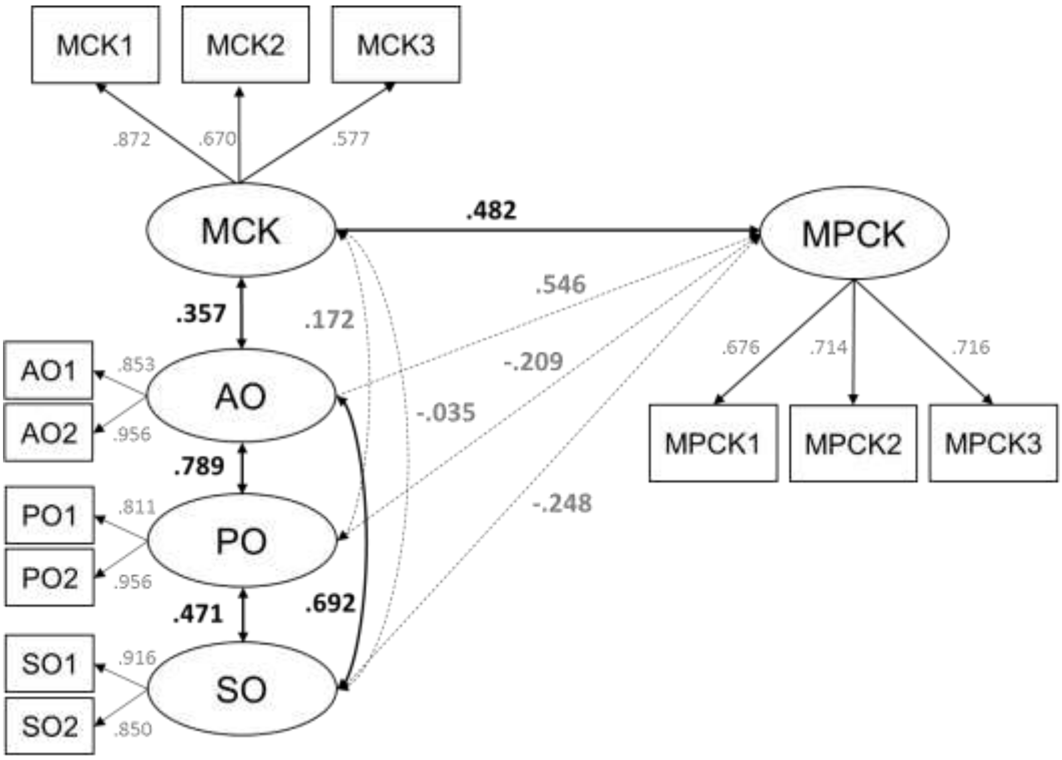


Figure 4: SEM results model A. Solid lines represent significant coefficients ($p > .05$), dashed lines represent non-significant coefficients.

The structural equation modelling results show a direct effect of MCK on MPCK while there is no significant relationship between the beliefs facets and MPCK. Furthermore, MCK is related to the application orientation but there is no significant relationship between MCK and the process-related orientation or between MCK and the static orientation. The beliefs facets are closely related. Even between the static orientation and the application orientation as well as between the static orientation and the process-related orientation a high positive correlation was found.

DISCUSSION

Previous research on the math-related competence structure of early childhood educators came to different results concerning the relationship of MCK, MPCK and the affective-motivational competence facet of in-service early childhood educators and prospective early childhood teachers. However, due to different constructs of MPCK and beliefs as well as different measuring instruments in each study the results are not directly comparable. In order to examine whether the different results can be related to the different samples or the different measuring instruments, we first examined possible differences between the samples. Therefore, we used the instruments developed by Dunekacke and colleagues (2016) on a sample of in-service early childhood educators and tested to which competence structure found in previous research our data fits best.

The results show that our data fits well to the model of Dunekacke and colleagues (2016) but not to the model of Oppermann and colleagues (2016). This indicates that the different results of the previous studies can be ascribed to the different instruments or constructs rather than the different samples. Looking at the instruments it seems reasonable to argue that Dunekacke and colleagues (2016) focus on a more cognitive facet of MPCK while the project of Oppermann and colleagues (2016) measures a more performance-related competence facet. Based on the different professional competence models (Gasteiger & Benz, 2016; Jenßen et al., 2015) this facet could be described as math-related perception. However, the professional competence models assume that the math-related perception is related to the MPCK as well as the MCK and the beliefs. In order to verify the assumption that the both projects actually measure different competence facets and that these facets are related as suggested in the competence models further studies should use the instruments of both studies and model the relationship between these measurements.

Concerning the relationship of the competence facets our results show a direct effect of MCK on MPCK. This fits to the results of Dunekacke and colleagues (2016). Looking at the beliefs we found that AO, SO and PO of in-service early childhood educators are strongly correlated. This fits partly to studies looking at secondary school teachers, as for example Grigutsch and colleagues (1998) did. However, studies with secondary school teachers typically found a negative correlation between AO and SO or PO and SO whereas our results indicate a positive correlation between these facets. As these results fit to the results of Dunekacke and colleagues (2016) who examined prospective early childhood educators with the same instrument, it could be argued that early childhood educators hold a static, a process and an application orientation at the same time. Concerning the relationship of knowledge and beliefs we found a correlation between MCK and AO, but not between MPCK and the beliefs. These results do not fit to the results of the studies with prospective early childhood educators (Dunekacke et al., 2016), where a relationship between the process-related orientation and MPCK as well as between the application-orientation and MPCK was found. This can be seen as an indicator of differences between the competence structure of prospective early childhood educators and in-service early childhood educators even though the general assumption of model A could be confirmed.

As a conclusion it can be pointed out that early childhood educators have rather undifferentiated beliefs towards mathematics in general. Perhaps the early childhood educators in our sample had unsystematic experiences with mathematics as a process or as applications and therefore are more likely to build up an undifferentiated beliefs structure (see also: Benz, 2012; Wittmann, Bönig, Levin & Schuler, 2016). Furthermore, on the basis of our results it can be assumed that MCK is a precondition for MPCK for prospective early childhood educators as well as for in-service educators if it is measured with the described instruments. Assuming that these instruments actually measure MCK and MPCK, this result indicates that MCK is a relevant competence facet for early childhood educators that should be further studied. Overall, it can be stated that we still know too little about the math-related knowledge of early childhood educators to make reliable statements. This is also shown by the differences in the competence structure between prospective and in-service early childhood educators found in this study. Hence, in our further research we would like to look at the knowledge facets more closely by using different measurements and examining their relationship to early childhood educators' performance. Thereby, we could also expand research results on the math-related competence structure to more performance related competence facets.

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PRE-SERVICE PRIMARY TEACHERS' PROFILES OF NOTICING STUDENTS' PROPORTIONAL REASONING

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The aim of this research is to characterise how pre-service primary teachers notice students' proportional reasoning. 83 pre-service teachers analysed students' answers to 12 problems related to fractions, proportional and non-proportional situations and ratios in comparison situations. Students' answers show different characteristics of proportional reasoning. Four profiles of pre-service primary teachers have been identified according to how they described and recognised students' reasoning.

INTRODUCTION AND THEORETICAL BACKGROUND

Recent research has shown that being able to identify relevant aspects of teaching and learning situations and interpret them to take instructional decisions (Mason, 2002) is an important teaching skill (professional noticing) (Mason, 2002; Sherin, Jacobs, & Philipp, 2010). Jacobs, Lamb and Philipp (2010) characterise the skill of noticing students' mathematical thinking as three interrelated skills: (i) attending to students' strategies that implies identifying important mathematical details in students' answers; (ii) interpreting students' mathematical reasoning taking into account the mathematical details previously identified; and (iii) deciding how to respond on the basis of students' reasoning.

Recently, some studies have focused on characterising the skill of noticing students' mathematical thinking in different domains such as pattern generalization (Callejo & Zapatera, 2016), derivative concept (Sánchez-Matamoros, Fernández, & Llinares, 2016), early numeracy (Schack, Fisher, Thomas, Eisenhardt, Tassel, & Yoder, 2013) and ratio and proportion (Buforn, Fernández, Coles, & Brown, 2015; Son, 2013). These previous studies have shown that noticing students' mathematical thinking is not an easy task for pre-service teachers, showing a complex relationship between the skills of attending to, interpreting and deciding and the different domains of pre-service teachers' knowledge (Ball, Thames, & Phelps, 2008).

Our study is embedded in this line of research and focuses on analysing how pre-service teachers notice students' mathematical thinking in the specific domain of proportional reasoning.

The development of proportional reasoning

The development of proportional reasoning is complex since it implies the understanding of the fractional scheme, the interpretation of ratios in comparison

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situations (both qualitative and quantitative), the ability to stablish multiplicative relationships between two quantities and extend this relation to another pair of quantities, and the discrimination between proportional and non-proportional situations (Kieren, 1993; Lamon, 2005; Pitta-Pantazi & Christou, 2011).

The fractional scheme consists of five sub-constructs: fraction as *part-whole* is defined as the relationship between the number of congruent parts in which a continuous quantity (or a set of discrete objects) is partitioned and the whole; fraction as a *measure* can be considered as a number which conveys the quantitative personality of rational number, its size; it is also associated with the measure assigned to an interval (the number line can be used as a representation of this measure assigned to an interval); fraction as *quotient* can be seen as a result of a fair share; fraction as *operator* is seen as a function applied to a number, object or a set. Finally, *reasoning up and down* that can be seen as the cognitive mechanism of representing a fraction from other fraction given.

Ratio comparison situations (both qualitative and quantitative) imply the ideas of: *covariance* that determines the relationship between two quantities in such a way that when one quantity changes, the other also changes (covariance) in a particular way with respect to the first quantity, *ratio* as a comparative index and *unitizing process* as the cognitive process of mentally chunking or restructuring a given quantity into familiar or manageable or conveniently sized pieces to operate with that quantity (Lamon 2007).

Furthermore, previous research has shown that proportional reasoning implies not only understanding the multiplicative relationship between quantities in a proportional situation, but also the ability to discriminate proportional and non-proportional situations (Cramer, Post, & Currier, 1993; Van Dooren, De Bock, Janssens, & Verschaffel, 2008). Therefore, students should be able to solve missing-value proportional situations and discriminate between missing-value proportional and non-proportional situations.

Our research question is:

- How do pre-service teachers recognise characteristics of students' proportional reasoning?

METHOD

Participants and the task

The participants were 83 pre-service primary teachers (PTs) in the third year of an initial teacher education program at the University of Alicante (Spain). In previous years, pre-service teachers had attended a subject focused on numerical sense (first year) and a subject focused on geometrical sense (second year). In the third year, they were attending a mathematics method course related to the teaching and learning of mathematics in primary school. One of the units of this course was about teaching and learning of the fraction concept and proportional reasoning. The aim of this unit is to focus pre-service teachers' attention on primary school students' fractional and proportional reasoning. Data were collected after this unit.

Pre-service teachers had to interpret three primary school students' answers to 12 primary school problems related to the fractional scheme (6 problems), the discrimination between proportional and non-proportional situations (2 problems) and the interpretation of ratios in comparison situations (4 problems). Students' answers show different characteristics of fractional and proportional reasoning. Pre-service teachers answered two questions related to the problem and students' answers (Table 1).

Questions	Aim
a) About the problem: What mathematical concepts should a primary school student know to solve this problem? Explain your answer.	Identifying the mathematical elements of the problem
b) About students' answers: What are the characteristics of students' mathematical reasoning involved in each answer? Explain your answer.	Recognising characteristics of students' mathematical reasoning

Table 1: Questions for pre-service teachers

Figure 1 shows the three students' answers to the *parte-whole* problem and the *reasoning up and down* problem. In the *part-whole* problem, the answer 1 shows the idea of fraction as parte-whole; the student of answer 2 uses the fraction as operator to obtain how many spots are 2/3 of 18 spots; and the student of answer 3 has difficulty with the part-whole relationship. In the *reasoning up and down* problem, the student of answer 1 does not recognise the whole (three small rectangles); the student of answer 2 recognises the whole but he does not obtain the fraction that represents 4 small rectangles; and the student 3 recognises the whole and obtains the fraction that represents 4 small rectangles.

Analysis

Data of this study are pre-service teachers' answers to the two questions (Table 1). The answers to each question were analysed individually by three researchers and agreements and disagreements were discussed. We focused on the mathematical elements of the problem identified by preservice teachers (the identification of the mathematical element was coded with a 1, and when the element was not identified with a 0). With regard to the second question, we focused on how pre-service teachers interpreted students' reasoning: if they used the mathematical elements of the problem to describe the students' answers and recognise characteristics of students' mathematical reasoning (it was coded with a 1) or if they provided general comments based on the correctness of the answer (it was coded with a 0). We carried out a Cluster Analysis using the SPSS. From this analysis, we have inferred four profiles of pre-service teachers that differ in how they had identified the mathematical elements involved in each problem and how they had used these mathematical elements to describe and recognise students' reasoning (Figure 2).

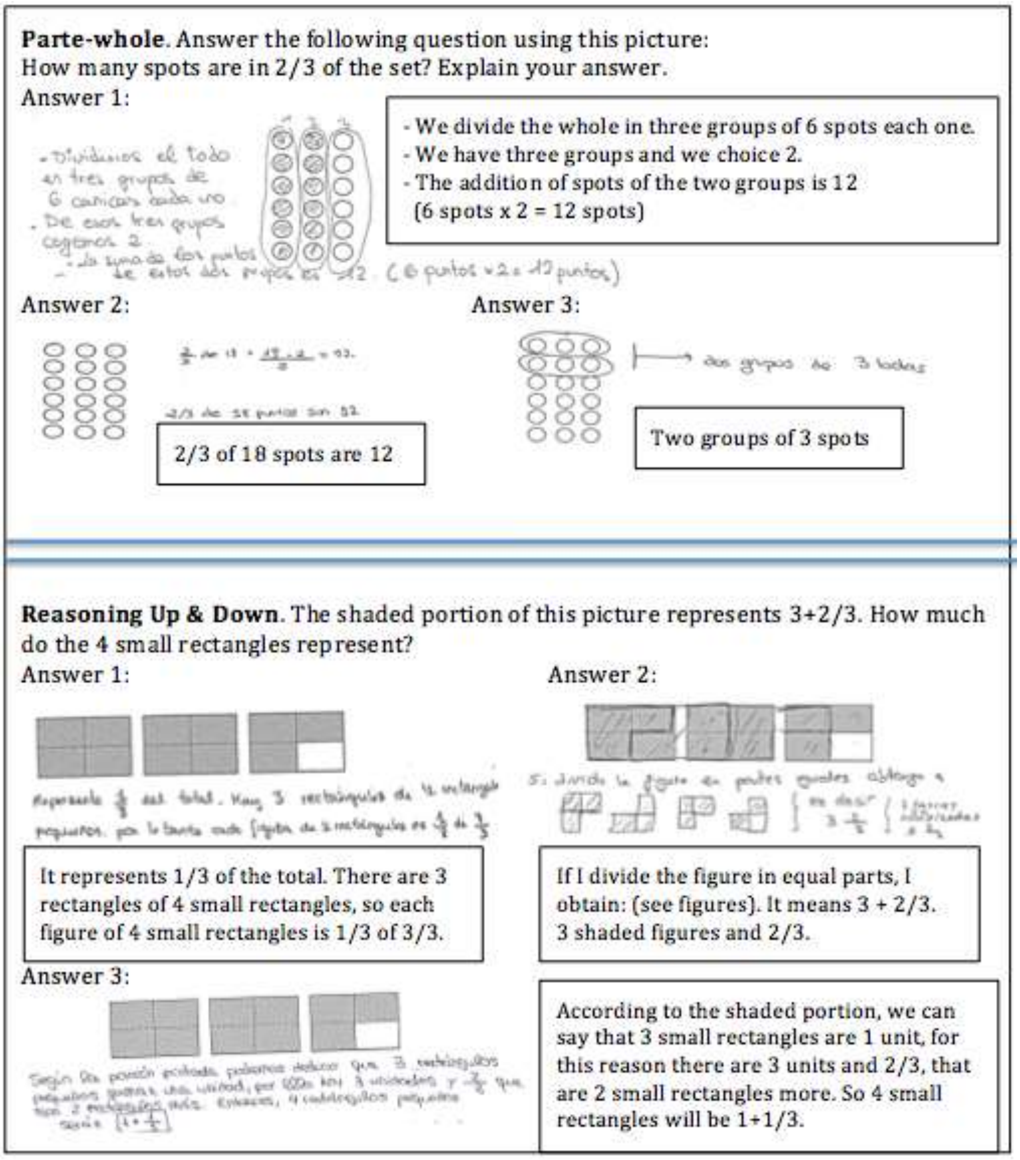

Figure 1: Students' answers to the *part-whole* and *reasoning up and down* problems

RESULTS

From the four profiles of pre-service teachers inferred (Figure 2), we can underline two main results. Firstly, the relationship between identifying the mathematical elements and recognising students' mathematical reasoning is not linear. Secondly, the mathematical elements implied in proportional reasoning were not identified by pre-service teachers in the same way and this influenced how they described and recognised students' reasoning.

Only 70 out of 82 pre-service teachers were grouped in the four profiles inferred from the Cluster Analysis. Thirty-six out of these 70 pre-service teachers had difficulties in recognising characteristics of students' reasoning (Profiles 0 and 1) and provided general comments based on the correctness of the answer such as: "*answer 1 is* *correct; answer 2 is correct; answer 3 is not correct, the student doesn't understand the concept*". Sixteen out of these 36 pre-service teachers identified the mathematical elements of the problems related to the fractional scheme (except *reasoning up and down*) and started to recognise some characteristics of students' fractional reasoning (Profile 1). For instance PT008 identified the mathematical elements involved in the problems of the fractional scheme (except *reasoning up and down*) and recognised characteristics of students' reasoning in the *part-whole* problem: *"Answer 1: he uses the fraction as a part-whole because he identifies 2 groups of the set. Answer 2: he uses the fraction as operator and it is correct because he interprets that 2/3 is what is required and he has 18 spots. Answer 3: he doesn't consider the whole"*. The characteristics of profile 1 suggest that identifying the mathematical elements of the problems is not enough to recognise characteristics of students' reasoning.

Profile 0 (20 PTs) Pre-service teachers who did not identify the mathematical elements and did not recognise characteristics of students' reasoning	**Profile 1** (16 PTs) Pre-service teachers who identified only the mathematical elements of the **fractional scheme** (except *reasoning up and down*) and started to recognise some characteristics of students' reasoning related to the **fractional scheme**	**Profile 2** (18 PTs) Pre-service teachers who identified the mathematical elements of **fractional scheme** and the **discrimination between proportional and non-proportional situations** and recognised characteristics of students' reasoning related to these problems	**Profile 3** (16 PTs) Pre-service teachers who identified the mathematical elements of **fractional scheme, discrimination between proportional and non-proportional situations and ratios meaning in comparison situations** and recognised characteristics of students' reasoning related to these problems

Figure 2: Profiles of pre-service teachers inferred from the Cluster analysis

However, the characteristics of profiles 2 and 3 suggest that identifying the mathematical elements of the problems let pre-service teachers recognise characteristics of students' reasoning. For instance, a pre-service teacher of the profile 2 (PT025-Table 2) identified the mathematical elements related to the fractional scheme and the discrimination between proportional and non-proportional situations and also recognised the characteristics of students' reasoning related to these problems. However, this pre-service teacher did not identify the elements related to the interpretation of ratios in comparison situations and did not recognise characteristics of students' reasoning in these problems, providing general comments based on the correctness of the answer (see the answer to the ratio problem in Table 2).

Problem	PT's answer
Reasoning up and down	*"a) Whole and unit fraction b) Answer 1: the student recognises the whole and the unit fraction, so then, he can represent and recognise the fraction that represents 4 rectangles. Answer 2: the student recognises the whole and the unit fraction, but he cannot represent the fraction of 4 rectangles. Answer 3: the student doesn't understand the problem, he cannot recognise the whole"*
Missing value proportional problem	*"a) Proportionality [...] relationship between quantities. b) Answer 1: he solves the problem using external ratios because he relates multiplicatively the screws of both machines. Answer 2: the student uses a building-up strategy because he adds 120 for each 40 screws of machine R until getting how many screws will have the machine J if R has 200. Answer 3: the student doesn't recognise the problem as a proportional problem [...] he solves it using an additive strategy."*
Missing value non-proportional problem	*"a) It is not a proportional situation. It is an additive situation. b) Answer 1: he understands the problem and he solves it identifying the difference between both companies. Answer 2: the student calculates the difference in company A, and then this difference is added to the first quantity of company B. Answer 3: the student solves the problem through a multiplicative strategy (double) so, the interpretation and the result are incorrect."*
Ratio	*"a) Ratios between quantities b) Answer 1: he interprets correctly the relationship between quantities. He compares and gives the correct answer. Answer 2: he interprets correctly the relationships between quantities. Answer 3: The answer is correct."*

Table 2: Answers of PT025 (PT of the Profile 2)

Our results also provide evidence that pre-service teachers identified the mathematical elements involved in proportional reasoning and recognised characteristics of students' reasoning differently. In fact, the mathematical elements of *fractional scheme* (except *reasoning up and down* sub-construct) were identified easier than those related to the *discrimination between proportional and non-proportional situations* and, the identification of the mathematical elements involved in *ratio comparison situations* was a difficult task for pre-service teachers.

DISCUSSION AND CONCLUSION

The four profiles describe how pre-service teachers recognise characteristics of students' proportional reasoning. Profile 1 shows that pre-service teachers can identify the mathematical elements in the *fractional scheme* problems but cannot recognise characteristics of students' reasoning in these problems. Furthermore, profiles 2 and 3 suggest that when pre-service teachers identify the mathematical elements of the problem, they are able to recognise characteristics of students' reasoning. This data points out that identifying the mathematical elements of problems is necessary but not sufficient to recognise characteristics of students' proportional reasoning, suggesting that the relationship between the skills of identifying and recognising is not linear, and showing preservice teachers' difficulties in recognising characteristics of students' reasoning. Our result is in line with previous research showing the complex relationship between the knowledge of mathematics and the knowledge of mathematics and students (Callejo, & Zapatera, 2016; Sánchez-Matamoros et al., 2015).

Furthermore, our results show that this complex relationship could be linked to the specific mathematical elements of the domain in this case, proportional reasoning since these elements were not identified and recognised by pre-service teachers in the same way. In fact, there were pre-service teachers who only identified the mathematical elements and recognised some characteristics of the students' reasoning in problems related to the fractional scheme (Profile 1); pre-service teachers who identified the mathematical elements and recognised characteristics of the students' reasoning in problems related to the fractional scheme and the discrimination between proportional and non proportional situations (Profile 2); and pre-service teachers who identified the mathematical elements and recognised characteristics of the students' reasoning in problems related to the fractional scheme, the discrimination between proportional and non-proportional situations and the interpretation of ratios in comparison situations (Profile 3). These profiles suggest that recognising characteristics of students' reasoning is more difficult in ratio comparison situations than in the discrimination between proportional and non-proportional problems. Finally, recognising characteristics of students' reasoning of fractional scheme was the initial step in recognising students' proportional reasoning.

This information provides data to conjecture a pre-service teachers' trajectory of noticing students' mathematical thinking of proportional reasoning that can be useful for teacher education programs.

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A LEARNING TRAJECTORY FOR LENGTH AS A MAGNITUDE AND ITS MEASUREMENT: USAGE BY PROSPECTIVE PRESCHOOL TEACHERS

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Abstract

This paper characterises the use made by Prospective Preschool Teachers (PPT) of a learning trajectory for length as a magnitude and its measurement, in order to interpret the characteristics of learning displayed by Early Years Education pupils and to take appropriate decisions and action. A total of 64 PPT enrolled on a teaching module about length and its measurement in Early Years Education were asked to analyse the responses of various children to a task. The results show that only 23 PPT were able to identify the understanding of one or more pupils. Of these, only 10, displayed a structured perspective on the responses of the Early Years Education pupils, by proposing activities based on inferred understanding.

THEORETICAL FRAMEWORK

This paper examines the professional perspective of prospective preschool teachers with regard to using a learning trajectory for length as a magnitude and its measurement. Various studies have examined this issue from the perspective of the pupil (Sarama & Clements, 2009; Van den Heuvel-Panhuizen & Buys, 2005), but few have done so from the perspective of the teacher (O'Keefe & Bobis, 2008).

Research into the professional development of mathematics teachers has highlighted the importance of the teaching competence of analysing the teaching and learning of mathematics from a professional perspective (Mason, 2002; Jacobs, Lamb & Philipp, 2010). The development of this teaching competence is one of the learning goals of teacher training programmes and an important strand of recent research dedicated to the teaching of mathematics. Jacobs, Lamb and Philipp (2010) characterise this teaching competence using three interrelated skills: (a) identifying relevant elements in pupils' responses; (b) interpreting pupils' understanding; and (c) deciding which actions should be developed in class. Some research has used learning trajectories to develop this competence, focusing on different topics (Fernández, Llinares & Valls, 2011 for problems of addition and proportions; Sánchez-Matamoros Fernández & Llinares for derivatives; Schack et al., 2013 for the study of early arithmetic; Wilson, Mojica & Confrey, 2013 for equal partitioning), but none of them has examined the topic of length and its measurement. These studies have shown the potential of learning trajectories to develop the three skills encompassed by the professional perspective.

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Sarama and Clements (2009) described a learning trajectory for length and its measurement in Early Years Education. This trajectory consists of: (a) a learning goal; (b) progression in the learning described based on relevant elements of length as a magnitude (recognition of length, conservation and transitivity) and the measurement of length (equal partitioning, unit of measurement, uniqueness, iteration, accumulation, universality of measurement, and the relationship between the number and the unit of measurement) and the transitions between the different levels of learning (one particular note is the inclusive nature of these levels); (c) instructional activities over the course of the sequence. A professional perspective on pupils' mathematical thinking supported by a learning trajectory implies having a structured perspective on this trajectory that entails: identifying the relevant elements, interpreting pupils' understanding within the framework of progression in learning, and finally, making instructional decisions in accordance with the inferred understanding.

Within this framework, the research questions tackled here are as follows: Which relevant elements do Prospective Preschool Teachers (PPT) identify in the responses of pupils (5-6 years old) in an activity about length and its measurement? To what extent do PPT have a structured perspective on pupils' responses?

METHOD

The participants were 64 PPT enrolled on the subject 'Learning of Geometry', in the sixth term of the 'Teacher Training Degree for Early Years Education'. One of the modules on the subject was the study of "Length and its measurement in Early Years Education". The PPT were given a theoretical document with the learning trajectory adapted from Sarama and Clements (2009) (Figure 1) and they were set professional assignments that involved analysing teaching situations related with the three skills encompassed by the professional perspective: identifying relevant elements, interpreting the understanding shown by the children, and proposing decisions and actions to allow the children to further their understanding. This paper focuses on how the participant PPTs in this study analysed the responses given by 5 and 6-year-old children in an activity involving making necklaces (Figure 2).

Level	Development progression
1	● Recognise the concept of length as a magnitude: - Identify the qualities of length. ● Make direct comparisons, considering length intuitively.
2	● Recognise the conservation of length: - Make direct comparisons through the displacement of objects.
3	● Use the transitive property to: - Make indirect comparisons - Order objects. - Measure lengths.

4	● Make equal partitions of objects. ● Identify a unit and make iterations of it: - Recognise the property of accumulation.
5	● Recognise the universality of the unit of measurement. ● Recognise the relationship between number and unit of measurement. ● Start to make estimations

Figure 1: Progression in the learning of length and its measurement (adapted from Sarama and Clements 2009)

Alicia is an Early Years teacher in a state school. Her pupils are between 5 and 6 years of age. A week ago, she began working with them on the concept of length as a magnitude and its measurement. Today, in art class, she invites the children to make necklaces using different materials (coloured beads and different types of pasta tubes) and different lengths of string (A, B and C):

Accessories for the necklaces

Once she has explained the task, the children choose their pieces of string and accessories and start making necklaces. When all the necklaces are finished, Alicia asks the children:

Teacher: *Who's made the longest necklace?*

Mario: *I've made my necklace with the piece of string that looks like a stick* [string C] *and I've used 13 pasta tubes* [he has used different types of pasta tubes].

Almudena: *Miss, I've made a necklace with the pink string* [string A] *and I've used 15 stars* [the stars are very spaced out].

Luis: *Mine has 12 pasta tubes* [he has used all the same type] *and I chose the string that looks like a spiral* [string B], *but it's longer than Mario's because the piece of string is longer.*

Elena: *I also used the pink string* [string A] *and I used 20 stars* [the stars are close together].

Almudena: *So Elena's necklace is the longest one of them all.*

Based on these responses, Alicia asks the children:

Teacher: *Do you agree?*

Mario: No *miss, I don't agree with Luis, because mine has more pasta tubes.*

Figure 2: Responses given by Early Years Education pupils in the necklace-making activity

The data from this research are the responses given by PPT to the following questions:

Question 1. Indicate the mathematical elements that, from the teacher's perspective, are necessary in order to compete the task.

Question 2. At which level of understanding would you place each of the children in the dialogue? Give reasons for your answer based on the characteristics shown and provide justification using the children's interventions.

Question 3. Imagine you are Alicia. Decide which child you consider to have the lowest level of understanding, and which child you consider to have the highest level of understanding, and then suggest an activity to further their respective understandings of length and its measurement.

The characteristics of the children's responses and the goal of the activities that might be proposed to advance their learning are given in Figure 3. Mario and Almudena are at the lowest level of progression (level 1) and Luis and Elena are at the highest level (level 4).

By means of an inductive process, we were able to identify certain characteristics in the responses of the PPT by applying the method of constant comparison (Strauss & Corbin 1994). To ensure the validity and reliability of the analysis, a group of five researchers first analysed a small sample, on the basis of which they discussed the encodings and their relationships with the evidence, in order to create several categories. Once they reached an agreement, new data were added with a view to verifying the system of categories created initially and confirming its validity.

<table>
<tr><th>Children</th><th>Level</th><th>Characteristics</th><th>Learning goals</th></tr>
<tr><td>Mario</td><td rowspan="2">1</td><td>There is evidence that they
• DO NOT understand the conservation of length (magnitude).
• DO NO consider the uniqueness of the quantity taken as a unit</td><td rowspan="2">Appreciate the conservation of length</td></tr>
<tr><td>Almudena</td><td>There is evidence that they
• DO NOT understand the conservation of length (magnitude).
• DO consider the uniqueness of the quantity taken as a unit
• DO NO consider the iteration of the unit of measurement</td></tr>
<tr><td>Luis</td><td rowspan="2">4</td><td>There is evidence that they
• DO understand the conservation of length (magnitude).
• DO consider the uniqueness of the quantity taken as a unit, along with iteration and accumulation. There is no evidence that they establish a relationship between number and measurement
There is evidence that they
• DO NOT make use of the inverse relationship between number and measurement</td><td rowspan="2">Acquire the universality of the unit of measurement

Establish the relationship between the number and the unit of measurement</td></tr>
<tr><td>Elena</td><td>There is evidence that they
• DO consider the uniqueness of the quantity taken as a unit, along with iteration and accumulation. There is no evidence that they establish a relationship between number and measurement</td></tr>
</table>

Figure 3: Characteristics of the children's responses and learning goals

Analysis of the PPTs' responses to the three questions asked was divided into three stages. The first stage involved analysing questions 1 and 2 together, and verifying whether the PPT identified the relevant elements in the learning progression of length as a magnitude and its measurement. This stage identified four groups of students according to whether they identified the relevant elements or not, and whether they were related with magnitude, measurement, or both. The second stage analysed whether the PPT from each of the groups used these elements to interpret the understanding of Early Years Education pupils (question 2). The third stage analysed whether the PPT who had interpreted the learning characteristics of any of the pupils had proposed instructional decisions based on inferred understanding (question 3).

RESULTS

This section describes (1) the relevant elements identified by the PPT to interpret understanding, and (2) the interpretation of understanding and instructional decisions proposed on the basis of the understanding identified.

Relevant elements identified to interpret understanding

17 of the 64 PPT did not identify any relevant elements or simply named them or used them incorrectly. The remaining 47 participants identified relevant elements pertaining only to length as a magnitude (group 1), only to the measurement of length (group 2), or both to length as a magnitude and its measurement (group 3). The most frequently identified elements were 'conservation', 'uniqueness' and 'iteration', followed by 'accumulation'.

Group 1 comprised 8 PPT. The common characteristic of this group was that they only identified the element 'conservation' of length as a magnitude. These PPT should be able to interpret the understanding of Early Years Education pupils with the lowest level of comprehension (Mario and Almudena), since there is evidence that these pupils do not understand the conservation of length (Figure 3). To identify the understanding of the other pupils, they would need to use relevant elements of measurement. Six of them interpreted the understanding of Mario and/or Almudena (Table 1).

Group 2 comprised the 29 PPT who identified elements of measurement. Their common characteristic was that they identified the elements 'uniqueness' or 'iteration': 20 identified both elements, 2 the element 'uniqueness', and 7 the element 'iteration'. 11 participants identified 'accumulation'. These PPT should be able to interpret the understanding of Early Years Education pupils with the highest level of understanding (Luis and Elena) who have acquired these elements, but not those with the lowest level of understanding, because they have not identified elements of magnitude. Two of them interpreted the understanding of Luis and/or Elena (Table 1).

Group 3 comprised 10 PPT who identified elements of magnitude and of measurement. Of these, 8 identified the 'conservation' of length: 6 'uniqueness' and 'iteration', and 1 'iteration'. Other elements identified were the 'recognition' of length as a magnitude (4) and 'accumulation' (3). These PPT, with the exception of the 2 participants who did not identify 'conservation', should be able to interpret the understanding of all the Early Years Education pupils in the task set. Nine of them interpreted the understanding of one or more of the pupils (Table 1), 3 interpreted the understanding of pupils with the lowest level of understanding (Mario and Almudena), 2 interpreted the understanding of pupils with the highest level of understanding (Luis and Elena), and 4 participants interpreted the understanding of pupils with the lowest and the highest levels of understanding.

Interpretation of understanding and instructional decisions

Only 10 of the 23 PPT who identified the characteristics of understanding proposed activities that were coherent with the inferred understanding (Table 1). Regarding the activities proposed, the 3 PPT in Group 1 who proposed activities based on the pupils' understanding focused on the acquisition of the ‘conservation’ of length. The 2 PPT from Group 2 proposed activities related with ‘iteration’ and ‘uniqueness’. Of the 4 PPT who interpreted the understanding of pupils with the lowest and highest levels of understanding, only 3 proposed coherent activities with the inferred understanding.

		Correct interpretation of understanding	Activities proposed based on understanding
Group 1. Magnitude (N=8)	Low level	6	3
Group 2. Measurement (N=29)	High level	8	2
Group 3. Magnitude and measurement (N=10)	Low level	3	1
	High level	2	1
	Low and high level	4	3
Total (N=47)		23	10

Table 1: Students who interpreted the understanding of some of the Early Years Education pupils and proposed activities that were coherent with said interpretation

Below we present two cases of PPT: one who did not propose activities that were coherent with the interpretation given and one who did.

E2-8 (Group 2) identified characteristics of the understanding of Luis and Elena (high level of understanding) based on the elements ‘uniqueness’ and ‘iteration’, indicating that: “Luis carries out equal partitioning [referring to uniqueness], he chooses the same pasta tubes, so he’s at level 4”; “Elena always uses the same stars, so she knows how to carry out equal partitioning [referring to uniqueness] and also iterates correctly, without leaving any gaps”. However, the decision made regarding further instruction did not favour their progression (see Table 2): “For children with a higher level of understanding, which in this case would be Luis and Elena, I would work again with the beads, perhaps using a ruler so that they can see that not all the beads, lengths of string, and pasta shapes measure the same.”

PPT E1-19 (Group 3), on the other hand, was able to identify the characteristics of Mario’s understanding based on ‘conservation’, indicating that: “Mario is at level 1, since he does not recognise conservation; he has not noticed the measurement of the lengths of string [different shapes and sizes]; he only focuses on the number of pasta tubes: ‘mines has more pasta tubes’”. For Mario, this participant proposes an instructional decision that would allow him to acquire conservation: “To work with Mario, I would make a direct comparison, both static and using movement, by varying them [the lengths of string of different shape and size], to recognise the importance of the length of the string and in order to work on the concept of conservation and transitivity. To do this, I would put the pieces of string in front of him [referring to the string used in the task]”

Furthermore, E1-19 identified the characteristics of Luis' level of understanding based on the elements identified: 'uniqueness', 'iteration', 'conservation' (level 4), indicating that: "Luis is at level 4, because he correctly understands equal partitioning [referring to uniqueness, since he has threaded the string with the same sized pasta tubes], with no gaps or overlapping [iteration]… he knows that the pieces of string are different lengths and that his is longer than Mario's [referring to conservation] "it's longer than Mario's because the piece of string is longer". He's not guided by the accessories". For Luis, the instructional decision proposed is that he should now acquire the relationship between number and measurement: "I would give them necklaces with different accessories and with the same ones, and I would see if they were able to count them like that and see that the measurement remains the same [even thought the number of beads does not]".

CONCLUSION

In this study, the following questions were posed: Which relevant elements do PPT identify in the responses given by Early Years Education pupils (5 years of age) to an activity? To what extent do the PPT have a structured perspective on the pupils' responses?

Firstly, we should point out that the PPT found it easier to identify the elements of 'conservation', 'uniqueness', and 'iteration', although on some occasions they used the term 'equal partitioning' to refer to 'uniqueness". Using these elements, 23 PPT were able to identify the understanding of one or some of the children. Of these, only 10 demonstrated a structured perspective on the responses given by the Early Years Education pupils, since they were also capable to propose activities based on the inferred understanding. This structured perspective was either partial (4 PPT had a structured perspective on the first part of the trajectory referring to length as a magnitude, and 3 on the second part referring to measurement) or total (3 EPM). This seems to show that the learning trajectory is difficult to assimilate in its totality, and that PPT gradually learn it 'bit by bit'.

The PPT who had a structured perspective on the first part of the trajectory only, based on the element 'conservation', were capable to identify understanding and made adequate instructional decisions to a greater extent than those who had a structured perspective on the second part of the trajectory only and who had identified more relevant elements. This could be due, on the one hand, to the fact that interpreting the understanding of pupils with a low level of understanding requires the use of fewer relevant elements than interpreting the understanding of students with a higher level of understanding. On the other hand, it is easier to propose instructional decisions for Early Years Education pupils who have difficulty resolving the proposed activity than for those who did not have difficulties.

Furthermore, the majority of the PPT (29 out of 47) only focused on the second part of the trajectory related with measurement, which shows that they have not perceived the inclusive nature of the different levels of learning progression.

The PPT found it easier to identify certain relevant elements, probably due to the characteristics of the task. For this reason, it is important to propose that PPT analyse the responses of children to different types of activities that show the diversity of elements involved in the understanding of length as a magnitude and its measurement.

It is also necessary to emphasise the type of instructional decisions associated with each of the levels of understanding, given that the PPT found it difficult to make appropriate decisions based on inferred understanding, particularly for pupils with a higher level of understanding.

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A HYPOTHETICAL LEARNING TRAJECTORY FOR SPANNING SET AND SPAN

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In this paper we interpret how a hypothetical learning trajectory (HLT) promotes the learning of spanning set and span. The focus of interest is an HLT in linear algebra to contribute to learning in this course. The theoretical basis of our research is based on the instructional design heuristic of emergent models (Gravemeijer, 1999) and mathematical modelling (Julie & Mudaly, 2007). The refinement of the HLT was part of a research design process of 3 cycles of experimentation. The results show the potential of our HLT to support the learning of these linear algebra concepts.

INTRODUCTION

In recent years, the research aimed at the development of learning trajectories has been the object of much attention (Weber & Lockwood, 2014). A learning trajectory, according to Confrey and Maloney (2015), is a conceptual model of how students transition from their informal knowledge to more sophisticated knowledge when they engage with a carefully sequenced set of tasks. For Simon (1995), a hypothetical learning trajectory (HLT) is composed of: the objective of the learning, the instructional tasks, and the hypothesis about the learning process of the students.

Recently, there have been studies on HLTs for linear algebra. Trigueros and Possani (2011) have designed an HLT for the concepts of linear combination, linear dependence and linear independence, and they conclude that it contributes to their learning. On the other hand, Wawro, Larson, Zandieh and Rasmussen (2012) have presented a HLT on the concept of linear transformation and its relation with the multiplication of matrices. They show that it helps overcome the difficulties of the conceptual relationship between linear transformation and matrices. Considering the results of these studies, in this research, a HLT is designed and experimented for spanning set and span. These concepts are chosen because of their relationship with important contents of this course, such as: base and dimension (Stewart & Thomas, 2010).

The objective of this research is to interpret how an HLT, based on the heuristic of emergent models and mathematical modelling, promo learning of spanning set and span concepts.

With this study, we intend to contribute in the design of HLTs for linear algebra to help to their learning. The intention is that this learning trajectory design supports teachers in creating models of thinking of their students. These will serve as a basis for seeking

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pedagogical responses that will help students to transit to their learning in the realm of linear algebra.

THEORETICAL BASIS

The focus of interest is a HLT in linear algebra. The theoretical basis of our research is based on the instructional design heuristic of emergent models (Gravemeijer, 1999) and mathematical modelling (Julie & Mudaly, 2007).

To initiate the intended learning, an initial task is chosen that is experientially real for the student (Gravemeijer, 1999). In our case, we chose a task within a real-life context, in which mathematical modelling serves as a tool to help the study of mathematics (Julie & Mudaly, 2007). We consider the modelling cycle proposed by Blum and Leiss (2007) to guide students in solving this task.

Once the initial task is defined, tasks are designed or selected that will allow students to achieve learning. In this research, the design of the tasks is guided by the instructional design heuristic known as emergent models (Gravemeijer, 1999), that seeks to create a sequence of tasks in which students first develop models-of informal mathematical activity that later become models-for their more sophisticated mathematical reasoning.

To progress from a model of informal mathematical activity to a model of formal mathematical reasoning Gravemeijer (1999) established four levels of activity: situational (interpretation and solution of the problem in a particular setting), referential (involving models, descriptions, concepts and procedures that address the problem of situational activity), general (developed through exploration, reflection and generalization as seen in the previous level but with a mathematical focus on the strategy without making reference to the problem), and formal (working with conventional methods and notations).

METHODOLOGY

The methodology for this study is the design research. This research aims to investigate the possibilities of educational improvement through the creation and study of new forms of learning (Gravemeijer & van Eerde, 2009). In the first phase, a HLT was elaborated that comprised the previous knowledge of the students (Simon, 2014). In the experimental teaching phase, three cycles were developed in which the initial HLT was refined. Of the participants in the third cycle, 3 students (18-19 years) were chosen who were representatives of an average level of the class, who had not previously worked with the mathematical modelling, nor had previously studied the concepts of spanning set and span. The objective of the learning was to understand the concepts of spanning set and span with an instructional design designed ad-hoc based on the heuristic of emergent models and mathematical modelling. The teacher guided the 3 students (S1, S2, S3) in the resolution of the tasks and, he led the discussion of topics that he considered relevant to favour learning. A synthesis of the HLT is presented in the Table 1.

Objective	Major task features	Major conjecture of learning trajectory
Students use vectors and the modelling cycle to create a mathematical model.	Task 1: create a generator password generator based on vectors.	(1) Students read information from the secure passwords; (2) students created a generator password by following the steps of the modelling cycle and using vectors.
Students identify features of spanning set and span; and they distinguish between them;	Task 2: make an analogy table between their password generator and the concepts of spanning set and span.	(1) Students find two sets from their password generator (one which has all the vectors that allow the generate numerical passwords to be generated and the other which contains the vectors that, after creating the linear combination, is obtained by the vector for each numeric password with them); (2) students identify common features between two sets connected to their password generator and the concepts of spanning set and span; (3) students distinguish between spanning set and span with the analogy table.
Students deduce properties of spanning set and span.	Task 3: Conjecture what the range of the array should be that has as its rows vectors of a set of R^2 for that set to generate R^2.	(1) Students explore particular cases and identify some regularity; (2) students relate spanning set and span with other concepts; (3) students conjecture the number of vectors of a set to generate R^2; (4) students determine that the number of vectors is not determinative to indicate if a set generates the span of R^2.
Students apply spanning set and span.	Task 4: Indicate if the set C={(1,0,0,1),(0,1,0,0)} is a spanning set for span W={(x,y,z,w)/x=w}.	Students to pose a solution: (1) they explore possible routes for resolution; (2) they find a spanning set or span (according to the resolution they decided); (3) they verify if the set C is a generator of W.

Table 1: Synthesis of the HLT

RESULTS

The results show the potential of our HLT to support the learning of these linear algebra concepts and how the mathematical modelling was a tool to support the study of mathematics, as posited by Julie and Mudaly (2007).

Students' actual learning trajectory (ALT) was reconstructed, and their degree of matching with the HLT (see Table 1) was analysed to interpret their learning progress of spanning set and span. The following results are presented: the activation of previous conceptions of vectors, the requirement for a greater cognitive demand, the real context to progress towards a more abstract level and the application of the concepts of spanning set and span.

The activation of previous conceptions of vectors

In the task 1, the students' ALT was a close match with the HLT (see row 1 in Table 1) because they were guided through the mathematical modelling cycle (Blum & Leiss, 2007) to create their password generator. Also, they activated their previous conceptions of vectors to propose a mathematical model, given that the student S3 asked to his companions "*What operations are we going to do with the vectors? Are we going to add them up, multiply them or something like that?*". Student S1 suggested "*A number that multiplies each vector and then, add the vectors*". That is, implicitly, student S1 proposes to make a linear combination. Then, student S3 asked about the number of vectors of his model to which student S2 responded "*three vectors*". The ideas proposed by the three students led them to propose a linear combination of vectors of the span of R^3 as a mathematical model, $a(1,0,1)+b(1,1,0)+c(0,1,1)$. This model allowed the students to relate to it, in the task 2, with spanning set and span.

Specifically, we consider that mathematical modelling contributed to the understanding of spanning set and span because it activated the students' previous conceptions in the task 1.

The requirement for a greater cognitive demand

The analysis of task 2 shows that the ALT was a close match with the HLT (see row 2 in Table 1) because they made a suitable analogy table (see Figure 1) between their password generator and the concepts of spanning set and span.

Name given in your password generator	How it is written in mathematical language	Mathematical name for this concept
Spanning set	*Vectors that generate numeric passwords.*	$\{(1,0,1), (1,1,0), (0,1,1)\}$
Span	*Describes the operations to find the numerical vector that generates a password*	$\{(a,b,c)\in R^3: a(1,0,1)+b(1,1,0)+c(0,1,1)\}$

Figure 1: Written answer to task 2, agreed upon by all 3 students

To respond to the relationship between the password generator and span, student S1 indicated that the span "*describes all the vectors that can be made*". This comment from student S1 is very general because it does not specify what the purpose of those vectors is and does not mention the link with the passwords. Perhaps, for this reason, student S3 specified that the span "*describes the operations to find the numerical vector that generates a password*". Student S3 mentioned that the span describes operations. He linked this concept to the context of passwords when he specified that these operations served to find the numeric vector that generates a password. From the written response, agreed on by the three students, it follows that when student S3 spoke of operations, he referred to those that have a linear combination (addition and multiplication). The answer was the set $\{(a,b,c)\in R^3: a(1,0,1)+b(1,1,0)+c(0,1,1)\}$ that included his mathematical model, $a(1,0,1)+b(1,1,0)+c(0,1,1)$, to generate passwords and associated it with span. These students did a similar process to relate spanning set to the context of the passwords (see Figure 1).

The 3 students, through the analogy table (see Figure 1), linked the real context with the mathematical one. This required a greater cognitive demand regarding task 1, and was important to initiate the understanding of spanning set and span, since they made a first example of both, and established a distinction between these concepts when having to relate them to sets derived from a real situation.

The task 2 corresponded to the level of referential activity (Gravemeijer, 1999). In this, we consider that not only should reference be made to the real context of task 1 (level of situational activity), as Gravemeijer (1999) points out, but it was also important that a greater cognitive request was demanded from the students to help with their progress towards the more formal reasoning of spanning set and span.

The real context to progress towards a more abstract level

It is made evident that the 3 students progressed towards a more formal knowledge of the concepts by the fact that they left the context of the passwords and made conjectures about the properties of spanning set and span. For example, in the task 3, they were asked to conjecture what the range of the matrix was by vector rows of a set of R^2 so that it would generate the span of R^2. For this, they were presented sets of R^2 (A, B, C y D) and the matrix whose rows were the vectors of each set (M_1, M_2, M_3 y M_4). This is shown in Figure 2.

Sets	Matrix whose rows are the vectors of set	Range of the matrix	**Conjecture:** *If the range is 2, the set can generate R^2.*
$A=\{(0,-3)\}$	$M_1=(0\ \ -3)$	*1*	
$B=\{(5,0),(7,0)\}$	$M_2=\begin{pmatrix}5 & 0\\ 7 & 0\end{pmatrix}$	*1*	
$C=\{(1,0),(1,-1)\}$	$M_3=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$	*2*	
$D=\{(-1,-4),(2,8),(0,-1)\}$	$M_4=\begin{pmatrix}-1 & -4\\ 2 & 8\\ 0 & -1\end{pmatrix}$	*2*	

Figure 2: Written answer to task 3, agreed upon by all 3 students

Student S1 indicated that the range of the matrix M_1 "*is 1*" and then, referring to the other matrices, noted that *"this is also 1 (showing M_2). Yes, this is 2 (indicating M_3). One, one, two, two"*. When student S1 mentions *one, one, two, two*, he alludes to the ranges of the matrices M_1, M_2, M_3 and M_4 respectively, for that is what appears in the written answer (Figure 2). From this, the 3 students surmised that *if the range is 2, the set can generate R^2*. From their response, it is inferred that they noticed that a spanning set of R^2 can have more than 2 vectors (such as set D), but that the minimum must be two vectors that are linearly independent (such as set C). That is, it is not enough that the set has 2 vectors (such as set B). From this it follows that the students observed that the number of vectors is not determinant to indicate if a set is generating a span of R^2, because apart from the fact that they are 2 vectors, they must be linearly independent.

The analysis of task 3 suggests that ALT was close to HLT (see row 3 in Table 1). The formation of this conjecture was fundamental, because it made the 3 students relate spanning set and span with the concepts of the range of a matrix and linear independence. In addition, they identified a property that characterizes the sets that generate the span of R^2. It served to allow them to progress towards a more abstract level of these concepts, as it was observed when they realized conjectures of properties of the concepts (see Figure 2).

The application of the concepts of spanning set and span

Task 4 presented a high cognitive demand for the 3 students, because it required their understanding of the concepts of spanning set and span. To solve it, they needed to: analyse it, resort to spanning set and span, explore procedures for their resolution, find a spanning set (or span) and verify if C was a spanning set of W. For example, they answered the question: indicate if C={(1,0,0,1), (0,1,0,0)} is a spanning set for the span W={(x,y,z,w)/x=w}.

The 3 students agreed that *"it is not a spanning set of W"*. To conclude this, they followed a process to determine a spanning set of W. Student S2 expressed (looking to the set W) that "*condition x is equal to double v*" and immediately, student S1 told his colleagues "*we are going to replace it*". Student S1 referred to substitution, the condition that was indicated by student S2, in the generic vector of the span W, since that was what they did and wrote the linear combination *w(1,0,0,1)+y(0,1,0,0)+z(0,0,1,0)*. On seeing it, student S1 stated that the set {*(1,0,0,1),(0,1,0,0)*} is not a spanning set for W by "*component z*" and added that "*this (indicating the set*{*(1,0,0,1), (0,1,0,0)*}*) is not a spanning set of this (showing W). Look (showing them w(1,0,0,1)+y(0,1,0,0)+z(0,0,1,0))*". Student S1 proposed to his colleagues that they compare the expression of the 3 vectors with the set that gave them 2. From this, they concluded that it is not a spanning set of W.

From the process followed by the 3 students in the task 4, it is inferred that they can find a spanning set for the span W and verify if the set C generated to W. This, approached the one posed in the HLT (see row 4 in Table 1).

Finally, the application of these concepts by the students corresponded with the level of formal activity (Gravemeijer, 1999), and allowed it to be inferred that they had understood them. Specifically, because they were able to solve a problem in the task 4 where they showed that they were able to: distinguish between spanning set and span, to follow a process to obtain a span and to verify if a set was generating a certain span.

The HLT proposed and used by the teacher gave the group of students the possibility to understand spanning set and span from a task that made sense to them, because it is within a real and everyday context: creating passwords. The goal was not to teach students strategies or techniques that had already been done but to help them develop their own methods to understand these concepts.

CONCLUSION

The THA proposed for the construction of spanning set and span stands out for being useful to teach this type of content because it provides a strategy to promote the understanding of spanning set and span. This strategy, provided by the HLT, allowed student S2 to activate his previous conceptions of vectors, since it suggested that the mathematical model to generate passwords had to have "*three vectors*". Meanwhile, it caused student S3 to relate the context of passwords with span by indicating that this set "*describes the operations to find the numerical vector that generates a password*". Later, it gave student S1 the opportunity to show his understanding of the concepts under study by having to verify if a set was generating a certain span (task 4). This strategy led student E1 to perform a process to find a spanning set, and then to compare its result with the information given in the task, to conclude that the given set *was spanning set* of a certain span. These events give indications that the designed tasks allowed the students to progress from their informal mathematical reasoning (associated with their previous conceptions) to a more formal mathematical knowledge (of spanning set and span).

This HLT may have some implications for the design of tasks in universities, because it provides a way of designing a sequence of tasks for concepts which present difficulty in their learning due to their high level of abstraction, as is the case with, for example, those of spanning set and span studied here, or others of linear algebra. In addition, it addresses a question of equity because the sequence of tasks starts with a problem in context that is accessible to everyone and increases its level of complexity as students make progress in it. In this way, it made it feasible for students to construct the concepts.

The results of this study show that this HLT contributes to the learning of linear algebra and is expected to serve as a guideline to design other trajectories that will help students to advance with their learning in this course.

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MATHEMATICS PRESERVICE TEACHERS' ARGUMENTATION

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This research deals with the preservice teachers' dialogic argumentations when presenting geometry tasks to their colleagues, during discussion sessions previous to teaching to children. An argumentation analysis tool is used that complement Toulmin's analysis proposal and that includes features related to mathematical logic, rhetoric and dialectic features. We propose both a representation for the dialogic argumentation and a way to identify its structural qualities.

INTRODUCTION

Due to the complexity of teachers' argumentation in the classroom, that do not let 'uniquely' follows the deduction rules of Aristotelian logic but recurs to 'persuasion' (Perelman, 1997), it is required diverse skills, specifically, to argue during teaching (Ufer, Heinze & Reiss, 2008). Several studies showed, that not only students have problems in this field (Reiss, Heinze, Kessler, Rudolph-Albert & Renkl, 2007), but also prospective and in-service teachers (Barkai, Tsamir, Tirosh & Dreyfus, 2002). The interest of the paper is to study preservice teachers' argumentations when explaining geometry tasks.

FRAMEWORK

In this research 'dialogic argumentation' is assumed as "social and collaborative process necessary to solve problems and advance knowledge" (Duschl & Osborne, 2002, p. 41). The dialogic argumentation is close related to 'communicative acts' that give and ask for reasons (Habermas, 1999; Toulmin, 2007) that includes not only logic-substantive features but rhetoric and dialectic put into play by preservice teachers while presenting geometry tasks to their fellow colleagues to explain, to teach and to convince.

The dialogic argumentations are analyzed in regard to structural qualities: logic-substantive, rhetoric and dialectic (Habermas, 1999); for representing the structure, it is used Knipping (2008) proposal. The warrants used by preservice teachers are presented as: a priori, empiric, institutional and evaluative (Nardi, Biza & Zachariades, 2011). The argumentative sequence, either progressive or retro-progressive, (Van Eemeren, Grootendorst & Henkemans, 2006) considers the natural way in which teachers give and ask for reasons.

CONTEXT AND METHODOLOGY

The research context is the course 'Teaching Practice', offered to preservice teachers in the program of mathematics in the School of Education, Antioquia University,

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 217-224. Singapore: PME.

Medellín, Colombia. This course spans for a year and a half. During the first year, the preservice teachers design and choose geometry tasks, solve and present them to their fellow colleagues, who criticized the presentations; in the remaining term, the teachers acted as teachers in the classroom. This paper informs about the first year. The preservice teachers were interviewed just after they presented the tasks to their colleagues. Interviews were recorded, transcribed and analyzed searching for: (1) formal argumentation structure (Knipping, 2008); (2) epistemological and pedagogical nature of reasoning (Nardi, Biza & Zachariades, 2011), and (3) argumentation sequences and interaction patterns (Clark & Sampson, 2008).

ANALYSIS AND RESULTS

We discussed two argument segments belonging to two preservice teachers who presented the solutions of two geometric tasks to their colleagues. The first argument responds to: How would you explain to your colleagues how to find the value of angle h? The second argument responds to the question: How would you teach the Pythagorean Theorem to ninth graders? In what follows the two preservice teachers' arguments are presented, the first argument belonging to Jhoanne (J), the second to Maria (M). It is shown the questions (Q) and the ensuing answers. The first question includes the graph as data. The pieces of the argument are signaled with a numeral located to the left (L1 means line one corresponding to the argument segment).

L1-L2 Q: How would you explain to your colleagues the way to find the value of angle h?

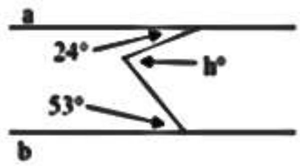


Figure 1: Graph for the question (Berg, Fuglestad, Goodchild & Sriraman, 2012, p. 682)

Jhoanne proceeds as follows:

L3-L4 J: In order to find out the value of angle h, first I prolong the lines in such a way that cut in A and D, the parallel lines a and b.

The argumentation motivated by the questions requires geometry knowledge organized in a sequence with the intention to explain. Figure 1 presents some data: the values of angles, two parallel lines and the nomination of the angle whose value is to be found. These data underline geometric knowledge that Jhoanne must know. In L3 Jhoanne uses an argumentative indicator -*first*- followed by a narrative in first person. Additionally, uses the modal qualifier: *in such a way that* (L4-5), uses the apriori epistemological warrant to extend the lines toward A and D. Next Jhoanne affirms:

L5-L7 J: Then by the properties of angles among paralell lines, I look for the angles alternate-interior as A and F, D and B, by the property of the sum of the measures of the internal angles of a triangle, which is 180°.

In this segment Jhoanne uses two a priori-epistemological warrants (Nardi, et al., 2011) corresponding to a property: alternate-interior angles among parallel lines and a theorem: the sum of the measures of internal angles in a triangle is 180º. Jhoanne uses an argumentation indicator -*then*- (L5), and a modal qualifier -*which is*- (L7). She proceeds:

L8-L9 J: We find the measure of angle C by the definition of plane angle we have that C°+ h°= 180° where C°=103° thus h = 77°.

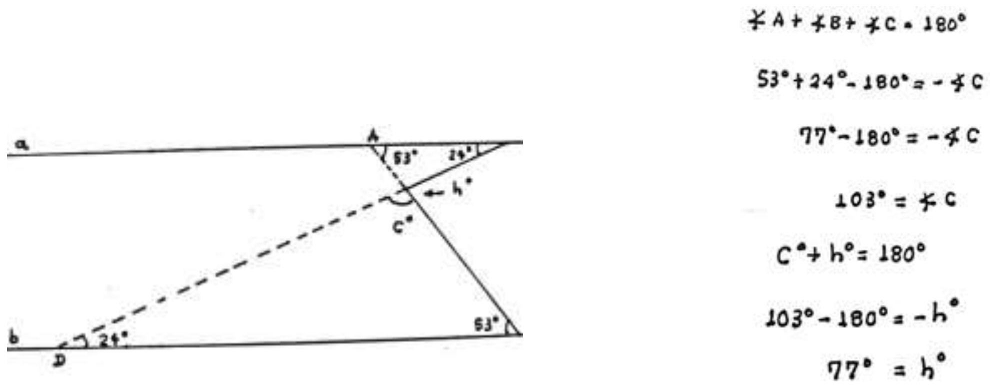


Figure 2: Ilustration and computing proposed by Jhoanne

Jhoanne uses manifold argumentation indicators: *we have that*, *where* and *thus* (van Eemeren, et al., 2006). When passing from L5-L7 to L8-L9, she uses an ilustration as a rhetoric resource (Perelman, 1997), and the procedure to find the unknown value is algebraic in nature. The second argumentative segment, corresponding to Maria, refers to the design of a class related to the Pythagorean Theorem.

L10 Q: How would you teach the Pythagorean Theorem to ninth graders?

Maria says:

L11-L15 M: What I understood […] is that I have to, more or less, propose a draft about planning an activity with ninth graders to teach them the Pythagorean Theorem, then I planned the activity as a guide, then I proposed a puzzle [tamgram like], […] and to arrive […] to the formal features of the Theorem.

In this segment, Maria first establish her argument conclusion, that deals with the teaching the Pythagorean Theorem to ninth graders (9º) using a puzzle, then she states the objective about discussing 'formaly' such Theorem. The sentence 'What I understood…' -first person- supposess *a communicative understanding* and *an action* as well (Habermas, 1999). Additionally, she uses '*more or less'* -a modal qualifier- and assumes her proposal as 'possible' and not as a definite statement.

L16-L20 M: […] Initially, as Carlos did, I would begin with some history, even though they [the kids] are in ninth grade, a story can be told to them about the Pythagorean Theorem […] because it is believed that Pythagoras discovered the theorem, but it was also known to ancient civilizations in Babylon and Egypt, the Pythagorean triads were also known to them…

In this segment, Maria says that she took into consideration her colleague Carlos`s proposal that accounts for the use of a rhetoric resource as the model (Perelman, 1997). Additionally, L3 serves as intersubjective evidence that refers to meaning negotiation by their colleagues. Within the first reasons expressed by Maria, appears an apriori institutional warrant (Nardi, et al., 2011), because she uses the history of the Theorem. Additionally, she expresses the modal qualifier '*though*' that refers to the likelihood of using a story as a resource to teach the kids. Telling a story is an evidence of the practical rationality or reasonableness (Habermas, 1999; Toulmin, 2007). The preservice teacher continues arguing:

L21-L26 M: […] As it is said there, a man called Pythagoras discovered an amazing fact regarding triangles, if a triangle as a right angle, so to speak, an angle whose measure is 90º and a square is constructed on each one of its legs; then the biggest square [refering to the square constructed on the hypotenuse] has exactly the same area as the other two squares together […]. The triangle's biggest side is called hypotenuse.

On one side, this segment manifests an a priori-epistemological warrant (Nardi, et al., 2011), because it resorts both to the statement of the Pythagorean Theorem and to the definition of a right triangle. On the other side, it uses the modal qualifier '*exactly*' (L25), because she is certain about her statement. Every fragment -L3 to L9- offers evidence on the use of the theoretical rationality in the dialogic argumentation (Habermas, 1999), which complements the practical rationality (Rigotti & Greco, 2009), and links the actions that are epistemological, teleological and communicative (Habermas, 1999) to the future teachers' argument.

L27-L30 M: So, I would begin with some templates more or less …the handouts would be the templates, that they [pupils] have to cut and they themselves can verify if the two squares are 'put togheter'; those that I constructed on the triangle legs, I would obtain the area [or the square constructed] on the hypotenuse.

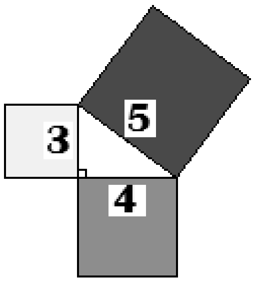
Maria begins her Pythagorean Theorem teaching proposal by using the puzzle to shape the rectangles over the legs and over the hypotenuse of a right triangle. The use of the puzzle puts into practice the practical rationality and manages to persuade her colleagues (Perelman, 1997), which links the puzzle activity to the Pythagorean Theorem.

L31-L34 M: Just before finishing […] the work about what I just said, I would reach the formal definition, that I would do by showing them […] the triangle, thus the square's area constructed on one leg plus the square's area built on the other leg would be equal to the square's area of the hypotenuse.

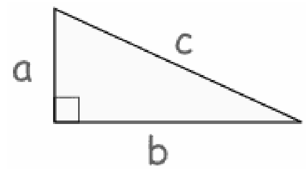
Later she performs Pythagorean Theorem verifications for particular cases. Maria employs the examples as a rhetoric resource, which let her to generalize (Perelman, 1997).

L35-L39 M: I would perform some verifications, if it works with an easy example, algebraically assigning numbers to the lenghts of both sides for the students to calculate the area with a simple operation, then one square measures three (3), the other square measures four (4), and the resulting square would measure five (5), there we show the solution.

L40-L44 M: Why would it be useful? If we know the side lengths of a right triangle, the Pythagorean Theorem would help us to find the length of the third side, but I would make them notice that it is only true for right triangles, that it is not true for every triangle. Then I would write it as an equation and there we would perform algebraic procedures using equations, just as it is shown in Figure 3.



Let's see if the areas are the same:
$3^2 + 4^2 = 5^2$
Computing we get:
$9 + 16 = 25$
Yes, it works!



$a^2 + b^2 = c^2$

Figure 3: Verification of the Pythagorean Theorem using an example

The tasks are formulated using questions, whose structure responds to: how is it argumented, whom the arguments are directed to and what is argumented. It can be said that the preservice teachers argue in first person (Habermas, 1999), and assume the 'leading' role. The roles of protagonist and antagonist give account of the 'progressive and retrogressive' arguments, based on 'argumentative indicators' (Van Hemeren, Grootendorst & Henkemans, 2006). On the other side, an advantage of the dialogic argumentation is that it favors solving the dichotomy between analytic and substantive argumentations (Toulmin, 2007), because preservice teachers not only use logic inference rules to communicate their knowledge but also rhetoric resources linked to illustrations, examples and models (Perelman, 1997). The latter can be seen between the first and the second segment of the arguments by the two preservice teachers which show 'density' in the dialogic argumentations that surpasses Toulmin's model (2007). A drawback of this report is that preservice teachers' dialogic argumentations, while teaching in the classroom to real pupils, are not discussed. On the other side, the argumentative indicators used let us to identify the warrants chosen by the preservice teachers. Table 1 shows the relationships among the argumentative indicators, modal qualifiers and warrants used by preservice teachers. When questions are used to generate a class planning, the preservice teachers use no absolute modal qualifiers, for instance: *more or less, such as, even though, it does not work for every case*. In terms of a pattern of interaction, we identify a protagonist role in Jhoanne dialogic argumentation, and the use of both explicit graphic reasoning (Figure 2) and verbal reasoning. The reasoning would not be explicit if logic-formal inference rules would be applied in an analytic argumentation. In the segment corresponding to Maria's argument, the argumentative sequence was not only accompanied by warrants linked to the geometric knowledge -a priori epistemological warrants-, but also by warrants linked to the history of the Pythagorean Theorem -institutional a priori warrant-.

		Argumentation indicators	Modal qualifiers	Warrants
Argumentative segment 1	L3	First	In such a Way	A priori-epistemological
	L4	And		
	L5	Then, I look for		
	L6	And, by		A priori-epistemological
	L7	Which is		
	L8	We find, by, we have	Where	A priori-epistemological
	L9	Thus		
Argumentative segment 2	L11		More or less	A priori-institutional
	L13	Then		
	L16	Initially	Though	A priori-institutional
	L22	So to speak		
	L23	And		
	L25		Exactly	
	L27		More or less	Empiric-personal
	L30	I would obtain		A priori-epistemological
	L31	I just said	Be equal to	A priori-epistemological
	L32	Thus		
	L34			
	L35		If it work	A priori-epistemological
	L41	But		
	L43	And		

Table 1: Relations among argumentation indicators, modal qualifiers and warrants

According to the relations, presented in Table 1, it can be stated that the first argument is devoted to explaining. The modal qualificators are sparse and express not only likelihood for the task proposed but also Maria's confidence in her solution to it. Meanwhile the second argument uses manifold modal qualificators that express Jhoanne' stance in regard to the likelihood for her solution to the task proposed. The auditorium was not an issue for the first argument, but it was for the second. In regard to the rhetoric resources, Jhoanne's argument uses only the illustration, while Maria's uses: illustration, model and example. The first argument use only of a priori epistemological warrants, while the second use a priori epistemological, empirical-personal as well as a priori-institutional. Additionally, the argumentation indicators facilitate the identification of the Toulmin's model components. If the question is linked to knowledge, Jhoanne's argument case, the modal qualificators give account of a structure close to the formal logic, but if the question is guided by a teaching intention, Maria's argument case, the modal qualificators point out to a doubt and establish a link to the reasonablennes.

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LEARNING RESEARCH IN A LABORATORY CLASSROOM: ISSUES AND SOME RESOLUTIONS

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This paper uses an international study of student collaborative learning to highlight and address methodological issues associated with conducting research in a laboratory classroom in which 10 built-in video cameras and more than 15 audio inputs recorded the interactions of intact classes of students and their teacher as they engaged in purposefully developed mathematical activities. Such laboratory classrooms offer possibilities for structured, rigorous, fine-grained investigation of the social aspects of mathematics learning. This paper discusses some of the issues raised by the use of these technologically innovative facilities and offers possible resolutions with the intention to inform the future use of such research facilities for investigating mathematics learning and its promotion in classroom settings.

LEARNING FROM VIDEO-BASED CLASSROOM RESEARCH

As a focus of educational research, learning and its promotion require investigation at dimensions that extend from the neurological to the sociocultural, in a variety of settings both institutionalised and personal, and with respect to all conceivable attributes, inclinations and skills, from aspects of recall within specific knowledge domains to strategies for self-regulation. Sophisticated research approaches and tools can help researchers to investigate the complex processes involved in learning in various settings. Janik and Seidel (2009) reported how video technology supports more sophisticated research designs, requiring complex theoretical frameworks modelling mediating processes between teaching and learning outcomes using multi-level analyses (Clarke, et al., 2012; Ulewicz & Beatty, 2001).

The laboratory classroom at The University of Melbourne is able to record classroom social interactions with a rich amount of detail. It was purposefully designed and built to allow simultaneous and continuous documentation of classroom interactions using multiple cameras and microphones. The facility has been utilised by several research projects since its launch in March 2015, one of which is the Social Unit of Learning project, which aims to examine individual, dyadic, small group (four to six students) and whole class problem solving and learning in mathematics and the associated/consequent learning. This paper discusses the new insights and challenges that the laboratory classroom has provided with illustrative examples from the Social Unit of Learning project. A brief overview of the laboratory classroom and a description of the project are provided, followed by a reflection on the methodological issues.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 225-232. Singapore: PME.

THE LABORATORY CLASSROOM

The laboratory classroom is a 129 sq. m. teaching space that resembles a typical classroom but is fitted with high definition audio-visual recording equipment and physically connected to an adjacent Control Room via a one-way window. Lessons given in the research classroom can be recorded through up to 16 high-definition video channels and up to 32 fixed and portable microphones. In the Control Room are screen monitors and computer equipment that allow a technical team to control and monitor the data generated by the recording equipment in the research classroom. Researchers can also observe the activities within the research classroom from the Control Room as the lesson progresses, either by direct observation through the one-way viewing window or on any of the monitors displaying the images recorded by the different video cameras and by listening to any of the audio channels. Observation is also possible via live streaming of selected video outputs to remote locations. The laboratory classroom affords a variety of collaborative research modes.

THE SOCIAL UNIT OF LEARNING PROJECT

The Social Unit of Learning project investigates the social phenomena that characterise learning processes in a mathematics classroom. The project uses the facilities of the laboratory classroom to record the interactions of an intact class of students (20 to 26 students per class) and their teacher as they engage in purposefully developed mathematical activities. A typical investigative session takes 50 to 60 minutes, where students attempt tasks individually, in pairs, and in groups of four to six. Figure 1 illustrates one of the activity configurations utilised in the project. The sessions are designed to facilitate recordable (visible and audible) social interactions, necessitated by the obligation to solve content-specific, open-ended mathematical tasks collaboratively.

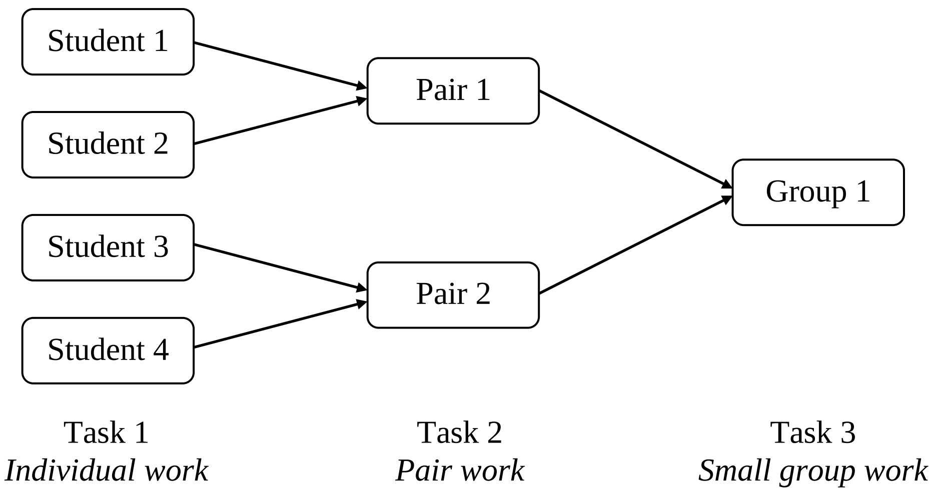

Figure 1: Student grouping for three tasks within a single investigative session.

The laboratory classroom allows the generation of continuous video and audio records of problem solving and associated learning for every student in the class in addition to digitised copies of student written work from the activities. A team-based multitheoretic approach has been employed which involves parallel, complementary analyses of data generated by the facility as described above.

As a general principle, technical innovation must be scrutinised for its methodological consequences and implications. The laboratory classroom provides a useful example because of the complexity and scale of the data generated. Four methodological issues are raised in this paper with suggested resolution. The issues include: the compromise between data prescription and claims of comprehensive documentation; the information provided by different data types and the challenge of their interconnection; the tension between authenticity and control; and the authority that may be claimed for any conclusions arising from such research. These issues are not unique to the laboratory classroom, and warrant the attention of all researchers in mathematics education, whatever methodology or design they employ.

Data prescription and claims of comprehensive documentation

Despite the richness of the records generated in the Social Unit of Learning project, we are very aware that the types of data being generated, the means of data generation, and the methods of analysis are predicated on the research design prescribed by the research team (Clarke, Mitchell, & Bowman, 2009) and can never claim to be comprehensive in any absolute sense. There is no intention to advocate either that more detailed records bring us closer to some spurious "classroom reality" nor that more detail is intrinsically better. Researcher choice is central.

Issue: Data generation is a process of selection from available information, particularly when the data source is as rich as the laboratory classroom. This selection runs the risk of being self-fulfilling with respect to the research questions posed (Clarke, 2011).

Resolution 1 – Juxtaposed Accounts. The same detail that demands these acts of selection, provides the opportunity for multiple parallel accounts that can then be juxtaposed for purposes of cross-validation and elaboration (Clarke, et al., 2012).

Corollary: In studies where the data source is less extensive (e.g., a questionnaire, an interview, or a single camera observation), the process of selection just occurs earlier (e.g., at the point of instrument development). In the laboratory, those acts of selection are both more visible and able to be addressed more effectively, through the juxtaposition of "complementary accounts" (Clarke, 1998).

Resolution 2 – Supplementary Data. In addition, analysis of classroom activity can often involve the attribution of intention and motivation to the participants. While classroom observation, via video or otherwise, is one way to examine a person's intentions and motivation, other sources of information, such as video-stimulated post-lesson interviews, can also be employed to inform inferences of intention of the person from such observation (Clarke, et al., 2009). Such interviews offer insights into participant motivation and the participants' interpretations of each others' actions. The juxtaposition of different interviewees' accounts of their actions and their interpretations of the actions of others can reveal important differences (and similarities) between participants' perceptions of the activities of the mathematics classroom, greatly assisting the interpretation of their documented actions.

Connections between multiple data types

The project has generated several forms of data: digitised copies of all written material produced by the students; copies of PowerPoint presentations and written instructional material used by the teacher; video footage of all of the students working individually, in pairs, in groups, and in whole class discussion during the lesson; video footage of the teacher tracked throughout the session; and transcripts of all teacher and student speech. Activities recorded in the laboratory classroom generate a great deal of information, potentially translatable into many types of data (e.g., visual, audio, or textual), with each data type offering a distinctive perspective.

Issue: How can these multiple data types be connected in a coherent descriptive or even explanatory account of the phenomena being studied?

Resolution 1 – Structured Interconnection to Facilitate Explanation. With access to these multiple data types, when examining a particular piece of written work from a group task, we can do more than just speculate about how the students constructed their response – we can also replay the video footage and study the discussions that the students engaged in during the task. Strategic use of individual and dyadic modes in the research design allows group interactions to be cross-referenced to pair and individual problem solving immediately preceding the group work, allowing the tracing of the processes that lead to particular learning products. This connective record can document the origins or development of a specific idea or intention of the students during a session and connect the group product to the established problem solving skills and inclinations of the participating students, whether acting individually or as dyads.

Resolution 2 – Account Connection through Common Data. By selecting each element or combination of elements, the researcher foregrounds a particular aspect of the classroom activity, while relegating other elements to the background. It is possible to generate parallel data sets from the video (and other) records and to conduct parallel analyses of these data sets. In this way, the researcher generates complementary accounts (Clarke, 1998), each using a different theoretical lens and a different analytical approach (Clarke, et al., 2012). Since these complementary accounts relate to events in the same setting, possibilities are created for connections between both the accounts and the theories from which they were generated. Emergent commonalities and tensions between the complementary interpretive accounts raise questions about analogous commonalities and tensions between the underlying theories and the methodological assumptions arising from those theories. It is the detailed records generated through the laboratory classroom that make possible the parallel analyses of the common set of records of classroom activities and artefacts; parallel analyses of the same students engaged in the same mathematical activities at the same time in the same setting.

Tension between authenticity and control

Since our focus was on recording and analysing student-student interactions with minimal distracting interruption, the teacher protocol minimised the teacher's role, even though this reduced the extent to which the recorded practice replicated the group's normal classroom behaviour. When designing the sessions for implementation in the laboratory classroom facility, care is required to balance the need for control in an experimental environment and the freedom for the participants to interact and behave as they would in a naturalistic classroom setting. Conscious decisions are required to specify the particular mathematical activities that the students are going to engage in and the kinds of responses and feedback that the teacher is permitted to provide to the students. In order to maximise the clarity of the documentation of the social interactions within the classroom, even specific seating arrangements are prescribed. Each instruction that is given to the teacher and the students by the research team essentially constrains some aspect of their freedom to behave and interact, but with the purpose of maximising the visibility of some valued behaviour. The distinction between experimental and naturalistic conditions will always be a consideration in an environment like the laboratory classroom, which must seem "artificial" from the perspective of the visiting teacher and students.

Issue: We are studying student-student social interactions for the purpose of understanding social learning processes. This is a legitimate focus for an experimental research study. However, we would like the social processes recorded to correspond as closely as possible to the students' usual practice, despite their occurrence in an "artificial" laboratory classroom; a delicate balance.

Resolution – Maximise Engagement through Strategic Task Selection. Although the teacher and students seemed to be aware of the presence of the cameras and the microphones in the laboratory classroom and the one-way window adjacent to the room, rather than being totally inhibited, they appeared to be able to re-establish a way to interact and engage with the prescribed activities in this new environment drawing from existing social norms and patterns of behaviours. As indicators of this lack of inhibition, some students used inappropriate language during group discussion, while others spontaneously raised their hands to seek clarification from the teacher, and there was evidence of humour and lively spoken interaction (excerpt below). This suggests that the research design has not totally stifled behaviours and interactions that would be typical in their normal classroom environment.

Natalie and Aruna were trying to work out the age of and the relationship between five members of a household if their average age were to be 25.

Natalie: … 27, 5, 25. Twenty something - 20 something, okay. Twenty something, okay? Okay. How old should your brother be? Like …

Aruna: My sister's younger than me.

Natalie: I have three older brothers so my brothers - if I'm in Year 7, my brother could be like 16. My brother is 16 so - wait. You want a sister or brother?

Aruna:	Any.
Natalie:	Okay. We'll just do brother.
Aruna:	There - I'm pretty sure they're the same except for the body parts.
(Laughter)	
Natalie:	Okay. You don't have to say that.
(Laughter)	

The conversation between Natalie and Aruna illustrates the ease that the students displayed when completing the problem solving tasks during the investigative session while being filmed. In this particular case, student engagement with the mathematical tasks acted to minimise the distraction that might have been caused by unfamiliar elements in the environment.

Authority claims

Researchers in any research setting, experimental or naturalistic, need to be very careful when making claims that extend beyond their research studies. By its nature, an experimental study in a laboratory classroom cannot claim to replicate the activities and relationships that one might find in the students' accustomed classroom at school. Questions regarding generalisability of findings might be asked of any clinical study where the focus of investigation is the identification of attributes assumed to be so innate to the functioning individual as to be generalisable simply from their repeated occurrence (e.g., clinical studies of brain function).

Issue: To what extent can findings based on data generated from such a laboratory classroom facility be extrapolated beyond the research context? In other words, what warrant can the researcher claim for the legitimate extrapolation of their findings to other settings or other individuals?

Resolution. Any claims of generalisability beyond the research setting must be aligned with the purpose of the research. If the purpose of the project were to characterise typical mathematics classroom practice as it occurs in schools in a given community, then the laboratory classroom would not serve such an exploratory purpose well. However, in the case of the Social Unit of Learning project, it is our claim that the particular phenomenon on which the study is focused (student social interaction) can be re-created in the laboratory classroom in ways that legitimately resemble those occurring in settings to which the findings might be extrapolated.

In the case of the Social Unit of Learning project, the essential requirement is that the social interactions between students when engaged in the collaborative solution of mathematical tasks resemble those that would pertain in other settings, such as school classrooms. The physical similarity between the laboratory classroom and classrooms with which the students were familiar was quite high. The teacher has confirmed that the tasks were not ones that the students would find unusual. The people with whom each student must interact were all familiar: the student's usual teacher and classmates.

The presence of these familiar elements encourages confidence that the function served by social interaction was highly similar to the students' normal mode of social interaction. The lack of inhibition in the students' behaviour provided further encouragement in the authenticity of their social interactions.

On these grounds, it can be claimed that findings arising from the research project provide evidence of possible links between certain conditions that would afford particular learning processes and associated learning products. Only through repeated records of the same "chain of actions" for a variety of individuals and group combinations can we be confident of the claims being made (Makel & Plucker, 2014). Because of the capability of simultaneously documenting the activities of up to six groups of students responding to essentially the same stimuli (instructions, task and environment), the data generated through the use of the laboratory classroom immediately affords comparison of such chains of action across the different participant student groups. Repetition with other classes provides further confirmation or refutation of any emergent hypotheses.

The use of the laboratory classroom provides the opportunity for a controlled re-construction or simulation of the phenomenon of interest, as it might occur in a "real-life" mathematics classroom. It can be readily acknowledged that other phenomena may occur very differently in the students' normal classroom. The authority for the validity and utility of any findings rests on the validity claim specifically for the phenomenon of interest, in this case, student-student interactions when engaged in mathematical problem solving. Analysis of the successful re-construction of the phenomenon of interest generates speculation, hypotheses and insights into the function of social interactions in other classroom situations. This may in turn lead to testing of hypotheses in further experiments within the laboratory classroom or in brain-imaging facilities, or outside the research facility in naturally-occurring classroom settings.

FINAL THOUGHTS

It is a paradoxical aspect of the use of a laboratory classroom that its purpose is to accommodate the complexity of social phenomena, while limiting that complexity through the manipulation of the conditions framing the participants' activities. Constraint on complexity at the macro (whole class, whole lesson, whole curriculum, whole school) level is accompanied by the need to replicate complexity at the "micro" level; that is, with respect to the phenomenon being studied, in such a way that its occurrence is a valid representation of the form it might take in other non-laboratory settings. It is precisely this balance between authenticity and control that has made learning research so difficult to undertake in "authentic" classrooms.

A laboratory classroom offers possibilities for the structured, rigorous, fine-grained investigation of mathematics classroom practice. Its research use necessitates decisions concerning what to constrain and what to emulate within the laboratory classroom setting, and places in juxtaposition the availability of multiple data types and the constraints inherent on the inferences possible from each data type. Our purpose in this paper has been to draw attention to the nature of some of these decisions. The challenges and possibilities identified in this paper require careful consideration if laboratory classrooms are to usefully contribute to mathematics educational research and to research on mathematics learning in particular.

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IMPROVING YEAR 4 PUPILS' PROFICIENCY IN MEASUREMENT FORMULAE THROUGH THE CONCRETE-PICTORIAL-ABSTRACT APPROACH

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The study aimed to improve Year 4 pupils' proficiency in measurement formulae (perimeter, area and volume formulae) through the Concrete-Pictorial-Abstract (CPA) approach. The study employed the non-equivalent control group design and two intact Year 4 classes in a National School were randomly assigned to the experimental group (37 pupils) and control group (35 pupils). The experimental group learned measurement formulae through the CPA approach while the control group learned the formulae through the conventional approach for two weeks. The results of the one-way ANCOVA showed that there are significant differences in the post-test mean scores of proficiency in measurement formulae between the experimental group and the control group with the pre-test scores as covariates favouring the experimental group.

BACKGROUND OF THE STUDY

The study of measurement formulae is important because it offers opportunity for learning and applying other mathematical concepts and skills such as number operations, geometrical ideas, and notions of function (National Council of Teachers of Mathematics, 2000). As such, measurement formulae namely perimeter, area and volume formulae form an important part of the Malaysian primary school mathematics curriculum starting from Year 4 in which pupils begin to learn how to find the perimeter of squares, rectangles, triangles and regular polygons in the new Standard Curriculum for Primary Schools. They also begin to calculate the area of squares, rectangles and triangles using square grid and formulae as well as calculate the volume of cubes and cuboids using 1 cm^3 unit cubes (Ministry of Education Malaysia, 2013). At the end of Year 6, primary pupils' proficiency in measurement formulae is assessed in the Mathematics papers of the Primary School Achievement Test, which is a national examination taken by all pupils before they leave for secondary school.

However, the Malaysian Examinations Syndicate (MES) reported that Year Six pupils lacked proficiency in measurement formulae as assessed in the national examinations. For instance, in calculating the perimeter of a whole diagram consisting of three congruent right-angled triangles with two of the triangles having a common hypotenuse to form a rectangle, the common mistakes of Year Six pupils were: (i) adding the common hypotenuse which is located inside the diagram; or (ii) calculating the perimeter of one right-angled triangle and then multiplying the perimeter by three (MES, 2005). In calculating the area of a shaded region consisting of two right-angled

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triangles, the common mistakes of Year Six pupils were: (i) calculating the area of a triangle using the area formula of a rectangle; (ii) calculating the area of one of the two right-angled triangles; or (iii) calculating the perimeter of the shaded region (MES, 2007). In calculating the volume of a cube with edges 6 cm long, the common mistakes of Year Six pupils were: (i) adding the length of 6 edges (6 + 6 + 6 + 6 + 6 + 6); or (ii) multiplying the length of 2 edges (6 x 6) (MES, 2010). In addition, Malaysian Form 2 students' proficiency in measurement formulae was unsatisfactory in the Trends in International Mathematics and Science Study 2011. For example, for the released item (ID_M052084) on calculating the area of a square with a given perimeter of 36 cm, only 40% of Malaysian Form Two students were able to answer it correctly. As a result, their performance was ranked 27th and the percent correct was significantly lower than the international average of 47%. For the released item (ID_M042201) on finding the length of a rectangular box with a given volume of 200 cm^3, only 42% of Malaysian Form Two students were able to answer it correctly. Consequently, their performance was ranked 23rd and the percent correct was slightly lower than the international average of 43%. For the released item (ID_M032116) on finding the perimeter of a square with a given area of 144 cm^2, only 43% of Malaysian Form Two students were able to answer it correctly. As a result, their performance was ranked 25th and the percent correct was slightly lower than the international average of 45% (Foy, Arora & Stanco, 2013).

Hence, there is an urgent need to improve Malaysian primary pupils' proficiency in measurement formulae starting from the first year they learn the formulae that is Year 4. One potential approach to improving their proficiency in measurement formulae is the Concrete-Representational-Abstract (CRA) sequence or Concrete-Pictorial-Abstract (CPA) approach which is based on Bruner's (1964) three modes of representation (enactive, iconic and symbolic). The CRA sequence has been reported to be effective with students who have difficulties with mathematics (Jordan, Miller, & Mercer, 1998), in remediating deficits in basic mathematics computation (Morin & Miller, 1998), in the teaching of place value (Peterson, Mercer, & O'Shea, 1998), and subtraction with regrouping (Flores, 2010). The CRA sequence has also been reported to have positive effect on low achievers in fractions (Butler, Miller, Crehan, Babbit, & Pierce, 2003), word problems (Maccini & Hughes, 2000), simple linear functions (Witzel, 2003) and advanced linear functions (Witzel, Mercer, & Miller, 2003). Apart from these studies, the CPA approach has been reported to be effective with students who have difficulties in quadratic factorisation (Leong, Ho, & Cheng, 2015; Leong, Yap, Thilagam, Karen, Quek, & Tan, 2010). According to Witzel (2005), the CRA sequence is beneficial for students with and without learning difficulties and from small-group settings to whole-class instruction. In addition, Fuchs, Fuchs, and Hollenback (2007) advocate the use of the CRA sequence to teach place value, geometry, and fractions for first and third graders. Moreover, according to Leong et al. (2015), the use of the CRA sequence to teaching mathematical concepts, particularly at the elementary level has been shown to be effective. But, to date research on improving primary pupils' proficiency in measurement formulae based on the Mathematical Proficiency Model (National Research Council, 2001) through the CPA approach is sparse.

Theoretical Framework

The CPA approach is based on Bruner's (1966) three modes of representation namely enactive, iconic and symbolic representations. It is a three-stage learning process in which pupils learn measurement formulae through the physical manipulation of concrete objects, followed by learning through pictorial representations of the concrete objects, and ending with learning through abstract symbols. Proficiency in measurement formulae, which is based on the Mathematical Proficiency Model (National Research Council, 2001) consists of five intertwined components or strands: (1) conceptual understanding which refers to comprehension of measurement formulae; (2) procedural fluency which refers to skill in carrying out procedures involving measurement formulae flexibly, accurately, efficiently and appropriately; (3) strategic competence which refers to ability to formulate, represent and solve problems involving measurement formulae; (4) adaptive reasoning which refers to capacity for logical thought, reflection, explanation and justification of solutions to problems involving measurement formulae; and (5) productive disposition which refers to habitual inclination to see measurement formulae as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy.

Objective of the Study

The objective of this study was to improve Year 4 pupils' proficiency in measurement formulae through the CPA approach. Specifically, this study aimed to answer the following research question:

Is there a significant difference in the proficiency in (1) perimeter formulae of a square, rectangle and triangle, (2) area formulae of a square, rectangle and triangle, and (3) volume formulae of a cube and cuboid between Year 4 pupils who learned the formulae through the CPA approach and Year 4 pupils who learned the formulae through the conventional approach?

Methodology

The research design of this study is the non-equivalent control group design which is a quasi-experimental research design. Two intact Year 4 classes were randomly selected from a National School in Penang, Malaysia and each class was randomly assigned to the experimental group and the control group. There were 37 pupils in the experimental group and 35 pupils in the control group. A multi-strand test for assessing Year 4 pupils' proficiency in measurement formulae was developed by the researchers based on the National Research Council's (2001) Mathematical Proficiency Model and the Malaysian Year 4 Mathematics Curriculum and Assessment Standard Document of the Primary School Standard Curriculum (Ministry of Education Malaysia, 2013). The multi-strand test consists of eight subtests, namely Perimeter Formula of a Square, Perimeter Formula of a Rectangle, Perimeter Formula of a Triangle, Area Formula of a square, Area Formula of a Rectangle, Area Formula of a Triangle, Volume Formula of a Cube and Volume Formula of a Cuboid. Each subtest comprises five items for assessing the five strands of proficiency in measurement formulae, respectively. The first, second, third, fourth and fifth items in each subtest assess the strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition, respectively. In addition, the fifth item in each subtest consists of five sub-items for assessing the five aspects of productive disposition, namely sensible, useful, worthwhile, diligence and one's own efficacy. The test was validated by a panel of three experienced National School primary school mathematics teachers. The scoring rubric was subsequently developed by the researchers and it was validated by the same panel of experienced primary school mathematics teachers. The validated multi-strand test was then piloted in a National School to determine its reliability. Table 1 shows the values of Cronbach’s alpha for the subtests. The high values of Cronbach’s alpha indicate a high degree of internal consistency of the items in the subtests and suggest that they are reliable to be used in the actual study.

Subtest	Cronbach’s Alpha
Perimeter Formula of a Square	.97
Perimeter Formula of a Rectangle	.88
Perimeter Formula of a Triangle	.90
Area Formula of a Square	.83
Area Formula of a Rectangle	.84
Area Formula of a Triangle	.85
Volume Formula of a Cube	.89
Volume Formula of a Cuboid	.92

Table 1: Values of Cronbach’s Alpha for the Subtests

A workshop was conducted to train the Year 4 mathematics teacher in the experimental group to teach the measurement formulae using the CPA approach. Following this, a pre-test was administered to the pupils in both the experimental and control groups using the multi-strand test of proficiency in measurement formulae. After the pre-test, the experimental group learned measurement formulae through the CPA approach while the control group learned measurement formulae through the conventional approach for two weeks. Lastly, a post-test was administered to the pupils in both the experimental and control groups using the parallel form of the multi-strand test of proficiency in measurement formulae.

Results

For the perimeter formulae, the results of the one-way ANCOVA showed that: (1a) there is a significant difference in the post-test mean scores of proficiency in perimeter formula of a square between Year 4 pupils who learned the formula through the CPA approach (M = 11.57, SD = 6.06) and Year 4 pupils who learned the formula through the conventional approach (M = 2.57, SD = 0.74) with the pre-test mean scores of proficiency in perimeter formula of a square as a covariate, F (1, 69) = 69.38, $p < .05$, partial $\eta^2 = 0.50$; (1b) there is a significant difference in the post-test mean scores of proficiency in perimeter formula of a rectangle between Year 4 pupils who learned the formula through the CPA approach (M = 12.59, SD = 5.61) and Year 4 pupils who learned the formula through the conventional approach (M = 6.46, SD = 2.89) with the pre-test mean scores of proficiency in perimeter formula of a rectangle as a covariate, F (1, 69) = 14.00, $p < .05$, partial $\eta^2 = 0.17$; and (1c) there is a significant difference in the post-test mean scores of proficiency in perimeter formula of a triangle between Year 4 pupils who learned the formula through the CPA approach (M = 10.43, SD = 5.48) and Year 4 pupils who learned the formula through the conventional approach (M = 2.49, SD = 1.27) with the pre-test mean scores of proficiency in perimeter formula of a triangle as a covariate, F (1, 69) = 30.10, $p < .05$, partial $\eta^2 = 0.30$.

With regards to the area formulae, the results of the one-way ANCOVA showed that: (2a) there is a significant difference in the post-test mean scores of proficiency in area formula of a square between Year 4 pupils who learned the formula through the CPA approach (M = 14.54, SD = 5.21) and Year 4 pupils who learned the formula through the conventional approach (M = 7.74, SD = 1.99) with the pre-test mean scores of proficiency in area formula of a square as a covariate, F (1, 69) = 34.76, $p < .05$, partial $\eta^2 = 0.34$; (2b) there is a significant difference in the post-test mean scores of proficiency in area formula of a rectangle between Year 4 pupils who learned the formula through the CPA approach (M = 12.89, SD = 4.58) and Year 4 pupils who learned the formula through the conventional approach (M = 6.94, SD = 2.36) with the pre-test mean scores of proficiency in area formula of a rectangle as a covariate, F (1, 69) = 20.64, $p < .05$, partial $\eta^2 = 0.23$; and (2c) there is a significant difference in the post-test mean scores of proficiency in area formula of a triangle between Year 4 pupils who learned the formula through the CPA approach (M = 13.32, SD = 5.78) and Year 4 pupils who learned the formula through the conventional approach (M = 4.51, SD = 2.45) with the pre-test mean scores of proficiency in area formula of a triangle as a covariate, F (1, 69) = 45.96, $p < .05$, partial $\eta^2 = 0.40$.

For the volume formulae, the results of the one-way ANCOVA showed that: (3a) there is a significant difference in the post-test mean scores of proficiency in volume formula of a cube between Year 4 pupils who learned the formula through the CPA approach (M = 14.38, SD = 6.90) and Year 4 pupils who learned the formula through the conventional approach (M = 4.91, SD = 2.61) with the pre-test mean scores of proficiency in volume formula of a cube as a covariate, F (1, 69) = 45.24, $p < .05$, partial $\eta^2 = 0.40$; and (3b) there is a significant difference in the post-test mean scores of proficiency in volume formula of a cuboid between Year 4 pupils who learned the formula through the CPA approach (M = 14.11, SD = 5.97) and Year 4 pupils who learned the formula through the conventional approach (M = 4.66, SD = 2.53) with the pre-test mean scores of proficiency in volume formula of a cuboid as a covariate, F (1, 69) = 57.11, $p < .05$, partial $\eta^2 = 0.45$.

Discussion and Conclusion

The results of this study showed that there are significant differences in the post-test mean scores of proficiency in measurement formulae (perimeter, area and volume formulae) between Year 4 pupils who learned the formulae through the CPA approach and Year 4 pupils who learned the formulae through the conventional approach, with the experimental group outperforming the control group in all cases. These results seem to provide further evidence to support the effectiveness of the CRA sequence as reported in previous studies (Butler et al., 2003; Flores, 2010; Jordan et al., 1998; Maccini & Hughes, 2000; Morin & Miller, 1998; Peterson et al., 1998; Witzel, 2003; Witzel et al., 2003) and the CPA approach as reported by Leong et al. (2010) and Leong et al. (2015).

According to Bruner (1966), the process of moving through the three modes of representation (enactive, iconic and symbolic) in the CPA approach theoretically reflects the "usual course of intellectual development" (p. 49) of the pupils in acquiring higher proficiency in measurement formulae. In other words, the CPA approach seem to facilitate the experimental group pupils to comprehend the measurement formulae, carry out the procedures involving measurement formulae, solve problems involving measurement formulae, explain their solutions to problems involving measurement formulae, and see measurement formulae as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy. In sum, the results suggest that the experimental group pupils are more able to understand, compute, solve, reason and possess a productive disposition toward the measurement formulae as compared to the control group pupils. In addition, according to Bruner (1966), the CPA approach allows the experimental group pupils to "fall back" to the concrete or pictorial representations of the measurement formulae if they cannot understand the meanings of the notations in symbolic representation or if they cannot recall how to solve problems involving measurement formulae.

In conclusion, the results of this study suggested for this sample of 72 Year 4 pupils from a National School that the CPA approach significantly improved their proficiency in measurement formulae. Because this study employed the non-equivalent control group design which is a quasi-experimental research design, we acknowledge our limitations in making any generalizations from the results of this study.

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NOVICE TEACHER KNOWLEDGE ON FACTORIZATION

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This study examined how teachers utilize their college mathematics knowledge in the context of school mathematics. Focusing on the unique factorization domain (UFD) in college mathematics and polynomial factorization in school mathematics, we explored teacher knowledge of UFD and how teachers' factorization concepts occur in the teaching context. We conducted semi-structured interviews with eight novice teachers. The interview tasks were developed to investigate how teachers deal with number factors when factorizing in school mathematics. The intentions of teachers who showed an appropriate UFD differed with regard to teaching polynomial factorization in the teaching context. Teachers who did not demonstrate an appropriate knowledge of UFD, could not explain the basis for their responses consistently. The result of this study can serve as a resource for teacher educators when teaching UFD in abstract algebra in the future.

INTRODUCTION

What role does college mathematics knowledge play as teacher knowledge for teaching mathematics? To answer this question, much research has been devoted to mathematics teacher knowledge (see e.g. Ball, Hill & Bass, 2005; Evens & Ball, 2009; Buchholtz et al., 2013). While these studies have been useful in providing us with the general features of mathematical knowledge for teaching, there is a lack of specific research on how college mathematical knowledge may contribute to teaching school mathematics. This study examined how teachers utilize their college mathematics knowledge in the context of school mathematics using the concepts that intersect between college mathematics and school mathematics. We focus on the concept of unique factorization domains (UFD) in college mathematics and polynomial factorization in school mathematics. We examine the understanding of polynomial factorization of novice teachers with relatively vivid advanced mathematical experience, in order to investigate what influence these teachers' advanced mathematical experience has on their teaching of polynomial factorization. The research questions are as follows: How do novice teachers understand polynomial factorization from a college mathematics perspective? How do novice teachers' polynomial factorization concepts emerge in a teaching context?

THEORETICAL BACKGROUND

Ball, Thames, and Phelps (2008) explored the different types of knowledge required for mathematics teachers and classified the domains of subject matter knowledge and

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pedagogical content knowledge into three sub-domains each, as shown in Figure 1, to conceptualize six types of teacher knowledge.

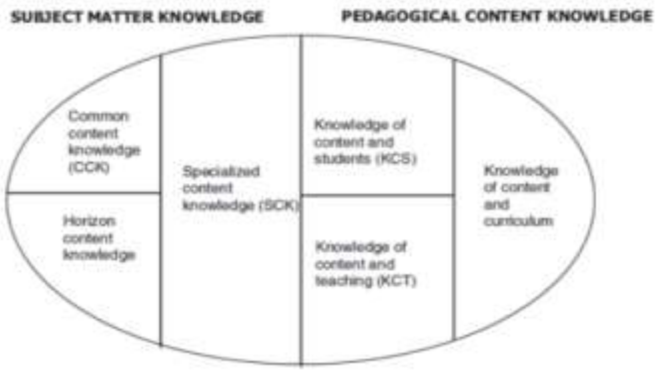

Figure 1: Mathematical knowledge for teaching (MKT) (Ball et al., 2008, p. 403).

They explained the meaning of the six types of knowledge as follows. Common content knowledge (CCK) is the mathematical knowledge that well-educated people acquire. In order to do the work that teachers assign to their students, teachers need this knowledge. Special content knowledge (SCK) is mathematical knowledge that goes beyond that expected of any well-educated adult. It involves decompressing mathematical knowledge in order to make particular aspects of it perceptible to students or to identify the source of students' difficulties. Horizon content knowledge (HCK) is knowledge about the way in which mathematical themes are related in mathematical curricula. Knowledge of content and students (KCS) is pedagogical content knowledge that compounds knowing about students and knowing about mathematics. For example, knowledge of an error typically derives from experience with students and knowledge of their thinking would be one aspect of KCS. Knowledge of content and teaching (KCT) is pedagogical content knowledge which is needed to teachers to know flows of particular content for teaching, and to decide which examples to start with and which examples to use to take students deeper into the content. Teachers need to evaluate the instructional advantages and disadvantages of the representations used to teach a specific idea. Knowledge of content and curriculum (KCC) is pedagogical content knowledge regarding how the contents of mathematical curricula are described and how to extend content throughout mathematical curricula.

The vast majority of these studies (see e.g., Hill at el., 2004; Davis & Simmt, 2006) focus on teachers' preparation and knowledge in elementary and lower secondary level mathematics, where the distance to university level mathematics is evident. In this paper, our focus will be on how college mathematics may contribute knowledge that is relevant to the task of teaching school mathematics.

METHOD

Since we considered novice teachers with the most recent college mathematics knowledge to be the most suitable participants for understanding how related college mathematics concepts are being used to understand school mathematics concepts, eight novice teachers who have been teaching in Korea were selected as our research participants. Six participants' teaching careers is four months, and the other two participants had less than three years of experience. The most participants work in high school, and only one teacher works in middle school.

We presented novice teachers with polynomial factorization tasks shown in Figure 2, and then conducted semi-structured interviews for one to two hours each. All study participants provided informed consent. All interviews were transcribed, and teachers with the same responses were categorized. In order to ensure the validity of the analysis, three researchers crosschecked the categories of teacher responses.

The following tasks are related to polynomial factorization.

1. Considering the polynomial $2x^2 + 4x - 6$ as an element of the polynomial rin g Z[x], where the coefficients are integers, factorize it. If we consider this pol ynomial as an element of the polynomial ring R[x] of which the coefficients ar e real numbers, does the result of the polynomial factorization change?
2. If a middle school student has asked you to what extent he or she needs to fact orize the polynomial, how would you explain it?

Figure 2: Polynomial factorization tasks

Tasks are developed by the researchers. Task 1 was developed to explore how teachers used their knowledge on UFD to solve a polynomial factorization task. Task 2 was developed to examine what kind of teacher knowledge about polynomial factorization is revealed in the teaching context, and to explore the linkage between their college mathematics knowledge and taught knowledge. In mathematically, the polynomial factorization depends on polynomial ring. However, in Korea, there are no clear descriptions of the domain of polynomial factorization in the curricula or textbooks. For a detailed analysis of curricula and textbooks, see the submitted manuscript by authors (2016). In the polynomial ring with integer coefficient (Z[x]), the number factor becomes an irreducible polynomial as it is not an invertible element. So, considering number factor, it is factorized to $2(x-1)(x+3)$ uniquely. In the polynomial ring with rational number coefficient (Q[x]) or the polynomial ring with real number coefficient (R[x]), because all constant except 0 are invertible elements, it is able to be ignored in uniqueness of factorization. So, both $(2x-2)(x+3)$ and $(x-1)(2x+6)$ are uniquely factorized in Q[x] or R[x].

RESULTS

Teacher knowledge from a college mathematics perspective

Novice teachers' responses to task 1 were categorized into four groups, and teachers were classified according to their responses regarding factorization in each polynomial ring. These categories of teachers' responses are shown in Table 1.

Teachers' responses to task 1		Teachers
Different	2(x-1)(x+3) in Z[x], and '2' does not matter in R[x]	Teacher 1, 2, 3
	(x-1)(2x+6) in Z[x], and 2(x-1)(x+3) in R[x]	Teacher 4
Same	(x-1)(2x+6) or 2(x-1)(x+3)	Teacher 5, 6, 7
	2(x-1)(x+3)	Teacher 8

Table 1: Teachers' responses regarding factorization

Among the four teachers who answered that the results of factorization in two polynomials had to be different, three teachers answered using the definition of UFD. They considered that the polynomial factorization in Z[x] should be 2(x-1)(x+3), and that it is unnecessary to extract the 2 in R[x]. All three teachers considered whether or not 2 is invertible in polynomial rings. They revealed appropriate mathematical knowledge and skills that most people who have learned abstract algebra acquire. Ball et al. (2008) refer to the mathematical knowledge we would expect a well-educated adult to know as common content knowledge (CCK). Thus, we analyzed them to have appropriate CCK.

The other teacher, teacher 4, said, "After factorization of polynomials in Z[x], it can be factorized more in R[x]," and considered that it is factorized to (x-1)(2x+6) in Z[x], and to 2(x-1)(x+3) in R[x].

> Teacher 4: If we factorize in the real number coefficient, <u>we can further decompose</u> the integer. (…) I think that "2" should be extracted in real number polynomials. (…) These elements need to be irreducible elements, but I don't know whether it should be different or not.

It is necessary to pay attention to teacher 4's understanding of factorization in bigger rings. Teacher 4 considered that there might be more factors in the bigger ring (in our study R[x]), than in the smaller ring (in our study Z[x]). It seems that teacher 4 knew about the irreducible polynomial (non-constant), but made the mistake of treating constants as irreducible factors when factorizing in a larger domain. We analyzed that teacher 4 did not have proper CCK. Although a limited understanding of polynomial factorization is seen in teacher 4's response, this teacher recalled the significance of irreducible elements, so we cannot claim that this teacher lacks knowledge related to college mathematics in factorization. However, we can explain that teacher 4 only vaguely recalls what was learned in college. It seems that teacher 4 relied on partial memory of college mathematics knowledge.

The three teachers who answered that the results of factorization in two polynomial rings is the same, said the factorization in these two rings does not need to be different because they are factorized as multiples of two linear expressions. It seems that they did not attempt to link the task with definitions of UFD. In this case, whether we can conclude that these teachers did not have proper CCK, or whether it was simply not revealed, is unclear. However, the results of interview with these three teachers found that they could not grasp compounded forms of KCS and SCK, which comprise the knowledge to deal with constants in factorization problems at the school mathematics level (KCS), and the specialized mathematical content knowledge that was required to teach when applying college mathematics UFD in a school mathematics context (SCK).

Teacher 8, who answered that the result of factorization in two polynomial rings was the same as $2(x\text{-}1)(x\text{+}3)$, considered that the 2 is a factor and explained it as follows:

> Teacher 8: I think. (…) the 2 can also be a factor in Z[x]. <u>I think we can also consider the integer as a factor</u>, (…) <u>I remember that the factor of the polynomial corresponds with the divisor in (natural) numbers.</u>

Teacher 8 considered 2 as a factor in Z[x]. Because he gave the result as $2(x\text{-}1)(x\text{+}3)$ in both rings, we can conclude that he considered 2 as a factor in R[x]. Teacher 8 naturally expanded the meaning by linking polynomial factorization with the prime factorization. The number 2 is a prime factor in prime factorization and he expanded the meaning in R[x]. This led to misunderstanding 2 as a prime element (irreducible element) in R[x]. Because it is known that polynomial factorization and prime factorization can be similarly expressed, it seems that teacher 8 revealed SCK. However, we were unable to examine understanding regarding the meaning of irreducible polynomials in polynomial rings.

Teacher knowledge in teaching context

Teachers' responses to task 2 were divided into two groups, and Table 3 shows a comparison of the responses to task 2 and task 1. Specifically, teacher 3 in one group, who responded $2(x\text{-}1)(x\text{+}3)$ in Z[x] and that 2 does not matter in R[x], argued differently from the other teacher in the same group. Furthermore, his response shows that the result of polynomial factorization in a school context is different from the result at college level. This pattern, whereby different claims were given in relation to the given polynomial factorization between the school context and college context, also appeared for teacher 7. We mainly describe the group containing teacher 3 and teacher 7, who differed in their responses regarding the knowledge that emerged in a teaching context and the knowledge from their college mathematics perspective.

Responses to task 2	Teachers	Relationship	Teachers	Responses to task 1	
'2' does not matter in school mathematics → $(x\text{-}1)(2x+6)$ or $2(x\text{-}1)(x+3)$, both are possible	Teacher 1 Teacher 2 Teacher 4 Teacher 5		Teacher 1 Teacher 2 Teacher 3	$2(x\text{-}1)(x+3)$ in Z[x], '2' does not matter in R[x]	Different
			Teacher 4	$(x\text{-}1)(2x+6)$ in Z[x], $2(x\text{-}1)(x+3)$ in R[x]	
'2' should be considered in school factorization → only $2(x\text{-}1)(x+3)$ is correct in school mathematics	Teacher 3 Teacher 6 Teacher 7 Teacher 8		Teacher 5 Teacher 6 Teacher 7	$(x\text{-}1)(2x+6)$ or $2(x\text{-}1)(x+3)$	Same
			Teacher 8	$2(x\text{-}1)(x+3)$	

※ If teachers' responses to task 2 correspond to their responses to task 1, lines (—) are used; if not, dots (···) are used.

Table 2: Relation between responses that middle school students are required to factorize and teachers' responses regarding factorization

Teachers 1, 2, and 3 clearly understood the meaning of UFD, and shared the knowledge that the results of polynomial factorization depends on polynomial rings. First, teacher 1, who responded that students do not have to pull out 2, considered the polynomial ring in the school context as Q[x]; therefore, this teacher mentioned that there was no need to pull out the 2 on the basis of college mathematics. We can identify that he was aware of how the content of school mathematics is related to that of college mathematics knowledge, which is termed HCK by Ball et al. (2008). On the other hand, teacher 2 did not clearly explain the domain of factorization that is handled at the middle school level. From teacher 2's response that "I don't think that students have to factorize uniquely," he seems to regard the polynomial ring in the school context as Z[x]. Thus, he explained that $2(x\text{-}1)(x+3)$ is mathematically correct. Although he knew the mathematically correct answer, he claimed that there is no need to pull out 2 in school mathematics. Teacher 2 argued that polynomial factorization in school plays a role as a tool to solve equations. The above results show that teacher 1 and teacher 2 understood the polynomial ring treated in the school context as Q[x] or Z[x], respectively. We analyzed that this knowledge influenced their understanding of the curricula as KCC. However, KCC emerged differently for teachers, because there are no clear descriptions of the domain of polynomial factorization in the curricula or textbooks. Teacher 3, who explained that the given polynomial in task 1 was factorized differently in the two polynomial rings, did not clearly express the domain of factorization dealt with in school mathematics. Teacher 3 considered that students had to be taught to pull out the 2 on the basis that it made the problem-solving process easier for them.

According to three teachers who had proper CCK, they have different intentions in teaching polynomial factorization in a school context; these include maintaining rigid mathematical knowledge, the instrumental characteristics of knowledge, and making the problem easier to solve. Three teachers identified pedagogical issues that affect student learning, and evaluated the instructional advantages and disadvantages of treating number factor 2 in polynomial factorization. According to the MKT framework, combined knowing about teaching and knowing about mathematics, which is KCT, were used by three teachers.

Teachers 5, 6, and 7, who understand polynomial factorization as multiples of two linear expressions such as $(x-1)(2x+6)$ or $2(x-1)(x+3)$ can be separated into two groups. Teacher 5 mentioned that it does not matter whether or not 2 is pulled out in school mathematics. Teachers 6 and 7 stated that 2 should be considered in school factorization. While teachers 5 and 6 did not explain the basis for their responses consistently and concretely, teacher 7 presented his reasoning as follows.

> Teacher 7: <u>School exams test how well a student has learned</u>; therefore, students should be taught to factor out integers. The fundamentals of factorization involve factoring out "m" from "ma+mb." <u>I think the reasons why students do not pull out 2 in the polynomial factorization are due to lack of understanding of the rule given at school.</u>

Teacher 7 responded that students must be taught to pull out the 2 in order to follow the procedure of explanation in mathematics textbooks. We analyzed that teacher 7 had a didactic intention that students should implement the rules given in school mathematics. Teacher 7 showed an understanding of polynomial factorization at the school mathematics level in task 1, while knowledge that interprets by connecting this to college mathematics could not be observed in this teacher's response. He explained that the common factor is first taught in the topic of polynomial factorization in middle school textbooks. Thus, he argued that students must factorize by pulling out the common factor 2. Additionally, he mentioned that students often make mistakes in ignoring the common factor 2. We analyzed that he showed an understanding of the common factor and the pedagogical flow of textbooks. In terms of MKT, combined KCT are identified. Furthermore, we analyzed that he grasped mistakes derived from his experience with students, and anticipated what students are likely to think. In terms of MKT, KCS are identified.

CONCLUSION

In this study, we examined how teachers are utilizing their college mathematics knowledge in the context of school mathematics. We identified that the knowledge taught by each teacher differed due to their individual PCK and didactic intentions. We have shown that there is a difference between knowledge that teachers consider mathematically correct and knowledge that students require. Teachers who knew about polynomial factorization at the college-level explicitly did not simply follow this knowledge to teach polynomial factorization in the school context. They sometimes modified that knowledge to suit the level of secondary school mathematics, but recognized how they were treating number factors in explaining polynomial factorization in a school context. Based on the results, we identified that HCK plays a significant role in understanding secondary school mathematics from a higher standpoint.

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NOTICING AFFORDANCES OF A TYPICAL PROBLEM

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Typical mathematics problems, such as examination-type questions, are often used in classrooms to develop students' procedural fluency. In this article, we describe and analyse what a secondary school mathematics teacher noticed about the affordances of such a problem, as well as how she orchestrated a mathematically productive discussion using the adapted problem in class. The findings suggest that a teacher's productive noticing of the affordances offered by typical problems can enhance the learning experiences of mathematics students.

INTRODUCTION

Using high cognitive-demand tasks is critical for orchestrating productive discussions (Smith & Stein, 2011) during lessons. However, besides the development of concepts, teachers are also mindful about the concomitant development of procedural skills to prepare students for tests and examinations. Hence, the practice of using *examination-type questions* with a more teacher-centred teaching approach is prevalent in Singapore classrooms (Foong, 2009; Ho & Hedberg, 2005). This preference for using typical problems—standard examination or textbook problems—may reflect teachers' belief that it is "important to prepare students to do well in tests than to implement problem-solving lessons" (Foong, 2009, p. 279), a classroom reality that cannot be ignored. Given teachers' strong preference for using typical problems in the classroom, it would be interesting to explore the use of such questions to orchestrate rich discussions. This paper is therefore framed by the following question: Whether, and if so, how typical problems, such as examination-type questions, can be used to orchestrate productive mathematical discussions? Drawing on our preliminary findings from a larger study, we present a case study of Ms. Alice, a proficient secondary school mathematics teacher, to highlight how her productive noticing of the affordances offered by typical problems could provide a mathematical learning experience, aimed at developing students' understanding of matrices.

THEORETICAL CONSIDERATIONS

Orchestrating Learning Experiences

With the aim of supporting teachers to plan for more skilful improvisation, Smith and Stein (2011) propose five productive practices—anticipating, monitoring, selecting, sequencing and connecting—as a way to make "student-centered instruction more manageable" (p. 7). Anticipating, which occurs during lesson planning, refers to predicting students' likely responses to the tasks. The other practices pertain directly to

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the actual work of orchestrating discussions after teachers set students to work on the task: monitoring students' responses while circulating in the classroom, selecting particular students' answers, and purposefully sequencing these selected answers for presentation. Last but not least, teachers support students in making sense of the mathematical ideas by connecting these responses to make a mathematical point. To enact these practices, teachers would need to draw on appropriate mathematical knowledge to interpret students' responses during lessons, and make the necessary pedagogical moves for advancing students' thinking (Smith & Stein, 2011).

Mathematics Teacher Noticing

According to Kilpatrick, Swafford, and Findell (2001), mathematics teachers who are proficient at orchestrating discussions should be able to examine the mathematical possibilities of instructional materials, adapt them for different student profiles, analyse students' reasoning, and respond to the different methods students use in their work. Doing this work of ambitious teaching requires developing a keen awareness of the mathematical connections and having a different act in mind (Mason, 2002). Therefore, developing teachers' eyes to see and the mind to make sense of these mathematical connections is critical for orchestrating learning experiences. Mathematics teacher noticing is an emerging construct that lies at the heart of these components of teaching expertise. It refers to what teachers attend to and how they interpret their observations to make instructional decisions (Mason, 2002; Sherin, Jacobs, & Philipp, 2011). Many studies (e.g., van Es (2011)) on teacher noticing used video studies and investigated what teachers noticed without giving explicit instructional aspects for teachers to direct their attention; other studies (e.g., Goldsmith and Seago (2011)) used teaching or learning artifacts to focus teachers' attention to specific features of instruction. More recently, Choy (2015) brings task design into the realm of teacher noticing and his findings suggest that an explicit focus for noticing, and an emphasis on pedagogical reasoning can increase the likelihood of teachers making instructional decisions which promote students' reasoning. Building on the Three-Point Framework by Yang and Ricks (2012), Choy (2015) highlights mathematics concepts, students' confusion, and teachers' courses of action as critical foci to facilitate productive noticing.

Affordances of a Mathematics Task

A typical problem, as described earlier, can certainly be used very procedurally by a teacher but can it be used in a more productive manner? What kinds of affordances do such problems offer to the teacher? In this context, using the perceptual psychologist, Gibson's (1986) ideas, we can emphasise that: (1) an affordance for using a typical problem exists relative to the action and capabilities of the teacher, (2) the existence of the affordance is independent of the teacher's ability to perceive it, and (3) the affordance does not change as the needs and goals of the teacher change. Gibson also highlighted that affordances in relation to an observer could be positive or negative which in our context may lead to productive or less productive use of the problems in class by the teacher. Hence, to perceive the affordances of a typical problem means to be able to *notice* the characteristics of the task in relation to the particular understandings of the related concept in order to adapt the task for use in classrooms. But what should a teacher notice about a task so as to recognise its affordances? In this paper, we adopt Choy's (2015) notion of productive noticing to investigate what a teacher noticed about the affordances of a typical mathematics problem.

METHODOLOGICAL CONSIDERATIONS

The data reported in this paper came from a larger study on orchestrating learning experiences in a secondary school mathematics classroom in Singapore. The study followed a design-based research approach to develop a toolkit for teachers as a means of supporting their orchestration of learning experiences, as well as to develop a theory about teachers' noticing in the context of orchestrating learning experiences. In this paper, we examine the practices of Alice (pseudonym), one of the three teachers who took part in the study. Alice is a Senior Teacher at Coventry Secondary School (pseudonym), which is a government-funded school performing slightly above average in the national examinations. As a Senior Teacher, Alice has a strong mathematical background and has been actively involved in mentoring novice teachers in her school.

Data were collected and generated through voice and video recordings of the lesson, as well as voice recordings of a pre-lesson discussion and a post-lesson discussion with Alice. The findings were developed through identifying themes related to what Alice noticed about the content, her students' confusion, and her own courses of action, with reference to the framework developed by Choy (2015). In addition, the lesson was also analysed by identifying segments which corresponded to Smith and Stein's (2011) five practices. In this paper, we present our preliminary findings through snapshots of how Alice noticed the affordances of a typical problem, and how she deployed the problem in class to orchestrate a mathematically productive discussion.

FINDINGS AND DISCUSSION

In this section, we present our analysis of Alice's lesson on Matrices for Secondary Three (Grade 9) students. Her students had learnt how to multiply two matrices prior to this lesson. The learning experience stipulated in the curriculum document was for students to apply matrix multiplications to solve contextual problems, and for them to justify if two matrices can be multiplied by checking the order of the matrices. During the introductory phase of the lesson, Alice used a modified version of a typical problem (See Figure 1 for the typical problem) and orchestrated a mathematically productive learning experience using students' responses to the modified problem (See Figure 2).

Perceiving the Affordances of a Typical Mathematics Problem

Alice selected a typical problem (See Figure 1) from a past examination paper as the source of the introductory problem to be used during the lesson. This is a typical examination-type contextual problem involving matrices. There are two parts to the problem: the first part requires students to perform a routine matrix multiplication that involves pre-multiplying a 3×1 matrix by a 2×3 matrix to obtain a 2×1 matrix $\begin{pmatrix} 185 \\ 184 \end{pmatrix}$ as the solution; the second part requires them to explain the meaning of the product which in this context represents the total points gained by Theresa and Robert for the awards.

Example 1

[Nov 2013] Teresa and Robert attend the same school. They keep a record of the awards they have earned and the points gained. The matrices show the numbers of awards and the points gained for each award.

$$\begin{array}{cc} & \begin{array}{ccc} \text{Gold} & \text{Silver} & \text{Bronze} \end{array} \\ \begin{array}{c} \text{Teresa} \\ \text{Robert} \end{array} & \begin{pmatrix} 29 & 10 & 5 \\ 30 & 6 & 8 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \text{Points} \\ \begin{array}{c} \text{Gold} \\ \text{Silver} \\ \text{Bronze} \end{array} & \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \end{array}$$

(a) Find $\begin{pmatrix} 29 & 10 & 5 \\ 30 & 6 & 8 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$.

(b) Explain what your answer to (a) represents.

Figure 1: The Typical Problem used during the Lesson

Instead of presenting the problem as it was given, Alice made two modifications to the problem (See Figure 2). First, she provided information about the awards obtained and the points for each award within the stem of the problem instead of representing them in matrices as in the original problem. Second, she asked for the total number of points obtained by each person instead of finding the matrix product directly. By doing so, Alice modified the problem in a way that required students to formulate their solution in terms of a matrix multiplication. Moreover, because the order of the matrices were not given, students had to decide on the order of the matrices before using the appropriate matrix multiplication to find the answer. In addition, the modified problem did not require students to use a matrix method, which then reduced the problem to a straight-forward arithmetic one. This provided opportunities for Alice to emphasise the connections between matrix multiplication and arithmetic which could potentially provide some meaning to matrix operations.

> Teresa and Robert attend the same school. They keep a record of the awards they have earned and the points gained. Teresa obtained 29 Gold, 10 Silver, and 5 Bronze awards. Robert obtained 30 Gold, 6 Silver, and 8 Bronze awards. They gained 5 points from each Gold award, 3 points for each Silver award, and 2 points for each Bronze award.
> Find the total number of points that Teresa gained.
> Find the total number of points that Robert gained.

Figure 2: The Modified Problem used during the Lesson

More importantly, Alice anticipated students might not use matrix multiplication, as intended in the original problem (Figure 1), because they had just recently been introduced to matrix multiplication. Instead, students could the answer by performing two separate matrix multiplications or they could use an arithmetic method. During the post-lesson interview, Alice revealed that she had considered students' confusion and anticipated their answers based on her knowledge of the students:

> Why I choose this question is because most of the exam style questions are based on solving problems involving matrices. And this question will extend their thinking and help them to transfer their mathematical thinking into other representations. This is what I find challenging amongst some students… I will know that certain students will give this [answer], exactly which students I don't know…

Hence, we see how Alice's reasoning for the modifications were made. These modifications afforded opportunities for her to build on students' less-than-optimal solutions to reveal students' reasoning, explain the procedure of multiplying two matrices, and connect students' solutions to the intended one.

Orchestrating Discussions with the Adapted Problem

Alice demonstrated her recognition of the affordances offered by the modified task when she orchestrated discussions during the lesson. After setting her students to work on the problem, Alice moved around in the class checking students' responses and helping out students who had queries:

1 Alice: (Walks around the class and comes to Student S1.) Can you write this for me on the board?

2 S1: Ok. (Walks to the whiteboard and writes the following.)

$T = 5 \times 29 + 3 \times 10 + 2 \times 5 = 185$

$R = 5 \times 30 + 3 \times 6 + 2 \times 8 = 184$ [*arithmetic solution*]

3 Alice: (Walks around while waiting for Student S1 to finish writing.) Ok. Most of you have written what [Student S1] has written. 5 points for 29 gold, 3 points for 10 silver and 2 points for 5 bronze. Most of you have written in this manner. The last few days, we have been talking about matrices, right? Would you like to convert this to a matrix problem?

(Student S2 raises his hands.) Have you written it in matrix form? (Student S2 nods and Alice goes over to take a look.) Okay. Can you write your answer on the board?

4 S2: (Walks to the board and writes the following.)

$$T = \begin{pmatrix} 29 & 10 & 5 \end{pmatrix} \times \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = 185 \text{ and } R = \begin{pmatrix} 30 & 6 & 8 \end{pmatrix} \times \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = 184$$

[*two separate matrix products*]

5 Alice: Any other answers from [Student S2's] answer? (Walks around the class and selects Student S3's answer) Can you write this on the board?

6 S3: (Walks to the board and writes the following.)

$$\begin{pmatrix} 29 & 10 & 5 \\ 30 & 6 & 8 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 29 \times 5 + 10 \times 3 + 5 \times 2 \\ 30 \times 5 + 6 \times 3 + 8 \times 2 \end{pmatrix} = \begin{pmatrix} 185 \\ 184 \end{pmatrix}$$ [*a single matrix produced*]

7 Alice: Thank you all three of you. [Student S1] has written using an arithmetic method. Most of you have written in this manner. This one comes very naturally to you, ok? [Student S2] has written Robert and Theresa's award separately. He has tried to use the matrix method, (points to Student S1's solution.) Something like this, ok? Let's check whether the order of matrix is correct or not.

(Alice goes through the method of matrix multiplication and gets the class to check the order of Student S2's matrices.)

… Ok. Student S3 has written Robert's and Theresa's together so that you only write this matrix once (points to the column matrix [5 3 2]). Don't need to write two times, correct or not? See. Over here. You have to write two times but here, [Student S3] only has to write it once. Let's check the order again…

Alice then asked the class why Student S3's solution was better compared to the other two students. She led the class to see that Students S3's method is a more economical process as the "points matrix" is written only once and that would be useful if there were, say, 100 students. She also highlighted the use of matrices to represent large amount of data. Following this, Alice initiated another short discussion:

8 Alice: I would like to bring this problem a little bit further. Notice that Student S3 presented the information this way. Is there another way to represent the same information?

(After some time, Student S4 highlights a possible way.)

9 S4: Change column and row. (Student S4 goes up and writes a 3 × 2 matrix.)

This response got students thinking about the order of the corresponding "points matrix". Another Student S5 went up and wrote the correct matrix product as a 2 × 1 matrix but pre-multiplied the 3 × 2 matrix to the 1 × 3 matrix. Alice then orchestrated a short discussion for Student S5 to realise his mistake, who then correctly wrote:

$$\begin{pmatrix} 5 & 3 & 2 \end{pmatrix} \begin{pmatrix} 29 & 30 \\ 10 & 6 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} 185 & 184 \end{pmatrix}$$

In this short vignette, we see how Alice orchestrated a mathematically productive discussion using Smith and Stein's (2011) five practices. Alice monitored students' answers to the questions carefully when she was circulating the classroom. Even though she asked for volunteers to answer the questions, it was clear that she was deliberate in her selection and sequencing of students' responses (See Lines 1 to 6). By beginning with an arithmetic solution, she was able to connect Student S1's arithmetic operations to how matrix multiplications are performed through the sequencing of Student S2's and Student S3's matrix solutions. Alice also highlighted the different ways to express the given information as matrices (Lines 8 and 9), which was an important idea for the lesson, and gave the motivation for using a matrix approach. The reason for writing the problem as a product of two matrices (Student S3's solution) was made explicit when Alice moved from Student S2's solution to the solution offered by Student S3.

ALICE'S NOTICING AND AFFORDANCES OF THE TASK

The two vignettes highlight how Alice went beyond solving the original problem procedurally and identified students' experiences that could be enhanced. She modified a typical examination question to emphasise at least three key ideas in matrix multiplication during the classroom discussion. First, Alice used the arithmetic solution to write out explicitly how matrix multiplication is performed (See Line 2). Next, she emphasised the order of matrices and when matrices could be multiplied (Line 7). Lastly, she highlighted how contextual information represented in matrices could be captured in different ways (e.g., using 2 × 3 and 3 × 1 matrices or using 1 × 3 and 3 × 2 matrices). Drawing from how Alice orchestrated the learning experience in class, we argue that she noticed the affordances of such a typical problem. More specifically, Alice attended to students' possible confusion that there was only one way to represent information using matrices, and used her understanding of the relationships between arithmetic and matrix operations to modify the problem. Moreover, Alice's orchestration of the discussion in class suggests she was more attuned to students' particular solutions and thus she was able to sequence the presentation to enhance students' learning experience.

In many ways, Alice's noticing can be classified as extended in that she modified the problem based on her interpretation of content requirements, and attended to particular strategies offered by various students. Her strong content mastery could be attributed to the professional development activities, such as mathematics-related and general pedagogical courses she had taken in recent years. Alice's use of the typical problem suggests that she was familiar with the syllabus requirements. Even though her post-lesson interview reveals a strong need to fulfil the requirements of the examinations, Alice tried to interpret and adapt the written curriculum to suit the needs of her students. She was reflective and was always ready to learn from her experiences with students. By "recognising possibilities" from her "three worlds of experiences" (Mason, 2002, p. 94), Alice was able to see beyond the given typical problem, had a different act in mind, and proposed another way to ask the question. Furthermore, given her focus on the concept, students' thinking, and how she orchestrated the learning experience of her students, we also classify Alice's noticing as productive (Choy, 2015) because she had recognised the affordances of a typical task that could potentially led students to gain new insights into the use of matrices.

CONCLUDING REMARKS

Despite a high-stakes examination-driven system, Alice's noticing had enabled her to use a typical problem beyond drill-and-practice for examination to emphasise conceptual understanding. Being able to perceive affordances or "notice possibilities" (Mason, 2002, p. 94), as Alice had done, could therefore provide a way to negotiate the murky territory between procedural and conceptual fluency. Adopting a pragmatic approach, she moved in between the two, and capitalised on the affordances of a typical examination problem to (i) reinforce procedural skills, (ii) emphasise examination requirements, and (iii) at the same time, develop conceptual understanding. Notwithstanding the limitations of our preliminary findings, Alice's modification and use of a typical problem highlight the potential of noticing affordances of such questions, and how even typical questions can be used for orchestrating discussions. Given the prevalence of using such typical problems in classrooms, supporting teachers to unlock these problems' affordances may hold promising implications for teaching and learning. It remains to be seen how we can support teachers to notice the affordances of tasks in our future work with them.

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MATHEMATICS TEACHER LEARNING AND DOING WITHIN PROFESSIONAL DEVELOPMENT

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In this theoretical research report, we aim to consider what is done within professional development activity and how it may or may not approximate to what is done in a classroom. We draw on enactivism to analyse what shifts are needed for a teacher, after engaging in a professional development activity, to make new and effective distinctions in their classroom. Drawing on our own experiences of organising professional development, we consider a range of scenarios, including being offered activities for the classroom and seeing someone else teach your students in your classroom. We conclude that it is a helpful tool in designing activities to consider what is invariant and what varies in mapping what is done within professional development onto what is done in a classroom.

INTRODUCTION

The literature on mathematics teacher learning through professional development has been categorised (Liljedahl, in Brown and Coles, 2010, p.377) into three strands: *content* (of teacher knowledge or belief); *method* (on specific models of professional development); and, *effectiveness* (looking at changes in practice). In this theoretical report, we consider the possibilities for teacher learning across different kinds of professional development activity. In particular, we are interested in what is involved for a teacher in mapping what is done, within a professional development (PD) activity, into their own mathematics classroom practice. We look at a range of PD methods and consider, from a theoretical point of view, what kind of translation or transformation is needed for a PD session be effective. We ask:

1. Who is doing what doing?
2. How does the doing in the professional learning activity approximate the doing in the classroom?
3. What is the role of the doer in the professional learning activity, compared to as a teacher in their own classroom?
4. What is significant about what is invariant and what is the variant?

ENACTIVISM

We draw on enactivism (Reid and Mgombelo, 2015) to help us consider the learning and doing of mathematics teachers, when they are involved in professional

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 257-264. Singapore: PME.

development activities, in relation to when they are teaching in their classroom. Enactivism is a perspective that is informed by systems thinking (Bateson, 1972), phenomenology (Merleau-Ponty, 1962) and a radical view of biology (Maturana and Varela, 1987) that all, in different ways, consider change and relationship as the basis of cognition. From the enactive perspective, the web of relations between components that constitute our being (including any tools we might use) is labelled our 'structure'. Every interaction in the world alters our structure and one of the enactive insights is that humans are 'structure-determined' beings. In other words, when an event occurs which provokes a response, the response we give is not a function of the trigger but a function of our structure. Furthermore, overtime, we become 'structurally coupled' with those people and contexts with which we have recurrent interactions. Each moment of interaction alters, however minimally, my structure and the structure of who and what I am engaging with.

So, in a PD session, each teacher is triggered by the other participants and the leader of the PD and the activities they undergo, changing their structure, making it possible (but not inevitable) for new behaviours to happen when they return to their own classroom. The cultural and the social are embodied in our very beings, in our structure. As a result of the history of our structural coupling, in most situations we make automatic responses, from driving a car to the small prejudices we may catch ourselves projecting onto others who are not like us. Skilled teachers have a vast array of automatic responses in their classroom, which can make it difficult for new behaviours to arise as possibilities unless those automatic responses come to be seen as ineffective for some reason (e.g., a change in school and therefore responses of the pupils, or the teacher becoming dissatisfied with the teacher they are becoming).

Within enactivism 'doing', 'knowing', 'being' are seen as synonymous: 'all doing is knowing all knowing is doing' (Maturana and Varela, 1987, p.27). What it means to know something is to act in an effective manner in a context. There are echoes of behaviourism in this statement but for enactivists there is no denial of an 'inner life', rather a more radical collapsing of the distinction between 'inner' and 'outer'.

In one of Bateson's famous examples (1972), he considered: where does the 'mind' of a blind man with a stick end? It seems clear that the blind man's attention is at the end of the stick – not in his hand, where the stick's vibrations are first 'felt' in the body. Our 'minds' do not stop at the edge of our skull, rather our whole 'structure' is embedded and enmeshed within countless arcs and patterns of interaction extending into the world. Learning is indicated by a shift in these patterns of interaction, by seeing differently, and therefore making new distinctions, in a particular context. In this report we look at varying professional learning activities through the lens of enactivism in order to consider the possible conditions of their effectiveness.

LOOKING AT PROFESSIONAL LEARNING ACTIVITIES

As we are adopting an enactivist lens our unit of analysis is activity and effective behaviour. In particular, we are interested in looking closely at what teachers can be paying attention to within professional development activity, what distinctions are available to be made for these teachers, and how these distinctions could map onto the distinctions that are necessary in a classroom. The set of professional learning activities we have chosen to examine, then, is neither exhaustive nor hierarchical. Rather, they are teacher-learning activities that afford us varying distinctions between what happens during the activity and what might happen in the classroom of the teacher. All three authors have been involved extensively in offering professional development to teachers of mathematics over one or more decades. We came to write this paper, partly through comparing our approaches to PD, and we draw on our experiences to consider the following range of activities: attending a lecture or course; watching a video recording of another teacher, or yourself; seeing someone else teach your class; being given an activity to try out in your classroom; being given a structure and an activity to try out in your classroom; being given a structure for activities to try out in your classroom (action research). In the next section, we consider each scenario, focusing on the questions from the Introduction.

PROFESSIONAL LEARNING ACTIVITIES

We consider each activity in turn, starting with a fictionalised example of what a teacher did, and then considering the 'doings' and distinctions that are in play.

Attending a lecture

> Maha attends a lecture during a conference in which she is told about teaching methods in East Asian countries. The lecturer discusses the use of 'variation' in teaching and learning new concepts. The intention is that Maha adopts new ways of working in her classroom as a result of being in the lecture.

Maha is listening and attending to the distinctions and words of the speaker. Of course it is impossible we ever share the same meaning for words that categorise complex and multi-faceted elements of practice and observation (e.g., 'variation'). Maha may recognise differences compared to her own practice, in what is being presented in the lecture. On one level, Maha cannot *not* change, however minimally, as a result of being in the lecture. But if she is to make new choices in her practice, after attending the lecture, she will need to recognise when there is an appropriate opportunity for making the distinctions offered. She will need to work to recognise in her own practice what is being discussed within the practice of someone else and perhaps work to 'suspend' (Varela and Scharmer, 2000) patterns of typical responses.

Watching a video recording of another teacher, or yourself

> Pippa attends a 'video club' for teachers of mathematics, which involves watching video recordings of others teaching and showing others video recordings from her own classroom. The intention is that teachers will focus on using activities that promote student reasoning and learn about effective teaching strategies for promoting reasoning.

There is significant interest at this time in the use of video in the context of teacher learning (e.g., to mention just a tiny sample: Sherin, 2007; Star and Strickland 2008;

Sherin and van Es 2009; Coles 2013). Typically, in a context of watching a video as part of a professional development course, there will be a facilitator who may guide or steer discussion. Teachers may be invited to share what they see in the video, in relation (or not) to a particular focus such as 'mathematical reasoning'. Pippa is, therefore, given an opportunity to share distinctions she observes in the video, which may be distinctions between actions she sees done by the teacher on the video and her own expectations or routines. These distinctions, perhaps clashes of expectations, are reported in many instances to lead to judgmental responses from teachers (Jaworski, 1990). It is reported that when discussion begins in a judgmental manner, it is hard for talk to be productive (Jaworski, 1990; Coles, 2013).

In both watching a video and being in a classroom, a teacher can notice students' actions and become aware of what they might do in that context, or (in a classroom) simply act. The video potentially allows the bringing into awareness of habitual ways of responding in one's classroom and, potentially, the awareness of alternatives.

Pippa may be involved in observing and evaluating other teachers as part of her job. In this case, there is a direct mapping from what is done in the video club to her evaluation role with other staff, i.e., in both cases she needs to observe another teacher and consider what to say about what she notices. If she is forced to observe in particular ways in the video club (e.g., following Jaworski (1990) and Coles (2013), she might have to start by just focusing on the detail of events and not any emotional judgments) there is the potential of her using a new way of observing in her observation work in school.

Seeing someone else teach your class

> [This example is based on a real experience involving two of the authors.] Alf was teaching mathematics in a school in London and had a high attaining grade 8 class (aged 12-13). Laurinda spent a day in his school and taught this grade 8 a lesson on algebra (on number sequences and algebraic rules), with Alf observing from the back. The topic was chosen by Alf and was what the class would have been doing had he taught them.

Dick Tahta taught one of Laurinda Brown's classes (in the 1970s) and Laurinda taught one of Alf Coles' classes (in the 1990s) as described above; Alf has since taught lessons in other people's classrooms, for example in the context of a primary school project (Coles and Scott, 2015).

We identify two sets of distinctions available from watching someone else teach your class. Firstly, there may be distinctions available around how the students in class behave differently to normal. Alf wrote, at the time of Laurinda taking his class (see Brown and Coles, 2008), of seeing his children 'thinking mathematically' and 'being algebraic' in a manner that he had never experienced before (in his own class or from observations of others). This set of distinctions is around seeing possibilities, in terms of what students can do and how they can be, that may never have even been around as things a teacher realised was possible.

The second set of distinctions mirrors ones available in some of the earlier activities, and these are around particular teaching decisions ‘I would not have done that’. In the case of someone teaching your own class of students, there is a close fidelity to the situation in which you would be in the position of making those teaching decisions.

Being given an activity to try out in your classroom

> Ben attends a professional development session on collaborative problem solving. During this session, the participants spend some time solving a mathematics problem posed by the facilitator of the session. The intention is that Ben will take this same problem and use it with his own class.

Unlike the watching of a video, where the activity is to observe others (or yourself) in the third person, solving a problem is a first-person activity. That is, teachers working individually or collaboratively to solve a mathematics problem are living the experience that is intended for students. Within this activity, the participants may become aware of the distinctions between the strategies that can be used to solve the problem and between the types of mathematics that can be used. If there is an opportunity to debrief this experience within the session then more of the same type of distinctions may be acquired. These distinctions need to be translated not only to a teacher’s own classroom, but also from experiences as student to their role as teacher.

Being given a structure and an activity to try out in your classroom

> Cathy attends a professional development session on collaborative problem solving. During this session, the participants are solving a mathematics problem in random groups working on wall-mounted whiteboards. After this the facilitator discusses his/her rationale for having them work in random groups and on whiteboards and the choice of task. After this they are given a new task to solve in new groups on whiteboards which is, again, debriefed. The participants are then told to try the same problem and the same structure within their own class.

As with being given an activity only, the work in the PD session is one of being a student, trying out the activities to be offered in the classroom. However, the difference is that, through being offered a structure and rationale for the activities, the teachers in the session are forced to split their attention and, simultaneously, to be in the action of working on some mathematics, and making distinctions about how their activity relates to its stated purpose. Mason (2002) discusses the layered awarenesses needed for engaging in activity while also noticing one’s engagement in activity.

As with the activity of watching someone else teach your own students, enacting an activity within a set structure affords the teacher the opportunity of behaving differently. This can allow the teacher to see the students being mathematical in a way that they may not normally be. As such, a set of distinctions is available around seeing what is possible, not only by the students, but by the teacher's own hand.

Being given a structure for activities to try out in your classroom: action research

> Nima attends a course, run at a University, that supports her to undertake action research in her own classroom. She chooses to focus on what she can do to make her students more

resilient and independent. From her readings and course meetings, she decides to try out a range of new actions in her classroom and evaluates their success.

Nima's course sessions support the making of new distinctions in the classroom through provoking and encouraging new or different (from what had been done in the past) actions on the part of the teacher (Brown and Coles, 2011). In some sense, the learning of the teachers is not mediated by the course leader, in that no one else is observing what takes place in the classroom. The distinctions shared in course sessions are about individuals' classrooms. Structure is provided by the action research model (e.g., Altrichter et al., 2003), which provokes Nima into experimenting with novel classroom activities and noting the reaction.

VARIANTS AND INVARIANTS

Looking across the scenarios that have been sketched above, and the consideration of the distinctions made both within the session and in the classroom, it is clear that there are some things that remain the same and some things that are different, in the move into the classroom. We summarise these invariants and variants below.

PD activity	**Invariant**	**Variant**
Attending a lecture	Intention that teachers will make the same distinctions being made in the lecture	From listening *to* acting (incl. recognising a context for a new distinction)
Watching a video recording of another teacher, or yourself	The classroom context and observations of student (and teacher) activity are shared	From observing a classroom and responding to teachers *to* observing a classroom and responding to students
Seeing someone else teach your class	The classroom and the students stay the same	From observing *to* acting
Being given an activity to try out in your classroom	The activity itself stays the same	Moving from acting as a student *to* acting as a teacher
Being given a structure and an activity to try out in your classroom	The activity and structure stay the same	Moving from acting as a student and (in parallel) as an observer of those actions *to* acting as a teacher and observer of student actions
Being given a structure for activities to try out in your classroom: action research	The focus of discussion is on distinction made by the teachers in their own classroom	Teachers need to implement actions discussed in sessions or suggested from readings

DISCUSSION

In several PD scenarios there is a passive to active shift required, from session to classroom, of moving from listener to actor (attending a lecture), or from observer to teacher (watching a video; seeing someone else teach your class). In other words, the doing in the PD session is quite different to the doing in the classroom. In other scenarios the doing in a PD session mirrors the intended doing of students (being given an activity / being given an activity and a structure) and a different translation is required. One phenomena we recognise is the offer in a PD session of an open problem to solve, which as a teacher we explore and solve in a particular manner – and the subsequent temptation to constrain the activity for students in the classroom to the particular method of solution we adopted, rather than the more open offer that we had received. If discussion in a PD session is focused on the distinctions of teachers (being given a structure for activities: action research) then the translation from session to classroom is one of noticing distinctions *after* classroom events to noticing them in-the-moment and using that awareness in acting differently.

Teacher learning and teacher change, from an enactivist perspective, are linked to the development of new habits in the classroom. For a change to occur, the teacher must act in a novel way in a given kind of scenario. One significant variant-invariant occurs when a PD session involves or leads to activity in the classroom that results in students in the classroom acting in a novel manner. Of course, this may be the result of any form of PD. However, there are certain forms of activity where it is likely that changes in student response will occur. Seeing someone else teach your class, will inevitably result in an observation of novel student behaviour and, where that behaviour is valued, such an experience provides a strong motivation to work on developing one's teaching. Seeing what is possible here and now with one's students, can be powerful. Being given an activity to try out can similarly result in students acting in novel ways. If the activity is far outside students' expectations and usual routines, it may be that the change is not perceived in a positive light. Being given an activity and a structure to try out, similarly can result in new student behaviours (for example, observing students working on wall-mounted white boards). Where the structure offered for the activities provides a rationale, that structure can provide a tool to allow a teacher to continue experimenting and exploring the possibilities of these new ways of organising the classroom.

CONCLUSION

If we accept the enactivist adage that doing is knowing, then the organisers of professional development need to pay attention to who is doing what, during a PD session and to the relationship between the doing now and the doing in the classroom. It is possible to approximate the classroom context in a PD activity in a range of ways, through: discussing it (attending a lecture); recording it (watching a video); being in it (seeing someone teach your class); making teachers the students (being given an activity and/or structure); and, by researching it (action research). Each approximation keeps some elements the same and changes others. Through each PD scenario, significant shifts occur when a teacher re-sees their context as offering new possibilities for acting and being in the classroom. It is clear to us that we cannot simply 'give' other people the distinctions we make. What we 'see' in a classroom is a result of our entire history of interaction. It is only through teachers articulating the distinctions they make (de-briefing after: watching a video; watching someone teach; engaging in an activity) that, as a leader of PD, we can become sensitive to those distinctions. We recognise that differences compared to expectations (e.g., what another teacher does or other students behave) can be experienced as ‘wrong’. Effective behaviour, within PD, may necessitate a letting-go of evaluative judgments and a re-directing of attention towards alternative behaviours and ways of being.

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REFRAMING MATHEMATICAL FUTURES II PROJECT: DEVELOPMENT OF A DRAFT LEARNING PROGRESSION FOR ALGEBRAIC REASONING

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Curriculum documents make a clear distinction between algebraic skills and algebraic reasoning, where the development of the former is far more readily articulated than the latter. While there are many studies of algebraic reasoning, these are usually topic specific and/or highly contextual. What are the big ideas of algebraic reasoning and is it possible to map their learning trajectory? This paper reports on the preliminary phase of a large national study in Australia which is designed to move beyond the hypothetical and to provide an evidence-based foundation for a learning progression. Using rich assessment tasks designed for middle years students of mathematics, this paper reports on the method of analysis used and some preliminary findings.

INTRODUCTION

This research is situated within the Reframing Mathematical Futures II (RMFII) Project (2014-2017) which is funded by the Australian Government through the Australian Mathematics and Science Partnership Projects. This competitive grant Project followed on from the Reframing Mathematical Futures (RMF) Priority Project (2013) that aimed to improve multiplicative thinking and proportional reasoning in Years 7-10 using the Scaffolding Numeracy in the Middle Years (SNMY) resources (Siemon et al., 2006). All participating schools in the RMFII Project also participated in the RMF Project, although some did this after having joined the second Project.

RMFII is aimed at building a sustainable, evidence-based, integrated learning and teaching resource to support the development of mathematical reasoning in Years 7-10. The Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2016) consists of three content strands (Number and Algebra, Measurement and Geometry, and Statistics and Probability) and four proficiency strands (Understanding, Fluency, Problem Solving and Reasoning). Three areas of mathematical reasoning, aligned to the content strands of the Australian Curriculum: Mathematics, were identified to be investigated. These areas were Algebraic Reasoning, Spatial Reasoning, and Statistical and Probabilistic Reasoning. This paper addresses the component of the Project that aims to identify and map the 'big ideas' in algebraic reasoning. For the purpose of the Reframing Mathematical Futures II Project algebraic reasoning encompasses:

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 265-272. Singapore: PME.

- *Core knowledge* needed to recognise, interpret, represent and analyse algebraic situations and the relationships and connections between them;
- *Ability to apply* that knowledge in unfamiliar situations to prove that something is true or false, solve problems, generate and test conjectures, make and defend generalisations; and
- *A capacity to explain and communicate* reasoning and solution strategies in multiple ways.

Four Phases of the Project were identified (in each of the three areas of mathematical reasoning):

1. Develop draft learning progressions from the research literature;
2. Develop, trial and validate assessment tasks;
3. Use the results to develop formative Learning and Assessment Frameworks (LAFs) and accompanying resources to support teaching and assessment;
4. Trial the above with partner schools, and evaluate in terms of student learning and shifts in teacher knowledge.

This paper will concentrate on the first and second Phases given above, that is, on the development of the draft learning progression (DLP) and the development, trialling and validation of the assessment tasks.

DEVELOPMENT OF DRAFT LEARNING PROGRESSION

The idea of developing a draft learning progression built on Simon's (1995) suggestion of constructing hypothetical learning trajectories as mini-theories of student learning. This was seen as a useful place to begin, as learning trajectories assist teachers to see where on the continuum students are and hence provide a starting point for teaching (Siemon, Izard, Breed, & Virgona, 2006). It should be noted here that there was discussion around the nomenclature of the construct. It was decided that the term "learning progression" would be more clearly understood by teachers in Australia. The distinctions between learning progressions and learning trajectories made by Ellis, Weber and Lockwood (2014) were not considered, as there was no intention to get tied up with semantics.

Although there has been much debate about the meaning and use of the terms learning progressions and learning trajectories, there are common elements of the varied interpretations and it is these commonalities that were used as the focus. One of the common elements is that learning takes place over time and effective teaching involves recognising where the learners are in their learning journey as a starting point to design challenging yet achievable learning experiences to support the students' progress. The second commonality is that learning progressions or trajectories are based on hypothesised pathways derived from a synthesis of relevant literature, the design and trialling of learning activities aimed at progressing learning within the hypothesised framework, and evaluation methods to assess where learners are on their journey.

In Australia, learning progressions have tended to take the form of learning and assessment frameworks such as the LAF developed and validated as part of the Scaffolding Numeracy in the Middle Years Project (Siemon et al., 2006). RMFII was designed along similar lines. By providing teachers with such a framework they are supported to recognise and understand students' learning needs, know what learning aspects should be targeted and how to assist students in their mathematical learning (Siemon et al., 2006). It is expected that by identifying and explaining the 'big ideas' involved in algebraic reasoning, as well as working with teachers to recognise and interpret student learning needs, will assist to improve learning outcomes for students in Years 7-10.

The process of developing the DLP for algebraic reasoning began with a comprehensive review of the literature about algebraic concept development and about learning trajectories and progressions. In this way, it was hoped to identify possible structures as well as for looking for what might sit within those structures. The first draft of the DLP was a synthesis of the research literature which was arbitrarily divided into eight zones of increasingly complex ideas and strategies. Although as researchers who actively work against pre-conceptions of what may be found so as not to influence what was found in the literature, inevitably when designing a DLP prior knowledge was used to group the ideas and strategies.

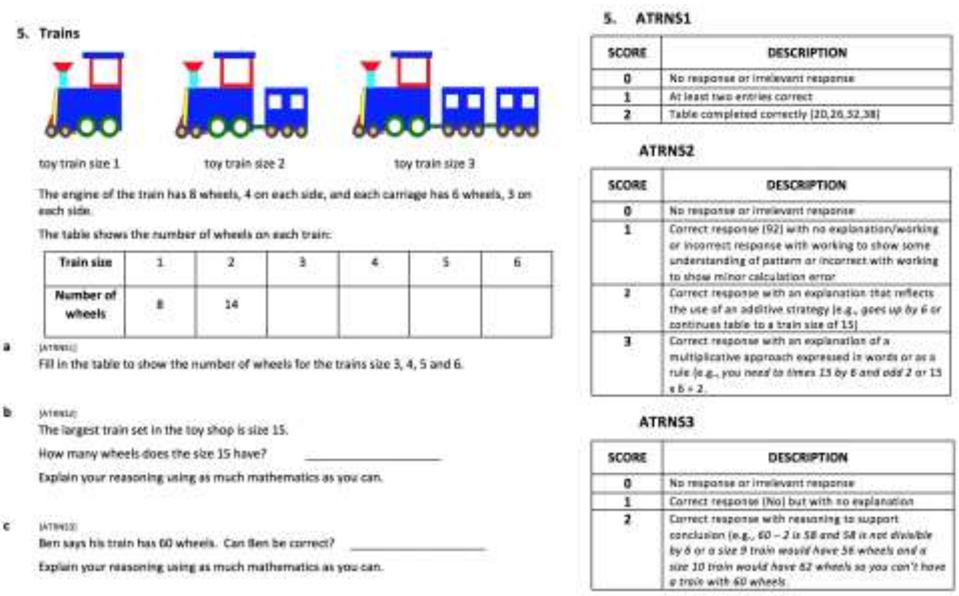
Once the first draft was in place, a thematic analysis was carried out to determine the 'big ideas' that were emerging. Five themes were identified: Pattern and Sequence; Generalisation; Function; Equivalence; and Equation Solving. There was a discussion about whether Equation Solving was part of the 'big idea' of Equivalence and it was decided to continue to separate them at that stage. The first draft was then examined to consolidate and condense the key ideas and then organised under the five 'big ideas'. This became the second draft of the DLP. An example of the Generalisation 'big idea' is provided in Table 1.

Zone	Generalisation	Sources
1	Explain a generalisation of a simple physical situation.	Carpenter, Franke, & Levi (2003); Panorkou, Maloney, & Confrey (2013); Perso (2003); Schliemann, Carraher, & Brizuela (2007); Watson (2009).
2	Explore and conjecture about patterns in the structure of number, identifying numbers that change and numbers that can vary.	Blanton, & Kaput (2011); Carraher, Schliemann, Bruzella, & Earnest (2006); Mason (2008); Miller, & Warren (2012); Panorkou, Maloney, & Confrey (2013); Perso (2003); Warren, Miller, & Cooper (2011).
3	Explain generalisations by telling stories in words, with materials and using symbols.	Blanton, & Kaput (2003); Mason (2008); Miller, & Warren (2012); Panorkou, Maloney, & Confrey (2013); Perso (2003); Tierrney, & Monk (2008); Warren, Miller, & Cooper (2011); Wilkie (2015).
4	Explain generalisations using symbols and explore relationships using technology.	Carpenter, Franke, & Levi (2003); Panorkou, Maloney, & Confrey (2013); Perso (2003); Stacey, & MacGregor (2001); Wilkie (2015).
5	Follow, compare and explain rules for linking successive terms in a	Kaput (1998); Kaput, Blanton, & Moreno (2008); Knuth, Alibali, McNeil, Weinberg, & Stephens (2005); Panorkou,

	sequence or pair quantities using one or two operations.	Maloney, & Confrey (2013); Perso (2003); Swafford, & Langrall (2000); Tierrney, & Monk (2008).
6	Use and interpret basic algebraic conventions for representing situations involving a variable quantity.	Kieran, & Sfard (1998); Perso (2003); Stacey, & MacGregor (2000), Wilkie (2015); Yerushalmy (2000).
7	Use and interpret algebraic conventions for representing generality and relationships between variables and establish equivalence using the distributive property and inverses of addition and multiplication.	Panorkou, Maloney, & Confrey (2013); Perso (2003).
8	Combine facility with symbolic representation and understanding of algebraic concepts to represent and explain mathematical situations.	Panorkou, Maloney, & Confrey (2013); Perso (2003); Yerushalmy (2000).

Table 1: The 'big idea' of Generalisation from the second draft learning progression.

The DLP was then used to select, modify and design a range of rich algebraic tasks which were trialled with 1550 students from Years 7-10 providing valid responses. The tasks that were designed contained some items that addressed one of the 'big ideas' while others addressed several of the 'big ideas' in a single task. Two assessment forms were designed containing only algebraic reasoning, two that included items of both algebraic and statistical reasoning and another two that included both algebraic and spatial reasoning. There were common items across all of the forms. Each of the assessment forms also included one of the validated extended tasks that had been used in the RMF Project, as a benchmark item. An example of a task, Trains (ATRNS) is shown in Figure 1.


5. Trains

toy train size 1 toy train size 2 toy train size 3

The engine of the train has 8 wheels, 4 on each side, and each carriage has 6 wheels, 3 on each side.

The table shows the number of wheels on each train:

Train size	1	2	3	4	5	6
Number of wheels	8	14				

a [ATRNS1]
Fill in the table to show the number of wheels for the trains size 3, 4, 5 and 6.

b [ATRNS2]
The largest train set in the toy shop is size 15.
How many wheels does the size 15 have? ______________
Explain your reasoning using as much mathematics as you can.

c [ATRNS3]
Ben says his train has 60 wheels. Can Ben be correct? ______________
Explain your reasoning using as much mathematics as you can.

5. ATRNS1

SCORE	DESCRIPTION
0	No response or irrelevant response
1	At least two entries correct
2	Table completed correctly (20,26,32,38)

ATRNS2

SCORE	DESCRIPTION
0	No response or irrelevant response
1	Correct response (92) with no explanation/working or incorrect response with working to show some understanding of pattern or incorrect with working to show minor calculation error
2	Correct response with an explanation that reflects the use of an additive strategy (e.g., *goes up by 6* or *continues table to a train size of 15*)
3	Correct response with an explanation of a multiplicative approach expressed in words or as a rule (e.g., *you need to times 15 by 6 and add 2* or 15 x 6 + 2.

ATRNS3

SCORE	DESCRIPTION
0	No response or irrelevant response
1	Correct response (No) but with no explanation
2	Correct response with reasoning to support conclusion (e.g., *60 – 2 is 58 and 58 is not divisible by 6 or a size 9 train would have 56 wheels and a size 10 train would have 62 wheels so you can't have a train with 60 wheels*

Figure 1. Trains question with rubric.

The task was designed to allow students fairly easy access at the start but to require explanation of reasoning in the latter parts of the question. The rubrics were designed to value algebraic reasoning over correct answers being provided with no explanation. In the ATRNS2 task a student scores a 1 if incorrect but with reasoning showing some understanding of the pattern. A score of 3, however, required a multiplicative understanding of the relationship with appropriate explanation, which may be in words, symbols or a combination.

RESULTS

Using Rasch analysis of actual student responses to these three ATRNS items, it was possible to rank the assessment items into eight zones. For example, in Table 2 below, ATRNS1.2 refers to the item ATRNS1 with an achieved score of two points. Each of the seven scores given in Table 2 is then matched with its associated Rasch zone.

ATRNS1.1	ATRNS1.2	ATRNS2.1	ATRNS2.2	ATRNS2.3	ATRNS3.1	ATRNS3.2
Zone 1	Zone 1	Zone 3	Zone 4	Zone 6	Zone 2	Zone 4

Table 2: Results of Rasch analysis on the above items

Responses to the seven assessment scales shown in Table 2 were scaled using Rasch analysis. For this group of items, Rasch scales range from Zone 1 to Zone 6. Completing some of the pattern in the table was the easiest task at Zone 1. Providing a correct answer to item a) with either no or a descriptive (e.g. I counted) explanation or using an additive strategy was the easiest to achieve at Zone 1. Whereas extending the pattern to a larger train (ATRNS2.1) scaled at Zone 3 and giving the correct answer to the larger train, as well as providing a mathematical explanation using a multiplicative strategy, was more difficult for students and was scaled at Zone 6. The sections of the rubrics that required elaborated explanations involving algebraic reasoning, that is ATRNS2.3 and ATRNS3.2, were the most difficult for students being scaled at Zone 6 and Zone 4 respectively. It is noticeable that it is when the students need to explain or provide reasons or even to give partial reasons for their answers that they have the most difficulty. Extending the experimental sample to include older students in the middle years may change these scores, but, even at this preliminary stage of data analysis, it is clear that many students lack confidence or experience when asked to provide explanations for their thinking. The challenge for teachers is to give more careful attention to supporting students' development and articulation of mathematical reasoning.

When analysing the data from the Rasch analysis and mapping it back to the DLP, it was decided that the distinction made between Equivalence and Equation Solving was unnecessary as was the distinction made between Pattern and Sequence and Function, as the Pattern work appeared to overlap with the lower echelons of the Function 'big ideas'. As a result, the original five 'big ideas' were collapsed into three 'big ideas', those of Pattern and Function, Generalisation and Equivalence. Relating the data from the Rasch analysis for this question back to the DLP suggests that rather than the students explaining the simple patterns at the lowest level they are really only identifying the pattern and for these students the explanations start much later in zones 3 and 4. This indicates the need for the teaching of algebraic reasoning and not just algebraic procedure.

A closer comparison of the zones from the Rasch analysis with the zones of the DLP shows some similarity and some difference. ATRNS1.1 and 1.2 required students to identify and complete, at least partially, a number pattern related to a real situation. This fits within the DLP zone 1 “explain a generalisation of a simple physical situation”. The second part of the question required the students to extrapolate the pattern to data beyond the figures provided in the table, which meant they needed to generalise and apply it. Doing this at a purely numerical level fitted into zone 3 while explaining it partially or additively equated to zone 4, which in the DLP was “explain generalisations using symbols”. The more sophisticated explanation involving multiplicative thinking was at zone 6 of the Rasch model, although it is a closer match to zone 5 of the DLP. This indicates that the Rasch data supports the DLP, at least to some extent, at the lower levels, but further data is needed for the higher zones.

Limitations of the Rasch analysis data

Although there were 1563 students in the database for algebraic reasoning, only 1550 provided valid responses. One of the limitations of using Rasch analysis is that it relies on student responses to assessment items. What was seen from the data was that the items that students perceived to be more difficult were often not attempted which meant that the more challenging algebraic reasoning assessment items were not able to be ranked. More trialling will be necessary, perhaps with older students in the Years 7-10 range, in order to incorporate the more challenging types of assessment items within the eight Rasch zones. As a result it would be expected that some of the data presented here would change zones once the upper zones include more challenging algebraic reasoning items.

CONCLUSION

The development of the DLP involved several stages. The first was an extensive review of the literature on algebraic reasoning and on learning progressions to identify both possible structures for the DLP and what might fit within those structures. Following the literature review a thematic analysis was carried out to identify the ‘big ideas’ that were emerging. Five ‘big ideas’ were identified and the DLP was structured around these headings. Appropriate algebraic reasoning tasks were found, modified or designed based on the DLP and then sent to schools all around Australia for trialling. Once the trial data were received and a Rasch analysis was applied, it was seen that the five ‘big ideas’ could reasonably be collapsed into three ‘big ideas’, those of Pattern and Function, Generalisation and Equivalence. It would appear that more extensive trialling of items that students perceived as difficult will be necessary in order to tease out the upper areas of the DLP. The results indicate a need for the teaching of algebraic reasoning and the encouragement for students to give explanations of their thinking.

As classrooms include more discussion and reasoning the results of such a Rasch analysis might move closer to the DLP which was initially proposed but at the moment there is a great need for targeted teaching of algebraic reasoning.

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ERRONEOUS ADDITIVE OR MULTIPLICATIVE REASONING: THE ROLE OF PREFERENCE BESIDES ABILITY

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Previous research has repeatedly shown that children erroneously reason additively in multiplicative word problems, while others erroneously reason multiplicatively in additive word problems. The present study aimed to investigate to what extent this erroneous reasoning depends on children's preference for additive or multiplicative relations, besides their abilities. A preference test, a word problem test, and a test measuring the (procedural and conceptual) additive and multiplicative reasoning abilities were administered to 246 third to sixth graders. Results revealed that a substantial percentage of the erroneous additive reasoners or erroneous multiplicative reasoners possessed all necessary abilities, and almost all of them had a more general preference for additive resp. multiplicative relations.

INTRODUCTION

Multiplicative reasoning and the erroneous use of additive reasoning

Learning to reason multiplicatively is a pivotal goal in upper primary mathematics education. One important way of teaching multiplicative reasoning is by means of multiplicative missing-value word problems, which consist of three given values and a fourth one that has to be found by identifying the multiplicative relation between two given values and applying this relation to the third given value (Kaput & West, 1994; Vergnaud, 1988). For example, in the problem "A car of the future will be able to travel 8 miles in 2 minutes. How far will it travel in 4 minutes?" (Kaput & West, 1994, p. 267) the correct solution is "16" (i.e. 2×2=4, so 8×2=16). Despite the omnipresence of those multiplicative missing-value problems in primary education, multiplicative reasoning is not achieved easily. Especially children in the lower grades often erroneously reason additively in multiplicative problems (i.e. answer "10" in the word problem above, as 2+2=4, so 8+2=10) (Kaput & West, 1994; Van Dooren, De Bock, & Verschaffel, 2010; Vergnaud, 1988). This kind of erroneous additive reasoning has been interpreted traditionally as evidence for an additive phase in the development of relational reasoning abilities, which would then require a transition to multiplicative reasoning. This transition has even been characterized as "one of the major barriers to learning mathematics" (Siemon, Breed, & Virgona, 2005, p. 1).

However, this interpretation of erroneous additive reasoning in multiplicative problems in terms of lacking *abilities* has been recently questioned, based, among others, on studies indicating that kindergartners are already able to correctly detect the

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multiplicative relations in a problem, and to follow through this way of reasoning in order to derive the correct multiplicative solution (e.g., Nunes & Bryant, 2010).

The erroneous use of multiplicative reasoning

Besides children's additive reasoning in multiplicative problems, the inverse mistake has been repeatedly reported as well. Children in upper primary education massively respond multiplicatively ("24 laps", i.e. 4×3=12, so 8×3=24) in additive missing-value problems such as "Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 4 laps, Kim has run 8 laps. When Ellen has run 12 laps, how many has Kim run?" (e.g., Van Dooren et al., 2010, p. 364). However, analogously to the erroneous use of additive reasoning discussed before, those upper elementary school children are assumed to possess all mathematical abilities necessary to correctly solve such additive missing-value problems. In particular, they should be able to correctly calculate the additive solution, since the arithmetical operations of addition and subtraction are taught and intensively practiced already in the first grades of primary education. Moreover, it seems unlikely that this kind of erroneous additive reasoning could be fully explained by children's lacking *ability* to characterize the quantitative relations involved in the problems as multiplicative, particularly because previous research indicated that primary school children score rather well when asked to classify additive and multiplicative word problems, and certainly better than when asked to solve these problems (Van Dooren, De Bock, Vleugels, & Verschaffel, 2010).

The role of preference for additive or multiplicative relations

It seems that children's erroneous use of additive or multiplicative reasoning does not merely depend on their *ability* to reason additively and multiplicatively. Resnick and Singer (1993) raised an additional explanation, by interpreting children's additive reasoning in multiplicative word problems as possibly indicating a *preference* for additive relations. Likewise, children's multiplicative reasoning in additive word problems may be based on a *preference* for multiplicative relations. The distinction between preference and ability is not new in the mathematics education literature (e.g., Bailey, Littlefield, & Geary, 2012; Pellegrino & Glaser, 1982; Resnick & Singer, 1993). The term preference is used to refer to the way of reasoning that "has precedence over" the other in a certain problem (Pellegrino & Glaser, 1982, p. 310), rather than to certain abilities that a child may possess. Preference and ability thus do not fully correspond, even though a child's ability may impact his preference, and vice versa, a preference may increase later ability (Bailey et al., 2012). In the present study too, preference and ability are hypothesized to be two interrelated but distinct characteristics that both impact children's word problem solving behaviour.

Although it has been suggested that a preference is at play in children's missing-value word problem solving (Resnick & Singer, 1993), one can argue that those classical word problems may not be best suited to capture children's preference, since they unmistakably contain an underlying additive or multiplicative mathematical model. Of course, children's abilities to reason additively or multiplicatively in line with the underlying model in the problem may be involved in solving those word problems too, apart from their preference. To validly measure children's preference, we need problems that do not contain any indication that additive or multiplicative reasoning is required. Those problems are thus entirely open to *both* ways of reasoning, i.e., both are equally valuable and correct. Several authors have suggested the usefulness of such open problems for both research and teaching (e.g., Pellegrino & Glaser, 1982).

The present study

The major goal of the present study was to investigate the impact of children's preference for additive or multiplicative relations – besides (lacking) abilities – on the erroneous use of additive or multiplicative reasoning in word problems. In doing so, both a procedural (*ability* to do the necessary additive or multiplicative calculations) and conceptual component (*ability* to analyse and characterize quantitative relations involved in problems as additive or multiplicative) of additive and multiplicative reasoning abilities (Nunes & Bryant, 2010) were measured, as well as children's *preference* and their word problem solving behaviour.

By administering those instruments, we aimed to *replicate* previous research results, to answer our first research question (RQ1): To what extent do children erroneously use additive or multiplicative reasoning in missing-value word problems, and how is this affected by grade? Second, and more importantly, we aimed to *extend* those results by means of our second research question (RQ2): To what extent could children's preference for additive or multiplicative relations help in explaining their erroneous use of additive and multiplicative reasoning, besides abilities?

METHOD

Participants and instruments

Four test instruments were administered to 246 children (68 third, 59 fourth, 58 fifth and 61 sixth graders; 128 boys and 118 girls) from three Flemish primary schools. To minimize the impact of specific problem characteristics, several versions of each test instrument, containing different numbers and word problem contexts, were used. The four instruments are presented below and illustrated in Figure 1.

First, the preference test consisted of six schematic problems. In each problem (see Figure 1a), three numbers were given and a fourth one was missing, and two arrows pointed out the relational structure between numbers (which was the same as the one underlying the word problems used in the word problem test and conceptual reasoning abilities test). As Pellegrino and Glaser (1982) pointed out, such problems are open to "several relations" (p. 302) that are equally valuable and correct, including additive and multiplicative ones. Previous research showed that those schematic problems validly measure children's preference (Degrande, Verschaffel, & Van Dooren, 2016).

The second test instrument, the word problem test (WPT) consisted of six missing-value word problems (see Figure 1b), of which three were additive and three were multiplicative. The additive and multiplicative word problems were similar in terms of context and numbers, but differed with respect to the underlying mathematical model. Those missing-value word problems were developed and extensively tested in previous studies (e.g., Van Dooren, De Bock, Vleugels, & Verschaffel, 2010).

A third test measured procedural additive and multiplicative reasoning abilities. It consisted of the same six schematic problems as the preference test, but this time these schemes were accompanied by addition or multiplication signs (three items of each, see Figure 1c). Children had to do the required calculations to find the missing number.

A fourth test, measuring conceptual additive and multiplicative reasoning abilities, contained the same six word problems as the WPT (three additive and three multiplicative ones). Children were asked which of both correctly completed solution schemes (additive or multiplicative) fitted the given word problem.

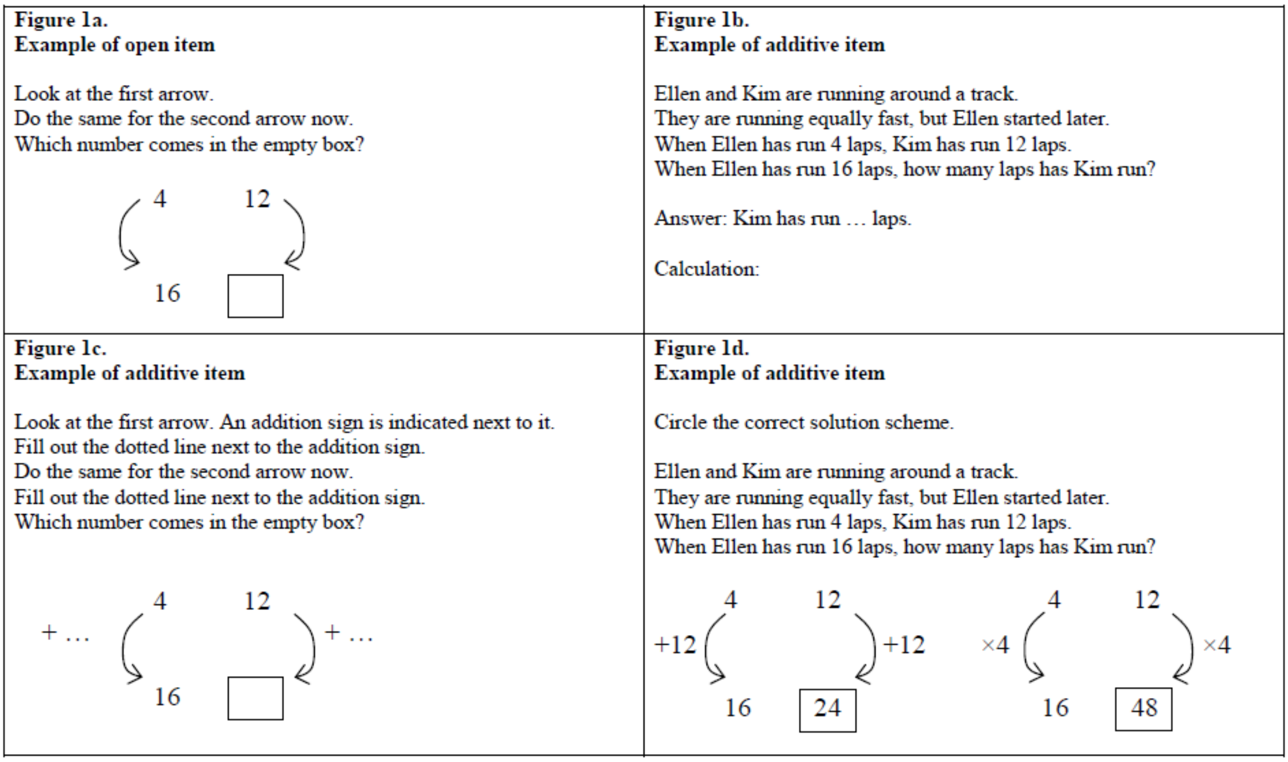

Figure 1: Example of test items of each of the four test instruments.

Procedure

The four paper-and-pencil test instruments were collectively administered in the aforementioned order, but on two separate moments. This was done to avoid response tendencies across test instruments, and thus to warrant the validity of each instrument. The preference test and WPT were first administered, and one week later, children's procedural and conceptual reasoning abilities were tested.

Analyses

Data were analysed by looking at children's answer profiles across each test instrument, rather than at answers to individual test items. More specifically, patterns of answers across at least 4 out of 6 open items of the preference test, and at least 2 out of 3 additive or multiplicative items on all other tests were distinguished. After that, logistic regressions were conducted to study the effect of grade on the occurrence of additive and multiplicative reasoners (RQ1) and the effect of preference on top of ability on the occurrence of additive and multiplicative reasoners on the WPT (RQ2).

RESULTS

We first report the distinct answer profiles on the WPT and their development across grades (RQ1). Only 7.3% of the children (see Table 1) reasoned correctly on the WPT, i.e., giving additive answers to additive word problems and multiplicative answers to multiplicative ones. Most children consistently gave multiplicative answers, which were correct in multiplicative but incorrect in additive word problems (48.4%). Other children consistently gave additive answers, which were correct in additive but incorrect in multiplicative word problems (26.0%). A final 18.3% did not belong to one of these three profiles, and was not considered in our further analyses.

In line with previous research, additive reasoning occurred most often in third grade (48.5%) and decreased across grades (8.2% in sixth grade, *Wald* $\chi^2(3) = 36.353$, $p < .001$, also see Table 1). The odds of third and fourth graders to reason additively were, respectively, 10.56 ($p < .001$) and 5.32 ($p = .002$) times larger than those of sixth graders, while the odds of fifth graders were only 1.54 times larger compared to sixth graders ($p = .486$). The occurrence of multiplicative reasoning, to the contrary, increased from 5.9% in third to 68.9% in sixth grade (*Wald* $\chi^2(3) = 86.588$, $p < .001$). The odds of fourth, fifth and sixth graders to reason multiplicatively were, respectively, 16.56, 45.87 and 35.37 times larger than those of third graders (all *p*'s < .001). In contrast, correct reasoning only rarely occurred in third (5.9%), fourth (5.1%) and fifth grade (1.7%). Even in sixth grade, only 1 out of 6 children reasoned correctly.

	Grade 3	Grade 4	Grade 5	Grade 6	Total
Correct	5.9	5.1	1.7	16.4	7.3
Multiplicative	5.9	50.8	74.1	68.9	48.4
Additive	48.5	32.2	12.1	8.2	26.0
Rest	39.7	11.9	12.0	6.6	18.3

Table 1: Percentages of third to sixth graders' answer profiles on the WPT.

Second, to answer RQ2, we now turn to the procedural and conceptual reasoning abilities and, finally, to the preference of children with distinct profiles on the WPT.

The *procedural* additive and multiplicative reasoning abilities were acquired by most children: 85.5% mastered the procedural skills necessary for doing correct calculations. When splitting up based on the profiles on the WPT, all correct reasoners (100.0%) mastered the procedural skills, and this was the case for the vast majority of the multiplicative (97.5%) and additive reasoners (76.6%) too. Hence, procedural ability could not be the crucial factor explaining children's erroneous additive and multiplicative reasoning on the WPT. In order to fully exclude this explanatory factor (since our main interest was the impact of preference besides abilities), only those children who possessed the procedural skills were included in further analyses.

In comparison to the procedural test, children experienced more difficulties solving the conceptual test. Among the students who mastered the procedural skills, only 38.4% also mastered the conceptual additive and multiplicative reasoning abilities. When considering this subgroup in relation to the way they solved the WPT, it becomes clear that 88.9% of the correct reasoners on the WPT had the required conceptual additive and multiplicative reasoning abilities, but this was also the case for a substantial percentage of the multiplicative (41.4%) and additive (26.5%) reasoners. For these students, it cannot be concluded that the conceptual abilities would explain why they consistently solved problems on the WPT additively or multiplicatively.

It also seems valuable to take a closer look at the children who did *not* master these conceptual abilities, as some tended to systematically choose the multiplicative (30.8%) or additive (20.4%) solution scheme in the conceptual test instead. In particular, 47.4% of all multiplicative reasoners on the WPT systematically chose the multiplicative solution scheme in the conceptual test, whereas 38.8% of additive reasoners on the WPT systematically chose the additive solution scheme (see Table 2).

		Conceptual Test			
		Correct	Multiplicative	Additive	Rest
Word Problem Test	Correct	88.9	5.6	0.0	5.6
	Multiplicative	41.4	47.4	6.9	4.3
	Additive	26.5	12.2	38.8	22.4
	Rest	14.3	10.7	57.1	17.9
		38.4	30.8	20.4	10.4

Table 2: Answer profiles on the word problem test and conceptual test of children who mastered the procedural abilities.

For the subgroups that mastered all required procedural and conceptual abilities, we can now link their behaviour on the WPT with their preference. Amongst the multiplicative reasoners on the WPT, 91.7% also showed a *preference* for multiplicative relations in the preference test. Likewise, 92.3% of the additive reasoners on the WPT showed a *preference* for additive relations on the preference test. Remarkably, there was no clear trend for the correct reasoners on the WPT: 56.2% seemed to prefer multiplicative relations, and 43.8% did not have a preference at all.

A logistic regression wherein procedural and conceptual ability were included as first predictors, and preference as an additional predictor, confirmed that preference indeed has an added value to predict the occurrence of additive reasoning on the WPT (*Wald*

$\chi^2(2) = 47.056$, $p < .001$). The odds of children who had an additive preference to also reason additively on the WPT were 6.07 times larger than for children without a preference ($p < .001$), when controlling for both abilities. Likewise, another logistic regression confirmed the impact of preference on top of both abilities on the occurrence of multiplicative reasoning on the WPT (*Wald* $\chi^2(2) = 49.767$, $p <.001$). More specifically, the odds of children who had a multiplicative preference to also reason multiplicatively on the WPT were 10.27 times larger than for children without a preference ($p < .001$), when controlling for both abilities.

Finally, the majority of those children who possessed the procedural skills but systematically indicated the additive or multiplicative solution scheme to the conceptual reasoning abilities test, either had a preference for additive (52.6%) or multiplicative relations (85.5%), respectively. Apparently, those children's preference was not only at play in word problems, but unexpectedly also in the conceptual test.

CONCLUSION AND DISCUSSION

Previous research has repeatedly shown that many children in lower primary education erroneously reason additively in multiplicative word problems, while many children in upper primary education erroneously reason multiplicatively in additive word problems. The present study aimed to *replicate* those results (RQ1) and to *extend* them by investigating whether a preference for additive or multiplicative relations could help in explaining their erroneous use in word problems – besides abilities (RQ2).

Results regarding RQ1 revealed that almost half of all children – mainly in upper primary education – reasoned multiplicatively, and another substantial group of children – particularly in lower primary education – reasoned additively in all word problems, irrespective of whether the underlying mathematical model was additive or multiplicative. With respect to RQ2, almost all those additive or multiplicative reasoners possessed the necessary procedural abilities, while this was not the case for their conceptual abilities. Thus, children's erroneous additive or multiplicative reasoning in word problems may – at least for a certain percentage of additive or multiplicative reasoners – be due to their lacking (conceptual) ability to determine the underlying mathematical model of the word problem. However, a substantial percentage of additive and multiplicative reasoners had acquired all necessary abilities (i.e., they were able to detect the underlying mathematical model and to correctly conduct the necessary calculations), and almost all of them had a preference for resp. additive or multiplicative relations. As expected, preference has an added value in explaining erroneous use of additive or multiplicative reasoning, on top of abilities.

Although further research is needed to get a view on the nature and origin of children's preference (i.e., How strong is it? To what extent is it deliberate and conscious? What processes underlie this preference?, similar to what Obersteiner, Reiss, and Bernhard (2016) did for additive and multiplicative reasoning in contingency tables), these results already indicate that getting a view on children's preference is indispensable to fully understand the development of additive and multiplicative reasoning, and to improve current educational practices. Instruction might, in particular, aim at preventing the development and uncontrolled impact of such an undesirable preference, by avoiding a stereotyped offer of problems in mathematics curricula (e.g., solely multiplicative missing-value problems in upper primary education, never being interchanged with additive ones) that may shape children's preference, and by using educational approaches wherein children discuss the considerations they make when deciding on the appropriateness of a solution method, especially in open problems.

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ANALYZING DIALOGIC TALK DURING MATHEMATICS PROBLEM SOLVING IN SMALL GROUPS

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Language is a crucial aspect of learning. Children working in small groups use language to justify their answer to mathematical tasks when they interact with each other. But not all types of interactions lead to effective learning. In this paper, I use a methodological tool to analyse the interactional events happening when a group of four children were solving an open-ended mathematical task. This type of analysis may offer a few pointers for researchers and teachers to distinguish what type of interactions are the ones that would produce the context for deep and meaningful mathematical learning.

INTRODUCTION

In 2012, I visited a school near Chicago to observe how a teacher with extensive experience in mathematics teaching was conducting her class. Her instruction was grounded in a collaborative learning approach. During the lesson, the teacher split the children into small groups. All groups had to solve the same task: to reach the number twenty, or close to twenty, taking cards out from a deck. The children had to multiply the numbers coming out from the deck. I went close to one of the groups: a child got a card with five points; then, he got a card with three points. He clasped both cards in his hands and said: “Five times three, twenty; I won!” All children in the group looked at each other; but they said nothing. That statement shocked all the other children, because they knew that it was the wrong answer; however, nobody dared to contradict (correct) him.

This episode illustrates the fact that not all interactions happening within the classroom lead to effective learning. Some interactions (e.g., the silence of the other students in response to the student’s claim that five times three is twenty) could reinforce erroneous concepts. Hence we need to identify which interactions are the ones that are more likely to produce real opportunities for learning. In this paper, I introduce an analytical tool to analyse social interactions among children as they solve an open-ended mathematical task, working in small groups.

THEORETICAL FRAMEWORK

Language (communication, speech acts, etc.) is a crucial aspect of learning. Drawing on a socio-cultural perspective, we can accept that language mediates learning in social settings. Vygotsky realized that when children talk with other [more capable] peers or with the teacher, sharing their thoughts about a particular task, the children are really

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internalizing the (mathematical) concepts. This is due to the fact that in that moment they link the abstract meaning of the concept with the practical task that they are solving using the speech act as a mediating tool.

Vygotsky (1978) coined the concept of Zone of Proximal Development (ZPD) to explain how this process of learning works. The ZPD highlights the guiding role of the *expert* in helping / supporting students to learn. Bruner later used the concept of scaffolding to explain the process of educational support between the "expert" adult and the child (Wood, Bruner, & Ross 1976). Bruner and his colleagues used problem solving to explore how children develop higher (cognitive) skills. He further developed this idea of scaffolding, which led him to focus increasingly on the inter-subjective aspect of interaction and talk (Bruner, 2012).

Theorists such as Vygotsky and Bruner argued that *language* and *communication* are central aspects of learning. However, research in many countries suggests that teachers tend to talk more than their students when supporting them in their process of learning. In fact, previous researches have established what we know as the initiation-response-feedback (IRF) scheme. Many researchers have criticised this pattern (Littleton & Howe, 2010; Waring, 2009). Lobato, Clarke and Ellis (2005) posed the question of "how much information should a teacher tell his/her students when conducting his/her lesson?" The researchers reformulated the telling/not-telling dilemma in mathematics education in three ways, "(a) in terms of the *function* (…) rather than *form* of teachers' communicative acts; (b) in terms of the conceptual rather than procedural content of the new information; and (c) in terms of its relationship to other actions rather than an isolated action." (Lobato & Clarke, 2005, p. 101). Their work claims our attention to the fact that is important *when* to tell, but also *what, why* and *how*.

More recently, Díez-Palomar and Cabré (2015), drawing upon previous work conducted by Soler and Flecha (2010), suggest that the crucial element in studying the learning process is to analyse *how* individuals use language to justify their statements. Teachers and/or students may use language to explain a particular mathematical idea using validity claims, or imposing their statements with coaction. Using one or the other opens up different opportunities for participants to learn. According to the researchers, not all interactions produce the same opportunities for learning: only interactions based on the use of validity claims by the participants would provide real opportunities for learning. In this paper I introduce an analytical tool to elucidate *interaction* in terms of dialogic/non-dialogic talk looking at the transcript of a video recording of a group of four students solving an open-ended task in mathematics.

METHODOLOGY

Data Collection

The data of this paper came from the *Social Unit of Learning* project, lead by Prof. Clarke at the International Centre for Classroom Research (ICCR) at The University of

Melbourne. The members of the ICCR designed a protocol to conduct the experiment. They selected a school in the Melbourne area and invited intact classes of Year 7 students (24 to 26 in a class) with their usual teacher to come to the university, and participated in a research experiment conducted within a laboratory classroom equipped with 10 built-in cameras and 15 radio microphones. In one configuration, the students were invited to solve a set of three problems: the first task was done individually, the second task in pairs, and the third task in small groups (four to six students in a group). The video data collected were transcribed and checked for accuracy.

For the purpose of this paper, I selected recorded data corresponding to the third problem (Fred's apartment), when the students were working in small groups.

The problem was:

> Fred's apartment has five rooms. The total area is 60 square metres. Draw a plan of Fred's apartment. 1) Label each room and 2) Show the dimensions, length and width, of all rooms.

My analysis focused on the interactions of one group with two girls and two boys.

Data Analysis

In order to analyse the data, I created an analytical tool (see Table 1) with the support of the ICCR members, grounded on the approaches of Soler and Flecha (2010), Mercer (2006) and Habermas (1984), to distinguish between dialogical (Type 3) and non-dialogical talk (Type 2) (Díez-Palomar & Cabré, 2015).

I analysed how students create validity claims to justify their statements/answers when solving a problem given to them. I used two levels (layers) of analysis: (1) the analysis of the illocutionary force of the statement used by the participant in the interactional event (IE); and (2) the propositional content of the statement. Level 2 only applies to Type 3 (dialogic) interactions, because I want to elucidate the nature of the claims used by the participants to create their justifications to the mathematical task proposed. I define interactional event (IE) as the set of sentences in a dialogue between individuals acting upon one another. IE starts with a query and ends when participants provide their answer to that question.

RESULTS

The third task started with the teacher introducing the task. He informed the students that they had 20 minutes to work together in small groups. He emphasised that all members of the group needed to provide a single solution to the problem. He did not make any more public statements after he read out the problem for everyone.

Anna (girl), Pandit (girl), John (boy), and Arman (boy) were in the same group. At the beginning, the boys were playing with the rulers that they had on the table. But after a couple of minutes, they started to focus on the task. The first thing they did was to decide on the shape of Fred's apartment.

	Definition	Example
LAYER 1		
Type 1: Exchange of Information	Shares, exchanges, particular information. The indicator of *illocutionary force* is **neutral**.	Student: It's 60 meters square.
Type 2: "Power" interaction	Makes a claim, a statement, or an assertion using his/her/their power position as a warrant to justify a claim. The indicator of *illocutionary force* is **coercion**.	Student 1: Average is like the most likely. Student 2: Why? Student 1: Because I'm telling you.
Type 3: Dialogic interaction	Makes a claim, a statement, or an assertion using validity claims as a warrant to justify his/her/their claim. Validity claims include conjectures, reasoning, proof, and participants may draw on school skills or practical skills. The indicator of *illocutionary force* is **consensus**. It could be a particular case (solidarity chain) when more than one participant engages in a dialogue to supports each other in understanding, solving, the task.	A conjecture: Fractions like 2/3, 45/46, etc. (where the numerator is bigger than the denominator) are always less than 1. An elaborated justification: When you divide two numbers *a* and *b*, with *a* always smaller than *b* ($a<b$), you need to split *a* among *b*, so no one gets a full piece of *a*. For example, if you have 24 candies and you need to split them among 25 friends, then no one would get a whole candy. You have to make them into bits and split them among your friends. Using a proof: $a/b = c$; $c<1$ if $a<b$; $c=1$ if $a=b$; $c>1$ if $a>b$.
LAYER 2 (To understand the nature of Interaction Type 3)		
Analysis regarding the "content"	Formal statement (formal math)	$a^2=b^2+c^2$
	Intuitive understanding	Prime numbers are infinite.
	Particular example	Five is prime number because you can divide it between 1 and itself.
	More than one particular example	Five and seven are prime numbers because you only can dive them between 1 and itself.

Table 1: Coding categories.

That was the first example of dialogic interaction. Anna made a suggestion to draw a square, but Pandit answered by asking, “why make it a square house?” Then, Anna re-thought her previous assumption about the shape for the apartment.

During the next interactions, dialogic (Type 3), power (Type 2), exchange of information (Type 1) and other types of interactions (coded as “other”) took place back and forth. Table 2 summarizes the total amount of IE for codes 1, 2 and 3 (according to our coding categories). I only focused on interactions of type 1, 2 or 3; not on other type of interactions that can be happening during the session.

	Number of IE	%
Interaction type 1: Exchange of information	36	51.4%
Interaction type 2: Power interaction	23	32.9%
Interaction type 3: Dialogic interaction	11	15.7%

Table 2: Distribution of the types of interactional events (IE) during Task 3.

We can see that half of the time Anna, Pandit, Arman and John were involved in exchanging information.

Pattern	Definition
Pattern 1 ($\overline{D} \rightarrow C_a$; $\overline{D} \rightarrow \overline{C_a}$). *Interaction Type 2*	A student takes the lead. S/he makes a claim. S/he does not justify his/her statement. His/her claim may either be a correct answer or a wrong answer.
Pattern 2 ($\overline{C_a} \rightarrow D \rightarrow C_a$; $\overline{C_a} \rightarrow D \rightarrow \overline{C_a}$). *Interaction Type 3*	A student questions someone else’s wrong answer. S/he asks or provides a different justification. In the end the pair or the group produces either a correct answer or a wrong answer (again). Justifications may be formal statements, intuitive, or particular examples.
Pattern 3 ($D \rightarrow C_a$; $D \rightarrow \overline{C_a}$). *Interaction Type 3*	Two or more students engage in the dialogue. They use validity claims. They may end with either a correct answer or a wrong answer. Justifications may be formal statements, intuitive, or particular examples.

* D means “dialogic”; $\overline{D}$ means “non-dialogic”; C_a means “correct answer”; $\overline{C_a}$ means “non-correct answer.”

Table 3: Types of patterns happening within the IE.

This seems to be the most common interaction type between the members of the group while they are working to solve the task. During the session, the four students started to try different possible solutions to the problem, while gaining greater understanding of the problem during the process, which involved several IE. I noticed several patterns of interaction emerging from the dialogues produced by the students during the session (see Table 3). For instance in the excerpt below, the four children are discussing which scale they will use to draw the blueprint of Fred's apartment on the worksheet. Anna decides to use 2 cm^2 = 1 m^2. I coded this excerpt as a Type 3 interaction, because the children are negotiating whether the scale they are using is all right or they need to change it. Anna wants to change the scale; Pandit affirms that they cannot change it. Anna explains how to do it. But Arman claims that the sheet of paper is actually *too small* to use that scale. I consider the words "too small" as a valid claim that Arman is using to justify his position in this IE.

Anna: Guys, let's actually change the scale.

Pandit: We can't.

Anna: Why not?

Pandit: We're not allowed to change.

Anna: You are. Let's make two centimetre square equals one metre.

Arman: (singing) Fred's house. Do - do do - do - do - do… The paper is too small.

Then Pandit complains that this is "so confusing". However, Anna replies that she just wants to use that scale (although it is confusing for Pandit). She does not add any other explanation, nor justification, to her claim. Hence, I consider this IE as a Type 2 interaction.

Pandit: No, don't do that. That's confusing.

Anna: Why not?

Pandit: Why do you want to confuse?

Anna: Because I want to.

Pandit persists with her effort to ask for a valid justification, since she is not happy with Anna's proposal. That forces Anna to move towards a dialogic interaction, since she has to justify her concept.

Anna: Let's make two centimetre - guys, let's make the two centimetre square one metre square in this, okay?

Pandit: Don't – don't. It's so confusing.

Anna: Why not? How is it confusing? You just double it.

Arman: Okay.

Pandit: Why do you want to change? Why can't you just make it one centimetre?

Anna: Because it's going to be too small.

Pandit: It's okay.

In this excerpt we can see how Anna uses the idea of "doubling" to illustrate how her concept of scale works. Arman seems to be happy with her justification. However, Pandit is still not convinced. She asks again why not use "one centimetre" instead of two. This question forces Anna to further justify herself. She says: "because it's going to be too small." The back and forth questioning and justification between Pandit and Anna indicates that the group wants to reach a *consensus* when solving the task.

Further along, the students decided to create a "square" that is 30 metres length and 20 metres width. However, Anna claims that this "won't work."

Pandit: This is…

Anna: Wait. Let's just say that's - no, Pandit, it won't work.

Pandit: It does. It does.

Anna: It doesn't. We have to get a 30 there and then look, up to there is 30. Do you have a brain?

Pandit: (laughs) I have a brain.

Arman: Oh wait, wait, wait.

Pandit: No. Wait, isn't that has to times?

Anna: Yeah.

Pandit: Twenty times 30 is like 600.

Anna: Six hundred.

Pandit: It has to be 60.

Anna: Yeah.

Pandit: You did it wrongly. That's why.

They discovered that 20 times 30 is not 60, but 600, which was not the correct answer. In this case, the dialogic interaction leads them towards a wrong result; but the necessity to justify that result was also the way for them to realize that actually they were wrong. Eventually, they created a feasible solution for the problem.

DISCUSSION

One of the recommendations of the Institute of Education Sciences (IES) via "What Works Clearinghouse" (WWC) website is to "help students recognize and articulate mathematical concepts and notation." This recommendation is based on evidence collected from previous studies. According to IES, "students will develop a better understanding of mathematical concepts when they are asked to explain the steps used to solve a problem." (Woodward, 2012, p. 41) Furthermore, a recommendation was put forward to "use small-group activities to encourage students to discuss the process used to solve a problem." (Woodward, 2012, p. 41) Teachers should guide students to make sense of the mathematical type of reasoning. However, this is not an easy task, as Lobato, Clarke and Ellis (2005) claimed more than a decade ago. It is difficult for teachers to know what to tell their students, while not dampening their curiosity, and help them move forward. Looking carefully at the students' justifications could be a way to better help students to succeed in their learning. Also, the effort that students need to make to justify their reasoning using valid claims that may be discussed and eventually accepted (or rejected) by their peers is also a good way for them to learn (or to consolidate their previous learning). The analytical instrument that I am using here to analyse the data collected may offer a few pointers for researchers and teachers to distinguish what type of interactions are the ones that would produce the context for deep and meaningful mathematical learning.

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THE MATHEMATICS IN THE TASKSTEXT

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This study focuses on textual features that can be demanding in the reading of mathematics tasks. Two types of qualitative analyses are conducted on a few tasks to explore and evaluate what some previous statistical relations between particular textual features and the tasks reading demand stand for. First, the type of progression between the content represented as natural language is analysed. Second, the interaction between all semiotic resources of the text (i.e. natural language, mathematical notation, and images) is analysed. Preliminary results indicate that a reading demand that is unwanted in mathematics tasks seems to be related to features of the natural language but not to interaction between words and images or mathematical notation.

BACKGROUND

Today there is a substantial agreement in the description of mathematical proficiency as consisting of several competences (e.g., Niss & Højgaard, 2011; OECD, 2013) and also about mathematical communication as one such competence. The conceptualization varies but it is agreed that the communicative competence includes both to interpret oral or written mathematics and to express oneself about mathematics (e.g., Niss & Højgaard, 2011). The current study seeks to enhance the understanding of which textual features in a mathematics task that can be demanding for the reader and what role those textual features have in a mathematical language, since such knowledge is important from both an educational perspective and in relation to assessments. The mathematical language is expressed using several different semiotic resources, and therefore being able to interpret text consisting of a combination of natural language, mathematical notation and different types of diagrams is essential as part of a communicative competence. Such a communicative competence is also tested in large international assessments such as TIMSS (Mullis & Martin, 2013) and PISA (OECD, 2013).

The aim to include aspects of a communicative competence in large assessments means that the reading and interpretation of the task text is part of what is assessed. This would however be the case independent of the aim of the assessment since the tasks are presented as printed text; the student must understand task text to solve the task. For a valid assessment it is important to avoid language demands in the task text that are irrelevant for the mathematics assessment. Several studies have revealed features of the task text that enhance the risk of the test to assess something else than mathematical ability. For example a study by Sato and colleagues reveal that linguistic modifications of mathematics test can make the mathematics task text easier to read to for second language learners without altering the mathematics that is tested (Sato, Rabinowitz, Gallagher, & Huang, 2010).

There are earlier studies that in different ways investigate particular features that are typical for mathematics task text. For example Turner, Blum and Niss (2015) base their investigation of item demand on pre-defined features that are important within the different mathematical competencies. This method enables an analysis of to what extent the tasks calls for the activation of a particular competence. Other studies analyse particular textual features that are said to be part of the mathematical language with other approaches. For example Duval's (2006) studies of difficulties that students experience when solving tasks with combinations of different semiotic resources such as mathematical notation and images is an important contribution to the field.

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From an educational perspective however, it is also important to distinguish which textual features that are not part of the mathematical language managed as part of a mathematical proficiency. In mathematics tests such textual features can threaten the validity of the assessment if they affect the reading of the text. In conclusion, an enhanced understanding of both i) which features that characterizes the mathematical language and, ii) which features that do not distinguish the mathematical language is important from an educational perspective. The first category since it provides guidance for teaching and assessments and the second category since it is important knowledge in the development of valid assessments. In the current study the focus is mainly laid on textual features with the potential to threaten the validity of assessments by causing a reading demand that is inadequate in a mathematics test.

PURPOSE

The purpose of the study is to enhance the understanding of which textual features in a mathematics task that can be demanding for the reader and what role those textual features have in a mathematical language. The purpose is fulfilled based on statistical and qualitative analyses of PISA mathematics tasks that have a high non-mathematics specific reading demand. The combination of these two types of results adds value to the current study since the combination make is possible to draw conclusions that would not have been possible based on the separate analyses.

METHOD

The method is designed to enable a combination of two types of results, statistical and qualitative, for the same type of data with the same focus. Initially the statistical results from four previous studies are synthesized. Thereafter two types of textual analyses are conducted on a few tasks that have either very high or negligible demand on reading ability (DRA). Eventually both types of results are interpreted together with a particular focus on whether one type of results can contribute to the interpretation of the other type of results. Before the two qualitative text analyses are described the measure DRA is explained, since an understanding of what that measure encompasses is essential in the interpretation of the results of the current study. The four studies focused in the synthesis all present results based on statistical analyses of textual features in tasks in relation to the tasks DRA, and for thee studies also to task difficulty. Results in relation to task difficulty are included since the possibility to contrast the results for to those two measures is valuable.

The quantitative measure for task demand on reading ability (DRA)

In all four studies (Österholm & Bergvist, 2012; Dyrvold, Bergqvist & Österholm, 2015; Dyrvold, 2016; Dyrvold, submitted), all Swedish students results on PISA mathematics and reading tasks is analysed in a principal component analysis (PCA). PCA is a method that reduces the number of features used to represent data. The result of the PCA on the PISA data is two main components for which each reading and mathematics task has loading values. Based on the pattern for the loading values on the tasks those components are interpreted as a mathematical ability component and a reading ability component. Each task entered in the analysis obtains a loading value for both those components, values explaining the unique contribution of that principal component on the solution frequency of that task. The loading values of the reading ability component is given the name demand on reading ability, and since the loading values represent the unique contribution of the component on the tasks solution frequency this DRA do not include a mathematical reading ability. Since the DRA do not explain reading demand that is reasonable in a mathematics task value for the tasks DRA represent something unwanted (see also, Österholm & Bergqvist, 2012).

Analyses of task text

Textual analyses are conducted on two tasks with high DRA and two tasks with negligible DRA. The tasks are chosen for analysis based on three criteria: very high or low DRA, includes several sentences and, have either expressions with mathematical notation or a diagram that is important for the solving of the task. Criteria two and three are used to ensure that the data is useful both to analyse important relations between several sentences and between different semiotic resources. The tasks are PISA mathematics tasks used in at least one of the years 2006 and 2012. The three criteria were not met for four tasks that are released and therefore the tasks are only described briefly in the bullet list. Odd numbered tasks are those with high DRA. Number of words in tasks is words not in images. Difficulty is (1– the fraction of all credits given on a task and all possible credits on that task), for all Swedish students who attempted to solve the task.

Task 1:task difficulty is 0.11, has mathematical notation and 54 words
Task 2:task difficulty is 0.78, has mathematical notation and 91 words
Task 3:task difficulty is 0.22, has a bar graph and 57 words
Task 4:task difficulty is 0.49, has four geometric diagrams and 47 words

The first analysis is directed at patterns for progression between themes and rhemes in the task text. In Swedish the theme is always the beginning of a sentence or a phrase and the rheme is, simply stated, the rest. The theme serves as a starting point for the message and it is the theme that orients the clause in relation to the rest of the text. The theme is developed in the rheme, and new information are often given there (Halliday & Matteson, 2014).

The analysis is chosen since a focus on natural language is reasonable based on previous results about both different aspects of vocabulary that can be difficult and aspects that has to do with meaning relations in the text. This choice means that the result of the analysis can add information to the previous statistical results since they concern similar but not the same features. The patterns for progression between the themes and rhemes give an understanding of how information given in phrases and sentences are tied together. The pattern illuminates where new information is introduced in the text and where some information earlier given is elaborated on. As illustrated in Table 1, the pattern for progression between the themes and rhemes can be categorized in three main types (see also, Danes, 1974). In linear progression the reader is gently lead through the text with new information given part by part. In progression with constant theme the same theme is elaborated on repeatedly in the subsequent sentences. In progression with new derived themes the new themes are derived from a superordinate item at the beginning of the text, for example if the first theme is quadrilaterals and the subsequent themes is rhombuses, rectangles and so forth. Linear progression is referred to as *simple* linear progression, and considered a logical means of creating text cohesion.

Table 1: Three types of progression between themes (T) and rhemes (R).

(1) Linear	(2) Constant theme	(3) New derived themes
T1 → R1	T1 → R1	T1 → R1
T2 R2	T2 → R2	T2 → R2
T3 → R3	T3 → R3	T3 → R3

The second analysis includes *all* semiotic resources of the text (i.e. natural language, mathematical notation, and images) and is about how the various semiotic resources function *together*. Such an analysis is justified since the study concerns the task text as a whole, and the different semiotic resources have different functions that together convey something that is not possible with the separate resources alone (Lemke, 1998). In the analysis, every sentence is divided according to three types of elements: participant (O), typically realized by a nominal group; process (P), typically realized by a verbal group; and circumstances (C), typically realized by an adverbial group or prepositional phrase (Halliday & Matteson, 2014). Thereafter all instances in the text as mathematical notation or images are divided in the same way based on what is present in natural language. The O for participant refers to concrete or abstract objects. The type of relations between natural language and any other semiotic resource is analysed regarding type of element (Halliday & Matthiessen, 2014) and type of relation (O'Halloran, 2005; Royce, 2007). Relations between the elements has not been analysed in detail yet, the analysis is mainly directed to a separation between congruent or non-congruent relations. The relation is congruent if the type of element (O, P, C) is the same in both semiotic resources and the relation is the intersemiotic counterpart for synonymy or repetition, namely intersemiotic complementarity. The word *complementarity* is used instead of synonymy or repetition since a word cannot express *exactly* the same thing as for example an image. For example, the word *angle* can be represented in different ways visually and the relation between *angle* and all those visualisations is of the type intersemiotic complementarity.

RESULT

The synthesis of the results of the previous statistical analyses reveals that the features related to task DRA have an emphasis on the natural language. The reviewed studies are marked A-D in the reference list. In the studies where the measure for the textual feature is based on correlations the measure is called *number* in Table 2 and in the study where the results are based on t-tests the measure is called *presence*.

Table 2: Textual features statistically related to task difficulty or task DRA.

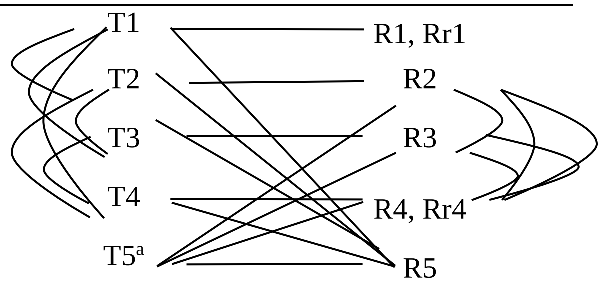
Study	Measure of textual feature	Textual features related to: high task DRA	Textual features related to: high task difficulty
A	number	word length, noun-verb quotient	NA
B	number	generally uncommon words	-
C	presence	-	images in combination with other semiotic resources
D	number	*less* of cohesive relations between represented objects	*more* of cohesive relations between represented objects

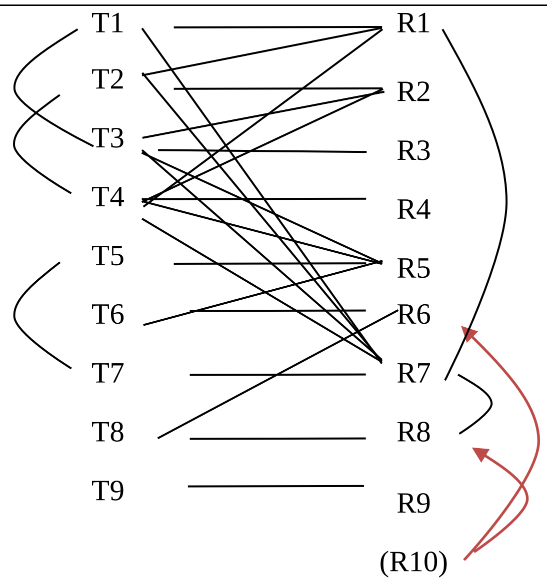
These statistical results indicate that different aspects of the vocabulary may play a role for how difficult the task is to read, but also that many cohesive ties between instances in the task text seems are more likely to be found in task with a low DRA. In study C-D natural language, mathematical notation and images are analysed.

The analysis of progression between themes and rhemes reveals a rather complex pattern for all four tasks. In Table 3-4 the leftmost tasks are those with high DRA. Response options and units given where the answer are to be filled in are also included since they are essential in the reading of the text. Those words have different roles than words in sentences and are therefore placed in parentheses and the relations are marked with arrows. The arrows illustrate in what theme and rheme the subject in the response options are mentioned. The lines in the tables represent progressions between themes and rhemes. No dotted line is used in the diagram since in all four tasks the progression between themes is between a constant theme.

Table 3: Progression between themes (T) and rhemes (R) in tasks with mathematical notation but no diagram.

Task 1 with high DRA	Task 2 with negligible DRA



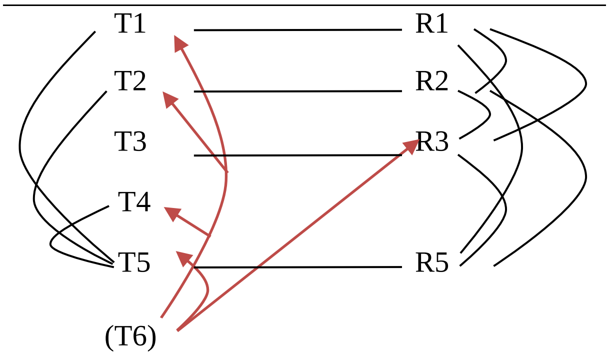


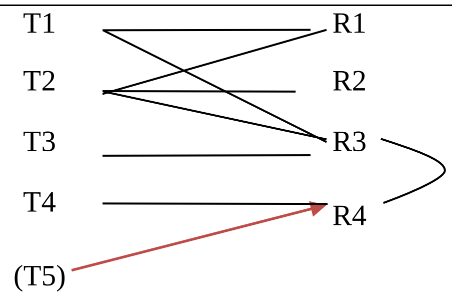
[a]Theme and rheme 5 is the question posed in task 1.

Linear progression is not very pronounced in any of the tasks, especially not in task 1 and 3. The last theme and rheme in task 1 is the question posed and besides that question the progression is only between themes and between rhemes (constant theme repeatedly). In task 1 there is a complexity in that two themes have rhemes that function independently. The double rhemes in task 1 are coded as two since the first rheme can be interpreted in relation to the theme independent of the other rheme. In task 2 and 4 the progression between themes and rhemes in different sentences can mainly be found in interaction between themes and rhemes. Task 2 has more sentences than all other tasks and therefore some instances of repeated themes or rhemes is expected, otherwise the task would consist of a substantial amount of different content. Considering that, it can be concluded that there is a tendency for the tasks with negligible DRA to have more of linear progression, but not only in the direction from rheme to previous theme.

Table 4: Progression between themes (T) and rhemes (R) in tasks with a diagram.

Task 3 with high DRA	Task 4 with negligible DRA





The results from the analysis of how the semiotic resources function together in the task text reveal that the interaction between the natural language and mathematical notation or some type of image is in several aspects more complex in tasks with negligible DRA. Tables 5-6 display two types of differences. Since most of the tasks are confidential the analyses are presented as numbers, but the purpose is not to quantify the analysis. To give some richness to the data some example of relations in Table 5-6 is given after a presentation of the content in the tables.

Table 5: Number and type relations between participants (O), processes (P), or circumstances (C) in natural language and mathematical notation in task 1 with high DRA and task 2 with negligible DRA.

	Task 1 Math. notation as			Task 2 Math. notation as		
Natural language as	O	P	C	O	P	C
O	**10**	-	-	**10**	-	2
P	-	**-**	-	1	-	-
C	-	-	**3**	6	-	-

Table 6: Number and type relations between participants (O), processes (P), or circumstances (C) in natural language and a diagram in task 3 with high DRA and task 4 with negligible DRA

	Task 3 Diagram as			Task 4 Diagram as		
Natural language as	O	P	C	O	P	C
O	**14**	-	-	**13**	-	1[a]
P	-	**1**	-	1	-	-
C	-	1	**7**	-	-	**4**

[a] In this relation the second semiotic resource is mathematical notation.

The relations between natural language and other semiotic resources in the task text are mainly obtained by relations between participants (O). The diagonals in table 5-6 represent congruent relations; that is relations between representations present as the same type of elements (O-O, P-P, C-C). In all cases in the analysis the relations in the diagonals (boldface) are of the type intersemiotic correspondence. That is the intersemiotic counterpart to synonymy or repetition (Jones, 2007). The number of relations that is *not* of the type intersemiotic correspondence is for task 1: none, task 2: eight, task 3: one, and task 4: one.

Examples: The congruent relations in task 1 and 3 are mainly relations between nouns and the same element represented as mathematical notation or in the diagram. The diagram in task 3 visualizes number of entities that have been sold. The substantial amount of non-congruent relations in task 2 (O-C) is descriptions given for particular criteria to be fulfilled (circumstance) something that in mathematical notation is represented as an object (participant as a variable).

Preliminary results indicate that a reading demand that is unwanted in mathematics tasks seems to be related to features of the natural language but not to interaction between words and images or mathematical notation. This conclusion is however only preliminary since the last part of the analysis has not been fully completed.

DISCUSSION

The current study reveals several features of the natural language that can enhance the risk of a task to assess a non-mathematics specific reading ability (measured as DRA). Aspects of how different semiotic resources interact in the text are on the other hand likely to be part of the explanation to mathematical difficulty of the tasks. The last step in the analysis, where the statistical result is evaluated in detail in relation to the textual analyses has not been performed yet, and therefore the discussion focus mainly on the results of the textual analyses and some aspects of the method. Some tentative conclusions based on an interpretation of the synthesis of the studies and the textual analyses together are however discussed.

Common for the textual features that characterize tasks with high DRA is that they regard the natural language. The analysis of how the semiotic resources interact in the task text reveals several aspects that are more complex in the tasks with negligible DRA. For example congruent relations are dominant in tasks with high DRA, whereas different non-congruent relations can be found in the tasks with negligible DRA. Since congruent relations are dominant in the tasks, the non-congruent relations are considered the distinctive feature. The synthesis of the statistical results do also reveal aspects related to the natural language that potentially explains a high DRA. The measure DRA is obtained through a principal component analysis where results on all reading and all mathematics tasks from PISA are entered into the analysis. Therefore, it may be tempting to excuse the current results about natural language away, based on the knowledge that the measure DRA stems partly from results on reading tasks focusing on natural language. The reading tasks is however not a reasonable argument for such a result, since PISA reading tasks assesses also non-continuous texts with parts of the necessary information in images.

Since the textual analyses are done on only four tasks it is important to reflect over the choice of tasks. The odd numbered tasks and the other tasks differ substantially in task DRA, which was an important criterion when the tasks were chosen. There are however also other properties of the tasks that differ. The task difficulty is higher for the tasks with negligible DRA, meaning that those tasks differ from the other two tasks from two perspectives; they do not demand a non-mathematics reading ability but they are more difficult to solve. This difference means that the textual features identified as pronounced in the tasks with high DRA may also be distinctive for task that are easy to solve. On the other hand, since the chosen tasks with high DRA are solved to a high frequency, that may mean the reading demand play a prominent role in those tasks since the DRA to a substantial extent contribute to the total difficulty. Task 2 does also consist of many more words than the other three tasks something that can add complexity to the text. The length of task 2 is evident in the analysis of thematic progression, but since number of words is presented the reader can take the length into account in the interpretation of the results and the length must therefore not in an inadequate way affect the conclusions.

The criteria for which tasks to choose was judged important for the analyses to be meaningful, something that lead to a sparse presentation of the results of the textual analyses since it was not possible to find a sample that fulfilled the criteria and at the same time was not confidential tasks. The tasks and the results are described and examples are also given to prohibit possible difficulties to interpret the results.

The synthesis of the statistical results will be further analysed in relation to the textual analyses, especially every textual feature presented in Table 2 will be evaluated in relation to the results of the textual analyses. Despite that this last analysis is still missing there is some educational implications of the results. There are some textual features identified in the tasks with negligible DRA that are valuable to focus on in teaching, especially since the analysed tasks have proven difficult to solve. Common for the results of the synthesis and both textual analyses is that if there is some intricacy in how the different constituents of the text is composed to a coherent whole, that may cause difficulties in the reading. The intricacy can be many meaning relations in the text (Table 2), or some complexity in the thematic progression between themes or rhemes (Table 3-4), or a complexity that have to do with whether there is congruence in how elements are represented in different semiotic resources (Table 5-6). Or, the interaction between these features may contribute to the complexity. Based on these results and previous research about difficulties related to how the text function as a whole (e.g., Duval, 2006) a focus on how the different constituents of the task text with different means interact to communicate the task is recommended in teaching. Apparently this recommendation is vague, and continued research is needed as a base for further educational guidance.

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PROOF FROM AN EMBODIED POINT OF VIEW

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Mathematical proof and logical deduction are often seen as abstract and unrelated to everyday thinking and experiences. However, an analysis utilizing the framework of embodied cognition argues that these capabilities are grounded in physical experience, including perceptual and motor experiences going back to childhood. Constructs and tools from cognitive linguistics are used to analyze evidence from a variety of sources, including written texts and videotaped interviews with doctoral students in mathematics. The embodied foundations for proof and logical deduction are identified, as well as several metaphors, some with specific cultural characteristics.

INTRODUCTION

As one of the central practices of mathematics, proof and proving have been studied from many perspectives (see Hanna, Jahnke, & Pulte, 2010 for an overview; also Hanna, 1995; Harel & Sowder, 1998). Mathematical proof and logical deduction are often seen as abstract and unrelated to everyday thinking and experiences. The purpose of this paper is to consider the nature of mathematical proof when viewed from an embodied cognition perspective, and to make a link between everyday thought and experience and proof. That is, we consider the questions: What are the conceptual underpinnings of proof? How have human beings constructed the activity of mathematical proving based on more basic cognitive capabilities? What are the ways in which proof is conceptualized by different kinds of people who utilize it? This paper will first describe the theoretical perspective of embodied cognition (Varela, Thompson, & Rosch, 1991). An analysis of proof from this perspective will be presented, drawing evidence from written and spoken texts, studies in infant cognition and linguistics, and research carried out by the author. Existing research relevant to the questions above will be integrated throughout.

EMBODIED COGNITION

The theory of embodied cognition focuses on the bodily basis of thinking, that is, "on the ways in which complex adaptive behavior emerges from physical experience in biologically-constrained systems" (Núñez, Edwards, & Matos, 1999, p. 49). As it relates to mathematics, embodied cognition holds that mathematical ideas are either grounded in physical experience (via grounding metaphors), or built up from existing ideas via linking metaphors (Lakoff & Núñez, 2000). The analysis presented in this paper focuses less on mathematical content and more on the kind of argumentation or discourse that is considered acceptable within the community of mathematics practitioners, that is, on mathematical proof and formal logic.

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One of the principles of embodied cognition has been labelled "continuity" by Mark Johnson (2007); it holds that all thinking is embodied, even the most abstract creations of the human mind, and that more complex kinds of thinking are built from the same basic conceptual building blocks as more mundane thought, including such elements as image schemata, metaphors and conceptual blends (M. Johnson, 1987, 2007; Fauconnier & Turner, 2002). Thus, if we are looking for the conceptual roots of mathematical proof, it makes sense to look at conceptually-simpler kinds of thinking that make proof possible, including images and processes derived from physical experiences.

WHERE DOES PROOF COME FROM?

Although there are different definitions of proof, we will begin by looking at two pieces of text that discuss proof, the first from a mathematics education researcher and the second from a mathematician.

> A proof is a transparent argument, in which all the information used and all the rules of reasoning are clearly displayed and open to criticism. It is in the very nature of proof that the validity of the conclusion flows from the proof itself, not from any external authority. (Hanna, 1995, p. 46)

> I'd like to spell out more what I mean when I say I proved this theorem. It meant that I had a clear and complete flow of ideas, including details, that withstood a great deal of scrutiny by myself and by others. (Thurston, 1994, p. 175).

Both of these definitions contain an interesting phrase that points to the embodied nature of our ideas about proof. Hanna states that "the validity of the conclusion **flows from** the proof itself" and Thurston refers to a "**flow** of ideas" (emphasis added). The use of the word "flow" indicates the existence of a conceptual metaphor, one that is based in our knowledge and experience of physical phenomena (M. Johnson, 1987, 2007). In the quotation above, Hanna also discusses the "source" of a conclusion's validity, namely "the proof itself" rather than an external authority (see Harel & Sowder, 1998 for a discussion of undergraduate proof schemes, one of which appeals to external authority). Both excerpts draw on the reader's understanding of the image/experience of "flowing," which is a physical characteristic of rivers, heads of hair, and other bodies of connected, ordered elements moving in the same direction. Thus, the definitions evoke an unconscious metaphor of a proof as an entity that starts at a particular source and "flows" toward an end point. This metaphor is related to what has been called the A PROOF IS A JOURNEY metaphor (Edwards, 2010), which will be examined in more detail below.

Logic and Causality: Evidence from Infant Cognition

If we drill deeper into how proofs are conceptualized, we find that they are made up of sequences of statements that are connected and supported via deductive logic. The Free Dictionary defines a proof as "a formal series of statements showing that if one thing is true something else necessarily follows from it" (http://www.thefreedictionary.com/ mathematical+proof). Yet even this definition is grounded in embodied experience, as seen in the use of the verb “to follow” (even the word “consequence” has the same base meaning: from Latin *consequentia*, from *consequent*- ‘following closely’). How, we might ask, does one thing “following” (or “flowing from”) another come to have the connotation of logical implication ("If A is true, then B is true")?

From the point of view of embodied cognition, this mapping is based on physical and perceptual experiences of causality. The development of an understanding of causality has been investigated by both psychologists and cognitive linguists, and is observed even in very young infants (Gopnik & Schulz, 2007; Sperber, Premack & Premack, 1996). For example, Leslie and Keeble (1987) found that 6-month-old infants who were habituated to a dynamic image of a causal sequence (one object apparently hitting another and causing it to move) paid attention longer to a physically-impossible version of that sequence than to non-causal stimuli. The ability to perceive the sequentiality of two actions is one step in constructing the notion of physical causality, along with the ability to note a recurrent pattern of an effect “following” a cause (the ability to construct such patterns is often called inductive reasoning). These abilities seem to develop over a period of time in infants, from around age 5 through 10 months (S. Johnson, 2003).

Thus, from a very early age, we are capable of noticing that one event may follow another in time, and building the concept that it was caused by the prior event. From an embodied perspective, our understanding of logical deduction (that one statement “follows from” another) is constructed based on this early experience, via the mechanism of conceptual metaphor. To make this metaphor explicit, let us consider a basic element of a logical argument or mathematical proof, the simple implication “If A then B” (where A and B are logical/mathematical statements). Table 1 spells out the metaphorical mapping from physical causation to logical deduction.

Logical Deduction IS Physical Causation	
Source Domain: **Causation via Physical Forces** **"This action caused that effect"**	**Target Domain:** **Logical deduction** **"If A then B"**
• Two entities	• Two declarative statements
• One is foregrounded or singled out (the "agonist" or target or effect)	• One is foregrounded as the "conclusion"' (the truth of which is at issue)
• The other is considered in terms of the effect it has on the agonist (the "antagonist" or cause)	• The other ("premise") is considered in terms of the implication that it has for the truth of the conclusion
• Physical force	• Logical necessity
• If the force of the antagonist is sufficiently strong, the result is motion of the agonist	• If the logical necessity connecting the premise to the conclusion is valid, then the truth of the conclusion is established

Table 1: The physical causation metaphor for logical deduction

Evidence from Cognitive Linguistics

Linguists have analyzed multiple languages and discovered constructions that map the notion of physical causality to both social situations and logical statements (Dancygeir & Sweetser, 2005). These mappings are based on cognitive primitives called image schemas, specifically, schemas concerned with force dynamics (Talmy, 1988). Image schemas are "recurrent, stable patterns of sensorimotor experience...[that] preserve the topological structure of the perceptual whole [and have] internal structures that give rise to constrained inferences" (M. Johnson, 2007, p. 144). Thus, image schemas involving physical forces, actions, and reactions are the raw materials used in the metaphor LOGICAL DEDUCTION IS PHYSICAL CAUSALITY. Within cognitive linguistics, force schemas share certain properties, including the following:

- Force schemas involve a force vector, i.e., directionality.
- Force schemas have sources for the force and targets that are acted upon
- Forces involve a chain of causality.

(Evans & Green, 2006)

These properties map to analogous properties within the discourse of mathematical proof, i.e.:

- Logical propositions have directionality ("If A then B" is not the same as "If B then A")
- Logical propositions have sources for the premises (previously proved propositions, postulates and/or axioms) and the conclusions (the premise and its sources)
- Mathematical proofs involve a chain of logical deductions (see additional analysis below about the chain metaphor).

Thus, from an embodied point of view, logical reasoning and proof are not constructed from a special kind of thinking; rather, they recruit conceptual capabilities that are deep-seated and based in physical experience (although of course additional constraints and requirements for acceptable proofs are added by the mathematical community). As Mark Johnson (2008) notes, "According to this view, we do not have two kinds of logic, one for spatial-bodily concepts and a wholly different one for abstract concepts. There is no disembodied logic at all. Instead, we recruit body-based, image-schematic logic to perform abstract reasoning" (p. 181).

RESEARCH ON PROOF

The analysis presented above will be augmented by results from a research study on proof carried out with a group of twelve doctoral students in mathematics. The students were videotaped in pairs for 90 minutes. The session had three parts: a general interview about their experience teaching and doing proofs; a 45-minute segment during which they worked together to prove a theorem (without the interviewer in the room); and a final segment during which they explained their proof and, if there was sufficient time, gave their opinion on whether a particular visual argument constituted a proof. The videotapes were transcribed and the students' speech as well as their gestures were analysed to look for evidence of how they conceptualized proof in general, as well as how they communicated through various modalities while constructing a specific proof.

Evidence from Doctoral Students' Words and Gestures

When asked directly to describe what a proof is, and whether there are different kinds of proofs, the doctoral students offered characterizations similar to those presented in the textual evidence above. For example (emphasis added):

> AC: It's a set of logical reasoning that begins with a premise and **leads to** a conclusion.

> AW: I would say it's just, you know, a well thought out **sequence of steps** that nobody would refute… In practice, it's just – it – a very, very solid argument in which **each step proceeds logically from the last**.

> AS: A rigorous proof would be based on the axioms of mathematics that we've set up…And based on a logic system that we, as humans, have [laughs]…Actually, **following it step-by-step** so that your **conclusion always follows** from some kind of logical steps.

These definitions include the idea of logical ideas which "follow" from previous ones, but the overall conceptualization goes further than this. The students' definitions often imply or state explicitly that these "steps" are part of an overall "journey," with a beginning, middle, and end. That is, a proof is understood in terms of the conceptual metaphor "A PROOF IS A JOURNEY," as spelled out in Table 2:

A Proof IS a Journey	
Source Domain: Physical journey	**Target Domain: Mathematical proof**
• Starting point (source) • Destination (goal) • Steps • Possible sequences of steps (routes or paths) • "Dead ends" or wrong paths • Obstacles to finishing the journey	• Premises (source) • Conclusion (goal) • Logical statements ("If A then B") • Possible sequences of logical statements (path) • Sequences that don't result in the desired conclusion • Obstacles to completing the proof

Table 2: The journey metaphor for proof

This metaphor is based on an image schema known as "source-path-goal" (M. Johnson, 1987), which is again based on embodied experience. In this case, the physical experience is again a very early one, that of moving oneself from a starting location to a destination, via a path. Terms consistent with this metaphor were used by all of the students, including the following representative sample:

> "destination," "the forward direction," "walking it back," "you want to end up over here," "you get kind of bogged down," "you get to a certain point," "I don't wanna go any further," "we'll try the other way," "maybe I don't know where I'm going," "at some point, maybe I can, like you know, see the goal," "there's so many ways you could go," "the better way to go"

Interestingly, the term "step" is used in more than one way when talking about proof. It is used metaphorically to refer to a single step in a journey. But it is also used more generically to mean one element of a procedure, or one written line in a formal proof. This latter usage is based on a different visual image, that of a proof written on multiple lines of a paper or blackboard. For example, at one point in an explanation, a student stated "so let's start back at the top" when referring to a written proof, referring to the way that proofs are traditionally written with premises at the top of the page. Another student displayed a gesture in which his hand was held palm down horizontally in front of the upper chest, and then moved iteratively downward while he said "And then the question is, well, can I fill in those steps that I have?" (Figure 1).

Figure 1: A gesture for one line or "step" of a proof

In this case, the gesture helps to clarify that the student is talking about filling in written lines in a proof, again moving from the top of the virtual page toward the bottom. This is an example of how a useful bodily-based image schema, that of taking a physical step with one's foot, can serve as the foundation for multiple related concepts.

The metaphor of a proof as a journey is not the only one utilized by the graduate students. They also discussed proofs as object or constructions which are made up of pieces or parts, as evidenced by the statements: "Sometimes you can show maybe two out of three **parts**" and "If it doesn't hold, then everything falls to **pieces**." A more specific version of this metaphor views proof as a building, with a foundation holding it up, and a certain degree of strength. Thurston (1994) exemplifies this metaphor when he states, "The kind of change I would advocate is that mathematicians take more care with their proofs, making them really clear and as simple as possible so that if any **weakness** is present it will be easy to detect" (p. 170; emphasis added).

Yet another metaphor emphasized the logical connections between statements within a proof, utilizing the metaphor of a proof as a chain made up of connected links:

NE: In a strict sense, I guess it's a **chain** of implications, from the hypothesis to the conclusion

Thus, we see that an analysis of proof from an embodied point of view does not yield only a single metaphor or image schema grounding its understanding, but rather several different mappings that each highlight a different aspect of this complex product of the mathematical community.

CULTURAL INFLUENCES

As a final example of the richness of an embodied perspective on proof, we will examine the cultural influences on a physically-grounded metaphor. Although embodied cognition looks to our shared physical experiences, it does not overlook the fact that these experiences are always situated within particular social and cultural contexts (Evans & Green, 2006; Fauconnier & Turner, 2002). The proof that the doctoral students were asked to create during the study involved the infimum of one sequence and the supremum of another, with a desired value between the two. In discussing this proof, the three researchers found that they each utilized a metaphor with a different surface structure, but with the same underlying image schema based on physical experience. One researcher, from the United States, described the students' solution as a "squeezing" proof; the second, from South America, used the metaphor of a sandwich; and the third, from Eastern Europe, called it a "policeman" proof, evoking the image of a prisoner held between two police officers.

Thus, each metaphor preserved the underlying physical situation in which one entity is located or held between two others, with the entities on either side exerting pressure on the inner one. However, as the researchers were from three different countries, this image schema was "dressed up" in different surface metaphors. It seems quite possible that these different metaphors developed within the different mathematical and cultural communities within each country.

DISCUSSION

We have presented an investigation of mathematical proof and logical deduction from the perspective of embodied cognition, drawing from a variety of kinds of evidence. Such an analysis may be useful to mathematics educators in helping to clarify the conceptual underpinnings of this important practice, and perhaps in illustrating metaphors that might be effective in helping students learn about proof. There are two directions for future research. One would be to investigate whether an analysis of speech, gesture, and other modalities would reveal an inner structure to the proof process. A second set of questions addresses cultural differences and similarities in gestures related to both proof and mathematical content, across different cultures and languages. In either case, the analysis will utilize the theoretical foundation of embodied cognition, and the powerful tools of cognitive linguistics and gesture analysis.

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TEACHERS' FORMS OF ATTENTION TO STUDENTS' WRITTEN RESPONSES IN PATTERN GENERALIZATION

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The purpose of this study is to explore teachers' forms of attention to students' responses in pattern generalization tasks. A survey consisting of ten students' responses to different types of generalization tasks was developed and then given to a sample of ninety-one in-service mathematics teachers. Data analysis showed five forms of attention: inability to attend to a student's response, gazing at the figural pattern, discerning details, recognizing relationships, and perceiving properties in a student's response. The findings showed that the forms of attention were mediated by pattern generalization type. In particular, as the generality demands of the task increased, "discerning details" decreased whereas "recognizing relationships" and "perceiving properties" increased. The findings also showed that teachers predominantly used "discerning details" in near generalization tasks whereas they predominantly used "perceiving properties" in far generalization tasks.

BACKGROUND

Pattern generalization is a core area in mathematics that is characterized by more strategic knowledge than mathematical content knowledge (El Mouhayar & Jurdak, 2015a, 2016). To date much of the research in the context of pattern generalization has focused on students' strategies and less on teachers' noticing of students' responses. One direction was the study of teacher ability to explain student reasoning in pattern generalization in terms of identifying the elements which constitute a complete explanation (e.g. El Mouhayar & Jurdak, 2013). Another direction in the literature focused on the lenses through which the teacher views and analyzes students' written responses in pattern generalization (El Mouhayar & Jurdak, 2015b). A third direction focused on teachers' strategies and reasoning approaches in generalizing patterns (Rivera & Becker, 2007). Recently, researchers (e.g. Callejo & Zapatera, 2016) addressed a new direction that focuses on teachers' noticing of students' thinking in the context of pattern generalization. The findings reported that even though teachers may have the ability to identify elements of students' strategies; however, this does not necessarily indicate that they have the ability to interpret and analyze students' understanding (Callejo & Zapatera, 2016).

The present study extends previous research on teachers' noticing of students' thinking. It attempts to identify and distinguish between different forms of teachers' attention to students' written responses in pattern generalization. Teacher attention to

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students' responses is one of skills of teacher noticing (Mason, 2011) and it is the most foundational (Star & Strickland, 2008).

TEACHERS' FORMS OF ATTENTION

Attention "*is* both observation and the medium through which observation takes place" (Mason, 2002, p. 45). For example, if someone is gazing at a diagram and he/she notices something familiar about a part of the diagram, this may lead him/her to start attending to that part. Mason (2011) distinguished between five forms of attention:

- "Holding wholes" is attending by gazing holistically at an object or a situation without discerning details. Gazing at a diagram is an example of holding wholes.
- "Discerning details" is looking for specific details and discerning the details by distinguishing "this" from "that".
- "Recognizing relationships" is looking at relationships that exist between discerned details in a particular situation.
- "Perceiving properties" is recognizing relationships that exist between specific details as particular cases of properties that could hold true in different situations
- "Reasoning on the basis of agreed properties" is using agreed properties shared by many examples to assert that those properties are to be the defining properties, leading to reasoning based on axioms, rules and definitions.

RATIONALE OF THE STUDY

Attention to student work is expected to be enacted in a regular manner by teachers and attending to students' mathematical reasoning is a key component of teaching expertise (Mason, 2002). There are few research studies in mathematics education that explore teachers' ways of attending to students' responses. Unpacking the forms of attention to students' work is particularly important in pattern generalization because of the strategic nature of reasoning in the latter. This study extends previous research on teachers' ability to explain students' responses in pattern generalization in two directions. First, the present study aims at exploring teachers' ways of attending to students' written responses whereas previous research focused on exploring teacher ability to identify and explain student reasoning in pattern generalization in terms of the elements of student response (El Mouhayar & Jurdak, 2013). Second, the present study aims at exploring the impact of pattern generalization type on teachers' forms of attention to students' responses whereas previous studies addressed the influence of pattern generalization type on teacher-used lens to explain students' responses (El Mouhayar & Jurdak, 2015b).

RESEARCH QUESTIONS

- What forms of attention do in-service mathematics teachers use to explain students' written responses in pattern generalization tasks?

- How is teacher attention to students' written responses influenced by pattern generalization type?

METHOD

Participants

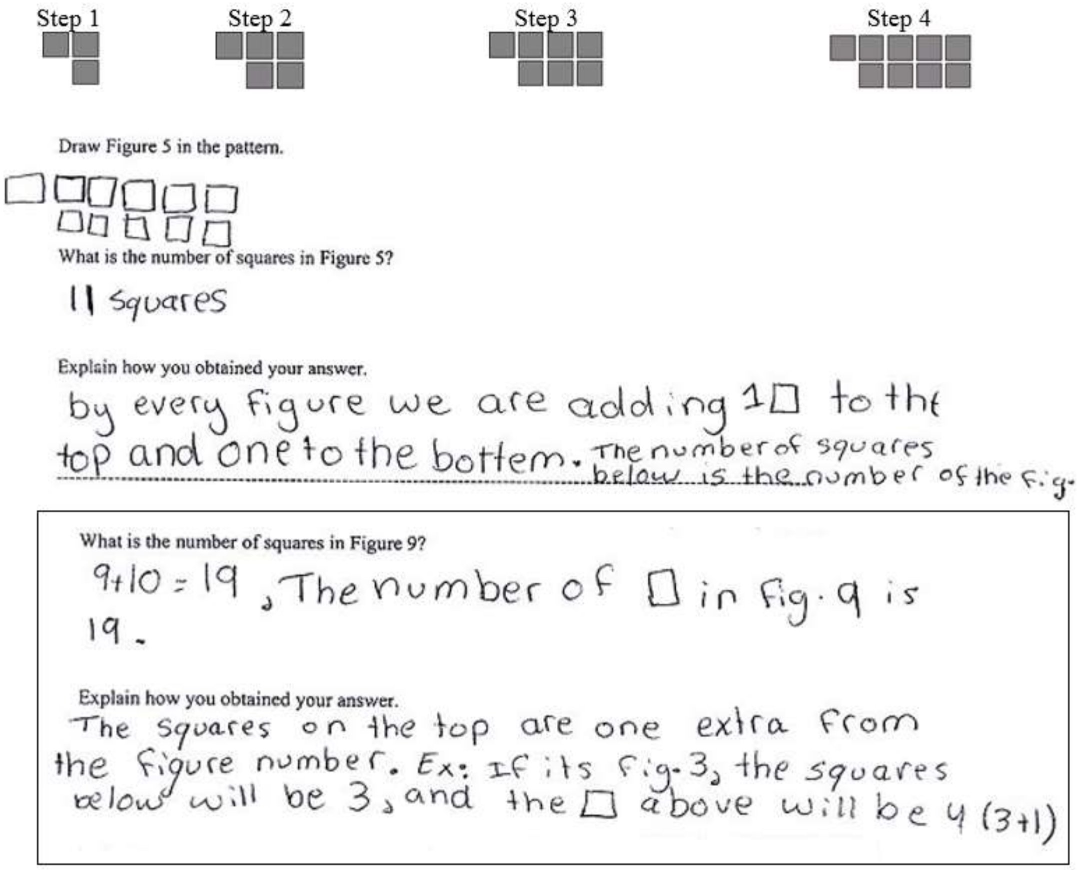
Ninety-one in-service school mathematics teachers from different grade levels were selected from twenty schools in Lebanon, particularly Beirut and its suburbs, to participate in the present study. The majority of the participants (75.8%) had five or more years of experience in teaching mathematics. Of the 91 participants, 79.8% were females and 20.2% were males.

Instrument

In the present study, in-service mathematics teachers filled out a questionnaire, in order to examine the ways that teachers use to attend to students' responses in pattern generalization. A sample of students' written responses were taken from a survey used in a previous study (El Mouhayar & Jurdak, 2015a) involving 1232 Lebanese students from grades 4 to 11. The survey consisted of 10 items representing different reasoning approaches and strategies used by students in near and far generalization tasks. Each of the items displayed the problem (a growing figural pattern showing the first four figural steps) and a student's responses to: (1) near generalization (predicting step 5 or step 9) or (2) far generalization (predicting step 100 or step n). For each item, participants were asked to explain students' responses by responding to the following question: "How did the student think to get the number of squares?". For example, participants were asked in item 1 of the survey to explain student's written response for a near generalization task step 9 (Figure 1). Participants filled out the questionnaire individually in the presence of the investigator in about 90 minutes.

Data Collection and analysis

The obtained data were subjected to a series of analyses. First, a constant comparative method of qualitative analysis (Glaser & Strauss, 1967) was applied to identify teachers' forms of attention to students' written responses. Two researchers coded the data independently and several meetings between them followed where the discrepancies in identifying and in coding teachers' forms of attention were negotiated until consensus was reached. Second, frequencies and percentages were determined for each of the categories of teachers' forms of attention to students' written responses. Third, a cross tabulation of teacher attention to student response by pattern generalization type was done to explore the possibility of significant differences in teachers' ways of attending to students' responses across generalization type. Fourth, percentages of teachers' forms of attention to students' responses across pattern generalization types were presented by bar graphs in order to identify the highest frequency of teacher attention within each generalization type and trends of variation of teacher attention across generalization type.


Step 1 Step 2 Step 3 Step 4

Draw Figure 5 in the pattern.

What is the number of squares in Figure 5?

11 squares

Explain how you obtained your answer.

by every figure we are adding 1□ to the top and one to the bottem. The number of squares below is the number of the fig.

What is the number of squares in Figure 9?

9+10 = 19, The number of □ in fig. 9 is 19.

Explain how you obtained your answer.

The squares on the top are one extra from the figure number. Ex: If its fig. 3, the squares below will be 3, and the □ above will be 4 (3+1)

Figure 1: Sample of a student's written response in a near generalization task (step 9)

FINDINGS

Forms of teachers' attention to students' written responses

Qualitative analysis resulted in five ways that the teachers used in attending to students' written responses. The five forms of attention are as follows:

- Inability to attend: The teacher did not attend to student's response by expressing inability to explain student response.
- Holding wholes: The teacher attended to student's response by gazing at the figural pattern (gazing at the diagram) without particularly discerning details in student's response.
- Discerning details: The teacher distinguished details of a student's response by attending to the student's mathematical thinking in particular step(s) of the pattern.
- Recognizing relationships: The teacher attention was directed towards relationships as objects that exist in particular steps of the pattern.
- Perceiving properties: The teacher recognized relationships as instances of properties that are independent of particular steps of the pattern and that are related to general aspects of the pattern.

The findings showed that the majority of teachers attended to students' responses by "discerning details" (48.8%) followed by "perceiving properties" (28.5%) followed by "recognizing relationships" (9.2%), Only 2.6% of teachers attended to students' responses by "holding wholes". 10.9% of teachers did not attend to students' responses and 4.1% of the explanations were coded missing. The excerpts in Table 1 are examples of different forms of teachers' attention to a student's written response (Figure 1) in a near generalization task (step 9).

Excerpt	Teacher attention	Sample teacher explanation
1	Holding wholes	Teacher: "In figure 1 there were 3 squares and they were increasing by 2. If we skip count by 2 we will reach 19. Steps: 1-2-3-4-5 – 6 –7 –8– 9 Number of squares: 3-5-7-9-11-13-15-17-19"
2	Discerning details	Teacher: "He added the cubes of the first row in figure 9 and the cubes of the second row"
3	Recognizing relationships	Teacher: "The student observed the pattern. He (she) counted the number of squares on the top and the one below. He noticed that we started by 2 squares up and 1 square down. We are adding 1 to the top and 1 square below. The number of squares in figure 9 is the same below and 9+1 = 10 above. He added 10 + 9 to obtain 19 squares"
4	Perceiving properties	Teacher: "The student compared the number of squares in the upper row and the lower row in all the figures. He found that in each step the number of squares is equal to the number of figure in the lower row and the number of squares in the upper row is equal to the number of figure + 1. Thus, the number of squares in figural step 9 is 9+10 = 19"

Table 1: Different forms of teachers' attention to a student's written response

Excerpt 1 is an example of "holding wholes" since the teacher generalized the pattern by gazing at different steps of the pattern without looking at the student's response. The teacher found the number of squares in steps 1 to 9 by looking at figural step 1, counted 3 squares and then skipped counting by 2 until reaching 19 squares in step 9.

Excerpt 2 is an example of "discerning details" where the teacher attended to particular details of student mathematical thinking by pointing out that the student added the number of squares in the top row to the number of squares in the bottom row in step 9.

Excerpt 3 is an example of "perceiving properties" since the teacher attended to particular relationships that exist in the pattern. Those relationships are instances of properties that the teacher perceived in student response in step 9 but that could also hold in other situations (those relationships exist in all steps of the pattern).

Excerpt 4 is an example of "perceiving properties" since the teacher recognized relationships between parts of the figure and any step number of the pattern.

Influence of pattern generalization type on teachers' forms of attention to students' written responses

The teacher's form of attention to students' responses was mediated by the type of pattern generalization task. A cross tabulation of teachers' forms of attention to students' written responses by pattern generalization type (near and far generalizations) were done. Findings show that chi-squared was significant ($\chi2$ (5) = 87.76, p = 0.00) indicating that teachers' forms of attention to students' written responses were significantly influenced by pattern generalization type.

Within generalization type, discerning details was most frequently used by teachers in near generalization tasks (60.3%) whereas perceiving properties was more dominant than other forms of attention in far generalization tasks (40%).

Across generalization type, as the generalization demands increased, that is changed from constructing of a step-by-step solution to finding a general formula, teachers' use of "discerning details" and "holding wholes" decreased whereas "recognizing relationships" and "perceiving properties" increased (Figure 2). Figure 2 shows that the teachers adopted discerning details (61.7%) and holding wholes (95.7%) most frequently in near generalization tasks whereas they used recognizing relationships (61.3%) and perceiving properties (70.3%) most frequently in far generalization tasks.

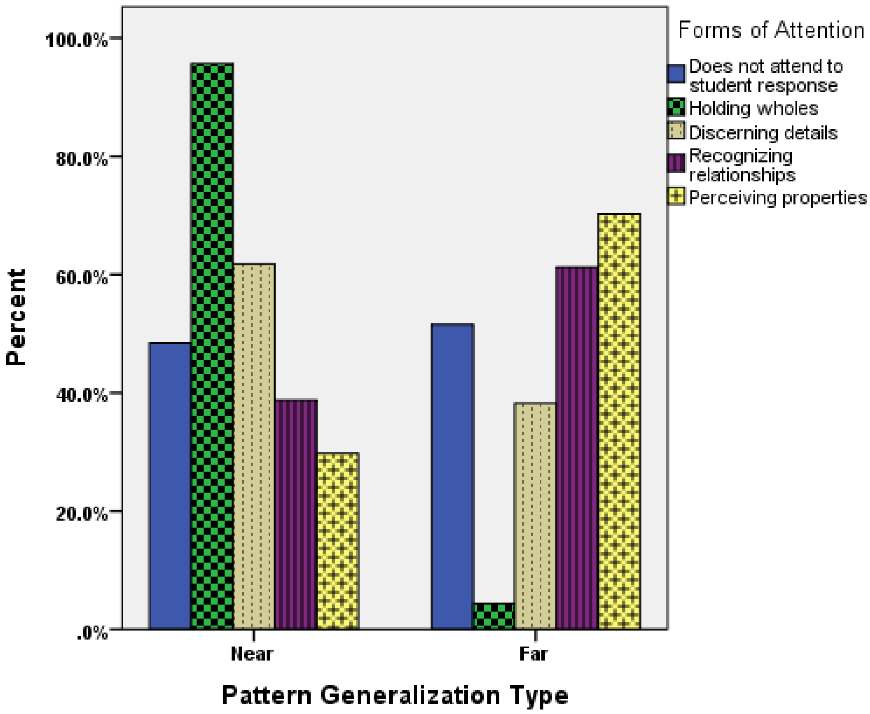


Figure 2: Bar graphs representing the percentages of teachers' forms of attention to students' written responses

DISCUSSION

One major finding in this study is that teachers predominantly used discerning details in attending to students' responses; however, teachers' forms of attention were mediated by pattern generalization type. The different ways in attending to students' responses significantly differed according to the generalization type. The frequency of "holding wholes" and "discerning details" decreased with the increase in the generality demands of the task (from near to far generalizations); whereas, the frequency of "recognizing relationships" and "perceiving properties" increased with the increase in the generality demands of the task.

The shifts in the structure of attention to students' responses may be due to different factors. Of those factors is the nature of near and far generalization tasks. As the step number becomes larger in a far generalization task, recognizing relationships and perceiving properties are more efficient compared to other forms of attention since both forms involve attending to relationships and commonalities that could hold in different steps of the pattern. On the other hand, as the step number gets larger, discerning details becomes less efficient because of its local nature since it involves looking at the details of student's response in particular step(s) of the pattern.

Another plausible explanation may be due to the interrelationships between the features of the strategies used by the students and their relationship to the generalization process, on one hand, and the pre-dominant use of student lens in attending to student response by experienced teachers on the other hand. Previous studies (e.g. El Mouhayar & Jurdak, 2015a) suggest that as the step number becomes larger for a far generalization task, the strategies that the students use to generalize the pattern become more advanced compared to other strategies in the sense that the advanced strategies allow grasping and generalizing the commonality that exists in a pattern to all the terms of the pattern. On the other hand, the participants in the present study are in-service teachers such that the majority (75%) had five or more years of experience in teaching mathematics. Previous studies reported that experienced teachers showed ability in noticing their students' thinking (Van Es, 2011) and in predominantly using students' lens in attending to their responses (El Mouhayar & Jurdak, 2015b).

A second finding of this study is that the teachers used "discerning details" more often than other forms of attention to students' written responses in near generalization tasks; whereas, they predominantly used "perceiving properties" in attending to students' written responses in far generalization tasks. Again, this may be may be due to the interrelationships between the structure of attention, the nature of the task (type of generalization) and the strategy being used to generalize the pattern.

In conclusion, the area of studying the responsiveness of teachers to students' work is underrepresented in the field of education of mathematics. It is hoped that more studies be carried out in this area in order to generate enough knowledge that may be incorporated in teacher education programs.

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WHAT MATTERS IN MATHEMATICS EDUCATION? AN ANALYSIS OF AN INTENDED CURRICULUM IN AUSTRALIA

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The 'intended curriculum' is where decision-making about the direction of teaching and learning in an education system takes place. This research used a content analysis process, informed by grounded theory, to identify the underlying priorities in the Victorian Curriculum: Mathematics (VCM), an authoritative document within the intended mathematics curriculum in Victoria, Australia. Results highlighted tensions in the VCM's messaging, regarding the purpose of mathematics education and how mathematics is expected to be taught and learned. The findings suggest the potential for confusion when using this component of the intended curriculum as a guide for teaching and learning.

INTRODUCTION

A curriculum provides the vehicle for expressing a society's ideal, acting as an intermediary force between the community of decision-makers and those in schools (see Ditchburn, 2012). The 'ideal' encompasses what has been deemed successful in the past and what hopes are held for the future (see Wong, Zhang & Li, 2014). As such, the significance of the curriculum cannot be understated or removed from any conversation that considers the realities of schooling.

The present research is concerned with what is described as the 'intended curriculum', which operates at the educational system level and is where strategic decision-making often takes place (see Travers & Westbury, 1989). Specifically, the analysis is focused on a prominent component of the intended mathematics curriculum in the state of Victoria, Australia, the *Victorian Curriculum: Mathematics* ('VCM') (Victorian Curriculum and Assessment Authority [VCAA], 2016). Aimed primarily at teachers, the VCM presents standards to assist with planning, monitoring and assessment. The intentions expressed through the document are accordingly interpreted and implemented by educators in schools. By presenting macro-level analysis of the VCM, this research therefore provides important insight into perceptions of mathematics education at a system level in Victoria.

Analysis undertaken for this research was guided by three research questions (RQ):

RQ1. Within the intended curriculum, what is the purpose of mathematics education?

RQ2. How is mathematics learning framed by the intended curriculum?

RQ3. How is mathematics teaching framed by the intended curriculum?

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LITERATURE REVIEW

The curriculum framework as outlined by Travers and Westbury (1989) has informed this research. It is underpinned by "the assumption that curriculum exists in different forms at different levels of the system" (Remillard & Heck, 2014, p. 706). Travers and Westbury's curriculum framework comprises three levels: the intended, implemented and attained curriculum. It is via the intended curriculum that the curriculum in its entirety is shaped. The scope of the intended curriculum includes "course outlines, official syllabi, and textbooks" (Travers & Westbury, 1989, p. 6), in addition to the rationale or goals for learning (see Burkhardt, 2014).

At its most extreme, what is laid out in the intended curriculum maps perfectly onto what gets taught and what is learned (see Wong, Zhang & Li, 2014). In reality, an interplay of factors, such as a teacher's content and pedagogical content knowledge and a student's readiness to learn, will impact the ability of students to demonstrate the expected skills, understanding and values.

Furthermore, the intended curriculum is not a static, unchanging entity, but gets "reconstituted according to evolving social priorities and criteria" (Brown, Hodson & Smith, 2013, p. 41). These priorities and criteria may be derived from, for example, national or even global pressures and/or advances in pedagogical and assessment techniques. If this is the case, then why use the intended curriculum for analysis? The value of focusing analysis on this component of a curriculum system comes in viewing the intended curriculum as a snapshot in time that provides insight into the context in which analysis takes place.

This research focussed on the intended mathematics curriculum in Victoria from the 2015-16 period. Literature reviewed for this analysis encompassed the period commencing from 2008, when the Australian Curriculum, Assessment and Reporting Authority (ACARA) was formed under an act of Federal Parliament, up to 2016, the year of release of the VCM.

The Intended Mathematics Curriculum in Victoria.

Examination of literature in the field of mathematics education has indicated a lack of analysis or evaluation of the curriculum at a system level in Australia. This observation has been noted by Atweh and colleagues, who suggest that research in mathematics education "often seems to be more concerned with how we can introduce a concept to maximise learning rather than why and to what purpose such learning is useful" (2012, p. 5). Although micro-level research is useful, others (e.g. Pais & Valero, 2012) have noted a problem with such research focusing on fixated points, in isolation and without reference to broader socio-cultural discussions.

By conceptualising mathematics education at a system level, it is possible to draw connections across and find meaning in specific aspects of the curriculum. Given the macro focus of this research, it is therefore important to consider the socio-political climate during the 2008-2016 period and how this may have shaped the intended mathematics curriculum in Victoria.

Indeed, as explained by Wong, Zhang and Li (2014) in reference to international testing programmes, “improving the position of one’s country/region in the ‘international league table’ is still a major goal in the current trend towards mathematics curriculum reform” (p. 608; see also Brown, Hodson & Smith, 2013; Pais & Valero, 2012). Others, such as Savage (2016), have argued that Australia’s national education reforms have been driven by global economic needs and labour productivity. These positions highlight mathematics education as being the means for developing economic strength and international competitiveness, as opposed to existing for the primary benefit of its recipients: students (see also Ditchburn, 2012; Swan, 2014).

Within Australia, each state and territory has historically held an autonomous and unique approach to the framing and operating of education. As such, nationalisation measures across the education system since the 1970s have sat alongside ongoing tensions in regards to state autonomy, combined with a blurring of lines of responsibility, as exemplified by the establishment of ACARA and subsequent development of the Australian Curriculum (AC) (ACARA, 2012). Where formerly states had had control over the strategic direction, writing and implementation of their own curriculum documents, an interplay of voices have since become involved.

In the case of mathematics curriculum documentation, this has resulted in the nationally developed AC being re-badged and tweaked by some states, including Victoria, despite the inter-jurisdiction agreement that took place to produce it. In 2016, a state-wide version entitled the Victorian Curriculum was launched. As with other subject curriculum documents in Victoria, the VCM provides a source of guidance and consistent reference point for understanding student development.

METHODOLOGY

Given that no well-defined frameworks seemed to be available with which to conduct this analysis, a grounded theory approach to data analysis was adopted (see Glaser & Strauss, 1967). Grounded theory describes both a methodology and a set of processes that are undertaken by the researcher, with the aim being "to find a core category, at a high level of abstraction but grounded in the data, which accounts for what is central in the data" (Punch, 2009, p. 205).

Data Source, Collection and Analysis

Two subsections of the VCM – the Introduction to the curriculum (VCAA, 2016, pp. 4–7) and the Content Descriptors within the Number and Algebra strand across the levels Foundation to 10A (pp. 8-68) – were analysed. As with the work of Kilpatrick, Swafford and Findell (National Research Council, 2001), the subsections were used "to illustrate what might be done throughout the curriculum" (p. xv).

Using a content analysis process, and informed by grounded theory, textual data relevant to the three Research Questions was systematically collected from the VCM subsections. This process involved three stages of coding – open, axial and selective coding – which progressively pulled apart and made meaning of the data collected. Collected data was not compared to a pre-existing theory or framework, but to itself.

RESULTS

Open coding for the VCM Introduction resulted in 11 categories, which relate to broader aims of mathematics and the structure of learning. These categories were then clustered into three themes during axial coding. Theme A1, entitled 'benefits for individuals', consists of the categories: appreciation of mathematics; benefits for individuals; capacity for problem solving; further study; and other disciplinary benefits. Theme A2, 'benefits for society', includes one category of the same name. Theme A3, 'structure of the curriculum', includes the categories: curriculum as a continuum; curriculum caters to diverse learners; expectations for learning; learning happens in different ways; and structured nature of the curriculum.

For the VCM Content Descriptors, open coding resulted in 13 categories, all of which are verbs to describe student action when learning mathematics. These categories were then clustered into five themes during axial coding. Theme B1, entitled 'concept development', consists of the categories: express and represent; introduce and define; and recognise. Theme B2, 'concept comparison and classification', includes the categories: compare and classify; and connect. Theme B3, 'procedural thinking', includes: apply; find and identify; graph; solve; and use. Theme B4, 'higher order thinking', includes: analyse; and create. Theme B5, 'confidence' includes a single category: develop confidence.

Theme names were developed by considering the connections between categories in that theme. As an example, the categories in Theme B1 underlie concept development. This includes the introduction and initial understanding that students develop about mathematical concepts, where being able to define and represent a concept leads to recognition of that concept. Categories in this theme precede the conceptual work involved in Theme B2.

Selective coding resulted in the creation of a core category to encapsulate the previous stages of coding. The core category was based on meaning underlying the categories and themes for both the Introduction and Content Descriptors, and is as follows.

The VCM frames mathematics education in two ways, such that there is a discrepancy between the messaging of the Introduction and the Content Descriptors. The Introduction focuses on the broad benefits of mathematics education to individuals and to society. It further acknowledges that there is value in having a structured curriculum, with this structure presented as a means for supporting the diverse needs of students. The Content Descriptors include a range of levels of thinking, however with a focus on skills related to procedural thinking and conceptual development. So, while axial coding themes are connected and to an extent build on one another, this same interconnectedness is not found in how the themes are distributed across individual content descriptors.

DISCUSSION

As the core category for the VCM has revealed, the intended mathematics curriculum in Victoria consists of competing perspectives. This may be indicative of the mediating influence noted by Remillard and Heck (2014) of "social, political, cultural [or] structural" (p. 714) factors that are present during curriculum development. Nonetheless, it is of concern that a prominent and highly utilised document such as the VCM contains contradictory messaging and is unclear in its articulation of the role of mathematics education. Indeed, themes from each section of the VCM are not easily synthesised and are presented below as if derived from distinct documents.

RQ1: The Purpose of Mathematics Education

So how is the purpose of mathematics education articulated across the VCM? The 'concept development', 'concept comparison and classification' and 'procedural thinking' themes (B1, B2, and B3 respectively), which emphasise the attainment of discrete skills and concepts across multiple levels, indicate that the Content Descriptors present mathematics as a fixed body of knowledge. The remaining themes' emphasis on higher order thinking (B4) and confidence (B5) show that mathematics education also includes opportunities to develop more complex skills and a positive disposition to learning. Mathematics education, in this way, exists for the development of individual students' knowledge, skills and dispositions towards an end-point that is defined by the curriculum.

While the Introduction also underscores the importance of mathematics education for students, it does two other things. Firstly, as indicated by Theme A1, mathematics is perceived as beneficial for students, not just within mathematics classes, but also more broadly. Secondly, Theme A2 suggests that mathematics education is advantageous for society due to its "fundamental role in… enabling and sustaining cultural, social, economic and technological advances" (VCAA, 2016, p. 4).

It is therefore apparent that the Introduction espouses a vision of mathematics education, whereby its purpose exists beyond the immediacy of the classroom and even the student. Thus, becoming educated and gaining mathematical knowledge is not the end-goal. Rather, mathematics education has been discursively positioned as a tool for a better life and societal outcomes (see Pais & Valero, 2012). This element of the intended mathematics curriculum in Victoria sits in conflict with the narrower purpose outlined by the Content Descriptors.

RQ2: How Mathematics is Expected to Be Learned

Across the VCM, two elements of learning were evident from the analysis: there is complexity to student learning needs, and to the content of what is then learned. The 'structure of the curriculum' theme (A3) in the Introduction and the structure of the VCM document itself highlight that diverse mathematical learning needs exist amongst students and that all needs should be catered to. Within the Content Descriptors, use of the term 'level' rather than 'year' to demarcate stages of progress implies that students are not expected to learn content that is matched to their year level. It is viable, for example, that a Year 6 student is learning Level 4 or even Level 10 content. Moreover, the inclusion of additional content, namely Level 10A at the upper end of the curriculum, is evidence that the learning trajectory for all students is not regarded to be the same. As with education policy-makers, curriculum-writers and researchers since the late twentieth century who have embodied the 'mathematics for all' discourse (see Pais & Valero, 2012), the VCM has suggested that all students can learn and have success in mathematics.

Despite this equity viewpoint, a one-to-one correspondence between the curriculum levels and school year levels is visible, rendering additional content as an add-on and deviation from the norm. The category breakdown of Theme A3 further highlights that set expectations for learning exist. So, in contrast to its messaging, the VCM via its structure appears to espouse a fixed notion of how student learning should progress. The onus thus rests with teachers to identify the need for – and implement – any differing levels of support for students beyond the year/curriculum level they are at. Similar concerns have been noted (e.g. Atweh, Miller & Thornton, 2012) in regards to the AC and the implications of its near-identical structure.

Within the VCM, the complexity of mathematical learning is highlighted by the open coding category 'learning happens in different ways' (see Theme A3). This is reinforced by the Content Descriptor themes 'concept development, 'concept comparison and classification', 'procedural thinking' and 'higher order thinking' (B1 to B4), which exist across the trajectory of the Number and Algebra strand.

Where the Introduction and Content Descriptors differ, however, is in the relative importance placed on different ways of learning. The Introduction has deemed four proficiencies – understanding, fluency, problem solving and reasoning – to be of equivalent value, noting that they "are fundamental to learning mathematics and working mathematically" (VCAA, 2016, p. 6). In contrast and as outlined by the core category for the VCM, there is an uneven distribution of Themes B1 to B5 across the Content Descriptors. Predominant weighting is given to conceptual understanding (B1 and B2) and procedural fluency (B3) over higher order thinking (B4) and confidence (B5). As such, the potential complexity and nuances that underscore mathematical learning have not been coherently carried through in the Content Descriptors.

Problematically, previous research (e.g. Burkhardt, 2014) has found that problem solving and higher order thinking are under-emphasised in Australian classrooms. Rather than addressing this issue, the VCM reinforces it, with the imbalance of skills in Themes B1 to B5 not made clear for teachers and others using the document. Thus, while Theme A3 of the VCM presents an intention of mathematical learning as rich and multi-faceted, much of this work is left to teachers in their implementation.

RQ3: How Mathematics is Expected to Be Taught

Across the VCM subsection used for analysis, the sole reference to teachers is in the Introduction:

> The curriculum sets out what students are expected to learn and is designed as a continuum of learning. The curriculum is being presented in a scope and sequence chart to support teachers to easily see the progression and assist in planning. (VCAA, 2016, p. 7)

The phrasing here suggests that successful curriculum implementation is achieved by the transfer of knowledge from the 'scope and sequence', also known as the Content Descriptors, to teacher to student. As the core category outlines, by adhering to this guidance, teaching practice would then emphasise the haphazard development of conceptual understanding and procedural thinking, over skills requiring higher level thinking.

Furthermore, the absence of explicit reference to teachers places little value on the development of pedagogical content knowledge and best practice. As Swan (2014) has argued, this exclusion "dodges our responsibility to help teachers apply the wisdom of research to daily practice" (p. 632). For new or out-of-field teachers or even for experienced teachers who are grappling with the diversity of students in their classroom, separating what should be taught from how presents a serious challenge.

CONCLUSION

Mathematics teaching and learning does not operate independently of the curriculum intent. Through analysis of a component of the intended mathematics curriculum in Victoria, this research has demonstrated that while there are consistencies in the priorities underlying the VCM, contradictions also exist. The divergent messaging that this communicates is thus problematic for those using the VCM as a source for developing a mathematics teaching and learning program. Given this context, it may even be expected that a similar phenomenon exists for the entirety of the VCM, including the 'Measurement and Geometry' and 'Statistics and Probability' strands.

The findings of this research present implications for those supporting teachers with mathematics curriculum implementation, including academics, school leaders and private service providers. Specifically, there is a need for professional development to focus on effective pedagogies underlying curriculum implementation, particularly in regards to areas that are typically challenging to teach.

Finally, this research has broader implications for curriculum writers of mathematics and other disciplines. The inconsistencies within the VCM emphasise the importance of reflecting on the perspective and approach taken during curriculum development. Curriculum documentation must be written in recognition of its position in the curriculum system: as an authoritative statement of what is valued in education and a source of guidance for teaching and learning.

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TALKING ABOUT CONCEPTUAL KNOWLEDGE: CASE STUDY ON CHALLENGES FOR STUDENTS WITH LOW LANGUAGE PROFICIENCY

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More and more research in mathematics education points to the importance of the discursive level of language for the learning of mathematics. The presented study theoretically builds on combining an Interactional-Epistemological Perspective from mathematics education with the linguistic theory of Interactional Discourse Analysis to further investigate the role of explaining and arguing for the learning of conceptual understanding. A qualitative analysis in remediating small group courses on fractions shows that students succeed in describing observations but need a lot of support for accomplishing tasks on explaining meanings and connections.

Large quantitative studies show repeatedly that students with higher language proficiency reach higher results in mathematics tests than students with lower language proficiency (Haag, Heppt, Stanat, Kuhl, & Pant, 2013; OECD, 2007). The role of language is a large field in mathematics education research that is considered from various perspectives (Barwell et al., 2016). Research focusing on the question how to foster language learners in mathematics classrooms point to the importance of considering the discursive level of language that goes beyond word and sentence level (e.g. Barwell, 2012; Moschkovich, 2013). The presented study aims at further unfolding how the learning of conceptual knowledge (Hiebert, 1986) is connected to the discursive level of language and to what extend students need help in accomplishing the challenging task of talking about meanings and connections. For this, theoretical background on the definition of discourse practices and their role in mathematics classrooms is presented, followed by presenting the research methods. The intertwined mathematical and linguistic challenges are exemplified in the empirical part by two excerpts from a remediate small group course on fractions in grade 7. The paper ends with a discussion of these insights and an outlook on further research.

THEORETICAL BACKGROUND: THE ROLE OF DISCOURSE PRACTICES IN MATHEMATICS CLASSROOMS

There is a broad consensus in mathematics education about the importance of participation in high level mathematical and linguistic practices for all students but especially for students who are still acquiring the language of instruction (e.g. Moschkovich, 2013). The basis of this consensus is a *participationist perspective* on mathematics classrooms (Sfard, 2008; referring to Vygotsky, 1978) that

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 321-328. Singapore: PME.

conceptualizes learning mathematics as "a process of enculturation into mathematical practices, including discursive practices (e.g., ways of explaining, proving, or defining mathematical concepts)" (Barwell, 2014, p. 332). In order to learn more about the role of discourse practices in mathematics classrooms, the Interactional-Epistemological Perspective (IPE) from mathematics education is enriched with the linguistic theory of Interactional Discourse Analysis (IDA) as introduced in Erath, Prediger, Heller, and Quasthoff (submitted). IPE focusses on knowledge construction and epistemic participation in the interaction of classroom microcultures (cf. Yackel & Cobb, 1996) and therefore investigates interactive processes while systematically bearing the subject in mind. Enrooted in the same ethnomethodological tradition (Garfinkel, 1967), IDA provides theories on discourse practices and discourse acquisition that facilitate a deeper understanding on the intertwinement of learning mathematics and language. For the presented study, especially the distinction between different discourse practices (e.g. explaining, arguing, narrating, reporting, describing) is used: Following IDA, discourse practices are defined as interactively co-constructed, contextualized pattern in a speech community (e.g. a mathematics classroom microculture) that are functionally oriented towards particular genres (Bergmann & Luckmann, 1995). This means that different discourse practices are distinguished by means of different 'problems' they solve in a speech community. For example, explanations solve the problem of conveying and constructing knowledge while arguments serve to negotiate divergent validity claims.

This way of differentiating discourse practices reveals why the practices of explaining and arguing have a special position in (mathematics) classrooms since these discourse practices meet the tasks of school: Knowledge needs to be constructed, connected, and demonstrated as well as for example different ways of thinking need to be argued. The theoretically expected dominance of explaining and arguing in classrooms can be empirically confirmed by a study on grade 5 whole class discussions in Germany (Erath et al., submitted): Explaining is the most frequently identified discourse practice in the observed mathematics classrooms followed by arguing and reporting on solution pathways. Furthermore, the interdisciplinary team points out that the linguistic demands rise with advancing in the process of knowledge construction. This means in particular, that talking about meanings, mental models, connections and other aspects of conceptual knowledge (Hiebert, 1986) is challenging for all students but especially for those with low language proficiency. But how do students with low language proficiency and teachers deal with situations in which talking about meanings and connections is necessary? Did they develop ways of bypassing the linguistic challenges? These questions are further investigated in the presented study by working on the following research questions:

Q1: Which discourse practices are used to talk about meanings and connections?

Q2: To what extend do students need support to participate in discourse practices linked with mathematical meanings and connections?

METHODOLOGY AND METHODS: QUALITATIVE ANALYSIS OF MODERATED SMALL GROUP PROCESSES

Methods of data collection. The presented data is part of the larger intervention study MESUT in which students' conceptual understanding on fractions was fostered by means of stimulating discourse related to 'pushed output' and 'relating registers' (cf. Prediger & Wessel, submitted). 186 German grade 7 students (aged around 13 years, mainly from underprivileged urban quarters, with comparatively low language proficiency, and weak in mathematics) worked in groups of 3 to 6 students for 5 lessons together with a teacher (researchers and trained student assistants). The analysis of this paper is based on video data and written products of 9 groups.

Methods of data analysis. For this study, 4 tasks in Lesson 2 and 3 were selected for transcription. Selection criteria was that they had shown a high potential for eliciting discourse practices in a first rough analysis of the video material. The transcripts were intensively analyzed in the tradition of discourse analysis as introduced in Erath et al. (submitted): First, all episodes were identified in which an explanation or argumentation became necessary in the interaction. In a second step, these sequences were analyzed guided by the following questions: (1) which mathematical aspects were dealt with (meanings, connections, representations, procedures …)? (2) How do students verbalize their explanations and argumentations? If not successful: which discourse practice do they choose instead or do they switch to single words or half sentences? (3) How does the teacher guide and help the students in accomplishing the task? In this paper, translated and simplified versions of the transcripts and the related task (Figure 1) are printed.

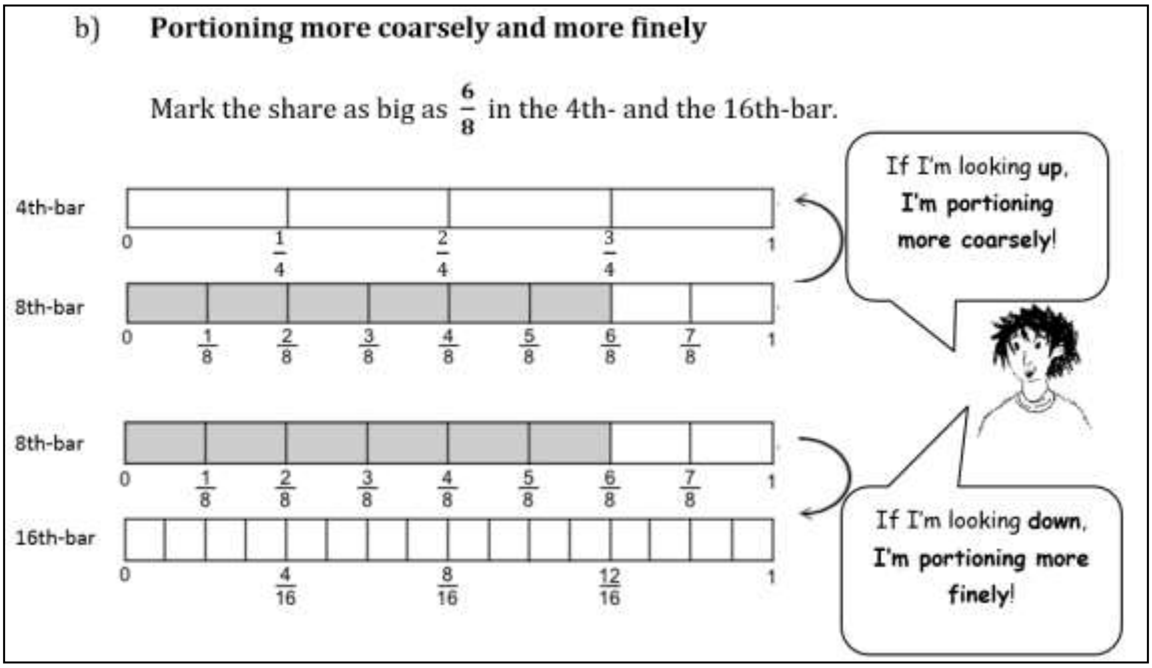

Figure 1: Task on portioning more coarsely and more finely

EMPIRICAL RESULTS: EXPLAINING MEANINGS AND CONNECTIONS AS MATHEMATICAL AND LINGUISTIC CHALLENGE

Two episodes from different groups are presented as examples in this section. Previously, students were working on comparing shares in the context of downloading movies and deciding equitably who scores best in a football competition. The fraction bar (see Figure 1) is an important representation that is used systematically in the intervention. In the phase of systematizing, students are working on the task printed in Figure 1 and are asked to explain what is meant by the meaning related expressions "If I'm looking up, I'm portioning more coarsely!" and "If I'm looking down, I'm portioning more finely!" with reference to the fraction bars. Therefore, the episodes focus on a crucial and challenging moment in the process of knowledge construction: The task aims at talking about the new mental model of finding equal shares (portioning more coarsely/finely) by referring to the familiar representation. This especially means, that it is necessary to verbalize connections since the bars must be considered comparatively.

Episode 1: Lowering linguistic and mathematical demands

Group H-BP consists of four students (three male, one female) from a lower secondary school. The teacher (Te1) and the students Dennis (Den) and Rahmiye (Rah) are speaking in Episode 1, starting after the teacher asks the students to explain the expression "If I'm looking up, I'm portioning more coarsely!":

22	Den:	I'd like to say something else
23	Te1:	W, What would you like to say?
24	Den:	The numerator has divided here, here is written eight and there four
25	Te1:	The numerator?
26	Den:	Or the denominator, no idea, down there, I don't know what it's called
27	Te1:	The denominator
28	Den:	Yes

In the first part of Episode 1 (#22-28), Dennis describes his observation of 'divided' denominators and succeeds in producing a description (#24), even though with a wrong term which is corrected by the teacher. In the second part (#29-38), the teacher builds on this observation by asking the students to connect it to the representation (#29/32/36) and thus making an explanation necessary that draws connections between denominators and fraction bars as well as between the two considered fraction bars. This seems mathematically and linguistically challenging for the students:

29	Te1:	Exactly, and what..the denominator um divided by two, right so in half, and what does it with the bar? [*points to the bar in the task, 8 sec. break*] whereby do I see at the bar that down here, the denominator is eight and above four
30	Den:	Because above#
31	Rah	#It doubles

32 Te1: Yes what doubles? Explain it

33 Rah: The denominator

34 Te1: The denominator from, from top to bottom it doubles, well, and how do I see it at the bar?

35 Den: Because the pieces are larger

36 Te1: Exactly, how many pieces and this is related to your doubling or halving, how many pieces are then one piece? If I'm looking the bottom up

37 Rah: Two

38 Den: Two

Rahmiye (#31/33) rephrases the observation without reference to the representation, Dennis (#35) comparatively refers to the size of the pieces but without explicitly drawing a connection between the two fraction bars and the denominators. The teacher supports the students in their verbalizations by asking clarifying questions that can be answered by single words (#36) and therefore simplifies the task linguistically (and at the same time lowers the mathematical requirements). This, as well as making explicit that an explanation (and not a description) is needed (#32) and constantly building on students' previous utterances, helps the students to remain active parts of the conversation. However, they do still not succeed in producing the pursued explanation. At the end (#39), it is the teacher who merges the students' observations in an explanation and makes the connections between the size of the pieces in the fraction bar and the meaning related expression of portioning more coarsely:

39 Te1: Correct yes, two pieces, make one lar um, make one large and this means Kenan with 'if I'm looking up I'm portioning more coarsely' […]

On the one hand, the teacher's explanation can be interpreted as keeping the students back from further attempts and hence learning opportunities regarding explaining meanings and connections. On the other hand, it can be seen as supportive since it can serve as a model for an explanation that also shows the expectations of the teacher.

Episode 2: Preserving linguistic and mathematical demands

Group M-SP consists of four students (two male, two female) from a middle secondary school. The teacher (Te2) and the students Cemil (Cem), Lorik (Lor), and Lisa (Lis) are speaking in Episode 2, starting after the teacher asks the students to explain the meaning related expressions printed in the task (see Figure 1):

74 Cem: fine is, well, small and more coarsely is large

75 Lor: Small but fine

76 Te2: [*To Cemil*] How do you mean that exactly?

77 Cem: Like just#

78 Te2: #Can one describe that again differently?

79 Cem: Yeeeeeeees, difficult [*smiles*]

In the first part of Episode 2 (#74-79) Cemil renames more finely and coarsely as small and large which the teacher asks him to explain in more detail, probably because small and large do not refer to comparatives (as smaller and larger). Also the students in this group struggle with verbalizing the required explanation as explicitly stated by Cemil in #79 while succeeding in describing observations regarding the printed shares: The second part of Episode 2 (#80-83) deals with Lisa's observation that the same share can be expressed differently:

80	Lis:	[*raises her hand*] ..But I know how it is correct, because here the share is written [*points to the task*] so 6/8 and with the others it's not written, it was then portioned by um different fractions, so ¾
81	Te2:	Mhm so I can express the shares differently, right?
82	Lis:	Yes

This is marked as correct by the teacher who nevertheless navigates back to Cemils utterance from the beginning. The teacher in this group does not lower the linguistic and mathematical demands, but continuously asks clarifying questions that still require answers longer than half sentences (#76/78/83).

83	Te2:	That's [true] in any case, and what was that with the more finely and more coarsely again?#
84	Cem:	#Yes for example this one here, um, which is larger [*points to a piece in the bar of 4th*] and that the finer, that two fit in there
85	Te2:	Ah okay! So this one piece becomes two, if I portion more finely and the other way round, if I make one out of two?

In the third part of Episode 2 (#83-85), Cemil succeeds in producing an explanation that compares the two fraction bars and verbalizes the connection between the sizes of the pieces backed by deictical means. The teacher amplifies his explanation by connecting it to the expressions more coarsely/finely given by the material.

DISCUSSION AND OUTLOOK

The two episodes are examples of phenomena that can be recurrently observed across the five sessions of the intervention as well as across different groups. Talking about meanings and connections in mathematics is (from a subject perspective as well as from a linguistic perspective) challenging for the students. The latter is even addressed explicitly by the students: Dennis (Episode 1, #26) struggles on word level, Cemil (Episode 2, #79) admits the difficulty in enhancing his utterance towards an explanation. Furthermore, both episodes show how the teachers try to support their students in order to help them verbalize their ideas as well as guide them towards the mathematical learning goal. The analysis thus shows once again how language and subject matter cannot be seen as separated. Furthermore, Episode 1 demonstrates how students learn vocabulary while participating in discourse practices.

Research question Q1 on which discourse practices are used for discussing connections and meanings in mathematics classrooms can be answered by pointing out that the mathematical learning goal of understanding the meaning related expression on finding equal shares and connecting it to the familiar representation was only addressed by explications in the presented episodes. The offered descriptions were used as starting point by the teacher in Episode 1 and were marginally approved in Episode 2, but these discourse practices did not succeed in verbalizing the connections between the representations and between the fraction bars and the meaning related expression. Thus, the theoretical and empirical importance of explaining and arguing (see above) is affirmed and shown on a micro level in this study.

The data shows that students need help to participate in the challenging practices of explaining and arguing in mathematics (research question Q2). Both teachers have to invest time and several moves in order to end the sequence with an explanation that meets the task. The two presented teachers deal differently with this challenge and of course adapted to their group and the ongoing interaction. From the data, we cannot conclude if staying on the initial high mathematical and linguistic level (e.g. also recommended by Moschkovich, 2013) would have also led to students' explanations in Episode 1. However, since in both episodes, students offer descriptions, it can be assumed that it might be effective to find a way of keeping the linguistic and mathematical demands (i.e. staying on a discursive level) and at the same time jointly converting students' descriptions to explanations. The question how this might work must stay unanswered here, but points to possible further research.

So far, we know little about teacher moves that support students on the level of discourse practices (cf. Erath, submitted). More research is needed to identify ways of enabling all students to actively participate in demanding discursive and at the same time mathematical practices since they are closely linked to important learning opportunities. Given the identified connection between explanations and argumentations and working on conceptual knowledge, these practices need to be considered in particular. This is further supported by first quantitative results (based on 9 groups) from the project MESUT (Nienhoff, 2017): The higher the share of explanation and argumentation in the total time of group discussion (excluding individual seatwork or working in pairs) the higher the gains in mathematical achievement. At the moment, this analysis is expanded to more groups. Nevertheless, it hints at a first quantitative support of the qualitative observation that including all students (especially the vulnerable) in these challenging discourse practices in particular means to offer important mathematical learning opportunities.

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STUDENTS' CLUSTERS OF CONCEPTS OF QUADRATIC FUNCTIONS

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This study characterizes children's understandings of quadratic function from a variation perspective in the context of a quantitatively rich instructional setting. We studied middle grade (ages 12-13) students' conceptions during a small-scale teaching experiment aimed at fostering an understanding of quadratic function as a growth situation with a constant difference in rate of change. We report five clusters in student thinking: (a) variation without coordination, (b) qualitative and/or implicit coordination of quantities, (c) explicit coordination of change in quantities, (d) attending to how change in quantities depend on other quantities, and (e) dependency relations and their symbolizations. This work contributes to an understanding of what students' rich conceptions of functions can be.

INTRODUCTION

Why is research on students' conceptual understanding of quadratic functions from a rates of change perspective important? Students often learn about quadratic functions through a method of finite differences, or by examining the steepness of a parabola through graphing (Ellis & Grinstead, 2008). In a parameter graphing approach there is often limited attention given to the meaning of the function rule—instead, the function rule is introduced as a way to connect symbolic, graphic, and numeric representations (Zbiek & Heid, 2008). Across these approaches, learning function can be introduced as calculation in the dependent quantity alone, or translation across representations, which can occur to the detriment of developing a richer conception of function as a relationship of coordinated change (Thompson & Carlson, in press).

We seek to address these issues by contributing to an understanding of the nature of supports for students' learning and the nature of students' learning processes in the context in which it was supported (Simon & Tzur, 2004). We situate our work within the paradigm of design research (Cobb et al., 2003), with the aim to demonstrate qualitative differences in students' changing conceptions of quadratic growth as they interacted with tasks to develop an understanding of function as a representation of co-varying quantities.

BACKGROUND AND THEORETICAL FRAMEWORK

A learning trajectory of children

The notion of a hypothetical learning trajectory—"the learning goal, the learning activities, and the thinking and learning in which students might engage" (Simon,

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 329-336. Singapore: PME.

1995, p. 133)—is a tool for supporting meaningful teaching and learning from a constructivist tradition. Hypothetical learning trajectories are hypotheses of intended students' learning and targeted supports for shifts in students' conceptions relative to mathematical goals and task-based activity. Others have elaborated the notion of a learning trajectory to include a path of students' thinking as they interact with tasks during instruction (e.g., Clements & Sarama, 2004). In this study we focus on retrospective analyses of children's activity during goal-directed instruction. Consistent with Steffe (2004) and as elaborated in Ellis et al. (2013), we take a learning trajectory of children to include students' concepts, observable changes in those concepts, and the tasks that students solved as part of the learning situation. We inferred students' conceptions from their speech, writing, gestures, and the representations they created and interpreted when solving problems.

Learning functions: Rates of change

In the symbolic forms of functions, a correspondence perspective is often championed wherein a function is conceived of as a relationship that maps an independent quantity to exactly one dependent quantity. An alternative way to introduce function is through covariation or coordinated change. Confrey and Smith (1994) introduced the notion of coordinated change in linked quantities as a sequential pairing of change from x_m to x_{m+1} and y_m to y_{m+1}.

Fonger, Ellis, and Dogan (2016) identified students' modes of reasoning about functions that supported students' abilities to symbolize function rules. They found that students conceived of function rules as representing both relationships of correspondence and relationships of coordinated change. In this paper we build on that initial study by identifying how students' conceptions develop over time in relation to their interaction with tasks during instruction.

METHODS

Teaching experiment and context of learning

We conducted a 15-session teaching experiment (TE) (Steffe & Thompson, 2000), each session lasting 1 hour, with 6 middle grades students (3 male and 3 female, ages 12-13). Author3 was the teacher-researcher (TR) for all sessions. We used this methodology to gain direct experience with students' conceptions of quadratic functions and how those conceptions changed over time; we tested and modified our hypotheses of students' learning in real time while engaging in teaching actions. Thus, we created and revised new tasks on a daily basis in response to hypothesized second-order models of the students' mathematics (Steffe & Olive, 2010). It is important to note that none of the students had experience with quadratic functions at the start of the TE, but had studied linear functions in their mathematics courses.

The TR fostered a learning environment that encouraged students to make and test conjectures, to make predictions and generalizations, and to explicitly attend to quantities and their relationships. All tasks were grounded in a growing rectangle context in which the height and length grew in a manner that maintained the proportional relationship between the sides. In this context, $y = ax^2$ expresses the area of a rectangle (y) as a function of the height (x), in which a is the ratio of the length to the height of the class of proportional rectangles. Tasks and instructional supports encouraged students to reason with the quantities height, length, and area. Many of the tasks involved creating tables of height, length, and area values and to identify trends and patterns in the tables. Thus, several conceptions described below refer to the students' attention to these changing quantities as instantiated in tables of coordinate pairs (or triples). Students were encouraged to refer to specific quantities and images of the growing rectangle context, as opposed to number patterns devoid of context. The participants often worked individually or in pairs before sharing with the group, and were encouraged to explain their reasoning and justify their solutions.

Data Analysis

All sessions of the TE were transcribed and enhanced to capture both the students' and TR's verbal conversation, as well as drawings on the board, student's written predictions prior to discussion, and their gestures while talking. Pseudonyms were assigned to all participants. We followed a constant comparative method of analysis in that codes were discerned from our analysis of the data, not from an a priori scheme (Glaser & Strauss, 1967). Each co-author independently analysed each day of the TE before discussing and reconciling code decisions and subsequently revising the emerging coding scheme. When coding differences occurred we refined, omitted, or added new codes; these discussions also catalysed the organization of relationships among code categories. After coding all 15 days of the TE, we reached a stable coding scheme in which no new codes emerged.

FINDINGS

In this section we elaborate five clusters of students' conceptual development as evidenced by their distinct ways of thinking while interacting with tasks in the aforementioned learning situation. For each cluster we highlight related conceptions, a sample task, and a data example.

In Table 1 we present the first cluster of student conceptions related to variation in single or multiple quantities. *Single-quantity variation* conceptions were prominent for students interacting with tasks in which the independent variable, height, changed by 1 (i.e., ΔH=1). For example, Jim focused solely on describing the second differences in the area of a growing rectangle as "going up by 18s" without attending to how the rectangle's height was growing. Some students began attending to variation across multiple quantities as isolated patterns, conveying a conception of *uncoordinated variation*. This was particularly prominent in far prediction tasks for which the height grew by more than 1 (i.e., ΔH>1). For example, Ally conceived of the difference in length as "going up by eight" and noted that "It's going up by 2s (in height)" yet did not attend to how the two quantities changed together.

Concept Definition	Sample Task and Data Example
Single-quantity variation. Student attends to a change in one quantity without coordinating this change with any other quantity change.	Create a table relating height and area of a proportionally growing rectangle; describe the patterns you notice. Jim: "This one is going up by 18s."
Uncoordinated variation. Student attends to a change in more than one quantity without coordinating simultaneous change; variation is treated as isolated sequences.	Given a table for the height versus the area, find the area a) when the height is 82; b) when the height is *x*. Ally: "I figured out it was going up by eight (in length)".

Table 1: Students conceive of single or multi-quantity variation without coordination.

The second cluster is that of *qualitative and/or implicit coordination*. Specifically, we found that students initially conceived of coordinated change in quantities from either a qualitative or implicitly quantified stance (Table 2) before they were able to describe explicit coordinated changes (see Table 3). For example, Daeshim explained, "If length were growing, area will be bigger," which is a qualitative description of both length and area becoming larger without quantifying how much larger. For implicit quantification, we discerned students' conceptions relative to growth or a difference in the rate of growth in area (i.e., 1st or 2nd differences, respectively). For instance, Jim conceived of the rate of growth in area as an implicit coordinated change, which he described as "how many new squares it's gaining every time it grows." Using the phrase "every time" leaves the change in the height implicit rather than explicitly quantified (for instance, by identifying the change in height as a particular value, such as 5 centimetres). In another example, Bianca expressed the difference in the rate of change of the area as "the area of the amount added to the previous area." Bianca understood that the second differences in area were linked to the first differences in area, yet both the amount of area and the change in area remained implicit.

Concept Definition	Sample Task and Data Example
Qualitative coordination of change in two quantities. Student links the change in two or more quantities, understanding that they change together, without quantifying the change.	What happens when the rectangle grows or shrinks? Daeshim: "Well, it's the length times height. If length were growing, area will be bigger".
Implicit coordination of change in two or more quantities. Students attend to growth in both quantities together, but the magnitude of change remains implicit for one or both quantities.	What does the rate of growth refer to? Jim: "How many new squares it's gaining every time it grows."
Implicit coordination of second differences with change in another quantity. Students coordinate the difference in the rate of growth of the area, but the magnitude of the change remains implicit for one or both quantities.	Identify the rate of rate of change in a picture; what does it mean? Bianca: "It's the area of the amount added to the previous area."

Table 2: Students conceive of qualitative and/or implicit coordination of quantities.

Table 3 presents the third cluster of conceptions, students' *explicit coordination* of linked quantities (i.e., numeric measures of magnitude of change). For example, Daeshim described how the rectangle's length and area grew together by explaining, "Length will be growing 1 centimetre and area will be growing 3 square centimetres." Students also quantified the second differences in area relative to differences in height. For instance, Tai explained, "Yeah, 20 and 5 would both work (for second difference in area) because we're going up by 2s (in height) and they're going up by 1s (in height)." In this example, Tai coordinated a change in height of 1 cm with a second difference in area of 5 cm^2 (likewise for 2 cm and 20 cm^2). We also found analogous student conceptions of explicit coordinated change in which one of the quantities was a multi-unit change (omitted here for brevity). It is also notable that in several cases, the explicit quantification of coordinated change emerged in relation to TR actions that encouraged students to clearly state the quantities and units in question. For instance, when a student would make a statement such as "each time it grows", the TR would press the student to clarify what "each time" meant, such as an increment of 1 cm.

Concept Definition	Sample Task and Data Example
Explicit coordination of two quantities. Student coordinates the change in two or more quantities together, and also quantifies the magnitude of both changes.	Make a table comparing length and area; what pattern do you notice? Daeshim: "Length will be growing 1 cm—and area will be growing… 3 square cm."
Explicit coordination of second differences with change in another quantity. A change in one quantity is coordinated with the difference in the rate of growth of the area.	For a 2 cm x 5 cm rectangle, make a prediction for what the difference in the rate of growth will be. Tai: "Yeah, 20 and 5 would both work (for second difference in area) because we're going up by 2s (in height) and they're going up by 1s (in height)."

Table 3: Students conceive of explicit coordination of change in quantities.

The fourth cluster of students' conceptions are shown in Table 4, which characterizes students' developed recognition of how one quantity changing affects how the other quantity changes—we call this *dependency relations of change*. For example, when describing the difference in the rate of growth of area, Jim said, "You can go like every 5 (in height), so your rate of growth can change no matter what." In this example Jim expressed the idea that a change in height affects how the rate of growth changes, but he did not clarify *how* one quantity's change affected the other. Later in the TE, Jim was able to clarify the nature of this influence, explaining, "But dirog (his term for the difference in the rate of growth of the area) divided by 2 is dil (his term for the change in length). So all you have to find is dil." Jim understood that the increment by which the length (and height) grew determined the second differences in area according to the relation $2 \bullet \Delta L = (\Delta\Delta A)$.

Concept Definition	Sample Task and Data Example
Recognition that change in one quantity determines change in the other. The magnitude of the change of one quantity determines the amount of change in another quantity. Student understands that there is a dependency relation without determining what that relation is.	What is the difference in the rate of growth of the area? Why is that number the difference in the rate of growth? Jim: “If you're going like every other one (in height), you can go like every 5 or whatever, So your rate of growth can change no matter what.”
Student conceives of the second difference as dependent on the other quantities OR on the change in other quantities (*such as* Δx).	Why is the difference in the rate of growth 5 for a 2 x 5 rectangle that grows by 1-cm increments in height? Bianca: “Hey, 2 times the original area of the rectangle equals the dirog.”

Table 4: Students conceive of dependency relations of change among quantities.

The final cluster of conceptions, summarized in Table 5, addresses students' understanding of *dependency relations and symbolized rules*. In this cluster students made sense of the relations between quantities by constructing multiplicative comparisons, and eventually expressing symbolic rules. For example, Bianca expressed a dependency relation between x and y (in this case, length and area) by articulating “The length is a third of the area.” Jim expressed a *quadratic rule* for a rectangle with height n as $4.5n^2$ and explained, “I put n times … 4.5 is your length times n again because … is your area.”

Concept Definition	Sample Task and Data Example
Dependency relation between x and y. Student conveys an understanding of a dependency relation between length, height, and/or area.	Examine the growth of a rectangle on a dynamic sketch; what patterns do you notice? Bianca: “I found that the length is 1.5 times the height.”
Quadratic rule. Quadratic growth can be represented with a function rule.	Here is a table for the height versus the AREA of a rectangle that is growing in proportion: (e.g., Height=2, Area=18, ΔH=2); what is the area when the rectangle is n units high? Jim: “I put n times … 4.5 is your length, times n again…is your area.”

Table 5: Students conceive of dependency relations and symbolize rules.

Returning to the driving mathematical goal of the TE, we also see evidence of students coming to understand quadratic growth as a constant rate of rate of change. For example, in describing the difference in the rate of growth of the area, Bianca explained, “It's going up by 2 between every time it's going up by a different number, so that makes me think that it's going up in a curve because it's like, staired…it's going up by the previous number plus 2.” Bianca has learned to imagine quadratic growth from a rate of change perspective—she understood that the difference in the rate of growth is constant, making the graph of this function curved. This conception is related to students' understanding of linear functions as having a constant rate of change.

DISCUSSION

We advanced five clusters of students' conceptions of quadratic function from a rates of change perspective. These clusters characterize several key shifts in student reasoning. Firstly, students' initial attention to variation was such that they did not coordinate change across quantities. Instead, they could only attend to variation in one quantity at a time, or, at best, they attended to two types of variation but only as a sequence of disconnected changes. The first shift occurred when students began to coordinate change in two quantities simultaneously. Their early forays into coordinated change occurred in a manner that was qualitative and implicit; often, change in one quantity was highlighted and quantified, while change in the linked quantity was backgrounded as something that happened "every time". The second shift occurred when students began to explicitly quantify increments of change in both quantities. Once they were able to do this, the students could then begin to attend to how change in one quantity affected change in the other, which represents the third significant shift in their reasoning. The manner in which this final understanding develops is the subject of ongoing analysis.

These shifts did not occur spontaneously. They were engendered by tasks and instructional actions deliberately engineered to foster and, ultimately, require students to explicitly identify the manner in which two quantities changed together. Effective tasks introduced multiple tables for the same rectangle, in which ΔH changed across the tables, thus forcing attention to the manner in which ΔH affected both ΔA and the difference in the rate of growth of the area. The TR also had students create tables with different increments of growth. When the students argued about their conflicting answers, they realized that it was necessary to attend to how all of the quantities changed together. Non-uniform tables also encouraged students to become more explicit in their attention to multiple changing quantities and to identify the links between them. Our findings suggest that explicit attention to coordinated change is conceptually challenging for middle-grades students, but possible to attain with the proper instructional supports. Further, a rates of change approach to quadratic growth offers a conceptually rich foundation to foster a dynamic understanding of function, which is critical for success in calculus and higher-level mathematics.

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AN INTERDISCIPLINARY APPROACH TO MATHEMATICAL MODELLING IN SECONDARY TEACHER EDUCATION

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The purpose of this paper is to describe and analyse the nature of an interdisciplinary approach to the development of an online learning module designed for secondary mathematics Initial Teacher Education Students (ITES). In developing a module on mathematical modelling, team members crossed both disciplinary and institutional boundaries. Semi-structured interviews were used to gain insight into the perspectives of team members on the collaboration and analysed through the frame of boundary crossing. The analysis revealed the process of collaboration was advantageous in a number of ways but brought with it complexities that required accommodation.

INTRODUCTION AND BACKGROUND

The performance of Australian students in international comparative assessment regimes such as the Programme for International Student Assessment (PISA) and Trends in International Mathematics and Science Study (TIMMS) is a source of increasing concern government, educational jurisdictions and the public at large. For example, across 2003-2015 PISA results, Australia was ranked 20th for mathematical literacy in 2015, down from 19th in 2012, 13th in 2009 and 8th in 2006. Further, PISA results show that 22% of Australian 15 year olds did not meet the international proficiency Level 2 for mathematical literacy – indicative of the level of competence necessary to use mathematics effectively in real-life situations. These results are paralleled by falling participation in mathematics, science and technology in Australia, raising serious questions about Australia's capacity to sustain a knowledge-based economy and society.

In a response to a report aimed at providing a blueprint for turning around such trends (Office of the Chief Scientist, 2012), the Australian government providing funding for a number of initiatives including the Enhancing the Training of Mathematics and Science Teachers (ETMST) scheme (2013-2017). A fundamental principle for the funding of projects under this scheme was that mathematicians, scientists, mathematics educators and science educators be brought together to develop programs aimed at strengthening Initial Teacher Education Students' (ITES) discipline knowledge. This principle was a challenging demand within the Australian context as there was little by way of existing culture related to this type of collaborative activity.

Under the umbrella of the ETMST scheme, *Opening Real Science: Authentic Mathematics and Science Education for Australia (ORS)* was developed and implemented over a period of 4 years through the support of seven universities and research institutions. The aim of the ORS project was to engage pre- and in-service

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teachers with “real” science and its authentic practice — dynamic inquiry and subsequent action related to real world phenomena. In support of pre-service teachers’ learning, the ORS teacher education program developed 25 on-line learning modules across mathematics and science, eight of which focused on mathematics, that utilised authentic contexts and enquiry-based pedagogical approaches.

The project’s approach focused on student-centred learning, employing problems in which students were genuinely interested, utilising investigative approaches, coupled with scaffolded applications of digital technologies.

The purpose of this paper is to describe and analyse the nature of the interdisciplinary collaboration that was integral to the design and development of the learning module on mathematical modelling – *Modelling the present: Predicting the future*. In attending to this issue we address the following research questions:

- Did the collaboration produce a quality outcome??
- What were the opportunities when collaborating across disciplines?
- Were there any limitations associated with interdisciplinary collaboration?

CONCEPTUAL FRAMEWORK

In his analysis of groups involved in shared practices within and across trades and professions, Wenger (1998) developed the notion of *communities of practice*. Within communities of practice, group members come together for the purpose of a mutual endeavour within which they contribute to each others’ learning by engagement in a common activity. Wenger proposed three dimensions of collaborative pursuit within such communities: mutual engagement, joint enterprise and shared repertoire. He also described different ways of participating within communities of practice:

- Engagement: doing things together, talking, and producing artefacts.
- Imagination: constructing an image of ourselves, of our communities, and of the world, in order to orient ourselves, to reflect on our situation, and to explore possibilities.
- Alignment: a mutual process of coordinating perspectives, interpretations, and actions so they realise higher goals.

(Wenger, 1988)

Communities, by their existence, are defined by boundaries that separate groups of participants and non-participants. Such boundaries can both divide and connect communities (Akkerman & Baker, 2011) but where it is advantageous, members of different communities will seek out opportunity for boundary encounters (e.g., Sztajn, Wilson, Edgington & Myers, 2013). Such encounters represent points at which coordinated and coherent shared action and interaction can be established.

According to Akkerman and Baker (2011), the concepts of *boundary crossing* and *boundary objects* are central to describing the ways in which different communities can engage with learning sharing, coordinated action and gainful interaction.

Boundary crossing refers to the transitions of individuals across communities and their interactions with new and different ideas and cultural norms. *Boundary objects* are those artefacts that act as bridging mechanisms by which a *crossing* is affected. The concepts of boundary crossing and boundary objects are of interest within educational contexts because of the potential for learning at intersections between communities who create and value different types of knowledge.

Suchman (1994) has argued that the term boundary crossing denotes the transition of an expert into an arena in which they are far less qualified. Such transitions have the potential for new learning and the development of new knowledge as those crossing boundaries must bring together their expertise with the unfamiliar knowledge and new ways of knowing and reasoning that exist within the community to which they have transitioned.

Within mathematics and science education, the ideas of boundary crossing and boundary objects have been utilised to analyse one-way transitions of different types including: school to work (e.g. Wake, 2014) and teachers who are required to work “out of field” (e.g., Hobbs, 2013). Additionally, these concepts have also been used to explore bilateral exchanges including: collaborations between educational researchers and teachers in-service (e.g., Goos, 2013); mathematics teacher educators and teachers involved in teacher professional development (Sztajn, Wilson, Edgington & Myers, 2013); and mathematicians and mathematics educators collaborating to strengthening initial teacher educations students discipline knowledge (Goos, 2015). Few, if any studies, however, have investigated how more diverse groups have collaborated on joint endeavors, such as in the case discussed in the sections which follow that involve mathematicians, scientists, mathematics educators and instructional designers.

CROSSING BOUNDARIES TO DEVELOP THE MODULE

Module development was carried out by a team of eight academics with backgrounds including biological evolution, financial mathematics, astrophysics and environmental science as well as mathematics educators with experience in the teaching and learning of mathematical modelling and instructional design. Members of the team either self-identified by responding to an expression of interest distributed to relevant staff (mathematicians, scientists, and mathematics and science educators) of participating universities or were invited on the basis of their expertise.

The process of module development began with introducing team members to the framework used to guide the development of the every module in ORS the Biological Sciences Curriculum Study (BSCS) 5Es Instructional model approach (Bybee, 2009). The 5Es enquiry-based approach to science education consists of five phases: engagement, exploration, explanation, elaboration and evaluation. Each phase has a role in developing students’ understanding of scientific and technological knowledge, attributes and skills (Bybee, 2009). There were then four additional phases consisting of: selection of content, identifying structure, and planning for subsequent phases; initial case study development; draft case study review; and finalisation of the module by linking of case studies. Case studies were based on authentic uses of mathematical modelling.

In order to identify potential case studies the module leader asked members of the Module Development Team (MDT) to talk about their personal research interests and how these were connected to mathematical modelling – to provide ideas about content and to provide opportunity for team members to share aspects of the communities in which they typically worked. Presented topics were diverse and included: evolution and transmission of disease-causing agents (epidemiology), effect of market forces on the stock exchange in relation to investment and risk (financial mathematics), nature of eclipsing binary stars (astrophysics) and impacts of pollution in waterways (environmental chemistry). After a discussion of these topics in relation to the module, the group came to the conclusion that each could be authentically represented as a case study from which students could gain an understanding of the use of mathematical modelling. This decision led to a subsequent discussion of how to organise the case studies within the module in a manner consistent with the 5Es model and within the constraint of 36-40 hours of study over 4-5 weeks allocated for a module. The outcome of this deliberation was agreement that the module would consist of: an introduction; a case study mandatory for all students; a second case study chosen from three options; and a final reflection tied to a capstone assessment. Consultation with, and review by educational designers took into account the views of the larger project team and selected teacher education student representatives in a cycle of review and development.

After the initial meeting, members of the Module Development Team (MDT) worked on developing draft versions of their case studies, some in teams and some as individuals, in collaboration with the instructional designer and the MDT leader. Draft case studies were presented at a second face-to-face meeting so that members of the MDT could provide critique and feedback. Comments and suggestions were accommodated into the existing drafts and then finalised.

EVALUATION OF THE COLLABORATIVE PROCESS

After the design of the module and trial with ITES, semi-structured interviews were conducted with each member of the MDT. The instructional designer, a member of the MDT, conducted six interviews within one month of the completion of the module. Interviews were digitally recorded and later transcribed by an independent researcher for the purpose of analysis.

Interview Protocol

Interviews were based on a protocol developed by the larger project team consisting of three core open-ended questions. Relevant to this report is Question 3 that included response eliciting prompts as set out below:

Describe, from your perspective, the experience of working in a cross-disciplinary team to develop the module as a whole. For example:

- What do you believe was the value in including contributions from different disciplines? Describe advantages/disadvantages.
- Are you satisfied/happy/impressed with the module as an outcome of the collaboration?
- Outline the opportunities/advantages for educators/mathematicians/scientists working together in promoting STEM education.
- Describe any limitation/constraints/barriers for educators/mathematicians/ scientists working together in promoting STEM education.

The interviewer also made use of additional prompts when she saw it necessary to clarify a response or probes when seeking greater depth in a response. Interview duration was between 35 and 55 minutes.

PERSPECTIVES ON THE EXPERIENCE OF INTERDISCIPLINARY COLLABORATION AND DISCUSSION

Participants' transcribed responses were coded through a process of constant comparison (Strauss & Corbin, 1990).) against the research questions and a frame informed by Wenger's ways of participating in a community of practice and the concept of boundary crossing. While not all comments could be categorised against the elements of the model, all noteworthy episodes were documented.

Did the collaboration produce a quality outcome?

Participants were unanimous in their views that the outcome of the collaboration was of high quality:

> Leonard: Yes I'm happy with it, I've spent most of my time looking at the binary stars and looking at the epidemiology. I'm quite happy with them, part of me, the mathematician in me would like to take them both a little bit further mathematically but at the level they're aimed at that would not be appropriate, I think we stopped at the right level.

> Martin: I thought that the end product was fantastic…Whether you naturally attracted to maths or not, and the big problems on this planet, I don't think we can solve outside of, without modelling...We have to model to foresee the future and we are all resource limited.

While most participants indicated they were pleased with the finalised module, they also viewed the product of the collaboration from the perspective of their own discipline – as in the case of Leonard, a mathematician, in the excerpt above, who had to hold back from arguing for the inclusion of more sophisticated mathematics. An exception was Martin who could see the value of bringing aspects of another discipline (mathematics) to her teaching of first year biology; via a collaboration with a mathematician that would complement his expertise as a scientist.

> Martin: You know I think if I do first year Biology, I will also need to bring in the mathematical expertise into it and it's not with me, it will be with someone

that comes and helps me develop the maths behind it. But I know what the context is in which the maths is needed.

Participants’ comments indicate the module had acted as a boundary object that allowed team members from different disciplines to cross disciplinary boundaries, there was an understandable tendency to view the product of their collaboration from the perspective of the discipline in which they were expert. Thus, while boundaries were crossed during the process of module development most developers crossed the bridge back to their own discipline when viewing the final product.

What were the opportunities when collaborating across disciplines?

All six interviewees spoke about the advantages of the problem-based approach that embedded mathematical modelling and situated their disciplinary knowledge and practices in real contexts, satisfying a broader goal of solving real life problems.

Many raised the challenge of knowing enough about other disciplinary knowledge but in some ways saw this as an opportunity rather than a disadvantage.

John: The advantages…being able to use contexts that are really authentic and that they address real problems…[teacher] educators may not be quite okay with some of these current edge scientific problems such as the spread of disease if they haven't got an expert that can really help them inform how they should, or what datasets they should use and how they should be interpreting data.

This comment makes it clear that teacher educators, at least, can be advantaged through the input of discipline experts. Others commented on the usefulness of having a teacher educator’s perspective on the implementation of teaching ideas within science or mathematics a discipline.

James: So, I think we can do with a lot more learning support in academia [refereeing to science and mathematics disciplines] in general. I particularly liked that this module was collaboration, in the full sense, between scientists and educators.

Another interviewee looked at the issue more broadly.

Leonard: There are certainly advantages for people to work together to promote STEM [Science technology Engineering and Mathematics]…I think we should take every opportunity to promote it. If people can work together, then perhaps we can create things that have more depth and breadth.

These comments indicate that there was advantage to both teacher educators and mathematics and science experts by crossing discipline boundaries – both in a reciprocal sense but also for the broader STEM agenda.

Were there any limitations associated with interdisciplinary collaboration?

Participants commented that the pressure to meet discipline-based content outcomes in their teaching limited the way that scientific disciplines worked together let alone looking for synergies with education.

Martin: I think the limitations are if we think too specific and too small and if we go "we don't have room in our curriculum to link across because I need all my time to stuff it full of Biology knowledge".

John: Could this collaboration happen easily and effectively where there is no science or mathematics faculty attached to a university with a teacher education program?

These comments indicate that there are institutional constraints that make collaboration between different communities of practice more difficult. Such restrictions need to be acknowledged and accommodated for it collaborative boundary crossing is a desired outcome.

One respondent expressed concern about how students' would receive the explicit embedding of mathematics in her discipline of environmental science.

Irene: When I first heard about this, I thought, mathematics? Environmental chemistry? Ah, from my experience with dealing with classes both at university, high school and primary school, my experience is generally that the idea of doing the maths would turn students off straight away.

Thus, not only was there risk associated with interdisciplinary collaboration in terms of mapping out new relationships and approaches to teaching but also in how the product of their collaboration was received by end users – their students. This is a reminder that boundary crossing between two communities of practice is not a simple matter as the outcome may influence and have impact on other communities.

CONCLUSION

The means by which the MDT interacted in the development of the online learning module within the ORS was consistent with Wenger's ways of participating within communities of practice. There was *engagement* as members of the MDT worked together to produce an artefact in the form of a module on mathematical modelling. The way in which MDT members represented their own disciplines while exploring the potential benefits (and risks) of interdisciplinary collaboration was consistent with the *imagination* mode of working within a community of practice. The outcome of the collaboration, the modelling module, required *alignment* of perspectives and actions to realise the goal of producing a quality outcome. Thus, representatives of different communities across mathematics science and education came together work as a community of practice for the purpose of a tangible outcome.

While MDT members were unanimous in their view of the high quality of the product of their work together and acknowledged the advantages of interdisciplinary collaboration, they also identified a number of constraints. These included disciplinary demands within their teaching roles that required attention to a large body of content, leaving little opportunity to include aspects of knowledge and practice from other disciplines. Such challenges are reminders of the complexities that must be accommodated when crossing boundaries in search of interdisciplinary collaboration. Thus, if interdisciplinary collaboration is seen as a priority in mathematics and science education, further research is needed into how to best enable the necessary boundary crossings in realising this goal.

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TWO COGNITIVE DIAGNOSIS MODELS TO CLASSIFY PUPILS' ALGEBRAIC SKILLS IN LOWER SECONDARY SCHOOLS

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Cognitive diagnosis models (CDMs) are a relatively new technique to analyse pupils' competencies. Two CDMs are presented to describe pupils' algebraic skills: a stage model to locate a pupil's skill profile on the steps from proto-algebraic thinking to formal algebra and a model to distinguish the use of different aspects of variables. The results are from a representative study with 636 participants in grade 8 and 9, indicating that both CDMs are empirically applicable and leading to some substantial results: Practically no gender differences are detectible; only 63.7% of the pupils reach the formal level of algebra; the learning effects are mainly located in syntactical skills; learning effects related to other aspects of variables (like inserting values or references to objects and real-world situations) remain in the background.

INTRODUCTION: CLASSIFYING INSTEAD OF MEASURING

International comparative studies like PISA have defined the standards of how to investigate and analyse pupils' skills. The methods of choice are based on the item response paradigm, assuming latent metric variables – interpreted as the pupils' skills or competencies – to be the unobservable causes of the pupils' performance. They are measured on continuous scales and their values are used to compare countries, genders, or other subgroups of a study's population (e. g. OECD, 2014, pp. 217ff.).

For comparative studies, a metric approach might actually be the most adequate methodology. This could be different, if no comparative purpose is pursued or if the skills of interest are supposed to have a specific structure that refuses to be measured on metric scales. The latter may be the case, if there is evidence that a skill is of dichotomous nature: You either possess this skill (and then you are able to handle specific tasks) or you do not possess it (and then you are not able to handle the respective tasks). In such cases, it makes no sense to measure skills on continuous scales; it would be more advisable to classify each participant according to the skills he is supposed to have, given the results of his test. Exactly this is the purpose of cognitive diagnosis models (CDMs, cf. Rupp et al., 2010, pp. 31f.): For each item, the test designer declares what skills are necessary to solve it; and having conducted the test, the responses are used to assign individual skill profiles to each participant.

In this paper, I will present two CDMs that were simultaneously used in one test to analyse the algebraic skills of 636 pupils of grade 8 and 9. The test was carried out in the canton of Aargau, Switzerland, in spring 2016. The school system of Aargau is non-comprehensive. It is separated in three different school types, so that type 1

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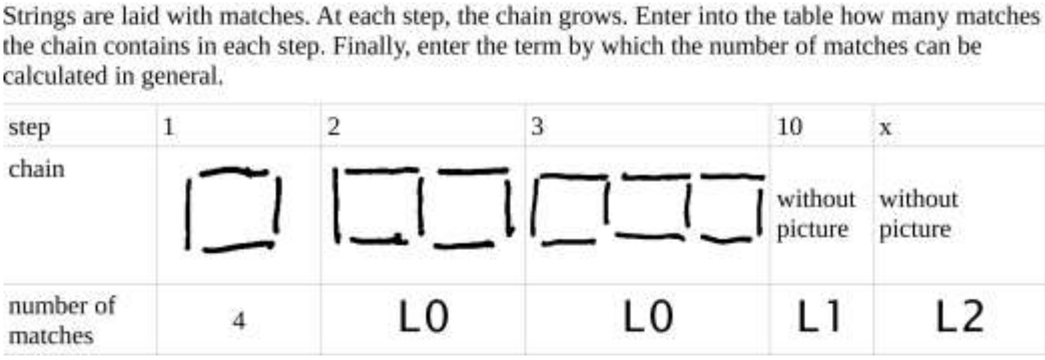
denotes the lowest level and type 3 the highest one (cf. Department Bildung, Kultur und Sport, 2014, pp. 15f.). 33 classes of type 2 and type 3 schools were chosen randomly. The schools of type 1 were omitted, since algebra is no central part of their curriculum. Concerning schools of type 2 und 3, the test is representative. Three kinds of covariates were collected: gender, grade, and type of school.

ALGEBRAIC SKILLS: THEORETICAL BACKGROUND AND TEST ITEMS

Algebraic skills are not restricted to formal algebra. As the debate on early algebra shows (e. g. Kaput, 2008), algebraic thinking emerges, before formal aspects are introduced. The core concept of algebraic is seen in the ability to generalise: "it is the making of general statements that algebra has its key role" (Mason et al., 1985, p. 2). Following this idea, the first bundle of test items was invented to define a stage model to describe the transition from non-formal algebraic thinking to a formal or symbolic stage. The theoretical background is based on the concept of proto-algebraic levels (Aké et al., 2013). The authors introduced six levels of algebraic thinking. Four of them are relevant to school mathematics; the first two level (level 0 und 1) are considered as proto-algebraic; the third (labelled as level 2) and the following ones are described as properly algebraic. Since the first CDM is only focused on the question if pupils have reached the formal level of algebra, it is not necessary to consider the upper three level of this model. The lower three levels where this transition takes place are sufficient. They are described as follows:

- **Level 0:** "Extensive objects, expressed by natural, numerical, iconic or gestural language, are involved. Symbols that refer to an unknown value can also intervene, but that value is obtained as a result of operations on particular objects" (Aké et al., 2013, p. 3).
- **Level 1:** "Intensive objects, whose generality is explicitly recognized by natural, numerical, iconic or gestural languages, are involved. Symbols that refer to the recognized intensive objects are used, but there is no operation with those objects" (Aké et al., 2013, p. 4).
- **Level 2:** "Indeterminate or variables expressed in literal-symbolic language to refer the intensive objects recognized are involved, but they are linked to the spatial or temporal information of the context" (Aké et al., 2013, p. 5).

This stage model is operationalised by three similar items. Each item is based on a real-world situation; the tasks connected to this situation are designed to be answered successively on level 0, level 1, and level 2. One example is shown in figure 1. This item allows answers on all three levels described above (labelled with L0 to L2). According to the nature of a stage model, it is assumed that there is a hierarchical dependency between the levels: Level 0 deals with concrete objects; additionally, you have to understand the situation to answer the questions; level 1 adds an implicit or informal generalisation; and level 2 can be seen as a part of formal algebra. It is implausible that pupils can answer correctly on a higher level, if they were not able to answer correctly on all the lower levels before. But this is only an a priori assumption and has to be validated empirically.


Strings are laid with matches. At each step, the chain grows. Enter into the table how many matches the chain contains in each step. Finally, enter the term by which the number of matches can be calculated in general.

step	1	2	3	10	x
chain				without picture	without picture
number of matches	4	L0	L0	L1	L2

Figure 1: One of the items to detect different levels of algebraic thinking (CDM 1)

The second pool of items is intended to investigate different skills on the formal level. The background theory is adapted from Malles's idea that the central concept of formal algebra is the notion of variables and that a profound understanding of variables is necessary to be algebraically successful. This "profound understanding" can be described by three aspects of variables (Malle, 1993, p. 45):

- **Object aspect:** Variables are understood as unknown or unspecified numbers – both purely mathematical or with reference to (variable) real-life measures (like quantities, lengths, surface areas, or statistical data).
- **Insertion aspect:** Variables are understood as place holders or blank spaces that can be replaced by numbers (leading to true or false statements).
- **Operator aspect:** Variables are understood as meaningless symbols with which one can operate according to certain rules.

Malle claims that pupils should be familiar with all three aspects; he also assumes that the object aspect is the basic one the other two should be built on; but he stresses that these are just normative postulates from a didactic point of view that may not be necessarily fulfilled empirically (Malle, 1993, p. 54f.). Hence, the second pool of items contains 34 tasks, each of them addressing exactly one of Malle's aspects, not presuming any dependency or hierarchy among them. Figure 2 shows some of the items used in the second part of the test. 16 items are related to the operator aspect. They demand syntactic operations pupils typically have to manage in lower secondary schools – like simplifying and expanding algebraic expressions, including some simple binomial theorems. The 6 items of the insertion aspect are connected to equations. All equations are designed in such a simple manner that pupils do not need to use any formal manipulations. They can just insert the given value and calculate the other one. The 12 items of the object aspect are quite diverse. Some of them stress functional dependencies (like the first example). This is supposed to be a purely mathematical application of the object aspect (expressing dependencies between unspecified numbers). The other two examples, on contrary, include references to "real-world" objects underlining the physical interpretation of Malle's object aspect. The last item is related to a typical misconception which occurs in the context of algebra word problems (Clement, 1982): e. g. “twice as many apples (a) as bananas (b)” is often translated into the equation 2a = b instead of a = 2b.

Operator aspect:

1) Simplify: $2a + 3ab + 4a + 3b + 5a =$ ____________________

2) Expand: $2(u – 2w) =$ __________________

3) Expand: $(2a + 3b)^2 =$ _________________

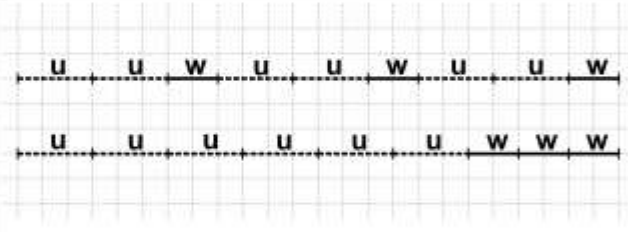
Insertion aspect:

1) Given the equation x = 4y. If x = 8 then y = ____________

Object aspect:

1) Given the equation 4y = 2x + 3. If the value of x becomes greater, what happens to y?
☐ y becomes larger; ☐ y becomes smaller; ☐ y does not change; ☐ nothing can be said about y.

2) The drawing shows a manipulation of an algebraic expression. Mark which manipulation is illustrated.



☐ $3(u + w) = 3u + 3w$
☐ $2uw + 2uw + 2uw = 6uw$
☐ $3 \cdot 2uw = 6uw$
☐ $2uw \cdot 2uw \cdot 2uw = 8u^3w^3$
☐ $3(2u + w) = 6u + 3w$

3) A situation is described on the left. Mark on the right which equation represents the situation correctly, where a is the number of apples and b is the number of bananas.

In a basket are apples and bananas. There are twice as many apples in the basket as bananas.	☐ 2a + b = 3 ☐ 2a = b ☐ 2a + b = a + b ☐ a = 2b ☐ a = b + 2

Figure 2: Selected items to detect Malle’s aspects of variables (CDM 2)

METHODOLIGAL REMARKS

CDMs belong to the non-metric part of the item response paradigm (cf. Rupp et al., 2010): The output of a CDM analysis is not a metric value, but a skill profile for each test participant. This profile indicates what skills a participant possess (with a certain probability), given his responses to the test items. Insofar, the purpose of a CDM analysis is a classification of the participants, not a (metric) measurement of their skills. The skill profiles can be used for individual diagnosis and consultation or (like in this paper) for identifying subgroups containing pupils of the same skill profiles. A CDM analysis is defined by two theoretical constraints: 1) For each item, the researcher declares what skills are necessary to master the item (the adequacy of this assignment can be checked empirically, cf. de la Torre, 2008); 2) it has to be defined how the skills are interrelated. The latter is declared by choosing a specific type of CDM. In the present test, both CDMs are of the DINA type. The DINA model is defined by the assumption that *all* the skills attributed to an item are necessary to solve it (Haertel, 1989). Concerning the first CDM, that means that a participant has to possess level 0 skills to solve L0 tasks, level 0 and level 1 skills to solve L1 tasks, and skills of the levels 0, 1, and 2 to solve L2 tasks – according to the hierarchic nature of this model. Additionally, the L0 questions are separated for each item, since all the three items are based on different situations and it is implausible that a pupil who can manage the first item related to matches (fig. 1) could necessarily managed the second item concerning a taxi driver and the third one concerning a candle. Hence, a skill profile of the first CDM consist of a sequence of five ones or zeroes like 11010, indicating that this participant could manage the L0 questions of the first two items, but not the ones of the third item and that he could answer the L1 questions of all the items he could solve on L0 level (the first and the second one), but not the questions of these items on L2 level. Overall, the sequence 11010 indicates that this person has reached level 1, but not level 2. The second CDM is much easier: Since there is no hierarchical dependency within the items, the items were just assigned to one of Malle's aspects. But unfortunately, the CDM with three skills did not pass the empirical validation (according to de la Torre, 2008). There was evidence that the items concerning the "Clement error" form a skill on their own. Hence, a skill profile of the second CDM consists of four ones and zeros like 1110, indicating that this participant can manage items concerning the operator, the object, and the insertion aspect, but not those items related to the Clement error.

The CDM analysis was performed using the CDM package (George et al., 2016) of the R environment (R Core Team, 2016). The fit indices of the first CDM are excellent (mean of RMSEA item fit: 0.037; max(χ^2): $p = 0.722$; SRMSR: 0.030); the fit indices of the second one are solely acceptable (mean of RMSEA item fit: 0.070; max(χ^2): $p < 0.001$; SRMSR: 0.072) – according to current standards, which are still in flux (cf. George et al., 2015, pp. 196-198).

RESULTS

A general result is related to the two CDMs themselves: The first CDM passed the empirical validation without any changes. That indicates that algebraic thinking is in fact ordered in the hierarchical manner Aké et al. (2013) assumed. The second CDM had to be extended to a fourth skill, related to the Clement tasks.

The skill distributions of the first CDM is as follows (all values has standard errors smaller than 0.035): 69.0% of the pupils can answer the first item on level 0, 81.1% the second item, and 86.3% the third item; nearly all of them (97.8%) are able to generalise the algebraic aspects on level 1 (concerning all three items); but only 63.7% of these pupils can express this generalisation on level 2. This is astonishing, since formal algebra is one of the most important subjects of mathematics in lower secondary schools and, additionally, since only schools of type 2 and 3 were chosen for the test as a positive selection. The pupils who have reached level 2 are assigned to class L2. This class is now investigated with respect to the three covariates gender, school type, and grade, using a χ^2 test to detect intergroup differences and applying Cramér's V as a measure of effect sizes, where a value of 0.1 is considered a small effect, 0.3 a medium effect, and 0.5 a large effect (cf. Liebetrau, 1983). Table 1 shows the results.

aspect	group 1	group 2	significance level	Cramér's V
gender	62.0% (male)	65.3% (female)	$p = 0.44$	0.034
school type	31.8% (type 2)	75.9% (type 3)	$p < 0.001$***	0.410
grade	64.4% (grade 8)	61.9% (grade 9)	$p = 0.42$	0.035

Table 1: Intergroup differences concerning algebraic levels (CDM 1)

There are no gender differences. The difference between school type 2 and 3 is significant and has a medium to large effect. This is remarkable, since – according to the official curriculum – no difference concerning algebra is intended between type 2 and type 3 schools. The most remarkable finding may be the observation that there is no significant difference between grade 8 and 9. That could imply the conjecture: Anyone who does not reach the formal level of algebra at an early stage will not reach it later. But this conjecture should be rechecked by testing the *same* pupils both in grade 8 and later in grade 9 (in this test, they were of different years).

The skill distribution of CDM 2 concerning Malle's aspects of variables is as follows (SE smaller than 0.038): 59.8% of the pupils master the operator aspect; 46.3% the object aspect; 77.6 the insertion aspect; and 28.9% the Clement tasks. It is remarkable that the insertion aspect is the "easiest" skill and that not even one third of all pupils can solve Clement tasks. A closer look on the skill pattern frequencies is interesting:

pattern	freq.	pattern	freq.	pattern	freq.	pattern	freq.
0000	15.2%	0001	1.2%	0110	7.1%	1101	0.0%
1000	2.9%	1100	1.5%	0101	0.0%	1011	3.3%
0100	1.0%	1010	10.0%	0011	5.4%	0111	4.7%
0010	15.1%	1001	0.5%	1110	18.2%	1111	13.7%

Table 2: Skill pattern frequencies concerning Malle's aspects of variables (CDM 2)

A remarkable amount of pupils (15.2%) master none of all skills (0000). Only the insertion aspect appears in a notable number (15.1%) as one aspect that is handled independently of all others (0010) and, additionally, in all the other cases that occur in a relevant quantity (>3%), the insertion aspect is involved (patterns of the type **1*). That may indicate that – in an empirical sense – the insertion aspect is the basic aspect of variables and not the object aspect as Malle supposed. On the basis of the insertion aspect, the operator aspect (1010 with 10.0%) or the object aspect (0110 with 7.1%) or more likely both of them (1110 with 18.2%) can occur. The "Clement aspect" seems to play the opposite role of the insertion aspect: It appears mainly when all the three other aspects are mastered (1111 with 13.7%) or at least in combination with the object aspect (0011 with 5.4%, 1011 with 3.3%, and 0111 with 4.7%). That may lead to the following conjecture: The “Clement aspect” presupposes the object aspect, and most of the time it is mastered only when the two other aspects are also present. This deliberation indicates a hierarchal dependency and could explain why the Clement tasks have the lowest solution frequency. Perhaps, the “Clement aspect” should not be considered as a separate aspect, but rather it seems to be that Clement tasks can only be solved if at least the object aspect is mastered and, additionally, if there is interplay with other, best all aspects. The covariates were analysed concerning the second CDM in a similar manner as in case of the first CDM. Table 3 shows the results.

	aspect	group 1	group 2	significance level	Cramér’s V
operator aspect	gender	50.0% (male)	50.3% (female)	$p = 1.00$	0.003
	school type	43.2% (type 2)	52.8% (type 3)	$p = 0.037$*	0.086
	grade	27.3% (grade 8)	72.6% (grade 9)	$p < 0.001$***	0.453
insertion aspect	gender	71.2% (male)	77.5% (female)	$p = 0.084$	0.072
	school type	51.7% (type 2)	83.0% (type 3)	$p < 0.001$***	0.321
	grade	72.4% (grade 8)	76.3% (grade 9)	$p = 0.295$	0.045
object aspect	gender	55.7% (male)	43.1% (female)	$p = 0.002$**	0.126
	school type	32.9% (type 2)	55.7% (type 3)	$p < 0.001$***	0.203
	grade	44.8% (grade 8)	53.9% (grade 9)	$p = 0.026$*	0.091
“Clement aspect”	gender	29.7% (male)	27.5% (female)	$p = 0.590$	0.025
	school type	11.9% (type 2)	35.0% (type 3)	$p < 0.001$***	0.228
	grade	28.9% (grade 8)	28.3% (grade 9)	$p = 0.974$	0.006

Table 3: Intergroup differences concerning Malle’s aspects of variables (CDM 2)

Again, no significant gender differences are apparent – with one exception: in case of the object aspect, but only with a small effect (0.126). Two other observations are more interesting: 1) With respect to the school types, there is a significant, but nearly negligible difference in managing the operator aspect (0.086), but larger differences with respect to all other aspects (0.321, 0.203, and 0.228); 2) concerning the grade, the largest difference is located in the operator aspect (0.453), whereas no significant differences can be found with regard to all other aspects. These observations might be interpreted as two consequences of the same cause: The focus of teaching algebra could be set on syntactical techniques – on the one hand leading to the result that the learning effect between grade 8 and 9 is directed to this topic and that the difference between the school types can be minimised; on the other hand, no remarkable progress is afforded between grade 8 and 9 and between the school types with respect to the other aspects of variables. But this supposed causal link to the focus of teaching is just a hypothesis that has to be investigated.

FINAL REMARKS

Both CDMs have led to new knowledge about algebraic abilities: The stage model could be validated and dependencies among the aspects of variables have been detected, suggesting the insertion aspect as being essential and the Clement tasks as demanding a bundle of aspects to be solved, at least the object aspect. These insights might be not detectible using a metric approach. Additionally, the classificatory nature of both CDMs allowed indentifying the class and amount of pupils possessing specific skills – mostly an amount less than expected –; and they helped to describe the learning effect – typically with an emphasis on syntactical operations.

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MIND THE GAP: CONGRUENCY AND GAP EFFECTS IN ENGINEERING STUDENTS' FRACTION COMPARISON

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Research on the cognitive processes and strategies underlying fraction comparison in mathematically-trained individuals has studied in recent years the role of congruency effects. Against this account, some works report higher accuracies for incongruent than for congruent items when the fractions to be compared have different numerators and denominators. Strategies such as gap thinking have been proposed as possible explanations for this unexpected outcome. We devised a fraction comparison task controlling for both congruency and gap and presented it to 57 Engineering students. Results confirmed a disadvantage for congruent items without common components and showed that this disadvantage may be accounted by gap effects, providing important constraints for models of the mental processing of fractions.

INTRODUCTION

The learning of fractions and rational numbers is an important milestone in middle school. But often these concepts are learned in a shallow manner, leading many adults to present misconceptions and difficulties in understanding and working with them. Nonetheless, to reach a deep understanding of how the mind works with fractions and rationals requires researchers to shed light not only on how challenged learners struggle with mathematical concepts, but also on how mathematically-trained individuals and experts work with them.

An important aspect of mastering fractions is the acquisition of basic intuitions about them, in particular of their many facets and representations and not only of their associated algorithms and procedures (Forrester & Chinnappan, 2010). A number of research studies in Psychology and Education in the last decade have investigated the strategies that students use for working with fractions (e.g. Clarke & Roche, 2009; Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; Pearn & Stephens, 2004; Stafylidou & Vosniadou, 2004) as well as their supporting cognitive and neural mechanisms (e.g. Barraza, Gómez, Oyarzún, & Dartnell, 2014; Gabriel, Szucs, & Content, 2013; Ischebeck, Schocke, & Delazer, 2009).

Many studies about fraction understanding have considered comparison tasks that are performed mentally (i.e. without pencil and paper). One early and relevant finding was the variability in strategies that learners deploy depending on specific item characteristics. For instance, students use different strategies to compare fractions that have a common component (numerator or denominator) versus comparing fractions where numerators and denominators differ (e.g. Meert, Grégoire, & Noël, 2009).

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Another aspect on which several studies have focused after the work of Ischebeck et al. (2009) is the match or mismatch relation between the magnitude of the fractions to be compared and the magnitudes of the natural numbers composing those fractions. Fraction comparison items where the larger fraction has also the larger numerator and denominator show a match between these dimensions and are called *congruent* (or *consistent*; e.g. 2/9 *vs*. 8/9, 1/3 *vs*. 5/7). In contrast, items where the larger fraction has the smaller numerator and denominator show a mismatch and are called *incongruent* (or *inconsistent*; e.g. 3/5 *vs*. 3/8, 1/3 *vs*. 4/9). A final category corresponds to items where one fraction has the larger numerator and the other the larger denominator, which are called *neutral* (e.g. 2/5 vs. 3/4). Table 1 summarizes and gives other examples of these item types. Research has shown that children's and adults' response accuracies and response times to fraction comparison tasks are modulated by congruency (e.g. DeWolf & Vosniadou, 2015; Gómez et al., 2014; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Eeckhoudt, 2013). But evidence has not always supported the predictions of the congruency account. A re-analysis by Gómez and Dartnell (2015) of several fraction comparison datasets showed that when fractions have no common components it is more likely that the data do not support the congruency account, namely that congruent items are more difficult and take longer to answer than incongruent items. In a similar line, DeWolf and Vosniadou (2015) found groups of university students showing significant congruency effects in opposite directions. Most interestingly, this reversal of the congruency effect has been observed in high achieving children (e.g. Gómez et al., 2014, p. 189) as well as expert mathematicians (Obersteiner et al., 2013). This raises questions about the theoretical basis of congruency, and highlights the need to consider more complex explanations involving, for instance, some of the commonly used strategies for comparing fractions.

	Congruent	Incongruent	Neutral
With a common component	$\frac{33}{64} vs. \frac{43}{64}$	$\frac{36}{71} vs. \frac{36}{47}$	
Without common components	$\frac{50}{83} vs. \frac{35}{68}$	$\frac{57}{73} vs. \frac{62}{91}$	$\frac{12}{59} vs. \frac{18}{49}$

Table 1: Examples of items of the five categories obtained by considering the presence or absence of common components as well as congruency.

One possible strategy, documented in qualitative research (e.g. Clarke & Roche, 2009; Pearn & Stephens, 2004), is known as *gap thinking*. Its name stems from the conception of fractions as parts of a whole, and consists in comparing two fractions indirectly by reasoning how many parts each fraction lacks to complete the whole (their *gaps*), and declaring that the fraction with the fewer parts missing is larger than the other. For instance, gap thinking applied to the comparison of 1/3 *vs*. 4/7 would compare both gaps (2 and 3, respectively) and conclude that 1/3 is larger because its gap is smaller. Whereas in this example gap thinking leads to the incorrect answer, there are many fraction pairs for which it leads to the correct one (e.g. 1/3 *vs*. 6/7) as well as cases in which it is uninformative because both fractions have the same gap (e.g. 1/3 *vs*. 5/7; although some children might conclude in this case that the two fractions are equal, see Pearn & Stephens, 2004). Further examples of these three cases are presented in Table 2. Interestingly, from the five item categories of Table 1, it turns out that gap thinking leads always to the correct answer in four of them: the only category for which gap thinking may lead to the incorrect answer or to no answer is that of congruent items without common components. We hypothesized that the use of gap thinking—whether explicitly or implicitly—may explain why congruent items without common components are associated to a worse performance than incongruent items in mathematically-trained individuals.

Gap thinking leads to the correct answer	Gap thinking leads to the incorrect answer	Gap thinking is uninformative
$\frac{67}{95}\,vs.\,\frac{45}{83}$	$\frac{51}{71}\,vs.\,\frac{22}{35}$	$\frac{37}{98}\,vs.\,\frac{11}{72}$

Table 2: Examples of items where gap thinking leads to the correct, incorrect, or to no answer.

METHODS

Participants

This work is part of a larger project on the cognitive bases of fraction comparison. Here we present data from a sample of 57 undergrad students of Engineering (39 men and 18 women; approximate ages between 18 and 25) recruited in Santiago, Chile. All of them gave informed consent prior to testing, and were paid for their participation.

Fraction comparison task

We designed a set of 180 pairs of fractions using numerators and denominators between 11 and 99. As in Obersteiner et al.'s study (2013), fraction pairs were divided into the five categories of Table 1 with 36 pairs each. All five categories were matched in terms of numerical distance between the fractions, and for each pair both fractions were always on the same side of 1/2 (either both larger or both smaller). In addition, we included (within congruent items without common components) items in which the use of gap thinking would lead to the correct answer, the incorrect answer, or be inapplicable (see Table 2).

Procedure

We used a standard fraction comparison task, where participants were presented with pairs of fractions on a computer screen and were asked to indicate which one is larger by using the keys Q (for the fraction on the left) and P (for the fraction on the right). To foster intuitive and conceptually-based answers, participants were not provided pencil and paper, and were given a time limit of 10 s to answer each item (after this limit, the item was considered as omitted and the next item was presented). Items were randomly ordered so that all item types were presented in an interleaved manner, and grouped into three blocks with self-paced pauses in between.

Data Analysis

Statistical analyses were conducted using R v3.3.2 (http://www.r-project.org/). Repeated measures regressions were done using the packages `lme4` v1.1-12, `car` v2.1-2, and `lmerTest` v2.0-33. The analysis of response times considered only the items that were answered correctly.

RESULTS

Three participants were discarded from analysis based on a Mahalanobis distance criterion. As a first analysis, we computed response accuracies and latencies for the five categories given by presence of common components and congruency. These results are presented in Table 3.

A repeated measures logistic regression for response accuracies showed a marginally significant interaction between the presence of common components and congruency ($\chi^2(1) = 3.15$, $p = .08$). Planned comparisons revealed that there was no significant difference between congruent and incongruent items with a common component ($OR = 0.0002$, $p = .99$), in contrast to items in which fractions had no common components, where congruent items had significantly lower scores than both incongruent ($OR = 0.43$, $p < .0001$) and neutral ($OR = 1.87$, $p < .0001$) items.

Histograms of response accuracy (Fig. 1) show no evident signs of bimodality in the data that could account for the lower results in the congruent with no common components condition, suggesting that participants responded to the task in a cognitive manner similar to one another.

A repeated measures linear regression for response times showed a significant interaction ($\chi^2(1) = 6.45$, $p = .01$). Planned comparisons showed quicker responses for congruent than for incongruent items sharing a common component ($b = 117$, $p = .02$), whereas congruent items with no common components were answered significantly more slowly than their incongruent ($b = -238$, $p = .003$) and neutral ($b = -558$, $p < .0001$) counterparts.

	Congruent		Incongruent		Neutral	
	ACC	RT	ACC	RT	ACC	RT
With a common component	98%	2695	98%	2812		
Without common components	86%	4271	90%	4033	97%	3713

Table 3: Average accuracies and response times for the five main item types.

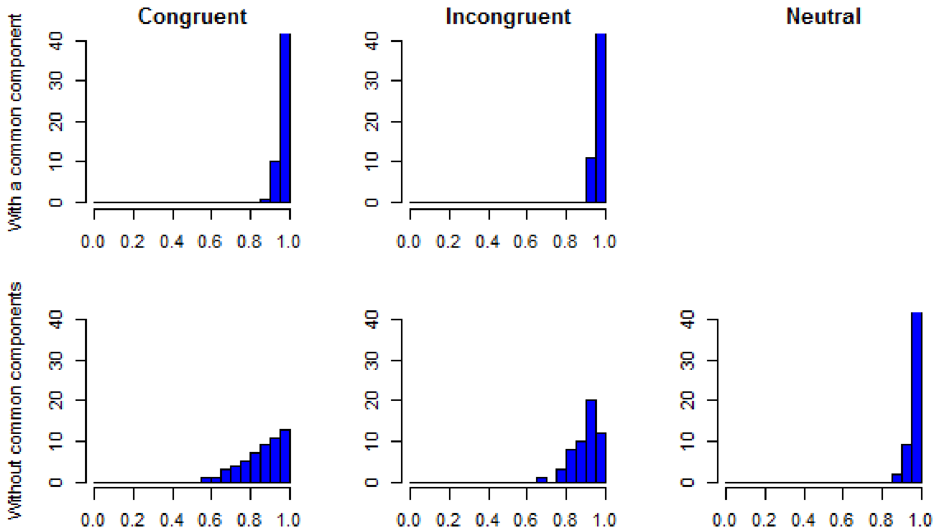

Figure 1: Histograms of accuracies per participant for the main five item types.

Items in which gap thinking…	ACC	RT
… leads to the correct answer	88%	4000
… leads to the incorrect answer	85%	4469
… is uninformative	86%	4393
Total average	86%	4271

Table 4: Average accuracies and response times for the three gap conditions.

Case	Variable	*B*	*p*
(A) Only congruency	Dummy [Incong-Cong]	-238	.003
	Dummy [Neut-Cong]	-558	< .0001
(B) Congruency + gap	Dummy [Incong-Cong]	33	.76
	Dummy [Neut-Cong]	-287	.009

Table 5: Regression coefficients for the effects of congruency on the response times of items without common components, either considering only congruency or both congruency and gap as predictors.

We then focused on congruent items without common components, the critical category where we could contrast the three gap-related conditions. A logistic regression for accuracies showed no significant effect of gap condition ($\chi^2(2) = 2.26$, $p = .32$). A linear regression for response times, in contrast, turned statistically significant ($\chi^2(2) = 18.00$, $p = .0001$). As shown in Table 4, responses to items in which gap thinking leads to the correct answer were significantly quicker than both those where gap thinking leads to the incorrect answer (b = -469, p = .0001) or is uninformative (b = -393, p = .001). These two types, instead, did not differ from one another (b = 76, p = .52).

As a final analysis, we explored whether the observed gap effects in response times could account for the observed differences between congruent and incongruent items without common components. Table 5 shows the regression coefficients associated to the factor congruency when explaining participants' response times in two scenarios: (A) when congruency is the only independent variable, and (B) when both congruency and gap are independent variables (notice that congruency in this context has three levels, so it is associated to two predictors). Results show that the difference between response times to congruent and incongruent items—as measured by the coefficient of the corresponding dummy variable—lost statistical significant after entering gap as a predictor (b changed from -238 to 33, p = .003 [bootstrap test]), whereas the difference between response times to congruent and neutral items had a significant reduction to about half of its value (b changed from -558 to -287, p = .01 [bootstrap test]). This shows that gap effects accounted for all the difference observed between congruent and incongruent items, and partially accounted for the difference between congruent and neutral items.

DISCUSSION

We have investigated the relevance of the gap difference between fractions in mathematically-trained adults' reasoning while performing a mental fraction comparison task. Their response accuracies based on common components and congruency showed a pattern that is reminiscent of what Gómez et al. (2014) found when looking at the group of highest achievers in a sample of 5^{th}-7^{th} grade children, where congruent items with no common components had lower accuracies than all other item types. Similar results were also obtained by DeWolf and Vosniadou (2015), who reported that Greek undergrads of Informatics and Mathematics departments performed better in the incongruent case. A study with expert mathematicians conducted by Obersteiner et al. (2013) also provided evidence in this direction: experts displayed an advantage for congruent items when fractions shared a common component, but this advantage reversed when they lacked it. This reversal has been replicated in several studies (see Gómez & Dartnell, 2015), although no possible explanations for it have been tested successfully so far.

We had hypothesized that the gap thinking strategy might play a role in explaining this disadvantage for congruent items without common components, and the data confirmed this conjecture. Gap-related conditions affected significantly participants' response times but, most importantly, their consideration as predictors for explaining response times reduced the differences between congruent and incongruent items to a non statistically significant figure. Moreover, the consideration of gap effects also reduced the magnitude of the difference between congruent and neutral items by about 50%.

Gap thinking has been observed in studies with children (e.g. Clarke & Roche, 2009; Pearn & Stephens, 2004) and is thought to stem from a naïve reliance on the schema of fractions as parts of a whole, in which the gap would be thought of as the number of parts missing to complete the whole. It is highly likely, however, that mathematically-trained individuals use fractions representations that are not only correct but also richer, and so it is reasonable to assume that the observed gap effects occur at an implicit level rather than an explicit level. In this context, gap thinking might be seen as comparing each fraction's numerator and denominator with a subtraction-based rule instead of the more complex, ratio-based one. This subtractive rule might arise as an intuition acquired with mathematical expertise (a *secondary intuition* in terms of Fischbein, 1987).

Although we originally hypothesized gap effects to explain the difference between congruent and incongruent items without common components only, they turned out to partially explain the difference between congruent and neutral items as well. Nonetheless, gap effects did not fully account for the latter. We conjecture that this is due to the fact that neutral items may also be answered by noticing that in neutral items, the larger fraction has both the larger numerator and the smaller denominator, both elements supporting a quick decision on the part of mathematically-trained individuals. This strategy is fundamentally different from gap thinking, and might explain the remaining difference between congruent and neutral items.

Altogether, these findings suggest that congruency is not the best predictor of mathematically-trained individuals comparing fractions. This does not mean that congruency is unrelated to performance (e.g. DeWolf & Vosniadou, 2015; Gómez et al., 2014; Obersteiner et al., 2013), but rather that it is a poor concept in terms of explanatory power at least for the reasoning of high achievers. Our data put forward gap thinking as a viable strategy that may partially explain the differences previously ascribed to congruency in some important contexts.

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LEARNING STRATEGIES IN ENGINEERING MATHEMATICS – EVALUATION OF A DESIGN RESEARCH PROJECT

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Like many other universities, Ruhr-Universität Bochum (RUB) has to deal with disappointing attainment in engineering mathematics and ensuing unnecessary dropout. A potential solution was identified in an interventionist project addressing first-years engineering students' learning strategies. In accordance with its Design Research background, the project procedures were modified over the years, taking different theories of learning into consideration. More than 2,000 data sets collected in four project cycles confirm the adaptations and emphasize the importance of motivational rather than cognitive or methodical learning strategies.

INTRODUCTION

University courses related to a technical subject such as engineering require a knowledge of mathematics. Compared to other subjects, the gap between school and university mathematics seems high and causes difficulties (cf. Dreyfus, 1995; Gueudet, 2008). Some predictors for success in university mathematics have been found: Apart from the mathematics competence acquired at school (Rach & Heinze, 2013), there is the use of learning activities offered by the university, cognitive activation, deep learning strategies, and motivation for or interest in mathematics (Blömeke, 2016; Trapmann, Hell, Weigand, & Schuler, 2007). The ensuing attempts to overcome the obstacles vary, and examples are numerous (Dunn, Lo, Mulvenon, & Sutcliffe, 2012; Hoppenbrock, Biehler, Hochmuth, & Lück, 2016). At RUB, a project for engineering mathematics, addressing learning strategies, was contrived (Dehling, Glasmachers, Härterich, & Hellermann, 2010). The idea was to support first-year students in their systematic learning, with the objective to use mathematics as an example discipline whose mastery would have an effect on other subjects.

THEORETICAL APPROACH

When describing changes in human behavior concerning understanding, i.e. cognitive development, the theory about its phases (Piaget, 1973) form a basis, concretized in the theory of the three worlds of mathematics (Tall, 2004). Both build on the rationale that knowledge has to be reconstructed by the learner. Apart from cognitive challenges, however, other factors are recognized, too (Gueudet, 2008; von Glasersfeld, 1991). Thus affective aspects have gained a growing role in the research on mathematics education (McLeod, 1992; Hannula, Evans, Philippou, & Zan, 2004). It is now generally accepted that "affect plays a significant role in mathematics learning and

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 361-368. Singapore: PME.

instruction” (McLeod, 1992, p. 575). So the perspective seems promising to explore general and meta-level skills in terms of learning strategies, whose investigation allows for revealing both the cognitive dispositions as well as affective barriers and pathways. How to sustainably encourage learning strategies, to identify successful interventions, and to understand the influence of motivation needs to be further explored.

As the aim of our project is to efficiently promote students’ learning outcomes, the choice of background structure fell on *Design Research* (Gravemeijer & Cobb, 2013, Brown, 1992), which combines two objectives: practically to design, conduct, and improve interventions, and theoretically to understand which interventions work and why that is the case. This approach fits our scenario, as it fulfills the conditions of being interventionist, iterative (the project is repeated in subsequent years, from 2010 until the present), process-, utility-, and theory-oriented, and it involves practitioners (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006; Plomp, 2013). The purpose is to support students to cope with the amount and depth of the mathematical contents they have to master in a given time. From appraisal of the literature on learning mathematics, it seems well-advised to design a learning environment that enables students to involve themselves with the subject matter in context (Dreyfus, 1995) and to discuss their approaches with peers and teachers (von Glasersfeld, 1991), as well as to address selected affective aspects, namely beliefs, attitudes, and approaches to learning (Goldin, 2002). The choice of methods should include sufficient resources and alternative scenarios of the learning process (hence learning strategies), and likewise treat the learner as an autonomous being in control of possible outcomes (Deci & Ryan, 1990).

This leads to two research questions, which address the evaluation of the project (for investigations of the specific design of the project and its development, see Griese, 2016).

RQ1 How does our project influence learning strategies (and motivation)?

RQ2 Which learning behavior is connected with academic success?

Project conceptualization and development

Our project was initially planned to distinguish between participants enjoying close and personal support in preparatory tutorials, and participants mainly relying on e-learning (and being offered a revision course in compensation). Both groups had access to a helpdesk and were asked to fill in a learning diary. After the first project cycle, the interventions were re-designed, according to feedback from participants, and taking affective aspects into more serious consideration. From then on, project procedures were more numerous and more closely linked. The revised procedures included a weekly focus topic, a weekly workbook, a special project helpdesk, weekly group meetings, a revised e-learning course, a learning log, mentor students, use of social networks, and a revision course, all referring to each other. The intertwined combination of interventions enabled the project staff to gradually withdraw their support, so that ideally, participants would rely on their own and peer resources in the long term. After the second project cycle, only minor changes were necessary, mainly concerning the digital tools. Because of the modifications, only data from the second to fourth project cycle was used for the calculations presented in this paper.

METHODOLOGY

LIST questionnaire

The LIST questionnaire (*Lernstrategien im Studium*, Wild & Schiefele, 1994) is a widely-used German questionnaire assessing learning strategies independent of subject. It is based on the same classification as the MSLQ (Motivated Strategies for Learning Questionnaire, Pintrich, Smith, Garcia, & McKeachie, 1993) and captures learning strategies in twelve categories (scales): the cognitive learning strategies *Organizing*, *Elaborating*, and *Repeating*, the metacognitive strategies *Planning*, *Monitoring*, and *Regulating*, and the resource-related strategies *Effort*, *Attention*, *Time Management*, *Learning Environment*, *Peer Learning*, and *Using Reference*. Each category is operationalized in three to eight items that are rated on Likert scales (four points, *very seldom* to *very often*). For our purposes, the cognitive category *Critical Checks* was omitted.

Data analysis

Data was collected in a pre-post design, at the beginning and at the end of the first semester, during the mathematics lecture. Apart from the LIST items, there was demographic data, and a code that enabled matching the survey data to project participation, to the result in a test a few weeks after the beginning of the university course (variable *Mini Test*), and to examination outcomes (*Written Exam*).

In order to have a sound basis for further calculations, the reliability of the LIST scales was tested via Cronbach's α, and an exploratory factor analysis (EFA) was conducted. For each data set, the separate item scores belonging to the same scale were combined to yield scale scores (from 0 to 100) used for further explorations. These consisted of calculating correlations, multiple linear regression with the outcome variable *Written Exam*, as well as pre-post comparisons via dependent *t*-tests or Wilcoxon ranked sign tests (depending on assumption of normal distribution of difference scores). These computations were carried out separately for project participants and non-participants.

RESULTS

The internal reliability of the LIST scales is mirrored in their Cronbach's α values, which should be >0.7 to be considered good, see Table 1. Together with the loadings observed in the EFA (orthogonal / varimax rotation, pairwise exclusion of cases, analogous results for oblique / direct oblimin rotation), where items from *Planning* loaded consistently on *Time Management*, and other metacognitive items loaded irregularly, the decision was taken to form a combined scale *Time Management / Planning* (α=0.790), and to omit *Monitoring* and *Regulating* (and one *Attention* item,

α_{Att}=0.858) in further calculations. This left nine scales for interpretation. Together,

LIST scale	α	LIST scale	α	LIST scale	α
Organizing	0.814	*Monitoring*	0.560	*Time Management*	0.756
Elaborating	0.766	*Regulating*	0.539	*Learning Environment*	0.700
Repeating	0.726	*Effort*	0.757	*Peer Learning*	0.783
Planning	0.642	*Attention*	0.749	*Using Reference*	0.765

Table 1: Cronbach's α for LIST scales, data from three project cycles

they explain 44.34% of variance, yield consistent loadings and show adequate fit (KMO=0.920, $\chi^2(1830)=29763.560$, p=0.000, KMO-values for individual items >0.8 but for two exceptions >0.7, $\chi^2/df \approx 2.49$).

LIST Scale	Mean pre	Mean post	*n* pre; post	*t*	df	Sig. (2-t.)	Effect Size *r*
Organizing	51.93	47.64	203; 207	2.811	170	0.006**	0.21
Elaborating	57.22	47.19	202; 197	8.773	160	0.000***	0.57
Effort	64.29	59.79	203; 200	4.676	163	0.000***	0.34
Attention	56.83	44.36	210; 220	3.770	188	0.000***	0.27
Time Man. / Pl.	42.08	45.06	210; 221	-1.717	184	0.088	0.13
Peer Learning	57.83	58.01	216; 201	0.705	174	0.482	0.05

Table 2: Dependent *t*-tests, **$p<0.05$, ***$p<0.001$, *Attention* items are reverse-coded

Regarding the development of learning strategies, only data sets that could be matched precisely (*n*=255) were used. The differences between the pre and the post scores were tested for normal distribution (Shapiro-Wilk test, values for skew / kurtosis, QQ-plots, distribution histograms with normal approximations). The scales whose difference scores could be classified as normally distributed (*Organizing*, *Elaborating*, *Effort*, *Attention*, *Time Management / Planning*, *Peer Learning*) underwent dependent *t*-tests, the others (*Repeating*, *Learning Environment*, *Using Reference*) were subjected to Wilcoxon signed-rank tests, see Tables 2 and 3.

LIST Scale	Median pre	Median post	*n* pre; post	*z*	Sig. (2-t.)	Effect Size *r*
Repeating	47.62	42.86	204; 216	-5.944	0.000***	-0.32
Learning Env.	61.11	66.67	210; 216	-0.832	0.405	-0.04
Using Reference	83.33	66.67	230; 232	-4.541	0.000***	-0.22

Table 3: Wilcoxon signed-rank tests for paired data sets, ***$p<0.001$

When conducting these tests separately for project participants and non-participants, it emerged that non-participants showed stronger decrease in the use of learning strategies than participants. Concerning *Time Management / Planning*, participants even showed a significant increase (mean diff. =-4.84, effect size r=0.34*), as opposed to a general non-significant decrease (mean diff. =-1.76, r=0.09). In their use of *Effort*, project participants revealed no significant change (mean diff. =2.46, r=0.17), whereas non-participants report a significant decrease (mean diff. =5.78, r=0.39***). With regard to *Attention* (reverse-coded), participants ($mean_{pre}$=51.59, $mean_{post}$=42.86) display a weaker decrease of concentration than non-participants ($mean_{pre}$=58.04, $mean_{post}$=44.47) and report an altogether higher level of concentration.

Concerning the influence of learning behavior on examination success, correlations (Pearson's r) were calculated between the scale scores, *Mini Test*, and *Written Exam*. The biggest (and significant, p<0.001) was found between *Mini Test* and *Written Exam*, r=-0.643. When following up with multiple linear regressions in the forward, backward, and stepwise method of including or excluding predictors from a model with the outcome variable *Written Exam*, a model containing *Mini Test* (with standardized β=-0.611) allowed only one other predictor, *Effort*, albeit with less impact (β=-0.138). As mathematical skill at the beginning of the course cannot be influenced by the project procedures, *Mini Test* was left out of subsequent models. A model with two predictors emerged, see Table 4. The influence of learning strategies operates as expected; more effort and higher concentration are connected to better grades.

Predictor	b	SE for b	β	Sig.
(Constant)	3.340	0.462		0.000
Effort	-0.013	0.005	-0.162	0.021**
Attention	0.009	0.004	0.149	0.033**

Table 4: Regression model with two predictors and outcome variable *Written Exam*, n=268, R^2=0.074, Durbin-Watson 1.946, **p<0.05

As a last step, project participation was tested for possible moderator qualities, i.e. it was investigated if project participation influenced the linear model predicting examination success. The results were weakened by the small numbers of data sets in these groups (n_{part}=66, $n_{non\text{-}part}$=197), but nevertheless the calculations showed that the contribution of *Attention* vanishes for the project group, and shows no significance (standardized β_{part}=0.003, p_{part}=0.982, as opposed to $\beta_{non\text{-}part}$=0.190, $p_{non\text{-}part}$=0.020).

DISCUSSION

In relation to students' learning strategies and their development (RQ1), the scales *Effort* and *Attention* are particularly interesting. They reflect the motivation items from MSLQ. The scores for these scales are lower in the post survey than in the pre survey, which means different things: For *Effort*, a lower score means less effort is made – or at the very least, that the students subjectively assess that they are making less effort, in comparison to an unspecified reference frame, which itself may have shifted due to experiences at university. The description is true both for the complete cohort and for the project participants, but project participants keep their level of *Effort*, whereas the average student's score decreases. For *Attention* (reverse-coded), a higher score means more distraction or less concentration. The fact that the *Attention* scores are lower in the post survey means that students report a higher level of concentration at the end of the semester. The change of *Attention* is more distinct among the non-participants, but the project participants start at a higher level of concentration and end at roughly the same level as the non-participants. So for both scales connected with motivation, project participants show advantages over the non-participants. Three other learning strategies show a meaningful decrease in use (*Elaborating*, *Repeating*, *Using Reference*), which is less pronounced for the project group for *Repeating* and *Using Reference*. This can be interpreted as another success of the project, as memorizing techniques as well as referring to apposite additional material was covered in the project procedures.

In view of the influence learning strategies can have on examination success (RQ2), different linear models were explored. They need to be viewed in connection to the fact that the LIST scales do not cover understanding, and our survey only asks how often a learning behavior is employed, and not how efficiently it is applied. When the variable *Mini Test*, which captures mathematical competence at the start of the university course, is entered into the model, the explained variance rises to over 40%. Other researchers, too, have found that the knowledge and skills students have acquired at school are a strong predictor for their success at university (Schiefele, Streblow, Ermgassen, & Moschner, 2003; Trapmann, Hell, Weigand, & Schuler, 2007). This serves as background to understand why learning strategies explain comparatively little variance. As our project aims at supporting mediocre or weak students, a little help may just be the assistance these students need in order to pass their examination in mathematics. The final model (Table 4) contains only the two predictors *Effort* and *Attention*. The importance of *Effort* is additionally stressed by the fact that it is the only remaining scale when restricting the model to project participants. The relevance of these psychologic aspects was not hypothesized in the conceptualization of the project, which has brought the insight that apart from the technical side of learning behavior, project procedures should be characterized by a focus on motivational aspects.

The constraints of the findings presented are not to be withheld: there is no proper control group. At the conceptualization of the project, a control group was intended, consisting of students who had applied for project participation but were not accepted. As this group hardly appeared in the post surveys and staffing improved, a comparison group was established in later project cycles (consisting of students with similar starting conditions in mathematics, according to *Mini Test*) and used for comparisons concerning pass rates. The decision was vindicated when the sole relevance of psychological aspects was realized. If the original control group had been retained, the experience of being expressly denied help might have influenced motivation negatively. Now that the relevance of motivation has become clear, we can conclude that this would probably have enhanced rather than diminished the differences to the project group, resulting in the opposite effect a control group should have.

In sum, the conclusion that *Effort* and *Attention* are the central factors in mathematics for engineering students is also a reason for optimism, as this is something that can be addressed. This is exactly where a project like ours can act out its strengths, and a strong factor in recommending this kind of project work.

Note: This paper presents selected findings from Griese (2016).

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A PUZZLING MISCONCEPTION OR A LOGICALLY PERSISTENT WAY OF UNDERSTANDING? EXAMINING STRUCTURES OF ATTENTION

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We use Mason's (2004) framework of structures of attention to offer an account of why a group of students continually posited a mathematically incorrect way of understanding (Harel, 1998) while working on graph theory task. Analysis suggests that despite continuous activity and discussion, certain aspects of the group's attention did not change over time, and the overall structure of their attention did not so much shift (in the sense of a change in direction or focus) as it did expand (in the sense of a change in size). We propose that the group's persistent attachment to the incorrect way of understanding is understandable in light of their structures of attention, and that a fine-grained application of Mason's framework can offer sympathetic insight into the persistence of incorrect ways of understanding.

INTRODUCTION

Mason (2004) proposed that when new knowledge emerges, what changes cognitively is the *structure of attention* (i.e., *what* one attends to, as in the mathematical objects, and *how* these objects are attended to). Recent studies (e.g., Palatnik & Koichu, 2015; Yoon, 2015) showed that shifts in students' structures of attention indicate radical changes in their mathematical thinking. Conversely, Mason (2015) posited that students get often stuck on a problem if they are unable to shift their attention beyond a *way of understanding* (Harel, 1998). We use 'way of understanding' to refer to the particular interpretation students give to a mathematical concept. In this paper, we apply Mason's (2004) framework of *structures of attention* on a piece of data in which a group of students persistently posited an insufficient way of understanding as they worked on a graph theory task. Studies on insufficient knowledge that students often possess are abundant in the literature as evidenced by the numerous terms associated with 'insufficient knowledge': misconceptions, alternative conceptions, obstacles, and so forth. The inability to revise insufficient knowledge can be a major hindrance to learning (or making progress during a task). However, revising insufficient knowledge, is problematic particularly because of their persistence, as Brousseau (1997. p. 99-100) explains about 'obstacles': "[they] withstand occasional contradiction, and even after inaccuracy is recognized they continue to crop up in an untimely persistent way." But such explanations are elusive on the matter of why and how insufficient knowledge persists. Further, it is often tempting to account for insufficient knowledge in terms of what students lack. Might a closer look at students' thinking yield a more sympathetic and elaborate understanding about the intricacies of

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students' "insufficient knowledge"? Our aim is to understand why a group of students persistently posited an incorrect way of understanding. Given the aforementioned findings of Yoon (2015), Palatnik and Koichu (2015), and Mason (2015) we hypothesize that some aspects of the group's structures of attention do not change over time, resulting in the persistence of the incorrect way of understanding. Our research questions are: what aspects of the group's attention do not change throughout the task? And, how do these unchanging aspects lead the group to constantly posit the incorrect way of understanding?

THEORETICAL FRAMEWORK—STRUCTURE OF ATTENTION

The structure of attention (Mason, 2004) framework comprises both *what* one attends to, as in the mathematical objects—nodes, arcs, functions; and *how* these objects of one's attention are attended to. The *how* refers to the *forms* of one's attention. Mason proposed five different forms of attention: (1) *holding wholes*: the form of one's attention when one gazes at whole objects (instead of particulars); (2) *discerning details*: the form of one's attention when one stresses particular aspects (that are almost always immediately apparent) of an object and ignores others (e.g., the number of nodes in a network, the number of arcs incident on a node); (3) *recognizing relationships*: the form of one's attention when one recognizes that certain objects are related (e.g., two nodes have an equal number of adjacent nodes); (4) *perceiving properties*: the form of one's attention when one conjectures that an object always possesses a certain feature (e.g., in any network the number of nodes with an odd number of arcs incident on it is even); (5) *deducing from definitions*: the form of one's attention when one attempts to deduce a general result, beginning with a conjecture and providing justifications based solely on perceived properties. These different forms and objects of attention lead to the proposition (Mason, 2004) that mathematical thinking entails shifting back and forth among different objects and forms of attention, and that learning something (or the emergence of a new way of understanding) is indicated by certain shifts in the overall structure of attention.

METHOD

We report on the group work of Lome, Chad, and Gil, who at the time of data collection were enrolled in a university-bridging mathematics course, which covers secondary school level algebra. The group worked on *The Jandals Problem*, which begins with a story about a means of ranking Hollywood actors through the 'Bacon number' (similar to the Erdös number), which is the smallest number of movie-links between an arbitrary actor and prolific actor Kevin Bacon (e.g., an actor who has co-starred with Bacon has a *Bacon number* of 1). After reading this story, the group worked on warm-up questions regarding Bacon numbers, which familiarized the students with graph theory terminology (e.g., a *node* in the network represents a person, and an *edge* between two nodes represents a relationship (e.g., friendship) between the two people). Then, the group read the problem statement, which describes a situation in which Xanthe, an American exchange student in New Zealand, picked up a number of colloquial New Zealand words, including “jandals” (equivalent to “flip flops” in the U.S). Upon returning home, Xanthe intends to spread the word, “jandals”, throughout her network of friends (shown below in Figure 1). The problem statement asks students to devise an algorithm to determine the first person with whom Xanthe must share the word, so as to reach everyone in the network in the smallest amount of time (assuming that two people are friends if there is an edge between them, and any person who hears the word will share the word with all of his/her friends in exactly one day). The problem also asks students to ensure that their algorithm would work for any similar friendship network. The group worked on the task for one hour in a quiet room in the presence of an interviewer, while being audio recorded and video recorded. The interviewer presented the task and answered clarification questions about the wording of the task, but did not give any mathematical hints about how to solve the problems.

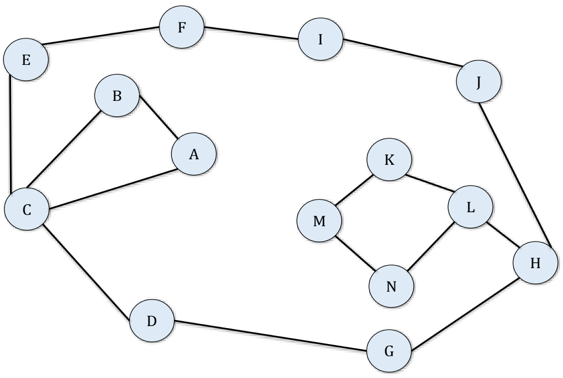

Figure 1: Friendship Network 1 (FN1)

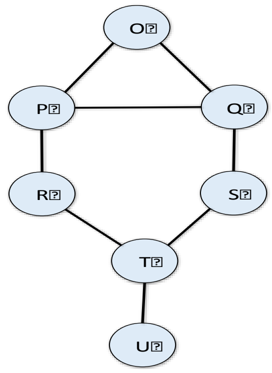

Figure 2: Friendship Network 2 (FN2)

We used Mason’s *structure of attention* framework to analyze the group’s work, focusing on several questions: (i) What mathematical objects is the group attending to throughout the task? (ii) How is the group attending to the mathematical objects? (iii) Why does the group shift their attention from one object to another, or from one form of attention to another? The *why* question was addressed by inferring the particular goals towards which the group was working when their attention shifts.

RESULTS AND ANALYSIS

We present three episodes from the group’s work on The Jandals Problem, documenting how the group constantly advanced a particular incorrect way of understanding, tracing at a fine-grained level the interactions between their structures of attention. We attempt to identify unchanging aspects of the group’s attention that might explain the group’s constant advancement of the incorrect way of understanding.

Episode 1: The starting person must have three friends

The group begins by agreeing that in order to find the quickest starting person (i.e., the person from which the word would spread throughout the network fastest), they need to figure out how many days it would take for the word to spread from each person in the network, then choose the person that corresponds to the minimum number of days.

The group's strategy for doing this is to pick a person, then identify the people that the word reaches on day one, day two, and so forth until all people have been reached. The group applies their strategy on *C, D, E, G, H, I, J, L, M*, finding the quickest to be *H*, which yields four days (aside: the group incorrectly applies their strategy on *G* and get 6 days, but it actually only takes 4 days, making it another quickest starting person). The group then attends to the problem statement, focusing on the need for an algorithm:

Lome: What is an algorithm?

Interviewer: An algorithm is like a method

Gil: It's like the way you got it.

Interviewer: Yes, so that you can use it for any other network... You've got to explain it as well as you can so that Xanthe can use it on any network…

After hearing this clarification about what an algorithm/method is, Gil focuses on *H*, notices that it has three friends, and subsequently proposes a method, henceforth referred to as *M1*: *share the word with someone who knows three people*:

Gil: So the algorithm would be...the person she tells first should tell three people, because if you told *H*, *H* would tell *L*, *G*, and *J*.

There are three main things to note from this episode: (1) Gil's focus of attention was person *H* (the quickest starting person); (2) the group was aware of the need to create an algorithm/method; (3) the group's strategy behind *M1* was to state a property of the quickest starting person. The property of person *H* that Gil perceived (*the starting person has 3 friends*) was the basis of *M1* and the group's way of understanding the concept of 'the quickest starting person'. One can easily verify that this way of understanding is incorrect, as it does not always equate to the quickest starting person in any network. But, as we will see in the following episodes, the group maintains this way of understanding in all the methods that they propose.

Episode 2: The group modifies *M1* but the initial way of understanding persists

After Gil proposes *M1*, the interviewer asks the group to re-read the problem statement. Lome then attends to *FN1*:

Lome: So the method is to share it with someone who is connected to at least three people. But if she starts at *L*, *L* is connected to three people but it doesn't work as fast [as *H*].

The group agrees that *M1* is inadequate for *FN1*, and Lome proposes a modification:

Lome: Maybe it [*H*] needs to be connected to three people [*L*, *J*, and *G*] but one of those three people [pointing to *L*] has to be connected to at least two other people [pointing to *K* and *N*]. That's why this person [*C*] doesn't work. It's connected to four people, but none of those people is connected to two other people. So that's an algorithm.

Chad and Gil approve Lome's proposal, and Lome writes down their new method (*M2*): *share the word with a person who knows at least 3 people, and one of those 3 people must know at least 2 other people*.

Note *M2* is merely the group's initial way of understanding – *the starting person must have three friends* – plus another property of person *H* (i.e., *one of its 3 friends must have at least 2 other friends*). So why does the group maintain their initial way of understanding in *M2*? At the beginning of the episode, Lome perceived that *L* (like *H*) also had three friends, which ultimately meant that *M1* was inadequate. The group's goal was then to modify *M1*. At this point the group was attending to: *M1* (and its inadequacy); *H* (and its property of being the quickest starting person in *FN1*); and the need for a method. We note that although the group had deemed *M1* inadequate, it neither changed the fact that *H* was the quickest starting person in *FN1*; nor changed the fact that *H* had three friends (this was a salient property of *H*). Furthermore, the group's strategy for creating a method was to state (distinctive) properties of the quickest starting person, which in this case was *H*. So by focusing on person *H* and their strategy for finding the quickest starting point, the group naturally maintained the notion that the quickest starting node has three friends. Moreover, Lome later recognized a relationship between *H* and *L* (both had 3 friends but *H* was a quicker starting person than *L*) that necessitated *M1*'s revision. However, Lome shifted his attention from *H* and *L* to *H* and *C*, and perceived (differing) properties of these two people, from which he deduced *M2*. Note *M2* also holds for *L* (which if noticed might have undermined *M2*) but the group did not notice this, as their attention had shifted from *H* and *L* to *H* and *C*.

Episode 3: The group modifies *M2* (twice) but the initial way of understanding persists

After Lome writes down *M2*, the interviewer gives the group a second friendship network (FN2; see Figure 2), and asks them to show how their new method (*M2*) would work on this network. The group tries to find the quickest starting person in *FN2*, employing the same strategy used for *FN1* (see Episode 1). They (incorrectly) determine that three days is the fastest it would take for the word to spread throughout *FN2*, with multiple quickest starting people: *P*, *Q*, *R*, *S*, *T* (aside: *R* and *S*, actually yield 2 days). Gil then directs the group's attention back to *M2*, and they try to apply it on *FN2*:

Gil:	So she has to tell someone who knows three people, so *Q* and *P* know 3 people. *Q* knows *S*, *P*, and *O*. *P* knows *R*, *Q*, and *O*. If you tell *Q* first, *Q* tells *S*, *P*, and *O*... Now, one person must know at least two other people.

Gil then gazes at *FN2* for a while, during which he has difficulty deciding whether one of *S*, *P*, and *O* actually know two other people (i.e., the second property in *M2*). After a while, Gil says:

Gil:	I reckon the method should be to tell someone who knows three people.
Lome:	No, but if we use that on this one (*FN1*)... *L* knows three people, but if you tell *L* first... it's a longer way...so that doesn't work.
Gil:	Yeah no, it [the method] should be start with someone who tells three people, and one of those three people must tell two people...
Lome:	No. Still if you tell *L*, one of these three people [*L*'s friends] does know two people. *H* knows two people. But *L*'s still a longer way...

The group agrees that *M2* is inadequate, and they discuss how to modify it:

Gil: So for this one [*FN2*], if you start on *T* who knows three people, these people [*R* and *S*] tell someone that's not in the same group...they don't tell the same person. That's why [points to *FN1*] *L* doesn't work here because the two people they told (*K* and *N*) only know one person. So of the three people, each will have to tell a different person...like *H*.

Lome: Okay. So out of the three people [that the first person tells], all three of them have to tell a different person. Can you write that down?

Gil: So you need to tell three people, with those three people telling a different person each time. So here [*FN2*], if you tell *Q* who knows three people…

Gil then has difficulty deciding whether *Q's* three friends each know a different person, so Lome makes a suggestion...

Lome: *P* and *S* tell two different people...

Chad: So can we say, at least two of the three friends tell different people?

Gil: I'm just thinking, if *Q* (in *FN2*) tells three people, and they each have to tell a different person... But *O* can't tell anyone else.

Lome: Yeah but *P* and *S* can tell different people. That does make sense, because for the networks we're given, the starting person tells three people...

Gil: And at least two of them need to tell a different person. Okay, so share the word with someone who knows 3 people, and at least 2 of the 3 people need to tell a different person.

In this episode the group maintained the notion that *the starting person must have three friends* through their modifications to *M2*. Again, we ask why the group did this? At the beginning of this episode the group had a goal to apply *M2* on *FN2*, which Gil tried to do but he experienced difficulty—he readily perceived that person *Q* had three friends, but found the second property of *M2* problematic (i.e., *one of the 3 friends of the starting person must know at least 2 other people*). Gil then resorted to *M1*, which Lome rejects as he did in Episode 2. Lome's rejection of *M1* shifted the group's attention back to *FN1*. Gil then re-proposed *M2*, justifying it using person *H*. However, Lome perceives a property of person *L* (one of its 3 friends (*H*) knows at least 2 other people), rendering *M2* inadequate (as it worked for *L*, but *L* was not a quickest starting person). After the group agreed that *M2* was inadequate, they aimed to fix *M2* using a simple strategy—keep what worked and change what did not work. Since it was the second part of *M2* that did not work (when Gil tried to apply it on *FN2*), it is understandable that the group still considered the notion that the *starting person must have three friends* favorably despite *M2*'s inadequacy. Gil then perceived a property of person *T* (i.e., its friends *R* and *S* each shared the word with a different person) – a property that Gil recognized holds for *H* but not for *L* in *FN1*. Gil then proposed the method (*M3*): *share the word with a person who knows three people, and each of those three people must tell a different person*. The group then tried to apply *M3* on *FN2*, starting with *Q*, but immediately experienced difficulty—again, they readily perceived that *Q* had three friends, but they could not verify the latter part of *M3* (*each of the 3* *friends must tell a different person each*). At this point, Lome and Chad perceived another property of person *Q*—*at least two of its three friends share the word with a different person*—which they had perceived was also a property of *H* in *FN1*. The group then modified *M3*, again employing their strategy for fixing an inadequate method—keeping what worked (i.e., *starting person must have three friends*) and changing what did not work (i.e., *each of the 3 friends must tell a different person)* to *two of the three people must tell a different person each*. This yielded the group's final method: *share the word with someone who has three friends, at least two of which must share the word with different people*.

DISCUSSION AND FUTURE DIRECTIONS

Our research questions were: (i) what aspects of the group's attention do not change throughout the task? and (ii) /how do these unchanging aspects lead the group to constantly posit the incorrect way of understanding? Our analysis shows three unchanging aspects of the group's attention that were influential in perpetuating the relevance of the incorrect way of understanding—*the starting person must have three friends*. These three aspects were: (i) person *H* (in FN1); (ii) the group's strategy for creating a method, by stating relatively distinctive properties of the quickest starting person; (iii) the group's strategy for fixing an inadequate method by keeping what worked and changing what did not work. How were these three unchanging aspects significant? We summarize their significance as follows: Person *H* was the quickest starting person in *FN1* (and having three friends was an invariant, relatively distinctive and salient property of person *H*). Further, the candidates for starting person in *FN2* that the group focused on (*P*, *Q*, and *T*) all had three friends. So whenever a method was deemed inadequate, the group naturally, and by virtue of their aforementioned strategies, maintained that the starting person had to have three friends, while changing the other properties of the method that did not work. These unchanging aspects of attention suggest that, on the whole, the group's attention did not so much *shift* (in the sense of a change in direction or focus) as it did *expand* (in the sense of a change in size). In other words, the group continually posited a particular way of understanding (i.e., *ID*) and added to it, by incorporating more and more information (even information that could potentially undermine it). This is reminiscent of Skemp's (1971) notion of a *self-perpetuating schema* which "while may be the most effective organiser of existing knowledge, it's very strength may be the source of it's potential downfall; a tendency towards the self-perpetuation of an existing schema...becoming an obstacle to adaptability" (p. 41). The notion of *obstacle* (Brousseau, 1997) also comes to mind, when thinking about the persistence of the group's way of understanding. An *obstacle* can be construed as a way of understanding that functions productively in a variety of situations, thus establishing it as a useful cognitive tool for interpreting some situations, but then breaks down and leads to errors in other situations. The defining characteristic of an obstacle is its persistence. Brousseau (1997) argued that an obstacle must be *overcome* (revised/modified) in order for learning to take place. But studies on obstacles (and overcoming them) in the short-term context of a task are rare and, still without an analytical framework, the notion of obstacle seems metaphorical at this point. In this light, we wonder whether the incorrect way of understanding proposed in our paper be thought of as an obstacle with respect to a task? And furthermore, perhaps unchanging aspects of students' attention are factors in the persistence property of obstacles? Such questions warrant further inquiry.

The main argument of this paper is that unchanging aspects of the structure of attention might cause the persistent advancement of an incorrect way of understanding. We have also shown, by examining at a fine-grained level their structures of attention, that the group's persistent focus on this way of understanding was understandable. We thus believe that Mason's framework has the potential to not only give us a more sympathetic outlook on the persistence of insufficient knowledge, but also gives us a more rigorous way of talking about the seemingly fortuitous persistence of students' so-called "insufficient knowledge".

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EXPLORING THE RELATIONSHIP BETWEEN ACCEPTED WAYS OF REASONING IN THE CLASSROOM AND INDIVIDUALS' WAYS OF REASONING

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While social theories have offered powerful ways to explore how social interactions can support learning, researchers are still grappling with how to coordinate these insights with what is known about the nature of individual learning. This study explores this topic using the emergent perspective (Cobb & Yackel, 1996) to investigate students' individual variation from ways of reasoning that were accepted in their class community. The results show that students can reason in ways that are qualitatively different in mathematically significant ways from established ways of reasoning, even after instruction has ended, and even if they continued to intellectually engage in classroom interactions. Perhaps more importantly, the results offer an image of ways this variation can occur.

INTRODUCTION

In the late eighties, mathematics education research took what Lerman (2000) called a "social turn" (p. 19). This meant that researchers began to conceive of knowledge as inseparable from the social context in which that knowledge was developed, to explore the semiotic and cultural mediation of thought, and investigate learning as enculturation into practice (Brown, Collins, & Duguid, 1989). Thus, mathematics educators began to expand the unit of analysis beyond the individual to more thoroughly explore the social aspects of learning.

However, as Lerman (2000) pointed out, while new theories advanced researchers' ability to study learning as it occurs in classroom contexts, it was not immediately clear how to account for the social nature of learning without losing nuances in individuals' ways of reasoning. He said,

> A major challenge for theories from the social turn is to account for individual cognition and difference and to incorporate the substantial body of research on mathematical cognition (p. 27).

In particular, while social theories advanced educators' understanding of how particular interactional patterns among participants support learning generally, the interpretative nature of learning was downplayed and it was not clear how different students might differ in what they learn after participating in the same social interactions.

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THEORETICAL PERSPECTIVE

One possible way to meet this challenge is to further develop ideas from the *emergent perspective* (Cobb & Yackel, 1996). This theory combines aspects of symbolic interactionism (Bauersfeld, Krummheuer, & Voight, 1988) and constructivism (von Glasersfeld, 1983) to coordinate social aspects of the classroom microculture with psychological features of individuals who participate in classroom activities. In this approach, the social and individual planes have equal weight.

The emergent perspective outlines three social aspects of the classroom—social norms, socio-mathematical norms, and classroom mathematical practices—and their individual psychological correlates. This study will focus on coordinating the last of these, *classroom mathematical practices* with its individual correlate, students' own mathematical conceptions and activity. Classroom mathematical practices are mathematical ways of reasoning and operating that become taken-as-shared. This means that in the classroom community, participants assume that other participants are familiar with and understand the way of operating.

According to Cobb and Yackel (1996) the relationship between individuals' conceptions and mathematical practices is indirect and reflexive. This means that individuals' ideas gives rise to classroom mathematical practices as individuals share and negotiate ideas. Then, as ways of reasoning become accepted in the community, they influence, but do not determine, students' further reasoning and conceptions. Because participation in emergent practices is not deterministic of further ways of reasoning, classroom participants may not share identical conceptions. This diversity of student ideas is acknowledged through use of the metaphor that students participate differentially in classroom mathematical practices. However, little is known about the nature of this differential participation.

EFFORTS TO COORDINATE THE SOCIAL AND INDIVIDUAL ASPECTS OF THE EMERGENT PERSPECTIVE

Despite the promise of the theory to coordinate individual and social constructs, most of the research conducted by those who have worked from the emergent perspective has focused on fleshing out and investigating the social constructs of social norms, socio-mathematical norms, and classroom math practices. This is understandable, given the need to operationalize these constructs in classroom-based mathematics education research. Indeed, the emergent perspective provides a powerful way to conceptualize social aspects of the learning environment generally, and in particular the mathematical progress of the class at the collective level. Studies that characterize the mathematical progress of the classroom are compelling because this is how the teacher experiences the classroom. However, this underscores the importance for coordination with individual ways of reasoning since it would be useful for teachers to know how the mathematical progress of individual students might differ from the mathematical progress of the class as a collective. This is consistent with the original goals of the theory as Cobb (1999) himself called for "the need to clarify the relation between individual students' reasoning and the collective practices in which they participate in” (p. 33).

Although understanding the complex relationship between individual interpretations and normative ways of reasoning is in its infancy, a few relevant studies have been conducted (Rasmussen, Wawro, & Zandieh, 2015; Stephan, Cobb, & Gravemeijer, 2003; Tabach, Hershkowitz, Rasmussen, & Dreyfus, 2014). The most comprehensive investigation was performed by Stephan, Cobb, and Gravemeijer (2003). They outlined how two first grade students, Nancy and Meagan, participated in the emergence of the classroom mathematical practices and how this participation affected their subsequent ways of reasoning. They found that while the students generally reasoned in ways that were consistent with the established practices, there were a few instances where Meagan’s mathematical conceptions were qualitatively different from an established practice. However, Meagan eventually reorganized her knowledge to be more consistent with the practices when her ways of reasoning became problematic for continued participation in the class. While this image provides a starting point in understanding the relationship between math practices and individual ways of reasoning, it is likely that the relationship described in this study is not the only possible relationship. Perhaps qualitatively different ways of reasoning can persist, even after instruction has ended. If unproductive ways of reasoning persist, under what circumstances does this occur? And how could this be addressed so a greater number of students reason in mathematically powerful ways? This gives rise to the research question addressed by this study: How are individuals’ ways of reasoning related to the progression of increasingly sophisticated ways of reasoning that function as if shared in the classroom?

METHOD

To answer this research question I studied a logarithm unit in a class of 26 prospective secondary teachers. An experienced mathematics education researcher taught this course and a wide variety of students regularly participated in classroom discussions. These discussions were the impetuous for the development of mathematical progress in the class.

To analyse progress at the classroom level, I used the documenting collective activity (DCA) method (Rasmussen & Stephan, 2008; Cole et al., 2012). Using this approach, I identified changes in the structure of classroom arguments over time. These changes gave evidence that particular ideas had become accepted in the class community (for more details on the criteria used see Rasmussen & Stephan, 2008; Cole et al., 2012). I called these accepted ways of reasoning normative ways of reasoning (NWRs). Following Rasmussen and Stephan (2008) and others who have used their method (Cole et al., 2012), I then grouped related normative ways of reasoning into collective mathematical practices. For this study, I then focused on one math practice, the point at which students were transitioning from linear reasoning to exponential. I choose to focus on this practice because previous research has shown that this is difficult transition for many students to make (Alagic & Palenz, 2006). As such, there was more potential for individual variation from the established practice.

After establishing the emergent mathematical practices and identifying one to focus on, I investigated individuals' ways of reasoning that were related to that practice by analysing clinical interviews administered after instruction to seven of the students in the class. When analysing their responses I used open coding and the constant comparison method from grounded theory (Strauss & Corbin, 1994) to develop and refine categories that described the students' ways of reasoning. The results of this analysis revealed the nature and extent of individual variation from the normative ways of reasoning.

INSTRUCTIONAL CONTEXT

The unit I studied focused on deepening students' understanding of exponential and logarithmic relationships. This centred on developing an exponential number line and then using that number line to explore exponential relationships. On the first day of the unit, the students were asked to create a timeline that represented the earth's history—from 15 billion years ago until today—and place several historic events on the line (Confrey, 1994). Three distinct approaches were presented in class. The first was a linear approach. The second was a piecewise linear approach in which they used several different linear scales. The third approach was different than the first two in that it used an exponential structure at a macro scale, meaning there were equally spaced tick marks that increased by powers of ten. However, the segments formed by those tick marks were subdivided linearly. Given the macro exponential structure of this approach, the teacher further explored it in class to encourage the development of a fully exponential number line—meaning one that had the macro exponential structure, but was also subdivided exponentially. After considerable debate, the students realized their linear method of subdivision was not consistent with the macro exponential structure of the line. This led to the acceptance of exponential ways of reasoning to subdivide.

RESULTS

The analysis of the classroom arguments using the DCA method resulted in identification of five Math Practices. Math Practices 1, 2, and 3 centred on creating a fully exponential number line, while Practices 4 and 5 focused making sense of exponential and logarithmic relationships, including using the exponential number line to do so. This study focuses on Math Practice 2, which consisted of two normative ways of reasoning that describe ways to exponentially subdivide the segments of the line (see Table 1). The acceptance of these two ways of reasoning eventually led to the rejection of the linear way of subdividing considered earlier.

The first of these, NWR 2.1: *Reasoning Linearly About Exponents*, is characterized by students focusing on exponents. Students would write the endpoints of the segment in the form a^b, ignore the base, and reason linearly with the exponents. For example, if they wanted to find the midpoint of 10 and 100, they would write two endpoints as 10^1 and 10^2, and then focus on the exponents of 2 and 1 by thinking of the "halfway point" of 1 and 2. This would tell them that the midpoint should have an exponent of 1.5. In the second way of reasoning, NWR 2.2: *Preserving Multiplicative Relationships*, students extended the times ten pattern that existed between segments to within the segments by treating same size subsections as representing multiplication by a constant factor. Thus, to find the midpoint of 10 and 100 using this method, students would notice that there is an increase by a factor of 10 between the two endpoints, and then recognize that to preserve the macro pattern of constant multiplication, one needs a number that when multiplied by itself yields ten. This is the square root of ten, meaning the halfway point should be ten times the square root of ten.

Math Practice 2: Subdividing the Segments
NWR 2.1: Reasoning Linearly About Exponents
NWR 2.2: Preserving the Multiplicative Relationship within the Segments

Table 1: The component NWRs of Math Practice 2.

It is important to note that even though both methods yield the correct answer, they are significantly different in terms of the conceptual understanding necessary to engage with them. In particular, it is possible to for students to proficiently use NWR 2.1, without engaging with exponential reasoning at all. These students could simply ignore the base and reason linearly. However, this method is useful to quickly determine placements on the number line. Thus, perhaps the ideal learning outcome for students would be that while they might use NWR 2.1 for efficiency's sake, they would also fully understand NWR 2.2 and its mathematical connections to NWR 2.1.

The results of the individual analysis were striking, in that more than half the students interviewed reasoned in ways that were not fully consistent with Math Practice 2, in that these students did not intellectually engage with both of the practice's constituent NWRs. In particular, only the three students whose reasoning was coded in the first category captured important nuances of Math Practice 2, which are critical to developing deep conceptual understanding of the exponential number line. In Category 1 were ways of reasoning in which students coordinated multiplicative relationships among the values on the number line with linear relationships that existed among the exponents. In contrast, a student whose reasoning was categorized in Category 2 was consistent only with NWR 2.1, meaning they reasoned solely about the exponents, with no connections to multiplicative relationships. Finally, reasoning in Category 3 was characterized by students still struggling to abandon a way of reasoning considered early on in the unit, reasoning linearly about the actual values on the line. Two students' reasoning was coded as Category 2 and two students' reasoning was coded as Category 3.

While the four students whose reasoning was coded as Categories 2 or 3 did not use multiplication to reason about what value lay at the midpoint, their ways of reasoning were recognizable from class discussions. Those who used Category 2 reasoning reasoned in a way that was consistent with normative way of reasoning 2.1. This correct way of reasoning was accepted in class. However, in class discussion NWR 2.2 was often used as a way of justifying the linear pattern. This means that in class, NWR 2.1 and 2.2 were used in conjunction as students fully justified their labelling. Category 3 was a way of reasoning that was discussed in class, though it was eventually abandoned as students pointed out the inconsistencies between the macro exponential structure of the number line and the linear subdivision.

This means that the ways of reasoning expressed in the post-interviews were not idiosyncratic interpretations of what was expressed in class. Rather, the ways of reasoning expressed in the interviews were consistent with ways of reasoning expressed in class (see Table 2), but importantly four of the seven students did not fully understand how the ways of reasoning discussed in class were related to one another, in that one was eventually rejected (Category 3) and the other (Category 2) had deep mathematical connections to multiplication.

Ways of Reasoning in Class	**Ways of Reasoning in Interviews**
Math Practice 2	Category 1: Multiplicative Reasoning Coordinated with Reasoning Linearly with the Exponents
NWR 2.1 Subdividing Segments by Reasoning Linearly About Exponents	Category 2: Reasoning Linearly with the Exponents
An Early way of Reasoning**: Linear Subdivision**	Category 3: Elements of Reasoning Linearly Among the Values

Table 2: The relationship between ways of reasoning that appeared in class and those that appeared in the interview.

DISCUSSION

Cobb and Yackel (1996) always maintained that researchers should not expect individual classroom participants to reason in ways that were identical to ways of reasoning that had become accepted in the class. However, previous research on the relationship between individual ways of reasoning and normative ways of reasoning has not illuminated the nature and extent of individual variation from normative ways of reasoning that can persist after instruction. In fact, previous work seemed to suggest that even though individuals can vary from normative ways of reasoning, problematic ways of reasoning will eventually resolve themselves through continued class participation (c.f. Stephan et al., 2003). However, this study demonstrated that it is possible for students to reason in ways that are qualitatively different in mathematically significant ways from an established practice, even after instruction has ended. This was striking given the continued participation in the classroom of the four students whose reasoning was not fully consistent with Math Practice 2, including giving arguments that contributed to subsequent mathematical development at the classroom level. This means that the way the teacher experiences the class, normally through the mathematical progress of the class as a collective, can belie students' individual understanding, even if those individuals are able to continue to productively participate in subsequent classroom interactions.

To understand why students were able to participate without shifting their thinking, it is productive to recall the impetus for Meagan's reorganization of knowledge in the study performed by Stephan et al. (2003). Importantly, Meagan ways of reasoning that were qualitatively different from established practices produced different answers than the accepted ways. This difference in outcomes caused problems for continued participation in class activities. This is in contrast to students who did not reason multiplicatively in this study, as reasoning linearly with the exponents yields correct placements on a number line. As such, it was possible for student to continue to participate without shifting their way of reasoning. However, these students did not seem to be able to coordinate the various ways of reasoning.

Thus, in retrospect, one way the teacher may have been able to encourage more students to coordinate NWRs 2.1 and 2.2 as they reasoned individually is to have more explicit class conversations about this relationship. In particular, it may have been productive to establish a third normative way of reasoning, as part of Math Practice 2, that centred on this relationship. While this relationship was implied in the classroom, it was not discussed in a way that qualified it as normative. A greater explicit focus on this relationship may have made it more problematic for continued intellectual participation in the classroom for students who could only reason linearly. This may have encouraged them to think about multiplication and its relationship to their linear way of reasoning

This study has provided insights into the nature of relationships that can exist between established practices and individual ways of reasoning. Exploring this topic further will yield more insights into the teaching-learning process in classroom environments. This greater understanding will in turn yield insights into how to structure classroom environments for students. For example, this study demonstrated that students could continue to participate in classroom interactions without engaging in multiplicative reasoning, because reasoning linearly with the exponents yielded correct answers. As such, it may be important for teachers to consider what conceptual understanding is necessary with various approaches to a problem and how to problematize ways of reasoning that may yield correct answers, but does not require deep conceptual understanding. This could be by encouraging students to think about relationships between ways of reasoning. This is just one example of how understanding the relationship between social and individual aspects of learning can inform teaching. More research is needed to fully understand the relationship and its implications for teaching.

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