



PME44 VIRTUAL

THE 44th CONFERENCE OF THE INTERNATIONAL GROUP
FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

July 19-22, 2021

Hosted by Khon Kaen, Thailand

Virtually hosted by Technion, Israel Institute of Technology, Isreal

Proceedings

of the 44th Conference of the International Group
for the Psychology of Mathematics Education

VOLUME 2

Research Reports (A-G)

Editors:

Maitree Inprasitha, Narumon Changsri
and Nisakorn Boonsena

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Cite as:

Inprasitha, M., Changsri, N., & Boonsena, N. (Eds). (2021). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (Vol.2). Khon Kaen, Thailand: PME.

Website: <https://pme44.kku.ac.th>

Proceedings are also available on the IGPME website: <http://www.igpme.org>

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ISBN 978-616-93830-1-7 (e-book)

Published by Thailand Society of Mathematics Education, Khon Kaen, Thailand

123/2009 Moo. 16 Mittraphap Rd., Nai-Muang, Muang District Khon Kaen 40002

Logo designed by Thailand Society of Mathematics Education

PREFACE

We are pleased to welcome you to PME 44. PME is one of the most important international conferences in mathematics education and draws educators, researchers, and mathematicians from all over the world. The PME 44 Virtual Conference is hosted by Khon Kaen University and technically assisted by Technion Israel Institute of Technology. The COVID-19 pandemic made massive changes in countries' economic, political, transport, communication, and education environment including the 44th PME Conference which was postponed from 2020. The PME International Committee / Board of Trustees decided against an on-site conference in 2021, in accordance with the Thailand team of PME 44 will therefore go completely online, hosted by the Technion - Israel Institute of Technology, Israel, and takes place by July 19-22, 2021. A national presentation of PME-related activities in Thailand is part of the conference program.

This is the first time such a conference is being held in Thailand together with CLMV (Cambodia, Laos, Myanmar, Vietnam) countries, where mathematics education is underrepresented in the community. Hence, this conference will provide chances to facilitate the activities and network associated with mathematics education in the region. Besides, we all know this pandemic has made significant impacts on every aspect of life and provides challenges for society, but the research production should not be stopped, and these studies needed an avenue for public presentation. In this line of reasoning, we have hosted the IGPME annual meetings for the consecutive year, July 21 to 22, 2020, and 19 to 22 July 2021, respectively by halting “on-site” activities and shift to a new paradigm that is fully online. Therefore, we would like to thank you for your support and opportunity were given to us twice.

“Mathematics Education in the 4th Industrial Revolution: Thinking Skills for the Future” has been chosen as the theme of the conference, which is very timely for this era. The theme offers opportunities to reflect on the importance of thinking skills using AI and Big Data as promoted by APEC to accelerate our movement for regional reform in education under the 4th industrial revolution. Computational Thinking and Statistical Thinking skills are the two essential competencies for Digital Society. For example, Computational Thinking is related to using AI and coding while Statistical Thinking is related to using Big Data. Therefore, Computational Thinking is mostly associated with computer science, and Statistical Thinking is mostly associated with statistics and probability on academic subjects. However, the way of thinking is not limited to be used in specific academic subjects such as informatics at the senior secondary school level but used in daily life.

For the PME 44 Thailand 2021, we have 661 participants from 55 different countries. We are particularly proud of broadening the base of participation in mathematics education research across the globe. The papers in the four proceedings are organized according to the type of presentation. Volume 1 contains the presentation of our Plenary Lectures, Plenary Panel, Working Group, the Seminar, National Presentation, the Oral Communication presentations, the Poster Presentations, the Colloquium. Volume 2 contains the Research Reports (A-G). Volume 3 contains Research Reports (H-R), and Volume 4 contains Research Reports (S-Z).

The organization of PME 44 is a collaborative effort involving staff of Center for Research in Mathematics Education (CRME), Centre of Excellence in Mathematics (CEM), Thailand

Society of Mathematics Education (TSMEd), Institute for Research and Development in Teaching Profession (IRDTP) for ASEAN Khon Kaen University, The Educational Foundation for Development of Thinking Skills (EDTS) and The Institute for the Promotion of Teaching Science and Technology (IPST). Moreover, all the members of the Local Organizing Committee are also supported by the International Program Committee. I acknowledge the support of all involved in making the conference possible. I thank each and every one of them for their efforts. Finally, I thank PME 44 participants for their contributions to this conference.

Thank you

Best regards

Associate Professor Dr. Maitree Inprasitha
PME 44 the Year 2021
Conference Chair

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RESEARCH REPORTS (A-G)

SECONDARY MATHEMATICS TEACHERS USE OF FACEBOOK FOR PROFESSIONAL LEARNING

Judy Anderson¹ and Blanca Swanson¹

The University of Sydney, Australia

Professional learning is critical for mathematics teachers to support reflective practice, and to learn about new ideas, resources and pedagogies. Online communities provide opportunities for teachers to engage with other practitioners, but to what extent do these sites enable shared understandings, mutual engagement, and the development of shared meaning-making resources? Little is known about how secondary mathematics teachers interact with online groups, and how such groups function for members of different levels of engagement. This study used an audit, questionnaire and interviews to explore levels of teacher engagement and to investigate the potential for the development of an online community of practice.

INTRODUCTION

Innovative reforms in practice, such as using inquiry-based pedagogies and alternative assessment approaches are less evident in secondary mathematics classrooms, particularly if teachers teach the way they were taught (Goos & Bennison, 2008). Changing practice is complex. Strategies such as policies which mandate new approaches to assessment have been used to change practice (Barnes, Clarke, & Stephens, 2000) although not always successfully nor sustainably. More successful reform programs employ high quality professional learning (Darling-Hammond, Hylar, Gardner, & Espinoza, 2017), frequently aiming to develop teacher ownership through shared understandings, mutual engagement and the collaborative development of resources.

Traditionally, teacher professional development involved attending courses, seminars and conferences (Lantz-Andersson, Lundin, & Selwyn, 2018), but there has been increased recognition of other opportunities for teacher professional learning, including through communities of practice, which allow for dynamic, collaborative and participant-driven learning (Goos & Bennison, 2008). Professional learning communities encourage teachers to ask questions focused on their practice, which facilitates growth in teachers' professional identities. Regardless of geographical location and potential isolation, social media sites provide new opportunities for teachers to share experiences and resources, and to create, develop and incorporate innovative pedagogies into their teaching practice. It is therefore worth investigating how an established

social media site such as Facebook could serve as a space for professional learning, potentially leading to an online community of practice.

This project investigated an existing Facebook group used by secondary mathematics teachers in one Australian state. The recent implementation of a new mathematics syllabus, including requirements for alternative, inquiry-based high-stakes assessment tasks for the final two years of schooling, positioned the group in a key period of implementation of new classroom practices. The project thus aimed to investigate the question: To what extent can an established Facebook group become a community of practice for secondary mathematics teacher members of the group?

LITERATURE REVIEW

The “community of practice” framework (Lave & Wenger, 1991, p. 10) is used to describe a group of professionals who use their social ties and common objectives to improve their practice, by building a body of related resources and knowledge (Goos & Bennison, 2008; Lantz-Andersson et al., 2018). The process of learning in such a community is inherently social; it is achieved through observation and participation within the community. Wenger (1998) described three key features of a community of practice: the formation of a joint enterprise held by the group; the practice of mutual engagement from members; and the creation of a shared repertoire of meaning-making resources. Wenger, McDermott, and Snyder (2002) noted there are three typical levels of participation in a community of practice: core members, who regularly initiate group interactions and energise the community; active members, who regularly participate in group interactions; and legitimate peripheral participants, who learn through observation of the interactions between core and active members. The legitimate peripheral participants have the potential to become core or more active members since through apprenticeship, they transition from novice to expert.

The rise of the Internet and its enhanced capacity to maintain stable infrastructure without external financial patronage, has provided new possibilities for teacher networking and collaboration. Consequently, there has been a multitude of initiatives designed to utilise online sites for professional learning, resource sharing and forming communities (e.g., Lantz-Andersson et al., 2018). Researchers have focused on Facebook groups as an opportunity for teacher professional learning (Rutherford, 2010), a means to promote teacher inquiry, collaboration, and adoption of innovative pedagogies (Goodyear, Casey, & Kirk, 2014), and as an “extended staff room” (Lantz-Andersson, Peterson, Hillman, Lundin, & Rensfeldt, 2017, p. 54). However, these studies have not addressed how Facebook groups can function independently as a community of practice, or to what extent participants’ contributions within the group might impact classroom practice. While online communities appear to

promote the resource and idea sharing that forms the development of innovative practice, how effective are they in promoting mutual participation and a shared teacher identity?

In addition, there is little information about how peripheral participants engage with online communities, as they often leave no digital trace of their presence. This limitation was recognised by Rutherford (2010), who concluded “there is no way of knowing if the knowledge of these ‘lurkers’ was affected by simply reading the posts of the active group members” (p. 68). Lantz-Andersson et al. (2017) argued that peripheral participants might view Facebook groups as networks rather than communities of practice, yet they suggested meaningful forms of passive engagement could still exist within the group. Also, there is limited research about how mathematics teachers engage in online communities, despite their potential to model the planning and pedagogy needed to bring new ideas and practices into the classroom (Goos & Bennison, 2008), hence the need for this study.

METHODOLOGY

A case study of teachers’ participation in the Mathematical Association of New South Wales (MANSW) Facebook group, a closed group with over 2000 members was conducted by the authors. Previous research on Facebook as an online teacher community has used a range of data collection methods such as participant interviews and collecting archival documents (Kelly & Antonio, 2016), surveys, online participant observations (Goodyear et al., 2014), and audits (Lantz-Andersson et al., 2017). Since the project aimed to investigate how a specialised Facebook group might support secondary mathematics teachers as a community of practice, the combination of an audit, questionnaire and interviews was chosen to explore the context. However, due to space constraints, this paper only presents data from the audit and questionnaire.

The audit of the group was conducted, focusing on the posts, comments and reactions within a one-year period. Facebook’s Group Insights tool was used to find the total number of posts, comments, reactions and members who viewed posts per day, as well as information on member demographics. Since the tool did not record which members posted to the group, one month was examined manually to record the frequency with which members posted to the group. A small sample of posts were then analysed with a process of open coding, resulting in the formation of 12 descriptive categories of post types. All posts within the one-year timeframe were then coded into these categories (see Table 1). The audit recorded the number of peripheral participants in the group, in comparison to previous studies that lacked such data (Lantz-Andersson et al., 2017).

After the audit, an anonymous questionnaire was posted to the discussion page of the group, seeking information about the underlying motivations of members

to participate in the group, as well as their perceptions about how their engagement impacted practice. The questionnaire was brief, asking respondents to identify: their reasons for visiting the group from a list of possible responses derived from the categories developed in the audit; how often they visited the group; whether they had seen any ideas or resources in the group that they would be interested in using in the classroom; whether they had used any ideas or resources from the group in the classroom; how long they had been teaching mathematics; and how often they commented or posted to the site. In comparison to the audit, surveying participants enabled a greater understanding of members' different levels of engagement.

For expedience, in contrast to Wenger et al (2002) definitions of core, active and peripheral members as representing experts or novices, we chose to initially classify members according to their engagement with the Facebook page through posting comments or reacting to the posts of others. This categorisation was further explored in the questionnaires and interviews suggesting that some experienced and potentially 'expert' teachers were peripheral participants. We argue that the focus of the study was to ascertain engagement and the potential for the development of a community of practice regardless of the level of expertise of participants.

RESULTS AND DISCUSSION

During the **audit**, posts were categorised into twelve purposes for engagement with the Facebook group (Table 1). The audit also provided evidence of the three levels of member involvement; over the year, there was an average of five posts, 45 comments and 108 reactions submitted to the group each day, showing the widening impact of core and active members. However, there was also an average of 1200 members each day who viewed posts, with approximately 13% of members who saw posts actively responding to them on a given day, and the remaining 87% of these members were considered peripheral participants. It is therefore crucial to consider how these peripheral users engaged with the group, as they appeared to comprise such a high proportion of members.

Category	Number of Posts	% Total
Sharing a link for discussion	327	18%
Sharing a link for humour	231	13%
Sharing resources/teaching ideas	230	13%
Asking for opinions on teaching	198	11%
Sharing experience or awards	173	9%
Asking for resources/ideas	141	8%
Asking questions about syllabus	135	7%
MANSW admin/conference information	131	7%
Offering/asking for employment	114	6%
Asking for solutions to a mathematics question	80	4%

Asking for assessment ideas/advice	51	3%
Other	16	1%
Total	1827	100%

Table 1. Categories of discussion posts over a one year period

The 120 **questionnaire** responses were collated and analysed to determine each respondents level of engagement. The levels of engagement were defined as: core members as those who regularly started discussions, commented on posts, and energised the community (11, 9%); active members as those who often commented or reacted to posts and occasionally started discussions (53, 44%); and peripheral participants as those who mainly read posts without actively replying (56, 47%). It is not surprising that more core and active members (53%) responded to the questionnaire than peripheral members (47%) given their more active engagement with the Facebook group. To investigate the extent to which the group supports mathematics teachers as a community of practice, the remaining data are presented under the three features of joint enterprise, mutual engagement, and a shared repertoire of resources. In each of the sections, attention is given to members of varying levels of engagement.

Joint Enterprise

Wenger (1998) describes the joint enterprise of a community as the purpose of a group as continually redefined and acted upon by its members. While the MANSW Facebook group (<https://www.facebook.com/MathsNSW/>) was developed for the exchange of information and ideas between members, it is important to consider how the group was used by its members. Within a one-year period, the 1827 posts submitted to the main discussion page of the group were analysed to determine the most common ways the site was used. From the 12 categories identified in Table 1, categories were grouped to provide the main uses presented here.

First, the group functioned as a place for intentional professional discussion. While discussions also arose in the comments of other posts, 29% of posts were intentionally aimed at either informing or starting discussions, by sharing an article related to teaching or asking for members' opinions on specific aspects of pedagogy. For example, the following post asked for opinions on a screenshot from one of the new syllabus documents, resulting in 32 comments and 66 reactions. The phrase "brains trust" was commonly used in such posts to the discussion group, evidencing a culture of collaborative discussion.

Brains trust, Asymptotes have been discussed here a few times. What do we think of this definition, from the Glossary attached to the new syllabus?

Second, the group acted as a space for individual teachers to seek specific help from a larger body of mathematics teachers – 22% of posts asked for resources, assessment advice, and clarifications about the syllabus, or solutions to specific

mathematics questions. The following post had 35 comments and 18 reactions, including teachers offering to share a Google Drive of resources.

I'm trying to come up with an alternative assessment task for my Year 12 Ext 1 class. Any ideas? I'm struggling!

Third, the group functioned as a social space for mathematics teachers, with 22% of posts used to share mathematics memes or jokes, personal experiences and pictures of conferences, or to celebrate members who had won awards. Fourth, the group was a place for individuals to offer resources or ideas to the community, as seen in 13% of posts. The group was also used by the MANSW executive to share information about conferences or administration (7% of posts) or for people to ask or offer employment (6% of posts). These posts generally had fewer interactions from other group members.

Each of these categories can be broadly considered to support the stated purpose of exchanging ideas and information in the group. However, members who posted directly to the group have expanded upon the set purpose to collectively start professional discussions, support teachers in need and share among other practitioners: essentially, to engage in collective professional learning. There was also a strong emphasis on posting for the benefit of the wider collective, rather than the group functioning simply as a get-help site for individual questions.

To gain further information about the use of the group from members who did not post directly to the discussion page, the questionnaire asked participants why they visited the site. From the 120 questionnaires, many provided more than one response but the most common selected categories were “to stay connected to other mathematics teachers” (83%), “to find resources” (79%), and “to ask a question about the syllabus” (47%). It is worth noting the passive nature of the two most common responses, which indicates many members visit the page regularly to benefit from reading existing posts.

Considering the responses of the peripheral members in more detail, there was a greater difference between the top two categories and the rest of the responses, as can be expected considering their observatory habits. Other differences for peripheral members indicated the top category was to “find resources/ideas” rather than “stay connected”. Ultimately, data from the audit and questionnaires reflected a common purpose of member participation with the group: to connect with like-minded practitioners for community and professional learning. The joint enterprise was thus evidenced among members of all levels of participation, despite their varying levels of engagement.

Mutual engagement from members

Relationships of mutual engagement are a key component to understanding how a network of people is united into a single social entity: in essence, how the

group functions as a community. In the MANSW Facebook group, engagement was expressed primarily through the main discussion page as users reacted to, and commented on, others' posts. Throughout the year, there was an average of eight comments and 20 reactions per post, which demonstrates a high level of community engagement. The nature of the Facebook group as a digital space also enabled mutual participation between members of different geographical backgrounds, although a clear majority (67%) of the members originated from the Sydney metropolitan area.

However, it should be acknowledged that many members of the group do not participate in mutual engagement to the same extent as the small group of core and active members. To obtain the frequency of members posting to the discussion page, March 2019 was chosen for detailed analysis because it was early in a new school year when a new syllabus was being implemented for the first time. In this month, there were 200 posts, 2429 comments and 4709 reactions submitted to the group. Each day, there was an average of seven posts, 78 comments and 152 reactions submitted, with 1485 members viewing posts at some point in the day. The posts submitted to the discussion page originated from 115 different members of the group, with 81 members only posting once in the month. This suggests that most members do not engage by actively posting to the group, and that even members who do post to the group do so infrequently. Yet, it would be unwise to underestimate the engagement of peripheral members. In a community of practice, all members, including peripheral participants, learn through watching the interactions between active and core members (Wenger et al., 2002). Indeed, online sites provide a powerful space for people to view these discussions, which are digitally preserved and visible to all members despite location or time.

The important practice of observation was exercised frequently by most members of the group, as evidenced by the questionnaire. Ninety-three percent of those surveyed checked the group multiple times a week, with 58% of respondents checking the group at least once a day. Facebook's Group Insights corresponded with the data, showing that for any given day, an average of 58% of the total members were viewing the group. Furthermore, frequent observation was common across members from all levels of engagement. In particular, 86% of peripheral members checked the group multiple times per week, with 46% viewing the page at least once a day. Notably, Wenger et al. (2002) argued if observation is frequent, peripheral members are not as passive as they appear, despite limited records of engagement. They explained, "like people sitting at a café watching the activity on the street, [peripheral members] gain their own insights from the discussion and put them to good use" (p. 56).

Shared repertoire of meaning-making resources

Sharing resources was a key function of the MANSW Facebook group; it was the third highest category (13%) of posts observed in the audit. The frequency of this category is especially notable when considering the comparative effort of each category of posts. Sharing links to Facebook is relatively easy; in contrast, resource sharing requires members to find or create a resource and take the initiative to share it, unprompted, with the group. An additional 11% of posts were submitted to ask for resources or ideas, which had a high level of response in comments by other members of the group. Furthermore, 79% of questionnaire respondents nominated “finding resources,” as the main reason for visiting the site. Evidently, the repertoire of meaning-making resources created by the group is important to core, active and peripheral members.

It is important to recognise that the resources created by a community of practice do not only consist of actual lesson plans or pedagogical ideas, but also the shared competencies and knowledge collectively produced by the group (Wenger, 1998). Shared competencies in the MANSW Facebook group, were evidenced through teachers contributing to knowledge on interpreting syllabus documents, marking solutions to mathematics questions, and textbook selection. Thus, the professional discussions held by the group, particularly to develop collective interpretations of the syllabus, should also be considered as part of the created shared repertoire.

CONCLUDING REMARKS

The qualities of a community of practice (Lave & Wenger, 1991): joint enterprise, mutual engagement between members, and the creation of a shared repertoire of meaning making resources, were all evidenced within the interactions of the MANSW Facebook group. However, in considering how the group supports mathematics teachers as a community of practice, there must also be an acknowledgement of the fundamental ongoing processes of observation and learning within the group that leave little digital trace. The project found evidence to confirm significant professional learning among peripheral members of the group, demonstrating that online communities should be considered as a powerful form of professional learning across members from all levels of engagement. It should be acknowledged that the project examined a single case study of a closed Facebook group for mathematics teachers in NSW, over the course of one year. Data from the interviews and questionnaire were also reliant on participant self-reporting, which may be affected by unconscious bias or deliberate self-censoring. However, its findings are relevant in recognising that social media sites can lead to an online community of practice.

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TENSIONS WITHIN TEACHERS' BELIEFS: IMPLICATIONS FOR TEACHER PROFESSIONAL DEVELOPMENT

Chiara Andrà¹, Peter Liljedahl², Ralf Erens³, & Annette Rouleau⁴

¹Università del Piemonte Orientale, Italy

²Simon Fraser University, Canada

³Freiburg University of Education, Germany

⁴University of Alberta, Canada

Beliefs that teachers hold about mathematics teaching and learning are the most investigated domain in affect-related research. These beliefs can be contradictory and thus lead to dilemmas that play a crucial role in shaping how a teacher changes her practice. In this paper, we give an account of how such contradictions have been addressed in literature and then propose a worldview lens to analyse the dilemmas of four teachers enrolled in Professional Development (PD) programs.

INTRODUCTION: BELIEFS AND PRACTICE

Beliefs are propositions about a certain topic that are regarded as true (Philipp, 2007), and tend to form clusters as they “come always in sets or groups, never in complete independence of one another” (Green, 1971, p. 41). According to Green (1971), belief clusters are coherent families of beliefs across multiple contexts. Thus, beliefs have a systemic nature. Affect-related research has provided evidence that beliefs have observable behavioural consequences (e.g., Di Martino & Zan, 2011), and a change in a teacher’s beliefs is likely to result in a change in their practice (Leder, Pehkonen & Törner, 2002), suggesting a dialectical relationship between change and beliefs in that one influences the other (Buehl & Beck, 2015). One of the challenges with this, however, is that such a dialectic relationship can lead towards the emergence of tensions between belief clusters. In this paper we are interested in looking closely at such tensions, to better illuminate the role of beliefs in shaping teachers’ behaviour.

THEORETICAL FRAMEWORK

The systemic nature of teachers’ beliefs can be understood in terms of “world views” (Grigutsch, Raatz & Törner, 1998), or epistemological beliefs about mathematics (Hofer & Pintrich, 1997), including its teaching and learning. According to Grigutsch *et al.* (1998), it is possible to outline four different world views (see also Liljedahl, Rolka & Roesken, 2007): a *process-oriented view* that represents mathematics as a creative activity consisting of problem solving using different and individual ways; an *application-oriented view* that

represents the utility of mathematics for real world problems as the main aspect of the nature of mathematics; a *formalist view* that represents mathematics as characterised by a strongly logical and formal structure; a *schema-oriented view* that represents mathematics as a set of calculation rules and procedures to apply for routine tasks. Even from the sketchily description, we can notice how world views are strongly linked to practice.

Each teacher's beliefs, thus, belongs to (at least) one world view (Erens & Eichler, 2019), as teachers' beliefs are organised in systems of beliefs (Fives & Buehl, 2012; Green, 1971; Philipp, 2007). One aspect of a belief system relevant for our research is that beliefs are organized in clusters that are not necessarily logically connected. The fact that beliefs can be contradictory (Fives & Buehl, 2012) allows the possibility for teachers to hold beliefs that belong to different clusters. Skott (2015) suggests, however, to interpret possible contradictions in teachers' belief systems not merely as incoherence, but rather to consider the different contexts in which beliefs are evoked. As "beliefs are expected to significantly influence the ways in which teachers interpret and engage with the problems of practice" (Skott, 2015, p. 19), they cannot be exhaustively described by one cluster of central beliefs. Given the complexity of teaching and the variety of stakeholders (e.g., students, parents, colleagues, the Ministry of Education), teachers usually show a coexistence of more than one cluster of beliefs (Erens & Eichler, 2019).

These considerations shed light on two intertwined features of teachers' beliefs: they are subjective in nature and individually held, but at the same time they are (or can be) socially and contextually shaped. The context plays a crucial role in evoking beliefs, for example a teacher, talking with a colleague (context 1), might show some beliefs that are different from, or even in conflict with, the ones she enacts in class (context 2) (e.g., Fives & Buehl, 2012). Our research hypothesis is that, even in the same context, contrasting beliefs may emerge. Namely, beyond Skott's (2015) findings, we aim at exploring the existence of beliefs that emerge in the same contexts but are in conflict with each other, almost like anti-clusters, and this reverberates in a teacher's practice, as change in a teacher's practice can be understood as an attempt to balance contrasting world views held by different stakeholders (Andrà, Rouleau, Liljedahl & Di Martino, 2019). In order to frame this, we refer to research on teachers' tensions.

Lampert (1985) understood tensions as problems to be managed, rather than solved, characterising teachers as "dilemma managers", who find ways to cope with conflict between equally undesirable (or desirable but incompatible) options without necessarily coming to a resolution. For Lampert (1985), the ongoing internal struggles presented by the tensions arise from and contribute to the developing identity of the teacher, and as such they have value in themselves. For Chapman and Heater (2010), "Meaningful change can occur

when the process is initiated and rooted in the teacher's experience based on a tension in self and/or practice that is personal and real to him or her" (p. 456). We further suggest that tension research applied to beliefs can offer a new insight into the frustrations and needs of the classroom and the changes that result. Furthermore, recognition of the tension inherent in teaching can help us as researchers in better understanding those apparently inconsistent behaviors we observe, and what might be construed as minimal or no change could be recast as a rational decision that weighed the practicality of the change against its potential consequences (Andrà *et al.*, 2019). Our aim with the research presented here is to understand the tension(s) between different world views. Tensions may emerge when teacher beliefs are challenged, for example during PD. Our research questions are as follows: When does a tension between world views emerge? How does a teacher cope with tensions? How does a tension reverberate in a teacher's practice?

METHODOLOGY

The participants for this study come from a set of more than 200 teachers who participated in PD sessions led by one of the authors in 2016. Of them, 26 volunteered to be interviewed at the end of the sessions. The relatively limited number of interviewees is due to the fact that researchers aimed at conducting extended interviews, which were semi-structured, lasted 30 to 60 minutes, were audio-recorded, and then fully transcribed. The structure of the interview aimed at letting beliefs emerge through the narrative rather than by direct questioning. For example, we invited the teachers to describe their school, the relationship with their colleagues, and with parents. Preliminary analysis of each of these 26 transcripts revealed that 19 expressed beliefs belonging to different clusters. To note, this confirmed Fives & Buehl's (2012) study that teachers often hold beliefs that can be contradictory. Further analysis revealed that the ways in which the teachers coped with this fell into one of four categories - (i) ignoring the conflict, (ii) internal struggling, (iii) balancing two worldviews, (iv) resolving the conflict. In what follows we present a deeper analysis of four prototypical cases, one selected from each of the aforementioned categories. Teachers' fictitious names are, respectively: Vicky for case (i), Julia for (ii), Ron for (iii) and Mary for (iv).

In analysing the verbatim transcribed interviews, we used a qualitative coding method (Mayring, 2015), based on Erens and Eichler's (2019) four deductive categories described in their coding manual. Examples of statements coded as *application-oriented view* are: "mathematics helps to solve tasks and problems that originate from daily life", "the ideas of mathematics are of general and fundamental use to society", and "a sound knowledge of mathematics is very important for students in their whole life". Examples of statements coded as *formalist view* are: "logical strictness and precision are very essential aspects in mathematics", "mathematics is a logically coherent edifice free of contradiction

consisting of precisely defined terms and statements which can be unequivocally be proven”, and “in mathematics students must use mathematical terms correctly”. Examples of statements coded as *process-oriented view* are: “there is usually more than one way to solve a task or problem in mathematics”, “in order to comprehend and understand mathematics, one needs to create or (re-)discover new ideas”, and “everyone is able to (re)invent or to comprehend the central ideas of mathematics”. Examples of statements coded as *schema-oriented view* are: “Mathematics consists of memorising, recalling and applying procedures”, “doing mathematics demands a lot of practice in adherence and applying to calculation rules and routines”, “nearly any mathematical problem can be solved by the direct application of familiar rules, formulas and procedures”, and “to solve a mathematics task, there is mostly a unique way of solution which needs to be found”. These examples are taken from Erens and Eichler’s research. Each teacher’s statement has been assigned a world view by one of the authors, and the other authors independently agreed or disagreed. In case of disagreement, discussion among the authors took place, until consensus has been reached.

RESULTS AND ANALYSIS

As teachers talk about (aspects of) their practice in their interviews, we analyse the tensions between worldviews that emerged. For Vicky and Julia, tensions emerge between two coexisting views, whilst for Ron and Mary the tension is provoked by an external agent. Julia and Mary significantly change their practice, Ron introduces a new practice but still employs the ‘old’ one, and Vicky does not show change.

Vicky: When asked to talk about her teaching method, Vicky commented that she does not “have a specific one: it is different for each class, because each one is different. [...] I propose problem-based group activities, where math and physics are applied to everyday life”. An *application-oriented view* emerges from Vicky’s words, as *mathematics helps to solve problems originating from daily life* (see examples of codes). Vicky, then, referred to one of her classes:

The characteristic of this class is that the traditional lessons annoy them, hence I started to propose group activities dedicated to the study of physical phenomena applied to real situations. The result has been excellent: the students have developed a high sense of critique and above all they have cooperated together for solving the given problems. Every activity has been welcomed with absolute enthusiasm.

In the last excerpt, a *process-oriented view*, which values solving problems in a creative way, emerges in one of Vicky’s classes. When talking about her teaching, and referring to her specific classes, two different views of mathematics emerge for Vicky, but there seems to be no tension lived by the teacher. It is as if they can coexist. Overall, Vicky’s teaching orientation could

be interpreted as being a means to an end to achieve application and process-oriented views. These two belief clusters coexist and the reason why Vicky does not live a conflict may reside in a lack of awareness about their differences, or more likely in a worldview that tries to accommodate these differences. Moreover, Vicky's teaching practice is a blending of problem-based activities originating from everyday life and solved in creative ways. Her reference to a specific class suggests that, in other classes, she may opt for a mostly application-oriented view, as she declares that she adopts different methods in different classes.

Julia: A completely different picture emerges from Julia's words. In her interview, she does not refer to a specific class or situation, but she makes a general statement about an uncomfortable internal struggle:

I really struggle when I see a student struggling to try and figure out a problem. I have a really hard time not giving them the answer as an example, and then letting them go from there, it's very — yeah. I really struggle with watching them struggle, I guess.

This excerpt can be interpreted in terms of a tension between a *process-oriented view* (struggling with new ideas, finding one's own path to solve a problem), and a *schema view*, according to which *nearly any mathematical problem can be solved by the direct application of familiar rules, formulas and procedures* and as such it may encourage a teacher to give the students the answer. Julia is well aware of the conflict. Like Vicky, she does not mention any external force that pushes her to act in a way that contrasts with her beliefs (e.g., she does not mention any PD session she attended, where she was faced with either alternative of teaching): rather, the two views, which are specific to the role of the teacher in problem solving, coexist in her belief system and the dilemma can be read mainly as her own, subjective elaboration. We can further see that, in her practice, Julia opts for the process-oriented view, as she tells us that she does not intervene.

Ron: After having attended a PD session, Ron referred back to his first experiences of teaching: "When you're a young teacher, you love having all the lessons and your notes set and all that and all this is great, got it all set." Ron seemed, from this quote, to adopt a *formalist view*, according to which *mathematics is a logically coherent edifice consisting of precisely defined terms and statements*. A formalist view blends with a *schema-oriented* one, as Ron further acknowledged that students like taking notes. However, also a non-formalist and non-schema view emerges, as he added: "I was getting tired of giving notes, giving lessons and just having them sit there and do it and observe. Because my thinking was they can get these notes anywhere". These words suggest that Ron came to PD with an emerging tension, seeking for a way to sort it out. In fact, Ron recognised that, "once you've been doing that for a short

while, you just, you realise it's kind of limiting". Ron's belief system was in motion, and the timing of the interview allowed us to capture this. A new view of mathematics was emerging:

Getting the students to do the work in class so that you know, even if they only get one or two problems, they really got it. And just so that, if they have to come in and think. I mean, I have to come in and think too because I don't really have to think if there's a [conventional] lesson. In a conventional lesson, I already know what to say and do.

A *process-oriented view*, according to which *in order to understand mathematics one needs to create or re-discover new ideas*, started to take form in Ron's orientations, and was valued. In Ron's words, not only the students have to "come in and think" during problem solving, but also the teacher has to do the same, whilst he does not "really have to think if there is a [conventional] lesson". However, Ron has not abandoned his previous, schema-oriented view as he mentioned:

A few of them [the students] would say to me that they like notes and so sometimes I would say, okay let's do that and then I would always tell them, see why I don't do this anymore. Some students said to me they liked the mini-lesson before, which is fair enough. But sometimes it's the questions that get you thinking in the first place, so I think it's fair enough to balance.

Ron uses the verb "to balance" to represent his way of living with the tension that is provoked by some of his students' preference for notes and formalism, which contrasts with his love for more engagement and thinking. Here, an important feature of tension emerges, that is: tensions are dilemmas that often cannot be resolved. In Ron's practice, this results in a mixture of teaching methods: sometimes students are exposed to 'mini-lessons' and take notes, while other times they 'come in and think'. As for Vicky, coexistence of different views mirrors the one of different practices.

Mary: Mary had been accustomed to strictly adhering to grade 1 curriculum in grade 1, and grade 2 in grade 2, without mixing up the content (a schema-oriented view). Participating in a PD session created a tension that caused her to change her mind. She acknowledged a change from before the PD, when she had a *schema-oriented view* of curriculum, to the present, as she now had a *process-oriented view* of mathematics, which involves a shift of attention to problem-based mathematical activities in her lessons, rather than being too much concerned about the constraints of curriculum. In Mary's words, the tension between these two views seems to be resolved:

It just freed up boundaries, I would say, like this is a grade one, this is grade two. You don't teach grade two in grade one. (*laughs*) It's just now that we're doing problem-solving activities it just naturally comes out and students that are ready will do it and students that are not ready just won't. The students can only learn at

their own pace or at their own development level and I'm okay with that. Before, I used to worry but now, it's just, — Okay.

The tension, currently resolved, initiated a change in Mary's practice and in certain belief clusters. Unlike Ron, for Mary there was not an external force prompting her to compromise, at least to a certain extent, between two worldviews, she abandoned the 'old' one and tension resolved. Differently from Vicky, Mary was aware of the conflict: she contrasted the two views explicitly in her account. Similarly to Julia, Mary makes a choice (her practice originating from that choice), but unlike Julia, Mary does not live uncomfortably with a struggle beyond her actions.

DISCUSSION AND CONCLUSIONS

The four prototypical cases allow us to exemplify some important features of tensions among belief clusters, and to attempt an answer to our research questions. Tension emerges when the teacher sees the conflict between different views, but is unable to resolve it. Teachers can live an internal struggle, or try to balance. There is no tension when the teacher ignores, or resolves, it. Tensions can be occasioned by PD, or emerge as the teacher encounters her classes and reflects upon her practice. An interesting case is Ron, who started to live a tension before PD and PD showed a way to (partly) solve it. For Mary, PD provoked a tension as it introduced a new worldview. Whilst Mary's case show that 'old' worldviews can be abandoned and the tensions can be resolved, resulting in a significant change in practice, Ron's case show that 'old' and 'new' views can find a way to coexist in a teacher's practice, as Ron's practice is a compromise between 'come and think' and 'take notes', since the schema-view has not been completely abandoned. We remark that, without a tension lens, Ron's choice would have been interpreted in a different way, namely as beliefs' resistance to change. For Mary's, Julia's and Ron's cases, we can say that we see a change in their practice, but we can also see the struggle behind it. For Vicky, we see no change and she blends different world views in her practice. In order to enrich the discussion, we summarise our results in Table 1, where we further distinguish between existence of external forces and 'pure' internal conflicts.

Table 1: Ways of dealing with beliefs belonging to contrasting world views

	<i>Internal contradiction</i>	<i>External force(s)</i>
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<i>There is tension</i>	...when the teacher values ideas, practices, behaviour that belong to different belief clusters and she is aware that they are in conflict (e.g., Julia). <i>For Julia, there is change.</i>	...when someone has different beliefs, and the teacher values the point of view of these people, she cares to have a good relationship with them and she sees the conflict (e.g., Ron). <i>For Ron, there is partial change.</i>
<i>There is <u>no</u> tension</i>	...when the teacher lives with ideas, practices, behaviour that do belong to contrasting belief clusters, but this is not a problem for her (e.g., Vicky). <i>For Vicky, there is no change.</i>	...when new experiences provoke a teacher change and tension between the old and the new is resolved (e.g., Mary). <i>For Mary, there is change (resolving tension in favour of the new one).</i>

Focusing on external forces, we notice a dual nature of world views: on one hand, they are subjective and internal to an individual person. They may conflict with external sources but are - in terms of cognition - cognitive traits (Erens & Eichler, 2019). On the other hand, however, if we consider the case of Ron, the formalist view which is tied to taking notes is also shared by Ron's students, and valued both by the teacher *and* the students. This view belongs to the teacher's beliefs system and to the 'external' source. Also, the process-oriented view, which resulted in breaking the boundaries among grade-specific curricula for Mary, was shared by the PD facilitator. This suggests that a teacher's world views can be altered by tension from external forces. Our data, thus, do not allow us to discard the central role of the social context not only in mirroring a person's belief system, but most importantly in dealing with contrasting world views and resolving (or balancing) the tension. This poses a question which deserves further investigation: Does an external force provoke a tension only if teachers hold the same view as the external force? Our preliminary results suggest the answer to this might be 'yes'. A follow up study will confirm this and it will reveal the incidence of each prototype in a much larger sample of teachers.

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LINKING AND ITERATION SIGNS IN PROVING BY MATHEMATICAL INDUCTION

Samuele Antonini¹ and Bernardo Nannini²

¹University of Pavia, Italy

²University of Florence, Italy

We focus on the role of signs in the process of constructing proofs by mathematical induction of high-achieving post-graduate students. Using a multimodal semiotic perspective, speech, written inscription (symbols, drawings, etc.), and gestures are analysed, and two particular categories of signs are identified and observed: linking signs and iteration signs. We analyse what these signs reveal and how the students use them to formulate a conjecture and to structure the proof by mathematical induction.

INTRODUCTION

The analysis of signs offers an interesting access to mathematical thinking and has promoted the discovery of interesting processes with important didactical implications. In the last decades the semiotic analysis has been integrated by the study of gesture that has enriched research in different areas of mathematics education and, recently, the studies on argumentation and proof (see, for example, Edwards, 2010; Arzarello and Sabena, 2014; Krause, 2015; Sabena 2018). In particular, Arzarello and Sabena show that gestures can contribute “not only to the semantic content of mathematical ideas, but also to the logical structure that organizes them in mathematical arguments” (Arzarello & Sabena, 2014, p. 76). Along the same line, Krause (2015) analyses the gestures produced during an activity involving reasoning by induction by grade 10 students who had not studied mathematical induction at school and states that gestures “give visual access to the structure of a reasoning action” (Krause, 2015, p. 1432).

The study presented in this paper is part of a wider research on proving by mathematical induction of post-graduate, undergraduate and secondary students. In particular, in this paper, we focus on signs in post-graduate students’ processes involved in the generation of a conjecture and of proof by induction.

THEORETICAL FRAMEWORK

In a multimodal perspective, we consider that thinking and learning processes involve simultaneously different kinds of signs (mathematical symbols, diagrams, sketches, language, gestures, etc.). Arzarello (2006) considers these different kinds of signs as an inseparable unit and defines a *semiotic bundle* as a

dynamic structure consisting of different *semiotic sets* and relationships among them. Two main types of analysis are carried out on a semiotic bundle: a *synchronic analysis* of relationships between different kinds of signs activated simultaneously and a *diachronic analysis* of evolutions of signs activated over the time.

In this paper, we analyse the semiotic bundle made of three semiotic sets - speech, written inscriptions (symbols, drawings, etc.) and gestures - in the production of a conjecture and of a proof by mathematical induction. The analysis of complex units of signs has enabled the identification of new interesting processes in argumentation and proof. In particular, Sabena (2018, p. 554) provides empirical evidence that “gestures may contribute to carrying out argumentations that depart from empirical stances and shift to a hypothetical plane in which generality is addressed”. Sabena, Radford and Bardini (2005) observe that a deictic gesture used by a grade 9 student to point at a figure on the sheet becomes a gesture in the air and identify a crucial role of a progressive detachment of gestures from a sheet in generalization processes. Similarly, Krause (2016) proposes a classification of gestures in three levels (*concrete*, *potential*, and *general*) according to their detachment from a concrete inscription. Gestures of *level 1* refers *concretely* “to something actually represented in a fixed diagram” (e.g. pointing to the sheet). Gestures of *level 2* *potentially* “depict new entities in an established diagram” but they need to be considered as embedded in it (e.g. gesture of rotating a figure). Gestures of *level 3* are *general* gestures performed in the gesture space. They are detached from a concrete level and their interpretation is general, i.e. not dependent on a “present referential frame” (Krause, 2016, p. 138).

In our study, we also refer to the classic distinction of gestures into *iconic*, *metaphoric*, *deictic* and *beats* (McNeil, 1992). We will use these classifications and synchronic and diachronic analyses to investigate processes of construction of a proof by induction.

Linking and Iteration Signs in Mathematical Induction

A proof by mathematical induction of a proposition $\forall n \in \mathbb{N}, P(n)$ consists in a proof of the base case $P(0)$ and of the inductive step $\forall n \in \mathbb{N}, P(n) \rightarrow P(n+1)$. Referring to the theory of natural numbers and to the logic theory, we know that the validity of the base case and of the inductive step guarantees that $P(n)$ holds for all natural numbers. Usually, a non-formal explanation is that from the propositions $P(0)$ and $P(0) \rightarrow P(1)$ it follows $P(1)$ by modus ponens; from $P(1)$ and $P(1) \rightarrow P(2)$ it follows $P(2)$, and so on. In other words, this process can be iterated to cover all the natural numbers. In this paper we aim to investigate signs that reveal and support the construction of the inductive step and the iteration in the generation processes of a conjecture and of proof. Constructing the inductive step requires the consideration of two cases ($P(n)$ and $P(n+1)$) and

their relationships. The iteration requires the consideration of the possibility to repeat the inductive step. Thus, in particular, we look for and analyse:

- signs produced or used to refer to two or more entities (objects, mathematical objects, problems, situations, etc.) and to their relationships, where these entities are seen in connection with two consecutive natural numbers. For these we use the term *linking signs*;
- signs that refer to iteration, or that are composed by a repetition (in time or in space) of linking signs, or that refer to a repetition of them. For these we use the term *iteration signs*.

Examples of linking signs can be found in usual algebraic manipulations. For instance, in the construction of the proof of the formula for the sum of the first n consecutive natural numbers it is common to use the sign $(1+2+\dots+n)+(n+1)$. This sign links the case n with the case $n+1$ and prepares the proof of the inductive step. Some examples of the iteration signs are the verbal “and so on”, or the image of falling dominoes.

In this study, our goal is to look for the presence of linking and iteration signs, and to investigate what they reveal, in the process of generating a conjecture and a proof by induction, and considering not only mathematical symbols but a wider variety of signs, as speech, written inscriptions, and gestures.

METHODOLOGY

This is a qualitative study based on interviews in which students were asked to solve 4 problems and then to speak about mathematical induction. Data consist of audio, video recordings, and of written inscriptions produced by the students. The subjects were 1 high-achieving post-graduate student in the Master’s course in Mathematics and 4 doctoral students in Mathematics. They were interviewed individually by the second author of this paper, for approximately 70 minutes each. They were neither aware of our interest about their written inscriptions and gestures nor of our focus on proof by mathematical induction. In this paper we will refer to the following problem:

“Consider a $2^n \times 2^n$ chessboard. What is the maximum number of squares which can be tiled with L-shaped pieces composed of 3 squares each?”

The solution is that it is possible to tile the entire $2^n \times 2^n$ chessboard except for one square, for any natural number n . This can be proved by mathematical induction on n .

CASE ANALYSIS

Giuditta is a post-graduate student in the Master’s course in Mathematics. In the first 10 minutes of the interview she produces some drawings and recognises that for reasons of divisibility it is not possible to completely tile any chessboards. By minute 10:00 she has sketched an 8×8 chessboard ($n=3$) and determined a tessellation which covers every square except one. The

interviewer then asks her if this property is also valid in other cases, for example in the case 16x16. In the transcript, Giu stands for Giuditta and with *italics* we describe gestures in the moments when they occur.

- 1 Giu: 16 by 16 (*with her left middle finger and the tip of the pen in the right hand she points to two vertices of the 8x8 chessboard drawing, Fig. 1a).*
- 2 Giu: but, then I have another three (*she keeps her left middle finger on the vertex, and with the pen in the right hand she indicates respectively to the right, upper right, and above the drawing of the 8x8 chessboard, Fig. 1b,c,d) of these (she points with the pen to the drawing of the 8x8 chessboard) squares here (she moves the tip of the pen along the perimeter of three imaginary squares in the three places she has indicated before, Fig. 2).*

The synchronic analysis of the bundle produced in line 1 reveals an interesting element. In this moment, on the sheet there is the drawing of the 8x8 chessboard and no other written inscriptions referring to a 16x16 chessboard. Giuditta says “16 by 16” and at the same time points to two vertices of the drawing of the 8x8 chessboard (fig. 1a). She refers to something through her speech and to something else through her gesture: this is a case of *speech-gesture mismatch* and Goldin-Meadow (2003) highlights the cognitive potential of a mismatch in the representation of a new idea. In this case, pointing at the drawings of the 8x8 chessboard is co-timed to saying “16 by 16”. The bundle and the mismatch offer Giuditta the possibility to represent simultaneously two different chessboards (8x8 and 16x16).

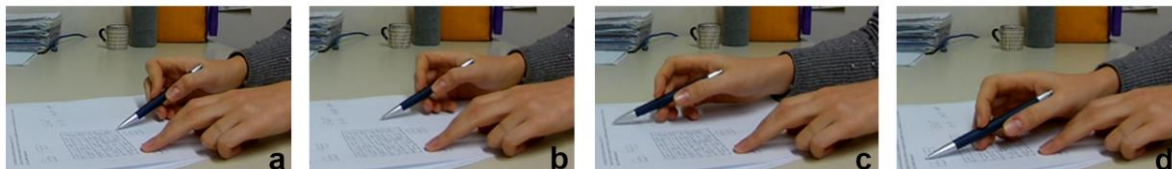


Figure 1: Gestures in line 2.

The diachronic analysis allows us to look at the evolution of signs. In line 2, Giuditta produces signs connecting the chessboards. She keeps the left hand still on the drawing of the 8x8 chessboard (deictic gesture of level 1) and with the right hand she points to three places on the sheet (fig. 1b,c,d). Then she moves the tip of the pen along the sides of three imaginary squares in the three places she has just indicated. In summary, four 8x8 chessboards are represented: one by a written inscription, and three by speech and gesture (fig 1 and 2). These gestures represent something new into the inscription and are therefore gestures of level 2. The bundle speech-inscription-gesture represents a 16x16 chessboard composed by four 8x8 chessboards and, as a unit, can be considered a linking sign referring to the two chessboards and to their relationships. This linking

sign, at this point, allows Giuditta to access the connections between the tessellation problem in the case $n=3$ (8×8) and in the case $n=4$ (16×16):

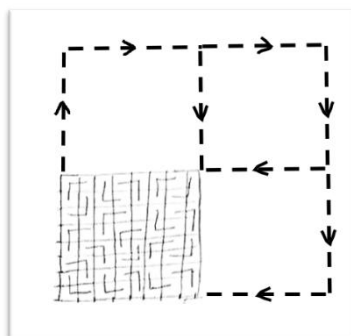


Fig. 2: Pointing with the left hand to the drawing of a 8×8 chessboard, Giuditta follows with a pen (without marking) the perimeter of 3 squares.

- 3 Giu: And then there would be left out one, one, one and one (*she points to the drawing of the 8×8 chessboard on the sheet and to the other three she has in mind*) [omissis]. And so I would think to put three of them together, somehow. And then, there would always be one left out?

Giuditta conjectures that the 16×16 chessboard can be tiled except for one small square (a square 1×1) and imagines doing it by using the tessellation of the four 8×8 chessboards. In each of them, one small square would be left out, thus 4 squares in total, but three of them can be covered with an L-shape tile. Therefore, also the 16×16 chessboard would be tiled except for one little square. Her linking sign has a crucial role in the conjecture generation. In particular it enables Giuditta to anticipate the fact that the 16×16 chessboard can be tiled using the tessellation of the smaller one “somehow” (she doesn’t know in which way and the conjecture is expressed as a question). At this point, Giuditta focuses on verifying her conjecture for $n=1$, $n=2$ and then for $n=0$. Differently from her reasoning in line 3, these cases are each tiled independently, without connections between them. Then she claims to be convinced of the truth of her conjecture. In argumentation process, new signs enrich the bundle:

- 4 Giu: So, what I was thinking (*the drawing of the 4×4 chessboard, Fig. 3a, is extended into a new drawing, Fig. 3b*) was that to come, to move forward from $n=1$ (*she makes an arc-shaped gesture in the air from left to right, Fig. 3c,d*) to $n=2$ (*with her left middle finger she points to a drawing of a 2×2 chessboard*) practically (*with the right hand she points specifically to three squares of the drawing of the 2×2 chessboard, see arrows in Fig. 3e*) I have to put another three identical little squares (*she draws two lines on the drawing in Fig. 3b obtaining the drawing of Fig. 3f*).

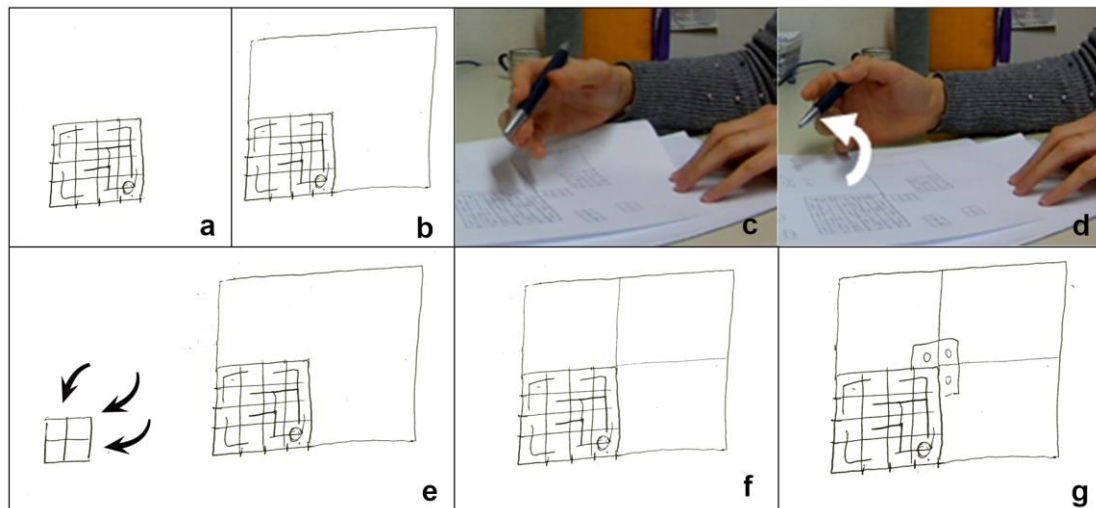


Figure 3: Gestures and written inscriptions in line 4 (a,b,c,d,e) and in line 5 (g).
Fig. 3e indicates where Giuditta points to on the sheet.

In this excerpt, Giuditta produces three linking signs that become the object of her exploration. The first is the drawing of a big square (fig. 3b) as extension of the drawing of the chessboard 4x4 (already on the sheet, fig. 3a). The second is the gesture in the air from left to right (fig. 3c,d). The third is the bundle composed by the deictic gesture with her left middle finger pointing to the drawing of the 2x2 chessboard and the gesture made by the right hand referring to the action of adding three small 1x1 squares to build a 2x2 chessboard up from a single square. The gesture from left to right is iconic and refers to a path, but can also be interpreted as a metaphoric gesture of level 3. This gesture is detached from a concrete inscription and it is co-timed to the verbal “to move forward from $n=1$ to $n=2$ ”. This gesture appears here for the first time and does not refer to any drawings, any chessboards or tessellations. With this, Giuditta doesn’t refer to the specific aspects of the relationship between a smaller chessboard and a bigger one, neither to the relationship between tessellations. Rather, the gesture represents metaphorically the transition between two cases, i.e. the inductive step. The structure of the argumentation is thus emerging. The analysis of the bundle shows the genesis of linking signs with different levels of generality and in reference to different cases: the verbal “from $n=1$ to $n=2$ ”; the written inscription linking the drawings of the 4x4 and the 8x8 chessboards (from $n=2$ to $n=3$, see fig. 3a,b,f); the gesture (level 2) linking the drawing of the 2x2 and 1x1 chessboards (from $n=1$ to $n=2$, see fig. 3e) and the metaphorical gesture (level 3, see fig. 3c,d). Giuditta is progressively shifting her focus from the tessellation of some specific chessboards to the links between these tessellations. Now, the produced linking signs allow her to establish the inductive relationship. In fact, at this point Giuditta shows how she could tessellate the 8x8 chessboard (except for one square) using a tessellation of the

4x4 chessboard and placing a tile in the central part of the chessboard (fig. 3g). After a few minutes, she concludes:

- 5 Giu: And this, I can do it in general (*after a circular gesture around the drawing of a 4x4 chessboard, with the right hand she makes a spiral movement that widens as the right hand rises and concludes with spreading both the hands, Fig.4a,b,c,d,e and Fig. 4f for a summary*).

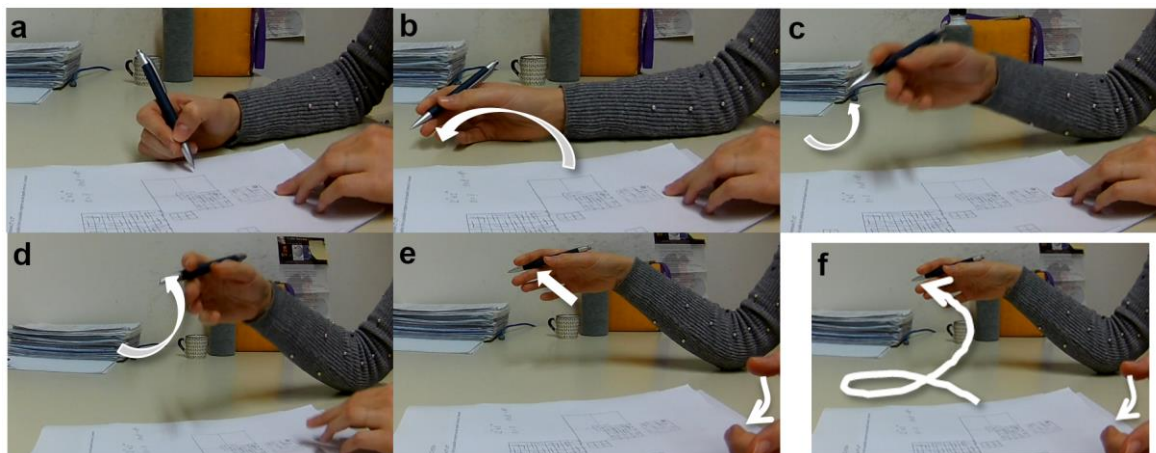


Figure 4: Gesture in line 5. The fig. 4f summarises the whole movement.

Giuditta does not write anything and she uses very few words: “and this, I can do it in general”. However, her gesture reveals the structure of argumentation and give us access to her reasoning. The gesture is articulated in four components.

The first component is the same gesture she has produced several times since line 1 when she linked the 8x8 and the 16x16 chessboards; now this gesture represents the action of constructing the 8x8 chessboard using the 4x4 chessboards.

The second component consists of contracting the previous gesture and moving away her right hand from the sheet in two directions: upwards and outwards. The upward direction takes the gesture from level 2 to level 3. It is the first time that Giuditta produces this gesture in the air. The shift through levels and her words indicate the generality of the actions of tessellation. Moreover, the gesture grows wider away from her body to indicate the construction of bigger chessboards (in mathematical terms, n is increasing). Until now, the left hand has remained still with a finger of the drawing of the 4x4 chessboards (which could represent the starting point of the recurrence; in fact she has already directly verified the cases of the smaller chessboards).

The third component consists in moving the right hand to the right - making the metaphoric gesture of a link, as seen in figure 3c,d - and moving the left hand to the left: the link between the chessboards of different sizes, represented before by an iconic gesture, here becomes an inductive step represented by a

metaphoric gesture. These first three components, consisting of a sequence of different linking signs, constitute a unique iteration sign, which in its complete form is a gesture of level 2-3: it starts on the sheet, in which the base of the induction is represented, and rapidly moves away from the sheet becoming a gesture of the level of the general (level 3).

Finally, the fourth component consists in keeping her hands still in the air, as if they contain the space in which the iteration gesture took place. This space, to use an expression of McNeil (1992, p. 173) when describing an iconic gesture that indicates a point in space, is not empty but “full of conceptual significance”. In our case, this space is the location that contains the argumentation and its logical structure.

CONCLUDING REMARKS

The multimodal perspective and the notion of semiotic bundle (Arzarello, 2006) has allowed us to identify and to analyse linking and iterative signs, and to observe and study the genesis of a proof by mathematical induction. Our analysis confirms the results presented in other studies (Arzarello & Sabena, 2014; Krause, 2015; Sabena, 2018) regarding the role of gestures in providing a logical structure to argumentation.

In the first excerpt, the speech-gesture mismatch (synchronic analysis) shows that the subject focuses simultaneously on two cases (8x8 and 16x16 chessboards). The bundle evolves and new signs are produced (diachronic analysis) to connect the two objects. The bundle is composed by different kinds of signs with mutual relationships. Only when we consider the bundle as a unit, we can see the linking sign representing a 16x16 chessboard as composed by 8x8 chessboards. This and other signs lead the subject to establish the connection between the problem of tessellating a chessboard and the same problem on a bigger chessboard, and then to construct the inductive step.

During the production of the argumentation, a repetition of linking signs produces an iterative sign and the complete detachment of the gesture from the sheet shows the transition to the general (Krause, 2016). The gesture contracts progressively, from iconic (referring to the extension of a chessboard into a bigger one) to metaphoric (referring to the inductive step), from level 2 (level of concrete) to level 3 (level of general). The iterative sign reveals that Giuditta constructs the entire recurrence even if it is not formally necessary (having proved the base case and the inductive step). The still hands at the end show the transition of argumentation from process to object.

The contraction of linking signs reveals a change of the focus. For Radford, “contraction is the mechanism for reducing attention to those aspects that appear to be relevant [...] We need to forget to be able to focus” (Radford, 2008, p. 94). The contraction of Giuditta’s gesture shows that she “forgets” the tessellation and focuses on the relationships between tessellations. Following

Radford (2003), the contraction of linking signs is a process of objectification of the inductive step.

Moreover, the repetition of linking signs is an example of *catchment*. According to McNeill (2005), a catchment is due to the recurrence of consistent visuospatial imagery in the speaker's thinking, and indicates and provides the discourse cohesion. Arzarello and Sabena (2014) show that catchments contribute to support the students in structuring a mathematical argumentation. Our analysis seems to confirm their results.

Finally, further research is necessary to identify linking signs in symbolic manipulation and to study the evolution of linking signs within the bundle from the proving process to the written proof.

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FAMILY BACKGROUND AND MATHEMATICAL MODELLING – RESULTS OF THE GERMAN NATIONAL ASSESSMENT STUDY

Ilja Ay¹, Nicole Mahler², & Gilbert Greefrath¹

¹University of Muenster, Germany

²Institute for Educational Quality Improvement, Germany

Family background predicts success in mathematics education in many countries – and particularly in Germany. Mathematical modelling with its authentic and realistic contents may be of importance for inequality research. Based on the German National Assessment Study, correlation comparisons, variance and regression analyses indicated that socio-economic status, migration background, and language use are more strongly related to mathematical achievement (excluding modelling) than to modelling achievement. Mathematical modelling might, therefore, contain facets which contribute to the reduction of social disparities.

INTRODUCTION

Family background determines educational success. Studies have repeatedly shown a connection between students' migration background, language use and socio-economic status (SES) on the one side and their achievement in mathematics on the other (e.g., OECD, 2013). This issue is of importance to the German education system. While on OECD average social disparities in mathematics sank over the last years, in Germany they rose again (OECD, 2013; 2019). Thus, Germany has a relatively low level of educational equity about mathematics. The German National Assessment Study – conducted by the Institute for Educational Quality Improvement (IQB) – identified a learning disadvantage in mathematics of almost three years for students from families with lower SES (Pant, Stanat, Schroeders, Roppelt, Siegle & Pöhlmann, 2013). Mathematics education should, hence, create conditions and provide learning opportunities that reduce social disparities. German educational standards describe mathematics education in which mathematical knowledge is applied functionally and flexibly in context-related situations. Therefore, in addition to content, general mathematical competencies are central to mathematics education in Germany (KMK, 2003). Mathematical modelling is one of those competencies. It includes solving realistic and authentic problems (Maaß, 2010) and thereby differs from dressed-up tasks with extra-mathematical contents.

However, empirical research does not come to consistent conclusions regarding background-related barriers and strategies in modelling processes. On the one hand, Cooper and Dunne (2000), amongst others, pointed out that lower SES students overemphasize everyday experiences while processing tasks with real-world content.

According to them, the SES is more important for realistic tasks than for purely mathematical tasks. On the other hand, Schuchart, Buch and Piel (2015) showed that the item context was not systematically related to the response rate of lower and higher SES students. The present study approaches this issue quantitatively by analyzing and comparing the effects of different family background factors on the response rates of modelling and non-modelling tasks.

THEORETICAL FRAMEWORK

For quite some time, research has been addressing the mechanisms through which family background is related to student's education outcomes. From a sociological perspective, according to Bourdieu's habitus theory (1984), individuals find themselves in a social space which limits their scope of action. This scope forms the way individuals think and act and it constitutes the foundation on which social inequality is built. It is passed on through socialization and by this, certain behaviors and values are being internalized. Empirical studies identified many factors that produce social disparities and then contribute to its consolidation and reproduction. Students from families with higher SES on average "benefit from a wider range of financial [...], cultural [...] and social [...] resources that make it easier [...] to succeed in school" (OECD, 2016, p. 206). Regarding family communication, higher SES parents place higher value on reasoning and discussing, whereas lower SES parents focus more on conformity (Heath, 1983). By different socialization, higher SES parents are comparatively better able to prepare children for educational requirements (Schuchart et al., 2015) and pass on their social advantages to their children. Besides, teachers tend to educate lower SES students' mechanical behaviors or give them routine instructions ('Do it this way'), while they tend to teach students with higher SES to think (Anyon, 1981). Further, teachers might underestimate the mathematical capacity of lower SES students, if they attribute students' problems to their cognitive ability and not to their background (Schuchart et al., 2015). Also, teachers may communicate differently with students of different social classes, since lower SES students are often less well equipped to interact with teachers and institutions (Calarco, 2011). This is accompanied by the tendency that higher SES students request and hence receive more help from teachers. They can use their working time more efficient (ibid.) and thereby create their own advantages. In this way, students, parents, and teachers contribute to the consolidation and reproduction of social disparities. Thus, it is hardly surprising

that PISA refers to socio-economic heterogeneity as being a challenge for teachers and education systems (OECD, 2016).

Mathematical Modelling

The German educational standards comprise six general competencies which students are expected to develop in secondary level: (i) Arguing; (ii) problem-solving; (iii) modelling; (iv) using descriptions; (v) dealing with symbolic, formal and technical elements, and (vi) communicating (KMK, 2003). Regarding mathematical modelling, students are supposed to translate the respective situation into mathematical terms, structures, and relations, to work within the mathematical model, to interpret and to check results with respect to the corresponding situation (ibid.). While modelling tasks in general contain solving realistic problems, they might differ in terms of authenticity, realism, involved modelling activities, level of openness, etc. (Maaß, 2010). An example for a modelling task that could occur similarly in the test described below is given by Figure 1.

In the picture you can see the historical city hall of Muenster.

How high is the city hall approximately?

..... m

Write down your assumptions and your approach.




Figure 1: Modelling problem “city hall”

The illustrated historical building really exists and estimating sizes by using reference values are part of everyday life. The mentioned problem is thus *authentic* and *realistic*. It is *open* since the students are free in choosing the object of comparison, for example, the man with the white shirt. The task involves *modelling activities*, including making assumptions about the average size of a person and interpreting the mathematical results in a meaningful way.

Therefore, there is not a single solution, but rather an interval of results that can be evaluated as correct. This ensures the comparability and evaluability of students' results.

DESIGN AND METHOD

The following research question derives from the current state of research:

- Is mathematical achievement in modelling and non-modelling differently connected to students' family background?

This study is based on data from the German National Assessment Study 2012, which was conducted by the IQB. In total, a representative sample of 24731 ninth-grade students across Germany participated in the mathematical part of this standards-based assessment (Pant et al., 2013). We compared students' achievement in mathematical modelling with their achievement in other mathematical competencies. We predicted achievement by students' SES. As SES is often related to migration background, language proficiency and language use (OECD, 2016; 2019), which in turn may affect mathematical achievement, we used these variables as additional predictors.

The test booklets were assembled under a multi-matrix design, so that each student worked on 24 to 60 out of 349 items. Based on specified evaluation criteria for each item, student solutions were coded dichotomously as 'correct' or 'incorrect'. A global score for mathematical competency – including all items – was estimated for every participant on a one-dimensional dichotomous Rasch Model (Warm, 1989). The estimation results in a metrical measure namely the Weighted Likelihood Estimate (WLE). For the present study, we further estimated person parameters (WLE) using the same statistical model for the achievement in modelling and the mathematical achievement excluding modelling. We will refer to them as modelling achievement and non-modelling achievement. The estimates are based on two nonoverlapping subgroups of items: Items targeting modelling according to the German educational standards and items targeting other mathematical competencies. The first subgroup contains 41 out of the 349 items.

Students' family background was assessed using student questionnaires. In this study, we used the HISEI (*Highest International Socio-Economic Index of Occupational Status of both parents*) to measure families' SES. It is determined by the professions of the parents and takes income and educational level into account. By means of the HISEI, it is possible to capture the SES of occupations by putting them on a one-dimensional hierarchical scale from 10 (e.g., kitchen helper) up to 89 (e.g., medical doctor), with a higher HISEI indicating higher SES (Ganzeboom, de Graaf & Treiman, 1992). Students' migration background was operationalized ordinally via the countries of birth of the parents. For the language use, students were asked how frequently they speak German at home (see Table 1). Though, the data on family background has missing values, since

in some German states it was optional for students to fill out the questionnaire. Further, part of the sample ($n = 14\,793$) completed a C-Test to measure their language proficiency in German (Robitzsch, Karius & Neumann, 2008).

Variable	Operationalization	Distribution
SES ($n = 17810$)	HISEI $\in \{10, \dots, 89\}$	$M = 51.40$ $SD = 20.47$
Migration background ($n = 18663$)	Both parents were born in Germany One parent was born in another country Both parents were born in another country	14710 (76%) 1780 (10%) 2713 (15%)
Language use ($n = 17276$)	Mostly/ only speaking German at home Sometimes speaking another language at home Or never speaking German at home	14025 (81%) 3251 (19%)

Table 1: Background variables

To answer the research question, we conducted linear regressions and single-factor variance analysis (ANOVAs), measured and compared the percentage of explained variation by means of η^2 and R^2 (Cohen, 1988). In order to do so, we compared dependent correlations with one common index (i.e., the correlation coefficients are calculated from a single sample and the correlations are overlapping with one common variable) according to Hittner, May and Silver (2003). They indicated that Type I error depends not only on sample size and population distribution, but also on the intercorrelation (between modelling and non-modelling achievement) r_3 and the discrepancy between predictor-criterion correlations r_1 and r_2 (see Figure 2).

RESULTS

Table 2 summarizes the results of the ANOVAs and linear regressions.

Dependent variable	Independent variable	Method	Explained variation	df	F	Significance
Modelling achievement	SES	Linear regression	$R^2 = .06$	1	1045.66	$p < .001$
	Migration background	ANOVA	$\eta^2 = .02$	2	190.60	$p < .001$
	Language use	ANOVA	$\eta^2 = .03$	2	224.63	$p < .001$
Non-modelling achievement	SES	Linear regression	$R^2 = .12$	1	2355.77	$p < .001$
	Migration background	ANOVA	$\eta^2 = .04$	2	389.55	$p < .001$
	Language use	ANOVA	$\eta^2 = .05$	2	474.76	$p < .001$

Table 2: Explained variation in modelling and non-modelling achievement by family background

It appears that all family background variables can explain variation in modelling and non-modelling achievement. Also, all background variables have a higher effect on non-modelling achievement than on modelling achievement.

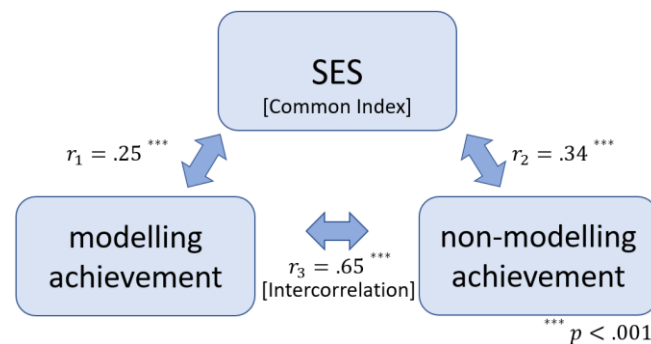


Figure 2: Comparison of the correlation between SES and modelling achievement with the correlation between SES and non-modelling achievement

Figure 2 shows the results of comparing the correlation coefficients. Analysis, based on Hittner's et al. (2003) correlation comparison yielded that the correlations r_1 and r_2 vary significantly from each other ($n = 15\,711$, $z = 14.8$, $p < .001$). The magnitude of the intercorrelation (IC) is high ($r_3 > .6$) and the effect size is .09. With a power of $1 - \beta = 1$, a statistically verified small correlation difference can be assumed. The same applies for migration background and language use with an effect size of .04 for both variables ($n = 16\,446$, $z = 6.7$, $p < .001$, $1 - \beta = 1$, $IC > .6$; $n = 15\,138$, $z = 6.3$, $p < .001$, $1 - \beta = 1$, $IC > .6$). Repeating the analysis controlling for language use, migration background and language proficiency still yields a significant difference in partial correlations between SES and the two mathematical achievement variables with a small effect size of .09 ($n = 7\,384$, $z = 7.7$, $p < .001$, $1 - \beta = 1$, $IC > .5$). Further analysis show that this difference cannot be explained by the fact that modelling tasks are, on average, more likely to contain extra-mathematical content. In fact, in our data SES correlates more closely with the response rates of tasks with extra-mathematical content compared to purely mathematical tasks ($n = 17\,810$, $z = 2.3$, $p < .05$, $1 - \beta = .75$, $IC > .7$).

DISCUSSION AND CONCLUSION

The current study shows that SES, migration background and language use are more strongly related to mathematical achievement (excluding modelling) than to modelling achievement. However, only for SES the correlation comparisons reveal an important, albeit small, difference. In addition, even when controlling for migration background, language use and language proficiency, SES is less closely correlated with modelling achievement. SES appears to be less important for modelling tasks than for non-modelling tasks. Considering that modelling tasks are on average more realistic than non-modelling tasks, one

could expect that SES is also less important for tasks with extra-mathematical content than for purely mathematical tasks. Though, in our data SES seems to be more important for tasks with extra-mathematical content than for purely mathematical tasks (see also Cooper & Dunne, 2000). Therefore, in our data the difference between r_1 and r_2 from Figure 2 cannot be explained by more realistic nature of modelling tasks. In conclusion, SES appears to be less relevant for modelling tasks, even though they contain realistic content.

At this point it remains uncertain which characteristics of the tasks cause these correlation differences and must be explored in further investigations. Moreover, even though SES is less closely correlated to modelling achievement than to non-modelling achievement, it is still important for the explanation of variation in modelling achievement. Methodologically, it must be mentioned that the contents of the extra-mathematical and the purely mathematical tasks differed from each other (in our study as well as in Cooper and Dunne's study). Further, considering the limitations of our study regarding the teaching of mathematics, the results should be interpreted with caution, since performance tests only provide very limited implications for mathematics teaching. Furthermore, our sample is representative only for ninth-grade students.

In sum, our study indicates that mathematical modelling contains aspects which may contribute to the reduction of social disparities in mathematics education. Since modelling plays a more underrepresented role in classroom practice than it would be desirable (Blum & Borromeo Ferri, 2009), these results may strengthen the importance of modelling in mathematics education. Future quantitative and qualitative studies should analyze these aspects in more detail, especially within the scope of classroom practice. This study does not aim to place mathematical modelling above other competencies. Rather, it should encourage to confront students from all social backgrounds with authentic and realistic mathematical problems. With a view to the empirical findings mentioned at the beginning, especially lower SES students might profit from mathematical modelling.

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EXPLORING TEACHERS' ENVISIONING OF CLASSROOM ARGUMENTATION

Michal Ayalon¹ and Samaher Nama¹

¹University of Haifa, Israel

This study explores how secondary mathematics teachers envision potential argumentation situations in the classroom. The data were collected by means of individual semi-structured interviews conducted with 31 secondary mathematics teachers. The participants were asked to express their views on argumentation for teaching mathematics, provide examples of argumentation as manifested in their own teaching, and formulate a script for the hypothetical implementation of a mathematical task in the classroom with the goal of engaging students in argumentative activity. Analysis of the teachers' responses yielded categories related to: (1) task characteristics, (2) teaching strategies, and (3) students' characteristics. From a cross-analysis of the teachers' statements, certain categories appeared more frequently than others. The findings are interpreted in light of theory and practice.

INTRODUCTION

In the last several decades, there has been a growing appreciation for the incorporation of argumentation in the mathematics classroom (Krummheuer, 2007; Yackel & Hanna, 2003). Firstly, argumentation is a valued mathematical practice whereby mathematicians socially construct knowledge through generating and evaluating alternative arguments. Secondly, existing literature suggests that participation in argumentation requires students to explore, confront, and evaluate alternative positions, voice support or objections, and justify different ideas and hypotheses, all of which promote meaningful understanding and deep thinking (Asterhan & Schwarz, 2016; Staples & Newton, 2016). Recent reform documents, in various subject domains worldwide, highlight argumentation as an important goal for students (e.g., Israel Ministry of Education, 2013). Nevertheless, argumentation in the mathematics classroom is not yet a commonplace practice (Bieda, 2010).

Research exists on many aspects of argumentation as it pertains to learning mathematics (e.g., Mueller et al., 2014; Staples & Newton, 2016; Yackel & Cobb, 1996); yet little work has focused specifically on teachers' understanding of argumentation (Ayalon & Even, 2016, Mueller et al., 2014). Considering that such an understanding impacts the way in which argumentation practices are implemented in the classroom (Conner et al., 2014), we deemed it important to

make it the focus of investigation. Hence, this study addresses the topic by exploring secondary teachers' views on argumentation. We asked teachers to provide examples of argumentation as manifested in their own teaching and to write a script for the hypothetical implementation of a mathematical task that would engage students in argumentation. Analysis of their responses yielded several dimensions of teachers' attention to potential classroom situations of argumentation, and these provide a lens through which we may learn about teachers' grasp of argumentation.

THEORETICAL BACKGROUND

A commonly accepted definition of argumentation is that of van Eemeren and Grootendorst (2004) who maintain that argumentation is “a verbal, social, and rational activity aimed at convincing a reasonable critic of the acceptability of a standpoint by putting forward a constellation of propositions justifying or refuting the proposition expressed in the standpoint” (p. 1). According to this definition, argumentation involves generating claims, providing evidence to support the claims, and evaluating evidence to assess their validity. This definition also posits argumentation in a social space, and, if incorporated into classroom discourse, it affords a venue for the articulation and critical evaluation of alternative ideas, eventually supporting collaborative knowledge construction (Asterhan & Schwarz, 2016). This definition forms the foundation in the literature for common descriptions of argumentation that are ‘fruitful’ for learning.

According to this definition, the present paper considers argumentation as having two important aspects – *structural* and *dialogic* (McNeill & Pimentel, 2010). The *structural aspect* of argumentation focuses on the feature of discourse whereby a claim, which can be presented as an idea, conclusion, hypothesis, solution etc., is supported by an appropriate justification. While mathematicians support claims using diverse justification types, specific types, such as deductive justifications, are valued in the mathematics discipline over others. In the mathematics classroom, the appropriateness of justifications is attained by negotiating socio-mathematical norms (Yackel & Cobb, 1996). The *dialogic aspect* regards argumentation as the interactions between individuals when they attempt to generate and critique each other's ideas. In mathematics classrooms, this is indicated by students listening to each other, building on each other's ideas, and critiquing ideas as the community moves toward consensus.

In this study, we explore secondary mathematics teachers' envisioning of potential classroom argumentation situations in both the structural and dialogic aspects of argumentation. We assume that teachers' attention to both of these aspects could help them better incorporate argumentation into their classroom instruction (McNeill & Pimentel, 2010). In mathematics education, research has

focused on teachers' attention as a topic for both investigation and development, upon the premise that it shapes teachers' actions and practices (Mason, 2015). For those researchers who focus on teachers' noticing (e.g., Jacobs, Lamb, & Philipp, 2010), attention is considered a fundamental skill. One important issue discussed in the research literature relates to how professionals attend to noteworthy aspects of complex situations: "We are interested in the extent to which teachers attend to a particular aspect of instructional situations" (Jacobs et al., 2010, p. 172). Research has shown that attention can be narrowly focused on one aspect of a situation at the expense of others, or, alternatively, it may be broad in processing a wide variety of details and aspects of the situation (Mason, 2015). Therefore, investigating what teachers attend to when envisioning potential classroom argumentation situations is important and can serve as an avenue for teacher educators to devise appropriate support, direction and guidance.

Mathematics teaching that encourages argumentation provides students with ample opportunities to take an active role in both structural and dialogic aspects; i.e., to construct arguments, share their ideas, consider others' ideas, and critically evaluate their validity, while adhering to normative aspects of mathematical discourse that are specific to the students' mathematical activity (Yackel & Cobb, 1996). Various factors associated with teaching generate opportunities for students to participate in argumentation. For example, teaching for argumentation is fundamentally associated with implementing appropriate tasks (e.g., Ayalon & Hershkowitz, 2018). In particular, open-ended tasks that invite multiple strategies for solutions are perceived as enhancing opportunities for argumentation (Mueller et al., 2014). In addition, teaching for argumentation is intrinsically linked with the teacher's actions, such as encouraging students' participation and thoughtful questions (e.g., Ayalon & Even, 2016). Moreover, teaching for argumentation requires teachers' sensitivity to their students' cognitive factors, such as prior knowledge, common ways of thinking, and argumentation skills, as well as to their students' affective characteristics, such as self-confidence, interest, and enjoyment (Knuth & Sutherland, 2004).

While recognizing that the three dimensions of task characteristics, teaching strategies, and student characteristics are only a subset of factors contributing to classroom argumentation, we view them as important initial steps for the successful integration of argumentation into classroom practice. These dimensions are naturally inter-related; however, focusing on each one individually allows us to discern each one and learn about its place in teachers' envisioning of class argumentation. Taking into account the two aspects of argumentation (structural and dialogic) across the three dimensions of argumentation (task characteristics, teaching strategies, and student characteristics), this study addresses the research question: *To what do*

secondary mathematics teachers attend when asked to envision argumentation in their classroom?

METHODOLOGY

Research participants

Thirty-one secondary mathematics teachers participated in this study. All of them had five years or more of teaching experience. The decision to focus on secondary-school teachers derived from the emphasis placed on argumentation in the curriculum of this student population in Israel (Ministry of Education, 2013).

Data collection

The data used for this study consisted of individual, semi-structured interviews of approximately 90 minutes that comprised two main parts. The first part involved questions about the place of argumentation in teaching mathematics and whether and how the research participants practice argumentation in their respective classrooms. The teachers were encouraged to explain their responses in detail and provide examples from their own teaching. In the second part, they were asked to select a mathematical task which, in their view, encourages argumentation, and to write a script for its hypothetical implementation, including the context of the teaching situation and the discourse among the participants while working on the task. Follow-up questions included: (1) What were you considering when writing the script, in terms of engaging students in argumentation? (2) In what ways do you find that ‘your manner of teaching’ within the script provides opportunities for students to engage in argumentation? (3) In your script, what factors contribute to shaping the argumentation? (4) What difficulties or inhibitors are you taking into consideration here? How do you deal with them?

Data analysis

The aim of the data analysis was to ascertain what secondary mathematics teachers attend to when asked to envision argumentation in their classroom. We used the teachers' responses as the main source of our systematic analysis. The teachers' written scripts served as a resource for us to better understand and interpret their discourse. First, we employed the three predominant dimensions found in the literature to in creating opportunities for class argumentation – task characteristics (TC); teaching strategies for argumentation (TS); and students' characteristics (SC) – as lenses through which we analyzed the teachers' statements. at the same time, we remained open to other dimensions emerging as well, although this ultimately did not happen. We then distinguished between statements in which the teachers' focal attentiveness was directed toward *structural* aspects of argumentation (i.e., responses pertaining to elements of arguments such as claims and justifications and what counts as an appropriate

justification) and those in which their focal attentiveness was devoted to *dialogic* aspects of argumentation (i.e., responses associated with students' interactions when generating and critiquing arguments). We then used inductive content analysis to devise sub-categories for each of the six categories (structural and dialogic across TC, TS, and SC). We iteratively checked categorizations against the whole data set. Since the analytical process was comparative, it required repeated analysis of the whole data set. Based on an in-depth discussion of the emerging categories, we reached a final consensus. We ultimately obtained 13 sub-categories, to be discussed in the upcoming findings section. Subsequently, we used these codes to re-analyze the transcripts of the interviews with the 31 secondary mathematics teachers for characterizing each teacher's envisioning of potential argumentation situations in the classroom. The analysis focused on classifying each response according to the previously received categories and sub-categories.

FINDINGS AND DISCUSSION

Table 1 presents the categories identified in the teachers' responses¹. For each dimension, we present categories that were found to focus on structural aspects (S) of argumentation and categories found to focus on dialogic aspects (D) of argumentation. Note that 'T' stands for a teacher. The third column presents the number of teachers found to attend to each category according to their interviews.

Table 1: Categories of teachers' envisioning of potential classroom argumentation situations

Category	Examples from the teachers' responses	#Teachers (n=31)
Task characteristics (TC)		
TC1. (S) Inviting the use of specific mathematical justifications	<i>I asked the students to justify the claim $n^3 - n$ is divided by 6 for each natural n. My goal in choosing this task was to expose the students to different kinds of arguments while distinguishing between their merits: an algebraic solution, which is valued; and other approaches, such as substitution of numbers into the expression, which are not.... (T13)</i>	5
TC2. (D) Affording various solutions as enabling students' participation	<i>I give the students multiple-solution tasks to encourage their participation.... Such tasks afford fruitful argumentation, with different points of views, allowing for disagreements among students which they will need to resolve. (T24)</i>	26
Teaching strategies (TS)		

¹ In the presentation we will elaborate the discussion on the categories and examples.

TS1. (S) Encouraging and scaffolding justifications	<i>I would use various scaffolding strategies that serve to generate justifications for their solutions. For example, as I wrote in my script, by providing real matches [from a matchbook] to help students develop a sense of the situation.... or by encouraging a student to use a table of values to support his efforts to justify his generalization. (T1)</i>	26
TS2. (S) Promoting adherence to standard disciplinary criteria for determining the truth of a claim	<i>In my script, in response to this student's argument, I, as the teacher, emphasized: "[That's] a very good argument. We have here a counterexample for the claim that the number of matches is the number of wagons multiplied by four. This is a mathematical method to show that the claim is incorrect." (T1)</i>	9
TS3. (D) Encouraging students to collaborate on constructing arguments	<i>I prompt students to collaborate on developing their arguments. For example, when a student suggests a solution, I ask the other students questions, for example: Who would like to explain the other student's idea? How can you build upon this idea? And when a student responds, I commend him for collaborating. (T28)</i>	26
TS4. (D) Prompting students to critically evaluate each other's arguments and search for alternative ideas	<i>I encourage my students to critically evaluate each other's arguments... I do that by asking them questions like: Do you agree or disagree? What do you think about it? (T8)</i>	24
TS5. (D) Encouraging attempts to reach a consensus	<i>During the activity, I wrote on the board all the arguments that the students raised... and discussed with them which are correct and which are incorrect... It is very important to me that all students will be convinced and then reach a consensus as to which arguments are correct and which are incorrect, and why. (T19)</i>	11
TS6. (D) Establishing a climate of mutual respect	<i>I explain to my students that there should be mutual respect within the classroom; they should listen to each other respectfully, not disparage the other's opinion, and acknowledge that different people have different points of view... I praise students who critique others respectfully or receive others' critiques in a polite and open-minded way. (T16)</i>	7
Student characteristics (SC)		
SC1. (S) Students' strengths and challenges in justifying and refuting	<i>In the script, I took into account familiar ways in which students' thinking about forming and justifying generalizations might be incorrect, such as employing empirical methods or using invalid proportional reasoning.... Here, in my script, I tried to deal with these tendencies by challenging the students with dilemma. (T1)</i>	8
SC2. (D) Students' skills of communicating arguments	<i>Some students may know and understand the correct answer but are unable to articulate and present it in class, which makes it difficult for others to evaluate it and thus impedes having a productive discussion. (T15)</i>	7
SC3. (D) Students' skills of critiquing each other's ideas	<i>...If students are asked to evaluate their peer's answers, they usually do so by saying 'right' or 'wrong', without discussing the weaknesses or strengths and how to correct the mistakes if any are found. (T17)</i>	6

SC4. (D) Students' sense of confidence	<i>Sometimes only a few students participate. This is because some students, especially the weaker ones, suffer from a lack of self-confidence which causes them to be awkward about expressing themselves and reluctant to give critical feedback to their peers. (T9)</i>	13
SC5. (D) Students' interest and enjoyment	<i>For my students, it is more interesting and challenging to work together on a task, try to convince their peers about their solutions' correctness, and critique each other's ideas, rather than to be assessed solely by the teacher. (T31)</i>	8

As seen in Table 1, whereas some categories were attended to by a large number of teachers, some were scarcely mentioned. In terms of *dialogic* aspects of argumentation, analysis of the teachers' responses revealed that, in their envisioning of potential classroom argumentation situations, the majority (26 out of 31), attended primarily to choosing mathematical tasks that invite multiple solutions as a means for students to discuss differences in viewpoint and critique ideas (TC2). Most teachers also attended to teaching strategies that encourage students to collaborate on constructing and critiquing arguments (TS3 & TS4, 26 and 24 teachers, respectively), and, to a lesser degree, encourage students to reach a consensus (TS5, 11 teachers). Still only a few teachers expressed sensitivity to student characteristics that enable or inhibit participation in argumentation, and those that did referred mainly to students' difficulties in communicating their ideas in a comprehensive and coherent way (SC2, 7 teachers). A relatively small number of teachers attended to affective factors such as students' lack of self-confidence or to students' interest and enjoyment when participating in argumentation (SC4 & SC5, 13 and 8 teachers, respectively). In terms of *structural* aspects of argumentation, the analysis of the teachers' responses revealed that most teachers (26 out of 31), in their envisioning of potential classroom argumentation situations, attended to teaching strategies that encourage and support students in their struggle to build justifications for their claims (TS1). Only a few teachers mentioned in-class teaching strategies which promote adherence to standard disciplinary criteria for evaluating the quality of arguments and which cultivate students' sensitivity to what constitutes acceptable mathematical arguments in the classroom (TS2, 9 teachers). In addition, few teachers referred to students' tendencies and possible difficulties when generating specific kinds of mathematical justifications, such as the tendency to use empirically based justifications, or the challenge in generating deductively-based arguments (SC1, 8 teachers).

Overall, in the teachers' envisioning of argumentation in their classrooms, we see much attention to social interactions that attempt to generate new ideas and those involving the critiquing of each other's ideas and solutions. To a much lesser degree, we see teachers' attention to the specific normative aspects of mathematical argumentation and to students' characteristics in relation to their

engagement in argumentation. Research has indicated the importance of instructional practices that integrate both the dialogic and structural aspects of argumentation which are mathematically specific (Nathan & Knuth, 2003). Research has also suggested that teachers who are likely to support student participation in argumentation but do not emphasize distinctions between acceptable and unacceptable mathematical justifications, may limit students' opportunities to develop an understanding of what constitutes accepted mathematical justifications and thus act more autonomously when engaging in mathematics (Ibid., 2003). In the teachers' interviews included in our study, we found wide mention of providing students with opportunities to participate in co-constructing and critiquing arguments. At the same time, we witnessed rather limited attention to facilitating students' participation in classroom argumentation grounded in normative aspects of mathematical argumentation. Our study suggests, therefore, that the teachers' envisioning of argumentation in the mathematics classroom was partial, at least as far as can be inferred from their interviews. Hence, there is more to learn about teachers' understanding of argumentation.

CONCLUSION

While the interviews used in this study provided a snapshot of teachers' views at a particular point in time, research suggests that attention can be cultivated over time (Mason, 2015; Paparistodemou et al., 2014). Findings of the current study can serve as a foundation and a resource for enhancing teachers' attention to argumentation. The range of dimensions identified in this study can serve as an analytic platform for planning and facilitating professional development activities to promote teachers' awareness of, and enthusiasm for, argumentation. Examples of teachers' responses compiled in this study can serve as sources for other teachers to analyze, compare, and reflect on, so as to construct a broad range of 'attention to argumentation' aspects. The fact that some of the teachers participating in this study perceived both the structural and the dialogic aspects across the three dimensions as an integral part of enhancing the argumentation processes in the classroom, is encouraging. It is evident from our findings that teachers are at least partially open to adopting a new mindset with respect to the teaching of argumentation in the classroom.

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METACOGNITIVE BEHAVIOUR IN PROBLEM POSING – A CASE STUDY

Lukas Baumanns¹ and Benjamin Rott¹

¹University of Cologne, Germany

This investigation aims at developing a framework for identifying metacognitive behaviour in problem-posing processes and illustrating the potential of such a framework for assessing the quality of problem posing. For this purpose, 36 task-based interviews were conducted with pairs of student teachers. On these, an inductive category development has been carried out to identify problem-posing-specific metacognitive behaviour of planning, monitoring, and evaluating. Subsequently, the identified metacognitive behaviours were applied to a selected transcript fragment.

INTRODUCTION

At least since Flavell's 1979 seminal work, metacognition has been a central construct of research in psychology and mathematics education (Schneider & Artelt, 2010). In particular, research on problem solving has benefited from the consideration of metacognitive behaviour (e.g., Schoenfeld, 1987). Surprisingly, for the field of problem posing, a systematic literature review in high-ranked journals on mathematics education revealed that nearly no study explicitly considered metacognitive behaviour (Baumanns & Rott, 2021). Yet we are convinced that considering and analysing problem-posing-specific metacognitive behaviour may be a pivotal enrichment to the field. On the basis of this desideratum, we aim at (1) developing a framework for identifying metacognitive behaviour in problem-posing processes and (2) illustrating the potential of such a framework for assessing the quality of problem posing.

THEORETICAL BACKGROUND

Problem posing

The numerous definitions of problem posing conceptualise mostly equivalent activities. Silver (1994) defines problem posing as generation of new and reformulation of given problems which occurs before, during, or after problem solving. Stoyanova and Ellerton (1996) refer to problem posing as the “process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems” (p. 218).

Based on the categories by Stoyanova and Ellerton (1996), we distinguish between unstructured and structured problem-posing situations depending on the degree of given information (Baumanns & Rott, 2021). Unstructured situations are characterised by a given naturalistic or constructed situation in which tasks can be posed without or with less restrictions. Asking to pose many problems to a given geometric configuration would be, for example, an unstructured situation. In structured situations, people are asked to pose further problems based on a specific problem, for example by varying its conditions. As structured situations are used in this study, an example is shown in the section *Methods*.

Metacognition

According to Flavell (1979, p. 906), metacognition describes “knowledge and cognition about cognitive phenomena”, which roughly means thinking about thinking. Based on this understanding, two facets of metacognition are identified: (1) knowledge of cognition and (2) regulation of cognition. In this paper, regulative activities are investigated, therefore facet (2) will be focused.

Regulation of cognition refers to procedural knowledge with regard to processes that coordinate cognition, including *planning*, *monitoring*, and *evaluation* (Schraw & Moshman, 1995). Planning refers to the identification and selection of appropriate strategies or resources concerning the current endeavour. Monitoring refers to the attention and awareness of the comprehension concerning the current endeavour. Evaluating refers to the assessment of the processes and products of one’s learning. Cohors-Fresenborg and Kaune (2007) provide a category system for classifying teachers’ and students’ metacognitive (i.e. planning, monitoring, and evaluating) and discursive activities in class discussions. This approach is used in this study.

Research on metacognition in problem posing

In mathematics education research, metacognition is considered most prominently in problem-solving research which had an immense impact on this field (e.g., Schoenfeld, 1987). However, research on metacognitive behaviour in problem posing remains largely unresearched to date. Some studies contain few aspects of metacognition and self-regulation (Pelczer & Gamboa, 2009; Kontorovich et al., 2012), metacognition is rarely explicitly addressed, though. Yet, for example, Voica et al. (2020) mention that they found metacognitive behaviour in their study with students as they were able to analyse and reflect on their own posed problems and thinking processes.

RESEARCH QUESTIONS

The lack of conceptual and empirical insight into metacognitive behaviour in problem posing constitutes a desideratum from which the following research questions emerge:

- (1) Which problem-posing-specific metacognitive behaviour (i.e. planning, monitoring, and evaluating) can be identified in students' problem-posing processes?
- (2) To what extent can different degrees of problem-posing-specific metacognitive behaviour be empirically assessed?

METHODS

Research design for data collection

For this study, 32 task-based interviews were conducted, each with two pre-service primary and secondary mathematics teachers, working in pairs on one of two structured problem-posing situations (A. Nim game; B. Number pyramid). Situation A, the Nim game, reads: "There are 20 stones on the table. Two players A and B may alternately remove one or two stones from the table. Whoever makes the last move wins. Can player A, who starts, win safely? Based on this task, pose as many mathematical tasks as possible." In total, 15 processes of situation A and 17 processes of situation B that range from 9 to 25 minutes with an average length of 14.5 min have been recorded. The processes ended when no ideas for further problems emerged from the participants. 7h 46min of video material was recorded and analysed.

Data analysis – Assessment of metacognitive behaviour

To answer research question (1), we conducted a qualitative content analysis (Mayring, 2000). There are three main categories of the metacognitive behaviour, *planning*, *monitoring*, and *evaluating*, which stem from theoretical considerations on regulation of cognition presented above, especially Cohors-Fresenborg and Kaune (2007). Although their framework is developed for analysing classroom interaction, it has been used successfully in paired problem-solving processes (Rott, 2014). Problem-posing-specific sub-categories were obtained through an inductive category development, with the goal of identifying the activities of planning, monitoring, and evaluating within the 32 recorded problem-posing processes.

For research question (2), we analysed in detail several transcripts using the developed framework. The analysis of a selected process fragment is discussed in the results. In this transcript, the participants' statements are reproduced verbatim. For the analysis, the transcripts were first read iteratively in order for us to obtain a rough understanding of the text and to be able to better integrate finer sections of the text into the overall context of the text. The codes developed in research question (1) are then applied to the transcript. The quality of the coding was ensured through consensual validation in team discussions. The coding of metacognitive behaviour of *planning*, *monitoring* and *evaluation* are color-coded in *blue*, *red* and *yellow* the style of Cohors-Fresenborg and Kaune (2007).

RESULTS

Identification of metacognitive behaviour in problem posing

In Table 1, the observed problem-posing-specific metacognitive behaviours of planning, monitoring, and evaluating are summarized. In the following, the behaviours are commented and discussed.

Planning. Four different behaviours of planning have been identified in the students' processes. Code P1 denotes the focus on a starting point for problem posing from which new problems can be posed. This can be, for example, a certain condition, a certain context or even a certain solution structure of the given initial problem. The behaviour in P2 has been frequently observed and is reminiscent of the well-known "What-if-not"-strategy (Brown & Walter, 2005), in which a similar activity is suggested before the actual problem posing. Reflecting necessary knowledge (P3) was observed quite rarely. Nevertheless, participants have partly considered what knowledge they or the potential solvers of a posed problem need to have in order to be able to solve it. In some cases, a general procedure for the upcoming problem-posing process was also named by the participants, e.g. first vary the initial task in multiple ways, then solving the varied tasks (P4).

Monitoring. M1 characterises that metacognitive behaviour in which participants control the problem-posing process. Controlling the notation or representation of the posed problems (M2) refers to figures drawn to illustrate a problem, to the formulation of the specific question so that it becomes understandable and precise, or similar behaviours. We frequently observed that participants made a modification to the initial problem and analysed the consequences of this modification on the newly created problem (M3), for example for the solution structure or its difficulty. The code M4 was set when participants analysed the mathematical structure of the given situation in order to get to a new problem or analysed the structure of a posed problem in order to be able to characterise it, for example with regard to its solvability or appropriateness.

Planning		Monitoring		Evaluating	
P1	Focus on a starting point of the problem-posing situation to generate new problems	M1	Controlling the general procedure for problem posing	E1	Assessing and reflecting on the characteristics of the posed problems
P2	Capturing the conditions and identifying the restrictions of the given problem-posing situation	M2	Controlling the notation or representation of the posed problems	E2	Reflect on modifications of the posed problems

P3 Reflect necessary knowledge	M3 Assessing consequences for the problem's structure through the modified or new constructed conditions
P4 Express general procedure for problem posing	M4 Mathematical activities related to a posed problem

Table 1: Regulative processes in problem posing

Evaluating. Assessing and reflecting on the characteristics of a posed problem (E1) was a common behaviour of the participants. They often evaluated whether their problem is interesting, solvable, or appropriate for a specific target group. This behaviour was also mentioned in previous studies on the process of problem posing (Pelczer & Gamboa, 2009; Kontorovich et al., 2012). A reflection on modifications of the posed problems (E2) was frequently observed when the posed problem lacks a specific characteristic, for example it is too easy or too difficult, it is not very interesting, or it is too similar to the initial problem.

The case of Tino & Ulrich

In this section, we show the analysis of a process fragment by the students Tino and Ulrich, focussing on the metacognitive behaviours that have been developed in the previous section. The transcript starts at 19m 49s of their 33m 21s problem-posing process of the Nim game. Beforehand, they already posed, solved, and analysed several new variations of the Nim game such as: What if there are 21 stones on the table in the beginning? What if you could remove 1, 2, or 3 stones from the table? In the following fragment, they pose the problem that you are only allowed to remove 2 or 3 stones from the table.

1.	U:	So and now we make a next variation namely you may no longer take 1 to 3, but you may either take 2 stones or 3 stones	P1
2.	T:	What about the variation with number of stones is also a victory factor?	
3.	U:	Oh yes, we can do that too...	
4.	T:	At least we can notice for a moment, right?	
5.	U:	... But I would like to do that later, I would like to save that for a little, so this is definitely also a variation.	M1

In turn 1, Ulrich poses a new variation, in which only 2 or 3 pieces may be removed from the table. This new starting point is derived from a previous task (1 to 3 pieces may be removed). Since Ulrich sets a new focus for the upcoming problem-posing activity, this statement is coded as *planning* (P1). After Tino has thrown in what happened to one of the previous ideas, Ulrich directs the general procedure in turn 5 and thinks that this task can be dealt with later and that the task posed in turn 1 should be discussed in greater depth. Ulrich intervenes in the process and tries to guide it in a structuring way. Therefore, this statement is coded as *monitoring* (M1).

10.	T:	(referring back to the problem posed in Turn 1) So you can't just remove one tile, right? (writes down) Okay.	M2
11.	U:	Here is a scenario; at 4 nobody wins (5 sec).	
12.	U:	When I take 3.	
13.	U:	Yes. This is a new game. I think it's great, it's already totally good. There are situations where nobody wins. Yeah, it's like at the game...	M3 & E1
14.	T:	Tic-tac-toe...	M1
17.	U:	Yes, I can't remember exactly. Already interesting! It is already interesting.	E1
18.	T:	Yeah, it's definitely interesting.	E1

In turn 10, Tino tries to find a formulation for the problem that was posed in turn 1. He writes down this task as a negation that one may not just remove one stone from the table. His thinking about the formulation of the question represents a control of the notation or representation of the problem and is therefore coded as *monitoring* (M2). Ulrich says that this change results in a “new game”. This assessment of the consequences that their variation has for the Nim game was coded as *monitoring* (M3). Ulrich states that he likes the consequences that follow from their variation since they are different from the initial task. Therefore, this is coded as *evaluation* (E1). In turn 18, Tino agrees with Ulrich's positive evaluation of the game.

20.	T:	The question is, the question is whether one still admits that one also loses if one only has one stone left, but one can no longer move.	E2
21.	U:	Yes, but I would not do that.	E2
24.	T:	Because then you practically keep it up, right? The winning strategy.	M3
25.	U:	But I'd say it's a little lame somehow.	E1

26.	T:	Yes, of course, but just to think about how I can keep this system up, it would probably be a possibility.	E2
27.	U:	Yes, that's true. Then, exactly, then the system would also be upright. But let's move on to the next step. ...	E2 & M1
38.	T:	... What if one could remove 3 or 4?	P1

Tino interjects in turn 20 whether they should modify the new game due to this situation. Ulrich argues not to make this change. In both statements, the participants consider to modify the posed problem so that the game has a definite winner (E2). Tino states in turn 24 that this change would restore the original winning strategy of the initial task. By that, he assesses the consequences of his slight modification and compares it to the initial task. Therefore, this statement is coded as *monitoring* (M3). Ulrich does not seem to like this change, perhaps because it would bring him too close to the initial task. In turn 13, he seemed to like this new element very much. This statement is coded as *evaluation* (E1). Tino reflects in turn 26 that one could modify the game with his suggestion in order to maintain the original winning strategy of the initial task. This is a reflection on their modification and, thus, is coded as *evaluation* (E2). Ulrich initially agrees with Tino's previous assessment (E2). Then, he focuses on a solution strategy of the modified game again and thinks about the situation in which five stones lie on the table. With his statement, he is controlling the process which is why this statement is coded as *monitoring* (M1).

DISCUSSION AND CONCLUSION

The present study examined metacognitive behaviour, which has so far been widely disregarded in problem-posing research. Analyses of 32 problem-posing processes of student teachers were conducted to identify regulative behaviours, sorted into planning, monitoring, and evaluating. The results of this exploratory investigation are discussed in the following with regard to research questions (1) and (2):

(1) Table 1, summarises observed behaviours that are predominantly metacognitive. Some of these behaviours may be considered as cognitive. Yet, it should be noted that being able to intentionally use these kinds of cognitive behaviour is a sign for metacognitive abilities. For example, searching for a solution can be seen as cognitive behaviour, but considering the solution in order to get a better idea whether the posed problem is, for example, solvable or appropriate for a specific target group can be seen as metacognitive behaviour. Moreover, not all codes within the main categories of planning, monitoring, and evaluating are separable from each other. However, a clear separation between these main categories should be recognizable. It should be emphasised that even if the named behaviours are labelled as metacognitive, they should not be considered without cognitive behaviour.

(2) Tino and Ulrich show several and dense acts of metacognitive behaviour in their problem-posing process. In the analysed fragment (duration 3:12 min), 19 activities were coded as metacognitive, i.e. one code every 10 seconds. This value should not be interpreted as a fixed value for metacognitive behaviour. However, it allows to identify a tendency for quantity of metacognitive behaviour. In other fragments that have been analysed in this study, this value was strikingly lower. Coding using the developed categories is intended to support this assessment.

The framework developed in this study provides numerous opportunities for follow-up research. With a larger sample, maybe additional problem-posing-specific metacognitive behaviours can be identified. As in research on problem solving, a comparison between metacognitive behaviours of experts and novices could reveal metacognitive behaviour related to successful problem posing. This study uses structured problem-posing situations. Future studies could investigate whether there are different metacognitive behaviours in unstructured situations. Often, the ability to pose problems is measured by analysing the products of a problem-posing process (cf. Van Harpen & Sriraman, 2013). The analysis of metacognitive behaviour could be used to assess the ability to pose problems on a process-oriented level. Neglected in this study was the metacognitive facet knowledge of cognition. The importance of this facet of metacognition could also be the focus of future studies. In addition, the interaction and discourse of the participants in the transcript fragment also plays a central role in the quality of problem-posing processes. Future considerations could look more closely at this interaction as an additional aspect of (negative) discursivity similar to Cohors-Fresenborg and Kaune (2007). Overall, we believe that the so far largely neglected perspective of metacognitive behaviour can be a significant enrichment for problem-posing research in the future.

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REASONING ACROSS THE CURRICULUM WITH A MEASUREMENT UNIT THAT VARIES IN SIZE

Sybilla Beckmann¹ and Andrew Izsák²

¹University of Georgia, United States of America

²Tufts University, United States of America

Reasoning about covarying quantities in terms of both a fixed measurement unit and a measurement unit that varies in size is an overlooked but potentially valuable way to help learners make sense of a range topics that past research has demonstrated pose perennial challenges. We identify several such topics including developing and explaining linear equations, making sense of slope and average rate of change, interpreting geometric similarity and trigonometric ratios, and understanding the relationship between empirical and theoretical probability. We explain how a specific way of conceptualizing proportional relationships—the variable-parts perspective—relies on reasoning with both a fixed measurement unit and a measurement unit that varies in size, and make the case this perspective can be a foundational and productive way of reasoning about a critical swathe of school mathematics.

INTRODUCTION

A well-known concern in mathematics education is disjointed, incoherent treatments of topics that rely on isolated, single-purpose tools. Another concern is that many important topics are difficult for students and teachers, including linear relationships, rates of change, trigonometry, and the law of large numbers (e.g., see Cai, 2017). One possibility is that there are foundational ideas that students need to work productively across a variety of topics, but that these ideas have either not been emphasized or their importance has not been discovered in mathematics education. Also, when the same idea can be used repeatedly, across many topics, students may see how prior experiences can help them reason about new situations.

In this theoretical essay we propose that a specific way of conceptualizing proportional relationships—the variable-parts perspective (Beckmann & Izsák, 2015)—may be valuable, in part, because it includes an idea that is needed in many contexts: the idea of using both a fixed unit of measurement and a unit of measurement that varies in size to measure and describe covarying quantities.

USING VARIABLE PARTS TO GENERATE AND EXPLAIN EQUATIONS

Research has shown that the variable-parts perspective provides one specific way to conceive of how quantities can vary together, yet be in a constant linear (or proportional) relationship (Beckmann & Izsák, 2015). It is part of a coherent, connected landscape of multiplicative ideas at the core of a large swathe of important mathematics (Izsák & Beckmann, 2019).

To illustrate the variable-parts perspective, Figures 1a and 1b show 4 parts of red paint covarying with 3 parts of blue paint. Initially, each of the 7 parts is 1 Litre. Then the parts are allowed to vary in such a way that the 7 parts remain the same size as each other, but that size can be any number of Litres. If we view 1 part as a unit of measurement that varies in size, then we can interpret the quantities of paint as simultaneously fixed and varying. Measured in parts, the red paint is fixed at 4 parts and the blue paint is fixed at 3 parts. Yet measured in Litres, the numbers of Litres of red and blue paint vary. When mixed, the paint would always make the same fixed hue of purple, but in larger or smaller amounts, depending on how many Litres make 1 part.

We can use the above perspective on the red and blue paint to develop an equation. Let the red paint consist of X Litres and let the blue paint be Y Litres. Then because Y is always 3 parts and X is always 4 parts, Y must always be $\frac{3}{4}$ of X , and therefore the equation $Y = \frac{3}{4} X$ describes how the covarying quantities of paint are related.

If we rotate the 3 parts blue, as in Figures 1c and 1d, we can see the line through $(0, 0)$ and $(4, 3)$ from a variable-parts perspective (see www.geogebra.org/m/fe9q378s for these and other dynamic Geogebra sketches). The foregoing reasoning also explains why the line has equation $Y = \frac{3}{4} X$.

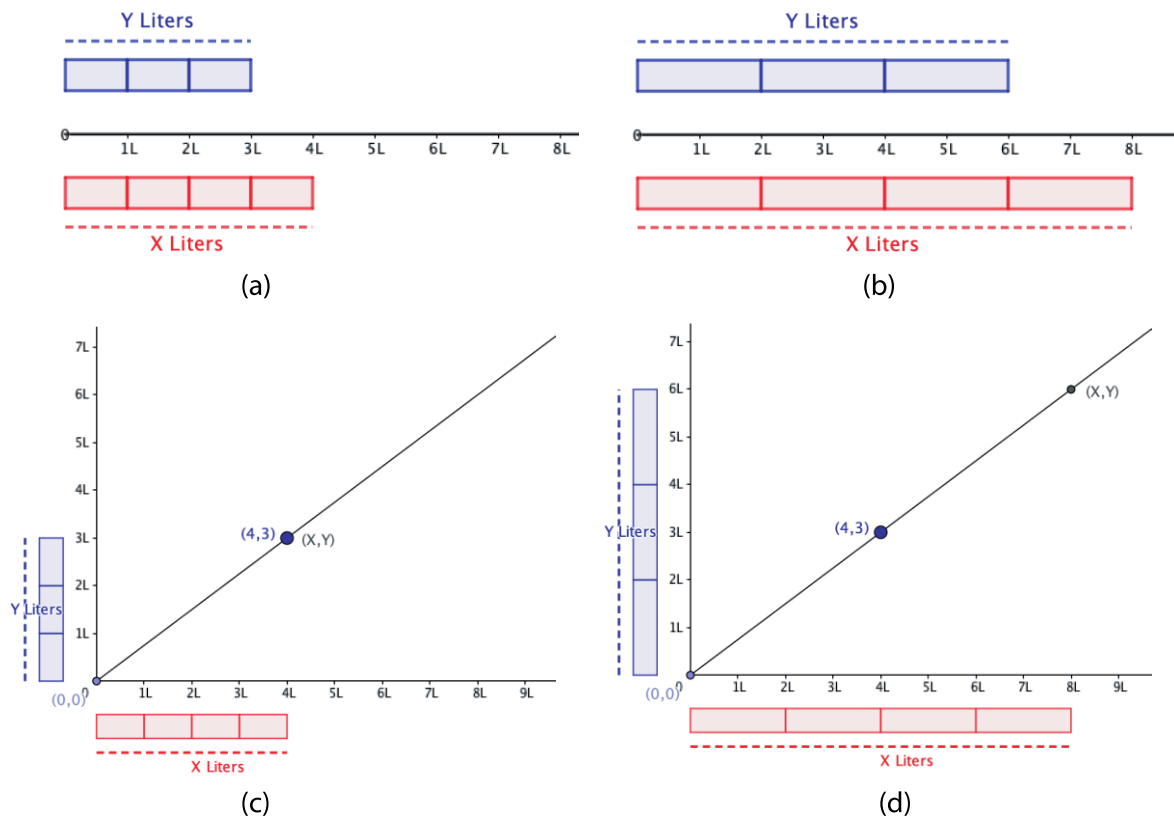


Figure 1: Covarying quantities of red and blue paint from a variable-parts perspective. (a, c) 1 liter per part. (b, d) 2 liters per part.

The variable-parts perspective is different from another “multiple-batches” way to conceive of the red and blue paint. For example, we might think of 20 Litres of red paint and 15 Litres of blue paint as composed of 5 batches, each consisting of 4 Litres red paint and 3 Litres blue paint. From this multiple-batches viewpoint, as we vary the numbers of Litres of red and blue paint, we imagine a fixed batch, consisting of 4 Litres red paint and 3 Litres blue paint, and we imagine varying the *number of batches* that we consider. But there is no *unit of measurement that varies in size*, and we do not describe the overall quantities of paint as fixed. In a coordinate plane, from this multiple-batches perspective we might view quantities of red and blue paint as obtained by repeatedly going over 4 Litres and up 3 Litres, or over 1 Litre and up $\frac{3}{4}$ Litres.

Past research has shown that middle grades students as well as future teachers often have difficulty justifying linear relationships (e.g., Rivera & Becker, 2007; Stephens, Ellis, Blanton, & Brizuela, 2017). Students and teachers have more success when they use visual strategies with figural patterns, but figural patterns are discrete and do not offer the opportunity to reason about a continuous context. Above, we showed one way to generate and justify linear equations in two variables by reasoning about how quantities are related in a continuous

context viewed from a variable-parts perspective. Beckmann and Kulow (2017) showed that future middle grades teachers enrolled in a mathematics course were able to use the variable-parts perspective to reason about covarying quantities and to generate and justify linear equations in two variables, including equations in non-standard forms. Thus, a variable-parts perspective might also be promising for helping middle grades students to reason quantitatively to generate and justify linear equations.

USING VARIABLE PARTS TO INTERPRET RATE OF CHANGE

Research has shown that the concepts of slope and rate of change are difficult for students and teachers. For example, in Lobato, Ellis, and Munoz's (2003) study, middle grades students interpreted m in $y = b + mx$ as a difference rather than a ratio. In a study of secondary teachers' meanings for measure, slope, and rate of change, Byerly and Thompson (2017) found that the majority of teachers interpreted a slope of 3.04 as meaning that for every change of 1 in x , there is a change of 3.04 in y , or as moving to the right 1 space and up 3.04 on a graph. When these teachers were asked how to interpret 3.04 if x changes by something other than 1, only 8% conveyed a multiplicative meaning for 3.04, such as x can change by any amount and y will change by 3.04 times the change in x .

The variable-parts perspective offers a way to view (average) rates of change and slope as the result of a measurement and therefore multiplicatively. In the example of Figure 1 discussed previously, the rate of change or slope, $\frac{3}{4}$, is the constant measure of Y Litres (3 parts) in terms of X Litres (4 parts); it is how much of X it takes to make Y exactly. This interprets the rate of change or slope multiplicatively, as how many times one needs to take one quantity to produce another, and is not limited to the case where the change in X is 1 unit.

More generally, the idea of using a unit of measurement that varies in size could be important for interpreting instantaneous rates of change in calculus. With a variable-parts perspective, we can interpret the average rate of change of a function in a way that makes sense even when both ΔX (the change in X) and ΔY (the change in Y) shrink toward 0. The average rate of change of a function over an interval is given by a difference quotient, namely ΔY divided by ΔX . To interpret this difference quotient as a measure, we can view ΔX as a measurement unit that varies in size, and we can use it to measure ΔY . The resulting measure—how many (or how much) of ΔX it takes to make ΔY exactly—is the value of the difference quotient $\Delta Y/\Delta X$. See Figure 3. For a differentiable function, as ΔX shrinks toward 0 (keeping the left end point of the intervals fixed, say), these measures are approximately constant, and approach the value of the derivative at the left end point.

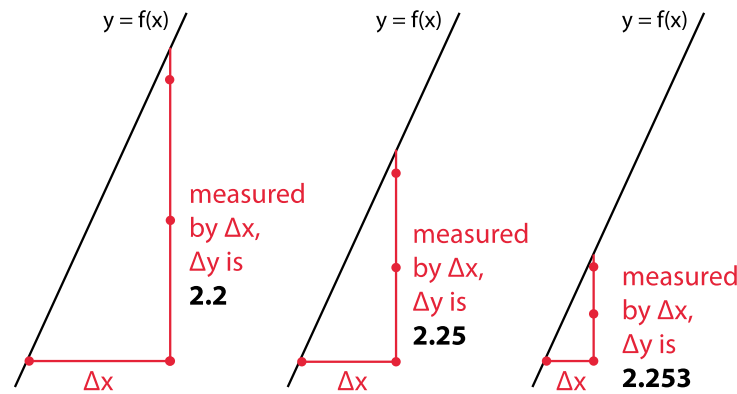


Figure 3: Average rate of change as the measure of ΔY by ΔX as ΔX shrinks toward 0.

A variable-parts perspective might build on approaches students have been found to use in past research. In a study of students' quantitative reasoning about covarying quantities, Johnson (2015) asked students to reason about how the height of liquid in a bottle varied with the liquid's volume. Even though height and volume are not the same kind of quantity and not measured in the same units, all three students compared changes in height with changes in volume. Johnson found that making such comparisons can be useful for interpreting covariation, but it does not foster attention to variation in the intensity of change. We propose that a productive next step for the students in Johnson's study might be to *measure* changes in the dependent variable by changes in the independent variable, given that they had just *compared* such changes. Such a next step would put students on a path to interpreting average and instantaneous rates of change as we described in the previous paragraph.

USING VARIABLE PARTS IN GEOMETRY AND TRIGONOMETRY

To see how the variable-parts perspective is useful for situations of geometric similarity, including trigonometry, consider dilations of 2-dimensional Euclidean space equipped with Cartesian coordinates. We may think of dilations that are centred at the origin in terms of two systems of coordinates on the same axes: one in which 1 unit of distance is fixed at 1 cm (say) and another set of coordinates in which 1 unit of distance—1 part—varies in size and consists of x cm, where x is the scale factor of the dilation. Given any point in the plane, its coordinates can be expressed in centimetres or in parts. For example, Figure 4 shows the effect of applying a dilation with scale factor 2. The light (grey) grid lines remain 1 cm apart and the heavy (red) grid lines remain 1 part apart even though 1 part changes from (a) 1 cm to (b) 2 cm. Expressed in terms of parts, the coordinates of the apex of the triangle are always (4, 3) even though the apex's coordinates expressed in centimetres vary as the scale factor of the dilation varies.

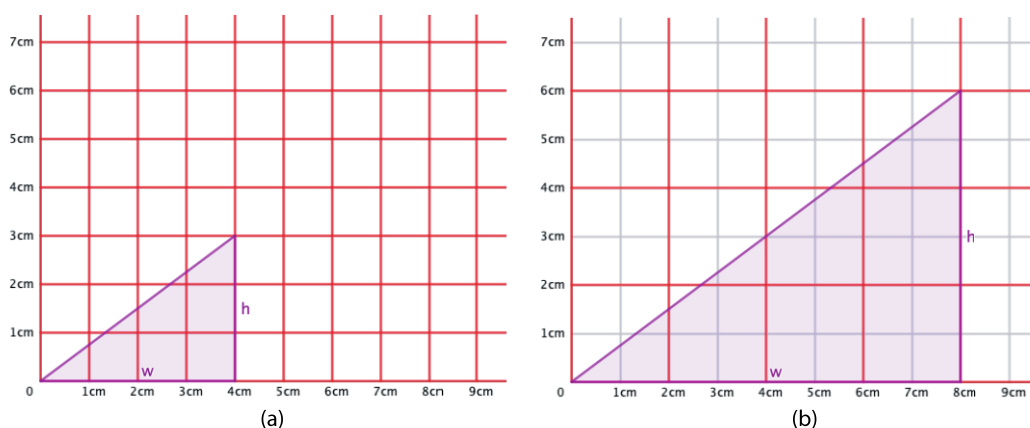


Figure 4: Heavy (red) grid lines are 1 part apart where 1 part is (a) 1 cm (b) 2 cm.

This variable-parts perspective highlights that the side lengths of a triangle remain in the same ratio even as the triangle is dilated. For example, in Figure 4 the height of the triangle, h , is always $3/4$ its width, w , and therefore h/w is always $3/4$ and is independent of the dilation that is applied to the triangle. The constancy of ratios of side lengths of right triangles is necessary and implicit in trigonometry.

To apply the variable-parts perspective to trigonometry, consider a right triangle inscribed in a circle of radius 1 part, which is r cm (say), where r is the scale factor of a dilation centred on the centre of the circle. With this view, the radian measure of an angle is the measure in terms of parts—i.e., in terms of the radius—of the arc subtended by the angle on the circle. The variable-parts perspective highlights that the radian measure of an angle does not depend on the size of the circle and that it is always how many or how much of the radius it takes to make the subtended arc.

A variable-parts perspective fits with the successful approach Moore (2013, 2014) took in his teaching experiments on angles and trigonometric functions. In particular, Moore's teaching experiments seem to have fostered the idea of measurement with respect to both a fixed unit and a variable unit (the radius). For example, Zac interpreted an arc length of 0.61 as 61% of a radius and explained that it is always the same percentage for each different circle. Zac was also able to interpret the sine and cosine as percentages of a circle's radius, regardless of the circle's size.

USING VARIABLE PARTS FOR THE LAW OF LARGE NUMBERS

In their review of the teaching and learning of probability and statistics, Langrall, Makar, Nilsson, and Shaughnessy (2017) noted that there has been particular interest in informal inference and a strong consensus that “informal inference includes (1) making claims or predictions beyond the given data while

(2) using the data as evidence for any claims that are made and (3) acknowledging that there is uncertainty in any claims or predictions” (p. 516). In reviewing misconceptions of statistical inference, Castro Sotos, Vanhoof, Van den Noorgate, and Onghena (2007) found a number of empirical studies that documented misconceptions regarding the idea behind the law of large numbers. According to Castro Sotos et al., students’ difficulties may have their source in the misconception known as “the law of small numbers,” in which even small samples are assumed to be highly similar to the population from which they are drawn (Tversky & Kahneman, 1971). Tversky and Kahneman noted that in sequential games of chance “subjects act as if *every* segment of the random sequence must reflect the true proportion: if the sequence has strayed from the population proportion, a corrective bias in the other direction is expected. This has been called the gambler’s fallacy” (p. 106). Castro Sotos et al. called for further research to identify sources and possible solutions for misconceptions.

In situations of random processes, such as spinning a spinner, there are different ways students might informally conceptualize the law of large numbers. Consider a spinner that has 5 sectors of the same size, 3 purple and 2 blue (see Figure 6), and assume that every time one spins the spinner, each sector is equally likely to be landed on. One way to interpret the theoretical probability of landing on purple, $3/5$, is “we expect 3 out of every 5 spins to land on purple.” With this interpretation, one might interpret the law of large numbers in terms of multiple “batches” (sets) of 5 spins, expecting that in every such batch, 3 should land on purple, and that when batches deviate from this expectation, subsequent batches will adjust to compensate. Such a view seems similar to the “law of small numbers” ideas described by Tversky and Kahneman (1971). Although it seems reasonable to some extent for students to use ideas like “3 out of every 5 spins should be purple,” it also seems that this way of thinking could reinforce the “law of small numbers” and the gambler’s fallacy.

The variable-parts perspective provides a different way to think about the law of large numbers. In the context of the spinner discussed above, imagine spinning the spinner over and over, and think of measuring sets of spins in two ways: in terms of the fixed unit “1 spin” and in terms of the unit “all the spins so far,” which varies in size. (Alternatively, one could use “the spins that have landed in Sector 1 so far.”) One can then think of measuring various sets of spins, such as the spins that have landed in purple so far. If we measure the spins that landed in purple by the unit “1 spin,” the result is a number of spins. If we measure the spins that landed in purple by “all the spins so far,” the result is some fraction or percentage—the empirical probability. Because the spinner is equally likely to land in each sector, we expect approximately $1/5$ of the spins to land in each sector, and so we expect approximately $3/5$ of the spins to land in purple. As more and more spins are performed, we should expect the spins to become more

and more evenly distributed across the 5 sectors (see Figure 6). So when we measure the spins that landed in purple by the unit “all the spins so far,” we should expect the measure to be approximately $3/5$, with a better and better approximation as there are more and more spins. This is a way for students to see why we should expect the empirical probability to approximate the theoretical probability more and more closely as the number of spins increase. We propose that such an interpretation of the law of large numbers provides a more accurate image and a better foundation for informal inference than a “3 out of every 5 spins” idea.

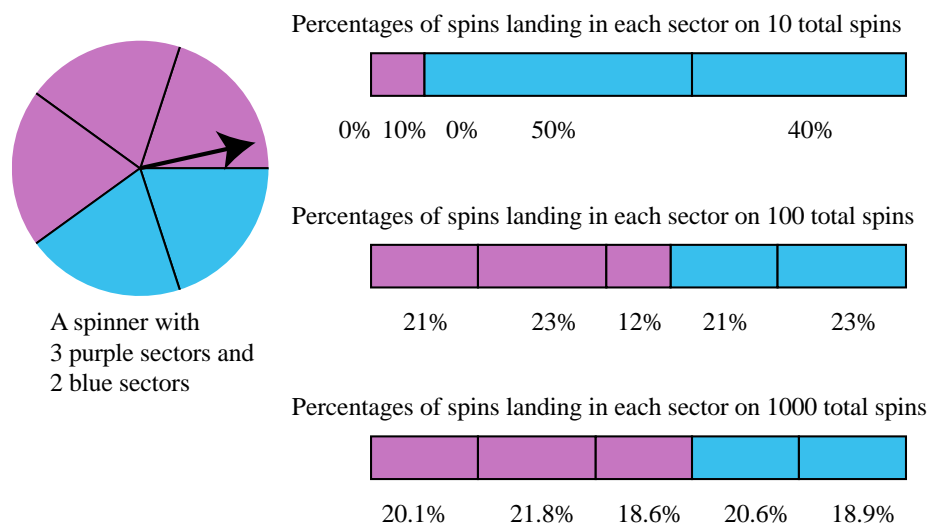


Figure 6: A spinner and percentages of spins landing in each sector on 10, 100, and 1000 spins.

In a study of future middle grades teachers who were enrolled in a mathematics course that taught the variable-parts perspective, Stevenson, Beckmann, Johnson, and Kang (2018) found that all 4 participants were able to reason about spinners using a variable-parts perspective, even though probability had not yet been discussed in the course. Three of the future teachers also used an interpretation like “3 out of every 5 times.” Two of them got stuck when using such an interpretation, but then made progress when they shifted to focusing on spins landing in parts (sectors) of the spinner and used variable-parts reasoning. Although further study is needed, these results suggest that a variable-parts perspective could be both accessible and useful for reasoning in probability and statistics.

Acknowledgment

This research was supported by the National Science Foundation under Grant No. DRL-1420307. The opinions expressed are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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THE APPROACH TO MATHEMATICS LEARNING AS A PREDICTOR OF INDIVIDUAL DIFFERENCES IN CONCEPTUAL AND PROCEDURAL FRACTION KNOWLEDGE

Maria Bempeni¹, Stavroula Pouloupoulou², & Xenia Vamvakoussi¹

¹University of Ioannina, Greece

²Athens University of Economics and Business, Greece

In the present study, we tested the hypotheses that: a) there are individual differences in secondary students' conceptual and procedural fraction knowledge, and b) these differences are predicted by students' approach (deep vs. surface) to mathematics learning. We used two instruments developed and evaluated for the purposes of the study which were administered to 463 students at seventh and ninth grade. We found four clusters of students corresponding to different ways of combining conceptual and procedural knowledge of fractions. Students' approach to mathematics learning predicted membership to some, but not all clusters.

THEORETICAL BACKGROUND

Procedural knowledge is commonly defined as the knowledge of algorithmic procedures, whereas conceptual knowledge as the knowledge of concepts and principles pertaining to a certain domain (Rittle-Johnson & Schneider, 2015). This distinction has been criticized (e.g., Star & Stylianides, 2013), a main issue of concern being whether it is possible for the two types of knowledge be separated, given that they are typically found to be highly correlated. Nevertheless, there are indications that the two types of knowledge can be separated both theoretically and empirically (Lenz & Wittman, 2021), and this distinction remains useful in the area of research on mathematics learning (Vamvakoussi, Bempeni, Pouloupoulou, & Tsiplaki, 2019).

Assuming that conceptual and procedural knowledge are distinct types of knowledge, the order of acquisition and their relation have long been an issue of interest. The currently predominant theory, namely the iterative model (Rittle-Johnson, Siegler, & Alibali, 2001), came to bridge the gap between two different accounts according to which one type of knowledge precedes the other (procedures-first and concept-first theories). The iterative model assumes that either type of knowledge can trigger the learning process, depending on the child's prior experience with the domain in question; and that, from then on, the links between the two types of knowledge are bi-directional and continuous, with increases in one kind of knowledge leading to gains in the other type of

knowledge. The iterative model explains many empirical findings, notably the well-established one that the two types of knowledge are positively correlated. However, such correlations found at group level do not accurately depict what happens at the individual level (Vamvakoussi, et al., 2019). Indeed, there is evidence that there are individual differences in the ways students combine the two types of knowledge. Hallett and colleagues (2010; 2012) investigated such individual differences in the area of fraction learning and identified different groups of students (Grades 5-8) with the one type of knowledge, conceptual or procedural, to be more developed than expected, given the other type. Similar individual differences in fraction knowledge have been found for older students, namely 9th graders (Bempeni, Pouloupoulou, Tsiplaki, & Vamvakoussi, 2018; Lenz & Wittman, 2021), and they may be even extreme (Bempeni & Vamvakoussi, 2015).

With the aim of explaining how these individual differences regarding knowledge in the domain of fractions, or other domains, arise, several hypotheses have been tested looking at various factors such as the amount of the prior knowledge in a domain (Schneider, Rittle-Johnson, & Star, 2011); differences in cognitive profiles, measured as general conceptual and procedural ability (Gilmore & Bryant, 2006; Hallett et al., 2012) or general cognitive abilities (Lenz & Wittman, 2021); and differences in educational experiences, measured as attendance in different schools or as school grade (Canobi, 2004; Hallett et al., 2012). No or limited support for these hypotheses has been found.

We have formulated the hypothesis that a possible source of individual differences in conceptual and procedural fraction knowledge is the individual's approach to mathematics learning. In the literature there is an overarching distinction between the deep approach to learning, associated with the individual's intention to understand; and the surface approach, associated with the individual's intention to reproduce. There are several ways of characterizing each approach, mainly adapted to tertiary education (Entwistle & McCune, 2004). In a qualitative study (Bempeni & Vamvakoussi, 2015) we adopted a model developed by Stathopoulou and Vosniadou (2007) and tested with secondary students. This model differentiates between the deep and the surface approach to learning along three axes, namely goals (personal making of meaning vs. performance goals); study strategies (e.g., searching for connections vs rote learning); and awareness of understanding (high vs. low). We interviewed in depth three 9th graders (A, B, C) who differed with respect to their fraction knowledge: A had strong conceptual as well as procedural knowledge; B had strong conceptual, but extremely weak procedural knowledge; and C had strong procedural, but extremely weak conceptual knowledge. We found indicators of the deep approach to mathematics learning for A and B, and indicators of the surface approach for C. We also traced differences among the students with respect to particular aspects of motivation

(e.g., enjoying vs. avoiding intellectual challenges in mathematics). In a second qualitative study, we further investigated the features of the deep approach to mathematics learning by studying exceptionally competent students in mathematics (Bempeni, Kaldrimidou, & Vamvakoussi, 2016).

These two qualitative studies, informant the development of an instrument assessing secondary students' approach to mathematics learning (deep vs. surface) along four axes, namely goals, study strategies, motivation, and self-regulatory behaviors (e.g., monitoring of understanding, regulation of study habits).

In the present study, we examined the hypotheses that there are individual differences in conceptual and procedural knowledge of fractions (hereafter, CKn and PKn) that become less salient but remain present up to Grade 9; and that these differences are predicted by students' approach to mathematics learning (surface vs. deep).

METHOD

Participants

The study had two phases. The participants in the first phase were 510 students at Grades 7 and 9, of whom 463 participated also in the second phase (262 ninth graders and 201 seventh graders). The participants came from seven Greek secondary schools.

Materials

Students' CKn and PKn was measured by an instrument that has been evaluated in a previous study with respect to reliability and validity (Bempeni et al., 2018). The instrument comprised 12 procedural tasks (e.g.: fraction operations, simplification of a complex fraction) and 14 conceptual tasks such as fraction representation, comparison, estimating the outcome of fraction operations (see Bempeni et al., 2018; Vamvakoussi et al., 2019 for a more detailed description of the instrument).

The new instrument assessing student's approach to mathematics learning comprised of 28 statements and 6 scenarios in which two hypothetical students presented two different views on an issue. Half of the statements were consistent with the deep approach to learning, and the other half with the superficial approach to learning. The students were asked to express the degree of their accordance in a scale of 1-4 (1=Totally Disagree, 2=Disagree, 3=Agree, 4=Totally Agree). The neutral choice "Neither Agree or Disagree" was not included because it has been proved problematic in similar studies (e.g.: Entwistle et al., 2015). Examples of such statements were the following: *"It's a waste of time to study for something that is not required for the exams"*, *"If I do not remember the particular strategy to solve a problem, it is meaningless to try*

to solve it”, “I prefer to solve new problems, than practicing with the ones I already know how to solve”.

Procedure

The students had fifty minutes to complete the first questionnaire with the fraction tasks, which was enough for them. The questionnaire for the approach to mathematics studying and learning was administered three weeks later. No time limit was imposed, but the students needed at about half hour to complete it.

DATA ANALYSIS – RESULTS

1st Phase of the study

The data of the first phase of the study were classified using the proposed hierarchical method of cluster analysis, and taking as variables the standardized residuals in the two types of tasks (Bempeni et al., 2018; Hallett et al., 2010, 2012). By following this method, we examined the *relative* difference between the two variables. Using a series of evaluation measures in R programming language (R project for statistical computing), we determined that the optimal number of clusters was 4.

In Figure 1, we present the average performance in conceptual and procedural knowledge by cluster. In a little more detail, the first cluster (*“Stronger than expected in CKn and PKn”*, N=163, 32%, 10% 7th Grade) performed better than expected in both types of tasks. The second cluster performed better than expected in procedural tasks based on their CKn (*“Stronger than expected in PKn”*, N=207, 40.6%, 28.6% 7th Grade). The third cluster performed better than expected in conceptual tasks based on their PKn (*“Stronger than expected in CKn”*, N=75, 14.7%, 6.9% 7th Grade). Finally, the fourth cluster (*“Weaker than expected in CKn and PKn”*, N=65, 12.7%, 8.4% 7th Grade), comprised of students with low performance in both measures. It is worth noting that despite the fact that the overall score of the cluster *“Stronger than expected in PKn”* was higher than the one of the cluster *“Stronger than expected in CKn”*, the CKn score was comparatively lower. Moreover, the average performance in PKn and CKn was better at 9th grade (69.5% and 49.2% respectively) than at 7th grade (66.9% and 32.8%).

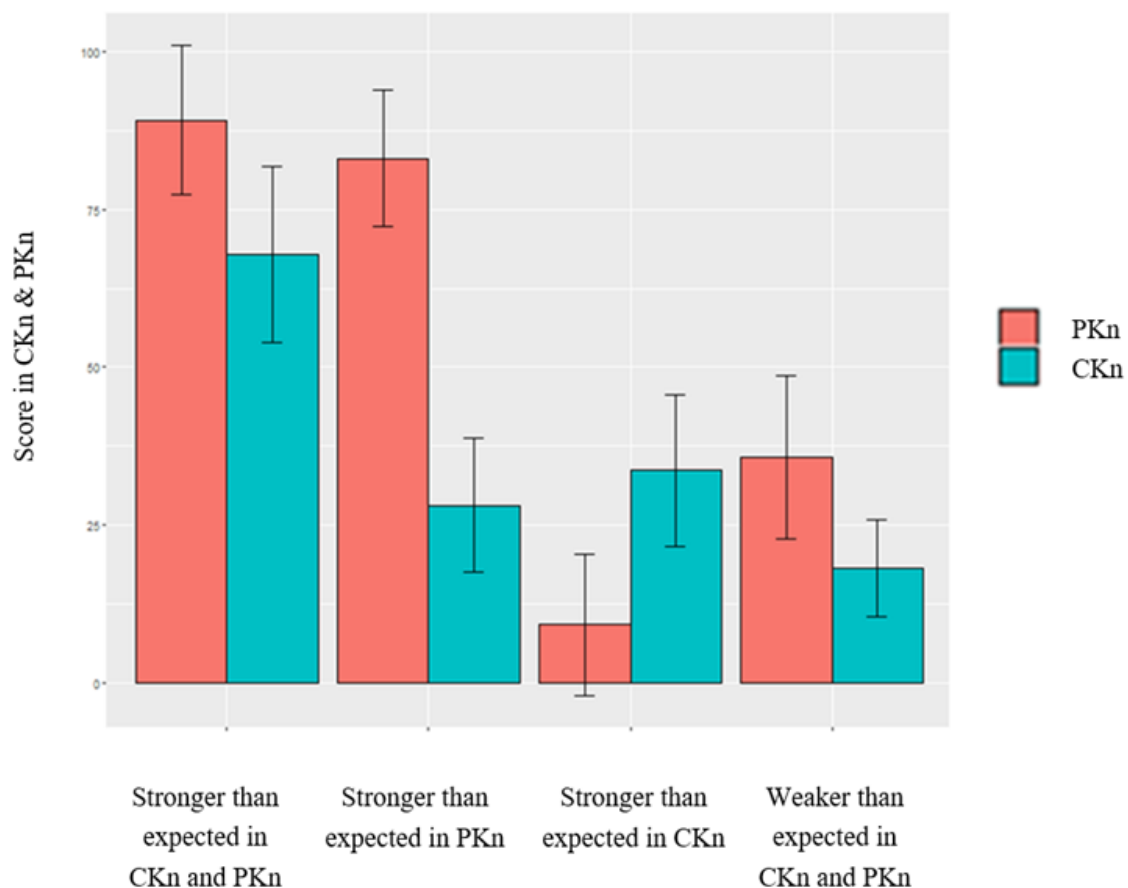


Figure 1: Average performance in CKn and PKn by cluster

2nd Phase of the study

In the second questionnaire, for the items consistent with the deep approach to learning, each choice (1-4) was taken to reflect the degree (low to high) of consistency of the response with the deep approach to learning. For the items consistent with the surface approach to learning the scores were ranked in the inverse order. The total score (hereafter, LA score) was calculated as the sum of the scores of all the items. For the analysis of the data, we used R programming language.

For the evaluation of the second questionnaire, we conducted a small pilot study. The participants of the pilot study were 120 seventh and ninth graders. In order to assess the internal consistency of the instrument, we calculated Cronbach's alpha. The value of Cronbach's alpha for two of the items had negative correlation with the scale, and as a result, these questions were excluded from our instrument. Finally, the value of Cronbach's alpha for the scale was $\alpha=0.821$. We also assessed the external consistency of the instrument over a period of 15 days with a test-retest method. Forty-one students completed the questionnaire for a second time. We calculated the value of intra-class correlation coefficient for each item separately. Five of the items displayed

intra-class correlation below 0.4 and thus we decided to exclude them from the final version of the instrument.

	Clusters	N	Mean	SD	Median	Range
1	<i>Stronger than expected in CKn and PKn</i>	158	2.987	0.414	3.037	(1.852 - 3.704)
2	<i>Stronger than expected in PKn</i>	194	2.830	0.397	2.923	(1.630 - 3.593)
3	<i>Stronger than expected in CKn</i>	52	2.636	0.275	2.633	(2.222 - 3.370)
4	<i>Weaker than expected in CKn and PKn</i>	59	2.593	0.367	2.630	(1.481 - 3.481)

Table 1: Mean LA score by cluster

The test of independence showed that there is a statistically significant correlation between cluster and approach to mathematics studying and learning ($\chi^2=60.396$, $df=3$, $p\text{-value}<0.0001$). As illustrated in the Table 1, the cluster “*Stronger than expected in CKn and PKn*” had the highest score with respect to the approach to mathematics learning, followed by the group “*Stronger than expected in PKn*”. The group “*Weaker than expected in CKn and PKn*” had the lowest score.

In order to test the hypothesis that learning approach and school grade are predictors of the level of students’ CKn and PKn, we conducted multinomial logistic regression (Table 2). The results showed that both learning approach and grade can predict cluster membership. With the cluster “*Weaker than expected in CKn and PKn*” as base level, for every unit that the individual’s LA score increases, it was 21.98 more likely for the student to belong to the cluster “*Stronger than expected in CKn and PKn*” and 4.77 more likely to belong to the cluster “*Stronger than expected in PKn*”. Using the same base level, a ninth grader is 8.35 more likely to belong to the group “*Stronger than expected in CKn and PKn*” than to the group “*Weaker than expected in CKn and PKn*”.

Predictor	<i>Weaker than expected in CKn and PKn</i>	B	OR= exp(B)	p-value
Vs.				
<i>Score in mathematics learning approach</i>	<i>Stronger than expected in CKn and PKn</i>	3.09	21.98	0.000
	<i>Stronger than expected in PKn</i>	1.56	4.77	0.000

	<i>Stronger than expected in CKn</i>	0.42	1.53	0.390
<i>9th Grade</i>	<i>Stronger than expected in CKn and PKn</i>	2.12	8.35	0.000
	<i>Stronger than expected in PKn</i>	0.18	1.19	0.606
	<i>Stronger than expected in CKn</i>	0.52	1.69	0.206

Table 2: Predictive factor testing

CONCLUSIONS – DISCUSSION

The results of our study confirm the hypothesis that there are individual differences in the way students combine CKn and PKn for fractions (Hallett et al., 2010; 2012). Although older students were more likely to have strong CKn as well as PKn, a considerable percentage of 9th graders belonged to the clusters “*Stronger than expected in PKn*” and “*Stronger than expected in CKn*”, indicating that individual differences remain present up to Grade 9. It is worth noting that the greater part of our sample was found in the group “*Stronger than expected in PKn*”, indicating that instruction favours mainly the development of PKn (see also Canobi, 2004).

In our attempt to detect the possible factors that are responsible for individual differences in CKn and PKn, we tested the hypothesis that the approach to mathematics learning predicts such individual differences. The LA score predicted the membership in the clusters “*Stronger than expected in CKn and PKn*” and “*Stronger than expected in PKn*”. This result only partially supports our hypothesis, due to the fact that the probability for a student to belong to the cluster “*Stronger than expected in CKn*” cannot be predicted; moreover, the mean LA score for this cluster was the second lowest one, lower than the mean LA score of the “*Stronger than expected in PKn*” cluster. A possible explanation is, that as a result of using residualized scores in the cluster analysis (Hallett et al., 2010, 2012; Bempeni et al., 2018), the “*Stronger than expected in CKn*” cluster includes students with *relatively* stronger CKn given their PKn, but not necessarily in absolute terms; and similarly, for students in the “*Stronger than expected in PKn*” cluster. A different method for clustering the students, differentiating between the low from the high performing students could be a viable solution (see Lenz & Wittman, 2021, for such a method).

Whilst the development of the two types of knowledge is not assumed to be symmetrical at any given moment (Rittle-Johnson & Schneider, 2015), our results put a challenge to the iterative model. More specifically, given the age and educational experience of the participants, we would expect a more

balanced development of the two types of knowledge which is not the case in our study.

The learning approach to mathematics deserves to be further investigated as a source of individual differences in CKn and PKn. The instrument that we developed is a contribution of some significance per se, since, to the best of our knowledge, there is no similar instrument targeting secondary students. An enrichment and refinement of our instrument, in view of the fact that several items had to be excluded from its final version following its evaluation, is worth-considering.

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GROWTH IN GEOMETRIC JOINT ROUTINES DURING MIDDLE-SCHOOL PEER INTERACTION

Naama Ben-Dor¹ and Einat Heyd-Metzuyanim¹

¹Technion – Israel Institute of Technology, Israel

In this paper, we draw on the commognitive framework to explore types of mathematical growth during middle-school geometry peer interaction. Comparing students' routines when working apart with their joint routines when working together, we identified four types of mathematical growth. Three types were object-level growth: applicability, refinement, and flexibility. One type was a meta-level growth consisting of a shift from a configural/visual procedure to a deductive one. Our study pinpoints the types of mathematical learning that can be achieved during peer interaction and shows the ways in which they can occur. Specifically, the study shows how different types of growth can be achieved by students building on their partner's procedure in different ways.

RATIONALE

Learning through peer interaction has come to be highly regarded not only as an important 21st century skill, but also as a means to improve learning (Kuhn 2015). Studies have shown that under certain interactional conditions, such as readiness of peers to listen to each other, problem-solving in pairs or small groups can be more conducive to students' learning than solving a problem alone (e.g., Schwarz and Linchevski 2007). Other studies have examined the types of learning that can occur in peer interactions. Phelps and Damon (1989), for example, have found that peer interactions are more effective for conceptual learning and reasoning than for rote kinds of learning. Pai and colleagues (2015) showed, through the examination of pre/post-tests, that peer interaction is conducive to learners' ability to apply or adapt prior knowledge to a novel situation. Although we learn from these studies about learning in peer interaction, we still know very little about the processes of mathematical learning that take place in these interactions and about *how* these different types of learning occur. In this study, our goal is to better understand how peer interaction promotes different types of growth in students' mathematical procedures used to solve a certain problem.

THEORETICAL FRAMEWROK

The theoretical framework which we use to pursue our goal is commognition (Sfard 2008). Commognition is a sociocultural discursive framework which has been productive in studying processes of peer interactions (Chan and Sfard

2020; Sfard and Kieran 2001) as well as processes of mathematical learning (Lavie and Sfard 2019; e.g. Lavie, Steiner, and Sfard 2019). The commognitive framework conceptualizes learning as a process of routinization of students' actions (Lavie et al. 2019). Routines - repetitive patterns of actions – are thus the commognitive basic unit for analyzing learning. A routine is a task-procedure pair; it is defined as “the task, as seen by the performer, together with the procedure she executed to perform the task” (Lavie et al. 2019:161).

By studying mathematical routines, commognitive studies have been able to track learning over time and identify different types of growth in learners' routines (Lavie and Sfard 2019; Lavie et al. 2019). Flexibility is one such type of growth. A routine grows in its flexibility when another procedure is used in response to the same task. For example, Lavie and Sfard (2019) showed a growth in a young child's routine for the task "where is there more?" when in addition to the initial procedure of visually estimating two piles of cubes, the child used another procedure of aligning these cubes. The child's routine thus grew in flexibility to offer two alternative procedures for accomplishing the task. Applicability is another type of growth. Growth in applicability is detected when after applying a certain procedure to a certain task, a learner applies the same procedure to a new unfamiliar task.

Much of the growth in children's mathematical routines happens at the object level. As they become familiar with certain procedures (e.g., adding, dividing) and certain objects (e.g., natural numbers), learners gradually apply the familiar procedures to different tasks, producing an increasing number of narratives about these objects. This growth constitutes object-level learning. Yet from time to time, as students gradually get introduced to more sophisticated mathematical discourses, a meta-level change is needed (Sfard 2007). Such a meta-level change can happen when rules for substantiating mathematical narratives change, or when new objects are introduced. For example, when students get introduced to rational numbers, the familiar arithmetic rules that had so far been successfully applied to natural numbers no longer apply.

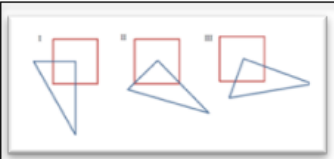
In this study, we wish to examine processes of peer interaction in junctures that afford object-level as well as meta-level learning. We pursue this goal by focusing on middle-school geometry, since a particularly critical transition is required from students in those years – the meta-level shift to deductive geometric procedures (Duval 1998). In this transition, students who are used to performing visual-configural procedures for substantiating claims about geometric objects (such as showing congruence by placing one triangle on top of the other) are required to shift to using new deductive procedures based on given data and geometric theorems (such as congruence theorems).

For examining mathematical learning in peer interaction, we add to our commognitive conceptual toolset the concept of a *joint routine* which we define

as the collection of procedures used by a group (or pair) of people working together on the same task. Based on this theoretical framework, we ask: in what ways did students' geometric joint routines grow during middle-school geometric peer interaction?

METHODOLOGY

The participants of our study were 10 middle-school students, six 8th graders (13-year-old) and four 9th graders (14-year-old), who took part in a one-hour geometric activity facilitated by the first author. The design of the activity was based on videotaped lessons of the VIDEO-LM project (Karsenty and Arcavi 2017) in which a geometric problem called *The three squares* was presented. The students in these lessons were asked to compare areas in three drawings. The canonical (correct) answer is that all areas are equal. Our design included: (1) a presentation of the geometric problem; (2) an individual session in which students worked on a worksheet (see Figure 1); (3) a dyadic session in which they worked on the same worksheet. Colored, half-transparent plastic shapes of a square and a triangle were given to the students as supporting tangible mediators.



1. In each drawing, colour the shared area of the square and triangle
2. Line up the drawings according to the size of their shared area (from the smallest to the largest)
3. Circle the correct answer in each of the following claims and explain it:
 - a. The shared area in drawing I is (smaller than /larger than/equal to/impossible to know) the shared area in drawing II
 - b. The shared area in drawing II is (smaller than /larger than/equal to/impossible to know) the shared area in drawing III
 - c. The shared area in drawing III is (smaller than /larger than/equal to/impossible to know) the shared area in drawing I

Figure 1: The worksheet

Data collected included students' 10 individual worksheets and 5 dyadic worksheets as well as footage from different cameras of both individual and dyadic sessions. Individual and dyadic sessions were fully transcribed (including non-verbal communication) and analyzed using footage from different cameras. Overall, 1530 transcription lines of verbal and non-verbal communication were analyzed.

Data analysis included the following steps: (a) analyzing students' visual mediators by adding to each line in the transcript a graphic representation of what they did, looked at and pointed to in the worksheet; (b) identifying students' procedures for the task of comparing areas when working alone, by examining students' written answers in individual worksheet as well as the

footage from their individual session and their communication at the beginning of dyadic session (c) tracking developments in dyads' joint routine for the task of comparing areas when working together, by analyzing their communication during dyadic session as well as their dyadic worksheet; (d) deductively and inductively identifying types of joint routine growth.

FINDINGS

Individually, the ten students used four main procedures for the task of comparing areas. These were: (1) the "Supplement procedure" – cutting and "moving" parts in order to supplement similar-looking shared areas; (2) the "Ratio procedure" –visually estimating the ratio of the shared area from the whole square; (3) the "Formula procedure" – visually estimating the relation between heights and bases of the shared areas and then applying to it an area formula (such as $\text{base} \times \text{height} / 2$); and (4) the "Given procedure" – examining the givens (or lack thereof) to assess if enough information is provided.

During the start of the dyadic session, the students within each dyad (dyad 1 to 5) compared their solutions with the solutions of their dyadic partners and tried to reach an agreement. Some of them used different procedures in their individual routines for comparing the shared areas.

Examining students' joint routines during dyadic session, we found four ways in which growth in these routines occurred. Three of these ways were object-level. In other words, the growth did not include a change in meta-rules. These categories of growth were: (1) applicability; (2) refinement; and (3) flexibility. Two of these growth patterns – applicability and flexibility – have been known from previous studies (Lavie and Sfard 2019; Lavie et al. 2019). Refinement is a new bottom-up category that we used to describe growth which included the refinement of specific steps in a procedure previously used by one of the students. The fourth type of growth was a meta-level shift to deductive procedures. Table 1 presents these types of growth, their description, and examples.

#	Type	Description	Example
1	Applicability (Object-level growth)	Extending application of an initial procedure to another task	The Supplement procedure, initially applied by one of the students only in relation to the comparison between shared areas I and II, was applied in dyadic sessions also to the comparison between shared areas I and III.
2	Refinement (Object-level growth)	Refining steps of an initial procedure	In the Ratio procedure, the step of visually estimating the ratio of the shared area from the whole square was refined into two separate steps: (a)

		visually estimating how many times the shared areas can fit into the square; (b) deducing the ratio of the shared area from the whole square.
3 Flexibility (Object-level growth)	Forming a new procedure based on an initial procedure (same meta-rules)	A new rotational procedure was formed based on the Ratio procedure. Both procedures, the original and the newly developed, relied on the same meta-rule of visual estimation.
4 Meta-level growth	Forming a new procedure based on an initial procedure (more developed meta-rules)	A new deductive congruence procedure was formed based on the Supplement procedure. The newly formed procedure relied on a more developed meta-rule (visual estimation is insufficient; justifications should be based on theorems and givens).

Table 1: Types of growth in joint routines during dyadic session

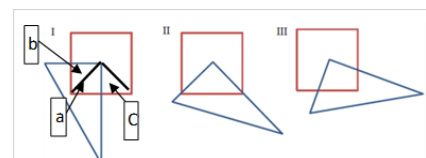
In what follows, we illustrate two of these types of growth – applicability and meta-level. We do so by focusing on the development of the most commonly used procedure – the Supplement procedure – through the case of dyad 1 (8th graders Noa and Eyal) and dyad 4 (9th graders Tamara and Orna).

Example of growth in applicability during dyad 1's session

Analyzing Noa and Eyal's initial processes in individual session, we found that Eyal only used the Ratio procedure, while Noa only used the Supplement procedure. Noa's use of the Supplement procedure was limited to the comparison between shared area I and II. Although they used different procedures to compare between shared areas I and II, they endorsed the same narrative, namely, that shared area I and shared area II are equal. Here is how Noa explained her procedure to Eyal at the start of their dyadic session:

Legend: (implied words); [parallel speech]; right column: representations of visual mediators.

- 36 Noa Look, these (shared areas I and II) are definitely equal 'cause... 'cause if you cut this, say, in half... here (draws line a), so what we have here (points to triangle b) you can move here (c), so we get a triangle (like shared area II) (in the picture to the right, Noa uses the plastic shapes to demonstrate more tangibly her procedure)



37 Eyal How did you think about that??



From her written answer in her individual worksheet as well as from her explanation in this excerpt, we deduced that Noa's procedure for comparing shared areas I and II included: (1) identifying the geometric shapes of the shared areas. This is evident in her reference to shared area I as "square" and to shared area II as "triangle"; (2) cutting the shape (line (a) cuts the square) of one area (area I) into sub-shapes (two triangles); (3) moving a sub-shape (triangle b) to another place (c) in the same drawing (I) so that it *supplements* a shape (triangle) similar to the other area (area II); (4) determining the relation between the shared areas (I and II) according to a visual comparison between the newly formed area (formed triangle in drawing I) and the other area (triangle in drawing II). The same procedure, with slight variations, recurred several times in students' answers, and was named the Supplement procedure.

Eyal's reaction to Noa's Supplement procedure, communicates not only that the procedure was new to him, but that he was surprised by and appreciated Noa's "thinking" (37). Following his reaction Noa suggested that they write her explanation in their shared worksheet. Eyal then said:

47 Eyal Yes, wait a second, you can cut also here (a), see? From here (a) and then put it here (b), We get this (c)

48 Noa Why?

49 Eyal To cut this, you can...

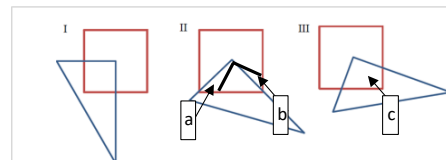
50 Noa [No but listen]

51 Eyal [take here this] small piece (a)

52 Noa [Aha]

53 Eyal [and then you] put it (a) here (b)

54 Noa But it's not enough for... [ahh right, o.k., you're right]



Here, Eyal applied Noa's Supplement procedure to the task of comparing between shared areas II and III. He suggested cutting the shape of one area into sub-shapes and moving a sub-shape to another place so that it supplements a similar looking shape. Therefore, Eyal did not only adopt Noa's Supplement procedure (starting his suggestion with "yes"), but also built upon it to suggest a new application (comparison between II and II) to the same procedure, an application which was not previously used by Noa. His words in line 47 communicate that he found ("wait a second") a new way of applying the same procedure ("also", "see?"). Therefore, Noa and Eyal's joint routine for comparing areas grew in applicability: from only applying the Supplement

procedure to the task of comparing areas I and II at the start of dyadic session (Noa's individual routine) to applying it also to the task of comparing between areas II and III (Noa and Eyal's joint routine).

Example of meta-level growth during dyad 4's session

Meta-level growth was found only in the interaction of Tamara and Orna, a pair of relatively high achieving 9th graders. This did not come as a complete surprise since only two interactions (Orna and Tamara's and one more dyadic interaction) were of 9th grade dyads; the other three were of 8th grade dyads, who were at the very initial stages of exposure to deductive geometric procedures. Both Tamara and Orna started out with configurally-based procedures. Orna used the Ratio procedure, while Tamara used the Supplement procedure. Toward the end of the pair's session, a meta-level growth in the dyad joint routine occurred when the girls discussed why shared area I and III are equal. The following exchange begins with Tamara suggesting using the Supplement procedure for comparing areas I and III:

- 132 Tamara ... you need to say that, like you move this part (a) to here (b) and then like [it will form a square]
- 133 Orna [I have an idea], if we, like, show congruence (between) this (a) and that (b), then... (given the context of previous utterances we interpret this as meaning: by showing that these triangles are congruent, we can show that their areas are "the same")
- ...
- 141 Orna No, No look, you need to say that this (a) is like (meaning congruent to) this (b) in order for it to be ok to move the...



In line 132 Tamara suggested her Supplement procedure: to move part (a) so that it covers part (b) and forms a square similar to the shared area in drawing I. In response to Tamara's suggestion, Orna proposed that they use congruence theorems to substantiate that the areas of the triangles (a and b) are the same ("I have an idea, if we, like, show congruence" [133]). In line 141, Orna further explained that in order to claim that triangle (a) can be moved on top of the triangle (b) in a way that exactly covers it, they need to show that they are congruent ("you need to say... in order for..."). In other words, she did not agree ("no, no...") with the meta-rule of the Supplement procedure that visual estimation is enough. Rather, she drew on the Supplement procedure to suggest a new deductive congruence procedure. The newly formed procedure relied on a more developed meta-rule (visual estimation is insufficient; justifications should be based on givens and theorems). By that, Tamara and Orna's joint routine for comparing areas underwent a meta-level shift.

DISCUSSION

Our goal in this study was to explore types of mathematical growth in peer interaction. Specifically, we examined developments in students' joint routines around a geometric problem that invited movement from purely configural/visual procedures to deductive ones. We found four ways in which students' routines grew during interaction. Three of these were object-level learning – applicability, refinement, and flexibility – while the fourth was a meta-level learning that included a shift from configural to deductive meta-rules. Our study contributes to commognitive research by extending the application of the study of routine growth (Lavie et al. 2019) from individuals' learning to peer learning. In addition, it adds on previous research on peer learning (Kuhn 2015) by pinpointing the types of mathematical learning that can be achieved during peer interaction, and showing the ways in which they can occur. Specifically, the study shows how different types of growth can be achieved in routines by students building on their partner's procedure in different ways.

The conclusions from this study are limited by the relatively small scope of cases, a regular limitation in studies that take such a micro-analytical look at students' discourse. Thus, future studies are needed to determine the relative frequencies of different types of joint routine growth in peer interaction. In addition, it is yet to be examined how much of what is developed jointly during students' interaction is later individualized by the participating students. Nevertheless, we believe that through our detailed theoretically anchored report, we are making progress in understanding the precise mechanisms of mathematical learning during peer interaction. A better understanding of these mechanisms of peer learning can aid educators in preparing, designing, and facilitating collaborative activities in the mathematics classroom.

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TEACHERS' INTERPRETATIONS OF THE CONCEPT OF PROBLEM—A LINK BETWEEN WRITTEN AND INTENDED REFORM CURRICULUM

Ewa Bergqvist and Tomas Bergqvist

Umeå Mathematics Education Research Centre, Umeå University, Sweden.

Over the last decades, there has been an on-going international reform for school mathematics, which has, not surprisingly, been difficult to implement. This study focuses on teachers' interpretation of formal written curriculum documents, especially whether their interpretations align with how a concept (the concept of problem) is conveyed in the documents (in Sweden). The results show that the formal written documents are vague, but that it to some extent conveys the concept of problem as "a task for which the solution method is not known in advance to the solver." The interviews show that about 53 % of the teachers interpreted problem as "any task," and that teachers' interpretations therefore are not aligned with how the concept is (albeit vaguely) conveyed in the documents.

INTRODUCTION

During the last 25 years, the descriptions of school mathematics have gradually changed all over the world. The main message of this reform is to complement content goals (such as algebra) with competency goals (such as problem solving) and this idea can be found in many international reform frameworks (e.g., NCTM, 2000; Niss & Jensen, 2002). In many countries the formal written (national) curriculum documents now use these kinds of competency goals to formulate goals for student learning in mathematics (e.g., in Singapore, SME, 2012). Many researchers argue that in the heart of doing mathematics you find problem solving (e.g., Schoenfeld, 1992) and problem solving is sometimes considered as the most important part of the reform. There is a lot of research on the implementation of educational reforms, for example, in Norway (e.g., Gundem, Karseth, & Sivesind, 2003), and in North America (e.g., Fullan, 2001). One main result is that educational reforms most often do not give the desired effect in schools (Hopmann, 2003) even when the teachers themselves believe that their teaching reflects the new ideas (e.g., Stein, Remillard, & Smith, 2007, p. 344). It is therefore important to understand how the different parts of the curriculum chain are connected. The purpose of this study is to deepen the understanding of the connection between written and intended curriculum in mathematics. The study will compare how a central standards-based reform concept is conveyed in the Swedish formal written curriculum (the

policy documents) with how it is interpreted by Swedish teachers', that is, the intended curriculum. In particular, we focus on the concept of *problem* and on Sweden, as one of the countries that has been part of the standards-based reform.

CURRICULUM CHANGE

The word curriculum has many different meanings in research. In this article we use a framework suggested by Stein et al. (2007), including the *written* (the printed page), the *intended* (as planned by the teachers), and the *enacted* (actual implementation in the classroom) curriculum. Research has shown many possible reasons that a reform does not result in change in teacher practice, that is, that change in the written curriculum does not result in change in the enacted. One possible reason is that the reform message is not clearly conveyed to the teachers (Fullan, 2001). Another is that the teachers are not supported enough to carry out the change (Fullan, 2001). Different parts of the chain between written curriculum and student learning have been studied extensively (see e.g., Stein et al., 2007), but in comparison there is not much research on teachers' interpretation of the formal written curriculum.

DEFINITIONS OF PROBLEM AND PROBLEM SOLVING

Problem solving has had an important role in many areas of research, for example, in cognitive psychology as the "paradigm for the higher cognitive processes" (Kintsch, 1998, p. 2). There are, however, many possible different definitions of problem and problem solving, and this has often been discussed (see e.g., Schoenfeld, 1992; Xenofontos & Andrews, 2014). In the words of Stigler and Hiebert (2004), "the word 'problem' clearly means different things to different people" (p. 13).

A traditional definition of the concept of problem is that it is *any task* including both routine and non-routine tasks (Schoenfeld, 1992, p. 337). This definition is in line with definitions presented in both English and Swedish dictionaries. Within mathematics education research, this traditional definition is often questioned: "In education it is important to distinguish a problem from a simple question to which the answer is known without any need for reflection" (Borba, 1990, p. 39).

Another definition that is more common today is to see a mathematical problem as *a task for which the solution method is not known in advance for the solver* (see e.g., Blum & Niss, 1991). In addition, this is a common definition in standards-based reform, which is central to this study (e.g., NCTM, 2000). Lester (2013) summarizes that although there have been many different research areas that have focused on problem solving, in general, "they all agree that a problem is a task for which an individual does not know (immediately) what to do to get an answer" (p. 247).

Another suggested definition of problem is *word task*, that is, a task with verbal text describing a situation or a context (see e.g., Borasi, 1986). A *real-world task*, that is a task with a real-world context or an applied task (see e.g., Chen, 1996) is also a suggested definition. In conclusion, even though most researchers presently define *problem* in line with *a task for which the solution method is not known in advance for the solver* there are many different definitions of and opinions regarding what a problem is.

TEACHERS' INTERPRETATIONS OF THE CONCEPT OF PROBLEM

That many mathematics education researchers use the same definition of what a problem is, does not necessarily imply that teachers would agree. Few studies focus on how teachers actually define what a problem or what problem solving is (Xenofontos & Andrews, 2014). Grouws, Good, and Dougherty (1990) interviewed 24 teachers and summarized their conceptions of problem solving into four categories: solving *word problems* (6 teachers), solving *real-world problems* (3 teachers), solving *problems* (10 teachers) and solving *thinking problems* (6 teachers). The third category is described as following a “step-by-step adherence to predetermined guidelines” and “involved computations or setting up equations” (p. 137), which we interpret as including *any task* and, perhaps in particular, routine tasks. Another study examined a representative random sample of 63 Finnish third grade elementary teachers' conceptions about mathematical problem and problem solving (Näveri, Pehkonen, Hannula, Laine, & Heinilä, 2011). On the multiple-choice question, “What is a problem?” most of the teachers (70 %) answered that it primarily is a word task. For a smaller group of teachers (24 %) “problem is a task for which the solution is not known” (p. 5). In conclusion, teachers' definitions of the concept of problem varies, and also vary between cultures, but are generally not in line with the most common definition within mathematics education research.

PURPOSE AND RESEARCH QUESTIONS

The purpose of this study is to deepen the understanding of the connection between written and intended curriculum in mathematics. The study will therefore compare how the concept of *problem* is conveyed in the Swedish formal written curriculum (the policy documents) with how it is interpreted by Swedish teachers. The research questions are:

1. What meaning of the concept of problem is conveyed in the Swedish formal written curriculum in mathematics?
2. How do Swedish mathematics teachers interpret the concept of problem when it is used in the formal written curriculum in mathematics?

METHOD

The method consists of an analysis of the written Swedish formal written curriculum, in relation to research question 1, and another analysis of teachers'

interpretations of curriculum documents, in relation to research question 2, as described below.

Categories for Analysis

The analyses use four categories of possible definitions of the concept of *problem*, chosen since they represent the four most common definitions within mathematics education research, as presented in the Background. The categories are:

1. *any task* (including routine tasks)
2. *tasks for which the solution method is not known in advance to the solver* (i.e., *non-routine tasks*)
3. *real-world tasks*, that is, tasks set in a context or applied tasks
4. *word tasks*, that is, tasks with verbal text describing a situation or a context

All these definitions make sense in a mathematics. However, note that the categories are not disjoint, since categories 2-4 are subsets of category 1.

Data Collection and Analysis of the Formal Written Curriculum

To answer the first research question, the Swedish formal written curriculum for mathematics in primary and lower secondary school and for upper secondary school valid at the time of the interviews (Utbildningsdepartementet, 1994) are examined. For upper secondary school, we analyze one text describing mathematics in general, common to all courses, and the text describing course A, since it is the only compulsory course for all students. We also include the official Commentary documents written by experts engaged in the writing of the formal written curriculum for mathematics for primary and lower secondary school (Emanuelsson & Johansson, 1997). There were no other official documents explicitly concerning mathematics valid at this time.

The formal written curriculum is searched for all instances where the word *problem* is used. The search includes the word *problem*, as well as any compound word including the word *problem*, such as *problem solving* (Sw. *problemlösning*). All instances are then analyzed in two steps. First, and most importantly, by examining each instance in search for definitions, explanations, and examples. Second, by examining whether the wording in the instances are in line with one or more of the definitions of problem (1-4) or if any instance has a wording that conflicts with any of these.

Data Collection and Analysis of Teachers' Interpretations

This part of the data collection was carried out within a larger project (see Boesen et al., 2014) in which almost 200 teachers were observed and interviewed. The selection of schools was “based on stratified random sampling

and was carried out by the Swedish Schools Inspectorate” (Boesen et al., 2014, p. 77). The data in this particular study consists of answers to one specific interview question from 126 upper secondary mathematics teachers and 61 primary and lower secondary school teachers, in total 187 teachers. During the interviews the teachers were presented quotes from the formal written curriculum and one quote included the word “problem”. The quote presented to the upper secondary school teachers was: “Pupils use appropriate mathematical concepts, methods, models and procedures to formulate and solve different types of problems”. The quote shown to the primary and lower secondary school teachers was similar. The teachers were then asked: “How do you interpret the word *problem*?”

The analysis was carried out in three steps. First, the researchers separately analyzed the answers from the upper secondary school teachers (126 answers) using the categories presented above. The researchers made the same categorization for 103 of these, which indicates a reasonable inter-rater reliability. Second, the researchers discussed the 23 answers for which they did not initially agree, which resulted in more detailed instructions regarding how to interpret the categories. Third, the remaining 61 answers) were analyzed by the second researcher.

RESULTS

The Concept of Problem in the Written Curriculum

The first research question is: What meaning of the concept of problem is conveyed in the Swedish formal written curriculum in mathematics? In the documents for primary and lower secondary school) the word *problem* is used 21 times as it is or in compound words. In the documents for upper secondary school, it is used 25 times.

First, and most importantly, examining the 46 instances, our main result is that there is no definitions, explanations, or examples of what a problem or problem solving is.

Second, that 37 of the 46 instances are compatible with all the definitions used in the analysis (1-4). Typical examples are instances saying that a problem can be *solved*, *understood*, *developed*, *formulated*, and that *different methods can be used to solve problems*, and all these are reasonable regardless of definition used. The other nine instances have wordings that are to some extent in conflict with one or more of the definitions. For example, the wording “mathematical problem solving is a creative activity” is in conflict with the definitions that include routine tasks. In summary, the concepts are undefined and used in a vague or even contradictory way. This is also the case for most other concepts in the Swedish formal written curriculum (Bergqvist & Bergqvist, 2017).

In the Commentary, the development of problem solving is described as a central purpose of all mathematics education (Emanuelsson & Johansson, 1997). The word *problem* is not explicitly defined but is used under the headline *Problem solving*: “Sometimes it is not even a genuine problem since the needed calculation method is given through the context or the chapter heading...” (Authors’ own translation. Emanuelsson & Johansson, 1997, p. 18). For a *genuine problem* “the needed calculation method” is **not** “given through the context or the chapter heading”, which indicates that a “genuine problem” is of type 2, *tasks for which the solution method is not known in advance to the solver*. Our conclusion is that in the Commentary a problem is conveyed as category 2, but that the wording is vague.

The answer to research question one is that the conveyed meaning of the concept of problem in these documents is unclear. The concept is not defined, explained, or exemplified in any text, but it is to some extent conveyed as being of type 2, *tasks for which the solution method is not known in advance to the solver* (or *non-routine tasks*).

Teachers’ Interpretations of the Concept of Problem

We present 187 teachers’ interpretation of the word problem in the written curriculum. Four categories (1-4) of possible interpretations were predefined and 151 of the 187 teachers gave answers that could be placed within these categories (see Table 1).

Interpretation of <i>problem</i>	Primary and lower secondary teachers (61)	Upper secondary teachers (126)	All teachers (187)
1. Any task	49% (30)	55% (69)	53% (99)
2. Task for which the solution method is not known in advance	10% (6)	15% (19)	13% (25)
3. Real-world task	3% (2)	8% (10)	6% (12)
4. Word task	10% (6)	7% (9)	8% (15)
5. Other	28% (17)	15% (19)	19% (36)

Table 1: Percentage (number) of teachers making interpretations of the concept of problem in line with each of the predefined categories.

The most common answer was that a problem is *any task* (99 teachers). This was expressed in a few different ways, but the most common answer (given by 61 teachers) was “uppgift”, which is Swedish for “task.” Other answers categorized as *any task* were “something to be solved” and “everything is a problem.” In category 2, 18 of the 25 teachers used expressions close to the

definition in this study, like “unfamiliar tasks”, “when you don’t know how to solve it,” and “when you can’t see the answer.” The remaining 7 used expressions that were not as close to the definition, for example, “many solutions”, but we chose to include them to avoid underestimating the category that is most common among researchers. Twelve teachers used expressions that were categorized as *real-world tasks*. In this category, statements like “applications”, and “real life tasks” were placed. Fifteen teachers said that a problem is a *word task*. They all used either the expression “text task” (Sw. *textuppgift*) or the expression “reading task” (Sw. *lästal* or *läsuppgift*). The expressions put in category 5, *other*, were of different types, for example, “problems are mathematical problems”, and “it can be on different levels, different for different students.” In general, these answers were hard to interpret. Three teachers in this group answered: “I don’t know what a problem is.”

The answer to research question two is that there is a large variation in how Swedish mathematics teachers interpret the concept of problem, but that more than half of the teachers interpret it as *any task*.

DISCUSSION

The purpose of this study is to deepen the understanding of the connection between written and intended curriculum in mathematics, and the study has a particular focus on the concept of problem. The results show that the formal written documents and the Commentary are vague, but that they to some extent convey that a problem is a *task for which the solution method is not known in advance to the solver*. The interviews show that about 53% of the teachers interpreted *problem* as *any task*, and that the rest of the teachers interpreted it in many different ways. The teachers’ interpretations are therefore not aligned with how the concept is (vaguely) conveyed in the documents.

In the formal written curriculum, *problem* is a very central concept, and it is implied that a significant part of the students’ work in mathematics should be devoted to solving problems. Different interpretations of the word *problem* could therefore lead to very different teaching practices. One example is that Swedish students spend a large part of their time (two thirds of the lessons) during mathematics classes working with the textbook (Boesen et al., 2014). Interpreting *problem* as *any task* means that the students already spend two-thirds of their time on problem solving. A teacher interpreting *problem* as *a task for which the solution method is not known in advance to the solver*, would have to examine the textbook tasks and probably add different kinds of tasks from other sources in order to ensure that their classroom practice meets the goals of the written curriculum. In this case, different interpretations of the written curriculum would result in large variation regarding both the intended and the enacted curriculum. Under these circumstances, the formal written curriculum cannot be said to clearly guide the teachers’ practice, a situation in line with

previous research (e.g., Hill, 2001). In this study we asked teacher to explain what a *problem* is, but not what *problem solving* is. Initially it was assumed that problem solving would be considered to be the same thing as solving problems. However, three teachers suggested that problems to be solved during problem solving are of a different kind than problems in general.

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MATERIAL ABILS AN IMPULSE FOR MATHEMATICAL ACTIVITIES IN PRIMARY SCHOOL — A SEMIOTIC PERSPECTIVE ON A GEOMETRIC EXAMPLE

Lara Billion and Rose Vogel

Goethe University of Frankfurt, Germany

In this paper, a semiotic perspective on mathematics learning is taken, focusing on diagrammatic work and thus on diagrammaticity. With this theoretical approach, action on diagrams, which include designing, manipulation and experimenting with diagrams on paper, on the computer screen or with physical material, are examined in more detail. It is assumed that the actions on diagrams show the mathematical interpretation of actors, which can be used to draw conclusions about their mathematical awareness. With the help of Vogel's (2017) adaptation of the context analysis according to Mayring (2014), mathematical interpretation processes from young learners are reconstructed using a geometric example of actions on physical material.

INTRODUCTION

The semiotic perspective on mathematical learning makes it possible to focus more strongly on materialised actions and to use these as a starting point for the identification of the learners' mathematical interpretation processes and thus to make them accessible for research in mathematics education. The material arrangement — often initiated by the formulated work task and the materialisations given therein (on paper, on the screen or in form of physical material) — is interpreted as a mathematical diagram and represents the beginning of diagrammatic work. In diagrammatic work, diagrams are interpreted mathematically, and rule-guided actions are performed in the diagrams. The implicit or explicit interpretation of diagram rules depends, among other things, on the selected material and problem arrangement. Which possibilities of reconstruction open up for research into mathematical interpretation processes and thus mathematical learning through this semiotic perspective will be presented in the following using a geometric example from primary school.

THEORETICAL FRAMEWORK

Mathematical Learning from Semiotic Perspective

From a semiotic point of view, learning of mathematics is seen as a perceptible action. These actions include dealing with diagrams, manipulating and

experimenting with diagrams as well as inventing new diagrams (Dörfler, 2006) “A Diagram is a representamen which is predominantly an icon of relations and is aided to be so by conventions.” (CP 4.418). The relations and conventions of a diagram become clear through the interplay of different inscriptions. Inscriptions can be signs on paper, illustrations on screen or consist of tactile material (Gravemeijer, 2002). In order to do mathematics with diagrams, the implicit and partly conventionalised rules in which the relation of the diagram are expressed must be interpreted by the learners. Only by this interpretation rules become usable for manipulations and the learners can experiment with the diagram. Through a rule-guided transforming of the diagram, a learning space is opened up for the learners, in which the learners can apply existing mathematical knowledge but also can gain new mathematical knowledge, and thus mathematical learning takes place. These insights include, for example, the determination of characteristics, the discovery of previously unknown relationships or the calculation of a result. Learning mathematics can thus be seen as interpreting and acting with diagrams (Dörfler, 2006).

Semiotic Perspective on Actions on the Material

The semiotic perspective on mathematical learning, especially the diagrammaticity described above (Dörfler, 2006), is another way to grasp actions on materials theoretically. The widespread view of material in mathematics teaching is that it is used to construct mental images (Lorenz, 1993; Dörfler, 1991). The material is usually assigned the function of representation. These representations of mathematical objects, which are concretely available through the materialisation, can be used for actions. Materialisations from a semiotic perspective do not stand for a mathematical object but allow to make mathematical experiences through manipulations and their interpretations. “[...] the number line does not represent \mathbb{Z} in an objective manner. However, the number line can be used to think ‘about’ whole numbers and their operations and relations.” (Dörfler, 2000, p. 103)

Therefore, in this paper actions are to be understood as what learners do in order to design diagrams (on paper, on the screen or in form of physical material), to manipulate them according to certain rules (also conventionally shaped) and to experiment with them. The central assumption for this paper is that actions on diagrams show the mathematical interpretation of the actors, from which their mathematical knowledge and mathematical cognitive processes can be deduced (Dörfler, 2000). Thus, the actions are the starting point for the reconstruction of the mathematical interpretations of the diagrams of the learners. Through actions, further inscriptions can be designed as part of the diagram on which the learners perform further actions and which they can take into interaction with other learners. Thus, the action itself is temporary, and the resulting inscription (manifestations on the material) can be interpreted as a diagram and

manipulated by the learners. In this way, further actions emerge from actions, which can lead to mathematical awareness (Dörfler, 2000).

RESEARCH DESIGN AND GOALS

The Study “MatheMat — Mathematical Learning with Materials”

The study “MatheMat — Mathematical Learning with Materials” focuses on primary school children’s actions on various material (digital and physical) (Billion, 2018). In four learning situations, primary school children deal with the representation of data, and in another four learning situations they deal with geometric quantities (e.g. volume and surface). Each learning situation is realised on the one hand with physical material and on the other hand with digital material. In total, 32 children (16 child pairs) from third and fourth grade participate in the study. Each child pair works on one geometric and one statistical problem, working once with digital and once with physical material. The processing time of one problem is about 45 minutes. The processing of the primary school children was recorded with two video cameras. One camera records the long shot, and the second camera focuses on the actions on the material. Specially selected video sequences from the learning situations are transcribed in order to be able to analyse them qualitatively. For this paper, the geometric learning situation “Relationship between surface and volume of similar cubes”, which is realised with physical material, is selected.

Learning Situation “Relationship Between Surface and Volume of Similar Cubes”

As in all learning situations, prompts are available to the fourth-graders. Prompts are challenges or short questions that activate learners’ mathematical concepts and knowledge, induce the execution of processes and stimulate cognitive and metacognitive strategies (Bannert, 2009). The learning situation starts with the same prompt for all learners. This prompt intended to stimulate with a question to produce similar cubes using an edge model. In this way, the concept of similar cubes can be clarified at the beginning.

In order to structure this first approach, at the beginning the learners are asked to consider a similar but larger cube and then to build it. In this way, learners intuitively but also systematically generating rules for the construction of similar cubes. Plastic sticks of different lengths are available to the learners to build the edge model, which they can plug together with corner connectors. Furthermore, in the arranged learning environment (see Fig. 1) they can use a wooden cube, which is introduced as a unit cube, and a flat square grid with the grid size of one side area of the unit cube. After processing the start prompt, further prompts are available to the learners. The order of processing is determined by the children.

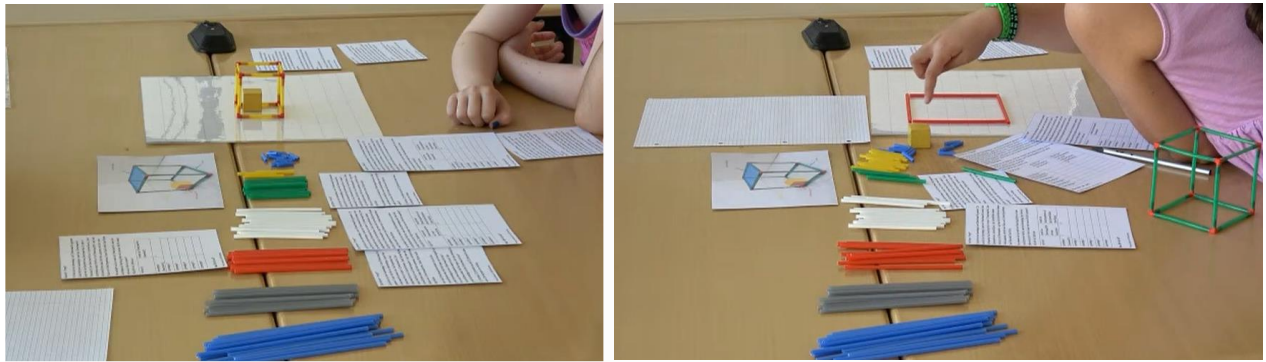


Figure 1: Prompts and the material arrangement from the selected learning situation

The prompts are written on paper cards that are spread out on the table in front of the learners. Each prompt contains the same information text, which clarifies basic concepts, a work assignment usually in the form of a question and a request for the children to reflect on their learning process or to note down results. The learners can flexibly decide which prompt they want to work on. If they do not understand the question of the prompt, they can put it in the back and work on another prompt first. Using the prompts, learners are asked to check how many unit cubes fit into and how many unit squares fit on all sides of the edge models of similar cubes and what patterns can be discovered.

Volume - Prompt*

A cube with edge length 1 is called a unit cube. The cube with edge length 1 fits into the unit cube once. The volume of the unit cube is therefore 1 unit cube. The unit cube has 6 side areas. Each side area of this cube is a unit square. If all side areas of the cube are put together, you get an area of 6 unit squares. This area is called the surface.

How does the number of unit cubes change when the edge length of the cube increases and the cube still looks like a cube?
Fill out the table:

Cube with	Number of unit cubes (Volume)
Edge length 1 (unit cube)	1
Edge length 2	
Edge length 3	
Edge length 4	
Edge length 5	
Edge length 6	

What do you notice?

*Prompt means request

Figure 2: Prompt with which the children work in the transcribed scene

To determine this, learners can use the square grid and the unit cubes, indicating the volume and surface in unit cubes or squares. The learners have the instruction to record their observations, findings and results either verbally or in writing e.g. in the form of a table.

DATA ANALYSIS

For the analysis of the data, selected sections of the video material are transcribed. For this purpose, those places in the data material are selected where the child pairs working with digital or physical material use at the same place in the order of processing (e.g. the third place) the same prompt. In the

transcripts, all action on the material and gestures of the children are reproduced in detail.

On the basis of the theoretical explanations, the following research question will be pursued in this paper: Which mathematical interpretations of the learners can be reconstructed on the basis of their actions on physical material during their processing of the learning situation “Relationship between surface and volume of similar cubes”?

Methodological Approach — Analysis of the Mathematical Interpretation

The basis of the qualitative reconstruction of the learners’ mathematical interpretations of the diagrams and their actions on them is the adaption of the context analysis (explication) according to Mayring (2014) for mathematical learning processes made by Vogel (2017). Here, the explication of a linguistic expression is transferred to the reconstruction of mathematical concepts. This adaption (Vogel, 2017, pp. 68–69) is specified for the reconstruction of learners’ interpretations as follows.

Step 1 – Determination of evaluation unit: As a starting point for the context analysis, a transcript passage is selected in which a mathematical (diagrammatic) action is described that is significant in the situation and that matches the research question and in this case is interesting for the reconstruction of the learner’s individual interpretations.

Step 2 – Explication 1 — mathematically and diagrammatically intended actions of the evaluation unit: (E1.1) Determining mathematically and diagrammatically intended actions by prompts and chosen material based on mathematical contents. (E1.2) Analysis of the transcription passage with regard to the shown actions and the interpretation of the actor expressed therein by contrasting them with the intended action. (E1.3) Compilation of the previous findings.

Step 3 – Explication 2 — narrow context analysis: (E2.1) All actions which are directly related to the transcript passage to be explained are compiled. (E2.2) Pursuing actions are searched in the transcript, which provide further dissociations for the actor’s interpretations. (E2.3) These transcript passages are the starting point for in-depth analyses. The description of the mathematically intended actions from Explication 1 as a frame of reference may need to be extended at this point.

Step 4 – Explication 3 — broad context analysis: Further explanatory material of the transcript is compiled, such as non-transcribed sections of the videographed learning situation. These will be used for a more in-depth continuation of the reconstruction.

Step 5 – Conclusion: Now, the reconstructed aspects of the mathematical interpretations of the selected actor during the different phases of the analysis are described in summary.

The following context analysis of the learning situation “Relationship between surface and volume of similar cubes” cannot be shown completely due to lack of space. Therefore, the broad context analysis (step 4) is not explicitly shown here. Selected results from this analysis are integrated into the conclusion (step 5).

Analysis of the Individual Interpretation of a Child

A transcript (scenes 01 to 29) of the editing of the prompt “volume table” by two fourth-graders is created (see Fig. 2), in which the learners are to determine the volume of similar cubes. The learners have already determined the volume of cubes with edge length two and three (scenes 05 to 07). The selected passage of the transcribed dialogue of the child couple (scene 15, 33:17 min) reflects exactly the actions of Mia to be explained in this analysis.

- 1 Mia: No no no no stop stop
- 2 *Mia places the plastic stick with length 4 back on the square grid perpendicular to the edge of the table.*
- 3 *She still touches the stick with the index finger and thumb of her left hand.*
- 4 *She removes her fingers from the stick.*
- 5 *She takes three more sticks with length 4, lying between the green and red sticks, between thumb and index finger of the left hand.*
- 6 *She places the first stick from her hand perpendicularly at the back end of the already lying stick as seen from the girl.*
- 7 *She places the second stick from her left hand again perpendicularly at the end of the stick she just placed.*
- 8 *She places the last stick from her hand perpendicularly on the first stick lying on the square grid and the last stick placed on it.*

Step 1: In this scene, Mia places a square of sticks with length 4 on the square grid. In the further analysis, we will focus on Mia.

Step 2 – Explication 1: (E1.1) In the learning situation, edge models of similar cubes are considered. The focus of the selected prompt (see Fig. 2) is determining the number of unit cubes (volume determination) that fit into similar cubes of different sizes. A suitable action for processing would be the construction of edge models for cubes of different sizes. By positioning the edge model on the square grid and using the unit cube, the volume can be determined. For example, the squares on the square grid can give orientation how often the unit cube fits into a row, a plane and finally into the complete edge model. To build an edge model of a cube, sticks of equal length must be selected for the twelve edges. In perpendicular prisms, the three edges that meet

in a corner are aligned at right angles to each other. The so-called spatial tripod (Müller, 2004, p. 30), which stands for the three-dimensional coordinate system, is materialised in the form of a plastic corner connector. The edge lengths can be measured using the unit cube or the square grid. It is also possible to determine whether the different sticks are of the same length by placing them next to each other. (E1.2) By selecting sticks of the same length, it becomes clear that Mia considers that edges of equal length are necessary to make a cube. In total, Mia selects four sticks of equal length, which she places on the square grid. She places the sticks on the square grid so that the ends of the sticks meet at a 90° angle. It can be assumed that due to the square grid and the available corner connections, Mia can interpret the conventionalised materialisation of a right angle in the plane and in space and use it for the construction of cube edge models. It is not clear at this point whether she uses the square as the basic side area for the edge model or whether she just does not extend what is lying. (E1.3) In the actions from this transcript, it becomes clear that she deliberately selects sticks of the same length and places them on the square grid in such a way that they are at right angles to each other, which becomes clear in the square grid. It can be assumed that Mia has interpreted the convention of sticks of equal length and the observance of right angles for the construction of a square and uses it in her actions.

Step 3 – Explication 2 — narrow context analysis (Analysis in sections): In scene 20, Mia grabs four sticks of length 5, and in the following scene, she places these four sticks on the square grid at right angles to each other, creating a square. At this point, it is still not clear if she will extend the square further. In scene 22, Mia taps the square grid five times with her stretched finger, moving her finger to the right after each tap. Then, she taps the square grid five times again and moves her finger down after each tap. Meanwhile, she counts from one to five twice. In comparison to the intended actions, it becomes clear that Mia does not use the unit cube to determine the area or volume but works with the square grid. In the following scene, Mia expresses the calculation five times five is twenty-five and then twenty-five times five. Already in the narrow content analysis, it becomes apparent that Mia recognises rules in the material arrangement, uses them to work on the mathematical problem and transfers her two-dimensional actions to the space.

Step 5 – Conclusion: It can be seen in Mia's actions that she uses the available material (sticks, corner connections, unit cubes and square grids) diagrammatically. The implicit rules and conventions for building cubes of different sizes as a basis of determining the volume are used by Mia to process the problem. Including the broad context analysis (see Fig. 1, left picture), it is noticeable that she reduces her actions in the course of the situation and still can make the same interpretations. No longer does she have to build a cube, nor does she move the unit cube in the built edge models, but she can infer the

volume of the cube from the base area using the internalised rules and conventions.

DISCUSSION

Based on the actions, Mia's individual interpretations of the diagrams and the interpretations of the actions on the diagrams can be reconstructed. By using different rules, such as the necessity of equal edge length for the representation of a cube, it becomes clear that Mia recognises this rule and uses it for processing the problem. Working on the problem, this diagrammatic work has to be applied several times and is incorporated into the work with other diagrams, e.g. determining the volume or filling in the table (see Fig. 2). It is noticeable that the actions decrease with the internalisation of the conventionalised set of rules. Thus, in the broad context analysis it becomes clear that Mia initially executes the actions, such as building an edge model, completely (see Fig. 1, left picture). Later, she only lays the base area of the cube and extrapolates from this area to the volume. For this purpose, further analyses will be carried out to see whether the actions and thus the interpretation of the diagram for the same problem situation, but with digital material, differ from the interpretations reconstructed here.

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SYMMETRY-ART: A STEAM TRAINING WORKSHOP FOR PRIMARY SCHOOL TEACHERS

Teresa F. Blanco¹, Valeria G. Roel¹, Roberto Capone², Laura Branchetti³, & Aaron Gaio³

¹University of Santiago de Compostela, Spain

²University of Salerno, Italy

³University of Parma, Italy

The aim of this paper is to analyse a symmetry and art workshop from a STEAM perspective. The theoretical framework of the Meta-Didactical Transposition is taken as a reference. The sample consists of seven Primary School teachers. A qualitative methodology is followed that is developed in four phases: learning, planning, implementation and reflection. The results show that the teachers are not flexible in dealing with the different conceptions of symmetry and the creative aspect of the workshop. In general, there is a positive attitude towards the interdisciplinary character of the workshop, despite the fact that they were not able to connect both disciplines in a balanced way.

INTRODUCTION

Recently, the ‘A’ of art has been included in the acronym STEM (Science, Technology, Engineering, Mathematics). The main goal of STEAM education is to make the students grasp the connections between different pieces of knowledge incorporating an artistic vision into the activities from a creative and emotional point of view (Henricksen, 2014; Yakman & Lee, 2012).

In particular, what is the relationship between Visual Arts Education and Mathematics? One reason for asking this question is that “on the one hand, mathematics is art, and on the other hand, working in art has a mathematical basis” (Hickman and Huckstep, 2003, p.1). Mathematics and art are two disciplines that have a close relationship since immemorial times. In order to motivate students to study mathematics, the connections between art and mathematics, in particular geometry, have been exploited in many works in mathematics education (Fenyvesi, K. & Lähdesmäki, T., 2017; Lavizca, Z. et. al., 2018; Portaankorva-Koivisto, P. & Havinga, M., 2019) showing them that these have been used for aesthetic reasons in the history and modern art.

Recently, the recommendations for including the arts and creativity in the teaching of mathematics significantly increased all over the world along with demands to move from paradigms of teaching concepts and methods in a purely disciplinary way to an interdisciplinary and integrated education that shows

connections, is based on complex problems and promotes critical and creative thinking (Council of the European Union, 2018). These recommendations come, in general, from outside the school. In particular, from EU and other transnational institutions and from labour market. That recommendations oblige the curriculum developer who wants to meet such promising but ambitious goals to take the issue of teacher training education seriously. Indeed, in order to make this new approach become a structural innovation in schools, a change of perspective would be necessary, first of all in teacher education: the teachers need to be prepared to carry out properly the classroom activities, becoming aware of their non-renounceable features and pursuing their goals with their more traditional ones in the complexity of the real classrooms.

In this paper a STEAM training workshop for Primary School teachers is analysed, emphasizing the disciplines of mathematics and art. The aims are to attend how the teachers react to the activities proposed and how they implement them in the classroom. Moreover, the process of personal transformation of the proposal made by some teachers is observed.

RESEARCH FRAMEWORK

The framework of the Meta-Didactical Transposition (MDT) (Aldon et al., 2013; Chevallard, 1999) is considered as a main reference. In particular, in this paper, the construct of praxeology is used. “The praxis or ‘to know how’ includes different kinds of problems to be studied as well as techniques available to solve them; and the logos or ‘knowledge’ includes the discourses that describe, explain and justify the techniques used and even produce new techniques” (Garcia et al, 2006, p.226). Within the MDT approach, the praxis is didactical and the logos not only concerns the knowledge of the discipline, but also of didactical and pedagogical research results. On one hand, in a teacher training activity, researchers’ and teachers’ praxeologies meet each other and members of two communities of practice have to find a common ground in order to allow the teachers to appropriate of the researchers’ proposals and effectively modify their praxeologies.

The transition from individual to shared praxologies is very delicate and requires the action of a ‘broker’, a subject that is a hybrid between the two communities who acts as a hinge between the two fields, the school itself and the academic. The broker has the difficult role of creating new connections and encouraging creations of meaning and learning (Rasmussen et al., 2009).

To analyse the teachers’ choices, when they plan and implement the activities of the symmetry-art workshop, the goal-oriented decision-making theory by Schoenfeld (2010) is relied on. This framework deals in particular with choices of the teachers in real-time. As Schoenfeld (2010) stated clearly, when the teachers move from the design to the implementation, something that changes even completely the goals of the designed activities often happens. Indeed, they

are only partially aware of their resources, goals and orientations, and these might remain invisible in the design phases, but appear clearly in the way they react to students' questions or unexpected happenings. Tensions appear between the planned and the implicit goals and orientations (Liljedal et al., 2015) and oblige the teachers to make real-time decisions according to their priorities. This point is crucial: a deep innovation requires the teachers to become aware of their knowledge and assumptions and seriously reconsider in a conscious way their goals and priorities.

RESEARCH METHODOLOGY

The training symmetry-art workshop was designed for Primary School teachers and was carried out in two Italian cities. In this paper, a sample of seven Primary School teachers is analysed. The objective is to answer the following research question: what is the general impact of the symmetry-art workshop on the teacher's design and implementation in their classrooms?

The research methodology is qualitative and from a STEAM perspective involves working the two disciplines together in a balanced way, both in terms of concepts procedures and procedures and attitudes. It was organized in four phases that are described below: (i) learning; (ii) planning; (iii) implementation; (iv) reflection.

(i) Learning phase. In this phase, the researchers present the STEAM methodology. Then, the teachers carry out the different workshops by interacting with the researchers. In accordance with the MDT, a PhD student graduate in Primary Education Sciences took on the role of broker, mediating the delicate passage of the interweaving of the praxeologies of the teachers with those ones of the researchers.

(ii). Planning phase. The objective is that teachers develop this proposal to the classroom, after a careful co-design shared between teachers and researchers. To this end, they should decide which tasks they are going to implement, whether and how they want to modify them, in which order, the time they are going to use for each task, the links with their curricular teaching plan and the methodology they are going to carry out (group or individual work, classroom discussions and the educational environment where the students would do the activities).

(iii). Implementation phase. In this phase, the teachers implement the symmetry-art workshop tasks as they have designed them in the previous phase. The aim of the research is to compare the decisions taken in the planning phase and the teachers' actual praxeologies in the classroom.

(iv). Reflection phase. Here, both researchers and teachers reflect on the entire instructional process. In this way, following the theoretical framework,

researchers' praxeologies should change interacting with the teachers to make the proposal more suitable from the cognitive and institutional points of view.

To collect the data the following instruments were used. In the planning phase, individual and group interviews with teachers were recorded. In addition, they were given a grid to fill in different sections regarding the organization of the tasks. In the implementation phase, video recordings were made of the observations of teachers and students in the classroom. Moreover, an observation tool was also designed which comprehends thirteen items. Within these items, special attention was given to those that refer to, among others, the good use of mathematical vocabulary, the mastery of the artistic techniques and the methodology carried out in class.

The tasks that were carried out in the STEAM training workshop are described below.

Description of the Tasks

Training Symmetry-Art workshop is made up of four tasks to carry out in two sessions of two hours. The tasks of this workshop are aimed at Primary School students (six to twelve years old). In mathematics education, the difficulties in the learning of this topic have been investigated in many studies (Bulf, 2011; Chesnais, A. & Munier, V., 2013, Bohorquez et. al., 2009), and it has been shown to be more complex as it might seem. These difficulties might affect the teachers' resources, both on the side of disciplinary knowledge and of the anticipation of students' difficulties. Within this proposal, a balance is sought between the two subjects of mathematics and art. Following a STEAM perspective, the objective is to work these two subjects in an equal way, that is, these tasks form a cycle starting from art (task 1) and coming back to art (task 4), with a renewed conceptualization of the everyday conception of symmetry (Chesnais, 2012) triggered by the artistic work and supported by research-based mathematical tasks (2 and 3).

Task 1: Artistic folding paper

This activity is designed with the intent to create a symmetrical artwork from the blank paper and without mentioning the concept of symmetry. The aim is to bring students closer to the study of symmetry and its elements, starting from the original artistic creation of each of them through the manipulation of different resources, in this case, thread, tempera and sheets. The contents that are worked on in this task are the concept of symmetry, the axis of symmetry, the types of lines, the equidistance, the concept of shape and dimension, the horizontal and vertical meaning, the manual work and the use of colour and its possible mixtures.

Task 2: TEPs.

Following to D'Amore and Maier (2003), the objective is, for each student, to create a TEP (Textual Eigen Production), which is an autonomous textual production, in this case, of the concept of symmetry and its characteristics based on the artistic work and the discussion carried out in the previous task. The contents worked on here are the use of the mathematical vocabulary to elaborate the definition, the written expression and, again, the concept of symmetry with some of its elements as the axis of symmetry, the equidistance of each point to that axis and the concept of form and dimension.

Task 3: Schematization

This task consists of drawing, on the grid sheet, the figure that the students made in the task 1. The aim is to make them work on symmetry and its characteristics through the elaboration of a scheme with drawing instruments as the rule or the compass. The students also work on the reproduction of a figure to scale, since at the moment of drawing the figure in a schematic way, they are transferring the figure to the grid sheet, taking the little square as a unit.

Task 4: Symmetrical figures with coloured threads

The last task is designed to finish the proposal with an artistic activity that gathers everything learned in the previous tasks. The activity consists of recreating, with coloured threads and pins, the figure made in task 1, and then outlined in the task 3. By stretching the threads and tightening them, the students create another artistic work in a different format in which the main theme is symmetry.

RESULTS AND DISCUSSION

The results are presented according to the aims set, derived from the research question presented in the previous section: to observe how the teachers react to the activities proposed in the symmetry-art workshop and how they implement them in the classroom.

Teachers Reactions

In terms of STEAM methodology, the teachers initially stated that they dealt with mathematics and art topics always separated. Although they had already dealt with the topics proposed in their classes, they did not realize that they could make an interdisciplinary lesson by drawing inspiration from artistic creations to get to the formalization of mathematical concepts. Moreover, it could be observed that the reactions of some teachers consisted on not considering the STEAM activities truly mathematical didactical activities, since the contents and the kind of tasks were different from the text-books exercises, that are their institutional reference. Some teachers perceived these activities as extracurricular motivation, since they emphasize their artistic character and gave importance only to the aesthetic aspect, that is, they did not consider them 'mathematical' (learning phase).

For most of the teachers, the tasks seemed to be not so far from their usual practice and the mathematical contents and artistic skills were considered easy. However, some of them did not feel confident to carry out the activities in the classroom observed by researchers and, in many cases, they had some difficulties to pursue the planned goals in the implementations. For example, a teacher somewhat insecure, asked “how I should start the lesson? Are we going to carry out the activity together?” (planning phase).

In the implementation phase, two of the seven teachers said “Do we have to carry out the lesson? But we can’t do it, we don’t know how to do it”, revealing to be unsure at the beginning of the class. Another teacher renounced to lead the activity and asked the researchers to do it. Part of the problem could be due to the presence of the researchers in the classroom or to the insecurity of applying the STEAM methodology.

Implementation in the Classroom

Of the seven teachers who planned to carry out the art and symmetry workshop in the classroom, six did so. Of those six, four implemented it autonomously while the other two needed further assistance from the researchers. Although the planning phase allowed them to modify and adapt the proposal to their classroom and students, only one of the teachers changed the order of the tasks and dedicated more time to the discussion that is carried out in task 1.

Paying attention to the mathematical aspect of the workshop, several facts are considered important. When the students commented on their TEPs for the rest of the class (task 2), the teachers corrected those who talked about important aspects of symmetry such as distance to the symmetry axis, because they identify the term symmetry only with the definition they know, which is the same one that appears in the textbook. Therefore, their goals were far from ours and were influenced by the textbook definition in a negative way for the students' mathematical processes.

For some teachers there is a total identification between the concept of symmetry and the fact that half of a figure could be superposed to its other half folding a piece of paper containing the picture; the paper folding activity helped them to feel comfortable but in some cases the symmetry-art workshop was not effective in enriching their concepts moving from the everyday to the mathematical concept. In some cases, the teachers did not take care properly of the students' spontaneous mathematical processes and interrupted the students who were carrying out their own reasonings in terms of symmetry. For instance, many students interpreted correctly the request of explaining with their words how to draw a ‘symmetric figure’ that is, a figure admitting (at least) one axis of symmetry while their teacher expected the students to use formal words and define the symmetry in the way the teacher had suggested and started limiting them without helping them in their developmental zone. This may be due to

teachers' lack of flexibility in conducting a group discussion with students on the concept of the symmetry (ibid., 2012). On the other hand, in many cases the teachers declared that their insecurities were due to unexpected difficulties with the mathematical contents, and emerged when the students were working and proposing their ideas in a manner that was different than the usual (reflection phase).

Focusing on the artistic part, it should be pointed out that it was the main aspect that motivates the teachers to implement the mathematics and art workshop. However, initially, most of them limited the creativity of the students, especially in task 1. This limitation could be due to the fact that the teachers showed a perfectionist attitude when they performed the workshop by themselves (learning phase) and wanted their students to obtain similar results to theirs, imposing some criteria like the colours they should use or indicating that the artwork should be 'beautiful' and 'well done' (implementation phase). Between these two phases, it could be seen that teachers' praxeologies (Schoelfeld, 2010) changed, since they were forced to make decisions just in time. For example, because of the motivation students to do this workshop, many of the teachers spent more time experimenting with more colours and creating more artworks. In addition, some of them left the students total freedom when performing the schematization (task 3) allowing them to use different colours and shapes.

CONCLUSIONS

Taking into account one of the aims of this paper, it could be observed that teachers' reactions to the proposed STEAM workshop were positive. In the reflection phase, all teachers valued the importance of proposing activities with an interdisciplinary character. Adding the planning phase was intended to give teachers flexibility and creativity in implementing the workshop in their classrooms. However, the changes that were observed were very specific and only one of the seven teachers modified the tasks by adapting them to her classroom context. In this case, the intersection between the teacher's and the researcher's praxeologies was obviously no longer empty.

On the other hand, the tasks of the workshop have an intrinsic complexity that makes students act in unpredictable ways. Although many of the teachers stated that the schematization (task 3), specifically, was very difficult, the students performed it very effectively obtaining great results. In some cases, however, teachers were not flexible to adapt the activities to just-in-time happenings.

The fact that more than one teacher has declared that they want to continue experimenting with mathematics and art workshops means that some practices have changed and that the symmetry-art workshop has been successful. It is therefore desirable that a dynamic process of professional evolution has been triggered in which some components external to the teachers praxeologies, such

as the use of interdisciplinary teaching through appropriate tasks, become internal as an effect of the process of meta-didactic transposition. The meta-didactic transposition, in our case, has its strength in the use of innovative tasks and the adoption of interdisciplinary teaching. Therefore, we propose to continue carrying out workshops and to focus on the relationship between mathematics and art encouraging a balance between these two disciplines.

Acknowledgments

This work is funded by the Spanish Government under the research project EDU 2017-84979-R

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“WHY DON’T YOU MAKE A DRAWING?” MOTIVATION AND STRATEGY USE IN MATHEMATICAL MODELLING

Judith Blomberg¹, Stanislaw Schukajlow¹, Johanna Rellensmann¹

¹University of Münster, Germany

Motivation is important for students’ learning and strategy use. However, we do not know much about the relations between motivation and the use of strategies such as the drawing strategy. In this study, we assessed the mathematical and strategy-based motivation of 194 ninth- and tenth-grade students using expectancy-value questionnaires. Further, we measured the spontaneous use of drawings for solving geometric modelling problems. We found a positive relation between mathematical and strategy-based expectations of success as well as between mathematical and strategy-based attainment value. Furthermore, mathematical and strategy-based motivation differed in their relation to the use of drawings. These results indicate the importance of both mathematical and strategy-based motivation for strategy use and modelling.

INTRODUCTION

Mathematics as an applied science is part of many other disciplines, such as the natural sciences, computer science, and the social sciences. An application-based view of mathematics is reflected in mathematical modelling. Mathematical modelling involves the use of mathematics to solve real-world problems (Niss, Blum, & Galbraith, 2007). Because of the importance of applications for life and work, countries around the world recommend that mathematical modelling be promoted in mathematics education, and it is included in the mathematics curriculum of different countries. However, prior research has repeatedly demonstrated that students have trouble solving modelling problems (Niss et al., 2007). The use of strategies such as self-generated drawing is considered to have a beneficial effect in overcoming the difficulties involved in solving modelling problems (Galbraith & Stillman, 2006; Hembree, 1992). Positive effects of drawings have been shown for students who made drawings spontaneously. However, why do learners rarely make drawings spontaneously? One possible factor that influences the spontaneous use of drawings is motivation. In the present research, we targeted mathematics and the drawing strategy as the objects of motivation because mathematical and strategy-based motivation might both be important for the spontaneous use of drawings. In this paper, we aimed to examine the relation between mathematical and strategy-related motivation and their importance for the spontaneous use of drawings in mathematical modelling.

THEORETICAL BACKGROUND

Self-generated Drawings in Mathematical Modelling

By making a self-generated drawing for a mathematical modelling task, the learner visualizes a problem described in the task by representing the objects and their relations to each other in an iconic way. By applying the strategy of making a drawing, we understand both the drawing process and the drawing as a product (Rellensmann, Schukajlow, & Leopold, 2017). As a strategy for learning and problem-solving, making drawings can support various activities in mathematical modelling such as constructing a mental model of the text, discovering errors in the mental model, structuring and simplifying the given situation and constructing a real model, mathematizing the real model, or validating the mathematical result.

Spontaneously making a drawing for a given mathematical word problem has already been shown to be a potentially performance-enhancing strategy for learners (Hembree, 1992; Uesaka et al., 2007). This strategy was found to be more helpful than improving mathematical vocabulary, verbalizing important concepts, or applying other strategies (Hembree, 1992). Thus, making a drawing might also be helpful for solving geometrical modelling problems. Despite the expected positive effects of generating a drawing in mathematical modelling derived from the analysis of modelling activities such as mathematizing, students rarely use this strategy spontaneously. One reason for this result might be students' motivation. For example, in Pressley's (1986) model of a Good Strategy User, motivational beliefs are suggested to predict the spontaneous use of strategies. Pressley further suggested that if students are motivated to use a strategy, they will use it more often.

Expectancy-value Theory of Motivation

In a broader definition, Middleton and Photini (1999) specified motivation as a reason for human behavior in a specific manner and in each situation. At the core of many theories of motivation are expectancy-value models such as the one by Eccles and Wigfield (1995). These models propose that performance-related decisions (e.g., using a specific strategy) are essentially influenced by two subjective beliefs: expectations of success (ES) and the value attached to the different options that are available. In research, expectations of success have often been estimated via self-concept or via general self-efficacy, which have repeatedly been found to be closely connected to each other (see the overview by Marsh et al., 2019). The value component includes three sub-components: the interest and enjoyment gained from the task (Intrinsic Value, IV), the personal importance of being able to do it well (Attainment Value, AV), and the perceived utility from solving it (Utility Value, UV). Similar to other affective constructs, motivation can target different objects (Schukajlow, Rakoczy, & Pekrun, 2017). The objects of motivation can be learning in general, a specific

topic, or even a specific problem. The present research involves mathematical motivation because the object of motivation is mathematics. Motivation that targets a specific strategy or its characteristics as its objects can be called strategy-based motivation. In the present research, we assessed strategy-based motivation by using the drawing strategy because of the importance of this strategy for problem-solving (Hembree, 1992).

Prior research hypothesized a positive relation between expectations of success that targeted different objects in one domain such as mathematics. The reason for this positive relation is that problem-solving activities within mathematics require related abilities and skills. Furthermore, students acquire different abilities and skills in mathematics in parallel in their mathematics lessons or in mathematical activities that they participate outside of school. These considerations were confirmed empirically by Marsh et al. (2019), who demonstrated a positive relation between mathematical expectations of success (that were asked about by referring to mathematics in general) and to specific mathematical problems as objects of motivation. Likewise, a positive relation can be expected between values within the same domain such as mathematics. The expectation that values for different objects in mathematics can be related has been supported by empirical results. For example, the utility value of modelling problems was found to be positively related to the utility value of intra-mathematical problems (Krawitz & Schukajlow, 2018). However, prior empirical results should be interpreted with caution because the differences in the objects of motivation are essential for the relations between the constructs. The relation between mathematical and strategy-based motivation is still an open question.

Motivation and Strategy Use

Many studies have demonstrated the positive effects of expectations of success and value on the use of cognitive and meta-cognitive learning strategies. For example, Virtanen, Nevgi, and Niemi (2013) showed that university students who reported high expectations of success and high intrinsic value were also more likely to report that they organize the learning content in their discipline. Focusing on the relation between mathematical motivation and self-reported learning strategies in mathematics, Berger and Karabenick (2011) found that both expectations of success and value predicted elaboration and metacognitive strategies. However, in these studies, researchers used self-reports to assess the strategies, and the validity of assessing strategies via self-reports has often been criticized in the past. Because of research on the relation between mathematical motivation and self-reported strategies, we suggest a positive relation between mathematical motivation and the use of the drawing strategy.

Moreover, we found only a few studies that analyzed the relation between motivation and the spontaneous use of the drawing strategy. A case study of an

eighth-grade girl who did not use a drawing strategy spontaneously at first but used it successfully after being instructed to do so suggests that spontaneous strategy use depends on the perceived efficiency of the strategy and thus also on motivation (Ichikawa, 1993; Uesaka, Manalo, & Ichikawa, 2007). Furthermore, Uesaka et al. (2007) demonstrated that the benefits attributed to learner-generated drawings reported by students were significantly related to the use of drawings. These findings indicate that strategy-based motivation might be important for the spontaneous use of drawings.

RESEARCH QUESTIONS AND HYPOTHESIS

Based on theoretical considerations, we conclude that the spontaneous use of a drawing strategy is related to motivational factors. However, there is a research gap regarding the relation between mathematical and strategy-based motivation as well as to the relation between motivational factors and the use of the drawing strategy. Moreover, we did not find any research that investigated the relation between motivation and making a drawing to solve modelling problems. Therefore, we addressed the following questions in this study:

(1) How are the mathematical motivational constructs (ES, IV, AV, UV MATH) related to the corresponding strategy-related constructs (ES, IV, AV, UV DRAW)?

We expected a positive relation between mathematical and strategy-based expectations of success because the development of the strategic skills involved in making drawings takes place within mathematical learning. We also expected positive relations between the different values of the mathematical and strategy-based constructs. However, as the relations between motivational constructs strongly depend on how close the objects of motivation are to each other, and only a little research has been conducted on strategy-based motivation, these expectations were based mostly on theoretical considerations.

(2) How are mathematical and strategy-based motivational constructs (ES, IV, AV, UV) related to the spontaneous use of the drawing strategy while students solve modelling problems?

Based on the expectancy-value theory, we expected both mathematical and strategy-based motivation to be important for the spontaneous use of drawings. An empirical indication for the positive relation between mathematical motivation and the use of the drawing strategy comes from research on self-reported strategies. One case study and one cross-sectional study carried out with school students on the use of the drawing strategy supported the expectation that students' strategy-based motivation might be related to spontaneous strategy use.

METHOD

Participants and Research Design

Two hundred twenty German ninth- and tenth graders (49.5% female, $M = 14.93$ years) of 10 comprehensive classes participated in the study. At the first occasion, the students answered a questionnaire about motivational constructs. After two weeks, they were asked to solve eight geometric modelling tasks. The analysis of students' solutions allowed us to assess their spontaneous use of the drawing strategy. Some students could not participate on both occasions for various reasons. In sum, 194 students participated on both occasions and were included in our analysis.

Measures

The 22-item survey was applied to assess mathematical motivation (MATH, 10 items) and strategy-based motivation with respect to the use of drawings (DRAW, 12 items). Students rated each statement on a 5-point scale (1 = "not true at all" to 5 = "completely true").

Mathematical motivation scale. The mathematical motivational items were adapted in accordance with Eccles and Wigfield (1995). *Expectations of success* (ES MATH) were assessed with three items (e.g., "I am very good at mathematics"). The three components of *value* are *intrinsic value* (IV MATH; 2 items; e.g., "In general, I find working on mathematics assignments very interesting"), *attainment value* (AV MATH; 3 items; e.g., "It is very important to me to be able to solve mathematical problems very well"), and *utility value* (UV MATH; 3 items; e.g., "Mathematics in school is very useful for my professional future after graduation"). The reliabilities of the subscales were mostly good to satisfactory ($.55 < \alpha < .89$). The confirmatory factor analysis revealed that the model with four factors fit the data adequately ($\chi^2/df = 1.72$, SRMR = .04, RMSEA = .06, CFI = .97).

Strategy-based motivation scale. The strategy-based motivation scale with respect to the use of the drawing strategy was assessed with four subscales: *expectations of success* (ES DRAW; 3 items; e.g., "I believe I can make very good drawings for any word problem"), *intrinsic value* (IV DRAW; 3 items; e.g., "I like to make a drawing for a difficult word problem"), *attainment value* (AV DRAW; 2 items; e.g., "It is important to me to be able to make a drawing for a difficult word problem"), and *utility value* (UV DRAW; 4 items; e.g., "Making drawings is important to me because it helps me solve difficult word problems"). The reliabilities of the subscales were mostly good to satisfactory ($.58 < \alpha < .86$). Confirmatory factor analyses showed acceptable values for the model ($\chi^2/df = 3.27$, SRMR = .04, RMSEA = .07, CFI = .95).

Use of drawings. The use of drawings was measured dichotomously for each of eight modelling tasks that could be solved by applying the Pythagorean

Theorem. A code of 0 was assigned to solutions without a drawing and a code of 1 to solutions with a drawing. The measurement showed good reliability (Cronbach's $\alpha = .866$).

RESULTS

Relations of mathematical and strategy-based motivation. As expected, the analysis of the correlations between mathematical and strategy-based motivation (Table 1) showed moderate positive correlations between ES MATH and ES DRAW as well as between AV MATH and AV DRAW. These results indicate that students who have high expectations of success and ascribe a high attainment value to mathematics are confident that they can use a drawing strategy to solve problems and feel that this strategy is personally important to them. However, we did not find a positive relation between intrinsic value or utility value for mathematical and strategy-based motivation. For example, students who ascribed a higher utility value to mathematics did not differ in their estimation of the utility value of the drawing strategy.

		MATH			
		ES	IV	AV	UV
DRAW	ES	.289**	.255**	.377**	.234**
	IV	-.041	.010	.233**	.087
	AV	.007	.104	.351**	.117
	UV	-.018	.010	.278**	.038
	V				

Note. ** $p < .01$, p two-tailed. MATH: mathematical motivation, DRAW: strategy-based motivation, ES: expectancy of success, IV: intrinsic value, AV: attainment value, UV: utility value. Correlations between the same constructs in different domains are presented in grey.

Table 1: Correlations between mathematical and strategy-based motivational constructs

Motivation and the use of drawings. Our analysis of the relation between mathematical motivation and the use of drawings confirmed our expectation for IV MATH (Table 2). Students who attributed high intrinsic value to mathematics used the drawing strategy to solve modelling problems more often. Mathematical expectations of success, attainment value, or utility value in mathematics were not related to the use of drawings. The analysis of the relation between strategy-based motivation and the use of drawings while modelling revealed a more consistent picture and confirmed our expectations. We found positive correlations for all strategy-based sub-constructs IV DRAW, AV DRAW, UV DRAW, and ES DRAW with the use of drawings. These results

indicate the importance of strategy-based motivation for the spontaneous use of the drawing strategy. Students who have high expectations of success for the use of drawings and who ascribe high intrinsic, attainment, and utility value to the drawing strategy more often used this strategy spontaneously.

		MATH				DRAW			
		EX	IV	AV	UV	EX	IV	AV	UV
USE	<i>r</i>		.180			.164	.212*		
		.047	*	.088	.098	*	*	.138 ^a	.172*

Note. ^a $p < .10$, * $p < .05$, ** $p < .01$, *** $p < .001$. p : two-tailed. MATH: mathematical motivation, DRAW: strategy-based motivation, EX: expectancy, IV: intrinsic value, AV: attainment value, UV: utility value, USE: spontaneous use of drawings.

Table 2: Correlations between mathematical and strategy-based motivational constructs and the spontaneous use of drawings

DISCUSSION

Based on expectancy value theory (Wigfield & Eccles, 2000), we investigated the relation between mathematical and strategy-based motivation and the importance of motivation for the use of drawings while solving modelling problems. As expected, the analysis of the relation between mathematical motivation and the strategy-based motivation to make drawings showed that mathematical and strategy-based expectations of success were positively related. However, the relation was weak. One reason for this result may be the cognitive structure of the activities: Although the making of drawings as a visual strategy is part of the mathematical curriculum, formal symbolic procedures usually predominate in students' learning in mathematics. Another reason may be the different categories of focused objects (the domain of mathematics vs. the strategy of drawing). As mathematics is a more general object and the drawing strategy is a more specific object, this difference might have an impact on the strength of the relation between the constructs (Marsh et al., 2019). The relation between the personal importance of being good at mathematics (AV MATH) and the personal importance of making good drawings (AV DRAW) was moderate in size. This result revealed that the personal importance of mathematics is closely related to the personal importance of making a drawing to solve mathematical problems. By contrast, the intrinsic and utility values of one object were not related to the values of other. The perceived utility of drawings for solving problems did not depend on whether mathematics was considered useful or not.

The strategy- and mathematics-based motivational constructs differed in their relations with the spontaneous use of drawings during mathematical modelling.

Whereas only the intrinsic value of mathematical motivation was correlated with the use of drawings, all four strategy-based motivational constructs were positively related to the use of the drawing strategy. We suggest that future studies conduct deeper investigations of the relation between mathematical and strategy-based motivation on the one hand and the use of drawings and performance on the other hand. One interesting research question might be whether mathematical motivation has an indirect effect on the use of strategies and performance via strategy-based motivation. In line with results from learning strategy research (Berger & Karabenick, 2011; Virtanen et al., 2013), intrinsic value with respect to mathematics was found to be related to spontaneous strategy use. In addition, as suggested by expectancy-value theory, we found a positive relation between strategy-based expectations of success and the use of drawings in our research. Positive relations between strategy-based values and the use of strategies indicated the importance of values for students' strategy use. Thus, our results confirmed the validity of expectancy-value theory for strategy use.

The results revealed intrapersonal differences when comparing mathematical motivation and strategy-based motivation with respect to making a drawing in mathematical modelling and in problem-solving. Effects of strategy-based motivation on learning outcomes should be addressed more often in future research because it can explain why some students make drawings spontaneously and others do not. Research on strategy-based motivation can be applied not only for the use of the drawing strategy but also to other strategies. Finally, for the practice of teaching, it is important to investigate which teaching interventions improve strategy-based motivation and students' strategic and achievement-related learning outcomes.

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WHEN TEACHER-STUDENT DISCOURSE REACH IMPASSE: THE ROLE OF COMPUTER GAME AND ATTENTIVE PEER

Orit Broza¹ and Yifat Ben-David Kolikant

Levinsky College of Education, Israel

Researchers traced the learning processes of 26 low-achieving students studying subtraction of decimal numbers, as they worked in small groups within a rich learning environment involving a computerized game, play money, peer interactions and teacher mediation. Data sources were videotaped sessions, worksheets, observations, and pre- and post-program teacher evaluations. Results indicate that low achieving students can build new significant knowledge, to participate in a reflective mathematical discourse, and benefit from it. Yet, the setting of computer games with an attentive peer served a fertile platform for strategies to emerge and consolidate.

INTRODUCTION AND THEORETICAL FRAMEWORK

Applying mathematics into real life is considered as an essential component for professional life (OECD, 2016). This might be the reason for mathematics educators to actively engage students with mathematical knowledge building, based on meaning, and avoid routine procedural learning. Insignificant learning base on drill and memorization, especially in early years, might lead to under-achievement among students who do not have any identified disability. This phenomenon is reflected in PISA findings which show that around 20% of OECD students with normal cognitive skills do not reach a minimum level of basic skills in mathematics (OECD, 2016).

Trying to explain these students' poor performance, the literature focuses on cognitive deficits and on behavioral manifestations of their failure (e.g. participation patterns). Low achieving students (LASs) find it difficult to retrieve basic mathematic facts (and knowledge) from their memory (Gray, 1991). Craik (2002) referred to this difficulty as '*fragile memory*': a product of superficial data processing in the brain. Other explanations points on affective reasons such as frustration, anxiety, and passivity (Ramirez, Gunderson, Levine, & Beilock, 2013).

Although the population of low-achieving students is heterogenic, some cognitive difficulties and behavioral characteristics are common. For example, such students find it difficult to retrieve basic mathematic facts from their memory (Geary, 2004) and to use effective computation strategies based on meta-cognitive skills (Goldman, 1989). They are sensitive to the learning

context (e.g., written and oral arithmetic practices or every day and formal mathematics), and find it much harder than other students to solve simple and complex addition and subtraction problems (Linchevski & Teubal, 1993). These difficulties may lead them to use less sophisticated strategies, and thus commit more errors. As they repeatedly experience failure and cannot keep up with the class, their motivation and self-esteem decrease. Therefore, they might have a weak sense of internal responsibility, be passive and/or rely on external authority (Geary, 2004; Linchevski & Tuval, 1993; Haylock, 1991).

Adding to that, teachers do not always take into account socio-emotional aspects of LAS, neither going beyond the cognitive and subject matter aspects, and look into socio-emotional aspects of the teacher-student interaction that could affect learning (Broza & Ben-David Kolikant, 2015). Instead of increasing LAS ability to build on past successes, and fostering a sense of internal responsibility for their advancement, some teachers typically conclude that the most effective way of promoting mathematical performance in low-achieving students is to ‘drill and kill’ (Anderson, Reder, & Simon, 2000), that is to focus more on the mathematical algorithms than on the mathematical meaning.

Digital game-based learning is considered as an effective means to overcome negative implications of learning mathematics. They are fun, meaningful and inspiring by their nature, thus, they allow disengaged students to gain interest for mathematics, enhance motivation to perform difficult tasks and maintain effort, and help children to overcome anxiety (OECD, 2016). Digital game-based learning theories (Squire, 2008; Gee, 2003), emphasize the potential of games to engage and motivate students in becoming active rather than passive, by enabling experiments and explorations without fear of failing in front of the entire class. Through active participation in a meaningful and authentic learning environment, mathematical strategies can develop naturally, as the concrete context is served as a cognitive scaffolding (Wood, Bruner, & Ross, 1976). Therefore, the use of games for teaching may thus be particularly beneficial for low-achieving students.

The current research examines learning processes of LAS who learn mathematics with a digital game and a teacher who was trained to attune her support to LAS cognitive and emotional needs. The learning environment was designed according to three theoretical lines: (a) ‘Learning in Context’ in which mathematical concepts and procedures are presented in a context relevant to a child’s day-to-day life (Gravenmeijer, 2004); (b) game-based learning (Gee, 2003; Squire, 2008), and (c) ‘Accountable talk’, which focuses on the role of the teacher to create a safe and constructive space for building new knowledge by establishing norms and provide opportunities to talk mathematics, as well as share thoughts and ideas with group members (Chapin, O’Connor & Anderson, 2009).

Researchers aimed at engaging students in significant learning by transforming their social and socio-mathematical norms (Cobb, 2004) from passive to active, from isolated to social collaboration, from impulsive to thoughtful. Group discussions were focused on reporting student' mathematical strategies (built by tools and teachers' scaffoldings), and establishing shared norms (e.g., examining students' strategies by approval and disapproval, and optimizing ineffective strategies).

Researchers were aware of LAS's tendency to impulsivity; thus, students were asked to learn in dyads, in front of a computer. Researchers hypothesized that the collaborative setting will trigger two types of interactions: Computer-student and student-student; and that peers will explain their calculations to each other, and question other's action, bringing about reflective and thoughtful interactions (Dillenbourg & Ficher, 2007).

In a previous work (Broza & Ben-David Kolikant, 2015), researchers endeavored to characterize the meaningful and complex learning processes among LAS in a rich supporting environment in general and at the different levels of progress. In the following section researchers present the importance of the presence of computer game in the environment with peers' discussions for progression.

METHODOLOGY

A total of 26 low-achieving fifth grade students took part in the above-mentioned extracurricular program, for one weekly hour, for the duration of eight weeks in tow iterations. They studied subtraction with decimal fractions prior to the topic being studied in their parent mathematics classes, learning in small groups (up to four students), with a teacher trained by the researchers. The instruction framework emphasized a delicate transition from the realistic environment to formal mathematics. For this reason, for example, in the first four lessons, subtraction was presented only through monetary simulations and problems, with no formal exercises. From the fifth lesson onward, the formal representation of operations was interwoven into the learning situations, while maintaining the focus on authentic contexts.

When playing the learning environment's "ice-cream shop" game (http://kids.gov.il/money_he/glideriya.html), the students acted as sellers: They received orders, prepared ice-cream, and then calculated and gave change. In addition, students were asked to work in supplementary online study units, which concerned the transition between money and formal representations, as well as change calculations. Students also enacted game-like situations with mock Israeli money (shekels and agorot).

While students engaged in computerized activities, the teacher stayed in the background, observing their work and difficulties, taking notes for the following discussion, and intervening when needed. Much of class time was devoted to

pair and group discussions. The teacher's interventions did not include direct corrections of students' strategies, but rather meta-scaffolding questions that encouraged the students to use the tools in the environment to build their own strategies.

Our primary data source was the transcripts of eight videotaped, 45-minute-long learning sessions, accompanied by eight screen captured computer sessions video screenshots (about 20 minutes each). Other tools included pre-program student interviews focusing on mental computation strategies, observation of the parent mathematics classes, student evaluations filled in by their parent mathematics class teachers' pre-and post-program, and individual worksheets each student filled in during the extracurricular lessons. According to a design-based research, data were collected in two iterations: Pilot study and main iteration. The transcripts were coded twice by two researchers. Using micro genetic approach (Siegler, 2006) researchers analyzed their knowledge building trial by trial. Utterances were segmented into episodes, so that each episode began with the presentation of a new task (Broza & Ben-David Kolikant, 2010). Each episode was classified according to the problem it deals with, and examined: (i) who participated in it; (ii) the tools that were involved; (iii) the knowledge pieces that emerged, and (iv) the difficulties that arose, including whether they were solved, and if so how and by whom.

After identifying the episodes in which constructing occurred, researchers searched for historical evidence, i.e. indications in previous episodes, that could hint about the specific ways this new piece of knowledge could have been constructed. This integrative analysis enabled to focus on the developmental changes in the student's thinking and behavior chronologically, as well as to examine it with respect to the literature of LASs.

RESULTS

Eighty two percent of the students in the main iteration significantly changed their discourse participation, and actively built their own strategies to solve mathematical problems and exercises. The learning process was complex or inconsistent with regressions and progressions alternately due to LAS fragile memory. Therefore, the teacher found it difficult to calibrate her support in accordance with students' prior experiences. However, despite the difficulties, 55 percent of the students in the main iteration exhibited stability in their knowledge during at least three continuous lessons. Additional 27 percent of the students exhibited short progressions with localized consolidations (within a specific lesson and not between lessons).

Students' Behavior While Playing the Game

As researchers hypothesized, the computerized environment, encouraged the students to be active as well as engaged in their task. During the play, researchers observed that the students were very focused on the task in hand. In

fact, students continued working (or playing) after the class had ended. The students reported in the interviews and ad hoc conversations that “it was fun...not a regular class”, “playing with the computer provides a sense of fun, [vs.] a blackboard, where you just sit and solve exercises”. Each student solved many subtraction exercises, manifested by the need to give change to customers in the shop. Students usually worked in turns: The one on the keyboard gave ice-cream, calculated the price, the change, and returned change.

Failures in this context did not discourage them. On the contrary, this is when researchers observed mathematical discussions with their peers and with the teacher. Usually, when they received a response from a “customer” indicating that the change they gave was incorrect, they paused to think and sometimes they turned to their peers and verbalized their solution process. Sometimes this verbalization occurred after their peers asked them how they had worked. The discussion often helped them to correct themselves. This behavior was dramatically different from the observed (and reported) passivity (or impulsivity) in the regular classes. Moreover, in this context, the students generally welcomed the teachers’ intervention and cooperated with them. Hence, the computer and the peers often generated a synergetic effect on the students.

The next two examples (to be reported at the conference) illustrates knowledge building next to the computer when the teacher find it hard to build on previous experiences due to the fragility of the knowledge. In both cases the successions of success were in lesson or between two lessons in front of the computer and at the next writing task. In both cases the strategy was not consolidated in the long term.

Li 's Example

In Lesson 3, Li was able to easily use borrowing to subtract decimals with halves from integers, yet in Lesson 4, she found it difficult to extend this to subtrahends with different decimals (e.g., 7.70). It took the teacher several attempts to identify the problem. Then, rather than explicitly teaching the procedure, the teacher elected to create opportunities for Li to build her own knowledge and made many attempts to support her in this process. Amongst her attempts were her suggestions and guidance to use play money, the verification procedure, the conversion procedure, and the linking of subtraction exercise in the task to the monetary terms of the problem story (the price, the change). Her suggestions were reasonable, given that Li previously experienced success with these activities and procedures. However, Li was apparently unable to remember or apply this past knowledge to the situation at hand.

It was only in the next lesson that Li was able to construct a conversion strategy. It was in the subsequent computer session when Li managed to solve a

succession of tasks as demonstrated in her explanation to her peer. The task was $20-12.20 =$

- 1 Li: 20 minus 10 equals 10...minus two equals] eight. Look, seven [NIS] and 20 agorot, right? [Gets feedback from the computer that the answer is correct].
- 2 Nina: Ah!! I got it, I got it...
- 3 Li: Understood?

In another task $20-15.50 =$ Li explains to her peer: "First do not pay attention to this [Agorot], look at the integers. Then do 20 minus 10 is 10, minus five is five. And five minus fifty [agorot]. And then you continue with the agorot..." after they got a positive feedback from the computer Li added to her peer: "You see, you are learning!".

Li could even apply her strategy to written individual tasks (as shown in Figure 1).

The image shows a piece of paper with four subtraction problems written vertically, each followed by a checkmark:

$$\begin{array}{r} 20 - 15.20 = 4.80 \quad \checkmark \\ 50 - 25.20 = 24.80 \quad \checkmark \\ 20 - 1.30 = 18.70 \quad \checkmark \\ 50 - 22.10 = 27.90 \quad \checkmark \end{array}$$

Figure 1: Li's writing performance

Yar's Example

The following excerpt is from the fifth lesson, solving the exercise $20-7.70$. After the teacher collected all the answers, she saw that Yar got a wrong answer, 13.30, and turned to him for an explanation:

- 59 Yar: It can be done vertically. 20 minus 7.70
- 60 Teacher: How shall I write it? I really do not know...
- 61 Yar: As if 20...[pause]
- 62 Teacher: 20, yes...[writing on the board]
- 63 Yar: Minus
- 64 Teacher: Vertical minus?
- 65 Yar: Now, you should do...[thinking]
- 66 Teacher: Come [to the board], tell me exactly where [to write 7.70]?
- 67 Yar: [goes to the board] eh, here [points right under the 0 of the 20] here...no, no...it is impossible.
- 68 Teacher: Impossible...

Yar thought that a vertical-solving procedure might help. However, it was the first time he wrote decimal numbers vertically, and he was unsure where to put

the decimal point. The teacher let him struggle with writing, repeating his conclusion, “impossible” (Line 68).

The next lesson opened with a computer session. For the first six of the ten exercises presented on the computer, Yar quickly typed a response in what seemed like a trial-and-error fashion, responding to "customer" feedback from the computer and correcting, as necessary. Then he was observed "just thinking". The exercise at hand was 20-12.80. He solved it, got positive feedback from the computer, and explained to his peer, Ron: “[20 minus 12 equals] 8, [changes one shekel to 100 agorot on the computer] 7 and 20 agorot”. Namely, he subtracted the integers, then subtracted one more integer and added the right amount of agorot. He solved the remaining three exercises in this computer session straightforwardly, employing the same strategy.

Apparently, the computer immediate feedback (and probably its non-judgmental nature) and the presence of a peer, to whom Yar verbalized the strategy he has just constructed, not only helped him construct a strategy.

In the next written individual task, Yar also succeeded:

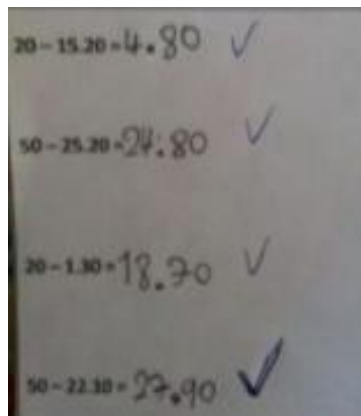


Figure 2: Yar's writing performance

DISCUSSIONS AND CONCLUSIONS

Both examples illustrate knowledge building next to the computer when the teacher find it hard to build on previous experiences due to the fragility of the knowledge. In both cases the successions of success were in lesson or between two lessons in front of the computer and at the next writing task. Probably, the experience while playing and explaining to attentive peer strengthen their fragile memory in the short term. Although the computer changes their learning experience, the strategies were not consolidated in the long term.

This complex picture is perhaps a result of the tension between LASs' active engagement in mathematics and their weaknesses. It is no surprise that teachers frequently conclude that LASs fail to acquire mathematical thinking and therefore minimize situations that require such thinking (Metz, 1978). Still the

change in their capacities and behavior points on potential of the environment. A longer research might conduct to observe longer-time stability.

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THE INFLUENCE OF ANALYTIC MODEL ON CRITICAL REFLECTIVE THOUGHT OF PRE-SERVICE MATHEMATICS TEACHERS FOR ELEMENTARY SCHOOL

Orit Broza and Ariel Lifshit

Levinsky College of Education, Israel

A total of 23 mathematics pre-service teachers learning process was examined as a result of using an analytic model designed for discourse protocols' analysis. The model contains three lenses to analyze discourse: (i) Examines the pre-service teachers' dominance in discourse; (ii) maps the types of questions, and (3) focuses on learners' reactions and comprehension performance. Results revealed that an active and dynamic process occurred, modifying teacher practice, and developing critical reflective thinking among pre-service teachers. The change occurred in two "ripples of influence": (i) Improving discourse to one promoting learning by demonstrating hypothetical scenarios and (ii) perception of the role of teachers and class management.

INTRODUCTION AND THEORETICAL FRAMEWORK

One of the challenges in teaching mathematics in general and teacher education is the existence of meaningful discourse that will lead to generalization and justification processes. Data collected in the past two years in the framework of work practicum lessons in a college of education demonstrate a difficulty among pre-service teachers to establish meaningful developing mathematical discourse for the purpose of constructing mathematical knowledge. Existing discourse is generally characterized by closed questions (e.g. IRF) and consequently, answers that do not lead to generalizations or justifications.

Michaels, O'Connor, and Resnick (2007) used the term "accountable talk" (to express the desired classroom mathematics discourse and the importance of teachers as leading the discourse. This approach was meant to involve pupils and create discourse situations whereby participants listened to one another, built ideas on one another's and asked questions to clarify or broaden any opinion. The participants create links between statements voiced in the discourse and provide reasons and justifications when disagreements arise. The teacher's role is to encourage conversation with questions such as: "Has anyone got anything to add?" or "Can someone say what he (a colleague) said in other words?", to request clarifications and explanations for what was said, to give time to think, to encourage learners who do not participate by asking to hear their opinion and to encourage agreements or disagreements about a common idea that arose in the group.

In recent decades, attempts have been made to characterize and define the concept justifications. Research literature deals mainly with high school. For example, Harel and Sowder (2007) defined justifications as a process carried out by a learner so as remove any doubt about a given hypothesis, a process made up of two secondary processes: Persuasion and becoming convinced. In 'persuasion' a learner removes the doubts of others. In 'becoming convinced' a learner (with the help of others) removes his own doubts. In elementary schools today, referring to justification as a process is very common to help processes of structuring knowledge and promoting meaningful learning. The expectation is for justification to occur within the framework of tools learners have and in accordance with their developmental stages, in other words, employing explanations for how something is solved, using supporting examples, using non-examples to refute arguments, employing definitions, rules and law and not complicated processes of proof.

Employing reflection in teacher education promotes teachers' abilities to learn from experience, initiate changes and be more aware of their understandings (Fox et al., 2011; Shulman & Shulman, 2004). Many studies have employed joint video observations to analyze teacher-learners interaction or transcripts of teachers' lessons to characterize diverse teaching styles, examine congruence between content and executing lesson aims or in order to understand unrealized teaching opportunities (Santagata & Yeh, 2013; Spitzer et al., 2011). A reflective process that combines in-depth research analysis contributes to understand processes of situational understanding (Korthagen, 2010). Hence, work experiences become not only a place to practice these teaching skills but a field in which to examine theory. Furthermore, reflective writing improves self-regulation, cognitive and meta-cognitive qualifications as well as motivation.

The aim of this research is to examine the learning that occurred among pre-service teachers who employed a reflective model developed especially for the practicum research course. The assumption is that analytical analysis will develop pre-service teachers' awareness of the way in which they conduct discourse, will reflect barriers in developing discourse, will lead to the development of optimal scenarios for situations that were not exploited, to finding possible leverage to improve discourse during research lessons and at the end of the day improve mathematical discourse in work experience classes.

The Model

Researchers constructed an analytical model containing three different lenses for analyzing discourse protocols focusing on diverse episodes of the discourse conducted in a lesson, examining the types of pre-service teachers' questions and answers in the discourse, and the connection between these and their learners' comprehension performance. The work stages of the model were as follows:

Stage A: Mark and quantify only pre-service teachers' expression in a discourse protocol and the frequency of these expressions in various episodes. Using this lens, the dominance of pre-service teachers in the discourse was examined (for example, IRF).

Stage B: Map the types of pre-service teachers' questions and answers in discourse: Closed, procedural, open, challenging questions or high thinking order questions that awaken thought and investigation (Bozo-Schwartz, 2011). Learning promoting feedback was defined as prolonging conversation through clarification questions, challenging learners to discuss with and explain to one another, repeating what learners say, linking learners' ideas in a discussion of mistakes (Bozo-Schwartz, 2011; Chapin, O'Connor & Anderson, 2009).

Stage C: Code learners comprehension performance in the discourse: providing an explanation, bringing examples, application, generalization, or justification (Perkins, 1998).

METHODOLOGY

The research was conducted within the framework of a “practicum research”, which is an integral part of the 23 pre-service teachers' practical experience in schools in the second and the third year of their studies. The course is annual and addresses improving the quality of teaching and self-examination of teaching/learning processes using questions regarding adapted teaching in general and in the field of mathematics. The researchers served as pedagogical instructors for the research group.

Research tools included 46 transcripts of complete lessons analyzed according to the three lenses of the model (23 from each semester), 46 lesson plans and 23 complete reflections on the research process.

Thematic qualitative analysis was carried out on the research work results and complete reflections of each pre-service teacher, a total of 23 pieces of work. The works were coded twice by two researchers, each separately, and there was a 95% match. The following aspects were analyzed: (a) Examining coding of types of questions asked by pre-service teachers at two points in time; (b) examining coding of learners comprehension function at two point in time; (c) discussion of link between type of question pre-service teachers asked, learners reactions, and their comprehension performance; (d) pre-service teachers' explanations and interpretations of the change occurring, if at all, and (e) pre-service teachers' ability to develop hypothetical scenarios at times when they were not satisfied with discourse progress.

RESULTS

An analysis of finding and comprehensive reflections pointed to a proactive process-taking place that led to a change in views of teaching/learning processes over and above the fundamental hypotheses of the model that sought to improve

the quality of discourse. In fact, two “ripples of influence” were created: The one at the level of awareness of classroom discourse, the role of a teacher as a mediator in structuring mathematics knowledge in class, designing and openness to hypothetical scenarios in situations where discourse did not promote learning. This type of effect will be called “local ripple”. The other ripple in a broader and more generic circle, is the influence on pre-service teachers' perceptions of the effect of discourse on classroom management norms (developing socio-mathematical norms, employing a range of interactions for learning). This type of effect will be called “expanded ripple”. Pre-service teachers who showed development of an expanded ripple also demonstrated a change in local ripple, as can be seen in Figure 1, and therefore, expanded ripple is also contained within local ripple.

Finally, the works of about 13% of the remaining pre-service teachers (3 works) did not testify to a meaningful process and analysis were paltry. Their group was defined as “no change” (Figure 1).

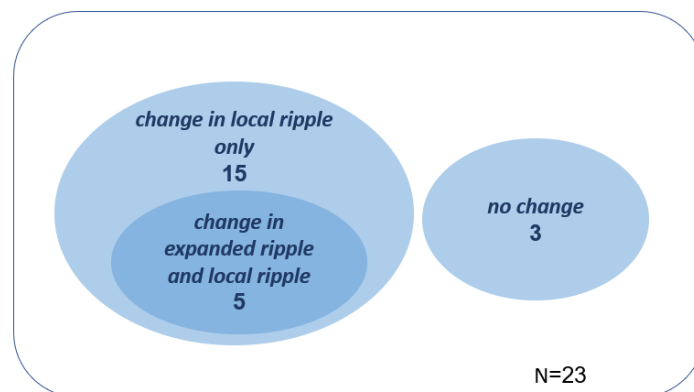


Figure 1: Types of effects on pre-service teachers

Figure 1 shows that the most frequent change was in the local ripple as researchers expected. However, among five of the 23 pre-service teachers, in addition to a change in the local ripple effects of an expanded ripple were found, a result that researchers did not expect would emerge. The following section will demonstrate episodes taken from the research works and reflections for each of the ripples and will discuss the challenges and difficulties described by pre-service teachers throughout the process.

In the next section, three presentative examples demonstrate the two ripples. Further examples will be presented at the conference.

“Local ripple” Effect: Awareness of Importance of Using Open Questions

The following episodes demonstrate how analytical analysis helped pre-service teacher N acquire insights regarding the questions she asks in her transcript analysis:

"I don't ask enough open questions. However, the open questions I do ask are mainly two types. The one is questions asking for an explanation, mainly the question "how"? – "How did you solve it?" (Line 5). "How did you get to 9 ½ ?" (Line 8), "How did you get to the whole?" (Line 85). "The second type is questions asking for other ways of solving, "Is there another way" (Line 87) "Did everyone solve it the same way?" (Line 10).

Later, N (referred as a "teacher") explained the implications of asking closed questions on the discourse with her learners.

- 80 A: 10 whole and a half less 9 and five tenths or 9 and a half.
- 83 Teacher: How many does that equal?
- 84 A: A whole
- 85 Teacher: How did you get to a whole? "I don't understand" Explain it to me.
- 86 A: Half less a half is zero, so it is nothing and 10 minus 9 equal 1 so it is whole.
- 87 Teacher: Is there another way?
- 88 B: I did 9 and a half and then I tried to add some wholes so it would reach 10 and a half and it comes out 1.
- 91 L: I did it another way. I did a half plus another half and it came out whole.
- 92 A: How did you get to a half plus another half? But what do you do with the 9?
- 93 D: How do you do it? But why exactly did you choose the half.
- 94 L: Because I know that there is 9 and a half so I added the half.
- 95 D: Ah! I understand.
- 96 L: And then another half and then it comes out a whole. Do you understand?
- 97 A: Yes.

N analyzed the above episode as the following:

"One can see that in Line 83 I asked a closed question: "How many?" And in Line 84, A gave me a fitting succinct answer. In contrast in Line 85 and Line 87 I asked open questions. Line 85 is a question requesting an explanation and the question in Line 87 encourages learners to offer further ways of solving the question. Accordingly, in Line 86, Line 88, and Line 91 there is comprehension of the explanation by the learners. In addition, one can see in Line 92- Line 93 that when learners did not understand how L solved it, they also asked "how?" and requested an explanation, like I ask for in lessons. In Line 94 and Line 96, L responded appropriately in giving an explanation."

This episode above shows the connection made by the pre-service teacher (teacher) between types of question and learners' comprehension performance, in other words, a closed question leads to a short and concise answer that actually testifies more to the existence of knowledge and less to comprehension. In the transition to a discussion of open questions, the pre-service teacher identifies the importance of using open questions to creating discussion norms among pupils who use the word "how" among themselves (Line 92, Line 93). Moreover, the pre-service teacher mainly supports a discourse being conducted among learners without her intervention but does not develop the topic around the various ways' learners raised but suffices purely with their presence in the discourse. She does not employ the practices of repeating and/or reasoning to leverage this opportunity to a discussion about the similarities and differences between the ways presented and verifying that discourse participants comprehend how the others solved the problem.

"Local ripple" Effect: Frequent Use of the Question "Why" as Feedback Promoting Learning

One of the criteria for feedback promoting learning is extending dialogue with learners and asking clarification questions requiring an explanation. Pre-service teacher K illustrated the importance of using the question "why" to encourage causality in learners' arguments and urging explanations from them.

K: "In that lesson I gave the learners a card containing a comparison between two different lengths of chains. In addition, the learners were asked to answer who had a longer chain. For this purpose, the learners had to convert the unit of measurement from centimeters to millimeters and then compare between two chain lengths."

Below is the evidence from the "centimeter" lesson held on 16 March 2016, Lines 39-40 and 46-49.

- 39 Teacher: Why is a centimeter longer than a millimeter?
- 40 HV: Because every 10 millimeters is one centimeter.
- 46 Teacher: Girls, why in your opinion is Yossi's chain longer than Daniel's?
- 47 A: Because a centimeter is longer than a millimeter and Yossi has one centimeter.

Pre-service teacher K (Teacher) summarized the importance of asking the "why" question:

"When I ask the group questions that demands reasons, I am in fact forcing them to use their existing knowledge so that they can base and explain their answers why a centimeter is longer than a millimeter and the like."

“Expanded ripple”: Changing the Discourse about Norms and Classroom Management

The following excerpts illustrate the effect of analysis on the interactions and norms by which pre-service teachers choose to manage learning.

T. “In lesson number 1, which took place in February, although most of the questions I asked were closed questions, I also posed a lot of open, reflective and meta-cognitive questions, and questions based on a high order of thinking. However, because of the nature of the lesson tasks, given to learners as personal tasks, there was almost no discourse between learners and their colleagues, but mostly reactions to questions I asked ... In the second half of the year, I used more group and pair tasks, so as to encourage mathematical discourse between learners, and indeed, it was possible to see in lesson no. 2 many more conversations between learners working in pairs, more reactions to what the other said, reasons and explanations they gave to each other, mediated by questions that I asked and also without mediation.”

T moved to group tasks instead of personal tasks to allow learners to talk among themselves. A change testifying to a different view of the teacher as a facilitator striving to structure knowledge by creating interactions between learners and not seeing herself as the source of knowledge. A perception promoting interpersonal discourse instead of IRF discourse with the teacher attests to a change in the teacher’s professional identity.

T added the effect of her learning process on organizing interactions and times within a lesson.

“...I shortened the opening part of the lesson with frontal acquisition for all, I prolonged the part of independent work experience and discussion following it and I planned a range of activities for the whole lesson that constituted demonstrating different levels of comprehension.”

DISCUSSION AND CONCLUSIONS

As mentioned, the aim of the research was to examine how employing an analytical model to analyze discourse promoting meaningful learning among pre-service teachers. The results of this research are compatible with the need for teacher education to turn work experience not just to a place to experience these skills but also to a field of theoretical research (Korthagen, 2010). In fact, what occurred here was an active process of changing views of teaching/learning processes expressed by awareness of the quality of discourse and their role as teachers in mediating teaching.

The results testify to a development in pre-service teachers’ reflective ability as expressed by critical observations of the discourse they conducted in lessons they taught and its influence on them as teachers. The model developed here led

to a significant step in pre-service teachers' ability to connect between theory and their personal teaching practices and to move to and from practice to theory and vice versa in their ambition to advance their teaching. However, from the testimonies about the first wave of influence it emerged that in most cases partially considering discourse exists characterized by practices to encourage discourse such as: Do you agree? Who wants to add? Whilst adhering to preplanning and without authentically relating to learners' answers and without deepening the discourse and promoting commitment to all participants. The change, therefore, is firstly on the level of questions alone.

The case of the expanded ripple teaches us that a pre-service teacher can metaphorically distance herself from the conversation and observe the group discourse from the side and plan steps that perhaps were not considered in lesson planning. Distancing allows them to develop the ability to listen to the developing authentic "here and now" discourse between learners, detachment from original planning that is likely to fixate and re-enter the conversation when they feel more confident.

The key conclusion emerging from this research is that using an analytical model to analyze discourse among pre-service teachers has great multi-directional potential, which is simple and clear and demonstrates how it can be integrated into the curriculum in an empowering and structured manner, and as an integral part of the work experience. As such it meets the need for a link between theory and practice in teacher education.

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PRACTICES ON THE DISCRETE RANDOM VARIABLE PROPOSED IN THE MATHEMATICS CHILEAN CURRICULUM OF SECONDARY EDUCATION

Belén Carrera¹, Luis Pino-Fan¹, Hugo Alvarado², & Jesús G. Lugo-Armenta¹

¹Universidad de Los Lagos, Chile

²Universidad Católica de la Santísima Concepción, Chile

The present research report takes one of the key notions of statistics and probability as an object of study: the random variable, studied from its discrete character. Supported by the theoretical-methodological tools from the Onto-Semiotic Approach (OSA) of mathematical cognition and instruction, it was possible to define the reference meaning that diverse authors have built upon this mathematical object, in order to study the representativeness of the institutional meaning and the types of mathematical practices expected and fostered by the Chilean mathematics curriculum for secondary education, to learn the discrete random variable. The context of the proposed tasks plays a key role, and in our work the possible relations between these and the meanings of the discrete random variable promoted in textbooks, are also analyzed.

RANDOM VARIABLE AS A FUNDAMENTAL IDEA

The advances in science and technology, the exponential growth in data collection systems, a globalized world that bombards day by day its citizens with information through figures and graphs, have generated the need for new analytic tools for the people, that could help them in the correct interpretation of the information surrounding them. A key tool in this process is the so-called statistical culture. Batanero (2002) explains that statistics have had a fundamental role in the development of modern society, as it has provided a battery of methodological tools to analyze variability, relations among variables, design of studies and experiments, and improve the predictions to make decisions in situations of uncertainty.

Because of the foregoing, the need to count with citizens culturized on statistics have become an objective for leaders of diverse nations, who have promoted the incorporation of statistics and probability in formal education. In this sense, researchers and teachers have contributed to define curricular lines that allow addressing these topics. The teaching of stochastic ideas throughout the education process, began to be conceived by Bruner (1959; cited in Ruiz, 2013), who in September of 1959 in the Woods Hole Conference, proposed the idea of a spiral curriculum consisting of a series of possible fundamental ideas to teach in different levels of complexity from preschool to university. Years later,

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Heitele (1975) boldly proposed ten fundamental ideas in stochastic, based on psychological and epistemological reflections, that is to say: Expressions of belief, the probability field, independence, the addition rule, equidistribution and symmetry, combinatorics, urn model and simulation, stochastic variable, the law of large numbers, and sample.

Heitele established the random variable as a fundamental idea from three perspectives: the epistemological in which plays a basic role in the mathematization of probability through history; the psychological in which the intuition of magnitudes where chance participates, arises earlier than that of random experiment; and as an explanatory model in which plays a key role in three aspects, its distribution, its expectancy and operations between random variables. Nevertheless, even when the importance of the random variable is well-known, how does the mathematics curriculum and textbooks in the Chilean context address the study of this notion? The present research report presents the advances of a developing study about the meanings of the (discrete) random variable, expected and promoted by the mathematics Chilean curriculum (understood as the duo <Plans of study and textbooks>) and the representativeness of those meanings regarding the reference meaning of the random variable.

THEORETICAL FRAMEWORK

The present work uses some theoretical-methodological notions of the Onto-Semiotic Approach (OSA) of mathematical cognition and instruction. To study a mathematical concept, it is necessary to comprehend its characteristics, scopes, fields of action, among other elements that might compose it, and thus having a deeper understanding of that intended to be observed; it is necessary to know the meaning of such mathematical object. It is possible to determine the meaning or meanings of a given mathematical object from the historical development of it through time. In this sense, Pino-Fan, Godino and Font (2011), propose that the reference meaning is understood as the systems of practices that are used as reference to elaborate the meanings that are intended to be included in a study process. For a concrete educational institution, the reference meaning will be a part of the holistic meaning of the mathematical object.

In the OSA, the notion of mathematical practice is of great relevance, which refers to any performance or manifestation (verbal, graphic, etc.) carried out by someone in order to solve mathematical problems, to communicate the solution to others, to validate the solution and generalize it to other contexts and problems (Godino and Batanero, 1994, p. 334). The practices can be idiosyncratic of a person (personal practices) or shared within an institution (institutional practices). Furthermore, in the OSA the anthropological premise of socio-epistemic relativity of the system of practices, of the emergent objects

and the meaning, is assumed. Thus, the meaning of a mathematical object is understood as the system of practices that a person makes (personal meaning) or shared in the heart of an institution (institutional meaning) to solve a type of situations-problems.

Pino-Fan, Godino and Font (2011) indicate that the partial meaning of the mathematical objects (that constitute the global reference meaning) have associated epistemic configurations (situations/problems, linguistic elements, concepts/definitions, properties/propositions, procedures and arguments) that are mobilized when solving certain problems situations, in given historical problems, and that gave rise to the emergence, evolution, formalization and generalization of a given mathematical object, in this case, the random variable.

REFERENCE MEANING OF THE RANDOM VARIABLE

Based on the study of diverse historical stages of the random variable evolution, according to different authors (e.g., Ruiz, 2013; Alvarado, 2007; Ortiz, 2002; Heitele, 1975), it is shown that the mathematical object variable is the result of numerous generalizations made through an evolution of more than 800 years. Thus, it was possible to identify four meanings of the random variable, which are described below.

Meaning 1: The Random Variable as a Variable of Interest

One of the first problem areas in which the idea of random variable is observed, is the one linked with games of chance. However, the more formal mathematical analysis of them, appeared in relatively recent times (García, 1971). The ideas depicted in these works are not very formal, as the existence of variables or distributions in a general form, is not mentioned. Nevertheless, variables are defined for particular cases and in certain cases their distributions are considered. Different mathematicians were attracted by the problem of estimating the equitable wager in the game of chance, which led them to implicitly consider random variables and distribution. In modern terms, their main interest was the mathematical expectation of the variable. Such was the case of Fournival, Cardano or Galileo, who motivated by their interest to find the best wager in games of chance, were devoted to study the possible outcomes for rolling three dice. At a later stage, Pascal and Fermat, based on the ideas of Fournival, Cardano and Galileo, started with the probability theory in search of the solution for the equitable wager, further on, is Huygnes who manifests the need to think about a variable of study, that is to say a *variable of interest* in consideration of the context. In the analysis of his solution, Huygens makes explicit the needed variable to analyze: “IN the first place we must consider the number of Games still wanting to (win) either Party” (Huygens, 1714/1657, p.4), for that, he situates in the context of the problem.

Meaning 2: Random Variable as Magnitude

De Moivre (1756), established a change regarding previous books of probability. Latin began to be replaced by writing in or simultaneously translating into English or the native language of the author, which made that a specialized vocabulary would develop faster by working with a living language. Furthermore, it showed a different conceptual approach, in which he clearly separated the probability of an outcome from its value or the expectation. In its third edition (De Moivre, 1756) established the paradigm of mathematical probability, leaving behind the philosophical problems and forming the theoretical basis to all his propositions (Sylla, 2006).

According to Pearson (1924), De Moivre wrote the first treatment of the probability integral and the essence of the Normal Curve, contributing with diverse tools for the field of probability. In that age, scientists used the idea of variable connected with the study of mathematical analysis. It was commonly called quantity or variable magnitude, which evidenced its character linked with measurement, process in which, the quality could take different values.

Meaning 3: The Random Variable as Statistical Variable

In parallel to the development of the probability theory, through the resolution of game problems, emerged the birth of statistics through the gathering and description of social or economic data. The human has had the need to do counts and representations that could be considered simple statistical recounts from time immemorial. The need to know and plan, in the sense of knowing what is at hand and make accessible and manageable that information to take decisions, caused that little by little politicians, traders and militaries would carry out increasingly sophisticated census and counting.

Thus, the statistical variable is associated with the observation and description of a sample from a dataset. Following this idea, Ríos (1967) proposed that the statistical variable describes the set of values obtained in the data by making the experiment a concrete n number of times, then, if we consider a random experiment S and make a certain n number of tests relative to the same, we obtain a set of observations called *random sample of extension n* . This set of results will provide a statistical table in which certain values of the variable correspond certain frequencies. To such “variable, that only represents the n results of n executions of the S random experiment will be referred as statistical variable” (Ríos, 1967, p.70).

Meaning 4: The Random Variable as a Function

Hawkins and cols. (1992), consider the concept of random variable as a function with numerical values which domain is a sample space. Borovcnik and cols. (1991) indicate that a variable is random when its value is determined as a result of a random experiment; it also establishes that to characterize a random

variable we need to know the set of all its possible results and the probabilities associated to each of them.

Then, a random variable is defined as a function of the sample space E in the set of real numbers R . Not any function can be a random variable. It is necessary that, for each interval I , the set should be an event of the sample space and, thus, should have a well-defined probability. This guarantees that the random variable would carry the P probability that is defined over the E sample space to the real line.

On the basis of that, Ortiz (2002) identifies the following elements of the meaning of the random variable as a function:

RV 1: The random variable takes its values depending on the results of a random experiment.

RV 2: It is a function of the sample space in R .

RV 3: Is characterized through the distribution of probability, along with the values that takes with its probability.

RV 4: It is required that, for each I interval of R , the original set would be the event of the sample space.

RV 5: A random variable defines a measurement of probability over the set of real numbers.

RV 6: For each random variable we can define a function of distribution in the following way:

$$\begin{array}{ll} 1) R: & \longrightarrow [0,1] \\ 2) x: & \longrightarrow F(x) = P(\xi \leq x) \end{array}$$

RV 7: The function of distribution of a random variable is a real function of real variable, monotonous non decreasing.

RV 8: The function of distribution of a random variable determines on a biunivocal form the distribution of probability.

RV 9: Be $(x_i, p_i) i \in I$ the distribution of probability of a discrete random variable. The media or mathematical expectation is defined as $E[\xi] = \sum_{i \in I} x_i p_i$. This concept expands the idea of media in a random variable.

RV 10: The mode is the most likely value of the variable.

RV 11: The median is the value of the variable by which the function of distribution takes the $1/2$ value. Then, the probability that one random variable would take a lower or equal value to the median is exactly $1/2$.

METHODOLOGICAL ASPECTS OF THE STUDY

The sample selected corresponds to the mathematics textbook of Chilean secondary education. Secondary education in Chile considers 6 levels, from 7th grade (12 years old) to 12th grade (15 years old). Each year the Chilean Ministry of Education (MINEDUC), provides textbook for free to all the students from public institutions. The elaboration of such textbooks is awarded on a tender basis, thus throughout secondary education it is observed that different editorials oversee the elaboration of them, as we can see in Figure 1.

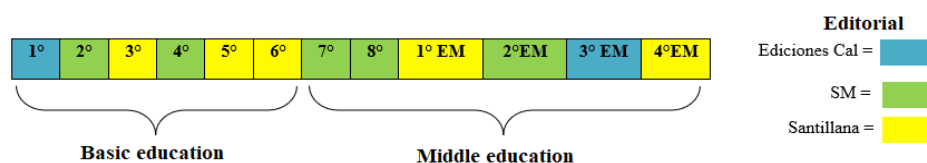


Figure 1: Representation of the editorials in charge of the edition of textbooks in Chile for each educational level

For the purposes of the present work textbooks from secondary education were selected, excepting those of 11th and 12th grade, as they are outdated in relation to the national curriculum published by the end of 2019. Along with the mathematics textbooks of 7th, 8th, 9th, 10th grade, the 8th and 11th grade history textbooks were analyzed, because of the relationship between the axis of statistics and probability with the objectives set by the history subject around the development of skills such as critical thinking.

EXAMPLES OF DEVELOPMENT OF THE ANALYSIS

To facilitate the analysis, a database which user screen we can see on Figure 2, was created. In such database the pictures of the proposed tasks in textbooks are uploaded and further analyzed. First, the general information of the task, the level, the subject, the code of the task, the section analyzed and the page of the original document from which it was extracted, are entered.

Figure 2: Database for analysis of typologies of tasks and meanings of the R. V

After that, the context present on the task is categorized. Based on a historical study 7 possible contexts were determined: (a) games of chance, considering every task involving dices, cards, coins, picking from a bag and others; (b) census and records, considering every task related with the counting of a population and its characteristics; (c) natural and biological sciences, considering any task related with natural environment, health, flora and fauna; (d) physics and astronomy, taking into account every task concerning stars and physical processes such as sound, speed, among others; (e) observation and interpretation of data from polls, entails every task in which interpretation of poll data is not determined by a particular population and which size is lower than that of a census, as well as, the data recording in matches of different types of sports, is involved; (f) formal, considering tasks which context is the use of axioms and formal definitions of the variable; and (g) without context.

Once the context is defined, the meaning which the task is trying to address is identified, this is done through the statement itself of the task and of the elements of the epistemic configuration intended to be used in the practices that solve the task. These meanings are: (S1) as variable of interest: (S2) as magnitude: (S3) as statistical variable; (S4) as function. Additionally, problems without classification were contemplated for such cases in which the task mobilizes more than one or any meaning, with or without context.

Once the context and meaning are identified, the types of activated representations or the ones expected to be activated by the task are analyzed, say: verbal, graphic, symbolic, tabular or iconic. Moreover, a differentiation between the previous representation, which we understand as the ones that should, originally, interpret and decode the student (or subject) with the aim of comprehending and facing the task; and emergent, seen as those that emerge as part of the subjects answers (or expected answers, if seen from an institutional point of view), is made. Depending on the type of task, it is possible that apart from a previous representation and an emergent one, may arise a transitory, necessary to address before the emergent representation.

Finally, and particularly for meaning four (S4), random variable as function, the intentional elements present in the task are identified, as well as, the typology of problems, based on the previously mentioned proposal of Ortiz (2002).

FINAL REFLECTIONS

From the analysis performed so far, we have determined that the intended meanings of the mathematics Chilean curriculum about the notion of random variable seem not to be representative of the holistic meaning of reference. While it possible to distinguish tasks that promote the S1 and S3 meanings, in the earlier stages of secondary education (7th and 8th grade) S2 meaning cannot be observed. On the other hand, despite 10th grade provides a complete section entitled random variable, in which this function is defined as that which takes

values according to the results of an random experiment, promoting S4, in 9th grade there is no visible definition of the variable provided, in detriment of an adequate transition between meanings. It appears that the existent relation between the statistical variable and the random variable is not promoted, restricting the first to a mere characteristic of a population, omitting the conception of this as the description of an n number of experiments, which would allow favoring a better transition of the students from meaning 3 to meaning 4. Finally, regarding the contexts of work, games of chance continue to be present in greater extent, followed by the observation of polls and census and records. Concerning the variables in study there is a tendency towards discrete variables in lower levels, it is worth noticing that, although there are tasks that promote the distinction between variables of discrete and continuous kinds in the first levels, this distinction seems to be lost as the higher levels are reached.

Acknowledgements

This work has been funded by Project Fondecyt 1200005 granted by the Comisión Nacional de Investigación Científica y Tecnológica (CONICYT) of Chile.

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RATIO COMPARISON PROBLEMS: CRITICAL COMPONENTS AND STUDENTS' APPROACHES

Salvador Castillo¹ and Ceneida Fernández¹

¹University of Alicante, Spain

This study focuses on examining secondary school students' approaches in ratio comparison problems. Two hundred forty-eight secondary school students (12-16 years old) solved two ratio comparison problems that can be interpreted as a couple of expositions or compositions. Three main students' approaches were identified according to whether they identified the relative quantities: relative comparison, relative trend, and non-relative comparison. Furthermore, the subcategories identified in the relative trend and non-relative comparison approaches showed students' difficulties with critical components of the problems: difficulties in interpreting the referent in the comparison, in identifying the multiplicative relationship, and with the norming techniques.

THEORETICAL AND EMPIRICAL BACKGROUND

The understanding of the concepts of ratio and proportion and the development of proportional reasoning have been broadly studied since the 80s (Cramer & Post, 1993; Lobato & Ellis, 2010; Tourniaire & Pulos, 1985). Many studies have reported students' difficulties in distinguishing proportional from non-proportional situations and the effect of some variables of the problem (such as the context, and the nature of ratios) on the students' success levels and strategies (Alatorre & Figueras, 2005; Van Dooren, De Bock, & Verschaffel, 2010). Most of them has used missing-value problems (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2012; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005) where three quantities of a proportion are known and the fourth must be found. However, little is known about how primary and secondary school students understand and use the ratio concept when solving ratio comparison problems (Alatorre & Figueras, 2005; Nunes, Desli, & Bell, 2003) where two ratios are given and have to be compared.

One of the challenges in these problems is that they involve the understanding of intensive quantities. Nunes et al. (2003) showed that when primary school students construct an understanding of intensive quantities, they have to face two challenges: thinking in terms of proportional relations and understanding the connection between the intensive quantity and the two extensive quantities which are related to it. These authors also show that primary school students have difficulties solving ratio comparison problems that involve intensive quantities.

In the understanding of the ratio concept, Freudenthal (1983) highlights the importance of considering situations in which the ideas of “relatively” and “norming” are required. The idea of “relatively” in the sense of “put something in relation to” involves the use of the term ratio as a relational number that relates two quantities in one situation and projects this relationship onto a second situation in which the relationship between the two quantities remains the same (Smith, 2002). Norming describes the process of reconceptualising a system in relation to some fixed unit or standard (Lamon, 1994).

Ratio comparison problems involve both ideas, relatively and norming. In these problems, the multiplicative relationship that exists between the quantities can be equal or unequal, and represents “relative quantities”, that is, “quantities put in multiplicative relationship with other quantity of reference” (called “the referent”) (Gómez & García, 2015, p.267). These problems can be interpreted as couples of expositions or compositions (Freudenthal, 1983). For instance, given the following ratio comparison problem: *In the greengrocer A, for each 2 kg of apples paid you get 3 kg. In the greengrocer B, for each 3 kg of apples paid you get 4 kg. If the price of a kilogram is the same in the two greengrocers, which offer is more advantageous?*

If it is interpreted as a couple of expositions, there is a set of greengrocers $\Omega = \{\text{greengrocer A, greengrocer B}\}$ and two functions ω_1 and ω_2 which assign a magnitude to each element of the set. The function ω_1 can assign the amount paid to each greengrocer (2kg in greengrocer A and 3kg in greengrocer B) or the amount free (1kg in greengrocers A and B). The function ω_2 assigns the amount purchased to each greengrocer (3kg in greengrocer A and 4kg in greengrocer B). The ratios that can be compared are: *amount paid (P) / amount purchased (PU)* (Table 1) and *amount free (F) / amount purchased (PU)* (Table 2).

	Greengrocer A	Greengrocer B
$\omega_1: \Omega \rightarrow \text{Amount paid}$	$P_A = 2$	$P_B = 3$
$\omega_2: \Omega \rightarrow \text{Amount purchased}$	$PU_A = 3$	$PU_B = 4$
Compared Ratio	$\frac{P_A}{PU_A} = \frac{2}{3}$	$\frac{P_B}{PU_B} = \frac{3}{4}$

Table 1: Couple of expositions: amount paid (P) / amount purchased (PU).

	Greengrocer A	Greengrocer B
$\omega_1: \Omega \rightarrow \text{Amount free}$	$F_A = 1$	$F_B = 1$
$\omega_2: \Omega \rightarrow \text{Amount purchased}$	$PU_A = 3$	$PU_B = 4$
Compared Ratio	$\frac{F_A}{PU_A} = \frac{1}{3}$	$\frac{F_B}{PU_B} = \frac{1}{4}$

Table 2: Couple of expositions: amount free (F) / amount purchased (PU).

As a couple of compositions, it is interpreted as a class partitioning $\Omega_A = \{\text{amount free, amount paid}\}$ and $\Omega_B = \{\text{amount free, amount paid}\}$ of two universes (greengrocer A and greengrocer B) attained according to the same principle, and two functions ω_1 and ω_2 , each function representing a magnitude. The function ω_1 assigns their respective kg to the amount free and the amount paid of greengrocer A; while ω_2 assigns their respective kg to the amount free and the amount paid of greengrocer B. The ratio that can be compared is *amount free (F) / amount paid (P)* (Table 3).

	Amount free	Amount paid	Compared Ratio
$\omega_1: \Omega_A \rightarrow \text{PU}_A$ (Greengrocer A)	$F_A = 1$	$P_A = 2$	$\frac{F_A}{P_A} = \frac{1}{2}$
$\omega_2: \Omega_B \rightarrow \text{PU}_B$ (Greengrocer B)	$F_B = 1$	$P_B = 3$	$\frac{F_B}{P_B} = \frac{1}{3}$

Table 3: Couple of compositions: amount free (F) / amount paid (P).

In the ratio comparison problems, the norming techniques allow “the unification of the antecedents (numerator) or consequents (denominator) of ratios in order to favor the comparisons” (Gómez & García, 2015, p.267), what can be done by procedures such as unit rate (obtained by quotient), fraction strategy (equivalence of fractions), cross product, or building-up (Cramer & Post, 1993).

In this study, we consider as critical components of ratio comparison problems: the multiplicative relationships, their equality or inequality, and the quantities used as referents (Gómez & García, 2015). We are interested in situations that can be interpreted both as a couple of expositions or compositions and that involve the necessity to apply norming techniques. Previous studies have focused on ratio comparison problems showing students’ success levels, strategies and the effect of some variables, such as the context or the numerical structure on students’ strategies (Alatorre & Figueras, 2005; Nunes et al., 2003). However, studies focused on how secondary school students solve ratio comparison problems examining the relationship between the critical components of the problems and students’ performance are scarce (Gómez & García, 2015; Monje & Gómez, 2019, both studies with pre-service teachers). The research question is: which are the secondary school students’ approaches when solving ratio comparison problems?

METHOD

Participants were 248 secondary school students from 7th grade (n=68), 8th grade (n=52), 9th grade (n=64) and 10th grade (n=64). There was approximately the same number of boys and girls in each age group, and students were from mixed socio-economic backgrounds. Participants solved the following two ratio comparison problems (problem 1 has been described above) that involve

intensive quantities and can be interpreted as a couple of expositions or a couple of compositions:

Problem 1 (Sale). In the greengrocer A, for each 2 kg of apples paid you get 3 kg. In the greengrocer B, for each 3 kg of apples paid you get 4 kg. If the price of a kilogram is the same in the two greengrocers, which offer is more advantageous?

Problem 2 (Mixture). To obtain chocolate shake, you need milk and chocolate. John used 450 ml of milk and got 600 ml of shake while Mary used 750 ml of milk and got 900 ml of shake. If both used the same grams of chocolate, which shake would have a stronger chocolate taste?

Three researchers, independently, analysed the students' answers to identify categories of students' approaches, considering:

- The idea of *relatively*. If students identify the relative quantities, i.e., quantities on a multiplicative relationship with another quantity of reference:
 - Identification of the multiplicative relationship.
 - Identification of the referent in the comparison.
- The idea of *norming*. If students use the norming techniques properly to compare ratios.

Then, agreements and disagreements were discussed until we reached an agreement with regard to the final categories of students' approaches. Final categories identified are shown and exemplified in the results section.

RESULTS

In this section, students' approaches are described and exemplified. Then, the frequencies of these categories in each problem and grade are shown.

Students' approaches

Three main categories of students' approaches were identified according to whether students identified the relative quantities: relative comparison, relative trend and non-relative comparison.

Relative comparison

In this category, students identified the relative quantities: "quantities put in multiplicative relationship with other quantity of reference". They were able to obtain and compare ratios, applying a norming technique correctly. The subcategories identified differed in the ratio and referent used.

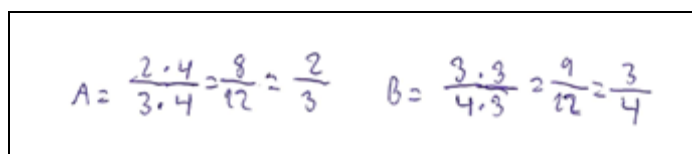
Some students interpreted the problem as a couple of compositions using the ratio *amount free / amount paid* in problem 1 and *chocolate amount / milk amount* in problem 2. For instance, a 9th-grade student used a building-up procedure in problem 1 looking for a common multiple: "*A is better than B*

because if you pay 6kg in A you get 3kg for free, but if you pay 6kg in B you only get 2kg”.

Other students interpreted the problem as a couple of expositions using the ratios *amount paid / amount purchased* or *amount free / amount purchased* in problem 1 and *milk amount / shake amount* or *chocolate amount / shake amount* in problem 2. These ratios differ in the referent used for the comparison. For instance, a 10th-grade student established the price of 1€ per kg paid, and calculated the price paid for 1 kg (unit rate) in each greengrocer regarding the kg purchased: “ $2€/3kg = 0.67€/kg$ in the greengrocer A and $3€/4kg = 0.75€/kg$ in the greengrocer B, so A is the better option since a kg is cheaper”.

Relative trend

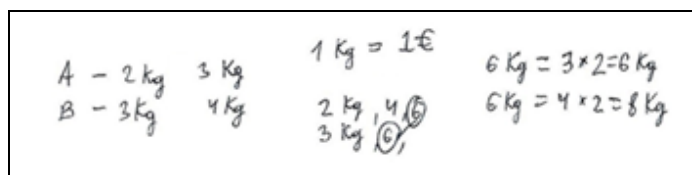
This category includes students’ approaches that showed evidence of identifying the relative quantities, but they had difficulties in some critical components. Two subcategories were identified: difficulty with the referent and difficulty with norming techniques. In the first subcategory, students were able to obtain the ratios applying a norming technique correctly, but the comparison according to the referent was incorrect. For example, an 8th-grade student obtained the ratios correctly using fraction equivalences (Figure 1), but had difficulties in interpreting the antecedents concerning the consequents (referent), since he said that “B is cheaper”. In this approach, the difficulty was the loss of meaning of the referent when they applied norming techniques (Gómez & García, 2015).



$$A = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12} = \frac{2}{3} \quad B = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12} = \frac{3}{4}$$

Figure 1: Example of a student who had difficulties with the referent.

In the second subcategory, difficulties are related to norming techniques. For example, a 9th-grade student used a building-up strategy to find a common multiple for the amounts paid (6 kg), but he did not extend it correctly to the amounts purchased (Figure 2): “B because in case you want 6 kg, with the same price, you will get more quantity of apples”.



$$\begin{array}{lcl} A - 2\text{ kg} & 3\text{ kg} & 1\text{ kg} = 1\text{€} \\ B - 3\text{ kg} & 4\text{ kg} & 2\text{ kg } 4\text{€} \\ & & 3\text{ kg } 6\text{€} \end{array} \quad \begin{array}{l} 6\text{ kg} = 3 \times 2 = 6\text{ kg} \\ 6\text{ kg} = 4 \times 2 = 8\text{ kg} \end{array}$$

Figure 2: Example of a student who had difficulties with a norming technique.

Non-relative comparison

This category includes students’ approaches that did not show evidence of identifying the relative quantities (they did not identify the multiplicative

relationship). Five subcategories were identified: ignoring data, additive answers, affective answers, incomprehensible answers, and blank answers.

Students who ignored data paid attention only to some data of the problem. For instance, some students compared only the amounts paid in problem 1, ignoring the relationship with the amount free or the amount purchased. The following 10th-grade student wrote: *“The cheapest offer is the greengrocer A because you pay only 2 kg of apples while you pay 3 kg in greengrocer B”*. In the additive answers, students related the quantities in absolute terms. For example, an 8th-grade student answered in problem 1: *“It is the same because in both greengrocers you can save 1 kg”*. In the affective answers, students based their answers on personal interpretations. A 7th-grade student said: *“the choice depends on the number of apples that you want to buy”*. Incomprehensible answers are those in which students did operations without sense.

Students' approaches by problem and grade

Table 4 shows the percentages of each category by problem and grade. Globally, students were more successful in problem 2 (mixture; 81.1%) than in problem 1 (sale; 30.1%), due to their difficulties in interpreting the quantities in relative terms. The average for all the grades considering both problems was 55.6%. Particularly, in problem 1, more than 50% of the students' approaches from 7th to 9th-grade were non-relative comparisons. Furthermore, although the relative comparisons increased from 7th to 10th-grade in both problems, the average in 10th-grade was 67.2%, so difficulties with ratio comparison problems persisted at the end of secondary education.

Grade	Problem 1 (sale)			Problem 2 (mixture)			Total
	Relative comparison	Relative trend	Non-relative comparison	Relative comparison	Relative trend	Non-relative comparison	
7 th	25.0%	14.7%	60.3%	70.6%	7.4%	22.0%	47.8%
8 th	25.0%	19.2%	55.8%	86.5%	3.8%	9.7%	55.8%
9 th	20.3%	18.8%	60.9%	82.8%	4.7%	12.5%	51.6%
10 th	50.0%	15.6%	34.4%	84.4%	1.6%	14.0%	67.2%
Total	30.1%	17.0%	52.9%	81.1%	4.4%	14.5%	55.6%

Table 4: Percentage of each category by problem and grade.

Table 5 shows the percentage of each subcategory in problems 1 and 2. In problem 1, 30.1% of students' approaches were relative comparisons. Particularly, the 23.3% interpreted the problem as a couple of expositions, while the 6.8% interpreted the problem as a couple of compositions. In problem 2, 81.1% of students' approaches were relative comparisons. Specifically, 68.2%

interpreted the problem as a couple of compositions and 12.9% interpreted it as a couple of expositions.

Subcategories		Problems	
		1	2
Relative comparison	As a couple of compositions	6.8%	68.2%
	As a couple of expositions	23.3%	12.9%
Relative trend	Difficulty with the referent	5.6%	2.0%
	Difficulty in norming	11.4%	2.4%
Non-relative comparison	Ignore data	19.0%	4.3%
	Additive answers	8.1%	4.3%
	Affective answers	7.3%	1.6%
	Incomprehensible or blank answers	18.5%	4.3%

Table 5: Percentage of the subcategories in problems 1 and 2.

In problem 1, students' difficulties with the relative quantities are explained by the 17% of students who used a relative trend approach having difficulties with the referents or the norming techniques, and by the 52.9% of students who did not identify the relative quantities (providing a non-relative comparison approach). Of this last group, we can highlight the subcategories: ignored data, and incomprehensible or blank answers. In problem 2, only 4.4% of the students had difficulties with the referents or the norming techniques, and 14.5% of students' approaches were non-relative comparisons. Of the last group, the most frequent subcategories were: ignore data, additive answers and incomprehensible or blank answers.

DISCUSSION AND CONCLUSIONS

Results provide information about secondary school students' approaches when they solve ratio comparison problems with intensive quantities considering the critical components of the problems. Three main students' approaches were identified according to whether secondary school students identified the relative quantities: relative comparison, relative trend, and non-relative comparison. These approaches coincide with the results obtained by Monje and Gómez (2019) with pre-service teachers, extending them to secondary education. In addition, the subcategories identified in the *relative trend* and *non-relative comparison approaches* showed students difficulties with some critical components: difficulties with the referent in the comparison, difficulties in identifying the multiplicative relationship, and difficulties with the norming techniques.

Results about the percentages of each category along grades have shown that students' success levels increased from 7th to 10th-grade in both problems. However, difficulties with intensive quantities (in ratio comparison problems) persisted at the end of secondary education. Nunes et al. (2003) showed that

primary school students have many difficulties in relation to intensive quantities. Our study has shown that there is a positive evolution throughout secondary education, but difficulties persist.

Finally, students were more successful in the mixture problem than in the sale problem. This result contradicts previous research that has stated that mixture problems are more difficult (Alatorre & Figueras, 2005; Tourniaire & Pulos, 1985) while other research has not found differences in primary school students' performance (Nunes et al., 2003). The characteristics of our problems could explain this result. Both problems have one of the quantities unified. In problem 1, although this quantity is not given explicitly, the amount free was the same for both greengrocers (1 kg). In problem 2, this quantity is given explicitly in the formulation of the problem: the chocolate amount is the same in both shakes. If students identified this data, they only needed to compare the other quantities, without performing calculations. This raises a question: would have the success in the sale problem been greater if students had asked directly about the amounts free given?

The characterization of the students' approaches obtained in this study can provide information for the design of classroom interventions aimed at overcoming the difficulties encountered in ratio comparison problems.

Acknowledgements

This research was supported by the project EDU2017-87411-R from MINECO (Spain) and by the pre-doctoral grant PRE2018-083765 from the same project.

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LEARNING FROM LESSONS: A FRAMEWORK FOR CATEGORIZING DIFFERENT FORMS OF MATHEMATICS TEACHER IN-CLASS LEARNING IN AUSTRALIA, CHINA, AND GERMANY

Man Ching Esther Chan¹, Yiming Cao², Daniel Barton³, Milena Damrau³,
Chongyang Wang², & David Clarke¹

¹The University of Melbourne, Australia

²Beijing Normal University, China

³Bielefeld University, Germany

This paper reports on a coding framework for categorizing different forms of mathematics teacher in-class learning. Utilizing a research design that stimulates teachers' reflection on their lesson planning and teaching, a coding framework was developed as part of this international project to categorize teachers' interview statements in relation to their learning. This paper explains the theory of teacher learning which underpins this project and reports on the development and implementation of the coding framework with illustrative case study examples from three countries (Australia, China, and Germany).

BACKGROUND

Organized professional development programs or activities are increasingly relied upon in different education systems to enhance teachers' professional knowledge and improve classroom practices, with the ultimate goal of fostering student learning and achievement gains (Borko, Jacobs, Eiteljorg, & Pittman, 2008). Nonetheless, participation in organized professional development programs is not the only means for teachers to develop professionally. The Learning from Lessons project (Chan et al., 2017) was designed to focus on what we called teacher "in-class learning": Teacher learning that takes place as part of teachers' day-to-day practice, particularly in relation to their lesson planning and teaching.

Theories of Teacher Learning

Focusing on the mechanism of teacher learning, Boylan, Coldwell, Maxwell, and Jordan (2018) reviewed five theoretical models of teacher professional learning (Clarke & Hollingsworth, 2002; Desimone, 2009; Evans, 2014; Guskey, 2002; Opfer & Pedder, 2011). These models intend to have wide applicability and have variously been used to inform the design, analysis, and evaluation of teacher professional development activities. Boylan et al. found differences and inconsistencies between the models, particularly in terms of the

components and domains of change included (e.g., teacher practice, student outcomes, teacher beliefs and attitudes, and school and learning activity systems), scope (micro, meso, or macro), theory of learning (socially situated experiential, social constructivist, or cognitive), the location of agency in directing or facilitating professional learning (mainly within the teacher or involving broader structures, processes, or systems), and the philosophical foundation (e.g., sociological positivist, social constructivist, or complexity theory). Rather than providing a unified “meta” model of teacher professional learning, Boylan et al. argued for the need to seek multiple answers in understanding the complexities of teacher professional learning.

The current project draws from the Interconnected Model of Teacher Professional Growth (Clarke & Hollingsworth, 2002; Clarke & Peter, 1993; Peter, 1996) which suggests that the process of teacher professional growth is non-linear and recursive. A unique feature of the model is the emphasis on the processes of enactment and reflection in connecting and facilitating changes in teachers’ professional environment, where *enactment* involves putting into action a new idea, a new belief, or a newly encountered practice (Clarke & Hollingsworth, 2002, p.953) and *reflection* involves “active, persistent and careful consideration” (p.954).

A pilot study (Clarke, Clarke, Roche, & Chan, 2015) was undertaken in Australia to identify empirical evidence of teacher learning based on teachers’ reflection of their lesson planning and teaching. Two forms of evidence were found: Teachers’ declarative “claim to know” (epistemic claim) and an observable or recounted change in the individual practice (adaptive practice). Examination of further cases in Australia (Chan, Roche, Clarke, & Clarke, 2019) found different mechanisms of teacher learning evident in teachers’ epistemic claims – *consolidation* of existing knowledge and beliefs, and *realization* of new knowledge and beliefs. It is suggested that these two mechanisms both contribute to teacher learning, particularly in day-to-day teaching practice as teachers expand their existing knowledge base (consolidation) and form new knowledge and beliefs (new realization).

To further refine and validate the learning categories and investigate the nature of teacher professional learning, this research seeks evidence of these learning categories in cases beyond Australia. It addresses the research question: To what extent do the learning categories of consolidation and new realization, and adaptive practice, apply to teachers in Australia, China, and Germany? Answering this question provides an important step towards cross-country comparison of teacher learning in the project.

RESEARCH DESIGN

The case study data reported in this paper came from an international research project, which aimed to investigate the mathematics teachers’ in-class learning

in Australia, China, and Germany (Chan et al., 2017). The project combined focused case studies with an online survey of mathematics teachers' focus of attention and consequent learning in the three countries.

Participants

Case studies were undertaken of three teachers, teaching Year 4 in China, and Year 5 in Australia and in Germany. The reason for the difference in year levels was to accommodate the difference in lesson topics commonly taught in the three countries, where some of the Year 5 topics taught in Australia and Germany are taught in Year 4 in China. The three teachers were male and in their thirties. The Australian teacher (AU_T5) had 5 years' teaching experience and was trained as a generalist primary teacher. The Chinese teacher (CH_T4) had 18 years' teaching experience and was trained as a specialist mathematics teacher. The German teacher (DE_T5) had 6 years teaching experience and was trained as a secondary (grammar) schoolteacher.

Data Generation

The three teachers were separately given a different set of three researcher-designed lesson plans in the local language appropriate for their teaching context. Three lesson topics that were common across Year 5 Australia and Germany, and Year 4 in China were chosen for the researcher-designed lesson plans: i) division with two digit divisors; ii) introduction to decimals, and iii) parallelograms and trapezia.

For example, for the Year 5 lesson plan on division with two-digit divisors given to the teachers in Australia and Germany, students worked in pairs to solve division word problems. The word problems all involve the same numbers (1144 and 32, which do not divide exactly), but each word problem has a different answer (e.g., "A dairy farm produced 1144 liters of milk, and has 32 containers in which to store the milk. If the containers are filled exactly, how much milk should go into each container?"). The purpose of the lesson is to draw the attention of students to the meaning of the question, and that the context of the problem determines the way in which the remainder is best used and expressed (Clarke, Roche, Sullivan, & Cheeseman, 2014). For the Year 4 lesson plan in China on the same topic, students were asked to solve problems with three-digit dividends and two-digit divisors in various contexts (e.g., "There are 178 storybooks to share with different classes. Each class can get 30 books. How many classes will have books?"). An emphasis of the lesson was for students to correctly write the calculation steps. The content of each researcher-designed lesson plan was checked for suitability to the local context by each country team.

Each of the teachers was asked to adapt the researcher-designed lesson plan and then teach the lesson to their usual class (26 students in a class in Australia; 55 students in China, and 30 students in Germany). After teaching the adapted

lesson, the teachers were asked to design a follow-up lesson themselves and deliver this lesson to the same class a few days after the adapted lesson. This process was repeated for each lesson plan provided, resulting in the delivery of three adapted lessons and three follow-up lessons per teacher. Pre- and post-lesson interviews were conducted with each teacher on the same day as the adapted and follow-up lesson. All the interviews were carried out by the local team in the local language.

The project was designed to generate data on each teacher's adaptation of a pre-designed lesson, the teacher's actions during the lesson, the teacher's reflective thoughts about the lesson and, most importantly, the consequences for the planning and teaching of a second (follow-up) lesson. All the pre- and post-lesson interviews and the adapted and follow-up lessons were video recorded, with the video recording of the lesson just taught used in the post-lesson interview to stimulate the teachers' recall and reflection on the lesson. All the interviews were fully transcribed in the local language.

Data Analysis

The analysis reported in this paper drew on the interview data with the three case study teachers, and specifically, the teachers' responses to interview questions related to their learning. Seven questions across the four interviews (two interviews each, pre- and post-lessons, for the adapted and follow-up lessons) were included in the analysis which explicitly asked what the teachers thought they learned from the activities carried out as part of the project (lesson plan adaptation, adapted lesson teaching, creation of follow-up lesson plan, and follow-up lesson teaching). Example questions included: "Please describe anything you have learned because of participating in the task activity, and in reading and planning the lesson. Explain your response" (pre-lesson interview), "Was there anything that happened during the lesson that was really unexpected by you?" (post-lesson interview), "Which moments in the lesson do you think provided learning opportunities for you? What did you learn?" (post-lesson interview).

After collating the teacher interview responses to the above questions, the responses were partitioned into *idea units*, where an idea unit is "a distinct shift in focus or change in topic" (Jacobs, Yoshida, Stigler, & Fernandez, 1997, p. 13). Each idea unit was then coded for epistemic claim (consolidation or new realization) and any indication of adaptive practice by at least two researchers in each country team.

RESULTS

After reviewing the reflective statements of the three teachers, we found that all the teachers identified things that they thought they had learned in the course of participating in the project. We could find statements that indicate learning based on the three coding categories (consolidation, new realization, and

adaptive practice) for all three teachers. The following provides illustrative examples for the coding categories which were drawn from interviews where the teachers have each been given a researcher-designed lesson plan on the topic “division with two-digit divisors” for adaptation and teaching. The statements of the Chinese and German teachers are translated into English for reporting in this paper.

For the Year 5 Australian teacher, he thought the lesson topic on the context of a mathematical problem “reignited” his emphasis on the topic in his teaching (consolidation).

“I would’ve liked to have thought that it was a big priority in my teaching, but reading this, it’s probably reignited that light of realising that, “hey, the context of the problem is super, super, super important.” ... I certainly have got more appreciation of that. So, that would be learning out of it, for sure.” (AU_T5 preadapted lesson interview)

In the post-lesson interview of the adapted lesson, he learned that not many of his students applied a problem-solving strategy that was covered in the past (new realization):

“I was surprised that looking through the sheets that not many of them like physically sort of circled or highlighted key information, which felt like a problem-solving strategy we’ve done in the past.” (AU_T5 post adapted lesson interview)

He particularly reflected on task difficulty for his students and thought starting with smaller numbers for the division problems could have given students more confidence for the lesson (adaptive practice).

“I guess I’m still sort of learning in terms of differentiating the task. On reflection, maybe I could have done that better at the start, knowing that the Grade 5 cohort would have really struggled with the big numbers. Even though using smaller numbers does not change the thinking of the actual task, at least it sorts of gives them a bit more of a security blanket.” (AU_T5 post adapted lesson interview)

Similar to the Australian teacher, the Year 5 German teacher also found his knowledge consolidated in the teaching process, specifically about the importance of helping students to understand how to deal with decimal places in relation to units of measurement (e.g., liters vs milliliters in the problem that deals with the division of milk into containers described earlier).

“What I found confirmative again was how important the last zero was if it is 75 ml or 750. [...] For them (the students), the problem is about part-whole relationship. They are not aware that I now have steps of thousandths for the units.” (DE_T5 post follow-up lesson interview)

In the pre-lesson interview for the first follow-up lesson, he learned from the previous lesson to anticipate typical student mistakes, even if he does not think that his students would make them (new realization).

“When considering typical mistakes beforehand, and then you realize that they (the students) really do make them, so you can actually even expect them to happen and plan how to deal with them. That they really occurred and that it really did fit well, that was funny. That I have learnt.” (DE_T5 pre follow-up lesson interview)

On reflection, the teacher thought it was important to give students who are still working on the problem more chance to keep working rather than make visible other students’ completed work to them pre-maturely, and he suggested an alternative way to address this in the future (adaptive practice).

“Next time I would definitely turn them [the post-it’s with the student solutions that had been pinned to the blackboard] around right away [so that the students who are still working on the tasks cannot see them].” (DE_T5 post adapted lesson interview)

Unlike the Australian and German teachers who spoke specifically about what was reconfirmed for them in the teaching process, the Year 4 Chinese teacher often spoke in general terms about what teachers should learn from their teaching. The teacher’s comment can be considered as a form of confirmation as he voiced his belief about the need to keep a positive attitude when teaching.

“For many inexperienced teachers, when the start of the class does not start smoothly, their emotions get affected and the rest of the lesson doesn’t run smoothly. So, there is a need to keep a positive attitude when teaching – it is normal for children to make mistakes. How to adjust their mistakes is what we (teachers) learn.” (CH_T4 post adapted lesson interview)

Through analyzing the lesson topic, the teacher “discovered” its importance (new realization).

“I read through the later key points and discovered something. ... For all the later key points, such as division that is not of numbers that are multiples of 10, everything needs to be converted to multiples of 10 in order to calculate. When do we need to convert? For example, the later Example Question 2, to divide ... 21, a student needs to think of 21 as 20 to try to divide. ... Examples 1 and 2 are basically the foundation of the entire two-digit division method, so this lesson needs to be treated seriously, as it is basically giving (the students) the foundation today.” (CH_T4 preadapted lesson interview)

For the teacher, the unexpected responses of the students’ summaries drew his attention to his questioning, which he thought could be improved (adaptive practice).

“When summarising the similarities between the two example questions, some students concluded that they are both divisions, some concluded that they are

division of multiple digits by multiple digits, which are all very superficial summaries. These kinds of summaries were unexpected. I mainly wanted them to summarise the two vertical mathematical expressions, (but) maybe my questioning was too vague. If I had pointed to something clearer, I would let (the students) directly see the similarities between these two vertical mathematical expressions, maybe that would be better.” (CH_T4 post adapted lesson interview)

Accordingly, we found statements given by the case study teachers in the three countries that appeared to confirm the teachers’ already held beliefs and expectations, even though they thought that was also part of their learning (consolidation). Through these statements, the teachers expressed their existing knowledge or beliefs, and how the new situation, activity, or event had “reignited” or “confirmed” those knowledge or beliefs. These consolidation statements contrast other teacher statements that appear to suggest something that was unexpected, surprising, or new for the teachers, conveying a sense of novelty in what the teachers observed or realized (new realization). In addition, we also found statements given by the teachers which showed that they actively thought of ways to improve their practice by suggesting alternative practice and things that they may do differently (adaptive practice).

DISCUSSION

One of the aims of the Learning from Lessons project is to provide cross-cultural insights into teacher in-class learning. Using purposefully designed experimental mathematics lesson plans, teachers were asked in this project to adapt a researcher-designed lesson plan, teach the adapted lesson, and create and teach a follow-up lesson. The pre- and post-lesson interviews conducted in the research provided opportunities for the teachers to reflect on what changes in their knowledge and practice were evident, and how those changes occurred. Care was given to replicate the research design in the three countries (Australia, China, and Germany), while accommodating differences in local contexts.

While we found evidence of the learning categories developed based on the Australian case studies (Chan et al., 2019), we were unsure if teachers in other countries would express their in-class learning in similar ways. From the case study teachers’ responses to the learning questions in the three countries, we could distinguish two learning mechanisms in terms of consolidation (reinforcement of existing knowledge and beliefs) and new realization (realization of new knowledge and beliefs). We have also found that the case study teacher in each of the countries actively considered ways to improve their practice based on their teaching (adaptive practice). The presence of teacher interview statements that fit with the proposed learning categories suggests the research design allows similar evidence of teacher learning to be found in the three different countries. This is a significant result, as this allows the project to proceed with making comparisons between teachers in the case study and

survey data in Australia, China, and Germany in terms of their reflection of teacher in-class learning.

On a theoretical level, conceptualizing teacher learning in terms of consolidation, new realization, and adaptive practice poses new questions for further research. We can ask the questions: What characterizes teachers who have a greater tendency to experience learning as new realization? What characterizes those who have a greater tendency to experience learning as consolidation? What types of events or conditions trigger new realization or adaptive practice? What are such new realizations or adaptive practices about? These questions will be addressed in future papers, drawing from the survey and cross-cultural components of the project.

Acknowledgments

This research is supported under the Australian Research Council's Discovery Projects funding scheme (Project number DP170102540). We would like to thank the case study teachers, students, parents, and school staff for their invaluable support and contribution.

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MALAYSIAN SECONDARY SCHOOL STUDENTS' VALUES IN MATHEMATICS LEARNING: A PRELIMINARY ANALYSIS

Hui Min Chia¹ & Qiaoping Zhang¹

¹Department of Mathematics and Information Technology

The Education University of Hong Kong, Hong Kong

This study examined 380 secondary school students' values in mathematics learning in Malaysia using the What I Find Important (in mathematics learning) questionnaires. The preliminary analysis shows that Malaysian secondary students valued the attributes of "process", "fun", "effort", "objectism", "ideas and practice", "exposition", "recalling", and "openness". Among different ethnics' groups, Chinese students tended to value "process" and "application" more than other ethnic groups. Malay students valued "hard work" and "effort" more than their peers in learning mathematics. In terms of gender difference, the result shows that Malaysian secondary school boys and girls valued almost the same value attributes in learning mathematics. The results provide some insights into understanding mathematics teaching and learning from the multicultural classroom context.

INTRODUCTION

All the while, improving students' learning has always been the focus of (mathematics) education research. Many factors influence students' learning, including cognitive factors (such as students' knowledge, ability and skills) and affective factors (such as their attitudes and beliefs). As a deep-seated and personal affective factor, Bishop (2001) proposed that students' values can also influence their learning. This idea was further elaborated by Seah (2013), whereby values regulate how cognitive skills and emotion are used in learning by a learner. He defined values as the "convictions" that a person perceived as important or worthy (Seah, 2013, p.193). This implies that values can be implicit as it is internalised in nature.

Alan Bishop first proposed three pairs of complementary mathematical values in mathematics education, that is, values regarding the discipline of mathematics: *rationalism* and *objectism*, *control* and *progress*, and *openness* and *mystery* (Bishop, 1988). The three pairs of complementary values echoing the three components of culture proposed by White (1959), namely ideological, sentimental and sociological. Bishop (1991) explained that *rationalism* "is concerned with the logic of the relationship between ideas" and *objectism* "is about the genesis and phenomenology of those ideas" (p. 202) as the ideology

component. *Control* and *progress* as the sentimental component. *Control* refers to developing mathematical ideas through specific rules or procedures and *progress* refers to developing mathematical ideas through alternative ideas. The sociological component, *openness* stresses the demonstration of ideas in public; however, *mystery* emphasises the wonder or mystery of ideas. Later in 1996, he further proposed a framework of three intersecting sets of values: mathematical values (relating to the discipline), mathematics educational values (relating to mathematics pedagogy), and general educational values such as *honesty* and *law-abiding* (relating to the ethical and moral principles). At the same time, he proposed that values are “deep affective qualities” which last longer in people’s memories than conceptual and procedural knowledge (Bishop, 1996, p. 19).

Recently, Seah and Andersson (2015) suggested that the process of valuing can be conative in nature, which involves both cognitive and affective aspects. Specifically, values reflect what an individual perceives as important and valuable through their actions in learning and teaching mathematics. This suggests that values, including mathematical values and mathematics educational values, can influence an individual’s learning process. Thus, to identify what values related to learning mathematics are embedded in an individual, within a classroom and even a cultural group to improve mathematics learning, such study is needed.

This paper reports on the part of a study in The Third Wave Project. The Third Wave Project is carried out by a consortium of research teams that concern the influence of values and valuing on mathematics learning. This current study, named ‘What I Find Important (in my mathematics learning)’ (WIFI), aims to investigate what primary school and secondary school students value regarding mathematics learning. The WIFI questionnaire has been translated into different languages so that the student participants in the 19 economies could respond to the items within their respective medium of instruction.

In Japan, Shinno, Kinone and Baba (2014) reported that data from 605 primary school students and 711 junior secondary students had valued different attributes in learning mathematics. Japanese primary students tended to value *process*, *effort*, *exploration*, *fact*, *openness* and *progress* more than secondary school students. Zhang (2019) has found a similar result in the Chinese Mainland data, whereby different grades had valued different attributes. Besides, there was a gender difference reported in several value attributes (Zhang, 2019). Those findings suggest that students’ value might change over time.

Moreover, boys and girls can value different attributes in mathematics learning which require different teaching approaches. Furthermore, from the literature (e.g., Seah, 2018; Shinno et al., 2014; Zhang, 2019; Zhang et al., 2015), students value different value attributes even within the East Asian culture.

How will the situation be in the context of a multicultural classroom as value is culturally dependent?

The current study explored students' value in mathematics learning in three (Malaysia, Singapore, and Thailand) out of 11 countries in Southeast Asia. Furthermore, International Mathematical Union (2013) reported many countries in Southeast Asia (e.g. Cambodia, Indonesia and Laos) face challenges in improving students' mathematics learning. Hence, more research about values in mathematics education is needed in Southeast Asia. Value researches in Malaysia focused on the primary school level, such as value espoused and enacted in the primary school mathematics lesson (Lim & Kor, 2012) and Chinese primary school students' values in mathematics learning (Kor, Lim & Tan, 2010). Additionally, Ong (2014) analysed WIFI study data of 383 Malaysian Grade 5 students and reported differences in attributes of learning mathematics valued by different gender and different ethnics groups. There is a gap in what Malaysian secondary school student might value in mathematics learning. Therefore, this research explores what Malaysian secondary school students have valued as important in learning mathematics. The following research questions guided this paper:

- a) What Malaysian secondary students valued in mathematics learning in general?
- b) Do Malaysian secondary students from different ethnicities value mathematics learning differently?
- c) Is there any gender difference in Malaysian secondary students' values in mathematics learning?

METHODOLOGY

Respondents

In this paper, the result from 380 secondary students (Grade 9 and Grade 10) is shown in Table 1. The data was collected randomly from public schools in Northern Peninsular Malaysia through personal contact. The sampling was convenient sampling without any specific selection of criteria.

Gender	Ethnicity				N	Percentage (%)
	Chinese	Indian	Malay	Other		
Male	107	26	75	5	216	56.84
Female	73	24	69	1	167	43.16
Total	180	50	144	6	380	100

Table 1: The participants of the study according to gender and ethnicity.

Data Collection

Data were collected using the WIFI questionnaire developed and validated by the WIFI study team (Seah, 2013). In the Malaysian context, the original

English items were translated into Chinese and Malay to facilitate the students' response to the questions. It was a four-section questionnaire, whereby Section A consisted of 64 items with a rating of 5-point Likert scale, Section B comprised 10 items of the slider rating scale, Section C contained four items of open-ended question and Section D was pupils' personal information items. In this paper, our analysis focused on Section B, whereby the students were required to choose by marking "x" at any one of the five positions in between two values given. For instance, when given the description "How the answer to a problem is obtained" (on the left) versus the description "What the answer to a problem is" (on the right) on a horizontal line respectively, the students were asked to mark their preference accordingly.

Data Analysis

In this paper, data were analysed using One-way ANOVA to analyse the statistical differences in students' responses of different ethnicity and the independent t-test to analyse the statistical differences in students' responses of different genders. The scores were assigned from one to five from left to right according to the five positions on the horizontal line. The lower the mean of the item, the more tendency towards the description on the left and vice versa.

FINDINGS

Malaysian secondary students' values in mathematics learning

The findings show that overall Malaysian secondary students tended to value the process of getting an answer (*process*) more than the end product for a problem (*product*), as shown in Table 2. They emphasised that having fun when doing and learning mathematics (*fun*) more than *hard work*. However, whether doing mathematics required *abilities* or *effort*, students tended to select *effort*. Furthermore, students tended to value using a mathematical formula to obtain the answer (*rationalism*) rather than applying mathematical concepts in problem-solving (*objectism*). They believed mathematical ideas and practice in daily life (*idea & practice*) were more important than discovering mathematics facts and theories (*facts & theories*).

Moreover, the students preferred to learn mathematics by someone with direct teaching, explaining or telling them the concept (*exposition*) rather than exploring the mathematics by themselves, with their peers/others (*exploration*). Yet, they tended to value *exploration* with a concrete example given more than someone telling them. Besides, students tended to value remembering mathematics ideas, concepts, rules or formulae (*recalling*) than creating. The students had also chosen to demonstrate or prove the concept to others (*openness*) over to keep mathematics mystical (*mystery*). They believed that mathematics' purpose should be more relevant in development or progression (*process*) than predicting or explaining certain events (*control*).

Malaysian secondary students' values in mathematics learning according to ethnicity

Items	Total		Chinese		Indian		Malay		Other		<i>F</i>	η^2
	Mea n	SD	Mea n	SD	Mea n	SD	Mea n	SD	Mea n	SD		
66. Process versus product	2.64	1.024	2.38	1.141	2.98	1.070	2.81	.808	2.83	.408	9.545***	0.05
67. Fun versus hard work	2.97	1.241	2.65	1.273	2.94	1.345	3.35	1.084	2.83	.408	11.742** *	0.07
68. Ability versus effort	3.47	1.239	3.24	1.264	3.61	1.255	3.72	1.151	2.83	1.329	4.886***	0.04
69. Objectism versus rationalism	3.03	1.094	2.87	1.142	3.00	1.021	3.21	1.043	3.17	.983	2.691**	0.02
70. Facts & theories versus ideas & practice	3.08	1.101	3.12	1.102	3.33	1.107	2.96	1.084	3.17	1.329	1.492	0.01
71. Exposition versus exploration	2.64	1.158	2.59	1.293	2.86	1.000	2.65	1.033	2.00	1.095	1.462	0.01
72. Recalling versus creating	2.45	1.151	2.45	1.196	2.55	1.081	2.43	1.141	2.50	.837	.142	0.001
73. Exposition versus exploration	3.35	1.189	3.34	1.210	3.29	1.155	3.40	1.191	3.17	.983	.197	0.002
74. Openness versus mystery	2.52	1.138	2.53	1.121	2.69	1.294	2.42	1.098	3.00	1.265	1.102	0.01
75. Control versus process	3.20	1.000	3.11	.927	3.37	1.035	3.28	1.058	2.67	1.033	1.697	0.01

Note: $p < 0.001$ ***, $p < 0.05$ **, SD= standard deviation

Table 2: The participants' responses to section B according to ethnicity

As we take a closer look into different ethnicities, for item 66, the ANOVA result was significant, $F(3, 170.310) = 9.545$, $p < 0.001$, $\eta^2 = 0.05$, suggesting Chinese students tended to value the *process* more than their peers. The results for item 67, $F(3, 170.522) = 11.742$, $p < 0.001$, $\eta^2 = 0.07$ and item 68, $F(3,$

372) = 4.886, $p=0.002$, $\eta^2 = 0.04$ imply that there are significant differences among the means of the four groups for the two items respectively. The results suggest that Malay students believed the *effort* was more important in learning mathematics than their peers. In item 69, $F(3, 372) = 2.691$, $p=0.046$, $\eta^2 = 0.02$, suggesting Chinese students emphasised *application* more than other ethnic groups students.

Malaysian secondary students' values in mathematics learning according to gender

Further analysis in the secondary students' population revealed a significant difference in the mean score of two value attributes, as shown in Table 3. One of the attributes was item 69, $t(382) = -2.32$, $p=0.021$, 95% CI [-0.48, -0.04]. This implies the girls ($M=3.16$, $SD= 1.048$) preferred to use a certain formula to find the answer more than the boys ($M= 2.90$, $SD= 1.104$). While another attribute is item 74, $t(381) = 2.79$, $p= 0.006$, 95% CI [0.10,0.55]. This suggests that the girls ($M=2.33$, $SD=1.171$) tended to value *openness* in learning mathematics as compared to the boys ($M=2.65$, $SD=1.091$).

Items	Male		Female		t-test
	Mean	SD	Mean	SD	
66. Process versus product	2.63	.962	2.66	1.107	-.289
67. Fun versus hard work	2.93	1.134	2.91	1.362	.139
68. Ability versus effort	3.43	1.204	3.53	1.271	-.788
69. Objectism versus rationalism	2.90	1.104	3.16	1.048	-2.319*
70. Facts& theories versus ideas& practice	3.04	1.079	3.18	1.134	-1.228
71. Exposition versus exploration	2.66	1.197	2.59	1.088	.593
72. Recalling versus creating	2.54	1.176	2.40	1.146	1.109
73. Exposition versus exploration	3.27	1.114	3.45	1.269	-1.425
74. Openness versus mystery	2.65	1.091	2.33	1.171	2.788*
75. Control versus process	3.12	1.030	3.32	.928	-1.944

Note: $p<0.05^*$, SD= standard deviation

Table 3: Secondary students' responses for section B according to gender.

DISCUSSIONS AND CONCLUSION

The results show that overall, Malaysian secondary students tended to value the attributes of *process*, *fun*, *effort*, *rationalism*, *ideas and practice*, *exposition*, *recalling*, *openness* over the respective opposing dimensions *product*, *hard work*, *ability*, *objectism*, *facts and theories*, *exploration*, *creating* and *mystery*. In addition, there are several values attributes that students from different ethnic groups value differently. A similar result is reported by Ong (2014), whereby Chinese and Malay primary students had valued different values attributes. According to Lim (2003), this can be related to their previous learning experience in primary schools whereby different medium schools applied different cultural practices or parents' influence from the family.

In terms of gender difference, the result suggests that Malaysian secondary school boys and girls valued almost the same value attributes in learning mathematics except that girls tended to value *rationalism* and *openness* more than boys. The finding is consistent with both studies conducted by Zhang (2019) and Ong (2014), whereby boys and girls valued certain value attributes differently.

One of the limitations of this study is the representation of the data. Due to the randomly convenient sampling, the ethnicity proportion was slightly different from the actual population in Malaysia. The proportion of Malay students was 38% in the sample, which is less than the actual population cap at 60%. However, this exploratory study still provides a glimpse of what Malaysian students valued in mathematics learning. More data will be collected in the future to better represent the Malaysian secondary students' population.

In conclusion, the preliminary analysis has provided evidence that different ethnic groups value different values attributes in mathematics learning. Furthermore, such a study helps the teacher promote effective mathematics teaching and learning in a multicultural classroom. Teachers can structure their teaching to align with students' value in mathematics learning to facilitate their learning process. Future study is needed to investigate factors that influence students' values in mathematics learning, so that students' learning can be understood and facilitated more effectively.

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CONCEPTS IN ACTION: MULTIPLICATION AS SPREAD

Sean Chorney¹ and Nathalie Sinclair¹

¹Simon Fraser University, Canada

In this research report, we work with the novel, multitouch app TouchTimes, which was designed to develop multiplicative thinking in young learners through gesture-based interactions. One aspect of multiplication highlighted in this app is its functional, one-to-many relation, which several researchers have identified as key to developing multiplicative thinking. In this study, we use Balacheff's cKç model, which highlights the action/feedback control structure to describe how this relation is instantiated in children's use of TouchTimes. Through an analysis of a pair of 9-year-olds, we show how this relation evolved into a concept, which we call multiplication-as-spread.

INTRODUCTION

Typically, children's first encounters with multiplication in North America is in terms of repeated addition. The use of this model often persists throughout grades 3 and 4. While it may be an intuitive way of introducing multiplication, it becomes problematic as it encourages continued use of additive thinking. There exist several other models which can support a more robust conception of multiplication. In this study, we focus on the model of the one-to-many relation articulated by Confrey (1994), which she calls 'splitting', and which involves "... an action of creating simultaneously multiple versions of an original" (p. 292). Splitting can be visualized using a tree diagram, which highlights the one-to-many relation that simultaneously produces copies of the original. The centrality of this relation is also highlighted by Askew (2018) and Davydov (1992).

We will be presenting a touchscreen application *TouchTimes* (TT; Jackiw & Sinclair, 2019) that aims to provide students with an experience of multiplication that uses this one-to-many model. This is done through the gestural expression of multiplication, which involves a dynamic, visual and simultaneous production, rather than the sequential one typical of repeated addition. Using Balacheff's cKç model, which is a re-articulation of Vergnaud's (1990) notion of 'concept in action', that emphasises the essential feature of control as an essential aspect of the concept, we study the emergence of the one-to-many multiplicative relation both in the gestural interaction and then as an articulated concept.

BRIEF DESCRIPTION OF TOUCHTIMES

The initial screen of TT has a vertical line down the middle which creates two sides (Figure 1a). When one side is touched with a finger, or a group of fingers, discs will appear in a one-to-one correspondence with each finger. These discs are called pips and represent the multiplicand, or unit, that will be multiplied. Each pip will be a different colour. When the other side is touched with a single or group of fingers, each configuration (called a ‘pod’) of the multiplicand side (the side that was touched first), both in terms of position and colour, will appear in a one-to-one correspondence with each finger. Each pod will be identical to the finger configuration on the pip side. If three fingers touch the left side in a triangle-like pattern, three pips will appear under each finger and each of the three pips will be a different colour (Figure 1b). When the other side is touched, the triangular pattern of the multi-coloured pips will be copied under each finger in pod-groupings (Figure 2c). If another finger is placed on the pip side, each pod on the other side will grow, in a simultaneous copy—this effectively performs the one-to-many relation. When a pip-finger is lifted, the inverse occurs: each pod decreases in size. Fingers can be added to the pod side to make new pods. Similarly, a pod can be dragged to the trash. Whenever pips and pods are created on the screen, a multiplication statement appears on the top of the screen (Figure 1c).



Figure 1: (a) Initial screen of *TT*; (b) Creating 3 pips; (c) Creating 4 pods

THEORETICAL FRAMING

Concepts in action as described by Vergnaud (1990) are actions made that are correct and conceptually coherent, even though students may not be able to explicitly articulate this. As Vergnaud writes, “We take up information with the help of invariants (categories, relationships, and higher-level entities), without expressing or even being able to express these invariants. This is especially visible in students' mathematical behavior, as they often choose the right thing to do without being able to mention the reasons for it” (p. 20). Concepts in action stem from Vergnaud’s theory of conceptual fields in which multi-faceted concepts (like multiplication) are not unified by one overall mathematical idea but involve multiple conceptual experiences. In our case, we are interested in the one-to-many conceptual experience that we hypothesise *TT* can provide.

Vergnaud’s concepts in action directs the researcher’s attention to the behaviour of students—to their choices, their actions and their language—which is then used to make inferences about their concepts in action. In articulating

Vergnaud's ideas further, Balacheff's (2017) cK ϕ model of conceptualization draws attention to the action/feedback loop as an essential component of a concept in action. Balacheff argues that choices made by a subject based on feedback represent a necessary "control structure" (p. 9) that is a fundamental part of the concept. In the case of studying TT, in which gestures are a significant form of action, and in which the visual presentation of pips and pods provides important and immediate feedback on these actions, our analysis of behaviour will involve looking closely at the various hand movements made by the students on the screen, and taking these particular movements as mattering to students' developing conceptualisations, as per the tenets of theories of embodiment (c.f., Arzarello, Bairral & Danè, 2014; Sinclair & de Freitas, 2014). The research objective is to document the transformation of the concept in action developed in TT into an explicitly articulated concept.

METHODS

The study took place in an elementary school in western Canada. We conducted teaching experiments aimed to gain insight into the multiplicative thinking that might emerge from interactions with TT. For this paper, we have selected one episode that involves two grade 3 girls (9 years old), who had begun to study multiplication (as repeated addition). The girls were working on the following task in TT: skip-count by 3s in two different ways. One method involves changing the number of pods; the other involves changing the number of pips. The students worked in pairs and many of them were video recorded by two researchers who circulated in the classroom from pair to pair. We have chosen this particular pair for analysis because the shift from a concept in action to the concept of the one-to-many multiplicative relation occurred during a single video clip (for most other pairs, we only captured the concept in action or the explanation). In our analysis, we draw on Vergnaud's (1990) method, which is: to precisely describe the behaviour of the student; to identify the invariant properties of the situation; and, to trace the development and transformation of language and symbolic activity to highlight the way in which the student can explicitly describe the concept.

FINDINGS

The two girls, whom we will refer to as Jen and Jessica, were working together on the floor with one iPad that rested at an angle on Jen's lap. Jessica did not say anything during the entire episode but did touch the screen. In the video clip, the researcher asked the girls to show her what they had figured out about skip counting by 3s and Jen proceeded to place three pip-making fingers on the left side of the screen and then iteratively placed one pod-finger on the right. The researcher asked, "Did you have another way?", to which Jen responded "No, we couldn't figure out a second way yet" (0.26s). The researcher suggested they keep trying. Jessica started to touch the screen, she made four

pips and touched sequentially on the pod side to make 6 pods (effectively skip-counting by 4s).





	Voice	Hands	iPad
0:56	Jen. If we do 4, that's counting up by 4s.	Jessica makes four pips and six pods.	
0:58		Jessica lifts her index, left pip-making finger. (The expression goes from 4 x 6 to 3 x 6; the pods go from having four to three pips in them.)	
1:00		Jessica lifts her middle finger. (The expression goes from 3 x 6 to 2 x 6; the pods go from having three to two pips in them.)	
1:01		Jessica lifts her third finger. (The expression goes from 2 x 6 to 1 x 6; the pods go from having two pips to one pip in them.) Jessica lifted all her fingers.	

Table 1: A one-to-many concept in action.

When Jessica made 4 pips and 6 pods, she did not seem to know how this would enable her to skip-count by 3s. She then lifted each individual pip-making finger to go from 4x6 to 3x6 to 2x6 to 1x6 to 0x6. Each lift of her finger was almost exactly 2 seconds. She was very intentional in her actions, which may suggest that she was becoming aware of the effect of this finger lifting on the 6 pods (the product would have decreased by 6 at each lift and each of the 6 pods would have become smaller and changed configuration at each lifting of a pip). We thus hypothesise that Jessica was starting to develop a concept in action—lifting the pip-making finger one by one—that could be used in the skip-counting task, and that effectively instantiated one-to-many relation. We see this as an example of action/feedback described by Balacheff, whereby Jessica is continuing the same action of lifting her finger based on the feedback from TT. The girls continued to make different combinations of pips and pods, and to experiment with pip-making and pip-lifting. Then, at 2:45, Jen has four pips and one pod on the screen.



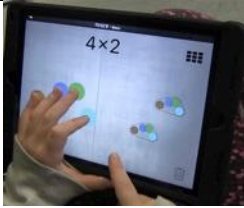


2:45	Jen. That's 4 but when I add another one.	Jen touches on the pod side to create another pod to make 4×2 .	
2:49	Jen. Oh wait.	Jen lifts her index pip-making finger on the left side, thereby getting 3×2 . (The number of pips in the pod goes from 4 to 3.)	
2:53	Jen. This.	Jen touches her index pip-making finger on the left side, thereby getting 4×2 again. She points at her pip-making finger with her right hand.	
2:56 - 3:03	Jen. Then that. R. How much is it going up by now? Jen. It will... it went up by 2.	Jen places another pip-finger on the left side. She continues to point with her right hand at her pip making fingers.	
3:07	Jen. Ya, it's going up by 2s.	Jen places one more pip-making finger on the left side after which Jessica adds a pip making finger on the left side.	

Table 2: The spreading effect in TT.

At 2:49 when Jen said, “Oh wait” she paused. Then she lifted her index pip-making finger. Over the course of the next 14 seconds, she made three pips and Jessica made one pip to produce $7 \times 2 = 14$. Jen stated at the end that the product was going up by 2. Although the relationship had not been fully articulated, the concept in action of spreading was emerging, as the girls became aware that each touch on the pip side was increasing each of the pods. Jen seemed to be “picking up” on the co-varying relation of the pips and pods. At 3:24 the girls successfully skip-counted by 3s by making three pods and then iteratively adding pips up to $5 \times 3 = 15$. At 4:00, the researcher asked, “So how is it doing that? How is it making it go up by three now?”



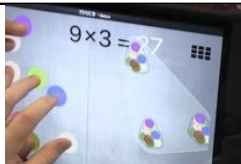




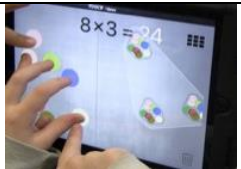
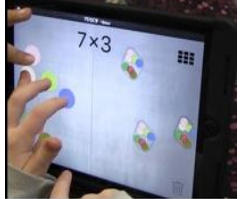
	Voice	Hands	iPad
4:00	Jen. Because there's three here.	Jen gestures to the pods on the screen.	
4:05	Jen. And then each time that we add one up it goes here.	Jessica places a pip-making finger on the screen. Jen places two more pip-making fingers on the screen. Jen points to the top pod.	
4:11	R. So if you add a finger, oh, that's a purple one.	Jessica places another pip-making finger on the screen.	
4:12	What happened to those purple ones?	Jessica lifts the pip-making finger.	
4:19	R. Oh now it's yellow. Jen. Um, a yellow one drops in there.	Jessica places her pip-making finger on the screen. Jen places two more pip-making fingers on the screen.	
4:21	R. Does it just drop in there?	R. points at the top pod.	
4:25	Jen. Ya. [Pauses] ...in every single one.	Jen points at all the pods in turn, starting with the lower left one, moving to the lower right pod, and then the top pod, as if spreading her hand to each pod.	
4:27	Jen. And say if we take away this one.	Without being prompted, Jen points at her own thumb which holds a yellow pip.	
4:28	Jen. Then that colour would disappear.	Jen lifts her thumb, points to the top pod and moves her hand towards the bottom of the screen. She then moves her hand in a circular motion, spreading her fingers to each pod.	

Table 3: Jen and Jessica skip-counting by changing the number of pips

When Jessica placed her pip-making finger on the screen, the concept in action of one-to-many was in play, where each pip addition made the product increase by three [4:05-4:19]. The researcher drew attention to this change when she said, “That’s a purple one...Oh now it’s yellow”. In response to the researcher’s question about the yellow pip, Jen explicitly articulated, by saying “every single one” and the gesture of pointing to the top pod and the spreading gesture, the relation between the change in pips and the ensuing *change in each of the pods*, which is the one-to-many relation (“dropping” one pip will change many pods).

At 4:51, the researcher asked, “How do you think you’ll make it do it by fours?”. Jen immediately tapped four times on the pod side. There were 9 pips on the screen, producing $9 \times 4 = 36$. Jen said, “and we add one, that’s 40 so...”. She placed another pip-finger on the screen, then counted on her fingers from 36 to 40, and said, “That’s by fours” and again gestured around to all the pods.

DISCUSSION AND CONCLUSION

The effect of seeing the pod change as Jen and Jessica touched and lifted their pip-making fingers, going from 4 to 3 and back up to 6, seemed to draw the girls’ attention to the relation between pips and pods and also encouraged them to repeat through their now developed control structure a particular gesture that required the coordination of two quantities—the pips on one side, changing, and the pods on the other, staying the same. Instead of remaining within the additive framework of repeated addition (sequentially adding one more pod, thereby focusing on just one quantity), the girls were expressing multiplication as a coordination of quantities. Once they had changed the number of pips, they could use this action again, with more pods; and this allowed them to see the simultaneous change in all pods that occurred when the number of pips changed. They had figured out how to count by 3s in a new way. Since this appeared to be a difficult task for all of the pairs in the classroom, we infer that it involves a new awareness both about how TT works, but also about the multiplicative relation.

In our analysis, we have shown the development of a concept in action, which was in response to the task of skip-counting by 3s, and which involved making three pods and then iteratively adding pips. We then showed how the researcher’s prompt occasioned an explicit articulation of this concept in action. The girls were then able to count by 4s—and it was perhaps the articulation of the concept in action that made this not only possible, but seemingly effortless. But more importantly, in terms of their multiplicative thinking, the girls experienced a particular aspect of multiplication, which is its one-to-many relation, which is instantiated in TT when a change in the number of pips (the unit) leads to a change in each and every one of the pods. Connecting back with Balacheff’s model of conceptualisation, we see that the girls have developed a

control structure in this setting with TT contributing and being a part of the knowledge they now have of multiplication-as-spread.

Acknowledgment

The work presented in this paper was supported by the Social Science and Humanities Research Council Insight Grant 435-2018-0433.

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WHERE TO PUT THE DECIMAL POINT? NOTICING OPPORTUNITIES TO LEARN THROUGH TYPICAL PROBLEMS

Ban Heng Choy¹

¹National Institute of Education, Nanyang Technological University, Singapore

It is challenging to design and structure lessons to maximize high-quality opportunities to learn mathematics in the classrooms. This paper presents a case study of Mary, a beginning mathematics teacher in Singapore, to illustrate how she noticed opportunities to learn during the planning and enacting of a lesson on decimal fractions for Primary 4 students. The case highlights the importance of noticing affordances of typical problems and opportunities to orchestrate productive discussions to provide quality opportunities to learn.

INTRODUCTION

All students should have access to high-quality mathematics curricular, effective teaching and learning, high expectations, and the support and resources needed to maximize their learning potential. To enhance students' learning experiences, teachers need to provide their students *opportunities to learn* from mathematically meaningful tasks. The notion of opportunities to learn was defined as the “amount of time allowed for learning” (Carroll, 1989, p. 26) and its conceptualization has broadened over the years. For example, Liu (2009) positions opportunity to learn as an “entitlement of every student to receive the necessary classroom, school and family resources and practices to reach the expected competence” (p. v). Although this entitlement has often been measured in terms of the amount of time (Carroll, 1989) given for a program, or the number of tasks with certain characteristics in a textbook (Wijaya, van den Heuvel-Panhuizen, & Doorman, 2015), Carroll (1989) highlighted that it is what happens during lessons that matters most.

With the aim of broadening the notion of opportunity to learn to examine other features of mathematics instruction, such as task implementation during lessons, Walkowiak, Pinter, and Berry (2017) re-conceptualized opportunity to learn in terms of teachers' mathematical knowledge for teaching, time utilization, mathematical tasks, and mathematical talk. This conceptualization puts teachers as the main orchestrator in the lesson to provide students these opportunities to learn. More specifically, Walkowiak et al. (2017) positioned teachers' mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) as a critical factor in relation to how teachers optimize time use during the lesson (Gettinger, 1989), how they design, select, and implements tasks (Mason &

Johnston-Wilder, 2006), and how teachers orchestrate discussions (Smith & Stein, 2011). This paper examines how Mary (pseudonym), a primary school mathematics teacher in Singapore, provided her Primary 4 students quality opportunities to learn mathematics by orchestrating the time, task, and talk for a lesson on decimal fractions.

Orchestrating Time, Task, and Talk

Although time allocated to teaching mathematics is important, Walkowiak et al. (2017) went beyond the number of minutes and investigated the amount of time spent in relation to the mathematical goal of the lesson. In particular, they examined whether teachers use “the majority of time in the lesson to reach the mathematical goal” and whether the lesson components are structured to “build on each other with explicit attention to the mathematical goal” (p. 12). This consideration is important for many classrooms because of the time constraints faced by teachers, especially in examination-driven education systems such as Singapore. In addition, many researchers suggest that it is crucial for students to have discussions around mathematically rich tasks as part of their learning experiences (Grootenboer, 2009; Smith & Stein, 2011). However, these tasks are usually time-consuming and pedagogically challenging to use in the classrooms. This raises the challenge of how teachers can optimize students’ opportunities to learn through mathematically meaningful tasks when given limited curriculum time. To this end, Choy and Dindyal (2018) highlighted how typical problems—standard examination or textbook-type questions—can be used to promote productive talk between students and teachers. While acknowledging the importance of using rich tasks, Choy and Dindyal (2018) not only suggested the possibility of using typical problems to orchestrate discussions, but also proposed how teachers can make connections between different representations of mathematics, which reflect a connectionist approach to teaching mathematics (Askew, Rhodes, Brown, Wiliam, & Johnson, 1997).

The Role of Teacher Noticing

Mathematics teachers, who use a connectionist approach to teaching mathematics, can notice and exploit the mathematical possibilities of instructional materials for different profile of students (Askew et al., 1997). Adopting a connectionist approach to teaching requires teachers to develop a keen awareness of the mathematical connections afforded by the tasks and use these connections to design opportunities to learn through orchestrating time, task, and talk during lessons. A key component of teaching expertise that enables teachers to do this ambitious work is mathematics teacher noticing, which refers to what teachers attend to and how they interpret their observations to make instructional decisions during lessons (Mason, 2002; Sherin, Jacobs, & Philipp, 2011). Most of the earlier studies on teacher noticing were centered about the use of video recordings of teaching episodes but Choy

(2016) brought task design into the realm of teacher noticing. His findings suggested that an explicit focus for noticing is useful, and an emphasis on pedagogical reasoning can increase the likelihood of teachers making instructional decisions that promote students' reasoning. In this paper, researcher extends and applies the notion of productive noticing (Choy, 2016) to examine what and how Mary noticed about the opportunities to learn during a lesson on decimal fractions. Vignettes of how Mary planned and implemented the lesson will be discussed in relation to the time, task, and talk during the lesson.

METHOD

The data reported in this paper were collected as part of a larger exploratory study on building a culture of collaboration and listening pedagogy in classrooms through Lesson Study for Learning Community in Singapore. The study involved a Lesson Study team comprising of 10 mathematics teachers in Quayside Primary School (pseudonym), a government-funded school. The vignettes feature Mary, a beginning teacher who had only six months of teaching experience at the time of this study. Although newly trained, Mary has a strong foundation in mathematics as she had studied mathematics as a university major. Data for this paper were generated through the voice and video recordings of the lesson, and the lesson plan designed by Mary with support from her colleagues. A thematic analysis approach (Braun & Clarke, 2006) was adopted for this study. Viewing the lesson plan as an instantiation of her thinking about the opportunities to learn, findings were developed through identifying aspects of the time utilization, tasks, and planned talk moves that provided opportunities for students to do mathematics. For the lesson, researcher analyzed the video and voice recordings by identifying segments, which corresponded to Smith and Stein's (2011) five practices for productive discussions, and highlighted aspects of the time, task, and talk that presented opportunities for students to learn.

NOTICING, DESIGNING, AND ORCHESTRATING OPPORTUNITIES TO LEARN

In this section, researcher first presented an analysis of Mary's lesson plan on Decimals for a Primary 4 class before researcher discussed her actual lesson implementation. Her students had previously learned about decimals and fractions, including the addition of decimals. The lesson of interest (an hour in duration) focused on developing students' relational understanding (Skemp, 1978) of multiplication of decimals with a whole a number. Up to this point, students had not learned how to do multiplication involving decimals. It is also important to note that the Singapore Mathematics Curriculum only covers multiplication and division of decimals by 10, 100, and 1000 in Grade 5. Mary started the lesson by recapping the idea of multiplication as repeated addition

before she set them the task of the day, which was to find the answer to 0.8×4 and orchestrated a lesson around the different solution methods developed by the students, both individually and as a group. The episode reported here started when a student asked a seemingly trivial question: How will you know where to put the decimal point? In the following discussion, researcher uses three of the four key dimensions of opportunities to learn—time, task, and talk—as developed by Walkowiak et al. (2017) to highlight what Mary might have noticed about the opportunities to learn for her students.

Designing Opportunities to Learn during Lesson Planning

In terms of time utilization, Mary and her colleagues planned 45 minutes (out of 55 minutes) of lesson time for students to work on two related forms of the question, 0.8×4 : (a) Solve 0.8×4 , and (b) How many ways can you think of to solve 0.8×4 ? Referring to Table 1, we see that Mary planned to spend most of the time in the lesson to reach the mathematical goal. The students first worked on the problem 0.8×4 on their own (10 minutes). This was followed by students working in groups on developing multiple solutions to the same question (How many ways can you think of to solve 0.8×4 ?). Moreover, Mary planned to have the students discuss the different solutions during the whole class discussion so that she could draw their attention to the linkages between the various solutions and the standard multiplication algorithm (See Figure 1). Hence, the time was structured so that the tasks built on each other, paying attention to the goal of understanding the idea behind the multiplication algorithm.

Components of Lesson	Time planned (min)	Actual time used (min)
Introduction	5	3
Understanding the Problem and Individual Work	10	7
Group work	15	15
Whole Class Discussion	20	25
Closure of lesson	5	5

Table 1: Planned and actual time utilization.

Next, the task “How many ways can you think of to solve 0.8×4 ?” was a modification of simply “Solve 0.8×4 ”, which opened up the solution space of a typical problem (Choy & Dindyal, 2018). Although this will not be categorized as a rich task, the design of Mary’s task provided students opportunities to use and translate among two or more representations so that they could make sense of the mathematics. In addition, Mary’s use of the typical problem highlighted that she was cognizant of how the question could support students in making connections between their prior knowledge and the new content. Therefore,

Mary and her colleagues demonstrated a keen awareness of the affordances of such typical problems beyond their usual usage (Choy & Dindyal, 2018).

More importantly, Mary did not plan to use the typical problem by simply explaining the procedure. Instead, as seen in Figure 1, she planned for students to explain their thinking and this could potentially allow students to move towards a better understanding of the multiplication algorithm. A closer examination of the lesson plan also reveals some planned talk moves similar to those proposed by Smith and Stein (2011). For example, she anticipated students' different responses to the question and planned for the sequencing of sharing by different students (See Figure 1). This corresponded to Smith and Stein's (2011) practices of anticipating, selecting, and sequencing.

As seen from Table 1 and Figure 1, Mary noticed the affordances of using a typical problem and modified the problem to open its solution space (Choy & Dindyal, 2018). Her planned use of time and planned talk moves around typical problem also provided students opportunities to learn about the multiplication of decimals, with a strong focus on mathematical reasoning.

20	<p><u>Sharing and discussion of ideas – Class Discussion</u> Student representative from each group to present their responses and explanations.</p> <p><u>Anticipated responses and corresponding sequence for discussion:</u></p> <table><tr><th>Anticipated responses</th><th>Sequence</th></tr><tr><td>- Students using repeated addition to get the answer. - $0.8 \times 4 = 0.8 + 0.8 + 0.8 + 0.8$</td><td>1</td></tr><tr><td>- Students using number bond to break the multiplication into simpler parts - $0.8 \times 4 = 0.8 \times 2 + 0.8 \times 2$</td><td>2</td></tr><tr><td>- Students make use of what they already know ($8 \times 4 = 32$) to make a list of possible answers. - 32, 3.2, 0.32 are possible answers - Since 0.8 is slightly less than 1, answer should be slightly less than 4, which gives 3.2</td><td>3</td></tr><tr><td>- Students make use of the place value of the decimals to help them multiply - 0.8×4 is the same as 8 tenths $\times 4$, which gives 32 tenths. - 32 tenths is the same as 3.2</td><td>4</td></tr><tr><td>Multiplication algorithm is used. $\begin{array}{r} 0.8 \\ \times 4 \\ \hline 3.2 \end{array}$</td><td>5</td></tr></table>	Anticipated responses	Sequence	- Students using repeated addition to get the answer. - $0.8 \times 4 = 0.8 + 0.8 + 0.8 + 0.8$	1	- Students using number bond to break the multiplication into simpler parts - $0.8 \times 4 = 0.8 \times 2 + 0.8 \times 2$	2	- Students make use of what they already know ($8 \times 4 = 32$) to make a list of possible answers. - 32, 3.2, 0.32 are possible answers - Since 0.8 is slightly less than 1, answer should be slightly less than 4, which gives 3.2	3	- Students make use of the place value of the decimals to help them multiply - 0.8×4 is the same as 8 tenths $\times 4$, which gives 32 tenths. - 32 tenths is the same as 3.2	4	Multiplication algorithm is used. $\begin{array}{r} 0.8 \\ \times 4 \\ \hline 3.2 \end{array}$	5	<p>Allow pupils to consider possible alternatives to solve a task and critically follow through a thinking process.</p> <p>Rational for Sequence</p> <p>1) Allow students to focus on multiplication as a repeated addition. Qn : What happens if the whole number gets changed to something bigger, e.g. 100?</p> <p>2) Highlight to the students that number bonds help to break the product into simpler parts that allow us to solve quickly</p> <p>3) Students are allowed to use their prior knowledge of whole numbers multiplication to form a list of possible answers, before using estimation to eliminate other answers.</p> <p>4) Progression to multiplying decimals by whole number by looking at the decimal place value of each digit.</p> <p>5) Link to sequence 4</p>
Anticipated responses	Sequence													
- Students using repeated addition to get the answer. - $0.8 \times 4 = 0.8 + 0.8 + 0.8 + 0.8$	1													
- Students using number bond to break the multiplication into simpler parts - $0.8 \times 4 = 0.8 \times 2 + 0.8 \times 2$	2													
- Students make use of what they already know ($8 \times 4 = 32$) to make a list of possible answers. - 32, 3.2, 0.32 are possible answers - Since 0.8 is slightly less than 1, answer should be slightly less than 4, which gives 3.2	3													
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Multiplication algorithm is used. $\begin{array}{r} 0.8 \\ \times 4 \\ \hline 3.2 \end{array}$	5													

Figure 1: Snapshot of Mary's lesson plan

Orchestrating Opportunities to Learn during Lesson

Mary also orchestrated several opportunities, as planned, for students to learn during lesson. While Mary circulated the classroom, she took notice of the strategies used by the different groups of students. Mary's attention to students'

strategies was demonstrated when she called upon different students to present their solutions according to the sequence planned as indicated in Figure 1. More importantly, Mary pressed the students for their explanation beyond giving the correct answers:

- 1 S1: So, first, we have four 0.8s, and after that we added all up, like 0.8 plus 0.8, then we get the answer then we plus 0.8 again then plus 0.8 again. Then...I thought that it will be easier if the number is smaller. But if the number is bigger, I think, then the multiplication method is easier?
- 2 Mary: Okay, so they [*referring to the group of students including S1*] did not choose this strategy as the most efficient one, but I ask them to present this strategy. Can anyone tell me why they did not choose this one as opposed to this one? S1 actually mentioned it, how about S2?
- 3 S2: Because is, when you get other bigger number, it will be hard for you to ...

As with the above exchange, Mary continued to press her students to explain their solutions to make their thinking visible to the other students throughout the lesson. This was so even for unanticipated responses from her students:

- 22 S3: ... First, you need to kick the decimal place away because you do not need it. And then you need to, er..., you need the time to, convert both the numbers you are multiplying into whole numbers and then you get the answer. And then you, you pick the decimal point, in between the tenth place and the ones place of your answer. And we chose this as the most efficient one because it takes only 2 steps...
- 23 Mary: Is that all? Okay, any questions for S3's method? S4?
- 24 S4: But the multiplication number reaches up to like a zillion, where will you know how to put the decimal point?
- 25 S3: Just put it between the tenth place and the ones place, and you are done.
- 26 S4: But how do you know which one [*cross talk*]?
- 27 S3: Yes, I checked it already.
- 28 S5: How do you know which number is in the tenth place?
- 29 S3: Because I checked just now.
- 30 Mary: How did you check? How did you know?

Here, S4's question at Line 24 was unanticipated. Instead of brushing aside the question, Mary stepped back and allowed students (S3, S4, and S5) to discuss S4's question. By doing so, Mary brought the question to the center of the whole-class discussion and these students' arguments were made available to all students, for them to think about and evaluate the validity of the points made:

- 31 S3: I put a big number times a big number and I tried it, and yes, it works.
- 32 Mary: So, it is always between the tenth and the ones place? Anyone disagrees?

As seen from Mary's response, she was comfortable in letting her students engage in a mathematical argument. The exchanges went on for several more turns before Mary tried to connect these responses:

- 55 Mary: ... Let's look at S3's method, he started with 8 times 4, 32. How many ways can we actually put the decimal point. How many ways, S8?
- 56 S8: Er... you can put the decimal point in front of 2?
- 57 Mary: In front of 2, in between 3 and 2? So, we can have 3.2. what else can we have?
- 58 S8: 0.32
- 59 Mary: 0.32. We can put it in front of the 2 numbers, we can have 0.32, 32.0, and anymore?
- 60 S9: 0.032
- 61 Mary: 0.032. So, you see we can have many ways to place the decimal point, but why are we so sure that this is the final answer, that this is the correct answer? ... Yes, S10?
- 62 S10: You could put it in between, because it's a, you know because 0.8 times 4, and then 8 is in the tenth place, so that 4 is actually the ones place, so it is like... since there is already a ones place that you need to multiply by, which is 4, it can't be a zero
... [After some discussion]
- 69 Mary: It cannot be zero in a ones place, because you are multiplying by 4 already. That is what he is (S10) trying to say. So, since you have 0.8 times 4, it should be more than 1, is that what you are trying to say? So, we eliminate which two answers? This one, and this one. But why can't it be 32.0? S11? Thank you, S10.
- 70 S11: Let us say the, since the question is 0.8 times 4, we can round 0.8 to 1...

In this series of exchanges between various students, Mary demonstrated her ability to orchestrate mathematically productive talk around the answers. Rather than endorsing or refuting the answers given by her students (See Line 32, and 69), she provided opportunities for her students to reasoning mathematically. She could have simply endorsed the students' answers and the discussion could have ended. Instead, Mary attempted to build on students' responses and moved the discussion towards strengthening the reasoning behind the answer. At the end, Mary used S11's answer that 0.8 is approximately one to highlight the importance of thinking about the reasonableness of an answer using estimation.

CONCLUDING REMARKS

When Mary's lesson plan and teaching moves are examined in terms of the dimensions of opportunities to learn (Walkowiak et al., 2017), it can be argued that Mary had optimised the time used during the lesson to orchestrate productive discussions around a modified typical problem. The lesson plan suggests her ability to notice the possibility of using typical problems such as 0.8×4 to create opportunities for students to reason, beyond simply explaining the procedure of multiplying decimals. Her teaching moves also suggested that she was able to notice opportunities to develop students' reasoning by engaging students to discuss the placement of the decimal point. Mary's instructional decisions during planning and lesson enactment reflect those of an experienced competent teacher, which is surprising given that she is a beginning teacher. What, and how, did Mary notice the opportunities to learn through the task? A more in-depth study of Mary's instructional decisions may yield some insights into her pedagogical reasoning processes, which will have implications for teacher professional development.

Acknowledgment

This paper refers to data from the research project "Building a culture of collaboration and listening pedagogy in classrooms through Lesson Study for Learning Community: An exploratory study in a primary school in Singapore" (OER 19/17 LKE), funded by the Office of Educational Research (OER), National Institute of Education (NIE), Nanyang Technological University, Singapore, as part of the NIE Education Research Funding Programme (ERFP). The views expressed in this paper are the author's and do not necessarily represent the views of NIE.

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INSTRUCTIONAL INNOVATION IN MATHEMATICS COURSES FOR ENGINEERING PROGRAMS – A CASE STUDY

Jason Cooper, Hadas Levi Gamlieli, Boris Koichu, Ronnie Karsenty, &

Alon Pinto

Weizmann Institute of Science, Israel

While the affordances of problem-based learning are broadly recognized, implementation of this innovative approach is not common, particularly in tertiary mathematics education. This study investigates early stages of an implementation of problem-based instruction in 1st year mathematics courses for engineering students, within a project encompassing 12 universities and colleges across Europe. Twenty-three lecturers from participating institutions took part in a preparatory workshop. Framing the project as a case of diffusion of innovations, we analyze post-workshop questionnaires to reveal the participants' conception-of and attitudes-toward the innovation. We highlight some challenges that the innovation entails, and how they relate to participants' general attitude toward implementing the innovation.

THEORETICAL BACKGROUND

First year university mathematics courses are considered challenging in general, and for engineering students in particular (Jablonka, Ashjari, & Bergsten 2017). In addition to general issues of transition from high school, the abstract approach to the discipline that is common in mathematics departments may not be appropriate for students who will eventually use mathematics as a practical tool for solving problems.

Problem-based learning (PBL; Savery, 2006) is an educational approach by which authentic real-world problem situations provide the impetus and the context for studying disciplinary content. Often this approach is implemented as project-oriented PBL (PO/PBL), where authentic problems emerge in the context of a long term project. While such approaches have been studied mainly in the context of pre-college education, PBL (project-oriented or otherwise) is common in some universities (e.g., Aalborg University, Denmark), and has, in particular, been found to be suitable for engineering education (Perrenet, Bouhuijs, & Smits, 2000). While the potential gains of such an approach are undisputed, implementing instructional innovation can be challenging (Begg, Davis, & Bramald, 2003). In this article we investigate challenges related to implementing PBL in 1st year mathematics courses for engineers.

Processes of adoption of innovation have been studied for many years, and the adoption model elaborated by Rogers in his 1995 book *Diffusion of Innovations* (2003) has been used extensively in many contexts, including educational innovation (e.g., Sahin, 2006). Rogers has recognized four main elements in the diffusion of innovations – the *innovation* and its perceived consequences, *communication channels* of diffusion, evolution of the diffusion over *time*, and the *social system* in which the diffusion takes place. Rogers has focused on the innovation decision process, which he describes as “an information-seeking and information-processing activity, where an individual is motivated to reduce uncertainty about the advantages and disadvantages of an innovation” (Rogers, 2003, p. 172). While approaches to teaching may often be prescribed by educational institutions, the details of what ultimately takes place behind the closed doors of classrooms and lecture halls are up to individual instructors. Hence, this decision process is highly relevant in any process of instructional innovation.

Early stages of the decision process are influenced by three factors (see table 1): Prior conditions (in particular *previous experiences* related to the innovation); *Knowledge*, including *how-to*-knowledge about the implementation and use of the innovation and *principles*-knowledge about how and why the innovation “works”; *Persuasion*, whereby adopters develop an affective attitude toward the innovation, influenced by the characteristics of the innovation as perceived by individuals. These characteristics include: A. *Relative advantage* compared to the current state of affairs; B. *Compatibility* with past experiences and with existing conditions and values; C. *Complexity* – the degree to which the innovation is perceived to be difficult to understand or to use; D. *Trialability* – the extent to which the innovation can be experimented with on a limited basis.

Prior conditions	<u>Previous practice</u>	Need for innovation	Innovativeness	Norms
Knowledge	Awareness	<u>How to</u>	<u>Principles</u>	
Persuasion:	<u>Advantage</u>	<u>Compatibility</u>	<u>Complexity</u>	<u>Trialability</u> Observability

Table 1: Rogers’s model of diffusion of innovations (relevant aspects are underlined)

We conducted our research in the context of an international project whose objective is to improve teaching, learning and understanding of 1st year mathematics among engineering students in European countries. Innovative

teaching methods, in particular PO/PBL, are the main vehicle for achieving this objective. We focus on a point in time immediately following a preparatory PBL workshop for participating lecturers. Eventually, these lecturers will decide whether, how and to what extent to implement the innovation. Their decision will be influenced by their perception-of and attitude-towards the innovation. Hence our research questions are:

Following a preparatory workshop on PBL, (1) What are the participants' perceptions of PBL as an instructional innovation to be implemented in their teaching? (2) What are their attitudes towards the innovation?

SETTING AND METHODOLOGY

iTEM – innovative Teaching Education in Mathematics – is a Capacity Building project for higher education funded by the Erasmus Plus program (EU) as of 2019. Its main objective is to improve teaching, learning and understanding of 1st year mathematics among engineering students in Europe through the implementation of PBL. Sixteen academic institutions are partners in the project. Twenty-three mathematics lecturers from partner institutions participated in a workshop organized by the University of Aalborg in Copenhagen, whose goal was to inform and inspire participants on how to integrate PBL-oriented ideas in their teaching. The workshop was preceded by online individual preparation, which included reading assignments on PBL approaches to teaching and learning and written exercises. The 2-day workshop comprised group-work and plenary sessions on the following topics: Real-life problems and strategies for their integration in university teaching; the special nature of assessment in PBL; challenges and opportunities of the approach; active learning and group work on problem solving. At the end of the workshop participants submitted an anonymous questionnaire (see table 2), followed by a plenary discussion on some of the questions. While the primary purpose of the questionnaire was to provide formative assessment of the workshop, the participants gave their written consent for using their responses for the research reported herein.

Data for the research consists of 17 completed questionnaires (6 were too incomplete to be useful). Pre-workshop submissions and video recordings of all the sessions including a plenary discussion following the submission of the questionnaire – were used as complementary data sources. Rogers's (2003) model of diffusion of innovations was used as a conceptual framework for analyzing participants' responses and utterances. To each response we ascribed one or more aspects of the model. Coding was for the most part consistent with Table 2, where we indicate for each question the categories of the framework that we expected respondents to attend to, though in some cases respondents attended to additional categories. The aspects of the innovation that respondents

chose to attend to were very different, providing a rich qualitative image of different conception of the innovation and the challenges it poses.

Reviewing respondents' questionnaires, it seemed clear that some had a more positive attitude toward the innovation than others. We operationalize this attitude as follows: Some questions invited a positive response, some a negative response, and others were phrased neutrally. Each response that was more positive/negative than the question invited scored ± 1 , and the sum of these scores over all questions constitutes the overall attitude. For example, responses to q1 (see Table 2) that only listed main ideas, but did not consider one that *appealed* to the respondent scored -1, and responses to q4 that did not list any ideas that the respondent will *not* implement scored +1. Our aim in this was to reveal which aspects of the respondents' perception of the innovation correlate with a general attitude toward the innovation.

	Question	Diffusion categories
1	List the main ideas raised in the course. Consider one idea that appealed to you, and elaborate or exemplify it.	Knowledge (neutral) Advantage (positive)
2	Is there any idea discussed in the course that you already implement in your current teaching? If so, please indicate which and describe how.	Prior practice (neutral)
3	Was there an idea discussed in the course that was novel to you? What do you think of this idea?	Innovation- (dis)advantage (neutral)
4	Was there an idea discussed in the course that you do not think you will implement in your teaching? If so, why not?	Innovation- disadvantage/ compatibility (negative)
5	Of all the mathematical examples discussed in the course, which did you most connect to?	Innovation-advantage (positive)
6	Which resources are needed in order to implement the ideas discussed in the course, or some of them, in your teaching? Which of these resources are already available to you in your institution and which would need to be developed?	Innovation- compatibility (neutral)
7	To what extent do you feel ready to implement the course ideas in your teaching? What, if at all, are you missing in order to do that? What kind of support are you expecting?	Knowledge-how (positive); Compatibility (negative)
8	Towards summing up the ideas in a Teaching Methodology Document, which ideas in your opinion still need to be added or elaborated upon?	Knowledge-how (negative)
9	In the next course, which idea(s) would you like go deeper into?	Knowledge (neutral)

- 10 Please add any comments that you feel are not covered by your previous responses. Neutral

Table 2: Questionnaire items mapped to framework categories, classified as inviting positive/negative/neutral response

ANALYSIS AND FINDINGS

We begin with a report of the diversity in respondents' perception of the PBL innovation, organized according to the categories derived from Rogers's model. We then focus on eight respondents whose attitudes were most/least positive.

Prior Conditions: Previous Practices

Many respondents reported prior practices pertinent to the PBL innovation, mainly in response to question 2. Practices included the use of real-world problems in homework, tutorials or lectures (7 respondents), some project work for course credit (4), and assessment practices that are consonant with those presented in the workshop, including varied and frequent testing, often based on real-life problems (4). One respondent's response was particularly revealing in this respect: "I thought I did [implement ideas discussed in the course], but now I understand that I did not. I feel this is a big step forward for me". This serves to remind us that self-reports of prior practices are highly subjective, and further indicates that at least this respondent gained some principles-knowledge regarding PBL, as discussed in the following subsection.

Knowledge: Principles-knowledge and How-to-knowledge

Eight respondents attended in their responses to principle-knowledge that they acquired during the course. Some responses were vague (e.g., attending to the general nature of PBL and recognizing its challenges), while others were more specific, attending to the independence of students (3) and the role of the teachers (1) where problems do not necessarily rely on prior teaching (1), the relationship between PBL and courses based on project work (1), and the ability of PBL-based instruction to attend to variance in student background and prior learning (1).

Nearly all respondents (15/17) attended to aspects of how-to-knowledge that were addressed in the course, including how to construct or find appropriate real-world problems (6) and integrate them in their mathematics courses (8), how to implement novel assessment methods (8), organize group work (2), motivate self-learning (2) and make use of visualization software (2). Nevertheless, many (11) attended to how-to-knowledge that was still lacking or inadequate. For the most part, they felt a need for more detailed guidance for implementing aspects of PBL, including assessment methods (6), teaching methodologies (6) and in particular those that can motivate students' independent learning (4). Some (2) attended to the challenge of connecting the innovation (real-world problems and novel teaching methods) with current

practices (prescribed curriculum and traditional teaching methods). This need for more practical knowledge was further elaborated in the plenary discussion. It was recognized that while the workshop could teach principles, the details of implementation in mathematics courses for engineers are quite unique, and will need to be figured out and shared by the partners. One respondent, who recognized the benefits of PBL principles, felt a need to be introduced to alternate non-PBL ways of achieving them. This demonstrates that the persuasion stage of diffusion, discussed in the following subsection, is conceptually separate from the knowledge stage, as Rogers has claimed.

Persuasion: Perceived Characteristics of the Innovation

Relative advantage: Nine of the 17 respondents referred to advantages of the innovation over their current practices. The advantages they attended to were: the active nature of students' participation in learning and their increased responsibility (5), increased applicability and relevance of student learning for their future studies and employment (3), utilization of existing technologies for enhancing learning (2), advantages of novel assessment methods, including frequent tests and oral peer-assessment (4). Some respondents recognized inherent advantages in the way PBL learning is organized, whereby problem solving precedes instruction (1) and there is a stronger connection between lectures and homework (1).

(In) compatibility and complexity: All but two respondents (15/17) attended to aspects of perceived compatibility and complexity of the innovation, mainly in response to questions 6 and 7. We do not distinguish between these categories, because they were often intertwined (e.g., incompatibility due to complexity). Once again, some responses were vague (2, e.g., I can't implement full PBL), while others attended to specific issues, including a rigid curriculum (4), large numbers of participants in courses (4), student maturity and willingness to cooperate (3), university policy (1), rigid lecture-hall arrangement that does not support group-work (1), staff devotion (1) and time constraints (1). Few respondents attended to positive aspects of compatibility, stating that their institution is basically flexible (1), that changes in the way material is presented in courses (2) and in the way the budget is managed (1) can be accommodated, and that the necessary hardware, software and facilities are in place (2). Mainly in response to question 6, respondents indicated some resources that would be required for implementing the innovation, including: curricular and assessment material (7), an increase in internal and external (mentor) staff (4) and additional time (4), which would entail an increase in budget (2), classroom facilities (3), IT facilities and support (5), instructional resources – both printed (1) and internet-based (2). Two respondents noted that communication channels between the project partners would be a crucial resource.

Trialability: Six respondents attended to their ability to try out the innovation on a limited scale. Five spoke of implementation with small pilot test groups, while one spoke of small scale integration of real-world problems across many courses.

Attitude toward the Innovation

According to our coding scheme, attitudes ranged between +6 (very positive) and -2 (somewhat negative). We now focus on the four most positive and four least positive respondents, and describe some commonalities and differences.

Positive attitude: Focusing on the four respondents whose attitudes were the most positive, measured at +6 (hereafter respondent A), +5 (B), +4 (C) and +3 (D), we find that all of them had some kind of previous practice related to the innovation – innovative projects integrated into coursework (A, D), use of mathematical software (A), diverse and frequent formative assessment (A, C), group work (A). The three most positive respondents (A, B, C) felt that their institutions (B), departments (A, C), and their students (C), would be supportive of the changes that the innovation would entail. In other respects, this group was indistinguishable from the rest.

Negative attitude: Focusing on the four respondents whose attitudes were measured at -2 (P, Q) and -1 (R, S), we find that only one of them (P) attended to principle-knowledge. Three of them listed significant incompatibilities of the innovation, including a large number of students per course (Q, S), and three of them (Q, R, S) felt that they lack how-to-knowledge for implementing appropriate assessment. In other respects, this group was indistinguishable from the rest.

DISCUSSION

We have analyzed what the respondents – 17 lecturers from participating institutions who attended a PBL workshop – attended to in their responses to a post-workshop questionnaire. There is a dialectic relationship between the data and the conceptual framework that guided our analysis. We first discuss what we have learned about the respondents' perception-of and attitude-toward the innovation, relying on the framework. This part of the discussion has practical implications for the implementation of instructional innovation of a similar nature. We then discuss implications in relation to the conceptual framework based on Rogers's (2003) categories and about its applicability in the context of tertiary instructional innovation.

Elaboration on Decision-process in PBL as an Instructional Innovation

All the participants in the workshop belong to educational institutions that have opted to participate in the iTEM project, and as such are expected to be committed to the instructional innovation that it entails. Yet the extent and nature of their implementation of the innovation is ultimately up to them, thus it

makes sense to analyze their responses to the post-workshop questionnaire as revealing a decision process about adoption of innovation, which Rogers's model proved useful to analyze.

One interesting finding of our study is the relationship between the participants' prior experience with related innovative practices and their attitudes towards the current innovation. The four most-positive respondents all reported some prior experience, while only one of the four least-positive respondents reported such experiences. This suggests that prior exposure may be a significant factor in the decision process.

How-to-knowledge was an explicit focus of the workshop, and nearly all the respondents attended to aspects of such knowledge in their responses. Of the 11 respondents who attended to how-to-knowledge they were lacking, three were among the least-positive respondents, while only one was among the most-positive. In contrast, the two groups were indistinguishable regarding principles-knowledge (1 of 4 attended to this in each of the groups). This suggests that a clear vision of the practical details of implementation may be a significant factor in the decision process.

Regarding the perceived characteristics of PBL as an innovation to be accepted or not, more than half of the respondents attended to advantages of the innovation, nearly all attended to incompatibilities of the innovation, and one third attended to its trialability. Perhaps surprisingly, the most-positive and the least-positive respondents were not distinguishable in any of these respects. This suggests that a positive attitude to an innovation does not necessarily rely on optimal conditions for its implementation.

Additional Implications

Perceptions-of and attitudes-toward the PBL innovation discussed above have practical implications for the implementation of instructional innovation. A careful reading of the findings section may yield quite specific implications for similar contexts (tertiary mathematics for engineering programs), while the more general patterns that we have discussed in the previous subsection may have implications in a broader context.

From a theoretical perspective, we have considered implementation of an instructional innovation as a case of diffusion of innovations, and have shown how Rogers's (2003) model of decision-processes can serve as a conceptual framework for the analysis of emerging attitudes to the innovation. We have operationalized a notion of positive-attitude regarding implementation, and have shown how it is constituted in the framework. We now discuss some peculiarities of the framework in the context in which it has been applied.

Of the four main elements of diffusion of innovations we have attended primarily to *innovation*. We now turn our attention to *time* and *communication*

channels, and suggest some extensions of the model. While for Rogers (2003) time is relevant primarily as a factor in the rate of innovation-adoption, the respondents in our investigation referred to the time element in varied contexts: as a crucial resource for implementation (scarce lecture and tutorial time, and the need for more preparation time), yet also as an important element in the decision process (“it takes time to process and think”). While for Rogers, the relevance of communication channels is primarily in creating and changing *adopters*’ (i.e., lecturers’) attitudes, one respondent drew attention to the role of *students* in this process – “this year’s students will advertise the change to the next year ones” – highlighting a systemic aspect of the diffusion. Additionally, some respondents viewed channels of communication as crucial for sharing knowledge and resources between project partners (examples of real-world problems, knowledge about educational software and its use, etc.), and also for sharing expertise within universities (help from the engineering department in formulating real-world problems that target particular mathematical content).

In conclusion, our study has put forth the importance of an emerging infrastructure for collective and mutually supportive adoption of instructional innovations in tertiary mathematics education. We believe that such infrastructures can be developed within consortia of institutions, as the case of the iTEM project suggests.

Acknowledgment

The iTEM project is supported by Erasmus Plus grant 2018-3265.

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THE RELATION BETWEEN ELEMENTARY STUDENTS' CONDITIONAL REASONING AND ALTERNATIVES GENERATION: THE CASE OF MATHEMATICS

Anastasia Datsogianni¹ and Stefan Ufer¹

¹University of Munich (LMU), Germany

Alternatives generation can be considered as knowledge-based prerequisite for conditional reasoning in elementary school students. This study examines the role of alternatives generation skills on conditional reasoning within an everyday and a mathematical context. A total of N=102 students from 2nd, 4th, and 6th Grade in Cyprus were interviewed. Alternatives generation skills predict correct conditional reasoning in both contexts, but interesting differences occurred. The results from the everyday context mirror previous results, predicting correct Acceptance of the Consequent and Denial of Antecedent reasoning and inhibiting correct Modus Tollens reasoning. In the mathematical context, alternatives generation predicted correct reasoning in all forms. The study points to the specific role of mathematical knowledge in conditional reasoning with mathematical concepts.

INTRODUCTION

Logical reasoning is considered as a key component of advanced thinking amidst human species (Markovits & Barrouillet, 2002) while if-then statements form the basis of scientific mathematical thinking (Markovits & Lortie-Forgues, 2011). Reasoning with if-then statements (e.g. 'If Anna breaks her arm, then it hurts') refers to conditional reasoning. Current theories describe conditional reasoning in younger students as a process that is based on semantic representations of the statements involved. Thus, it is an open question, to which extent domain knowledge influences conditional reasoning skills. In everyday contexts, reasoners' ability to generate multiple alternative models for a given condition (e.g. 'For which other reasons might Anna's arm hurt?') has been found as a predictive factor to draw valid inferences even from early age (e.g. De Chantal & Markovits, 2017). For conditions about mathematical concepts, this generation of multiple alternatives is similar to generation of alternative solutions for mathematical problems which according to Leikin and Lev (2007) is considered as an indicator of students' creativity and mathematical knowledge. However, our knowledge about the connection between alternatives generation skills and conditional reasoning in the context of elementary school mathematics is still weak.

CONDITIONAL REASONING

Conditional reasoning tasks are formed of a conditional rule “if p, then q” as a major premise, and a minor premise (e.g. “q is not true”). The traditional interpretation of conditionals considers p as sufficient, but not necessary for q (Evans & Over, 2004). Four different minor premises lead to four possible logical forms of inference: *Modus Ponens* (MP; “p is true, so q is true”), *Modus Tollens* (MT; “q is false, so p is false”), *Denial of Antecedent* (DA; “p is false, so q or not q”) and *Acceptance of the Consequent* (AC; “q is true, so p or not p”). Thus, the uncertain logical forms AC and DA do not allow for definite conclusions about p and q respectively. The other two forms (MP and MT) allow valid definite conclusions.

According to Mental Model Theory (MMT) inferences are drawn through the construction of mental models (Johnson-Laird & Byrne, 2002). MMT has been found to describe conditional reasoning accurately not only in adults but also in the age group of primary school children (e.g. Markovits, 2000). Mental models are semantic representations of the possibilities, given the truth of the premises (Johnson-Laird & Byrne, 2002). To derive conclusions, individuals reconstruct the meaning of premises based on their knowledge, to represent what is possible given the premises (Nickerson, 2015). Based on working-memory considerations, Barrouillet and Lecas (1999) proposed an evolvement of individuals’ conditional reasoning skills starting from a conjunctive-like interpretation (only one model ‘p and q’; correct MP reasoning), to a biconditional (‘p and q’; ‘not-p and not-q’; correct MP and MT reasoning), and then a conditional interpretation (‘p and q’; ‘not-p and not-q’; ‘not-p and q’; correct reasoning in each logical form). This evolvement shows up in increasing solution rates for MT, followed by later changes towards a conditional interpretation with increased solution rates for DA and AC. ‘Alternatives’ are mental models of the type ‘not-p and q’, which are necessary to arrive at the indefinite conclusions in the AC and DA forms. Beyond model generation, other authors also state that MT and DA are more cognitively demanding compared to MP and AC forms due to the negation statements involved (Johnson Laird & Byrne, 1993).

Conditional Reasoning & Alternatives Generation in Everyday Contexts

According to MMT, generation of mental models, and in particular of alternatives, is based on knowledge about the content of the conditions. Alternatives generation for a given condition is considered as a crucial prerequisite to draw valid inferences (Johnson-Laird & Byrne, 2002; Markovits & Barrouillet, 2002).

Studies on reasoning with conditions from an *everyday context* (e.g. De Chantal & Markovits, 2017) have shown that alternatives generation skills predict conditional reasoning even from pre-school age onward. In these studies, alternatives generation skills are associated with correct AC and DA reasoning, in particular (Cummins et.al, 1991; Markovits & Vachon, 1990). In addition to alternatives generation, individuals might also generate *disablers* (mental models of the form '*p and not-q*', contradicting the major rule) describing inhibitory factors which might prevent q from occurring, even in the presence of p (e.g., 'Anna took a painkiller, so her arm does not hurt, even though it is broken'), according to Cummins et al. (1991). Disablers might lead to the rejection of valid conclusions for MP and MT inferences (Janveau-Brennan & Markovits, 1999). Many studies report a positive correlation between the numbers of generated alternatives and disablers (Thompson, 2000; De Neys, Shaeken, & D'Ydewalle, 2002).

In this line of reasoning, studies with university students revealed that correct DA and AC reasoning correlates negatively with correct MT reasoning (Newstead et.al, 2004). Hence, it is likely that alternatives generation is positively linked with AC and DA reasoning, being not or negatively related with MT reasoning. In prior research with young students, disabler generation is considered less relevant for logical reasoning than alternatives generation (Janveau-Brennan & Markovits, 1999; De Chantal & Markovits, 2017).

Conditional Reasoning and Alternatives Generation in Mathematical Contexts

While the role of alternatives for conditional reasoning is well-studied in the everyday context, it has not been studied for conditions that involve mathematical concepts (mathematical context; e.g., "*If I arrange three rows of four squares each, then I need 12 squares.*"). In this case, alternatives generation concerns the mental construction of mathematical objects that fulfill '*not-p and q*' (e.g., "*12 squares could be constructed by six rows of two squares each*"), beyond those that represent '*p and q*' (or '*not-p and not-q*'). Generating such alternative perspectives to mathematical situations is often discussed in research on multiple solutions (Leikin & Lev, 2007). Based on this perspective, alternatives generation can be assumed to require mathematical knowledge of the conditional content (Leikin & Lev, 2007). Beyond a general link between mathematics skills and conditional reasoning skills (Attridge & Inglis, 2013), this could lead to a specific influence of mathematical knowledge on AC and DA conditional reasoning in the mathematical context. Studies in the field of mathematics with university students show negative correlation between MT form and DA as well as AC form (Attridge & Inglis, 2013; Morsanyi, McCormack & O' Mahony, 2017). This backs up the assumption that conditional reasoning in this context is based on mental model construction, similar to the everyday context.

Overall, the existing literature in primary school pupils investigates the relation between alternatives generation and conditional reasoning only in the everyday context (with different levels of abstraction (Markovits & Lortie-Forgues, 2011). In the mathematical context, research with elementary school students either investigate conditional reasoning (Christoforides, Spanoudis & Demetriou, 2016) or multiple solution tasks (Sullivan, Bourke & Scott, 1997). Yet, to date research has not addressed alternatives generation in relation to conditional reasoning in two different contexts and this study aims to fill this research gap.

STUDY GOALS AND QUESTIONS

This study aimed to transfer results on the role of alternatives generation in elementary students' conditional reasoning from everyday conditions to conditions from a mathematical context. The following study questions are addressed:

(1) Is the influence of alternatives generation skills on conditional reasoning specific to the respective context (everyday vs. mathematical)? Based on the MMT account, we expected alternatives generation in each context to primarily predict conditional reasoning in the corresponding context (e.g. De Chantal & Markovits, 2017).

(2) Do alternatives generation skills predict correct reasoning differently across the four logical forms? Based on prior results and the MMT account, we expected that alternatives generation skills in the everyday context would predict correct AC and DA reasoning (Markovits & Vachon, 1990). However, considering that alternatives generation skills are related with the generation of disablers in the everyday context (Thompson, 2000; De Neys, Shaeken & D'Ydewalle, 2002), this might entail a decrease in MP and MT reasoning. For the mathematical context we expected alternatives generation skills to be related to correct AC and DA reasoning, but not to MT reasoning (Attridge & Inglis, 2013; Morsanyi, McCormack, & O' Mahony, 2017).

METHODS

In this cross-sectional study, $N=102$ students from 2nd, 4th, and 6th Grade in Cyprus (average age 12 years old) were interviewed individually. The feasibility of the instrument was piloted in a previous study, showing that it is accessible to this age group of students (Datsogianni, Ufer, & Sodian, 2018). Ethics approval, parental consent signed form, and students' individual oral assents were obtained.

Participants solved four conditional reasoning tasks on each context (Cronbach's α : .62. for everyday and .68 for mathematical contexts respectively). The everyday conditions referred to daily life situations. Conditions in the mathematical context referred to situations that involved

mathematical structures which were supposed to be familiar for the participants (e.g., “If a dwarf’s house has exactly 3 rows of 4 rooms each, then it has 12 rooms”). All forms (MP, MT, DA, and AC), were included in each task. The order of two contexts, the order of the conditions in each context, and the order of logical forms for each condition were randomized across students.

Alternatives generation skills were measured afterwards with specific tasks in each context, using the same situations as in the conditions (4 mathematical and 4 everyday). The reliability scores were good (Cronbach’s $\alpha=.86$ for everyday and $.76$ for mathematical contexts respectively). The experimenter (first author), asked students to find as many examples as they could that matched the model ‘not-p and q’ by drawing their ideas. Participants did not receive positive or negative feedback. The order of two contexts and the order of the conditions in each context were randomized across students.

Example of alternatives generation task in the everyday context: “Remember what Peter found out before. If a glass is dropped on the ground in the kitchen, then there is a sound. Peter is at home and hears a sound in his kitchen. Find as many reasons why a sound in the kitchen may occur, as you can.”

Example of alternatives generation task in the mathematical context: “Remember that dwarfs build their houses so that there are rooms which all have this form: The houses always have one or more rows of rooms which are all equally long. Remember what Peter found out before. If a dwarf’s house has exactly 3 rows of 4 rooms each, then it has 12 rooms. How could a dwarf house with 12 rooms look like? Draw as many different houses as you can.”

In the end of the interview procedure, students solved a working memory test (backward digit span). Separate linear mixed models for each context were used to analyze the data using the package lme4 in R, controlling for working memory skills. The factor logical form and the alternatives generation scores were included in the model. Insignificant interactions between logical form and the alternatives generation scores were removed from the model prior to the final analysis. In the mathematical context, the random factor controlling for individual differences explained no variance.

RESULTS

Overall, students solved 59.1% of the items correctly in the everyday context (ED), and 57.1% in the mathematical context (MA). In both contexts, MT was solved significantly less often than MP (ED: MP 88.9%, MT 73.7%, $p < .001$; MA: MP 84.3%, MT 61.5%, $p < .001$) and AC less often than MT (ED: AC 42.2%, $p < .001$; MA: AC 37.2%, $p < .001$). DA was solved less often than AC in the everyday context (ED: DA 31.5%, $p < .05$), while the difference was not significant in the mathematical context (MA: DA 45.65, $p = .16$).

Students generated more alternatives per task in the everyday context (range 1-11, $M = 4.25$) compared to the mathematical context (range 0-5, $M = 2.08$). It is worth noting that the number of possible alternative solutions was more limited in the mathematical context compared to the everyday context.

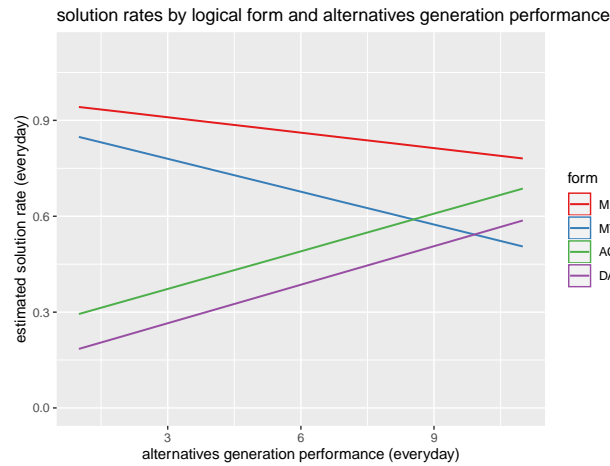


Figure 1: Estimated solution rates and 95% confidence intervals by alternatives generation and logical form for everyday context

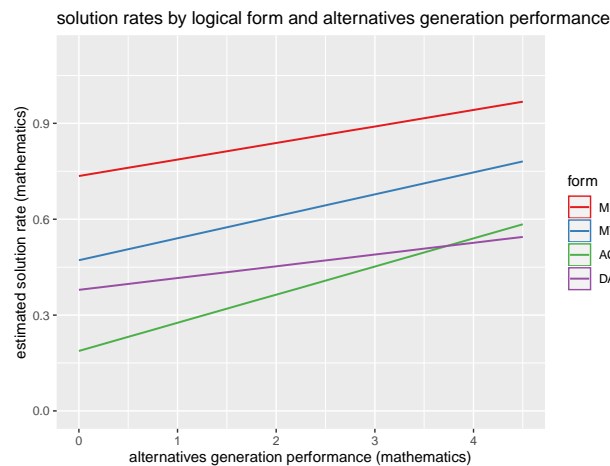


Figure 2: Estimated solution rates and 95% confidence intervals by alternatives generation and logical form for mathematical context

Regarding study question (1), mathematical alternatives generation ($F(1,398)=13.7$, $p < .001$), but not everyday alternatives generation ($F(1,398)=0.32$, $p = .57$), showed a significant effect on conditional reasoning in the mathematical context. Conditional reasoning in the everyday context was related significantly to mathematical alternatives generation ($F(1,98)=8.35$, $p < .001$) but – over all logical forms – not to everyday alternatives generation ($F(1,98)=0.70$, $p = .41$).

Regarding study question (2), we found a significant interaction between *logical form* and alternatives generation scores only for everyday conditional reasoning ($F(3,300)=6.57$, $p < .001$, fig. 1). In this context, everyday alternatives generation predicted correct reasoning significantly positively in the AC

($B = 0.039$, $CI_{95\%}[0.008, 0.070]$) and in the DA ($B = 0.040$, $CI_{95\%}[0.010, 0.071]$) form, but negatively for MT ($B = -0.034$, $CI_{95\%}[-0.065, -0.004]$). For example, $B = 0.039$ indicates an estimated increase in the conditional reasoning solution rate of 3.9% per generated everyday alternative. The non-significant interaction ($F(3,398)=0.75$, $p = .52$, fig. 2) between *logical form* and mathematical alternatives generation for mathematical reasoning indicates, that alternatives generation (positively) predicted conditional reasoning comparably strongly for all logical forms in this context.

DISCUSSION

Regarding study question (1), alternatives generation skills in the everyday context did not have a significant main effect on conditional reasoning in the same context in general (cf. De Chantal & Markovits, 2017). Given the significant interaction between everyday alternatives generation scores and logical form, this pattern is in line with prior results. Mathematical alternatives generation skills predicted logical reasoning in both contexts. Since alternatives generation is mainly based on prior knowledge of the respective content (Leikin & Lev, 2007), this is in line with previous evidence about the relation between logical reasoning and mathematical knowledge that has been found in the literature, before (Attridge & Inglis, 2013).

Regarding study question (2) and the results for the everyday context were similar to those found in prior studies. Alternatives generation in the corresponding context was predictive for correct AC and DA reasoning (Cummins et.al, 1991; Markovits & Vachon, 1990). It is also observed that alternatives generation (in this context) inhibits correct MT reasoning; probably students extend the strategy of generating antecedents to generating and (incorrectly) interpreting inhibitors (De Neys, Shaeken, & D'Ydewalle, 2002). However, as for the mathematical context, it seems that alternatives generation is generally predictive of conditional reasoning skills, mostly independent of the logical form. As mentioned above this might reflect a general relation between logical reasoning and mathematical knowledge (Attridge & Inglis, 2013). On the other hand, we cannot differentiate this explanation in this study – it might also be that knowledge about the mathematical content is necessary to generate a representation of mathematical conditionals and any kind of related mental model (not only of the type ‘not-p and q’). If the sole representation of mathematical conditions is indeed so strongly dependent on corresponding knowledge, this might cover a specific effect of alternatives generation for DA and AC in this context.

Overall, the results of this study indicate that reasoning with mathematical conditions is, overall, not substantially harder, or easier than reasoning with everyday conditions. However, the analysis of the relation to alternatives generation points to possible differences in the reasoning process. For example,

it might be that, in spite of early conditional reasoning skills in the everyday context, students are not able to activate the corresponding strategies in the mathematical context, due to restricted ability to represent the conditionals' meanings in mental models. If indeed problem representation turns out as the primary problem, this implies the necessity not only to practice conditional reasoning, but also to carefully consider students' mathematical knowledge before engaging with basic deductions about mathematical concepts in mathematics instruction. However, reflecting deductions and the meaning of conditionals about mathematical concepts during classroom instruction might also help to build up this prerequisite knowledge.

One possible limitation arising from this study is that alternatives generation tasks addressed only questions with given consequents, for which students had to create as many possible antecedents, as possible. Future studies might separately measure the generation of an initial mental representation of the conditional and alternatives generation. The negative relation between alternatives generation and MT reasoning, moreover, might be explained through investigating the generation of disablers in future research. However, alternative antecedents have been considered more central to conditional reasoning of young students than disablers (Janveau-Brennan & Markovits, 1999). Thus, this study provides first insights for the relation between alternatives generation and conditional reasoning with mathematical concepts, which can be extended in further research.

Summarizing, the role of conditional reasoning in mathematics can hardly be denied. Thus, the results of this study imply the instructional necessity to include and practice conditional reasoning tasks in elementary school within the context of mathematical statements by providing opportunities to students to interpret and discuss mathematical conditions, as well as generate alternative antecedents for these conditions. Even though open question remains, the study extends evidence, that knowledge of mathematical concepts and being able to reason about them (with conditionals) are strongly related.

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PRIMARY SCHOOL TEACHERS' MATHEMATICAL KNOWLEDGE FOR LAKATOS-STYLE PROOF INSTRUCTION

Dimitrios Deslis¹, Andreas J. Stylianides¹, & Mateja Jamnik¹

¹University of Cambridge, United Kingdom

In this paper we conceptualise Lakatos-style proof instruction (LSI) as a teaching approach based on the formulation and refinement of conjectures through the examination of supportive examples and counterexamples. We identify aspects of mathematical knowledge that LSI requires from teachers (MaKTeLaP) in relation to content, student perceptions and teaching practices, and we report findings of ten primary school teachers' MaKTeLaP based on in-depth, vignette-based interviews. Participants' responses indicate satisfactory content knowledge, intermediate knowledge of teaching practices, and weak knowledge of students. This study offers a theoretical basis for further research on the incorporation of LSI into the classroom and on the provision of support for the development of teachers' MaKTeLaP.

INTRODUCTION

Proof is considered as a key element of mathematics (Bundy, Jamnik, & Fugard, 2005), but also a focal point of modern school mathematics (Hanna, 1990; Stylianides, 2016). Many scholars have acknowledged the crucial role example examination can play in proving, including the philosopher Imre Lakatos. In his seminal book "Proof and Refutations", Lakatos (1976) used an imaginary classroom setting to narrate historic moments in the evolution of conjectures around Euler's theorem. Although his theory was not intended for use in actual classrooms, certain aspects of it have been considered in educational contexts (e.g., Balacheff, 1991; Komatsu, 2010, 2016; Larsen & Zandieh, 2008). These studies offered images of school and university students productively engaging in Lakatos-style proving activity, illustrating also the benefits of instruction that aims to engage students in this kind of activity. Despite their encouraging findings, teachers' mathematical knowledge for Lakatos-style proof instruction has received no research attention thus far. In this paper we take a step towards addressing this gap. We developed a conceptualisation of Lakatos-style proof instruction and of important aspects of mathematical knowledge for teaching (Ball, Thames & Phelps, 2008) that Lakatos-style proof instruction requires from teachers, and we used those as a theoretical basis in an interview study with ten primary teachers to address the

following research question: What is the state of primary school teachers' mathematical knowledge for Lakatos-style proof instruction?

THEORETICAL FRAMEWORKS

LSI: Lakatos-style Instruction

Considering Lakatos' (1976) original theory and studies describing attempts for the incorporation of certain aspects of it into classrooms (e.g., Balacheff, 1991; Komatsu, 2010, 2016; Larsen & Zandieh, 2008) we conceptualised LSI: a teaching approach inspired by, and based on, Lakatos-style reasoning. LSI consists of four phases: *P1- Formulation*, *P2- Validation*, *P3- Refutation*, *P4- Modification*. First, students are presented with a task and are asked to formulate a relevant conjecture (P1). Then, they examine several cases to check whether it is true. The discovery of supportive examples may lead them to assume that it holds for all the cases of its domain (P2). However, the discovery of counterexamples may indicate to them that the conjecture does not hold (P3). Reflecting on the examined examples, students may modify the original conjecture (P4), and then test and refine it, going through the same steps again.

Conjecture types and modification techniques, and student justification and refutation schemes are all relevant to LSI. We elaborate on them next.

MT: Modification Techniques

This aspect of our framework elaborates on P4 of LSI. To modify a conjecture, solvers may employ two Lakatosian techniques: *Restriction* and *Expansion*. As the names indicate, the domains of the conjecture can either be restricted to exclude the counterexample or expanded to include it as a supportive example. As an illustration, let us imagine that the conjecture "the digits of multiples of nine add up to nine" is refuted by 99. By employing restriction, one could examine whether this is applicable only for numbers up to 90, while by employing expansion one could conjecture that the digits of the sum should continue being added until they give a one-digit result.

CT: Conjecture Types

This is a categorisation of statements, as proposed by Tsamir, Tirosh, Dreyfus, Barkai, and Tabach (2008), that identifies three types of conjectures: *Always True (AT)*, *Sometimes True (ST)*, *Never True (NT)*. The type of conjecture in LSI is determined by the kinds of examples a solver may discover during its investigation: while both supportive examples and counterexamples exist for an ST conjecture, only supportive examples and only counterexamples exist for AT, and NT conjectures, respectively.

JS: Justification Schemes

This is a three-level taxonomy of students' perceptions about the role of supportive examples in proof based on Harrel and Sowder's (1998) framework

of *justification schemes* and their adaptation by Stylianides and Stylianides (2009). The students holding the least sophisticated perception believe that evidence from a few supportive examples suffices to validate general statements (*naïve empirical justification scheme*). Others continue to believe so, but demand that the examined cases be selected on the basis of a strategy or rationale (*crucial experiment justification scheme*). Finally, others realise that supportive examples cannot prove, no matter how many there are or how they have been selected (*nonempirical justification scheme*).

RS: Refutation Schemes

Aside from students' views about supportive examples, their understanding about counterexamples has also attracted researchers' attention (e.g., Balacheff, 1991; Stylianides & Al-Murani, 2010). Still, unlike supportive examples, there exists no widely accepted categorisation of student perceptions about counterexamples. Aiming to address this gap, we developed a three-level taxonomy of students' *refutation schemes* (Deslis, 2020). Starting from the least advanced level, some students resist to the idea that the existence of counterexamples can affect the validity of a convincing conjecture, treating them as exceptions (*naïve objection to refutation scheme*). Others acknowledge counterexamples' power to refute, on the condition that several are found, usually coming from a strategical selection of cases (*crucial experiment refutation scheme*). Finally, students may realise that even a single counterexample suffices for the refutation of conjectures (*refutation scheme*).

MaKTeLaP: Mathematical Knowledge for Teaching Lakatosian Proof

Previous research has attempted to extend Shulman's (1986) popular construct about teachers' professional knowledge, to outline the knowledge required for the instruction of specific subjects, including mathematics (e.g., Ball, Thames & Phelps, 2008) or even more specifically proof (e.g., Stylianides, 2011). Our Framework *MaKTeLaP* (Deslis, 2020) attempts to further specialise those constructs by describing the knowledge a teacher needs to implement LSI. *MaKTeLaP* consists of three interrelated types of knowledge: *CoLaP*, *StuLaP*, *TeLaP* (standing for knowledge of *Content*, *Students* and *Teaching*, respectively). *CoLaP* refers to knowledge of the role examples play in proving and refuting. It is expressed by the ability to produce valid example-based arguments or evaluate the validity of arguments raised by others. *StuLaP* focuses on knowledge of students' typical understandings about the interplay between examples and proof. In the classroom, teachers are expected to anticipate, recognise, and analyse students' relevant misconceptions. Finally, *TeLaP* refers to knowledge of appropriate instructional techniques that can effectively support students' engagement with Lakatosian methods and procedures. It is associated with the ability to provide guidance through feedback or questions to promote the productive use of examples. All three

components of MaKTeLaP revolve around two types of examples, which are the backbone of LSI: supportive examples (*SEs*) and counterexamples (*CEs*). Therefore, alongside with the previous analysis, we can also distinguish one strand of knowledge for each example type: *MaKTeLaP-SE* and *MaKTeLaP-CE*, which both run across the three components.

RESEARCH METHODS

Data were collected through in-depth, semi-structured, vignette-based interviews with ten in-service primary school teachers in Greece (2 males; 8 females; average teaching experience: 1y 9mo). The participants were recruited through a convenience sampling strategy, conditional upon teaching experience at fifth and/or sixth grade (10-12 y.o.). All of them held a bachelor's degree in primary education while all but two participants also held a master's degree in various education-related fields.

During the 50-minute open-ended interviews the participants were presented with 19 short classroom episodes (EP1-19) depicting students' exchange of arguments within groups about the "Count the Squares" proof task (Zack, 1997). The problem asked students to examine how many squares there are in square grids of various sizes starting from the 4-by-4 grid and up to the 60-by-60 grid, and to prove their answers. The vignettes reflected situations that are likely to emerge in the classroom covering all the possible combinations of LSI phases, modification techniques, conjecture types, justification and refutation schemes, as described in the theoretical framework. The vignettes were based on real classroom episodes reported by Zack (1997) and Reid (2002), although in some cases adaptations were necessary. To make the conversations more engaging and immersive, the vignettes were presented through comic-style representations. Like in Lakatos' original book, the protagonists were named after letters of the Greek alphabet. Aside from a tribute to his work, this (in conjunction with the comic characters' generic appearance) enabled the concealment of characteristics like ethnicity and gender, thus preventing teachers from judging on the basis of potential biases. As an illustration, parts of two sample episodes are given below:

EP6- CT: Sometimes True, LSI Phase: P2-Validation, JS: naïve empirical justification

Stud. Iota: I checked the 5x5 grid and found 55 squares. Like in the 4x4 grid, the result is again a multiple of five. So, it will be a multiple of five in all grids!

EP9- CT: Sometimes True, LSI Phase: P3-Refutation, RS: refutation

Stud. Theta: I think I've found a grid that breaks the pattern. Imagine an 1x1 grid: it has only one square. This is not a multiple of five. So, the rule is wrong!

The design of the interview protocol was guided by our conceptualisation of MaKTeLaP and its three components. After each of the 19 episodes, participants were asked to (1) evaluate the validity of students' arguments, (2) comment on the level of students' understandings and predict their upcoming moves, and (3) explain what they would encourage students to do next by suggesting feedback and/or questions they would use in the classroom. In this way, information about participants' perceptions was elicited, relevant to CoLaP, StuLaP, and TeLaP, respectively. After the transcription of the audio recordings, participants' responses were analysed for themes with a twofold focus: MaKTeLaP components and example types. The qualitative findings were supplemented by the calculation of descriptive statistics.

FINDINGS & DISCUSSION

The thematic analysis in conjunction with the examination of the guiding frameworks generated one set of three themes for each possible combination of MaKTeLaP components and example type. These six sets are given in Table 1, with the themes presented in descending order of sophistication (SE_2 - SE_0 and CE_2 - CE_0), which was determined by the degree of their alignment with the respective frameworks. To enable the calculation of statistics, every response was given a score ranging from 0 to 2, determined by the level of sophistication the respective response theme represented. Figure 1 visualises the variation of teacher participants' (T1-10) responses in the three MaKTeLaP components and the two example types, with reference to the themes their responses represented. The overall MaKTeLaP score was 1.5/2. The average performance was better in the SE-related episodes (1.57/2) than in the CE-related ones (1.43/2), although the respective differences varied within each knowledge type.

SE	CoLaP [SE]	StuLaP [SE]	TeLaP [SE]
2	Non-empirical justification	Awareness of SEs' misuse and consideration of it as such	Examination of SEs aiming at proper proof/refutation
1	Crucial experiment justification	Awareness but favourable consideration of SEs' misuse	Examination of SEs to terminate the investigation
0	Naïve empirical justification	Unawareness of SEs' misuse	Disregard of SE use
CE	CoLaP [CE]	StuLaP [CE]	TeLaP [CE]
2	Refutation	Awareness of CEs' misuse and consideration of it as such.	Examination of CEs aiming at conjecture modification
1	Crucial experiment refutation	Awareness but favourable consideration of CEs' misuse	Examination of CEs endlessly/ to terminate the investigation
0	Naïve objection to refutation	Unawareness of CEs' misuse	Disregard of CE use

Table 1: Response themes for each MaKTeLaP component and example type.

The formulation of response themes for CoLaP [CE] and CoLaP [SE] was informed by Frameworks RS and JS, respectively. As for the former (average score: 1.8/2), the wide acceptance of the idea that a single CE can sufficiently refute a statement (8/10 participants) was amongst the most positive findings of the study. Yet, although the majority (6/10) was also aware about the limitations of SE use in proving, a large proportion of participants were found to believe that empirical arguments can also sufficiently validate statements, resulting in a relatively lower average score of 1.6/2. Despite this difference between the scores in the two example types, the sample's overall CoLaP score was the highest among the three components of MaKTeLaP.

In contrast, the overall performance in StuLaP was the lowest of the three components (1.3/2), while the sample had, in general, better knowledge of students' perceptions around SEs (1.4/2) than those around CEs (1.2/2). As far as StuLaP [SE] is concerned, nine teachers showed an awareness of students' tendency to base their proofs on empirical arguments and expected that their next moves would reflect this erroneous belief. However, only five of them were fully aware that this constitutes a misconception. The remaining four evaluated favourably students who hastily validated the conjecture after the discovery of a few SEs and/or unfavourably others who legitimately questioned this practice. To illustrate this finding, commenting on a student's objection to example-based validation in EP8, participant T7 said: "This student has not understood how patterns work and how generalisations can come out of them. I believe that this view is unproductive."

Turning to StuLaP [CE], again nine participants were found to be aware of students' tendency to treat CEs as exceptions. However, the majority (6/10) critiqued students who argued that one CE suffices and/or misperceived other students' demand for additional CEs as productive or desirable. As an illustration, in EP16, where a student objects to the rejection of a conjecture after the discovery of one CE, T4 commented: "I like this student's critical attitude! It is always good to be reluctant and demand more evidence." Still, the other three participants consistently argued against the treatment of CEs as exceptions across all relevant episodes. In the same episode another teacher (T3) said: "This student has weak understanding. The conjecture had already been proved wrong; any additional checks would be superfluous."

Knowledge of	CE ₂	T8	T1, 6, 7, 10 T6	T2, 3, 4, 8 T2, 3 T4, 9
	CE ₁		T1, 7, 10 T1, 10	T5, 9 T4, 5, 9 T2, 3, 5, 7, 8

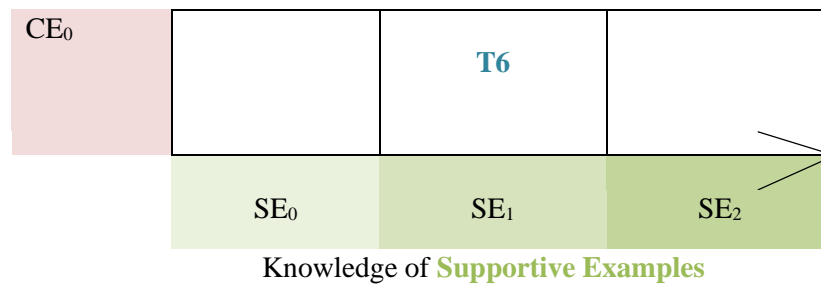


Figure 1: Comparative presentation of teachers' (T1-10) performance in **CoLaP**, **StuLaP**, and **TeLaP** in relation to the two example types.

The average performance in TeLaP was 1.5/2. In general, teachers promoted the search for, and examination of, both SEs and CEs, thus showing an appreciation of their role in proving. Overall, the responses indicate that participants are in a better place to support students' productive use of SEs (1.7/2), in comparison with the use of CEs (1.3/2), although difficulties relevant to both emerged. Three teachers encouraged students to conclude the investigation after the discovery of many SEs, like T10 in EP3: "The examples have shown that the rule works; I'd give the students a new problem". Furthermore, most teachers were either unable to suggest a sensible next step after the discovery of CEs, proposing a pointless never-ending examination of cases, or encouraged students to discard the faulty conjecture and replace it with a new one (unrelated to the starting conjecture). However, three participants took a step further giving responses that either implied conjecture modification in a generic sense or clearly referred to one of the Lakatosian modification techniques (usually conjecture restriction). T4's and T9's responses to EP10, and T6's to EP9 are indicative:

- Part. T4: The student can investigate under which conditions the rule can be correct.
- Part. T9: Clearly not all numbers are multiples of five. The student can re-examine the results [...] This could lead to a new rule... for example: "only grids with side multiple of five give a result that is also a multiple of five".
- Part. T6: The student should improve the conjecture by investigating whether the rule is correct only for a subset of the grid for example, only for every $n > 2$.

Responses of this kind indicate that some teachers are capable of spontaneously encouraging students to employ this less obvious, but crucial, step of the Lakatosian investigation, which constitutes an encouraging finding.

CONCLUSION

This interview study explored the thus far uncharted territory of primary teachers' mathematical knowledge about LSI, a style of teaching proof inspired by Lakatos-style reasoning. The vignette-based interviews shed light on ten

participants' knowledge in relation to (1) content, student perceptions and teaching practices relevant to LSI, and (2) two example types that constitute the backbone of the Lakatos-style investigation: supportive examples and counterexamples. In general, the sample's performance was better at situations relevant to supportive examples, although the results fluctuated within each knowledge type. Considering the performance in both example types, the participants' responses indicated satisfactory content knowledge, intermediate knowledge of teaching practices, and relatively weak knowledge of students. Although each teacher's strengths and difficulties were different, all teachers showed good intuitive perception of at least some aspects of LSI. No teacher performed excellently across all six scales, but for each scale there was at least one teacher who achieved the maximum score, despite the lack of any prior instruction. This shows that all aspects of LSI can lie within primary teachers' reach, and they can overcome the difficulties they encounter if offered adequate support.

A primary limitation of this study is the small sample size, due to which it is unknown whether any of the findings can safely be generalised. However, the theoretical frameworks and the exploratory findings of this study have laid the foundation for further research with larger samples and/or at different education levels. In future we will also explore appropriate ways of assisting teachers in refining the professional knowledge required for the implementation of LSI in classrooms, which may include the design of an interactive simulated environment based on choice-driven scenarios.

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DO QUALITATIVE EXPERIMENTS ON FUNCTIONAL RELATIONSHIPS FOSTER COVARIATIONAL THINKING?

Susanne Digel¹ and Jürgen Roth¹

¹University of Koblenz-Landau, Germany

Introducing functional relationships with experiments has proven to be beneficial for functional thinking (FT). While hands-on material elicits modelling schemes, simulations open up a dynamic view. Combining both seems promising, but the question on how remains unanswered. Prevalent approaches set a numerical focus through measurement, but research on the development of a functional concept strongly suggest a rather qualitative view to foster covariational thinking. This ongoing study compares two experimental settings (numerical vs. covariational) in a pre-post-test intervention. Preliminary analyses (N = 66) show that both settings lead to a significant increase in functional thinking, with higher gains in the covariational settings, indicating that a focus on covariation seems to be beneficial for 7th graders.

FOSTERING FUNCTIONAL THINKING

According to Vollrath (1989), functional thinking is based on three main aspects: the correspondence of an element of the definition set to exactly one element of the set of values; the covariation of the dependent variable when the independent variable is varied and the final aspect, in which the function is considered as an object.

Concept of function according to APOS

This differentiation is in line with the developmental perspective on students' conceptualization of functions derived by Breidenbach et al. (1992) using the Action-Process-Object-Scheme (APOS) theory. The action concept on the lowest level is limited to the assignment of single output values to an input. With the more generalized process concept students consider a functional relationship over a continuum, enabling the reflection on output variation corresponding to input variation. Finally, functions conceptualized as objects can be transformed and operated on. Students with an elaborate concept of functions are supposed to be able to use the action, process or object conception depending on the mathematical situation (Dubinsky and Wilson 2013).

Findings on experimental approaches to functional thinking

Experiments provide a basis to enable constructivist approaches, that lead to higher learning gains in combination with digital technologies (Drijvers 2020). And learning environments with experimentation activities have proven to be

beneficial for functional thinking (Lichti and Roth 2018). One possible explanation could be the proximity of functional thinking to scientific experiments as illustrated by Doorman et al. (2012): with a given variable as starting point, a dependent variable is generated in an experiment. Relating the output to the input clearly addresses the correspondence aspect and the action concept. Following manipulations of the input and concurrent observation of the output make the covariation of both variables tangible and enables a process view.

Lichti and Roth (2018) implement the scientific experimentation process – preparation (generate hypotheses), experimentation (test the hypotheses) and post-process (reflect results) – in a comparative intervention study to foster functional thinking of sixth graders with either hands-on material or simulations and report learning gains for both approaches (*ibid.*), but a closer look reveals disparities: while hands-on material promotes the correspondence aspect and the association to the real situation, simulations foster covariational thinking, the interpretative usage of graphs and lead to higher overall gains in functional thinking (Lichti 2019).

Theory of instrumental genesis

The instrumental approach (Rabardel 2002) and its distinction between artefact and instrument can be useful when interpreting these results: while the artefact is the object used as a tool, the instrument consists of the artefact and a corresponding utilization scheme that must be developed. This developmental process - the so-called instrumental genesis (Artigue 2002) - depends on the subject, the artefact and the task in which the instrument is used. Hence, different artefacts lead to different schemes. Artefacts that are more suitable for the intended mathematical practice of a task appear to be more productive for the instrumental genesis and facilitate the learning process (Drijvers 2020). In addition, embodied activities in a task seem to contribute to the instrumental genesis (*ibid.*). From the viewpoint of instrumental genesis, the results of Lichti (2019) can be interpreted as follows: when using simulations, schemes that develop are concerned with variation and transition, while measurement procedures of the hands-on material induce schemes that concentrate on values and conditions (*ibid.*). The students working with hands-on material associate their argumentation more often with the material, while the rationale of students using simulations frequently relates to the graph. Again, the instrumental genesis can explain these disparities: the hands-on material stimulates basic modelling schemes, relating the situation to mathematical description. Simulations already contain models of a situation and when used as multi-representational systems (Balacheff and Kaput 1997) illustrate connections between model and mathematical representation (e.g. graph and table) that evoke schemes for these representations and their transfer.

The study presented here attempts to make use of these beneficial influences on the instrumental genesis through an appropriate combination of hands-on material and simulations in experimental activities to foster functional thinking.

Setting 1: Experiments with hands-on material and simulations (numerical)

The learning environment is set in a story of two friends preparing to build a treehouse. The student activities are structured in six contexts (see below for details), each one laid out like a scientific experimentation process with preparation, experimentation and post-processing phase. Starting off with hands-on material to activate modelling schemes and enable embodied experience, students are asked to make assumptions about a pattern or relationship and on that basis, estimate values. During experimentation phase they take a series of measurements and data is recorded in a table within a simulation (GeoGebra). The simulation is designed in accordance to the hands-on material and provides the opportunity to create a graph concurrent with the context animation and to display the measurements of the hands-on material (and a corresponding trendline). This gives students the opportunity for systematic variation and parallel observation of the altering quantities, to induce schemes with a dynamic view and covariational thinking. Above, it facilitates the time consuming but little challenging representational switch from table to graph (Bossé et al. 2011). In the post-processing phase the students verify their measurements and analyse the graph (interpreting and interpolating). Subsequently they get back to the real material to check their estimations from preparation phase. Finally, they elaborate on the answer to the overarching task (calculate the amount of material needed to build the treehouse) based on the insights from experimentation activities, bringing together the modelling and representational schemes developed.

The learners go through these phases for three contexts subsequently, share their insight after each context with a partner and solve the overarching tasks as team.

Contexts

Both settings use a treehouse building story with identical overarching tasks. The contexts are implemented with the same hands-on material (see figure 1 and 2) and simulations, but different components of the simulations are visible in the settings.

The students work in pairs (A and B), each working on three contexts. The contexts are chosen to represent a linear and a quadratic relationship and one with varying change rate.



Figure 1: Hands-on material of the contexts for partner A

For partner A (see Figure 1) these are: the perimeter of a circular disc determined by its diameter, the number of cubes needed for a “staircase” determined by the number of steps and the fill height of a vessel determined by the volume of water filled into.



Figure 2: Hands-on material of contexts for partner B in both settings

Partner B (see Figure 2) examines the weight of a package of nails determined by the number of nails, the number of beams needed for a woodwork determined by the number of floors and the fill height of cylindric vessels with different diameters determined by the volume of water filled into. A bonus context for quick learning teams depicts the diameter of an unrolling tape determined by the length of tape that has been unrolled.

Setting 2: Combination of artefacts with a focus on covariation (covariational)

In setting 1 proposed above the measurement plays a dominant role, which sets a focus on the individual values of quantities and on single states of the relationship. This promotes the action concept of function and concentrates on the correspondence aspect (see above). In accordance with Breidenbach et al. (1992) and Dubinsky and Wilson (2013) it would be desirable to shift this focus to a process concept and to covariation, especially since possible sources of student’ difficulties with functional relationships are seen in the dominance of numerical settings in school (Goldenberg et al. 1992). Together with the close relation of covariation to the difficult concept of variables (Leinhardt et al. 1990), this led to the call for a qualitative approach to functions (Thompson & Carlson 2017) to facilitate the idea of covariation. Thus, in a second setting we explicitly choose a non-numerical approach for experimenting with immediate examination of covariation.

The learning environment of setting 2 is structured accordingly to setting 1, with modifications in the experimental structure of the contexts: in the preparation phases the students estimate subsequent values of a quantity represented in the hands-on material, before they use simulations to identify the relation between quantities. In the following experimental phase, the students observe the variation and covariation of the quantities in the simulations and verbally describe the relationships discovered. Subsequently graphs are generated within the simulations and in the post-processing phase students are asked to analyse the form of the graphs and connect their insights with the relationship described in the previous phase. Similar to setting 1 the students then team up with their partner and share their insights, but here they are asked to compare both contexts and identify similarities in the relations. In an additional phase they take measurements in the context of their partner, represent the covariation in the measurement table and compare this to the results reported by their partner. As a final task the partners are asked to group the contexts by their kind of covariation, i.e. build pairs of similar contexts based on their findings.

The settings can be accessed in digital classrooms: www.geogebra.org/classroom Code: HQX7 UZRQ for the numerical Setting (1) “Team of Engineers” Code: D3XM DDSB for the covariational Setting (2) “Team of Architects”.

STUDY DESIGN

A comparative intervention study (pre-post design) contrasts the two approaches with regards to their effect on students’ functional thinking to answer the following research question:

1. Do both settings (numerical and covariational) based on experiments with hands-on material and computer-based simulations (GeoGebra) lead to significant effects on the FT of seventh graders?
2. Does the covariational setting lead to a significant different effect on FT than the numerical setting?

The intervention is designed for six lessons (split into three sessions) in the seventh grade and comprises of. It is preceded and followed by a short test on functional thinking (FT-short, 27 items, Rasch-scalable, see Digel and Roth 2020, online version of test: www.geogebra.org/m/undht8rb), to compare the learning outcomes in both settings. Students work in groups of two pairs and two focus groups (low-/high-performer in FT-short) per school class are videotaped. For an in-depth analysis on the learning progression throughout the intervention it is planned to evaluate the student products and videotapes regarding the aspects of functional thinking in general (Lichti 2019) and covariational thinking in particular (levels of covariational reasoning, Thompson and Carlson 2017).

A pilot study (Digel and Roth 2020) verified the comparability of the two approaches in terms of processing time and difficulty. Due to the corona shutdown the intervention was adapted to an online classroom supplemented with a “math box” containing the hands-on material. The qualitative analysis of group interaction was replaced by an expert rating and a reflective analysis with student teachers.

METHOD

Here we present preliminary results of the ongoing main study. It is an intermediate quantitative analysis on the data collected so far. Four additional intervention groups (N~100) are scheduled from January to May 2021 and a control group is planned as well. A statistical power analysis (2 groups, 2 measurements, power .85, $\alpha = .05$) for a medium effect ($\eta_p^2 = .06$) in a mixed ANOVA gave a desired sample size of $N = 144$.

Data analysis was conducted according to Item Response Theory. The dichotomous one-dimensional Rasch model and the virtual persons approach were used to estimate an item difficulty for every item of FT-short ($N = 132$). The person ability was then estimated with fixed item difficulties. We applied a mixed ANOVA (between factor: numerical/covariational setting; within factor: time) after controlling data for normal distribution and homogeneity of variance. Pairwise t-tests were used to investigate differences of both settings.

RESULTS

The estimation of the Rasch-model, which was used to determine the person abilities, showed good reliabilities in the pre- and post-test: $EAP-Rel_{pre} = .73$ and $EAP-Rel_{post} = .77$ as well as $WLE-Rel_{pre} = .73$ and $WLE-Rel_{post} = .76$.

The mixed ANOVA (see Figure 3 left) resulted in one significant effect and one minor effect: First, there was a significant main effect for time $F(1.64, 0.42) = 45.54$, $p < .001$, $\eta_p^2 = .42$. The results in FT-short for the total sample (numerical and covariational setting together) increased significantly with a large effect from $M = -.61$ logits ($SD = .96$) up to $M = .21$ logits ($SD = .99$). The subsamples of both settings did not differ before the intervention ($t(64) = -.55$, $p = .132$), but results of both increased significantly from pre- to post-test (numerical: $t(66) = -2.61$, $p = .005$, $d = .32$; covariational: $t(62) = -3.84$, $p < .001$, $d = 0.50$). The mixed ANOVA also showed a non-significant interaction effect between time and setting ($F(1.64, 0.42) = 1.32$, $p = .256$, $\eta_p^2 = .02$).

The intersecting discrepancy in the effect sizes (pre/post) for both settings indicate a difference, that could possibly not be identify due to the lack of statistical power. Since there were large SDA SD for both groups a second analysis was performed on a subset of FT-short, using the 27 items assigned to covariation and object aspect of functional thinking, representing a concept of functions developing towards a process scheme.

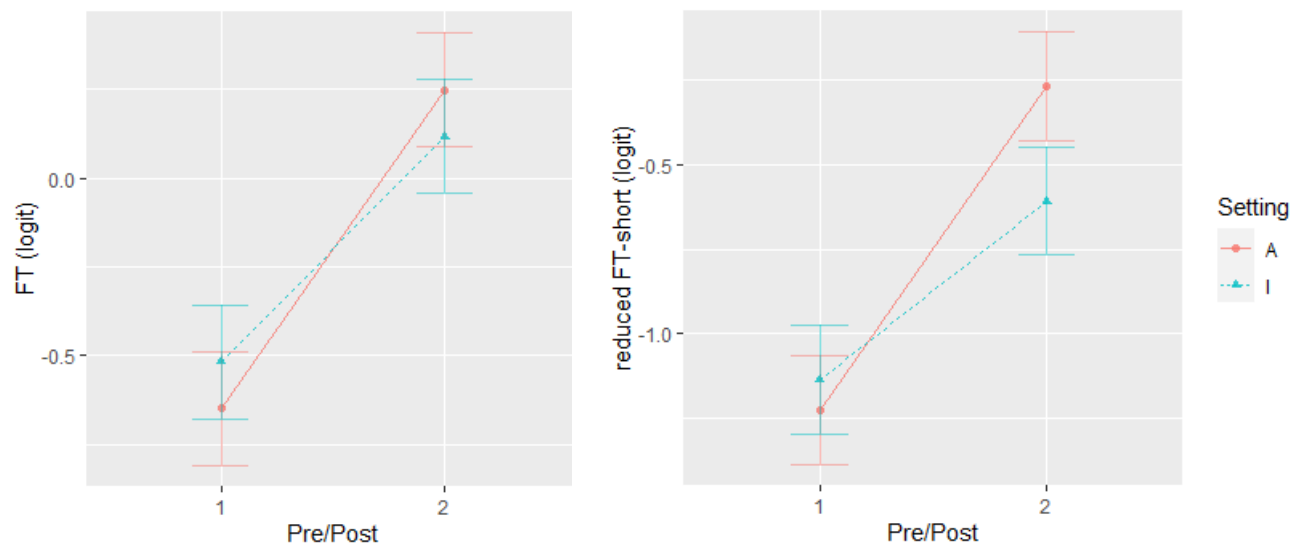


Figure 3: Increase in FT-short (left) and reduced FT-short (right): comparison of covariational (A) and numerical (I) setting in pre- and post-test

The estimation of the Rasch-model, which was used to determine the person abilities in the reduced FT-short showed acceptable to good reliabilities in the pre- and post-test: $EAP-Rel_{pre} = .71$ / $EAP-Rel_{post} = .74$ as well as $WLE-Rel_{pre} = .67$ / $WLE-Rel_{post} = .73$.

The mixed ANOVA (see Figure 3 right) resulted in two significant effects: again, there was a significant main effect for time $F(1.64, 0.42) = 43.11$, $p < .001$, $\eta_p^2 = .40$. The results in reduced FT-short for the total sample (numerical and covariational setting together) increased significantly with a large effect from $M = -1.23$ logits ($SD = 1.12$) up to $M = -.45$ logits ($SD = 1.08$). The subsamples of both settings did not differ before the intervention ($t(62) = -.33$, $p = .371$), but results of both increased significantly from pre- to post-test (numerical: $t(66) = -1.85$, $p = .034$, $d = .23$; covariational: $t(62) = -3.84$, $p < .001$, $d = 0.51$). The mixed ANOVA also showed a significant interaction effect between time and setting ($F(1.64, 0.42) = 3.58$, $p = .050$, $\eta_p^2 = .053$).

The sum scores of the six items that were excluded from the FT-short were not significantly different between both settings in the pre- and post-test (pre: $t(62) = .52$, $p = .696$; post: $t(64) = .59$, $p = .772$). Sum scores are reported here since the subset of these items was not Rasch-scalable.

DISCUSSION

The results show an increase of FT in both settings from pre- to post-test, with a small effect for the numerical setting (FT-short $d = .32$ / reduced FT-short $d = .23$) and a medium effect for the covariational setting ($d = .49$ / reduced $d = .51$). There seems to be an influence of both interventions on FT. Regarding the function concept and the intended covariational thinking, we found a first indication, that the covariational setting seems to be more suitable to foster this aspect: there was a small to medium interaction effect of time and setting

($F(1.64, 0.42) = 3.58, p = .05, \eta_p^2 = .053$). The results of FT-short in the items for covariation and object aspect increases significantly more in the covariational setting. Hence for our first research question, we can conclude that both learning environments, as designed in this study, lead to a significant increase of students' FT. Considering the second research question, there is no evidence on significant differences on FT in general, but our results indicate that the setting with a focus on covariation tends to be more suitable to foster the covariation and object aspects of FT than the numerical approach. Since no data for a control group is available at the moment and with regards to the sample size that does not match the power analysis, this conclusion must be handled with caution and needs to be verified. Results on the extended sample will be presented at the conference. Overall, our results are not generalizable, they depend on the concrete settings developed in the study as introductory course for sixth to seventh graders.

Nevertheless, with reference to our theoretical background, we can assume that in both settings hands-on material and simulations are combined in a supportive way for the instrumental genesis. The results for the reduced item set of FT can be interpreted as a first indication that the covariation aspect of FT is also accessible to learners in introductory courses on functional relationships and can be fostered through non-numerical experiments with simulations and hands-on material. Moreover, this benefit for the covariational aspect is not on the expense of the correspondence aspect.

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EXPERT NORMS FOR DEALING WITH STUDENTS' MATHEMATICAL THINKING IN DIFFERENT CULTURES

Anika Dreher¹, Anke Lindmeier², Paul Feltes¹, Ting-Ying Wang³, & Feng-Jui Hsieh³

¹Freiburg University of Education, Germany

²IPN – Leibniz-Institute for Science and Mathematics Education, Kiel, Germany

³National Taiwan Normal University, Taipei, Taiwan

There is a growing interest in the international mathematics education community in research on teacher noticing as an important component of teaching expertise. However, it is likely that often the researchers' understanding of good instructional quality influences what they expect teachers to notice. It is particularly not clear if and how different cultural norms of instructional quality influence how teacher noticing is operationalized in East Asian and Western cultures. Therefore, our cross-cultural research project on teacher noticing in Taiwan and Germany focuses in a first step on eliciting such expert norms. By means of a vignette-based online expert survey, we explored culture-specific norms regarding instructional quality. In this paper, we provide evidence of culturally different norms for dealing with students' thinking.

INTRODUCTION

Students' mathematical thinking is a focus that has been frequently used for investigating and developing teacher noticing – especially in the US context (e.g., Colestock & Sherin, 2015; Jacobs, Lamb, & Philipp, 2010). An underlying reason for this focus is the idea that instructional quality depends heavily on whether and how teachers attend to, interpret, and deal with students' thinking in the mathematics classroom. Corresponding research usually uses – at least implicitly – a frame of reference for what the teachers are expected to notice (so-called “target noticing”, Stockero & Rupnow, 2017). However, it is well known that Western and East Asian perspectives on what constitutes high quality mathematics classrooms are different in many aspects (Leung, 2001). Since such different norms probably influence how teacher noticing regarding students' thinking is assessed by researchers in different cultures, it is questionable whether such research can be cross-culturally valid (Clarke, 2013). Therefore, it is important in our inter-cultural research community to make such culture-specific norms, which may influence how teacher noticing is assessed, explicit and take them into account for the interpretation of findings. Consequently, this research report focuses on revealing how researchers' (i.e.,

experts') norms for dealing with students' thinking can be different from a Western and an East Asian perspective.

THEORETICAL BACKGROUND

Especially in the last decade, teacher noticing has been established as an important component of teaching expertise in the international research community in mathematics education. Although different conceptualizations of teacher noticing can be found in the growing body of research, essentially, they encompass the perception and interpretation of relevant features of instructional situations (Sherin, Jacobs, & Philipp, 2011). Hence, in line with Sherin (2007), we understand teacher noticing as attending to aspects of classroom situations that are relevant for instructional quality (selective attention) and interpreting them by drawing on corresponding professional knowledge and beliefs (knowledge-based reasoning). Similarly, many different operationalizations of the construct exist, but it is widely accepted that vignettes in the form of short videos, comics, or transcripts can be used as representations of practice. Furthermore, a common "operational trick" for assessing teacher noticing is to design or select vignettes in a way that in the represented instructional situation something occurs that does not meet the expectations of "good" teaching, that is, they include a *breach of a norm* regarding some aspect of instructional quality (e.g., Dreher & Kuntze, 2015; Herbst & Kosko, 2014). The teachers' reaction in response to these critical incidents is then used as an indicator for the specific noticing expertise.

This kind of operationalization makes particularly obvious that norms regarding aspects of instructional quality play a double role in teacher noticing research: Such norms are assumed to shape what teachers notice and they also form the frame of reference that is already implemented (more or less explicitly) in the operationalization by the researchers. In particular, researchers use the consistency of their own norms with what teachers notice as an indicator for noticing expertise (e.g., Stockero & Rupnow, 2017). Hence, it is not clear whether such research can be cross-culturally valid, since corresponding norms may be culture-specific (e.g., Louie, 2018).

Especially East Asian and Western cultures, it is well-known that different perspectives on mathematics classrooms exist. Leung (2001) contrasted for instance characteristics and underlying values of East Asian and Western mathematics education by means of six dichotomies: product versus process; rote learning versus meaningful learning; studying hard versus pleasurable learning; extrinsic versus intrinsic motivations; whole class teaching versus individualized learning; and competence of teachers: subject matter versus pedagogy. He emphasized that these distinct characteristics "are based on deep-rooted cultural values and paradigms" (p. 35) and thus influence the perspectives of mathematics educators on mathematics classrooms. He pointed

out for instance that although mathematics educators from both East Asian and Western countries would say that mathematics is both the product (a body of knowledge with distinctive knowledge structure) and the process (a distinctive way or process of dealing with particular aspects of reality), their position on the continuum between the two extremes is different: While the contemporary Western perspective is that the process of doing mathematics is more important than the content arising out of the process, the East Asian perspective is rather that ultimately the content and its correctness are essential (Leung, 2001, p. 39). Although there is also diversity within and among Western countries as well as East Asian countries (e.g., Clarke, 2013), Germany and Taiwan can be considered as representatives of Western and East Asian cultures in this respect (Yang et al., 2019). Against this background, it can be assumed that aspects of instructional quality, which are in the focus of research on teacher noticing are perceived differently from researchers in Taiwan and Germany. Specifically regarding the focus of students' thinking, Colestock and Sherin (2015) identified for instance different purposes for attending to students' mathematical thinking, which may depend on different overarching instructional goals, such as diagnosing student errors or misunderstandings that need to be addressed or looking for students' ideas that have the potential to serve as the foundations for new understandings. In their study, they explored different teacher-identified purposes for attending to students' mathematical thinking and found that the teachers focused on these purposes to various degrees. However, Colestock and Sherin (2015) did not take into account the perspectives of experts in mathematics education or different cultural contexts and hence it is still an open question whether there exist different cultural norms for attending to and dealing with students' thinking in the mathematics classroom.

OBJECTIVE

According to the need for research pointed out in the previous section, the objective of this research report is to illustrate how expert norms for dealing with students' thinking can be different from a Western and an East Asian perspective.

SAMPLE AND METHODS

The vignette that we focus on in this contribution (see Figure 1) is part of a larger bi-cultural instrument developed in a process comparable to the dual-focus approach (Erkut et al., 1999). The vignette was authored by the Taiwanese researchers in our team. Accordingly, from their perspective, the represented classroom situation contains a breach of a norm regarding how the teacher deals with students' thinking. In this case: The teacher does not address S1's misunderstanding and inadequate use of strategy (over-generalizing the strategy applicable in the case " $f \times g = 0$ ") properly.

Teacher T is reviewing the topic *solving quadratic equations* a month after its introduction. T writes the following task on the blackboard: "Solve the quadratic equation $(x+1)x=6$."
[Students work on their own for 1 minute.]

S1: I got the answer, it's 2.
T: Does anyone else have a different answer?
S2: My answer is different. x can also be -3 .
T: Good! This answer is correct!
S1: OK, but how can you find the second x ?
[Several students raise the same question.]
T: First of all, you must be very careful when you see a quadratic equation. Usually it has two solutions. In this case, you have to extend the equation [writes $x^2 + x = 6$ on the blackboard] and move 6 to the left [writes $x^2 + x - 6 = 0$]. Now, we can solve the equation. How do we usually do this?

Figure 1: Taiwanese vignette focusing on students' thinking

When the German researchers in our team saw this vignette, they agreed with the idea that the teacher should have asked S1 how he or she obtained the answer. However, they had problems to see a misunderstanding or an inadequate strategy. We figured that these different perspectives on the student's thinking in this classroom situation may not be restricted to our research teams and thus we anticipated underlying cultural differences between the perspectives of Taiwanese and German experts in mathematics education. To investigate whether this was indeed the case or whether this was just a matter of different perspectives in mathematics education in general, this vignette was presented to Taiwanese and German professors of mathematics education in an online expert survey.

This online survey was conducted in the native languages of the experts (Chinese/German). The necessary translation processes were carried out according to the ITC Guidelines for Translating and Adapting Tests (ITC, 2017). A sample of $n_1=19$ Taiwanese professors (6 females, 13 males) from 10 universities and a sample of $n_2=19$ German professors (5 females, 14 males) from 14 universities completed the survey. All of them were researchers as well as educators in mathematics education. Most of them had experience as school teachers (TW: 14, GER: 17) and some of them had also conducted research in mathematics (TW: 5, GER: 6). To capture the experts' frame of reference for investigating teacher noticing with a focus on students' thinking, the experts were asked to answer the same open-ended prompt that would be used to assess corresponding teacher noticing: "Please evaluate how the teacher deals with students' thinking in this situation and give reasons for your answer."

Their evaluations were analyzed with respect to two main aspects: 1) Did they see some breach of a norm regarding the teachers' dealing with S1's thinking?

And if so: 2) Which norm was breached from their perspective? Hence, in a first step, the answers of the participants were coded in a top-down process regarding the question whether the teachers' dealing with S1's thinking was evaluated as insufficient/inadequate. In a second step, the answers were analyzed regarding the question why the teacher's dealing with S1's thinking was evaluated negatively in order to infer which norm was breached from the perspective of the expert. This step was partly inductive: On the other hand, we coded whether the experts saw the same norm being breached as the developing Taiwanese researchers (S1's misunderstanding and inadequate use of strategy is not addressed). Likewise, as mentioned above, we expected especially the German experts may see different reasons. Hence, reasons indicating a different norm were also extracted inductively from the experts' answers.

To allow all authors to engage in the coding process of all experts' answers and to compare them directly across cultures, the answers were translated into English and all the language versions were included in the coding processes. Moreover, all of the answers were coded independently twice by all of the authors: In a first round, the coding scheme was complemented inductively and in the second round the resulting coding scheme was applied to all of the answers. In both rounds, the coding was first compared within the national research teams and discrepancies were resolved through discussion. Subsequently, the resulting national coding was compared again, in case of discrepancies, a consensus was reached through discussion.

In view of the aim to identify culture-specific or inter-cultural norms of instructional quality regarding the aspect "dealing with students' thinking", we finally took a look at how many of the experts in each country recognized a breach of a specific norm regarding this vignette. For interpreting this result, it should be considered that even if a specific norm exists, it cannot be expected that all the experts' answers indicate that they noticed the corresponding breach. There may always be individual experts who do not agree with commonly accepted norms in their culture. Furthermore, like teachers, the experts had to accomplish a process of noticing which becomes visible in their answer in way that we could code it accordingly. Thus, we assumed that if most of the experts from one country actively recognized the breach of a specific norm, then there was strong evidence for the existence of this norm in the corresponding culture.

RESULTS

As the Taiwanese team authored this vignette, we start by focusing on the answers of the Taiwanese experts, in the sense of a validation within a culture. Indeed, almost all answers (17 out of 19) indicated negative evaluations of how the teacher dealt with Student 1 (S1)'s thinking, suggesting these experts saw breach of a norm for dealing with students' thinking. A total of 11 experts' answers indicated that they assumed that S1's answer shows a problem to be

addressed (misunderstanding/inappropriate strategy), which was not done properly by the teacher. These experts recognized the breach of the norm, which was implemented by the Taiwanese research team. To get some insight into these kinds of experts' answers, we will now focus on two typical examples.

TW1: The teacher allowed two students to propose their answers. However, after detecting that one student's answer was incomplete, the teacher did not ask him further, how he got the answer to guide him to figure out where the problem is on his own.

TW2: [...] the teacher gave a correct method but did not bother to find out why S1 found only one solution.

Both experts criticize that the teacher did not ask S1 how he got his answers, indicating that they recognized a breach of a norm for dealing with students' thinking. While TW2 identified a problem in the fact that S1 found only one solution, TW1 makes even more explicit that there is a problem to be addressed regarding S1's thinking.

In view of the answers by the German experts, it became quickly obvious that the situation was different: While also most of the German experts saw some kind of breach of a norm for dealing with S1's thinking, only one answer indicated that a problem was seen in S1's thinking that should have been addressed. Instead, different reasons for why the teacher should have dealt differently with S1's thinking were mentioned. To provide insight into these kinds of answers, we will give three examples.

GER1: [...] The fact that S1 immediately saw a solution in the given equation, namely 2, is an expression of number sense or structure sense. However, this achievement remains completely without recognition by teacher in this situation. [...]

GER2: The teacher does not appreciate the achievements of the students to find solutions through thinking. However, it is appropriate to address other ways of solution as well.

GER3: The teacher ignores the students' abilities to use the method of looking closely (or Viéta's formula). The teacher wants the students to use the standard way of solution via the "Mitternachtsformel" or p-q formula. This hinders the development of flexible solution strategies by the students. [...]

Hence, from answers like these, another kind of reason for seeing the teachers' dealing with S1's thinking as inadequate/insufficient was extracted and added to the coding scheme: S1's answers hint at a valuable mathematical ability or strategy, which should have been appreciated and encouraged. To investigate further, whether these two perspectives reflected culture-specific norms of instructional quality regarding dealing with students' thinking, the resulting coding scheme was applied to all the experts' answers as described above. This

is allowing to distinguish the following cases: i) breach of originally implemented norm recognized; ii) breach of alternative norm recognized, and iii) breach of unidentifiable norm recognized or no breach of a norm recognized. For the first three cases, it was necessary that the teachers' dealing with S1's thinking was evaluated as insufficient/inadequate. Which of the three cases were applied depending on the kind of reason that was identified for this evaluation. The comparison of the number of these cases among the experts in Taiwan and Germany presented in Table 1 clearly shows the differences. Most of the Taiwanese experts actively recognized the breach of the norm implemented by the Taiwanese research team. Most of the German experts' evaluations indicated that they recognized a breach of a different norm corresponding to another kind of purpose for attending to students' mathematical thinking in this classroom situation ("mathematical strategy/ability to be valued").

	Taiwanese experts	German experts
Breach of "original" norm recognized	11	1
Breach of "alternative" norm recognized	0	10
Breach of unidentifiable norm recognized	6	4
No breach of a norm recognized	2	4

Table 1: Numbers of experts in each case

DISCUSSIONS AND CONCLUSIONS

Regarding a specific representation of practice, we illustrated how expert norms for dealing with students' mathematical thinking can be different from an East Asian and a Western perspective. While experts from both countries pointed out that the teacher should have attended to the student's thinking, different purposes for attending to students' thinking (Colestock & Sherin, 2015) were identified: The majority of the Taiwanese experts assumed that the student's answer shows a misunderstanding or inappropriate strategy to be addressed and the majority of their German counterparts assumed that the student's answer indicates a mathematical ability or strategy to be valued. On other hand, these results suggest that attending to individual students' thinking is considered as an important aspect of instructional quality in both countries. This may reflect the phenomenon that what is considered high-quality mathematics instruction in Taiwan today is not only shaped by traditional perspectives, but also by Western ideas of constructivist-based instruction, such as discussing individual students' solutions as well as focusing on individual students' thinking and misconceptions (Hsieh, Wang, & Chen, 2019). On the other hand, there appears to be a difference between Taiwanese and German experts regarding what is the most important frame of reference for the interpretation and evaluation of the

students' thinking: the content and its correctness or the students' processes of doing mathematics. This result may be interpreted as evidence for how the deep-rooted cultural values underlying Leung's (2001) dichotomy product versus process still shape the perspectives of researchers and educators in mathematics education on dealing with students' mathematical thinking in a specific classroom situation.

Before discussing possible implication for international research on teacher noticing, we would like to recall the limitations of this research, which suggest interpreting the evidence with care. Although the experts in the sample of this study were professors in mathematics education from many different universities and the response rate was about 60% in both countries, it is not entirely clear whether these experts' answers can fully represent the perspectives of mathematics education researchers and educators in Taiwan and Germany. Furthermore, the results of this research report are based on only one vignette. The analysis of our data regarding further vignettes will soon allow us to draw a broader picture. Moreover, further research should complement these findings by means of different methodological approaches.

Bearing this in mind, our findings give, however, insight into different expert norms for dealing with students' mathematical thinking in different cultures. In view of the fact that such different norms may influence how teacher noticing regarding students' thinking is assessed by researchers in different cultures, it is questionable whether and how such research can be cross-culturally valid (Clarke, 2013). Therefore, the question of how teacher noticing can be investigated in a way that is sensitive to different cultural context certainly merits attention in our international research community.

Acknowledgment

This study is part of the project TaiGer Noticing which is funded by the DFG – German Research Foundation (grant numbers DR 1098/1-1 and LI 2616/2-1) and the Ministry of Science and Technology (MOST, grant number 106-2511-S-003-027-MY3).

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THE SHARED KNOWLEDGE OF THE CLASS VERSUS INDIVIDUAL STUDENTS' KNOWLEDGE IN A COURSE ON CHAOS AND FRACTALS

Assaf Dvir, Tommy Dreyfus, & Michal Tabach

Tel-Aviv University, Israel

This study focuses on gaps between the knowledge which functions-as-if-shared in a class as a collective and the knowledge that is used by each individual student. We analyze to what extent knowledge and ideas that are shared by the class community are available to and have been applied by the individual students. The research is based on data collected in an introductory course on chaos and fractals. The course included challenging inquiry activities that led to genuine argumentation, and to the emergence of quite a few new (for the students) mathematical notions. Initial findings present important gaps. We investigate to what extent these gaps can be explained by individual students' problem-solving skills, in heuristics.

INTRODUCTION

Understanding learning in mathematics classrooms requires coordinated analysis of individual learning and collective activity in the classroom (e.g., Cobb, Stephan, McClain, & Gravemeijer, 2001). A considerable amount of research has been dedicated to documenting collective classroom activity or group activities without considering individual students (e.g., Conner, Singletary, Smith, Wanger & Francisco, 2014; Stephan & Rasmussen, 2002). The research presented here investigates whether there are important gaps between the knowledge of individual students and the knowledge shared by the class as a collective. We performed a coordinated analysis of class discussions, followed by individual problem-solving in an interview situation. The class discourse was analyzed using the Documenting Collective Activity (DCA) methodology (Rasmussen & Stephan, 2008). The problem-solving activity was analyzed based on existing methodologies of problem-solving (Schoenfeld 1992; Carlson & Bloom 2005).

The mathematical domain selected for our research was chaos, fractals, and dynamical systems. Chaos is a phenomenon wherein a deterministic rule-based system appears to behave unpredictably. It is characterized by mathematical ideas whose in-depth comprehension is challenging. A major reason for selecting this domain are the counter-intuitive situations it affords; these situations give rise to challenging problems, for which deep and thorough

knowledge is crucial; therefore, they are useful for the investigation of problem-solving behaviors.

THEORETICAL BACKGROUND

Collective activity is a sociological construct that addresses the construction of ideas through patterns of interaction (Rasmussen & Stephan, 2008). More specifically, this activity is defined as the normative ways of reasoning that develop in a classroom community. DCA methodology proposes a rigorous approach for analyzing this communal activity. It uses Toulmin's (1969) model which considers an argument as composed of data, claim, warrant, rebuttal, backing and qualifiers. DCA uses three criteria to identify when a mathematical idea or way of reasoning becomes normative and functions in the classroom *as if it is shared*. "Function-as-if-shared" (FAIS) means that particular ideas or ways of reasoning are functioning in classroom discourse "as if" everyone in the classroom community reasoned in a similar manner. It should be noted that only some mathematical ideas discussed in class become FAIS.

Knowledge accumulated in problem-solving (PS) has shed light on both, what mathematical thinking involves and how learners can construct robust knowledge in problem-solving environments (Schoenfeld 1992). In this research, we focus on two major aspects of PS: Firstly, the use of PS methodology for analyzing students' problem solution and knowledge reconstruction processes; secondly, PS heuristics, their variety, taxonomy and usage as fundamental means in students' PS processes. A heuristic is "a systematic approach to representation, analysis and transformation of scholastic mathematical problems that solvers use (or can use) in planning and monitoring their solutions" (Koichu, 2010).

The Multidimensional Problem-Solving Framework (MPSF) developed by Carlson and Bloom (2005) offers a method for investigating and explaining mathematical problem-solving behavior. The framework defines four phases during problem-solving, namely orientation, planning, executing, and checking. We used MPSF for analyzing how a student applies FAIS knowledge when dealing with the interview problems. We use the term gap to refer to all mathematical ideas which were observed to FAIS in the class but were not or incorrectly applied by a student during the interview.

Dynamical systems (DS) theory is an area of mathematics used to describe the behavior of a time-dependent system, usually by employing differential equations or difference equations. The subset of dynamical systems which is relevant to our research is that of iterated functions (Feldman, 2012). The process of repeatedly applying the same function is called iteration. In this process, starting from some initial number x_0 , a given function is iteratively applied, thus generating an infinite sequence called orbit. An initial number p satisfying $f(p)=p$ is called a fixpoint. An orbit might reach or tend to a fixpoint.

A fixpoint is called attractor (ATT) if it attracts any orbit in a small neighborhood. The behavior of dynamical systems can be analyzed analytically and graphically. The basic graphical tool is called Cobweb (see Figure 1).

Analytically, the value of the first derivative of f at the fixpoint x_{fp} indicates whether x_{fp} is an attractor ($|f'(x_{fp})| < 1$) or not. This is called the fixpoint stability theorem (FPST).

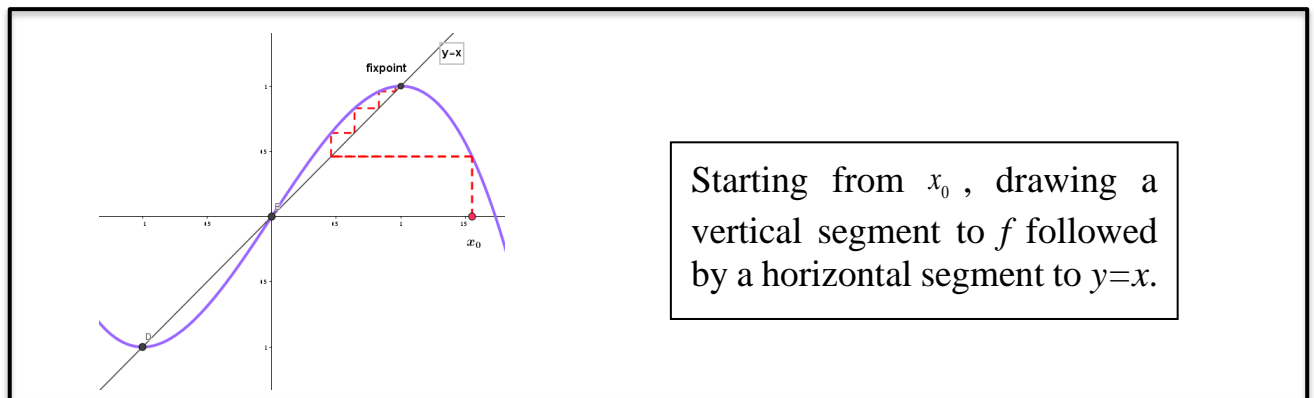


Figure 1: A cobweb plot (dashed segments)

RESEARCH QUESTIONS

1. In a class learning about chaos and fractals, which mathematical ideas related to attractors function-as-if-shared (FAIS) by the class?
2. Among the FAISes identified in 1, which ones are used, possibly after reconstructing them, by individual students in interviews, and which ones are not used, thus suggesting the existence of gaps?
3. If a student closes an initial gap by reconstruction during the individual interview, can the reconstructing process be explained by PS notions?

While this paper focuses on ATT, other notions of DS have been analyzed similarly.

RESEARCH DESIGN AND METHODOLOGY

To answer the research questions, we needed data from a class as well as data from students' individual interviews. To identify ways of reasoning that FAIS, we looked for a classroom where all members were actively engaged in producing, challenging, and modifying arguments. We chose an introductory course to chaos, fractals, and dynamical system for graduate level mathematics education students at an Israeli university. The course had 11 participants. Their degree program required a substantial mathematics component, and the chaos and fractals course fulfilled part of that requirement.

A typical course session consisted of presentation of a new notion, such as ATT by the teacher, followed by group work and whole class discussions designed to develop the notion's properties and relationships. Lessons were video-taped, transcribed and analyzed. Interviews on ATT were held with nine of the students about a month after the relevant lessons. The relevant part of the interview protocol dealt with orbits of an iterated function, cobweb diagrams, fixpoints and attractors.

The data analysis comprised three stages: Firstly, we analyzed the FAIS knowledge of the class using DCA analysis of the whole class discussions, resulting in a list of FAISes related to ATT. Secondly, we analyzed the interviews to identify which of these FAISes that each student mentioned or used; we categorized these uses into three levels: A - the student used the FAIS fully and correctly; B - the student used the FAIS partially; C - the student incorrectly used the FAIS or did not use it in spite of having an opportunity to use it. This resulted in a list of gaps: All cases of levels B and C. In the third stage, we focused on those cases in which a student reached either level A or level B by reconstructing their knowledge during the interview. On these cases, we carried out an MPSF analysis to identify the PS phases as well as the heuristics used by the student and examined how the heuristics supported the reconstruction.

FINDINGS

Stage 1: DCA Analysis

The DCA analysis resulted in a total of 51 mathematical ideas which functioned-as-if-shared by the class. Here we focus on the ones related to ATT:

- ATT term: An attractor is a term used in DS.
- ATT definition: A fixpoint p of the dynamic process generated by f is attractive, if there is a neighborhood of p such that for any point x in this neighborhood, the x -orbit converges (in a finite or infinite sequence) to p .
- ATT meaning: An attractor p means that if the orbit is slightly "bumped" away from p , the orbit subsequently moves back to p .
- ATT graphical solution: A fixpoint p is potentially ATT if a cobweb starting in a neighborhood of p moves back to p .
- ATT analytical solution: A fixpoint p is ATT if $|f'(p)| < 1$.

One important element of the analysis is our distinction between ATT term, its definition, its meaning, and its applications. This allowed us to refine our study of FAIS knowledge by students over a scale from lack of knowledge, remembering the term, via mastering the definition, up to competency in using and applying the knowledge in various scenarios.

Stage 2: Interview Analysis: Levels of Students' Use of FAISes

The interview analysis provided the proficiency level of every student per each of the five FAISes. Three levels were defined to evaluate the extent to which the student mastered a FAIS. Tables have been prepared for each notion; an example related to ATT is given in Figure 2.

In this case the aggregated results of mastering FAISes by level are: A - 41%, B - 17%, C - 22%, and undetermined - 20%. By undetermined we refer to FAISes which students did not have the opportunity to relate during the interview.

Student \ MKI	ATT term	ATT definition	ATT meaning	ATT graphical solution	ATT analytical solution
ADR	A	C	C	C	C
ALN	A	C	A	C	C
BZL	A	-	A	A	A
GDN	A	-	A	A	A
KOB	A	-	B	A	A
LIN	A	A	A	A	A
MUL	A	B	B	A	C
ORT	A	A	A	A	A
YAL	A	C	B	C	B

Figure 2: Student FAISes proficiency map for attractor (ATT)

Stage 3: Interview Analysis by MPSF Protocol

Our PS analysis protocol focused on students' problem-solving phases and their use of heuristics. Special attention was given to 11 major heuristics which were either classified as useful by experts (Koichu, 2010) or appeared more than three times throughout all interviews. This stage resulted in a description of the variety of the heuristics that the student used while solving the problems in the interview and the general work-flow of the student according to MPSF phases (see four examples in Table 1).

Heuristic	Description
Break down into modular sub-problems	When the problem is difficult, trying to decompose it and examining smaller independent parts
Use multiple representations	Representing a problem by means of a representational system different from the given representational system
Examine extreme cases	Choosing extreme cases (e. g. a function with a large slope) to observe attributes at extreme ends

Working backwards, reverse thinking	Imagining having solved the problem and work backwards from visualizing the solution to the problem.
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Table 1: Examples of major heuristics

In the following section we provide an in-depth analysis of one student, Bzl. We present his solution process for finding the ATTs for a given iterative function graph (Figure 5), which Bzl could not immediately answer. The analysis presents the mathematical solution behavior with special attention to heuristics which led to a successful response, bridging what initially was a gap. We selected this example since Bzl was a knowledgeable student (PhD in engineering) and very cooperative. He reflected deeply and shared his thoughts in a detailed manner.

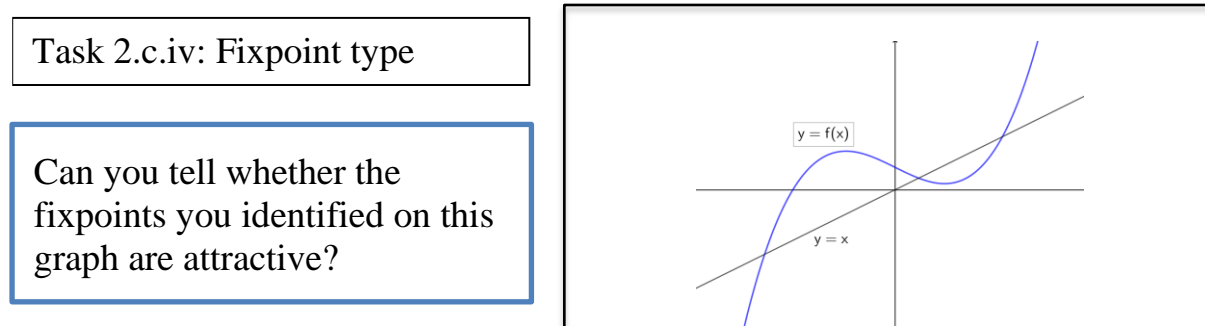


Figure 3: Interview task

Bzl identified the three fixpoints and then continued with a global analysis, although the question referred to local points. "We can say that the world is divided into four parts" (Line 21); he used the heuristic of breaking-down the problem into modules, the four regions between the fixpoints. Using this heuristic was unusual: Most other students started with a local fixpoint analysis or did not manage to continue at all.

Then, Bzl decided to navigate by random selection of a single region, selection of a point in this region, and using a cobweb diagram for graphically analyzing the orbit behavior around the middle fixpoint ("It maybe stated that the world is split into 4 regions and then I can randomly check what happen in each one of them", Line 22).

Bzl did not remember how to draw a cobweb, but he managed to reconstruct this FAIS, step by step. We describe his PS process in detail, pointing to the observed heuristics according to MPSF. Later, after completing this task, Bzl explained that in similar situations he regularly uses a general heuristic: instead of remembering a mathematical item, understand the logic behind it and redevelop it. So, he knew where to begin and selected a starting point close to the designated fixpoint. He used the heuristic of check by example. He drew a

vertical line to the f -graph since "if I start here on the x -axis, I know where $f(x)$ is" (Line 23). The next step is tricky, one must find the next iteration. So, Bzl continued with "now $f(x)$ turns into be the x for the next iteration". He thought for about twenty seconds and then drew a horizontal line to the auxiliary $y=x$ line (the heuristic: use auxiliary elements). Now he graphically managed to turn the y -coordinate into the x -coordinate for the next iteration. He completed the iteration by a similar move drawing again a vertical line and reached $f(f(x))$. He continued to contemplate the viability of his approach, by generalizing that "I can repeat that and see that we have some sort of a process" (the heuristic of generalization). He continued by drawing three more cobweb iterations. We may claim that Bzl managed to reconstruct the cobweb procedure, a FAIS which he did not immediately remember.

The cobweb convinced Bzl that this fixpoint is an attractor. This could be considered as a complete answer. However, right after the execution phase, Bzl turned to a checking phase by asking himself:

- Bzl25: Is it enough? Does one location suffice? Is it the same from both directions?
- I26: What do you mean by both directions?
- Bzl27: In principle, I do not think that the point selection really matters. I can see that the general behavior will be similar.
- I28: So?
- Bzl29: I start another one here if I want [*draws additional cobweb*]. I can see the general behavior, which means that I always move up, go here [*horizontally to $y=x$*] and I have the overall picture how the orbits behave.

From a PS perspective, Bzl used the check by example heuristic, followed by the generalization heuristic since he identified the pattern of the cobwebs about the middle fixpoint. We summarize that during his iterative execution-checking process Bzl managed to progress by using several common heuristics.

Bzl continued by providing a long explanation of what he meant by the geometrical patterns. In order to understand the dynamic process behavior in the neighborhood, he had to build auxiliary segments between the function graph and the graph of $y=x$ and analyze them graphically in combination with the algebraic meaning of moving between the graphs. He restored the cobweb algorithm by using multiple representations. In his own words: "I don't know why I was stuck here. When you move from this point, which is an intersection, you had to move up since you want to apply this value to $f(x)$. And then you move back to $y=x$..." (Line 37).

When summarizing his solution process, Bzl exposed an additional layer of thought. "*The split into regions was not random... I think that the guiding principle was to start by observing something, decompose it, and this way I can*

move from the micro to the macro and vice versa.". Evidently, Bzl had very good reflection skills, which significantly helped in understanding his mathematical thoughts.

We observed that Bzl used relatively large variety of heuristics and closed the temporary gap by reconstructing the relevant FAISes of ATT and cobweb. During the solution process he moved in cycles of execution-checking phases, skipping the planning, a solution behavior which resembles experienced mathematicians (Carlson, 2005). He was flexible and managed to change his focus and navigate between global and local views, which is prevalent among experienced mathematicians.

DISCUSSION

In response to the first question, the learning processes by group inquiries and class discussion in the specific chaos and fractals course resulted in a large number (51) of mathematical ideas which functioned-as-if-shared in the class. However, moving on to the second research question, we found gaps between what FAIS in the class and what students individually applied. Furthermore, we found interrelationships between a student's heuristic literacy and their ability to bridge initial gaps by reconstructing in the interviews. In particular, the heuristics proficiency, as shown in our research, might make the difference between students who manage to reconstruct and rebuild what they learned in class, and students who do not. The reasons for the gaps call for additional in-depth future research, but their existence is ground for caution since the teacher might assume that the class masters FAISes, although the reality is different.

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DIFFERENT CULTURE, DIFFERENT BELIEFS: THE CASE OF PRE-SERVICE TEACHERS' BELIEFS ABOUT MATHEMATICS

Andreas Eichler¹, Federica Ferretti², & Andrea Maffia³

¹University of Kassel, Germany

²University of Ferrara, Italy

³University of Pavia, Italy

Aiming at unfolding possible cultural differences concerning pre-service teachers' beliefs towards mathematics and its teaching, this paper presents a comparative study between nearby regions: Italy and Germany. The sample is composed by 460 pre-service teachers from three universities, one in Germany and two in Italy, one of them close to the Austrian border and multilingual. Using a clustering technique, we analyse responses to two multiple-answer questions, and we compare the composition of the obtained clusters in terms of linguistic background and origin. Relevant differences are evidenced and explained, at least partially, in terms of cultural differences.

INTRODUCTION

The importance of the effects of teachers' beliefs on their practice was the motivation for the extensive research on teachers' beliefs (cf. Fives & Gill, 2014). However, a major obstacle in interpreting locally obtained results about teachers' beliefs from an international perspective is given by the impact of local culture on teachers' beliefs (Felbrich, Kaiser, & Schmotz, 2012; Hofstede, 1986; Romijn, Slot, Leseman, & Pagani, 2020). Whereas Hofstede (1986) refers to cultural differences from a global perspective including the effect of high or low individualism that divides, for example, Europe from Asian countries, Romijn et al. (2020) refer to differences in beliefs of teachers from European countries. Thus, a seemingly homogenous cultural region may comprise cultural differences that are apparent in teacher' beliefs. Following this line of research, our research aims at providing a contribution to unfold possible cultural differences concerning teachers' beliefs towards mathematics and its teaching in two nearby regions, Italy and Germany.

Following Felbrich and colleagues (2012) we understand common experiences of a group of people that are shared through generations as the basis of culture. Furthermore, according to Hofstede (1986, p. 314), we conceive language as "the vehicle of culture". We investigate Italian and German pre-service teachers' beliefs as a specific expression of culture. Taken the linguistic

influence into account, we also consider Italian pre-service teachers from a border region where some teachers use German language and other teachers speak Italian at school and in their daily life.

THEORETICAL LENSES ON TEACHERS' BELIEFS

We refer to beliefs on the basis of two aspects: First, teachers' beliefs as part of teachers' mathematics related affect (Hannula, 2012) play an important role in teachers' professional lives (Calderhead, 1996). For example, Eichler and Erens (2014), starting from the definitions by Pajares (1992) and Philipp (2007), understand the term beliefs as an individual's personal conviction concerning a specific subject, which shapes an individual's ways of both receiving information about a subject and acting in a specific situation. Thus, beliefs strongly impact on the way teachers learn mathematics at universities and teach mathematics at school (cf. Philipp, 2007). As pointed out by Pajares (1992), teachers' beliefs are often already developed during pre-service university courses; hence many studies focus on beliefs of perspective mathematics teachers (cf. Hannula, Liljedahl, Kaasila, & Roesken, 2007).

The second one concerns the impact of cultural aspect on pre-service teachers' beliefs towards mathematics and its teaching. One of the main obstacles to the general interpretation of results about teachers' beliefs obtained at national level is the influence of social and cultural factors on teachers' beliefs (Felbrich et al., 2012). Some studies highlight that the process of learning and teaching of mathematics is dependent on the teachers' cultural background; this is evidenced both from global (Hofstede, 1986) and European (Romijn et al., 2020) perspectives.

Our research moves within this stream of thought and our aim is to investigate the cultural differences concerning teachers' beliefs towards mathematics and the teaching mathematics in two nearby regions, namely Italy and Germany. In details, in this paper we focus on pre-service teachers' beliefs about features that are decisive both for being successful in mathematics and performing well as teacher. As detailed below, we frame pre-service teachers' beliefs about mathematics within the model of *mathematical giftedness* by Pitta-Pantazi and colleagues (2011); we frame beliefs about mathematics teaching within the *Knowledge Quartet* (Rowland et al., 2005). Our research question is: What differences can we observe in beliefs manifested by primary pre-service teachers from different cultural and linguistic backgrounds?

METHODOLOGY

Sample

Our sample consists of students of the university of Bologna and the University of Bozen-Bolzano (Italy) and the university of Kassel (Germany). The University of Bologna is an historical big university in the northern part of Italy,

attended by students coming from many different Italian regions. The University of Kassel, in Germany, is younger and is attended mainly by students from the surrounding area. The University of Bozen-Bolzano is a small university located in the South Tyrol region (Italy), at the border with Austria. This region, originally Austrian, was annexed to Italy after World War I and it still is a bilingual region. There are both German and Italian schools for any school level. The university of Bozen-Bolzano provides two versions of each course, in Italian and in German. Among our 460 respondents, 40% are from Bologna and 39% from Kassel. Pre-service teachers who attended their courses in Italian (we will refer to this group as Bozen ITA) are 15% of the sample; the remaining ones attended classes in German (Bozen GER). Respondents received the text of the questionnaire in the same language of their courses. Translation of the questionnaire has been checked by all the authors and by a consultant speaking both languages.

Questionnaire

The data analyzed in this paper refer to two questions from a wider questionnaire (Ciani et al., 2019). We analyze the answers to two multiple-answer questions (Maffia et al., in press), corresponding to questions B2 (Fig. 1) and B4 (Fig.2) in the original questionnaire.

B2. Select <u>THREE</u> features that, according to you, are important for having success in mathematics.	
A. Fluency	G. Predisposition
B. Organized working	H. Analytic thinking
C. Language appropriateness	I. Confidence
D. Flexible thinking	L. Memory
E. Motivation	M. Control
F. Perseverance	N. Originality

Figure 1: Question B2 on beliefs about success in mathematics

We selected the answer-options to question B2 according to the model of mathematical giftedness described by Pitta-Pantazi, Christou, Kontoyianni and Kattou (2011). Following this model, mathematical ability is the result of *Learned Abilities* (like verbal, spatial, quantitative abilities, etc. – options B, C, and H) and *Creativity* (defined as a combination of fluency, flexibility, and originality – options A, D, and N). Both Learned Abilities and Creativity are supported by *Natural Abilities* (including working memory, control, and speed of processing – options G, L, and M). We integrate this model adding the dimension of *affect* (options E, F, and I).

B4. Select THREE features that, according to you, are important for being a “good” teacher of mathematics.

- | | |
|---------------------------------------|--------------------------------------|
| A. Planning with awareness | G. Giving feedback about errors |
| B. Knowing several teaching methods | H. Adapting lessons to contingencies |
| C. Giving effective explanations | I. Knowing students’ abilities |
| D. Valorising students’ interventions | L. Selecting appropriate examples |
| E. Knowing mathematics | M. Relating different topics |
| F. Using several representations | N. Using technical terms |

Figure 2: Question B4 on beliefs about mathematics teaching

Answer-options for question B4 were established according to the model of the Knowledge Quartet by Rowland and colleagues (2005). It is a theoretical framework for the analysis and development of mathematics teaching. From the perspective of the Knowledge Quartet, knowledge and beliefs evidenced in mathematics teaching can be seen in four dimensions: *Foundation* (options B, E, and N), *Connection* (A, I, and M), *Transformation* (C, F, and L) and *Contingency* (D, G, and H).

Data analysis

Answers to the multiple-answer questions have been clustered using an agglomerative hierarchical clustering algorithm on the whole sample of 460 respondents. In terms of the method, single linkage may function to determine the outliers in the data, and then performing the Ward algorithm classifies the remaining elements. While this algorithm usually results in a valid clustering, in this work its performance was reduced, due to the lack of isolated data points (Maffia et al., in press). The complete linkage rule was then chosen aiming to find compact clusters of similar diameters, avoiding chaining phenomena (Everitt, Landau, Leese & Sthal, 2011). The number of clusters is established minimizing the absolute maximum deviation from the median of the number of respondents per cluster (Maffia et al., in press).

RESULTS

In presenting our results, we dedicate a sub-section to each of the two abovementioned questions, that is B2 and B4, providing information about the obtained clusters and comparing the composition of clusters in terms of respondents having different origin.

Beliefs about success in mathematics

For question B2 we obtained six clusters and, even if they differ one from the other, their characterization depends on a few answer-options. In general, we can notice that Natural Abilities are considered as not important for succeeding in mathematics, while attention to affective factors is high. Clusters differ

mostly in terms of the percentage of selection of affective factors, being ‘Motivation’ and ‘Perseverance’ some of the most selected options in many clusters. Creativity is represented in the largest clusters by ‘Flexible thinking’, while ‘Originality’ is usually undervalued. In the same fashion, the most representative Learned Ability is ‘Analytic thinking’, while ‘Language appropriateness’ is rarely considered.

As it is shown in figure 3, the composition of the six clusters differ in terms of the origin of respondents having some clusters mainly composed by Italian-speaking pre-service teachers and other more populated by German ones.

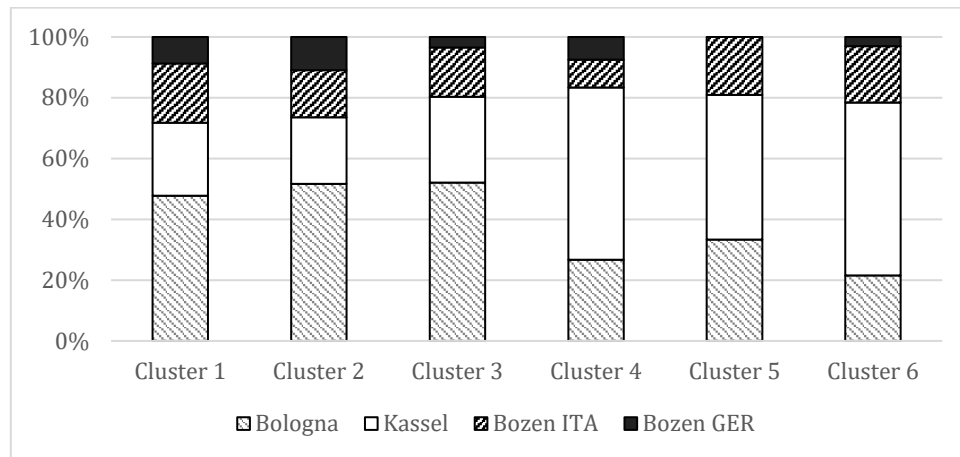


Figure 3: Composition of clusters (question B2) according to respondents' origin.

Respondents from Bologna are highly represented in the first three clusters where Affective factors are strongly considered. Clusters 1 and 2 comprehend respondents paying strong attention to ‘Motivation’. While respondents belonging to the first cluster (10% of the sample) also often select ‘Organized working’, those in the second one (20% of the sample) selects more often ‘Flexible thinking’, showing a preference for Creativity over Learned Abilities. The third cluster (25% of the sample) gives high credit to ‘Flexible thinking’ and ‘Perseverance’ (Fig. 4).

The percentage of pre-service teachers from Kassel is higher in clusters 4, 5, and 6 (respectively 26%, 14%. and 5% of the respondents), characterized by a high selection rate of options related to Learned Abilities (mainly options H or B, e.g. Fig. 4). Cluster 6 is the only one having a high percentage of members opting for ‘Predisposition’.

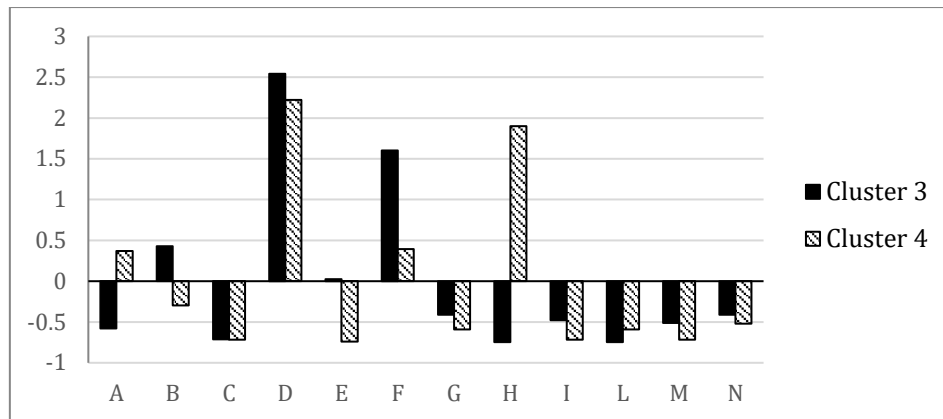


Figure 4: Standardized frequencies (1 unit corresponds to a difference of 1 SD from the average) for the answer-options to question B2 for the two largest clusters.

Respondents from Bozen-Bolzano are represented more evenly in the clusters, but we can notice that, among them, German-speaking pre-service teachers are more strongly represented in clusters 1 and 2. The percentage of Italian-speaking students from Bozen-Bolzano is higher in clusters where ‘Perseverance’ is considered one of the most important features. More generally, there is not a correspondence between clusters having the higher percentage of respondents from Kassel or Bologna and those having the higher number of respondents from Bozen-Bolzano speaking the same language. The only exception is cluster 3 that is composed by a large majority of Italian speakers.

Beliefs about mathematics teaching

The number of clusters obtained for question B4 is 11, much higher than the previous question. This result may suggest that pre-service primary teachers’ beliefs about mathematics teaching are more various than those towards mathematics itself. Participants are distributed unevenly in the clusters having the smallest ones representing each 4% of the sample (clusters 1 and 6) and the larger ones comprehending almost 15% of the sample (cluster 2 and 5). Clusters 3 and 4 count each 8% of the sample while other clusters include the 10% of the participants circa.

Even if the number of clusters is quite high, it is interesting to notice that most of them are characterized by four answer-options belonging to three of Rowland and colleagues’ (2005) dimensions: Foundations (knowledge about teaching methods and/or mathematics in particular), Transformation (mainly the effectiveness of explanations), and Contingency (mainly feedback on students’ errors) have a more relevant role than Connections in characterizing our pre-service primary teachers’ beliefs about the teaching of mathematics.

Figure 5 shows the composition of the eleven clusters in terms of the origin of respondents. We can see that there are extreme cases, where the cluster is

almost entirely composed by pre-service teachers speaking the same language, while other clusters are more evenly composed.

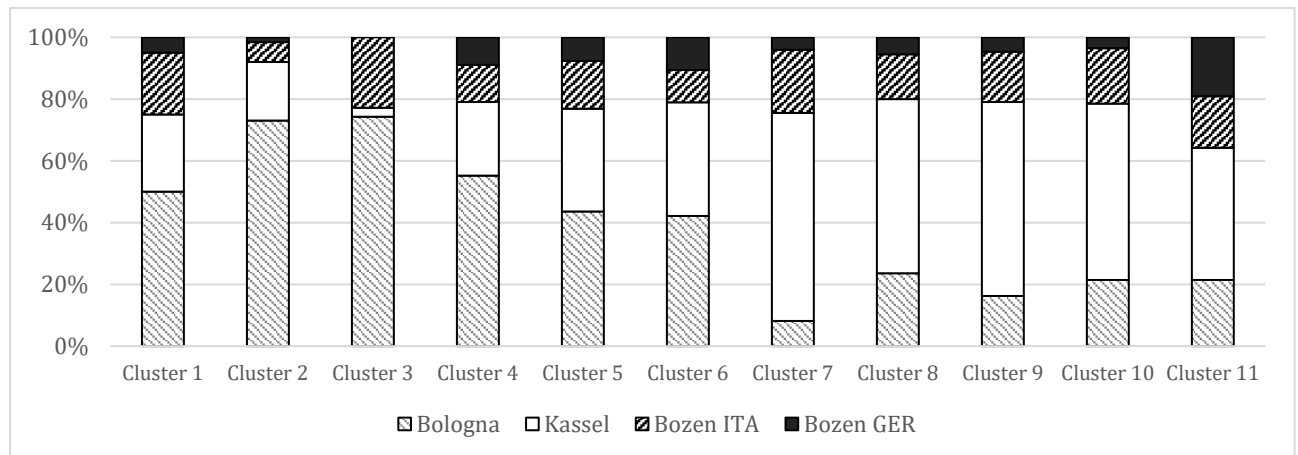


Figure 5: Composition of clusters (question B4) according to respondents' origin.

The first four clusters are all characterized by a high percentage of respondents from Bologna. In these clusters, there is a high rate of selection for the option 'Knowing several teaching methods', while other aspects of Foundation are often ignored. Cluster 3, the most "Italian" cluster, differs from the others since 'Knowing mathematics' is the most chosen option. Members of clusters 1 and 2 often select 'Giving feedback about errors'. However, these two clusters differ in their attention for Transformation: cluster 1 believes that 'Giving effective explanations' is as important as 'Knowing several teaching methods'. A high attention to effective explanations characterizes cluster 4 as well, but this cluster does not have a particular preference for options belonging to the categories of Connection and Contingency. On the contrary, Contingency is the focus for the last three clusters, where students from Kassel are more present. Members of clusters 9 and 10 often refer to 'Giving feedback about errors', while cluster 11 selects mostly 'Valorising students' interventions'. Cluster 7, 8, and 9 pay strong attention to effective explanations. Cluster 7 – the one with the highest percentage of respondents from Kassel – often opts for 'Planning with awareness' and its attention to Foundation is lower than many other clusters.

Clusters 5 and 6 reflect the composition of the whole sample. Cluster 6 is the one giving more credit to knowledge about mathematics (option B) and it is one of the two smallest clusters. Cluster 11 is characterized by a high presence of respondents from Bozen-Bolzano and by a high rate of selection for option D; this is also the only cluster paying a certain attention to answer-options related to Connections.

DISCUSSION AND CONCLUSION

Our analysis allowed to observe relevant differences in pre-service teachers' beliefs about features that are decisive both for performing well in mathematics

and for a successful mathematics teacher. We obtained six different clusters related to mathematics (question B2) and 11 clusters concerning its teaching (question B4).

Our results show that pre-service teachers from the three universities give less importance to natural abilities as basis of success in mathematics (B2). The clusters seem to show cultural differences. For example, the clusters more populated by students from Kassel are characterized by a strong attention to analytical thinking and creativity. By contrast, the percentage of pre-service teachers from Bologna is higher in clusters characterized by attention to flexible thinking and affective factors. Students from Bozen-Bolzano are almost distributed equally in all the clusters, suggesting a mix of beliefs. In relation to the basis for successfully teaching mathematics (B4), we observe that Italian speakers have a stronger attention to foundations than the German ones. Also, pre-service teachers from Bozen-Bolzano are characterized by the strongest attention to connections; a peculiarity of this border-location differing both from the German context and the context of another Italian university.

Considering together the results of both questions, we can state there are common features to all the linguistic and cultural contexts, but also peculiarities. These differences may depend on many factors related to the common experiences of the group of people attending the same university in the same city, that is what we have considered as their culture (Felbrich et al., 2012). Among these experiences we must certainly consider schooling and, in particular, the university courses attended by the pre-service teachers participating in the research. The organization of their university degree cannot be the only source of the observed differences. Indeed, in Italy, Primary Education degrees are regulated at national level and so pre-services teachers from the University of Bolzano-Bozen attend a degree course that is structurally similar to the course of the University of Bologna – the main difference being multilingualism. We are not assuming that all the observed differences could be explained in terms of the spoken language but, assuming that language is the vehicle of culture (Hofstede, 1986), we can claim that cultural factors can affect pre-service beliefs even in nearby regions, and not only at the global level as most of research has shown up to date (e.g. Felbrich et al., 2012). However, more research is needed to better clarify the nature of these factors. Furthermore, it is an open question if and how the observed differences correlate with other constructs that shape the teachers' professional lives, namely the teachers' knowledge, emotions, or motivation.

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WHEN GENDER MATTERS: A STUDY OF GENDER DIFFERENCES IN MATHEMATICS

Ferrara, F., Ferrari, G., Robutti, O., Contini, D. & Di Tommaso, M.L.

Università degli Studi di Torino, Italy

This paper addresses gender differences in mathematics at the early grades of primary school, based on a research study conducted in Italy, in the region with the largest gender gap in mathematics in the National panorama. Borrowing from the literature around gender and its different conceptualizations, we focus attention on the possible relationship between the gap and the cognitive demand, task and formulation of mathematical test questions. Restricting the analysis to the content area of numbers, the one with the largest gap, we will highlight some of the variables that seem to affect the gender gap, arguing for a more equitable mathematical practice.

INTRODUCTION

In this paper we want to contribute to the current discussion on gender differences in mathematics. Differences in mathematical performances in favour of boys exist and are considered as having implications on the fact that females are substantially under-represented in STEM university subjects and in highly innovative and technological careers (Miyake et al., 2010). We refer to the difference in mathematical performance between males and females as the gender gap in mathematics (GGM). Research has shown that the GGM is a matter of concern for policies that address equity both at school and in the labour market (Di Tommaso et al., 2018), especially at a time of social crisis, like the current one in regard to the pandemic. On the other hand, patterns of gendered inequity provide a sobering counterpoint to claims of an equitable mathematical experience, thus troubling and disrupting given gender performances within contexts and conditions does matter more than ever (Walshaw et al., 2017). As Walshaw and colleagues underline, other constructs of social difference such as class, race, ethnicity also become significant, as do histories of mathematical access, success, production, underachievement or exclusion. Speaking of GGM is therefore important in relation to a wider perspective of binaries between diversity and equity.

The latest international assessments of mathematics (like PIRLS and PISA) show Italy as one of the countries with the largest GGM. This emerges from the primary through upper secondary school test scores. In particular, Italy possesses the largest gap among the 57 countries taking part in TIMSS grade 4 evaluation (Mullis et al., 2016), and is in the second position in the case of 15-

year-old students (OECD, 2016). These results are further problematized looking at data from the National Institute for the Evaluation of the Education System (INVALSI) in Italy, according to which a GGM is observable since grade 2 and becomes more prevalent during secondary school. The primary purpose of this paper is to address issues concerning the GGM in grade 2 in Italy, starting from the results of the assessment of mathematics of years 2013 to 2017. We are particularly interested in studying variables that might affect the GGM in this context, and in designing classroom-based interventions to reduce it in mathematics. To this aim, the research team is interdisciplinary and involves mathematics educators and social economic researchers. In the next section, we frame the research study into the literature that we see as relevant to highlight and discuss differences between male and female performances in mathematics.

THEORETICAL HIGHLIGHTS

Much international literature shows unique achievement trends of males and females in mathematics and reading across a number of countries (e.g., Robinson & Lubienski, 2011; Ajello *et al.*, 2018). Math gaps favouring males were found to increase between kindergarten and third grade (Rathbun, West, & Germino-Hauskin, 2004). Also, the GGM is particularly pronounced among high-performing than among low-performing students and widens as children grow older even if it does not widen during lower secondary school (grade 4 through 8; Contini *et al.*, 2017). In the broader literature, developments in gender research endeavour to think differently about the GGM, with understandings of gender ranging from biological or cultural and environmental factors to family and teacher beliefs and biases, to girls' low self-confidence and self-efficacy in terms of mathematical ability and performance within gendered identity-work (Else-Quest *et al.*, 2010; Lubienski *et al.*, 2013). The role of stereotypes and other socio-cultural forces is well established (see Aronson & Steele, 2005 for a detailed review). Some available research studied gender differences in mathematics in relation to performance and highlighted that they seem to be related to the cognitive processes that are investigated by the question and linked to the type of question. For example, Bolger and Kellaghan (1990) discovered that while boys outperform girls in multiple-choice questions, girls outperform boys on open-ended questions. Other studies indicated strong association between aspects of reading and of mathematics tests (Marks, 2008; Caponera *et al.*, 2016). Robinson and Lubienski (2011) further claimed that given that gender patterns in math performance tend to run counter to those in reading, examinations of both subjects together provide a more complete picture of girls' and boys' learning. Ajello *et al.* (2018) claim that the reading burden of mathematics questions is associated with student performance in mathematics, independently of mathematical ability. Due to the fact that girls are better performers than boys when facing reading tests, they

seem to be advantaged in mathematics questions with a high reading demand, independent of their level of reading literacy. Questions with a low reading demand are instead more in favour of boys. According to Ajello and colleagues (2018), question difficulty and task can also be related to such differences, therefore further research should investigate the type of cognitive process involved in answering the task, for example whether a computation or problem solving. Other research stresses that variations on question formulation affect differently male and female performances and that this might be concerned with different strategies used by the two populations (e.g., Bolondi *et al.*, 2018).

Borrowing from these considerations, we shift attention to studying the possible relationships between the type of task, formulation and cognitive demand in mathematical questions and the existence of a GGM, as we have defined it above. In this way, the paper wants: (a) to contribute to current discussions on mathematical gender differences at primary school, in a double manner: by confirming findings from the literature, and by expanding these focusing on variables strictly related to the questions; and (b) to examine the local context of Piedmont, which shows to be the Italian region with the largest GGM in grade 2, supported by territorial funding for dedicated research. In the next section, we introduce context and method of the study.

CONTEXT AND METHOD

As mentioned above, in our research we take the GGM as the difference between average male and female scores in their mathematical performance. Our original data source is given by the scores of the National grade 2 assessment tests of mathematics over the period 2013 to 2017. In order to avoid possible bias related to cheating, the estimation sample was reduced to including only those classes that were supervised by external inspectors during the tests. In addition, the sample was further restricted to a sub-sample including only the classes in Piedmont, where we work with an active network of policy makers and schools.

The assessment test of mathematics delivered each year by INVALSI approximately contains 25 to 28 questions, each of which can be composed by more than one item, like in the case of True or False multiple choice questions. The scores to which we associate the GGM take into account all the items of the grade 2 assessment of mathematics in the period mentioned above, for a total of 6.732 observations. The items are associated to a content area, a dimension (the main cognitive process implied by the item) and a question intent (the item purpose). According to the Mathematics Assessment Framework of INVALSI (INVALSI, 2018), which follows the National Guidelines for the curriculum, three are the possible content areas for grade 2: *Numbers, Data and previsions, Space and figures*, and three the cognitive dimensions: *Knowing, Problem solving* and *Arguing*. The question intent is

concerned with typical forms of mathematical thinking, like text comprehension, calculation, use of different representations or measurement tools, reasoning, data research, and problem solving.

Table 1 offers the results of the initial descriptive statistics of our sample by content area, with average score and GGM, and the percentage of items for each area. The score provided for each student is measured as the percentage of correct answers over the total items. The results show that the average score is lowest in the case of Numbers for both males and females, but also contains the majority of items. On average, the total gap is 0.028 (2.8 percentage points, or p.p.): while females answer correctly to 53.9% of the items, for males we get 56.7%. Additionally, the area of Numbers has the largest GGM (3.7 p.p.), moving us to centre our investigation on this particular area. The number of items belonging to Numbers between 2013 and 2017 is 82 (the number of observations in the table; each observation was assessed on about 1340 subjects). Focus was on these items to better understand which of their characteristics could partake of the GGM revealed by the statistics. The analysis was centred on the study of constant differences in the GGM concerned with item characteristics over the entire period rather than on the trend over time. Therefore, we adopted a mixed method, both qualitative and quantitative. The qualitative part borrows from the literature we refer to and regards an initial search for variables that constitute each item formulation and structure, beyond those variables that are considered already by the assessment framework. The second part of the analysis involves descriptive statistics of all these variables. This allows us to study the relationships between the GGM and the type of task, formulation and cognitive demand of the mathematical items.

Variable	Overall	Males	Females	GGM (M-F)	% items
Average score	0.554	0.567	0.539	0.028***	100
Content area					
<i>Numbers</i>	0.517	0.535	0.498	0.037***	56.9
<i>Data and previsions</i>	0.614	0.620	0.608	0.012*	16.0
<i>Space and figures</i>	0.613	0.618	0.608	0.010**	27.1
<i>N. observations</i>	6,732	3,387	3,345		

* p -value < 0,10; ** p -value < 0,05; *** p -value < 0,01

Table 1: GGM: Average score (% of correct answers) by content area

QUALITATIVE ANALYSIS AND VARIABLE IDENTIFICATION

As anticipated above, we identified the variables that characterise item formulation and structure through a qualitative analysis of all the selected items. This process brought forth the following as relevant variables:

A. Cognitive dimension: Arguing, Knowing, Problem solving.

- B. Question intent: Calculation, Text comprehension, Reasoning, Different representations, Data research, Problem solving, Measurement tools.
- C. Type of item: Open-constructed response, Multiple choice.
- D. Item formulation: Situation, No Situation, Objective, No objective.
- E. Kind of figure: No figure, Drawing, Figure in context, Representation.

While the first three classes of variables (A, B, C) refer to INVALSI framing of the items, the other two classes (D, E) were added to account for: the presence or absence of a situation which provides the context of the task, or of an objective which gives the aim of the task (D); the absence or presence of a figure and the eventual kind of figure (E). We distinguish figures according to three kinds: drawings, figures in context, and representations (Fig. 1 shows three examples from specific questions of the 2017 assessment test). A drawing simply contains a number of objects to which the task refers (asking for example to count them, Fig. 1a). A figure in context implies an understanding of the sense to attribute to objects in specific contexts (like in the case of money, Fig. 1b). A representation requires a step forward to infer the relationships between objects (like when lengths of different objects need to be compared, Fig. 1c).

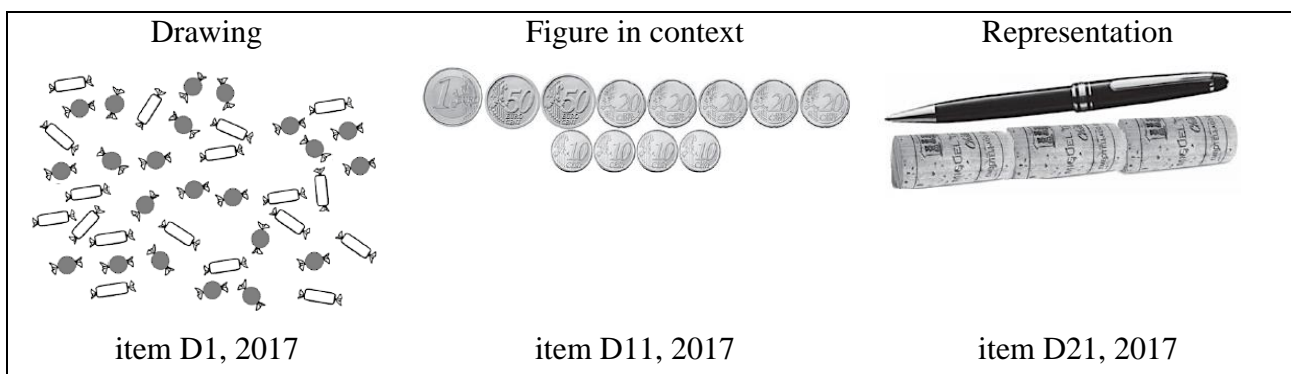


Figure 1: Examples of different kinds of figures

After this identification process, we selected the particular variables for each of the 82 items of our sample and created a table, in which each row refers to a specific item D_n while the column cells are targeted with value 0 or 1 depending on whether the corresponding variable is absent or present in that item.

QUANTITATIVE ANALYSIS AND RESEARCH FINDINGS

The attribution of values 0 and 1 to item variables was used to develop the new statistics for our quantitative analysis through simple linear regression, which allowed us to get some descriptive measure of the influence of particular variables on the presence of the GGM. In so doing, we focused on the difference across single items obtaining some information from which to begin:

the mean percentage of correct answers across items is 52.5%, while the gender gap across items is 0.039; there is large variability embedded in this gap, with the minimum -0.10 (in favour of females) and the maximum 0.23 (in favour of males). This relevantly suggested that, as a matter of fact, the nature of the items (briefly, their formulation and structure) actually affects the gap, although without saying in which terms. Investigating the variables above exactly allows us to see how and to which extent this occurs. Tables 2 to 4 below help to better explain this. In all the tables standard errors are in parentheses and the number of asterisks defines how significant the gap is (the lower the p -value the more significant the gap is). In particular, Tables 2 and 3 are concerned with the influence of the variables from the INVALSI framework, that is, cognitive dimension and question intent. Table 4 instead refers to the additional variables we identified.

Cognitive dimension	Item GGM
<i>Arguing</i>	0.018 (0.015)
<i>Knowing</i>	0.036*** (0.008)
<i>Problem solving</i>	0.052*** (0.010)
<i>Obs.</i>	82
R^2 adj.	0.348

* p -value < 0,10; ** p -value < 0,05; *** p -value < 0,01

Table 2: Item GGM: influence of Cognitive dimension

Question intent	Item GGM
Calculation	0.027** (0.011)
Text comprehension	0.021 (0.018)
Reasoning	0.012 (0.027)
Different representations	0.048*** (0.013)
Data research	0.067** (0.031)
Problem solving	0.050*** (0.011)
Measurement tools	0.038 (0.038)
<i>Obs.</i>	82
R^2 adj.	0.348

* p -value < 0.10, ** p -value < 0.05, *** p -value < 0.01

Table 3: Item GGM: influence of Question intent

Item GGM				
Open constructed-response	0.028***	(0.008)		
Multiple-choice	0.053***	(0.009)		
No situation	0.061***	(0.016)		
Situation	0.035***	(0.006)		
No objective		0.039***	(0.006)	
Objective		0.036**	(0.018)	
No figure			0.033**	(0.011)
Drawing			0.010	(0.013)
Figure in context			0.064***	(0.013)
Representation			0.045***	(0.010)
<i>Obs.</i>	82	82	82	82
<i>R² adj.</i>	0.364	0.345	0.327	0.386

p*-value < 0,10; *p*-value < 0,05; ****p*-value < 0,01

Table 4: Item GGM: influence of Item type and formulation

From Table 2 we see that problem solving is the cognitive dimension that affects the GGM the most. Table 3 shows that the use of different representations and problem solving are the two most problematic aspects implicated by the items concerning the GGM. Regarding item type and formulation variables (Table 4), our results confirm (in the local context) the findings of the literature according to which males perform better than females in answering multiple-choice questions, showing a gap of 53%. On the contrary, open constructed-response items are more favourable to females (in fact, the gap is 28%). Further, the absence or the presence of a situation does not seem to affect the GGM in any particular manner (both contribute to it to an almost equal extent), while the presence of an objective seems to act in the direction of reducing the gap with respect to its absence (two asterisks instead of three). The bearing of a drawing is marginal as regards that of a representation or (even more) of a figure in context, while the absence of figures affects the GGM on average. These findings move us to make didactical considerations. For example, more work with representations seems to be needed within the mathematics classroom, both in terms of the treatment of different representations and in relation to their meanings, with the aim to reduce the

documented GGM. Similarly, attention should be devoted to contextualising mathematical activity, like in the case that we use figures requiring knowledge of the context to be understood. The dimension of problem solving is another delicate one that calls for didactical intervention.

CONCLUSIVE REMARKS

Our study wants to contribute to existing discussions about gendered disciplines by shifting emphasis from available gender research to material, concrete experiences of gendered performances in mathematics. Borrowing from the existing literature and the findings from these performances, we suggest that lines of didactical intervention are needed to deeply engage both females and males in mathematical doings. This is particularly relevant in a time of social crisis like that of the pandemic, which also showed to intensify differences. A rethinking of educational practice is needed towards a more equitable mathematics, one that disrupts boundaries to overcome gendered identity discourses within the classroom, for example by de-centring consensus about practice mainly based on calculation and procedural knowledge and shifting attention to problem solving. Focusing on the local context of Piedmont, the Italian region with the largest GGM already at grade 2, we offered reflections about variables that seem to affect the presence of the gap and that we see as relevant to any discourse of mathematics teaching and learning. Future research is necessary to widen the horizon on possible interventions and efficiently inform policy making in these directions.

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WHO IS BEST IN MATHEMATICS? GRADE NINE STUDENTS' ATTITUDES ABOUT BOYS, GIRLS AND MATHEMATICS

Staffan Frid¹, Guri Nortvedt², & Lovisa Sumpter¹

¹Stockholm University, Sweden

²Oslo University, Norway

Sweden has a reputation for its equality work, but at the same time mathematics is still considered a male domain. We studied grade nine students' attitudes about who could be considered best in mathematics, both from an individual perspective and how they perceived different groups in society would answer. A questionnaire was used and the analysis showed that girls more often think that this is not a matter connected to biological sex, whereas boys more often state that boys and girls are equally good. Two groups are stereotyped as thinking that boys are better in mathematics both by girls and boys: boys in grade nine and boys in general. This is not reflected in their self-evaluation. Overall, the students showed an awareness of the concept of gender, including some intra-cultural dimensions of the concept.

INTRODUCTION

In many western countries, although there is no major differences in achievements in mathematics (OECD, 2013), the subject is often considered as a male domain; for instance, there are differences in enrolment in various STEM subjects, both at undergraduate level and at graduate level (Piatek-Jimenez, 2015), and stereotypical symbols have been attributed to boys and girls, such that boys are creative and girls insecure (Walkerdine, 1998; Sumpter, 2016). Another example is that boys express a higher degree of ability and self-confidence compared to girls (OECD, 2013). In this way, gender is an issue relevant for research and discussion. This is true for Sweden too, which is interesting given it is a country with reputation for its work regarding gender equality (Weiner, 2005). In the curriculum for Swedish school, we can read that teachers should actively work to enhance and develop students' critical thinking about gender stereotypes and this has been a central topic in governing school documents for over 50 years (Hedlin, 2013). Previous studies signal that students at different ages consider mathematics as a male domain (Brandell, Leder & Nyström; 2007; Brandell, 2008) including boys reporting higher levels in measures of self-evaluation (OECD, 2013; Sumpter, 2012), this despite that girls' grades are higher throughout secondary school (age 13-19). At the same

time, teachers state that gender is not an issue neither in their teaching nor for themselves as teachers (Gannerud, 2009).

Therefore, there is a paradox between the social, political norm and the symbols that individuals express including gender stereotyping. This paradox invites to further study how individuals perceive that different groups in the society view mathematics and gender, and how individuals would reply from their own perspective. Here, we would like to study grade nine students' expressed attitudes with a focus on attributed ability in mathematics. Our research questions are: (1) In what way do boys and girls attribution differ regarding ability in mathematics?; (2) How do they experience other groups attributions?; and, (3) To what extent do students express that this has changed over time?.

BACKGROUND

Our theoretical starting point is that gender is a social construction more than just a consequence of a biological sex, that gender is:

“a pattern of social relations in which the positions of women and men are defined, the cultural meanings of being a man and a woman are negotiated, and their trajectories through life are mapped out.” (Connell, 2006, p. 839).

These social relations include characteristics and traits that are cultural dependent, and in a longer time perspective, they create norms. This is a dynamic process meaning that the attributions, beliefs, identities, norms etc. are not static and as socially constructed differences, they support differences and inequality (Acker, 2006). In order to study attributed symbols, a further division of gender is fruitful. Here, we follow Bjerrum Nielsen (2003) and divide gender into four different aspects: structural, symbolic, personal, and interactional gender. The first aspect, structural gender, is about social structures alongside with other factors such as class and ethnicity. One example of structural gender is the ratio men/women in enrolments in mathematics. The second aspect is symbolic gender which appears in the shape of symbols and discourses. It informs us what is considered normal and what is deviant (Bjerrum Nielsen, 2003). One example is the idea of mathematics as a male domain (Brandell, Leder & Nyström, 2007; Brandell, 2008). Symbols as such can be very powerful; studies have shown that the main reason for gender imbalance at university level is the explanation for success that uses the two symbols ‘the hard working female’ (e.g. Hermione Granger) and ‘the male genius’ (e.g. Sherlock Holmes) (Leslie, Cimpian, Meyer & Freeland, 2015). The third aspect is personal gender which looks at how individuals perceive the structure with its symbols (Bjerrum Nielsen, 2003). Given it is a dynamic process, the structure and symbols can influence and change which in turn affects personal gender. The following quote illustrates the experience of not fitting in to the created norm:

An advantage of being male would be to have been more encouraged to pursue a career in mathematics/engineering/technology. I would also have fitted in at high school better than I did—my Years 9 and 10 were spent on an all-girls campus where it was supremely uncool to be good at maths and science (Leder, 2010, p.453).

The last aspect described by Bjerrum Nielsen (2003) is interactional gender which focus on interactions of individuals within the structure with its symbols. In the present paper, we are interested in how individuals perceive themselves in the structure (i.e. personal gender) and symbols including stereotyping (i.e. symbolic gender).

METHODS

The first step towards the data collection was a pilot study where a well-known questionnaire was used with the intention to reproduce studies of individual's attitudes about gender and mathematics (e.g. Gómez-Chacón, Leder & Forgasz, 2014). However, although following “good practices”, the results indicated several limitations and not just intercultural differences but also intracultural (Nortvedt & Sumpter, 2017). The feedback stressed that “you can’t ask question like this” meaning a revision was needed to make the questionnaire function in a Nordic context. A literature review showed that most prior research treat gender as a cultural-neutral construct and do not consider cultural dimensions: that questionnaires very seldom gave the respondents opportunities to demonstrate knowledge about gender beyond the classic male –female dichotomy or nuances in gender symbolism. (Sumpter & Nortvedt, 2018). We therefore applied Clarke (2013)’s seven dilemmas: (1) Cultural-specificity of cross-cultural codes; (2) Inclusive vs Distinctive; (3) Evaluative Criteria; (4) Form vs Function; (5) Linguistic Preclusion; (6) Omission; and, (7) Disconnection. One solution to meet some of these dilemmas were to apply vignettes. One example is the first question, Question 1a, “Who is best in mathematics, boys or girls?” with an vignette saying that different groups in the society might have different views of who is considered able in mathematics. By adding such a vignette, the question allow the respondent to express perceived gender stereotyping from others whilst expressing a personal attitude that might differ. The pilot study indicated that the questionnaire did allow students to demonstrate their awareness of a range of culturally rooted differences in attitudes towards boys’ and girls’ abilities to learn mathematics (Nortvedt & Sumpter, 2018).

To answer the research questions in the present paper, we will focus on Question 1a, “Who is best in mathematics, boys or girls?”, Question 1b, “Do you think this has changed over time?” where the latter also allowed qualitative responses. We also analyse the responses to one of the background questions which was a self-evaluation. The data comes from lower secondary school

students (grade 9; age 15; $n=241$) from seven schools in different locations in Sweden (north/south; rural/town/city). Given that online surveys have less response rate (Fan & Yan, 2010), the first author used personal contacts to find participating schools. Ethics rules provided by Swedish Research Council were followed. This means that those students who had not turned 15 before December 2019 could not participate, which according to Statistics Sweden should be around 6% of the population meaning two students per class. The statistical analysis of the replies used stated gender (boy/girl) as a factor ($n=222$) and we applied chi-squared test to analyse where girls' replies differ from boys. The qualitative responses were analysed using inductive thematic analysis (Braun & Clarke, 2006), and then compared to previous research as a second step. This means that we searched for similarities and differences in the written replies, gathering similar statements using a coding scheme. One example are statements that could be connected to a broader theme describing gender as a dynamic concept, where the codes were words like "change" or "difference". In this way, disjoint themes were created.

RESULTS

The first set of results focus on the attribution of ability in mathematics meaning the responses to the question "Who is best in mathematics, boys or girls?". In Table 1, G stands for Girls and B for Boys:

Groups	Girls are best	Boys are best	They are equally good	It is not about sex*	I'm not sure	<i>p</i>
Girls in grade 9	G: 27(24.5%) B: 57(53.3%)	7(6.4%) 5(4.7%)	19(17.3%) 14(13.1%)	50(45.5%) 27(25.2%)	7(6.4%) 4(3.7%)	<0.05
Boys in grade 9	G: 17(15.6%) B: 23(21.9%)	49(45.0%) 45(44.9%)	19(17.4%) 14(13.3%)	19(17.4%) 18(17.1%)	5(5.0%) 5(4.8%)	>0.05
Dads	G: 7(6.5%) B: 17(16.3%)	21(19.4%) 27(26.0%)	31(28.7%) 30(28.8%)	36(33.3%) 24(23.1%)	13(12.0%) 6(5.8%)	<0.05
Mums	G: 12(24.5%) B: 23(53%)	1(0.9%) 9(8.7%)	31(28.7%) 40(38.5%)	59(54.6%) 28(26.9%)	5(4.6%) 4(3.8%)	<0.05
Male teachers	G: 14(13.1%) B: 17(16.5%)	10(9.3%) 14(13.6%)	31(29.0%) 36(35.0%)	48(44.9%) 32(31.1%)	4(3.7%) 4(3.9%)	>0.05
Female teachers	G: 10(9.3%) B: 23(28.7%)	3(2.8%) 7(6.7%)	32(29.9%) 41(39.4%)	57(53.3%) 28(26.9%)	5(4.7%) 5(4.8%)	<0.05
Girls in general	G: 31(28.7%) B: 44(42.3%)	10(9.3%) 8(7.7%)	19(17.6%) 25(24.0%)	36(33.3%) 14(13.5%)	12(11.1%) 13(12.5%)	<0.05
Boys in general	G: 24(22.2%) B: 18(17.5%)	40(37.8%) 40(38.8%)	17(15.7%) 22(21.4%)	18(16.7%) 12(11.7%)	9(8.3%) 11(10.7%)	>0.05
You	G: 7(6.6%) B: 12(11.4%)	2(1.9%) 18(17.1%)	13(12.3%) 24(22.9%)	81(76.4%) 37(35.2%)	3(2.8%) 14(13.3%)	<0.05

Table 1: Responses to “Who is best in mathematics?”, *n*(%). Total responses differ from 106-110 (girls) and 103-107 (boys). *In Swedish, there is a difference between gender (‘genus’) and biological sex (‘kön’)

The majority of boys and girls attributes no gender, both regarding what they think other groups would answer but also in their own responses. It is interesting to note that one difference between girls and boys is that girls more often has their main response ‘It is not about gender’ more often than boys, whereas boys more often choose ‘they are equally good’. A few results stand out: both girls and boys reply that boys in grade nine and in general would reply that they are better. However, when responding as themselves (as ‘you’), this is not reproduced. Instead, the majority of boys (58.1%) think it is not a question about sex or that boys and girls are equally good. Here, there is a difference

between what is attributed to boys as a symbol and what could be considered as a personal view on a group level. Continuing with self-confidence and stereotyping, boys more often reply that girls in grade nine and in general would answer that girls are best in mathematics, a response pattern girls do not repeat. An interesting symmetry which is statistically significant appears in the responses about what the students think that mums and dads would reply: both boys and girls state that fathers would pick boys as better in mathematics, and for mothers to pick girls. This symmetry is not repeated regarding female and male teachers.

On the question whether this has changed over time, girls and boys differ in their responses, Se Table 2:

	Yes	No	I'm not sure	<i>p</i>
Girls	85(75.9)	9(8.0)	18(16.1)	<0.05
Boys	56(50.9)	22(20.0)	32(29.1)	

Table 2: Changed over time *n*(%)

Although the majority of both groups states “Yes”, girls do it more so. In the motivations why, the analysis generated three themes. The first theme is based on the idea that things do change over time, especially stereotypes:

I believe that before, one thought that boys were better. Women have always been oppressed and lads were the ones who got to show that they could do maths. Lately, I think that girls also have had a chance to show that they are good at maths and humans have realised that the difference is not so big [Girl 1]; I think that everything depends on the stereotypes what is male and [what is] not. We have [previously] related that men are often best in mathematics since they used to be [Boy1].

Both these motivations show an awareness of gender as a dynamic concept and that stereotyping is a part of the this changes of power. The second category is about boys and symbols attributed to boys:

I believe that boys normally are less interested [in school] than girls and therefore are looked upon as worse than girls. Guys live a life where you should not care about school to be considered cool. [Boy2]

In this response, there is an awareness about the relationship between symbolic gender and personal gender. The third theme is that biological sex is irrelevant:

Biological sex should not determine your knowledge in math and there is no sex better than the other. [Girl2]

Since Swedish language uses different words for gender and biological sex, the focus here is that biological sex is extraneous in this matter. That doesn't imply that gender is not relevant.

Table 1 indicates that both boys and girls more often connect boys with the reply 'Boys are best', but when looking at responses from a personal view, this is not repeated. As a final measure, we studied girls and boys responses regarding self-evaluation (see Table 3):

	Very good	Good	Average	Below average	Weak	<i>p</i>
Girls	13(11.7)	34(30.46)	39(35.1)	12(10.8)	13(11.7)	>0.05
Boys	19(17.1)	26(23.4)	40(36.0)	10(9.0)	16(14.4)	

Table 3: Self-evaluation *n*(%)

In Table 3, most responses are 'Good' or 'Average' and the results do not significantly differ. As a summary, the students participating in this study indicated that overall, gender is not a determining factor or there is no difference between boys and girls. In their written motivation, they showed great awareness of gender as a dynamic concept. However, their responses still signalled that boys, as a group, would think that they are better in mathematics, either as a sign of self-confidence or ability.

DISCUSSION

Here, grade nine students' attitudes about boys, girls and mathematics were studied with a focus on who could be considered better in mathematics: boys or girls, if they were equally good or if the question was not about biological sex at all. The majority of the respondents picked the latter two categories, but there were some differences in their response patterns. One pattern is that although the majority of responses, both from boys and girls, signal that neither boys nor girls are better at mathematics, boys more often answered that boys and girls are equally good and girls more often state that this is not about sex. When one take this result in comparison with gender theories (e.g. Acker, 2006; Connell, 2006), it could be seen as a difference between the level of understanding of gender; that boys more often signal that there is a gender division whereas girls more often state that such division is not fruitful. Both groups, however, turn to traditional stereotypical patterns when answering the questions from a group

perspective of boys in grade 9 and boys in general. Both groups are connected to the statement 'Boys are best'. This is in line with previous reports that boys more often than girls opt for higher levels in self-evaluations (OECD, 2013; Sumpter, 2012). This it is not repeated when boys answer from a personal perspective: girls and boys responses in the self-evaluation do not differ. Here, we have a variation between what is attributed and what is reported from an individual perspective. Boys also attribute similar gender stereotyping to girls, which girls do not repeat. This difference needs to be further investigated since it can inform us about intra-cultural tensions (e.g. Clarke, 2013; Nortvedt & Sumpter, 2017) or, in the light of Bjerrum Nielsen (2003) different aspects of gender, relationships between symbolic gender and personal gender.

When the students responded what they think their parents would reply, a symmetry appeared: fathers would say that boys are better in mathematics and mothers would choose girls. However, this symmetry should be viewed from the perspective that most of the students state that parents would express gender neutral attitudes. One possible explanation could be found in the written motivations where the main theme was that gender stereotypical views has changed in the society as a whole. The awareness of gender as a social construct, and not just a division of sex, among the 15 year olds participating in this study was impressive. When comparing to Gannerud's (2009) study where the teachers answered that gender is not an issue since the society is already equal, the students talked about an awareness of change including less oppression and how power has shifted (e.g. Acker, 2006). One possible explanation could be that this is a reflection of gender equality work in Swedish schools (e.g. Hedlin, 2013) or that progress has continued (e.g. Brandell, 2008). One implication is that if teachers want to fulfil the goals of the curriculum where it states that they should help students to critically analyse and discuss gender issues, they should be aware of that the students might have a developed gender view but that old stereotypes could still exists within this view.

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HOW DOES TEACHERS' ANALYSING OF CLASSROOM SITUATIONS DEVELOP IN THE FIRST YEAR OF TEACHING? A VIGNETTE-BASED LONGITUDINAL STUDY

Marita Eva Friesen¹ and Sebastian Kuntze²

¹University of Education Freiburg, Germany

²Ludwigsburg University of Education, Germany

Being able to analyse classroom situations forms an essential part of teacher expertise. Research into the development of what teachers identify and interpret as relevant for students' learning merits, consequently, particular attention. Building on our prior research, this longitudinal study investigates N=100 teachers' analysing regarding the use of multiple representations and its development in the first year of teaching (induction phase). A vignette-based test with 12 classroom situations from the content areas fractions and functions was administered at two points of measurement. For each of the situations, the participants were asked to evaluate the observed use of representations. The findings show little growth in the teachers' analysing as well as differences between analysing vignettes dealing with fractions or functions.

INTRODUCTION

Teachers' analysing of classroom situations informs teaching decisions and is therefore highly relevant for instructional quality and student learning (e.g., Kersting et al., 2012; Sherin, Jacobs & Philipp, 2011). In the mathematics classroom, the essential role of multiple representations draws specific attention to teachers' corresponding competence of analysing: Learning mathematics requires the use of multiple representations in a flexible and controlled manner (Acevedo Nistal et al., 2009). The complex cognitive demands related to changes between different representations can, at the same time, cause difficulties and hinder students' learning (Ainsworth, 2006; Duval, 2006). Consequently, the competence to *analyse the use of multiple representations* can be described as an important aspect of mathematics teachers' expertise (Friesen & Kuntze, 2016; Friesen, 2017).

The study presented in this paper builds on our research into teachers' competence of analysing and addresses in particular the *development* of such competence in the first year of teaching, the so-called induction phase in Germany. In most of the German federal states, an induction phase follows the university part of teacher education: For a period of mostly 18 months, the prospective teachers teach at a reduced level of hours and are supervised by a

mentor, usually an experienced teacher in the subject. One day per week, they attend courses on mathematics education. The Standards for Teacher Education (here: for the Federal State of Baden-Wuerttemberg) highlight the importance of developing mathematics teachers' competence of analysing during the induction phase and explicitly describe *analysing the use of representations in the mathematics classroom* as an essential learning goal. There is, however, not much evidence how that important aspect of mathematics teachers' professional competence develops in the first year of teaching. We addressed that need for research and describe the theoretical framework, the design of the study and selected results in the following.

ANALYSING THE USE OF MULTIPLE REPRESENTATIONS

The use of multiple representations plays a crucial role for the teaching and learning of mathematics. As mathematical objects are abstract in nature, they can only be accessed by using representations (e.g., Goldin & Shteingold, 2001). According to Duval (2006), representations can stand for mathematical objects and often use so-called representation registers (e.g., oral or written language, symbols, drawings, diagrams, graphs, etc.). Since changing between representation registers is often a key to solving problems and mathematical understanding (Ainsworth, 2006), the use of *multiple* representation registers can be regarded as indispensable for the teaching and learning of mathematics. Teachers and students generate and use multiple representations for introducing new topics, for explaining, for solving problems and for sharing ideas in the classroom, among others (Duval, 2006; Acevedo Nistal et al., 2009).

Numerous studies have shown, however, that using multiple representations of a mathematical object and changing between them involves high cognitive demands for the learners (Ainsworth, 2006; Duval, 2006). The changes between different representation registers, so-called conversions, can consequently lead to serious problems in understanding, e.g., when students fail to see how representations of the same mathematical object in different registers (e.g., verbal explanation, written symbols and drawing) are connected (Duval, 2006). For this reason, mathematics teachers have to be able to analyse classroom situations regarding the use of multiple representations to support their students in connecting different representation registers when conversions are carried out (Friesen & Kuntze, 2016). Based on the concept of teacher noticing (e.g., Sherin, Jacobs & Philipp, 2011), we described such ability as teachers' *competence of analysing the use of representations*, in particular regarding learning obstacles arising from conversions between unconnected representation registers (Friesen & Kuntze, 2016). In earlier research, we found that such competence can be described empirically through a one-dimensional Rasch model (Friesen, 2017).

Although analysing classroom situations as described above is regarded as highly relevant for the learning of mathematics, corresponding studies have found that both pre-service and in-service teachers often lack such competence (e.g., Friesen, 2017). Since it can be expected that practice-based learning opportunities can lead to further development in teachers' analysing (cf. Stahnke, Schueler & Roesken-Winter, 2016), this study aims at contributing to the field by addressing how teachers' competence of analysing focusing on the use of representations develops in their first year of teaching.

RESEARCH INTEREST AND RESEARCH QUESTIONS

In previous studies, we used cross-sectional designs to compare pre-service and in-service teachers' competence of analysing classroom situations regarding the use of multiple representations (e.g., Friesen, 2017). There is hence still a need for longitudinal studies allowing the assessment of teachers' analysing at several points of measurement to better describe its development; also the role of specific learning opportunities can thus be taken into account (specific course contents related to the use of representations as well as teaching practice regarding particular content areas). Most studies investigating teachers' analysing address only one particular content area, such as fractions, geometry, arithmetic or functions (cf. Stahnke, Schueler & Roesken-Winter, 2016). We were consequently particularly interested in comparing the development of teachers' analysing in two content areas (fractions and functions). Consequently, this study addresses the following research questions: (A) How does teachers' competence of analysing the use of representations develop during the first year of teaching (induction phase)? (B) Does teachers' competence of analysing develop differently in the two content areas fractions and functions? (C) What is the role of specific learning opportunities (course content, teaching practice in the two content areas during induction phase) in the context of questions (A) and (B)?

SAMPLE AND METHODS

The data analysed in this study was collected from $N=100$ teachers (61.0% female; $M_{age}=26.8$; $SD_{age}=4.3$) in their induction phase for teaching mathematics at secondary level (grades 5–10). The teachers' competence of analysing was assessed at two points of measurement using a vignette-based test instrument (pre-test: at the beginning of the induction phase; post-test: 12 months later). Vignettes can represent classroom practice in different formats (video, cartoon, text) and offer various possibilities for eliciting teachers' analysis of classroom situations in systematically designed research settings (Buchbinder & Kuntze, 2018). In our prior research (Friesen, 2017), we could show that different vignette formats are equally suitable for eliciting teachers' competence of analysing regarding the use of representations. Accordingly, we used twelve purposefully designed vignettes in the formats cartoon and text that

represented mathematics classroom situations with a similar narrative: A group of students struggle with solving a task, they show the teacher their work in a certain representation register (e.g., calculation, written symbols) and ask the teacher for help. The teacher tries to support the students by unnecessarily changing the representation register (e.g., by making a sketch or drawing). However, the teacher does little to connect the students' representations with this new representation and there is no specific support for the students to see that the different representations belong to the same mathematical object. Based on the theory of learning with multiple representations as outlined above, such unconnected conversions are very likely to cause further problems in students' understanding.

The vignettes (Figure 1; cf. Friesen, 2017) were administered in a paper-and-pencil test and the participants of the study were asked to evaluate the vignette teachers' teaching in the twelve different classroom situations (six situations each from the content area of fractions in grade 6 and functions in grade 8). Each vignette was followed by an open-ended question (*How appropriate is the teacher's response in helping the students to solve the task? Please evaluate the use of representations and give reasons for your answer.*) and four rating-scale items (e.g., *By using an additional representation, the teacher supports the students' understanding*).

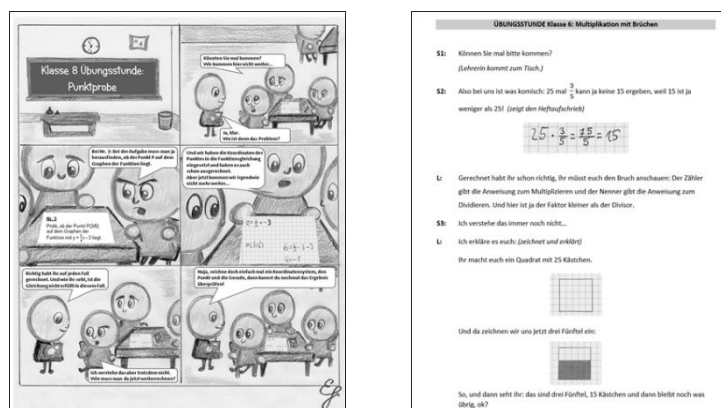


Figure 1: Sample vignettes (left: cartoon-based function vignette; right: text-based fraction vignette; drawings by Juliana Egete)

In a prior study with $N=175$ mathematics teachers (Friesen & Kuntze, 2020), the empirical item difficulties of the twelve vignettes were computed using IRT scaling. Accordingly, different booklets for the pre-test and post-test were designed (eight vignettes at each point of measurement, four anchor items) to be able to control for test repetition effects (see Figure 2 for the design of the booklets).

PRETEST (8 vignettes)							
Fractions				Functions			
C1	C2	T4	T5	T6	T1	C2	C3
POSTTEST (8 vignettes)							
Fractions				Functions			
T3	C2	T4	C6	C4	T1	C2	T5

Figure 2: Design of the test booklets in pre-test and post-test (T: text-based, C: cartoon-based; 1–6: vignette numbers in the content areas of fractions or functions)

To examine the role of specific learning opportunities for the development of the participants' competence of analysing during their induction phase, they were asked in the post-test: (1) if multiple representations and their use had been provided as a course topic during induction phase and (2) if and how long (instruction time per week) they had collected experience in teaching fractions and/or functions.

DATA ANALYSIS AND SELECTED RESULTS

The rating-scale items were scored dichotomously to examine if the participants of the study have perceived the potentially hindering changes of representations in the classroom situations in their analysis. This coding resulted in a maximum of eight points per measurement. Taking into account the design as shown in Figure 2, we analysed the data from pre-test and post-test using the joint calibration method (Wu, Tam & Jen, 2016). We could find a good compatibility with the Rasch model ($0.88 \leq wMNSQ \leq 1.12$; $-1.2 \leq T \leq 1.7$; cf. Bond & Fox, 2015), indicating that the scores from pre-test and post-test can be modelled on a joint scale. Since the study contained vignettes from two different content areas (fractions, functions), we compared a one-dimensional model (containing all vignettes) with a model including two subdimensions (subdimension 1: fraction vignettes, subdimension 2: function vignettes). The model with subdimensions takes into account the potentially higher local dependencies amongst vignettes from the same content area (cf. Hartig & Höhler, 2009). The comparison of the two models indicated no significant difference ($\chi^2(2)=3.70$, $p=.157$). Since good fit values for the Rasch model could be found, we were able to use the raw scores for the following analysis to compare the results from pre-test and post-test.

To answer research question A and B, we compared the participants' scores for their competence of analysing between pre-test and post-test. The findings revealed different developments of the participants' competence of analysing in the two content areas under investigation: Only in the content area of functions, a significant increase in the analysing scores could be found ($M_{pre}=2.17$, $SD_{pre}=1.11$; $M_{post}=2.57$, $SD_{post}=1.19$; $F(1, 99)=8.800$; $p=.004$), indicating a small effect ($d=.308$).

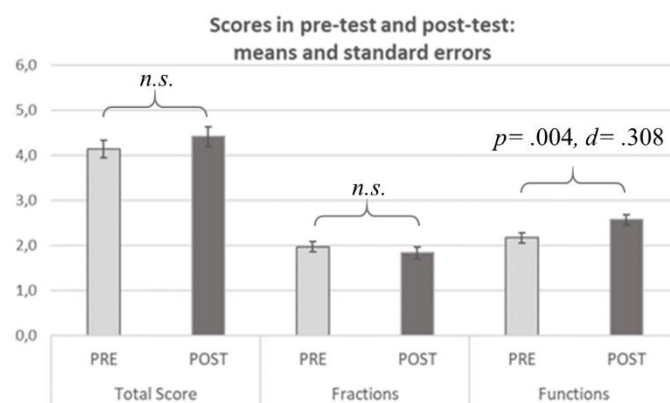


Figure 3: Scores in pre-test and post-test: means and their standard errors

Figure 4 and 5 illustrate these findings with sample answers to the open-ended items. As reported above, each of the vignettes was followed by the question: *How appropriate is the teacher's response in helping the students to solve the task? Please evaluate the use of representations and give reasons for your answer.* Figure 4 shows a participants' responses to a fraction vignette (C2 in Fig. 2) from pre-test and post-test. At both points of measurement, there is no indication for a successful analysis regarding the use of representations: In the pre-test, the vignette teacher's reaction to the students' question is evaluated as "good". Although the change of representations is described, its potentially obstructing role for the students' understanding is not mentioned. In the post-test, the answer focuses only on the potential of the bar model used by the vignette teacher, the change of representations is no longer mentioned.

<p>Wie gut eignet sich die Reaktion der Lehrperson, um den Schülerinnen und Schülern weiterzuhelfen? Bitte beurteilen Sie im Hinblick auf den Umgang mit Darstellungen und begründen Sie!</p> <p>Gute Reaktion Bildliche Darstellung passt zur Vorstellung der Aufgabe</p>	<p>Pre-test</p> <p>Good reaction. Visual representation corresponds well to the task.</p>
<p>gut geeignet durch Streifenmodell wird das Verständnis für Anteile angeregt</p>	<p>Post-test</p> <p>Well suited. The use of the bar model activates the understanding for parts.</p>

Figure 4: Sample answers (pre and post) to the same fraction vignette

Figure 5 shows another participants' answers to a function vignette (C2 in Fig. 2) illustrating the pre-service teachers' growth in analysing this classroom situation: In the pre-test, the vignette teacher's change of representation from equation to graph is mentioned but evaluated as helpful for the students' understanding. The answer from the post-test indicates that the participant analysed both the students' problem in understanding and the teacher's reaction. The participant acknowledges the role of using another representation for helping the students in this situation, however, he also describes the lack of explanation when the change of representation is carried out. The missing connection between the students' question and the teacher's reaction is

highlighted additionally by mentioning that the students' question was not answered.

<p>Der Lehrer ermutigt die SuS dazu eine weitere Darstellung zu nutzen und so selbst hinter die Bedeutung der ersten Darstellung zu kommen.</p>	<p>Pre-test The teacher encourages the students to use another representation [here: graph]. This helps them to find out the meaning of the first representation [here: equation] on their own.</p>
<p>Dem SuS fehlt in dieser Zusammenfassung das Verständnis, dass eine nicht erfüllte Gleichung ein mögliches Ergebnis der Punkte ist und was das bedeutet. Diesen Verständnis kann über den Wechsel der Darstellungsform erreicht werden. Dafür fehlt noch meine Meinung etwas deutlicher. Die L. beantwortet die Schülerfrage nicht!</p>	<p>Post-test The students do not understand that the equation is not true and what that means when they check if a point is on a line. Changing the representations [here: from equation to graph] can help, but I think the explanation is not enough here. The teacher does not answer the students' question!</p>

Figure 5: Sample answers (pre and post) to the same function vignette

To answer research question C, the items addressing the specific learning opportunities during the participants' induction phase were examined. 58.0% of the participants reported that one of the weekly courses (lasting about two hours) focused on representations and their use in the mathematics classroom or that this topic was discussed frequently, regardless of grade level or content area. Asked about their *teaching experience*, 45.0% of the participants stated that they had taught fractions in grade 6 (or 7) and 23.0% stated they had taught functions in grade 8 (or 7) during their induction phase. This corresponded to a reported instruction time of four to five lessons per week. Examining relations between the reported learning opportunities and the teachers' post-test scores did, however, not yield statistically significant findings in terms of correlations or systematic relations in contingency tables.

DISCUSSION

Before we discuss the findings of the study, we would like to address its limitations. The vignette-based test is restricted to analysing the use of multiple representations in classroom situations from the content areas of fractions and functions. Since the sample is not representative for German mathematics teachers, conclusions should be drawn with care. Despite these limitations, we could find answers to our research questions: The results from the rating-scale items indicate that there was on average only little development in the pre-service teachers' competence of analysing during their induction phase. Furthermore, it could be shown that the development differed in the two content areas under investigation: Significant growth in the participants' competence of analysing (indicating a small effect) could only be found in the content area of functions. However, no systematic relations of the teachers' competence of analysing with any of the reported learning opportunities (teaching experience, course contents) could be revealed. Deepened analyses of the answers to the open-ended items might provide additional insight into the development of the teachers' competence of analysing in their first year of practice since they allow to explore participants' reasoning and also related difficulties in more detail.

The findings of the study encourage further research into effective learning opportunities for facilitating teachers' analysing during their induction phase, a phase during which teachers are particularly required to connect classroom observations with the professional knowledge developed at university. Our prior research related to developing such analysing in the context of teacher education courses (e.g., Friesen, Dreher & Kuntze, 2015) showed, for example, how (video) vignettes can be used to foster student teachers' growth in analysing. We expect further insight how prospective and early career teachers can be supported in analysing classroom situations from follow-up studies carried out in the ERASMUS+ project coReflect@maths (Digital Support for Teachers' Collaborative Reflection on Mathematics Classroom Situations).

Acknowledgements

This study was supported by Ludwigsburg University of Education and the Ministry of Science, Research and the Arts in Baden-Wuerttemberg as part of the project EKoL.

The project coReflect@maths (2019-1-DE01-KA203-004947) is co-funded by the Erasmus+ Programme of the European Union. The European Commission's support for the production of this publication does not constitute an endorsement of the contents, which reflect the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

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KNOWLEDGE AND ACTION FOR CHANGE TOWARD THE 4TH INDUSTRIAL REVOLUTION THROUGH PROFESSIONAL PRACTICE IN ETHNOMATHEMATICS

Linda H.L. Furuto

University of Hawai‘i at Mānoa, United States of America

Research in teacher education over the past ten years has led to policy and practice implications for learning and instruction, including institutionalization of the world’s first academic program in ethnomathematics. The ethnomathematics program at the University of Hawai‘i empowers teachers as leaders to align professional practices with state and national standards via the innovative design, implementation, and assessment of culturally-sustaining research and praxis in formal and informal, place-based contexts. The mission and vision of the program are inspired by its role in the worldwide voyage of the traditional canoe Hōkūle‘a, bridging Indigenous wisdom and 21st century interdisciplinary knowledge and action toward the transformational 4th Industrial Revolution. The underlying goal is a shared commitment to equity, empowerment, and dignity for all.

FOCUS AND OBJECTIVES

Three decades after the 1984 International Congress on Mathematical Education’s declaration of “mathematics for all,” we have come to understand that mathematics is undergoing one of the most critical periods in its recorded history (Bishop, 1988; NSTC, 2018). The U.S. National Science & Technology Council’s Strategy for STEM Education (2018) is to “provide all Americans with lifelong access to high-quality STEM education, especially those historically underserved and underrepresented in STEM fields and employment...[and] an urgent call to action for a nationwide collaboration with learners, families, educators, communities, and employers—a “North Star” for the STEM community as it collectively charts a course for the Nation’s success” (p. v). The current era emphasizes the “dreams, possibilities, and necessity of public education,” and the role of mathematics in influencing the equilibrium of achievement (Weiss & Miller, 2006).

The goal of this paper is to discuss how knowledge and action for change are achieved through professional practice in ethnomathematics, in an ongoing process of navigating toward the “North Star” in Hawai‘i and the Pacific (Furuto, 2018; Tuhiwai Smith, 1999). Specifically, our research has developed new theoretical insights into honoring and sustaining non-Western systems based on examples in mathematics teacher education. The premise is that

“Mathematics is powerful enough to help build a civilization with dignity for all, in which ethnomathematics practices encourage respect, solidarity, and cooperation...in the pursuit of peace” (Rosa, D’Ambrosio, Orey, Shirley, Alanguí, Palhares, & Gavarrete, 2016, p. ix). Our vision is grounded in a shared commitment to equity and empowerment, as required to responsibly and ethically navigate toward the 4th Industrial Revolution (Maynard, 2015; Rosa et al., 2016).

THEORETICAL FRAMEWORKS

Ethnomathematics is real-world problem-solving that empowers locally-minded, global citizens through interdisciplinary learning that is connected to the ecological, cultural, historical, and political contexts in which schooling takes place (Gutiérrez, 2017; Rosa et al., 2016). Over the past three decades, research in teacher education has emerged to promote development in the areas of equity, empowerment, and ethnomathematics, including: culturally relevant pedagogy (Ladson-Billings, 1995), engaged pedagogy (hooks, 1994), critical care praxis (Powell & Frankenstein, 1997), and culturally sustaining pedagogy (Paris, 2012). According to Paris (2012), “Culturally sustaining requires that our pedagogies be more than responsive of or relevant to cultural experiences and practices...it requires that they...simultaneously offer access to dominant cultural competence” (p. 95).

Research in Hawai‘i and Pacific communities demonstrates the importance of culturally sustaining pedagogy as we strive toward social justice and overcome deficit theories (Furuto, 2014; Kana‘iaupuni, 2005). Through interconnected work with educational institutions, research organizations, and community partners, we have created an ethnomathematics program to further a deeper understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof (Adler & Venkat, 2014).

A tradition that runs deep in Indigenous peoples of Hawai‘i and the Pacific for over 2,000 years is deep sea voyaging by celestial navigation without modern navigational tools (Finney, Kilonsky, Somsen, & Stroup, 1986). Traditional wayfinding is guided by the sun, moon, stars, winds, currents, and mathematical modeling. When the navigation renaissance began in the early 1970s by the Polynesian Voyaging Society (PVS), Native Hawaiian and others voyaged to prove that purposeful migration occurred across the Pacific (PVS, 2016). Now, with the tradition of wayfinding revived and thriving, the voyages allow new generations to honor and sustain knowledge, culture, and values through education. The PVS prototype canoe Hōkūle‘a has sailed over 160,000 nautical miles and spawned a legacy of more than 25 deep sea voyaging canoes birthed across 11 Pacific Island nations (Finney et al., 1986; Furuto, 2018). Hōkūle‘a serves as a powerful vehicle to draw on the strengths of our Pacific histories,

identities, and cultures, and broadens the participation of groups historically underrepresented in mathematics.

Hōkūle‘a’s most recent voyage circumnavigated the globe from 2013–2017 with a mission to mālama honua—to “care for Island Earth” and all people and places as ‘ohana (“family”). The lead author was an apprentice navigator and education specialist on the voyage, sailing with leaders such as the Archbishop Desmond Tutu, His Holiness the 14th Dalai Lama, and United Nations (UN) Secretary General Ban Ki-moon. From outside the UN Headquarters on World Oceans Day 2016, Ban Ki-moon stated, “I am honored to be part of the Mālama Honua Worldwide Voyage. I am inspired by its global mission, and support our common cause of ushering in a more sustainable future and a life of dignity for all through education.”

SETTING AND SIGNIFICANCE OF WORK

Knowledge and action for enduring, transformational change toward the 4th Industrial Revolution comes from working with and learning from the populations we are endeavoring to serve, and co-constructing thinking skills necessary for the future (Powell & Frankenstein, 1997). According to Jaworski, Wood, and Dawson (1999), “In-service providers cannot just ‘deliver’ a course or workshop. They must become part of learning communities” (p. 12). This is what we have strived to do by bringing the voyages back to land.

Hawai‘i’s population is among the most diverse in the nation. The breakdown is Caucasian (25%), Filipino (15%), Japanese (14%), Native Hawaiian/Pacific Islander (10%), and others (U.S. Census Bureau, 2010). There are a range of schools classified as urban, suburban, and rural. The Hawai‘i State Department of Education (HIDOE) serves many students in poverty, and 47% receive free and reduced lunch (HIDOE, 2020). Moreover, Hawai‘i is the only statewide school district in the nation, and operates a single public higher education system at the University of Hawai‘i (UH). The data and context make Hawai‘i a valuable study, and provide a significant lens into the future of diversity in the U.S. with global implications.

In Fall 2013, the PVS Promise to Children was authored by educational leadership in Hawai‘i and the Pacific, including the HIDOE Superintendent and UH system President who participated as crew members on the Mālama Honua Worldwide Voyage. This alliance spans early childhood education through graduate studies (P–20), public and private sectors, and invites new partners to achieve collective impact (Kania & Kramer, 2011). As a result of P–20 collaborations, the HIDOE created learning outcomes to inform policy at the statewide level. Nā Hopena A‘o (2015) is a framework to honor the unique context of Hawai‘i’s Indigenous language and culture. Similarly, interwoven in the UH System Strategic Directions 2015–2021 (2015) are key imperatives to

being a foremost Indigenous-serving institution and advancing sustainability, with the Mālama Honua Worldwide Voyage as a catalyst.

The College of Education at the UH system's flagship campus, UH Mānoa, is ideal to help achieve P-20 knowledge and action for change through ethnomathematics. The College of Education directs teacher preparation programs, curriculum design, and research projects in Hawai'i and U.S. affiliated Pacific Islands. It produces more than 65% of Hawai'i's teaching force and prepares professionals to contribute to a just, diverse, and democratic society across the Pacific (UH IRO, 2020).

RESEARCH METHODS AND DATA

The Ethnomathematics Institute was developed to bring together research institutions, cultural practitioners, and community-based organizations in support of undergraduate STEM majors at UH West O'ahu (2008-2013), and later, transitioned to UH Mānoa to strengthen professional development for P-20 STEM educators (2013-2018). Grant funding was provided over the years by the National Science Foundation and U.S. Department of Education, among others.

The main objectives of the institute were to: (1) explore promising practices in historically marginalized populations in alignment with national and state standards, such as Mathematics Common Core State Standards (CCSS), Next Generation Science Standards (NGSS), and Nā Hopena A'o (HĀ); (2) prepare teachers as leaders to provide ethnomathematics instruction and professional development in their schools and communities that are relevant, contextualized, and sustainable; and (3) strengthen campus-community partnerships to build sustainable networks within Hawai'i and the Pacific.

From 2013-2018, the participants in the Ethnomathematics Institute represented a diverse range of experience, disciplines, grade levels, and locations (Table 1).

Demographic Descriptions	No. of teachers
<i>Grade level taught during study year (n = 78):</i>	
Elementary school (Grades K–5)	12
Middle school (Grades 6–8)	18
High school (Grades 9–12)	28
District resource teacher (K–12)	4
Undergraduate students	6
Post-secondary teachers	6
Other (i.e., non-formal, informal educators)	4
<i>Ethnic background (n = 78):</i>	
Asian	24
Caucasian	24
Native Hawaiian	18
Hispanic	4
Pacific Islander	4
Other	4
<i>No. of years teaching (n = 78):</i>	
<1	2
1–4	30
5–10	11
11–15	13
>15	22
<i>School type (n = 78):</i>	
Public	58
Public charter	15
Private	5
<i>Disciplines taught during study year (n = 78):</i>	
English	4
Math	32
Science	24
Technology	6
All subjects (elementary)	12

Table 1: Demographic descriptions of participants

Within the Ethnomathematics Institute, learning occurred in formal and informal place-based contexts. Through real-world applications from mountains to sea, the project activities, assignments, and assessments bridged Indigenous wisdom and 21st century skills to classrooms and communities. For example, content areas included: wayfinding by geometrical angles of the sun, moon, and stars; algebraic studies of currents and water quality in nearby rivers and streams; and monitoring the impact of climate change on school gardens, among others (Furuto, 2018).

Project evaluation from 2013-2018 was based on mixed-methods grounded in strengths-based approaches. The evaluators collected both formative and summative data. Evaluation questions were aligned with the three objectives of the Ethnomathematics Institute (Table 2).

Goals	Evaluation Questions
Increase knowledge of content and pedagogy in culturally sustaining mathematics aligned with Common Core State Standards (CCSS), Next Generation Science Standards (NGSS), and Nā Hopena A‘o (HĀ)	1. To what extent did the participants perceive that the project affected their knowledge of culturally sustaining mathematics pedagogies aligned with CCSS, NGSS, and HĀ?
Prepare teachers as leaders to provide instruction and professional development in ethnomathematics in their schools and communities through high-quality learning that is relevant, contextualized, and sustainable	2. How did the participants perceive the process of their lesson plan development and implementation?
Strengthen campus-community partnerships within Hawai‘i and the Pacific for sustainable classroom and community networks	3. To what extent did the participants report the project cultivated a supportive, sustained community?

Table 2: Program goals and evaluation questions

Qualitative data was collected using semi-structured focus groups. Content analysis was utilized to evaluate constructed response questions with a grounded theory approach (Corbin & Strauss, 2014). Participants reported that the program increased culturally responsive pedagogy and emphasized the importance of understanding student culture, promotion of cultural understanding, and adjustment of teaching practice to reflect student culture using pedagogical strategies familiar to students. One participant commented, “I am making...a point to connect almost everything I teach to something the kids know about already, something that is in our community or environment and close to their hearts. It is making all the difference.”

Quantitative data analysis consisted of descriptive statistics. In general, participants perceived the Ethnomathematics Institute to be valuable and relevant to their teaching practice, as measured on a Likert scale from 1-5 with 1 = Do Not Agree and 5 = Strongly Agree (N = 78, M = 4.84, SD = 0.37). At the end of the Ethnomathematics Institute, participants were most likely to agree that they understood and could incorporate culturally sustaining pedagogy aligned with state and federal standards into their classrooms. The

disaggregated items had a reliability of 0.75 (Cronbach's alpha) and an overall mean of 4.15 (select results in Table 3).

Prompt	N	M	SD	Min, Max
1. The project helped me better understand and incorporate culturally sustaining pedagogy	78	4.74	0.45	4, 5
2. The project helped me better understand and incorporate mathematics content	78	4.42	0.69	3, 5
3. The project helped me better understand and incorporate Common Core State Standards	78	4.78	0.52	4, 5
4. The project helped me better understand and incorporate Next Generation Science Standards	78	4.52	0.65	3, 5
5. The project helped me better understand and incorporate Nā Hopena A'ō	78	4.78	0.56	4, 5

Table 3: Understanding and incorporation of pedagogy, content, and standards

RESULTS

Over the past ten years, the Ethnomathematics Institute has grown through successes and challenges. When the program was based at UH West O'ahu, performance measures included a 1400% increase in undergraduate students enrolled in mathematics courses, as the population grew from 940 students in 2007 to 2,361 students in 2013 (UH IRO, 2020). This led to the development of 11 new mathematics courses tied to institutional learning outcomes, accreditation, and graduation requirements, all of which are grounded in ethnomathematics. When the Ethnomathematics Institute transitioned into a yearlong professional development program for P-20 educators, the participants represented all HIDOE complexes and districts. This led to an integrated statewide network that extended to the Pacific and demonstrated a commitment to transform education.

The world's first academic program in ethnomathematics was institutionalized at the UH Mānoa College of Education in 2018, thus leading the way for mathematics education. The 15-credit program is designed to lead into the M.Ed. Curriculum Studies: Mathematics Education, providing an attractive option for graduate students. There are no master's degrees in mathematics education at any other UH system institutions or U.S. affiliated Pacific Islands.

Moreover, in an unprecedented move, the Hawai'i Teacher Standards Board, which licenses teachers throughout Hawai'i and U.S. affiliated Pacific Islands, officially approved ethnomathematics as a field of licensure in 2018. This approval indicates that program assessments, rubrics, and frameworks aligned

with the Council of Chief State School Officers' model core teaching standards (CCSSO, 2013).

CONCLUSIONS

A decade of research, theory, and praxis has ultimately led our voyage to the creation of a new academic program. This illustrates how ethnomathematics has empowered teachers as leaders, through equitable practices aligned with state and federal standards that bridge Indigenous wisdom and 21st century learning. The skills necessary for the 4th Industrial Revolution require innovative and interdisciplinary research-based practices that further our understanding of teaching and learning mathematics (Adler & Venkat, 2014; Maynard, 2015).

Three decades after the 1984 International Congress on Mathematical Education, we have increasingly hopeful responses to the challenge of re-examining the equilibrium of mathematics. "Mathematics for all" is not just a vision but a growing reality. As we reflect on our calls to action, we are inspired by the proverb, "'A'ohe hana nui ke alu'ia—No task is too big when done together by all" (Pukui, 1993, p. 18). Through storms and calm seas, we will remain steadfast in our firm commitment to follow the "North Star" to equity, empowerment, and dignity for all.

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FLEXIBILITY IN DEALING WITH MATHEMATICAL SITUATIONS IN WORD PROBLEMS – A PILOT STUDY ON AN INTERVENTION FOR SECOND GRADERS

Laura Gabler¹ and Stefan Ufer

Ludwig Maximilian University, Munich, Germany

Empirical studies have underlined students' difficulties with arithmetic word problems involving comparisons of sets. Although research has proposed strategies to reinterpret difficult word problems into easier ones, no corresponding interventions have been designed and evaluated. This study takes up the idea and aims at (1) replicating and systematizing former results on the difficulty of word problems and (2) investigating, if second graders are able to identify similar situation structures in pairs of word problems, and use this information to solve more difficult word problems. Results did only partially replicate prior research on the difficulty of word problems, and did not show that students transferred situation structures between pairs of tasks. This underlines the necessity of a corresponding intervention study.

INTRODUCTION

Prior research has shown that the way an arithmetic problem is presented influences the difficulty for learners: The same problem presented in numerical format (e.g., $3 + 5 = 8$) is solved 10 to 30% less frequently, if it is embedded in a word problem (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). This implies that factors other than arithmetic skills influence a word problem's difficulty. Current research often distinguishes between comprehension obstacles, which relate to reading comprehension, and conceptual obstacles, which relate to problems with the acquisition of the semantic problem structure of a word problem (Prediger & Krägeloh, 2015). In this article, we present a preparatory study for the design of an intervention program with a focus on the analysis of conceptual obstacles: After summarizing processes and difficulties occurring when solving of word problems, and presenting strategies to tackle these difficulties, we investigate, if students already make use of these strategies spontaneously.

CURRENT STATE OF RESEARCH

In the past, there has been extensive research on solving word problems (Stern, 1993; Vicente, Orrantia, & Verschaffel, 2008), in particular focusing on one-step arithmetic word problems on addition and subtraction. Common frameworks on solving word problems (e.g., Kintsch & Greeno, 1985) include two models that need to be constructed individually: the *situation model* and the

mathematical problem model. An alternative perspective focuses on structures that problem authors may have intended while writing word problems. According to this perspective, students need to reconstruct these structures in the form of situation models when solving the problem. The relationship between these two perspectives will be outlined in the following.

Problem authors can realize a word problem linguistically in different ways (*text base*). For instance, a description of the comparison of two sets requires relational terms. The authors can decide how they describe the relation (with terms like “more”, “less”, “bigger”, or “smaller”, etc.). (1) As a first step of solving a word problem, the learners decode this generated text base and integrate the information and prior knowledge into an initial *situation model* (Kintsch & Greeno, 1985). At best, the first “draft” of an individual situation model contains the basic components of the author’s intended *situation structure*. In course of the solution process, learners can enrich their situation model so that it also reflects alternative situation structures, using inferences based on their prior knowledge (Kintsch, 1998). (2) Furthermore, the learners need to transfer the situation model to a *mathematical problem model* by describing their situation model with mathematical concepts (Kintsch & Greeno, 1985). At best, this mathematical problem model corresponds to the author’s intended *mathematical structure* of the word problem. To find an adequate mathematical problem model based on the situation model, the learners need to know, which mathematical structures can describe a certain situation structure. The constructed situation model is essential during this process. Depending on the aspects of the situation structure, which are available in the individual situation model, the learners are assumed to activate different facets of their individual knowledge, which might be more or less helpful for the construction of a mathematical problem model.

Various prior studies have investigated, which factors influence the difficulty of solving word problems (e.g., De Corte & Verschaffel, 1987). One line of research indicates that the linguistic presentation influences the difficulty of word problems (Bailey & Butler, 2003). Beyond this, the intended situation structure of word problems has been found as another relevant factor. Different characteristics of this situation structure may influence the difficulty of a word problem on addition and subtraction: the *semantic structure*, the *unknown set*, and the *additive or subtractive wording*.

Regarding the *semantic structure*, research on word problems on addition and subtraction often distinguished between *change*, *combine*, *compare*, and *equalize* situations (e.g., Riley, Greeno, & Heller, 1983). While *change* and *equalize* word problems describe a dynamic action, *combine* and *compare* word problems are considered static situations. Past research has shown that students’ solution rates vary with the semantic structures in a word problem, and *compare* problems seem to be particularly challenging for learners (Riley & Greeno,

1988; Stern, 1993). One reason might be that the difference set in *compare* problems does not describe an existing quantity but a relationship between two sets and therefore is hard to represent in a situation model. In the sequel, we mostly exclude *combine* problems from our analysis, since they have a substantially different situation structure and are usually not among the more difficult problem types.

Moreover, one of the three involved sets of a one-step word problem is usually unknown initially (*unknown set*). In change word problems, this can be the *start set*, the *change set*, or the *result set*, whereas these options are called *reference set*, *difference set*, or *compare set* in compare word problems (Riley et al., 1983). Studies on the impact of the unknown set agree that word problems with an unknown reference set or start set are most difficult (Stern, 1993).

Additionally, *additive or subtractive wording* (a/s wording) (Fuson, Carroll, & Landis, 1996) can be seen as a part of the situation structure: It determines a certain direction of the action or a certain perspective on a compare situation, which can be represented in a situation model. While the a/s wording of dynamic change and equalize problems is expressed with action verbs (additive wording: e.g., “to get”, “to win”, subtractive wording: e.g., “to give”, “to lose”), the comparison of sets requires relational terms such as “more than”, “bigger than” (additive wording) or “less than”, “smaller than” (subtractive wording). Combined with the mathematical structure of a word problem, the a/s wording influences the difficulty of a task: If the a/s wording and the directly applicable mathematical operation do not match (e.g., additive wording in a problem that can be solved by a direct subtraction) lower solution rates for the word problem are observed. These word problems are called *inconsistent* (Lewis & Mayer, 1987).

Differences in performance caused by the various characteristics of the situation structure led to the idea that reinterpreting more difficult types of word problems by reinterpreting them in terms of easier situation structures could be useful for solving them. Greeno (1980) proposes to use easier accessible semantic structures: Instead of solving change word problems, he suggests that students could adapt their situation model by reinterpreting the situation as a combine word problem. Alternatively, Stern (1993) proposes to reinterpret a word problem’s a/s wording. Being able to transfer between different directions of mathematical relations (e.g., “Anna has two marbles more than Ben.” equals “Ben has two marbles less than Anna.”) or actions might help learners to reinterpret inconsistent word problems as easier, consistent word problems. For this, learners need to understand the equivalence of such symmetric relational expressions. Both approaches describe processes that require the enrichment of the initial situation model with an alternative situation structure that represents a different perspective on the situation. Being able to make such inferences towards alternative situation structures provides *flexibility in dealing with*

mathematical situations. We assume that this flexibility might support learners to find a mathematical problem model based on the enriched situation model. A lack of this flexibility could be one cause for the described differences in performance when solving word problems. Accordingly, an intervention supporting students in acquiring this flexibility might improve the students' performance in solving difficult word problems.

However, it is currently unclear why learners do not apply the reinterpretation of situation structures spontaneously. On the one hand, learners might not have gained flexibility in dealing with mathematical situations yet. This would imply the need for an intervention study to determine, whether the skill can be trained. On the other hand, learners might have acquired the required skill but fail to apply it. In this case, solving two consecutive, structurally similar tasks should elicit the application of this flexibility: Students might transfer features of the empirically easier situation structure (e.g., equalize situations) to the situation structure of the next, more difficult word problem (e.g., compare situations), if it is embedded in the same context situation. Finally, if providing a hint to use this structural similarity of consecutive tasks would trigger learners to apply the described strategy, an intervention would need to focus on the application rather than the acquisition of this flexibility.

AIMS AND RESEARCH QUESTIONS

To study if an intervention based on Greeno's and Stern's suggestions could be helpful, two main issues need to be resolved: The first aim of this contribution is to replicate prior results regarding task difficulty, which are fundamental for the intervention.

Q1: Which of the task features *semantic structure*, *a/s wording*, and *unknown set* cause differences regarding the difficulty of word problems on addition and subtraction?

Based on prior studies, we expected that compare word problems would be more difficult than equalize (H1.1) and change word problems (H1.2). Solution rates should be higher for consistent word problems than for inconsistent word problems (H1.3). Moreover, existing studies have varied surface features of the word problems (names, quantities, involved objects) together with the mentioned task features. Our study controls the variation of difficulty caused by these surface features. We expected only minor differences in solution rates due to surface features (H1.4).

The second aim of this study was to study the (spontaneous) use of the described strategy to reinterpret situation structures.

Q2: Do students use similar situation structures spontaneously for the solution of word problems? Does a hint support the use of the strategy?

The successful use of this strategy should cause higher solution rates for items for the second of two structurally similar, consecutive word problems (as compared to the first problem in the pair, H2.1). We expected stronger differences, when compare situations occurred after a dynamic situation, than in the reverse sequence (H2.2). In the case that students have already gained the required knowledge but fail to apply the strategy spontaneously, we assumed that the effects in H2.1 and H2.2 would be more pronounced, if the learners receive an explicit hint on the similarity of the situation structures (H2.3).

METHOD

To answer the research questions, paper-and-pencil based tests were used in a cross-sectional study with second graders from eight classrooms in Germany ($N = 139$). Each student solved twenty different word problems on addition and subtraction, which were selected from a larger collection of task variations. To examine Q2, the word problems were arranged in pairs. Each *pair of word problems* contained two structurally similar word problems with the same surface features, a/s wording, and unknown set, but differed only in their semantic structure. An exemplary pair of word problems could consist of task A: “Anna has 13 marbles, Ben has 8 marbles. How many marbles does Anna have to give Ben, so that she has as many as Ben?” and task B: “Anna has 13 marbles, Ben has 8 marbles. How many marbles does Anna have more than Ben?” In this example, the first word problem deals with the equalization of sets, while the second word problem describes a similar comparison of sets. For the compilation of all possible versions of pairs of word problems, we varied combinations of semantic structures (change and compare, equalize and compare), the a/s wording, and the unknown set systematically. These prototypical types were each embedded in twelve different situation contexts. In the end, we generated a second set of word problems by reversing the order of the two word problems within each pair.

Procedure: Each student solved ten randomly selected pairs of word problems in a random sequence. Each questionnaire also contained two distractor pairs of word problems, which had dissimilar mathematical structures in the two tasks. This was done to keep students from solving only the first task and automatically transferring the answer to the second word problem. Each page of the questionnaire showed one word problem and students were instructed to not move backwards through the pages to avoid them from adjusting their answers retrospectively. In half of the participating classrooms, students received an explicit hint, which aimed to encourage them to use similar structures for solving the following word problem.

Coding: Students’ solutions were coded in two different ways: The first option (*correct result*) classified the answer of a student as correct, if the numerical

result was correct. The second option (*correct operation*) classified the answer of a student as correct, if at least the calculation *or* the result was correct.

Statistical analysis: For the inferential statistical analyses, we used generalized linear mixed models for dichotomous data with a logit link function (Bates, Maechler, Bolker, & Walker, 2014), which predict the correctness of an operation or a result for each task based on individual person features and task features. Dependencies between answers of the same person were taken into account by including a random intercept. For the examination of main effects and interaction effects of the task features, we used likelihood ratio (LR) tests based on a chi-square statistic. To compare solution rates under different conditions, we calculated contrasts between the respective estimated marginal means. The reported regression coefficients can be interpreted as difference values on a log odds ratio scale similar to differences of item parameters in an IRT model. All calculations were executed in R with the packages *lme4* (Bates et al., 2014) and *emmeans* (Lenth, Singmann, Love, Buerkner, & Herve, 2018).

RESULTS

To answer Q1, we analyzed only the first task of each pair of word problems excluding the distractor pairs. As expected (H1.4), the variation of the situation context explained only a small proportion of variance (less than 0.01%). First, we analyzed the main effects of a word problem's semantic structure, unknown set, and a/s wording. There were no significant differences between the *semantic structures* concerning the frequency of correct results (change: 77.1%, equalize: 71.0%, compare: 72.1%; LR test $\chi^2(2) = 4.06$; $p = 0.13$). However, students identified the correct operation significantly less frequently ($B = -0.60$; $p = 0.03$) in equalize word problems than in change word problems (change: 82.4%; equalize: 75.4%; compare: 76.3%; LR test $\chi^2(2) = 6.71$; $p = 0.027$). Concerning compare word problems, there were no significant differences regarding the frequency of correct operations in comparison to change and equalize word problems. These results did not confirm H1.1 and H1.2. The main effects of *a/s wording* on the frequency of correct results (LR test $\chi^2(1) = 0.54$; $p = 0.46$) and the frequency of correct operations (LR test $\chi^2(1) = 2.70$; $p = 0.10$) were not significant. However, we found significant differences for the frequency of correct results (LR test $\chi^2(2) = 20.99$; $p < 0.001$) and correct operations (LR test $\chi^2(2) = 32.72$; $p < 0.001$) depending on the *unknown set*. Students gave the correct result significantly more often, if the result/compare set (78.1%; $B = 0.51$; $p < 0.001$), or the change/difference set (74.0%; $B = 0.81$; $p = 0.001$) was unknown, than if the start/reference set was unknown (66.8%). Similar effects occurred, when the identification of correct operations was analyzed. This matches with results by Stern (1993).

Second, we analyzed the interactions between the three main effects. The results showed a significant interaction effect of *unknown set* and *a/s wording* for the

frequency of correct results (LR test, $\chi^2(2) = 22.40$; $p < 0.001$) as well as the frequency of correct operations (LR test, $\chi^2(2) = 30.84$; $p < 0.001$). Furthermore, there was an interaction of *semantic structure* and *a/s wording* (LR test, $\chi^2(2) = 8.20$; $p = 0.017$). The triple interaction was not significant for both performance measures (results: LR test, $\chi^2(4) = 2.75$; $p = 0.60$; operations: LR test, $\chi^2(4) = 2.61$; $p = 0.62$). As expected (H1.3), correct results and operations occurred more frequently, if the a/s wording matched the operation necessary to solve the problem (consistent word problems).

For Q2, we analyzed both tasks of each word problem pair. The main effect of task position (first vs. second task in a pair) was not significant for both coding options and all variations of word problem pairs (e.g., LR test for pairs of compare and change word problems: correct results: $\chi^2(1) = 1.61$; $p = 0.20$, correct operation: $\chi^2(1) = 1.67$; $p = 0.28$). Consequently, the hypothesis that the processing of a structurally similar word problem supports at solving the following task was not confirmed (H2.1). Also, the interaction of *task position* and *semantic structure* was not significant in all cases. Thus, the assumption that the solution of change or equalize word problems improves the solution rates of structurally similar, subsequent compare word problems was not confirmed (H2.2). Finally, we included the effect of *hint* into the models. This main effect and its interaction with *task position* was not significant for both combinations of semantic structures. In addition, the interaction of *hint* and *semantic structure* and the triple interaction of *hint*, *task position*, and *semantic structure* was not significant. Consequently, the hint showed no effect on the use of structurally similar word problems (H2.3).

DISCUSSION

One aim of this contribution was to investigate if prior results concerning factors influencing the difficulty of word problems on addition and subtraction could be replicated (Q1). Results indicate higher, more homogenous solution rates compared to previous studies (e.g., Stern & Lehrndorfer, 1992 in grade 1). In particular, substantial solution rates for compare word problems contradicted prior results that classified this type as the most difficult type. This finding could be explained by the assumption that learners in grade 2 might already have gained experiences with all semantic structures. Additionally, students might benefit from the advancement of mathematics education since the reported studies were conducted (early 1990s). Results underline, that the a/s wording of a word problem is far more important for the identification of a correct operation or result. Consequently, an intervention supporting students in solving word problems should not only focus on the understanding of semantic structures, but also on equivalent statements concerning the a/s wording of a word problem.

Regarding the use of similar situation structures for the solution of consecutive word problems (Q2), results indicate that the participating students did not use preceding, structurally similar word problems to solve subsequent tasks in the same situation context, even if the students received a hint on their structural similarity. One explanation could be, that because of the similar difficulty of both tasks in each pair of word problems, learners might not have considered the transfer of situation features useful. Another reason might be that students were not capable of applying this strategy. Finally, it is also possible that the students did not apply the described strategy for other, unknown reasons, although they were able to apply it in principle. Each of these explanations speaks for further research to obtain more evidence whether and how flexibility in dealing with mathematical situations can support students in solving word problems.

Although the present study allows statements on causal relationships between task features and task difficulty because of its experimental design, some questions remain open. For example, the study provides valuable information for the design of an intervention, but cannot predict its potential effect. In order to understand mechanisms underlying the identified relations better, an in-depth analysis of individual problem solving and learning processes would be valuable. Another open question concerns the conscious restriction of task variety in this study, as only one-step word problems were considered. In which way learners apply the examined skills in more complex situation should be analyzed in further research. Nevertheless, this study provides an update on older results concerning factors influencing the difficulty of word problems on addition and subtraction, which need to be integrated in further investigations. Regarding an intervention study to support students, this study contributes essential implications for the focus of a training program. The result that students do not use strategies of dealing flexibly with situation structures underlines the need to analyze potential obstacles.

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GAINING FLEXIBILITY IN DEALING WITH WORD PROBLEMS – A QUALITATIVE ANALYSIS OF STUDENTS' DEVELOPMENT DURING AN INTERVENTION¹

Laura Gabler¹ and Stefan Ufer¹

Ludwig Maximilian University, Munich, Germany

Prior research confirmed language and situation structure as factors influencing a word problem's difficulty. Until now, instructional approaches to encounter these difficulties still need empirical foundation. This paper describes an intervention to develop second graders' flexibility in dealing with arithmetic situations. During ten sessions, two strategies to enrich students' situation models were introduced supported by macro-scaffolding. We investigated the development of four preselected second graders by applying qualitative content analysis and compared their development to the intended learning trajectory (LT). Results point to potential key processes when gaining such flexibility and to required adaptations of the LT.

Many learners struggle with solving additive (including subtractive) one-step word problems. Language skills play an important role during word-problem solving, since textually represented descriptions of arithmetic situations need to be decoded (Dröse, 2019). This process can be more or less difficult depending on the text's linguistic features, such as syntax or semantics (e.g., Stern, 1993). Such potential difficulties have led to ideas how students could reorganize their situation model by integrating different perspectives on the depicted situation. In this paper, we describe an intervention program that intends to support students with describing arithmetic situations displayed in word problems flexibly from different perspectives. Since it is an open question, if and how students respond to the intervention, this paper aims at the detailed analysis of four preselected students' development during the program.

PRIOR RESEARCH

Various studies have investigated the difficulty of additive one-step word problems and emphasized features of the problems' underlying situation structure, such as *semantic structure*, *unknown set*, and *additive or subtractive wording (a/s wording)*, as factors determining a word problem's difficulty (e.g., Daroczy, Wolska, Meurers, & Nuerk, 2015; Gabler & Ufer, 2020; Stern, 1993). For example, a problem's semantic structure can relate to either a change, combination, comparison, or equalization of sets (Riley, Greeno, & Heller, 1983). Research identifies problems on the comparison of sets as particularly challenging and assumes that the difference between two compared sets is a

main reason for students' difficulties (Riley & Greeno, 1988). Moreover, understanding quantitative comparison statements, such as "Susi has 2 marbles more than Max" is considered linguistically demanding. Fuson, Carroll, and Landis (1996) emphasize the importance of deriving from a quantitative comparison statement *which* quantity is more or less (qualitative information) and *how big* the difference between the two quantities is (quantitative information). The a/s wording of a word problem defines a certain perspective on a situation (e.g., expressed by "more than" as additive, "less than" as subtractive wording in compare problems). Finally, the difficulty of a problem's situation structure varies depending on which of the three involved sets is unknown. For compare problems, either the reference set, the difference set, or the compare set can be missing (Stern, 1993). Problems with an unknown reference set (e.g., "Susi has 5 marbles [compare set]. She has 2 marbles [difference set] more than Max. How many marbles does Max have [reference set]?") are considered the most difficult type. The unknown set also affects the directly applicable mathematical operation (addition or subtraction). In combination with the a/s wording, the variation of the unknown set results in either consistent or inconsistent compare problems: Problems, in which the directly applicable mathematical operation is inconsistent with the a/s wording (e.g., directly applicable *subtraction* but *additive* wording, like in the example above), are usually harder than consistent problems (Lewis & Mayer, 1987).

To address students' difficulties, a theoretical account of solution processes during word-problem solving is crucial. Common models on this matter assume that students decode the problem text into an initial situation model (Kintsch & Greeno, 1985) by reconstructing features of the situation structure as close as possible. To find a matching mathematical operation, they need to identify corresponding mathematical concepts that describe this model adequately. During these processes, students can extend and enrich their situation model with further information (Kintsch, 2018).

These theoretical foundations led to the idea of introducing strategies that aim at enriching the situation model by reinterpreting the problem's situation structure as an easier accessible problem type. Some authors assume that this reinterpretation can make it easier to mathematize an individual situation model: One suggestion originates from Greeno (1980), according to whom students could make use of easier accessible semantic structures. For example, they could reinterpret difficult compare problems as dynamic situations on the equalization of sets (Dynamization Strategy, see Fig. 1). Alternatively, Stern (1993) suggests to rely on the inversion of the a/s wording: By transferring between different perspectives on relations (e.g., "Susi has 2 marbles more than Max" and "Max has 2 marbles less than Susi"), learners could reinterpret inconsistent problems as easier, consistent problems (Inversion Strategy, see Fig. 1). We summarize these strategies to enrich the individual situation model

with further aspects of the situation structure under the term *flexibility in dealing with arithmetic situations*. Understanding and describing situations flexibly is tightly connected to language skills: While the Inversion Strategy requires well-connected vocabulary on relations, equalization within the Dynamization Strategy builds on action verbs (“to take away”, “to get”) and conditional sentences (“If..., then...”).

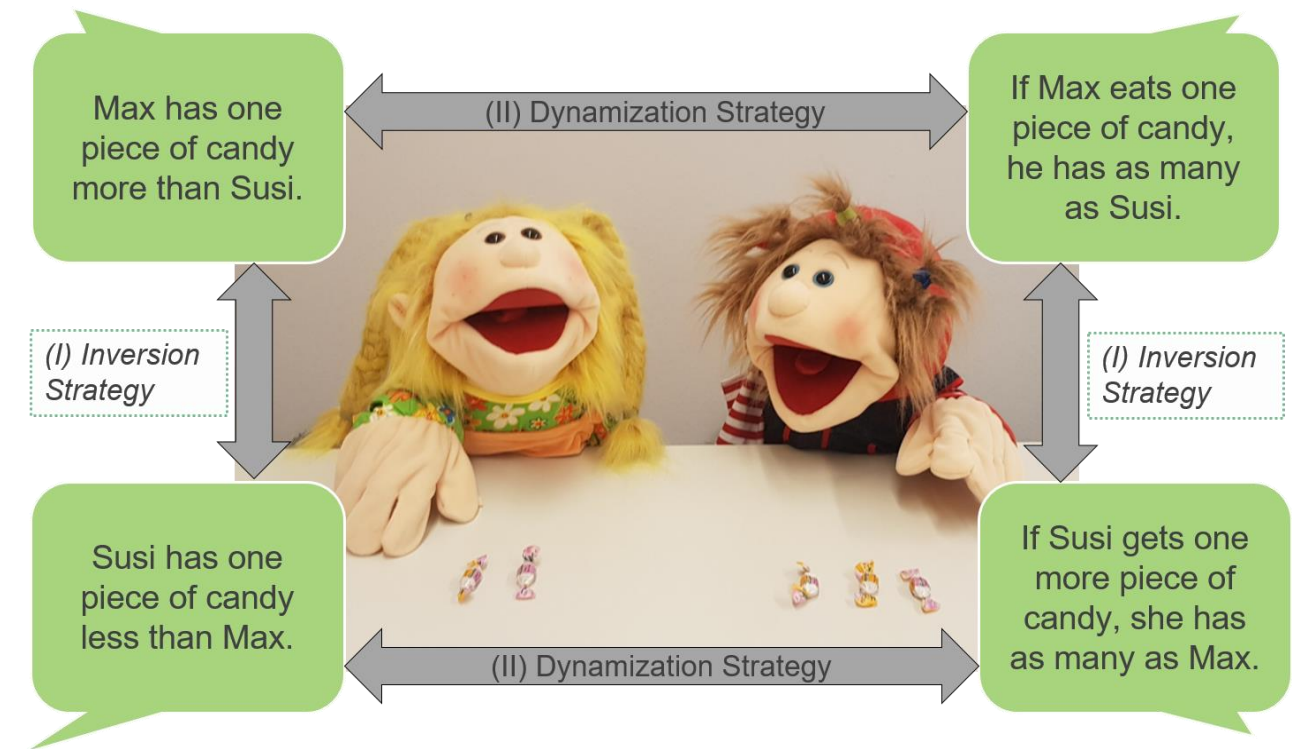


Figure 1: Examples for Inversion Strategy and Dynamization Strategy

CONTEXT OF THE CURRENT STUDY

To build up such flexibility, and to support students to progress from easier to more complex applications of the mentioned strategies, macro-scaffolding was included in the intervention as pre-organized support (Hammond & Gibbons, 2005). In addition, the intervention was guided by an intended learning trajectory (LT) describing a sequence of phases and activities, which were distributed over ten 40-50 min sessions over five weeks. After an initial phase of *Basics* in session 1 and 2 (e.g., on understanding quantitative comparison statements or the concept behind equalizing), students worked with *given* statements to encounter crucial linguistic means needed for the application of the two strategies. During *Verifying* in session 2, students decided if given statements on a situation displayed as text or as a picture were true and discussed their decision afterwards. In session 3 and 4, students matched statements to two situations with swapped concrete sets in the *Matching* phase (e.g., Susi has one piece of candy less in the first picture, and one piece of candy more in the second picture). Contrasting statements on these inverse situations

should systematize the provided linguistic means and raise the students' language awareness in the context of comparison and equalization. During the *Describing* phase, which spanned over the sessions 5 to 10, students were encouraged to make use of the provided linguistic means and describe situations flexibly. Tutors provided sentence templates, sentence starters and word cards ("more", "less", "If..., then...") oriented at the two strategies and removed these scaffolds gradually. Overall, the difficulty of the tasks progressed from empirically easier to more difficult compare situations. Explicitly *solving* traditional word problems was not part of the intervention.

AIMS AND RESEARCH QUESTIONS

Since the intervention was implemented for the first time, we were interested, if and how students made use of the learning opportunities in the intervention, and how this related to their development of flexibility in dealing with arithmetic situations. To this end, we investigated two questions:

Q1: Which differences in students' learning paths point to parts of the intervention at which the intended LT is not sufficiently adapted to individual students yet?

We expected that learners would respond differently to the offered learning opportunities of the intervention. Investigating these differences may help with the identification of typical patterns or obstacles and result in potential "key processes", which need to be considered when supporting students to develop the pursued flexibility. These key processes may provide first indications, which adaptations of the intended LT are necessary to meet the students' individual needs.

Q2: How does students' flexibility develop during the intervention?

Taking the key processes from Q1 under consideration, we analyzed how the students' flexibility in dealing with arithmetic situations developed during the intervention. We expected the learners to become familiar with the introduced strategies and to be able to describe arithmetic situations richly from different perspectives at the end of the intervention.

METHOD

Sixty second graders from elementary schools in southern Germany participated in ten different the intervention groups. For the qualitative analysis, four of the sixty students were selected based on the pre-test. Since we were particularly interested, if the intervention was helpful for students with lower language skills, we selected students with relatively low scores in a reading test (ELFE II). Valerie and Anna were selected from group 6, and Adrian and Emil from group 5. Both groups were trained by the same tutor.

Coding: Following the principles of qualitative content analysis (Mayring, 2014), the transcripts from all intervention sessions were investigated together with the students' answers on work sheets. Based on theoretical implications, we developed a coding manual to identify different manifestations of flexibility. While the first two categories address assumed prerequisites for the pursued flexibility, such as the verbalization of comparison and equalizing statements, the other two categories reflect the application of the two strategies. Each student statement counted as one coding unit. Phases of group work were omitted, since contributions could not be attributed to specific individuals.

Analysis: To identify key processes, we started from the coded data and proceeded to the raw data to check and enrich our initial interpretations. Additionally, we used coded data to substantiate remarkable observations in the raw data (Q1). The students' development of flexibility was traced by analyzing codes during activities, in which students should describe situations without explicit instruction (Q2).

RESULTS

Q1: During the analysis, three main differences in students' learning paths emerged (key processes, "KP"). KP1 showed in the raw data that students seemed to interpret difference sets in comparison statements differently. Thus, we investigated the codings for such situations systematically. In contrast to Adrian and Emil, Valerie and Anna's answers frequently related to concrete sets, although questions targeted quantitative comparison. This is exemplified in the following excerpt:

Session 1, group 6, Basics: *The students play a game with the tutor. After determining who has more chips, the tutor encourages Valerie to quantify the difference.*

Tutor: Valerie, what do you think, how many do I have more?

Valerie: You have six.

Tutor: I do have six, but how many do I have *more* than you? Think about it.

Valerie: Four.

Tutor: Four, exactly. So, how many [chips] am I allowed to take?

Valerie: Four.

With the help of the tutor, who contrasted the concrete set and the difference set verbally, Valerie determined the difference set correctly. However, the codings indicate that Valerie and Anna still related to concrete sets instead of difference sets occasionally during the intervention. The data suggest that they often seemed to understand statements on quantitative comparison, such as "Susi has two marbles more than Max" as two messages: "Susi has two marbles" and "Susi has more marbles than Max". This observation indicates that the Basics

phase, which should tackle such difficulties, was not adapted sufficiently to some of the students' needs and that the interpretation of compare statements might require more attention.

KP2 relates to the transfer of linguistic means from the Verifying phase to the Matching and Describing phase. Since the encounter with relevant linguistic means played a crucial role in the intended LT, we decided to investigate transcripts on Matching and Describing tasks with a specific focus on instances where students made use of language support. While Adrian and Emil had few problems to integrate the provided linguistic means in their active language use, Valerie and Anna received more language support by the tutor. For example, Valerie struggled with the expression of an equalization first, since she did not find an adequate action verb to complete her sentence (e.g., "add"). Session 5, group 6:

- Elisa: [reads aloud the provided sentence frame] If I ..., then my tower is as tall as yours.
- Tutor: What should she do? Valerie. Do you remember what we did there?
- Valerie: If I one... eh? From Sebastian?
- Tutor: So, try to think about it again.
- Valerie: If I one, then... at this tower... as tall as yours.
- Tutor: If you do what? "Then my tower is as tall as yours."
- Valerie: If I...
- Tutor: What can you do, so that the tower is as tall as this one?
- Valerie: One away?
- Tutor: Exactly! Let's do that.

Providing a sentence frame and the subsequent possibility to follow a peer's example helped her to broaden her vocabulary and to make progress in formulating equalization statements. This emphasizes the importance of individual language support especially during the transfer to the active description of situations.

In some transcripts from Matching tasks, students based their explanations why certain statements or pictures were similar or different on different aspects of the underlying situation (KP3). To back up these observations, we compared the codes for each student during such activities. While Valerie and Anna mostly referred to concrete sets (Valerie: 8 statements on concrete sets; Anna: 2 statements on concrete sets, 2 on equalization), Adrian and Emil reasoned with comparison statements frequently (Adrian and Emil: 4 resp. 5 statements on involved difference sets). It seems that such reasoning activities can uncover students' perception of situation structures and thus help the teacher to identify corresponding need of support.

Q2: To analyze the students' development of flexibility over the ten sessions, we selected three activities, which required describing situations freely without explicit support or instruction, distributed over the sessions 2, 5, and 10. The codes during these activities indicated that, despite relatively low language skills, all four students progressed in developing flexibility, but in different ways and different pace.

Adrian was the only one who already formulated a comparison statement spontaneously in session 2. Very low general language skills and comparably low arithmetic pre-test scores did not prevent him from quickly adopting the two strategies. In line with his consistent focus on relations, he preferred formulating comparison statements and applying the Inversion Strategy to equalizing statements and the Dynamization Strategy. In contrast to Adrian, Emil did not focus on the relation between sets initially. Although he missed session 5 and 6, Emil managed to adopt both strategies and gained flexibility with a strong focus on equalizing statements until session 10. Valerie developed flexibility more slowly and did not follow all parts of the intended LT: While her focus laid on concrete sets during session 2 and 5 (in line with KP1), she formulated an (incorrect) comparison statement and correctly applied the Inversion Strategy with equalizing statements in session 10. To make further progress in flexibility, an even stronger focus on supporting her with comparison statements might have been helpful. Anna had similar problems with understanding and formulating comparison statements. However, she progressed more quickly in developing flexibility and already focused on equalizing in session 5. In session 10, Anna attempted to formulate comparison statements and their inversion. Her answers indicated a misunderstanding of the strategy: Instead of inverting the a/s wording, she formulated the opposite of each comparison statement, which did not match the given situation. Similar to Valerie, a stronger focus on the Basics phase might have allowed her to benefit more from subsequent learning opportunities regarding both strategies. In particular, Anna's possibly superficial application of the Inversion Strategy deserves further attention when developing the LT.

DISCUSSION

The students' different learning paths point to parts of the intended LT, which require adaptation and reveal insightful implications in the context of fostering the pursued flexibility (Q1). KP1 emphasizes the importance of developing an adequate understanding of linguistic means to describe situations from different perspectives, in particular in the context of comparison statements. This issue could be addressed by providing tasks that emphasize the difference between statements such as "Susi has 2 marbles more than Max" and "Susi has 2 marbles, and more marbles than Max". KP2 underlines the necessity to not only encounter and understand linguistic means, but also to be able to use them in descriptions. Developing further ideas how to support students with this transfer

may be a next step in refining the intervention. KP3 delivers a tool to determine, which support might be helpful to encourage a specific student to enrich the situation model. If students focus on concrete sets in their descriptions and explanations, teachers could encourage students to consider other aspects of the situation structure, for example with word cards (“more”, “if..., then...”).

Despite different learning paths and learning paces, the students’ progress along the intended LT supports the assumptions that the intervention is a feasible way to foster students’ flexibility. This allows to study, if such flexibility supports word problem solving as has been argued, but not studied systematically in the past (Gabler & Ufer, 2020; Greeno, 1980; Stern, 1993). Beyond the mentioned adaptations to the LT, other factors than the fit of the LT may have caused different learning paths. For example, Adrian already had a tendency to focus on relations at the beginning (McMullen, Hannula-Sormunen, & Lehtinen, 2013), which might have given him a good starting point to adopt both strategies. We also cannot exclude that motivational aspects or the mathematical self-concept influenced the students’ development.

The analyses yield information on possible learning paths to develop flexibility in dealing with arithmetic situations and necessary adaptations of the intended LT. However, quantitative analyses are necessary to investigate, whether the intervention caused a substantial gain in students’ flexibility for some students or the whole sample, and if students could transfer the corresponding skills to word-problem solving.

¹ An extended version of this work is published in the journal *ZDM Mathematics Education*.

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GENERALIZATION AND CONCEPTUALIZATION IN A STEAM TEACHING LEARNING SEQUENCE FOR PRIMARY SCHOOL ABOUT AXIAL SYMMETRY

Aaron Gaio¹, Laura Branchetti¹, Valeria González Roel², & Roberto Capone³

¹University of Parma, Italy

²University of Santiago de Compostela, Spain

³University of Salerno, Italy

In interdisciplinary teaching, students' attitude to generalize mathematical knowledge to new contexts of application is encouraged naturally. Moreover, it fosters the development of creativity and critical thinking. In our research, we focused on integration of Mathematics and Arts in primary school. We designed and tested a Teaching Learning Sequence about axial symmetry, to develop mathematical skills through the execution of artistic techniques and reflections on products and actions carried out. In this paper, we present tasks and results about students' mathematical activity obtained analyzing classroom implementations with children in 4th and 5th grade in Italy. The generalization processes make interesting information about their conceptualization and schemes application and validation emerge.

INTRODUCTION

When students, by themselves or guided by teachers, search for new situations and contexts in which applying and revising their mathematical knowledge, they develop successfully key aspects of mathematical thinking, like problem solving and generalization; design research should “offer teachers an empirically grounded theory on how a certain set of instructional activities can work.” (Gravemeijer, 2004). In interdisciplinary tasks, students' attitude to generalize mathematical knowledge to new contexts of application is encouraged naturally. The European Union recently published recommendations (EU Council, 2018) to integrate all areas of the scientific disciplines with their applications in technology and engineering, and with artistic expressions (STEAM). Benefits would be, for example, the positive effects of art in interaction with different disciplines, including mathematics, from the affective and motivational point of view. Moreover, it fosters the development of creativity and critical thinking (ibid., 2018). Among the several possibilities to pursue such goals, we focused on integration of Mathematics and Arts in primary school. We decided to design and test a Teaching Learning Sequences (TLS, Psillos & Kariotoglou, 2016) about axial symmetry, where students were asked to “reinvent” mathematical concepts (Gravemeijer, 2004)

and develop mathematical skills through the execution of artistic techniques and reflections on products and actions carried out.

In this work, we present some tasks of this interdisciplinary TLS and some results we obtained analyzing classroom implementations. We focus particularly on the students' mathematical activity. We worked with children in 4th and 5th grade in different schools across Italy. We collected data through video and audio recordings, observation and materials produced during the lessons by the students. Results show that encouraging students to generalize make interesting information about their conceptualization emerge. Moreover, we show how linguistic practices and discussions are important to self-realize this generalization and conceptualization mechanism.

LITERATURE REVIEW

In many research, it has been shown that learning axial symmetry is not trivial for primary school students. First of all, the term “symmetry” might be used in different ways (Chesnais, 2012): (a) symmetry as a property of a given figure; (b) axial symmetry as a ternary relationship involving two figures and an axis and/or (c) symmetry as geometrical transformation involving points.

Moreover, axial symmetry is a mathematical concept but also an everyday concept (ibid., 2012). From a mathematical point of view, the geometrical transformation comes before symmetry as a property, being the property a result of the invariance of the figure under the transformation. On the other hand, in the everyday concept, the geometrical transformation could be seen only in the paper folding movement. If not expanded upon, it can lead to the main misconceptions about symmetry, that can be an obstacle to global characterization of the properties of a figure and of the geometrical transformation of the plane (ibid., 2012). It is possible for the teachers not to see these conceptions, since students will continue to produce results as constructing the mirror image of a figure or identifying axes of symmetry on a single simple figure. In general, students are more confident with tasks that require an *intrafigural* perspective (Piaget & Garcia, 1989), where attention is directed towards the internal relationships of figures, than with tasks involving *interfigural* demands requiring attention to the relationships between the figures and objects that are external to them (Healy, 2004). Relying on this review, we decided to orient the students' activity gradually towards the construction of the mathematical concept and an interfigural approach, encouraging them, by means of generalization and verbalization tasks, to reframe the everyday characterization of the axial symmetry.

THEORETICAL FRAMEWORK

In this study, we refer to generalization as *the process of applying a given argument in a broader context* (Harel & Tall, 1991). Generalization is classified as *expansive generalization* when the subject expands the applicability range of

an existing scheme without reconstructing it; *reconstructive generalization* when the subject reconstructs a scheme to widen its applicability range (ibid., 1991). A common trait is the need to change the applicability range of a given concept, extending it to a broader concept. In reconstructive generalization, the old scheme is changed and extended, to be embedded in a more general scheme, that still “contains”, or is a generalization of, the first schema.

According to Vygotsky (2012), concepts can be spontaneous or scientific, where the former are the result of a generalization process of everyday personal experience. Considering our tasks and our target grade, we refer essentially to the Theory of Conceptual Fields (Vergnaud, 1998) to frame the notions of concept and scheme. According to Vergnaud (1998; 2013), mathematical knowledge is centered and constructed around a *concept*; a concept results from a process of actions and perceptions. Concept is constituted by three components: the set of situations the concept is rooted in and has meaning on, a set of operational invariants and the set of different linguistic and non-linguistic representations used to represent it.

A scheme (Vergnaud, 2013) is defined as “invariant organization of activity and behavior for a certain class of situations” (p. 47); to tackle new situations extend the scope of application of the scheme. It is made of four categories of components: goals and anticipation, a set of rules of action, operational invariants and possibilities of inferences. Operational invariants, which make the scheme operate and often remain implicit, can be of two kinds: theorems-in-action and concepts-in-action (ibid., 2013). They can be expressed by words and sentences, but their original function is action and the application of schemes is based on them.

METHODOLOGY

We designed the TLS following these principles: a growing challenge level; to foster generalization (in the meaning given by Harel and Tall (1991), to promote linguistic practices that can be meaningful to connect the different activities and to build up to a gradual conceptualization (in the sense of Vergnaud’s Theory of Conceptual Fields, 1998; 2013), developing a more precise language and promoting argumentation.

In the first two tasks, students met the first two situations:

Task (1), artistic symmetry: folding the paper with colors, a “similar” figure is obtained (same shape, same, area, same colors).

Task (2), modelling the art: doing “the same things” on the left and on the right, at the same height and the same distance with respect to a line, a figure is obtained that resembles the figure obtained by folding.

We told the students that the line obtained folding and the line drawn in the second situation were both called ‘axes of symmetry’, that the figure obtained

by folding was ‘the symmetric figure’ with respect to the starting one and that the whole ‘figure is symmetric’. Thus, we introduced some terms and the relationships between different elements of a conceptual field named ‘symmetry’.

Task (3), TEP: “explain to a younger student how it is possible to build a symmetrical figure with respect to another figure”.

Here students are asked to produce a textual *eigenproduction* (TEP, D’Amore & Maier, 2002), i.e. texts produced by students in an autonomous way to describe some mathematical situation. The goal of TEPs is that of better understanding and exploring the true conceptualization of the student. We expected the students to find linguistic and/or not linguistic representations of their concepts and to start making explicit their actions that they should then organize to make them become schemes.

Task (4), square: “find, by folding, the axes of symmetry of a square”.

Students are expected to generate a first version of their concept of axis of symmetry including: three situations (1, 2 and 4), an operational invariant (*concept in action*: if, folding, the two parts are overlapping exactly, the fold represents an axis of symmetry) and graphic and linguistic representations of the axis. Meanwhile, since they have to solve a new task, they are also asked to start generalizing their previous actions to a scheme, composed by: one goal (to find axes of symmetry), the rules of action (correct procedure to build a fold that is an axis and a control procedure to check if it is an axis or not), an operational invariant (*concept in action* of axis of symmetry), a set of possibility of inference (conditions to carry out the procedure: possibility to fold the paper, possibility to check if the pieces of the figure have the same features).

Task (5) star: “find, by folding, the axes of symmetry of a regular 5-pointed star”.

Students are expected to enrich their previous concept, including another situation and to reinforce the previous scheme. Students are expected (and encouraged) to use their linguistic characterization of the *concept in action*, on which the scheme should be based (task 3), to validate their actions in the different situations (4 and 5).

Task (6), snowflake: “build, as you want, this snowflake” (see Figure 1).

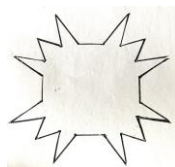


Figure 1: The snowflake to build from a blank sheet of Task 6.

Students are expected to recognize that the figure is symmetric, what are the axes of symmetry, and to decide to exploit this property to build the figure without retracing it, folding a sheet of paper (scheme 1) and/or using the distances from the axes (scheme 2). To do this, the students should: study the situation in terms of possibilities of inference; recognize the same goals of Task 4 and 5 (to find axes of symmetry) even if it is not mentioned in the description of the task; carry out a set of rules to identify the correct folds. Only after the application of the scheme, the students should draw the starting figure, reproducing it symmetrically, to have the most correct result.

Our research questions are:

1. How do the students face spontaneously tasks in which a concept is expected to be applied in a new situation? What kind of information can the observation of a process of generalization give about the students' conceptualization?
2. Whether and how the verbalization tasks and the classroom discussions lead the students to a refinement or a generalization of their personal concepts?

Context and participants

The TLS was implemented in classes of students 8 to 10 years old (two 4th grade and four 5th grade classes of primary school) as part of an in-service teacher training lasting one semester. Class context and formation are variable both geographically through the country and in terms of background of the students. The class teacher acted as main teacher for the TLS; one or more of the authors planned the lesson with the teachers involved, collected data about the students, assisted and helped, intervening occasionally, during all teaching blocks.

Data collection and analysis

The explorative nature of the study led us to use qualitative techniques for data collection towards an interpretative approach. The research data were collected over several sessions at school and consists of (1) audio and video recordings, (2) documents review, (3) researchers' field notes and (4) students' textual productions (TEPs, D'Amore & Meier, 2002).

In particular, (1) videos were analyzed by more researchers and transcripts were finally used as data which we present here. Video analysis (Powell et al., 2003) has been done in more phases: a first review of the videos, cataloguing their content and annotating some particular episodes; a deeper analysis with transcription of some episodes, that were flagged as occurring of generalization; connection of single episodes to consider the overall development of the students' conceptualization. Focus was, as said, on the understanding of the students' conceptualization of axial symmetry, analyzing data inside

Vergnaud's Theory of Conceptual Fields (1998; 2013) and with an eye on the generalization processes that took place (Harel & Tall, 1991).

RESULTS

In relation to our first research question, we observed, in the majority cases, in the tasks from Task 3 to Task 6, spontaneous application of previous knowledge to the new situations they are facing. However, is the procedure always correct? Re-applying the spontaneous concepts (in this sense, generalizing; Vygotsky, 2012) can lead the students to different situations. A spontaneous expansive generalization process can be correct but still lead to some non-correct conclusion, due to a concept in action that is either incomplete, and therefore not extendable to other cases without adding other conditions, or valid only in some situations, thus becoming not correct when the related scheme is applied to a new range of situations. Examples can be seen in Table 1.

We can observe that one of the main risks here is that students go on with what they think is a good property (concept in action), and apply it in a range where it will not work without realizing it will not actually be valid. However, without asking students questions that encourage them to apply their schemes in a new situation, these incomplete or situated concepts would not be identified and revised by the students.

From the video analysis, we could pinpoint also different cases in which correct generalization occurs, both expansive and reconstructive. Some students connect the two schemes, performing in this way a sort of reconstructive generalization. Viola and Andrea, for instance, in Task 6, overlapping the drawing with a folding, realize that "sides cannot be longer or shorter, they need to have the same measures!", connecting the two schemes and reconstructing

Scheme (2), which allows them to re-describe the concepts in action of the paper folding Scheme (1) in terms of measures and distances.

<i>Initial concept</i>		<i>Situation / Concept</i>	<i>Examples of students' sentences/indicators</i>	<i>What happens when re-applying the concept</i>
Incomplete concept	I1	Folding the paper	<p>“Axes of Symmetry are lines” (also “zigzag” lines)</p> <p>“Symmetry is just folding the paper”</p>	<p>Students identify every fold/line, or every line dividing the figure in two parts with the same area, with an axis of symmetry. The right answers based on this incomplete concept, are true but partial.</p> <p>There is a need for a strengthening of the <i>concept in action</i>.</p>
	I2	Two parts with the same area	<p>“a line that divides the paper in two halves with the same area”</p>	
Concept valid in some situation but that becomes not correct if changing the applicability range	S1	Axis has to be vertical	<p>“the axis of symmetry is a vertical line dividing the figure in two parts”</p>	<p>Students apply the concept they inferred from a particular example, but it is not working when changing the setting. More difficult to correct, there is a need to revise the <i>concept in action</i>, removing some features of the line (S1) or referring the concept to a given figure (S2).</p>
	S2	Axes not related to the figure	<p>when “finding all the axes of symmetry of a figure”, students iterate the procedure, with the new figures obtained by folding the first one.</p>	

Table 1. Examples of data analysis

Expansive generalization occurs in many more cases, in all tasks: Task 3 – Task 6, i.e. students keep one scheme they built, always applying the same to a new situation and expanding it, without seeing the connection between folding and overlapping on one hand, lengths and measures on the other hand. This is for example the case of Dora, who generalizes in every situation her scheme about symmetry as folding (1), even when it was easier to use Scheme 2, and never compare the two.

On some occasions, the attempt to generalize the concept will first lead to a non-correct conclusion in a broader situation, but it can also help realize the mistake and therefore adjust the concept and definition the students are trying to identify. For example, as in the transcript below, after an I2 occurring, Elin and then Sara realize there is something not working with their previously discussed definition of axis of symmetry as “a line that divides the paper in two halves with the same area” (Andrea I2 misconception).

- Teacher: Why is the diagonal of the square an axis of symmetry?
- Andrea: Because it is a line that divides the paper in two halves that are the same, the quantity is the same. [...]
- Teacher: So, if I do this, *showing a square that is folded in two parts with the same area, but where the fold is not an axis of symmetry*, I fold the square and obtain two pieces with the same area, are they the same? Is this fold representing an axis of symmetry?
- Class: Yes! No! Yes!
- Teacher: Why is it or why not? Please try to provide some arguments.
- Michael: Yes, because there is a line, anyways... [I1 misconception]
- Andrea: It works because there is the same half [on both sides – I2 misconception]
- Elin: I say no, because...because the figure is rotated. It is the same half on both sides, but one goes up and the other goes down... the same figure is turned one facing up and the other facing down [...]
- Sara: I say no, because...so, it looks like it is, because it forms a line that divides the sheet into two parts that are equal. But in my opinion, it is not an axis of symmetry because...it should have been like this” indicates the diagonal folding with the hands [...]
- James: “the angles are not corresponding...”
- Sara: Ok, if I try again with the colors experiment and fold the paper it will not work. If I do once more the thing with the thread, it could not work on the other side. The two sides are different [they will not overlap]”.

During the discussion, students realize their starting point was correct only if applied to the initial problem of a rectangle divided in two parts, but also that not all lines, even if dividing the figure in two equal parts with the same area, are axes of symmetry for a figure. Therefore, the discussion led to an enrichment of the concept, reconstructed to be adapted to the new situation.

DISCUSSION AND CONCLUSION

We observed that students facing tasks in which a concept is expected to be applied in a new situation re-apply their previous schemes and *concepts in*

action to the new situation. While this spontaneous generalization inclination does not surprise, as it seems to be in fact natural in the students, it is interesting to observe the complete process students are undertaking, to get information about their conceptualization. The kind of tasks proposed are revealing students' misconceptions (as in Table 1), which cannot always be observed with standard "textbook exercises" and which cannot be identified by the class teachers themselves, who were surprised by this discovery during the implementations.

While re-applying schemes is a spontaneous process, the same cannot be said of the processes of evaluation of the consistency between the *concept in action* and the linguistic representations and the control of the rules applied in the new situation. With an appropriate mediation by the teachers and encouraging discussion with peers and argumentation, the lack of a proper control or validation structure for the generalization process can be identified. Properly guided by the teacher, students can understand that their set of rules might not be applicable to every situation and revise their *concept in action* and scheme to adapt them to the new situations. In Task 3 and Task 6 students are encouraged to connect two schemes based on two different *concepts in action* and to carry out a reconstructive generalization by means of a verbalization task and a problem-solving activity. While in the first task this process of generalization never occurs, we observed it in the problem-solving activity, and other students did it during the discussion about their solutions, using one Scheme (2) to check the validity of the procedure carried out with the other Scheme (1).

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EARLY DROPOUT FROM UNIVERSITY MATHEMATICS: THE ROLE OF STUDENTS' ATTITUDES TOWARDS MATHEMATICS

Sebastian Geisler

Ruhr University Bochum, Germany

High dropout rates in mathematics during the first year at university underline students' difficulties in the transition from school to university mathematics. In this contribution, we present a quantitative study based on the three-dimensional model for attitude towards mathematics. Using questionnaires, we analyse differences concerning attitudes towards mathematics between dropped out students and students who continued their studies of mathematics. Our results show that dropped out students are less interested in university mathematics and report a lower mathematical self-concept than those students who continued their studies. Moreover, dropped out students report a decline of their mathematical self-concept during the transition from school to university mathematics.

INTRODUCTION

Dropout is a major concern in university mathematics. In Germany nearly 80 % of all mathematics students drop out or change their subject (Dieter & Törner, 2012) – most of them during their first year at university, so called early dropout. These facts reveal students' difficulties during the transition from school to university mathematics.

One obstacle during the transition are the major differences between mathematics at school and at university. These differences have been extensively discussed in the literature – for a detailed discussion see Ufer, Rach and Kosiol (2017). At school, new concepts are introduced with many examples aiming at an intuitive understanding. In contrast, new concepts at university are introduced via formal definitions. Whereas tasks in school mathematics are often focused on solving real-world problems and schematic calculations, typical tasks at university involve proofing (cf. Ufer et al., 2017). These tasks usually are not directly connected to the real world and cannot be solved by schematic calculations. In Germany, freshmen traditionally attend the courses Real Analysis and Linear Algebra which are focussed on formal definitions and deductive proofs (Halverscheid & Pustelnik, 2013). Both courses are usually accompanied by weekly tutorials and homework tasks with a strong focus on proofs.

According to theories of person-environment-fit (e.g. Swanson & Fouad, 1999) a sufficient fit between the characteristics of the students (e.g. attitudes, prior knowledge, learning behaviour) and the characteristics of the university (e.g. contents, learning environment) is necessary for a successful transition. Due to the already mentioned differences between mathematics at school and at university, this fit does not seem to be self-evident. However, a sufficient fit between the characteristics of the students and the characteristics of the university leads to satisfaction and appropriate achievements of the students (Swanson & Fouad, 1999) while an insufficient fit increases the risk for dropout. According to Haak (2017), an insufficient fit leads to a personal crisis. This crisis is frequently mentioned in the mathematics education literature (e.g. Di Martino & Gregorio, 2019). Haak (2017) proposes two ways to overcome this crisis: Students can either adapt their personal characteristics (e.g. their attitudes or their learning behaviour) or they can decide to drop out.

While early studies in the field of transition to university mathematics had mainly a cognitive orientation (Artigue, 2016), recent research pays attention to affect and the role of attitudes during the transition as well (e.g. Rach & Heinze, 2017; Di Martino & Gregorio, 2019). In this contribution, we focus on the role of students' attitudes towards mathematics for early dropout from university mathematics.

ATTITUDES TOWARDS MATHEMATICS

The question how to conceptualise attitudes and mathematics related affect has been frequently discussed in the mathematics education literature (Di Martino & Zan, 2011). Di Martino and Zan (2011) have proposed a Three-dimensional Model for Attitude (TMA) towards mathematics (comprising the dimensions *emotional disposition*, *vision of mathematics* and *perceived competence*), which is based on an analysis of school students' narratives about their attitudes to and experiences with mathematics. Following the TMA, we understand attitudes towards mathematics as an interplay between interest in mathematics (*emotional disposition*), beliefs concerning the nature of mathematics (*vision of mathematics*) and mathematical self-concept (*perceived competence*) (cf. Di Martino & Gregorio, 2019). The TMA has already been used in qualitative studies focussing on students' experiences and difficulties during the transition to university mathematics (e.g. Di Martino & Gregorio, 2019).

Interest in Mathematics

Individual interest is considered to be a rather stable relationship between an (abstract) object and a person, comprising of emotional (e.g. feeling of joy while engaging with the object of interest) and value related (e.g. personal esteem of the object of interest) components (Krapp, 2007). Since people are motivated to engage with the objects of interest, interest is considered to play a crucial role for successful learning processes.

Studies concerning the role of interest during the transition from school to university mathematics have reported contradictory results (cf. Ufer et al., 2017). Ufer et al. (2017) have argued, that a clear distinction between interest in school mathematics and interest in university mathematics is necessary, when dealing with interest during the transition. Otherwise it is not clear whether students have school or university mathematics in mind, while answering items measuring interest in mathematics. Kosiol, Rach and Ufer (2019) found that interest in university mathematics goes hand in hand with more satisfaction during the first term at university, while interest in school mathematics is connected with less satisfaction. We follow the argumentation of Ufer et al. (2017) and differentiate between interest in school mathematics and interest in university mathematics in this contribution.

Beliefs Concerning the Nature of Mathematics

Philipp (2007, p. 259) describes beliefs as “propositions about the world that are thought to be true”. Traditionally we distinguish between rather static and dynamic beliefs concerning the nature of mathematics (Grigutsch & Törner, 1998). Static beliefs are characterized by the view that mathematics is a static summary of different (unconnected) procedures, rules and formula. In contrast, dynamic beliefs highlight that mathematics is a creative process and field of research with applications in other domains and everyday life.

Regardless that university teachers hold static as well as dynamic beliefs (Grigutsch & Törner, 1998), dynamic beliefs seem to be more beneficial than static ones during the transition. Dynamic beliefs correlate positive with interest in mathematics during the first year at university (Liebendörfer & Schukajlow, 2017). Moreover, students with rather dynamic beliefs are more successful in exams than students with rather static beliefs (Crawford, Gordon, Nicholas & Prosser, 1994).

Mathematical Self-Concept

Bong and Skaalvik (2003) describe self-concept as a person’s perception about herself or himself with emphasis on the own skills and abilities. The self-concept is influenced by prior experiences especially mastery experiences and the feeling of competence and success in a particular domain (Bong & Skaalvik, 2003).

Di Martino and Gregorio (2019) found that most mathematics students start their studies with a high mathematical self-concept but report decreasing self-concept during the transition due to experiences of failure. Rach and Heinze (2017) found that students’ mathematical self-concept predicts their exam attendance in the first term at university. Students who do not attend their exams – according to Baars and Arnold (2014) an useful indicator for dropout – report a lower mathematical self-concept than those students who attend the exams.

RESEARCH QUESTIONS AND METHODS

In this contribution, we want to clarify the role of students' attitudes towards mathematics for early dropout during the first year at university. Since the character of mathematics changes during the transition from school to university, changes in students' attitudes during this transition phase are likely. Furthermore, an adaption of attitudes is one possibility to overcome the crisis that occurs if students' attitudes do not fit to the characteristics of university mathematics (Haak, 2017). Therefore, we consider students' attitudes at the beginning of the first term and during the first term at university. This leads to the following questions and hypotheses:

- 1) Do students who dropped out from mathematics and students who continued with their studies already differ concerning their attitudes towards mathematics at the beginning of the first term?
- 2) Do students who dropped out from mathematics and students who continued with their studies differ concerning their attitudes towards mathematics in the middle of the first term?

Overall, we expect the differences between the two groups of students during the first term to be larger but in the same direction as the differences at the beginning of the term. In detail, we phrase the following three hypotheses:

Since interest in university mathematics goes hand in hand with more satisfaction (Kosiol et al., 2019), we expect that students who dropped out report less interest in university mathematics than those students who continued their studies of mathematics (H1). We have no special hypothesis concerning the interest in school mathematics.

With regard to beliefs concerning the nature of mathematics, dynamic beliefs seem to be more beneficial for a successful transition than static beliefs (Crawford et al., 1994). That is why we expect that dropped out students tend to agree more to static beliefs and less to dynamic beliefs than those students who continued their studies (H2).

Based on the result that students with low mathematical self-concept often do not attend their exams (Rach & Heinze, 2017) – which is an indicator for dropout – we believe that dropped out students will report a lower mathematical self-concept than those students who continued their studies of mathematics (H3).

In order to answer the research questions, two questionnaires – one at the beginning of the first term (at the end of the second week, T1) and one in the middle of the first term (at the end of the ninth week, T2) – were used. At the end of the first year we checked whether students continued their studies or dropped out. The questionnaires have been handed out in the Real Analysis and the Linear Algebra lectures (which German freshmen usually attend during their

first term) at a large public German university. The instruments used in the questionnaires can be found in table 1.

Variable	Source	#	$\alpha(T1/T2)$	Example
Interest School Mathematics	Ufer et al., 2017	5	0,80/0,80	In school, mathematics was very important for me.
Interest University Mathematics		5	0,88/0,87	The kind of mathematics that is done at university is fun for me.
Beliefs: static	Laschke & Blömeke, 2013	6	0,52/0,64	Mathematics means learning, remembering and applying.
Beliefs: dynamic		6	0,71/0,76	Mathematics involves creativity and new ideas.
Mathematical Self-Concept	Kauper et al., 2012	4	0,84/0,82	I am very good in my study subject mathematics.

Table 1: Instruments used in the questionnaire with number of items (#) and reliability (cronbachs α)

All items had to be answered on a five-point likert scale (1= totally disagree; 5=totally agree). 271 freshmen (mathematics majors and mathematics pre-service teachers) voluntarily filled out the first questionnaire. 222 freshmen participated in the second survey. All used scales had at least satisfying reliability, except the static beliefs scale, which has therefore been excluded from the further data analysis.

RESULTS

To answer the research questions and to check the hypotheses, multivariate analyses of variance (MANOVA) were used – to avoid the cumulation of the α -error compared with single t -tests. In the following, we first describe the results concerning students' attitudes at the beginning of the first term (T1), before discussing the results concerning students' attitudes during the term (T2).

Attitudes at the Beginning of the first Term (T1)

A large group of students who dropped out during their first year did not attend the lectures at the middle of the first term (T2) anymore. Therefore, we compare three groups of students: students who continued their studies (no dropout), students who dropped out prior to T2 (very early dropout) and students who dropped out during the first year but after T2 (early dropout). Table 2 shows the results concerning the differences between these groups:

Variable	Very Early Dropout N=59		Early Dropout N=40		No Dropout N=172		η^2
	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>	
Interest School Mathematics	3,41	0,65	3,45	0,63	3,62	0,69	0,02
Interest University Mathematics	2,55	0,79	2,86	0,78	3,17	0,83	0,09***
Beliefs: dynamic	3,50	0,54	3,60	0,53	3,74	0,59	0,03*
Mathematical Self-Concept	2,41	0,62	2,76	0,58	2,97	0,70	0,10***

Table 2: Means, standard deviations and results of the MANOVA concerning the attitudes towards mathematics at T1; N=271; * $p < 0.05$; *** $p < 0.001$

As expected, dropped out students report less interest in university mathematics (H1), less dynamic beliefs (H2) and a lower self-concept (H3). No significant differences can be found with regard to interest in school mathematics. Due to the fact, that the static beliefs have been excluded from the analysis, H2 can only be confirmed partially.

Post-Hoc-tests (with Bonferroni correction) show that mainly the differences between the very early dropped out students and the students who continued their studies are significant. Furthermore, the very early dropped out students report significantly lower self-concept than the early dropped out ones. However, early dropped out students do not differ significantly in their attitudes from students who continued their studies.

Attitudes during the First Term (T2)

Since the very early dropped out students did not attend the lectures at T2 anymore, differences concerning the attitudes during the first term can only be compared between early dropped out students and those students who continued their studies. Table 3 shows these differences. As expected dropped out students report significantly less interest in university mathematics than the students who continued their studies (H1). We found no significant differences with regard to interest in school mathematics and dynamic beliefs. However, early dropped out students report significantly less mathematical self-concept than the students that continued their studies (H3).

Variable	Early Dropout N=56		No Dropout N=166		η^2
	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>	
Interest School Mathematics	3,62	0,72	3,70	0,72	0,00
Interest University Mathematics	2,97	0,79	3,27	0,81	0,03*
Beliefs: dynamic	3,40	0,60	3,42	0,64	0,00
Mathematical Self-Concept	2,51	0,63	2,95	0,66	0,06***

Table 3: Means, standard deviations and results of the MANOVA concerning the attitudes towards mathematics at T2; N=222; * $p < 0.05$; *** $p < 0.001$

This is remarkable because at the beginning of their studies, these two groups of students did not differ significantly concerning their mathematical self-concept. A closer look at the means at T1 and T2 reveal that the mathematical self-concept of those students who continued their studies remains nearly constant while the early dropped out students report a clear decline of their self-concept.

DISCUSSION

Our results indicate that dropped out students and students who continued their studies of mathematics differ mainly concerning their interest in university mathematics and their self-concept. The differences found depend on the time of measurement.

At the beginning of the first term, only the very early dropped out students report less interest in mathematics, less agreement to dynamic beliefs and a lower mathematical self-concept than the students who continued their studies. It seems that the very early dropped out students start their studies of mathematics with unfavourable attitudes that do not fit to university mathematics (in the sense of person-environment-fit). It seems that these students do not try to adapt their attitudes (as proposed by Haak (2017)) and therefore drop out very fast. It remains the question, whether some kind of supporting program would be beneficial for this group or if the very early dropout has to be understood as a fast correction of a wrong study choice. In this case it would be helpful to inform future students better about mathematics at university to enable them to make deliberate and appropriate study choices. Especially information about major differences between mathematics at school and at university should be given – preferably already during secondary school.

The early dropped out students did not differ from those students who continued their studies concerning their attitudes towards mathematics at the beginning of the first term. However, in the middle of the first term, the early dropped out

students report significantly less interest in university mathematics and a lower mathematical self-concept than the students who continued their studies. While the self-concept of the students who continued their studies remained nearly constant during the transition, the self-concept of the early dropped out students decreased. This is in line with the findings of qualitative studies like the one of Di Martino and Gregorio (2019) who found that the mathematical self-concept of many students – even those that continued their studies – decreases during the transition from school to university due to the unexpected experiences of failure in mathematics. Many experiences of failure during the transition are connected to students' problems with the weekly homework tasks (Liebendörfer & Hochmuth, 2017). Therefore, the design of these tasks could be reconsidered. Tasks that are challenging but offer experiences of success might help strengthening students' mathematical self-concept.

All in all, we found clear evidence for differences concerning the attitudes towards mathematics between students who dropped out and those who continued their studies. However, our study has some limitations. We only collected data at one university, thus our results might only reflect the local situation. The questionnaires relied on self-reports that can be biased. In addition, questionnaires were filled out during lectures. Students who do not regularly attend the lectures were not captured in our study. Our ongoing research will now focus on supporting measures that foster students' mathematical self-concept.

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DATA-BASED MODELLING WITH EXPERIMENTS – STUDENTS' EXPERIENCES WITH MODEL-VALIDATION

Sebastian Geisler

Otto-von-Guericke-University Magdeburg, Germany

Validating mathematical models is important yet challenging for many students. One pedagogical approach to foster validation is to use tasks that combine modelling with experimental data collection. In this paper we present a modelling-task with related experiment concerning the decay of beer froth. We analyse students' validation of their models using qualitative content-analysis. Our results indicate that even if students are aware of substantial deviations between their model and their experimental data, they struggle with the validation of their models. Furthermore, students seem to put more trust in their models than in the data they measured during the experiment. Therefore, students tend to suggest to improve the measurement during experimentation instead of revising their models in order to improve fit between model and data.

INTRODUCTION

According to Niss (1994), modelling is a central contribution of mathematics for a modern society. Accordingly, mathematical modelling is considered a key mathematical competence to be taught. This is reflected by several national curricular documents (e.g. National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010; KMK, 2012) as well as the PISA-framework (OECD, 2017). However, modelling is a complex cognitive process. Therefore, modelling tasks are challenging for many students. Specially the absence of validation of the formulated models is an often reported shortcoming in students' modelling processes (e.g. Blum & Leiß, 2007).

In the case of data-based modelling with functions, Engel (2010) criticises that some tasks in textbooks use unrealistic data which already fit well to the intended model. The necessity to validate ones' models becomes not apparent for students in this case. Therefore, Engel (2010) argues that real data – which usually does not fit perfectly to a mathematical model – is necessary for authentic modelling and to illustrate the relevance of model-validation. One possibility to integrate real data in the modelling process to stimulate validation is to combine modelling tasks with data which students gather themselves via (scientific) experiments (cf. Zell & Beckmann, 2009).

Even though this approach is often described in articles that offer concrete teaching-ideas for teachers at different educational levels (e.g. Ludwig &

Oldenburg, 2007), only a few empirical studies deal with benefits and constraints of modelling tasks with experiments. In the exploratory study reported here, we analyse students' validation in a modelling task concerning the decay of beer froth.

THEORETICAL BACKGROUND

Mathematical Modelling and Model-Validation

Even though different conceptualisations exist for mathematical modelling in educational contexts, most researchers assume modelling to be a circular process (e.g. Galbraith, Stillman, & Brown, 2010). We follow the conceptualisation of Blum and Leiß (2007) describing an idealized modelling-cycle with seven steps (see Figure 1).

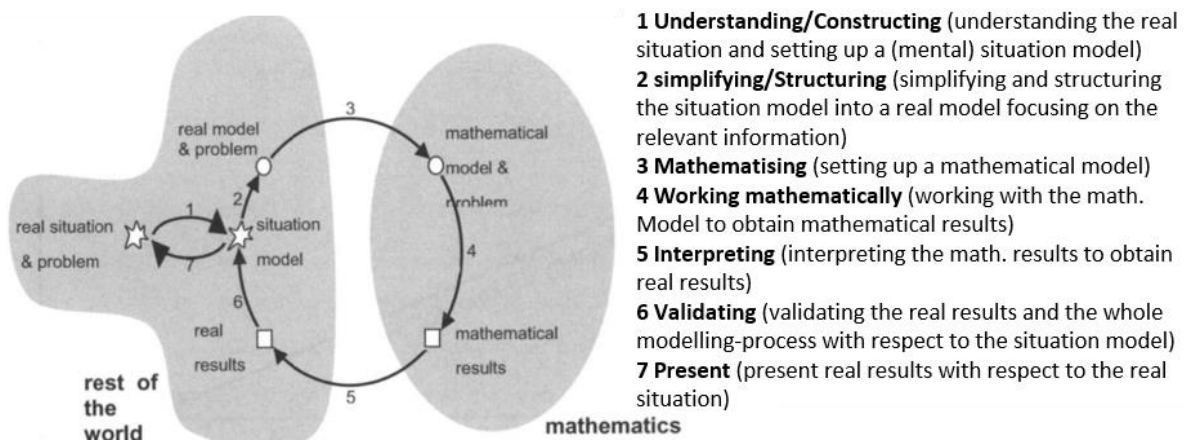


Figure 1: Modelling-Cycle following Blum & Leiß (2007)

During the *validation* step, students examine whether the fit between their model and their obtained results is adequate. According to Niss (1994, p. 369) validation is the “single most important point related to mathematical modelling”. For many students, validation is also a hurdle in the modelling process. Blum and Leiß (2007) report, that the model-validation is absent in many students' modelling-processes. Some students even seem not to see model-validation as necessary (Stillman & Galbraith, 1998; Hankeln, 2020). However, some scholars report, that students validate their models but sometimes rely on rather intuitive feelings that their model might be wrong (Borromeo Ferri, 2006). In an intervention-study aimed at fostering students' modelling-competencies, Borromeo Ferri, Grünwald and Kaiser (2013) found that ninth grade students' validation-competence was the weakest developed sub-competence. Furthermore, they found that validation-competence can be fostered even in short interventions.

Modelling Tasks with Experiments

Experiments related to mathematics find their natural place in the framework of mathematical modelling because they represent the “rest of the world” for which mathematical models are built. (Halverscheid, 2008, p. 226)

Carreira and Baioa (2018) describe that students see modelling tasks with experiments as credible. Furthermore, Ludwig and Oldenburg (2007) argue that experiment-based tasks tie the whole modelling-process to students' practical experiences and allow models to be validated with students' own measurements. Similarly, Zell and Beckmann (2009) see valuable opportunities for validation and reflection upon models when using measurements from real experiments:

Because of measurement errors the formula is never correct. So it is natural to talk about the correctness and the limitations of the model and its results. [...] Hence there is a strong emphasis on the validation process [...]. (Zell & Beckmann, 2009, p. 2216)

They report that secondary students were able to validate their models based on physical experiments during classroom discussions. Maull and Berry (2001) found that undergraduate students did not validate their experiment-based models on their own and that prompts from the teacher were necessary for the validation process. However, even if students notice divergences between their model and experimental data, this does not ensure useful validation: Carrejo and Marshall (2007) describe how students justify systematic shortcomings in their models by measurement errors in their data.

So far, empirical studies do not draw a consistent picture of the benefits and constraints of experiment-based modelling tasks. More research on how students approach modelling tasks with experiments focusing specially on how they validate their models with respect to their experimental data is needed.

THE CURRENT STUDY

The study at hand is part of the Design-Based-Research (DBR) project "Mathematical Modelling with Experiments" (MaMEx). MaMEx objectives are the design and evaluation of modelling tasks with experiments and research concerning the benefits and constraints of such tasks for fostering students' modelling- and especially validation-competencies. Before addressing the research-questions of our study, we give a brief overview of the modelling tasks used in the MaMEx-project.

Design-Principles and Modelling Task "Stale Beer"

The modelling tasks and related experiments used in the MaMEx-project satisfy the following design-principles (cf. Galbraith et al., 2010): (1) The context should be realistic and use only physical quantities that are familiar to students. (2) Experiments should be easy to conduct and not take too much time so that mathematical modelling is foregrounded. (3) Setting up a mathematical model should not be too complicated but the task should offer useful occasions for validation. (4) Since not all students validate their model spontaneously, a validation prompt should be implemented. We illustrate these design-principles for the task "Stale Beer". In this task, students model the decay of froth from

alcohol-free beer using data measured in an experiment. Similar tasks without the experiment can be found in several mathematics textbooks from Germany (Leike, 2002). The initial task was formulated as follows:

The quality of beer is (among other criteria) evaluated by the speed of decay of its froth. Since people drink beer at different paces, there are different opinions on how long froth should be stable. Model the decay of froth for the alcohol-free beer at hand and evaluate the quality of the beer.

The context is realistic (1) since the quality of froth is a quality-criterion of beer for both: consumers and breweries (Evans, Surrel, Sheehy, Stewart & Robinson, 2008). Furthermore, consumers from different countries prefer different characteristics of beer froth (Evans et al., 2008). Students were advised to conduct an experiment by measuring the height of froth from freshly poured beer for five minutes. The quantities time and height are familiar to students. The experiment takes only a few minutes and materials used (alcohol-free beer, measuring cylinder, ruler) are easy to handle (2).

After conducting the experiment, students were asked to set up a function serving as a model for the decay of froth. Mathematising is not too complicated (3) since the decay of froth can be approximated by exponential decay: $f(x) = b \cdot a^x$, $0 < a < 1$, $b > 0$. A reasonable solution is to use the first measured height (in cm) as an estimate for the parameter b . a can be estimated as the quotient of two consecutive measured values (e.g. $a = \frac{\text{height}(1 \text{ min})}{\text{height}(0 \text{ min})}$). Occasions for validation are given, since the decay of froth is not perfectly exponential (Leike, 2002). This subtask served as validation prompt (4):

Compare your function with your measurement-data. Does your function describe the data accurate enough? How could your model be improved?

One possibility to improve ones' model is to consider several pairs of consecutive measured values and using the mean of their quotients as an estimate for a .

Finally, students were asked to reflect upon their validation:

In which way was the former subtask relevant for evaluating the quality of the beer? Why does it make sense to ask yourself the questions in the former subtask?

Research-Questions

This paper reports results from the first DBR-cycle of the MaMEx-project. Within this cycle the task "Stale Beer" was implemented in German upper-secondary schools in order to analyse students' model-validation and – if necessary – to refine the task for the next DBR-cycles. In particular, we were interested in the following questions:

- How do students validate their models with respect to their experimental data?

- In which way do students explain the relevance of validating their models?

Methods

19 German upper-secondary students from two classes of a grammar-school serve as the sample for the study at hand. The students were familiar with exponential functions of the type $f(x) = b \cdot a^x$ as well as the typical characteristics of exponential growth and decay. However, so far they had not worked with empirical data and did not all already know how to compute the parameters b and a based on given values.

In order to answer the research questions, students' answers on the aforementioned validation prompt and the reflection subtask were analysed using qualitative content analysis (Mayring, 2010). Inductive categories have been derived from students' answers, resulting in a coding guide. Based on this guide, all answers were coded independently by two coders, reaching a good intercoder-reliability of $\kappa = 0.75$. Given the limited space, we cannot present the coding-guide here.

RESULTS

Except of two, all students were able to construct an exponential function as a model for the decay of beer froth. The students used their first measured value as an estimation for b . Ten students computed a as the quotient of consecutive measured values (see Fig. 2a). Only three of these students used more than two consecutive measured values and computed a mean as an estimate for a (see Fig. 2b for an example). However, three students indicated that they systematically tried different values for a and compared the resulting functions with their measurement data. Five students did not provide an explanation for how they set up their model.

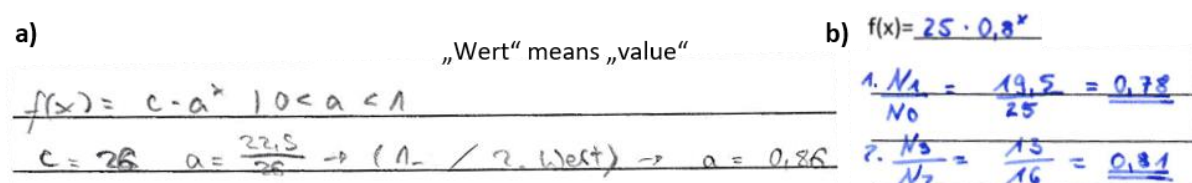


Figure 2: a) Solution using two consecutive measured values to estimate a , b) Solution using in sum four consecutive measured values and the mean of their quotients to estimate a

Model Validation

Nine students indicated that they identified no or only a small deviation between their model and experimental data. These students stated that their models described the data adequately enough (e.g. “*The function describes the measured values precisely. There are hardly any major deviations.*”). Even if they identified no significant deviations, five students proposed ideas for

improvement of the fit between model and data. Two of them proposed to measure the height of the froth more precisely during the experiment (e.g. “*by using more precise measurements*”) and three suggested to take into considerations more decimal places while computing a (e.g. “*You can make the function more precise by adding more places after the comma.*”). Four students described no concrete idea to improve their models.

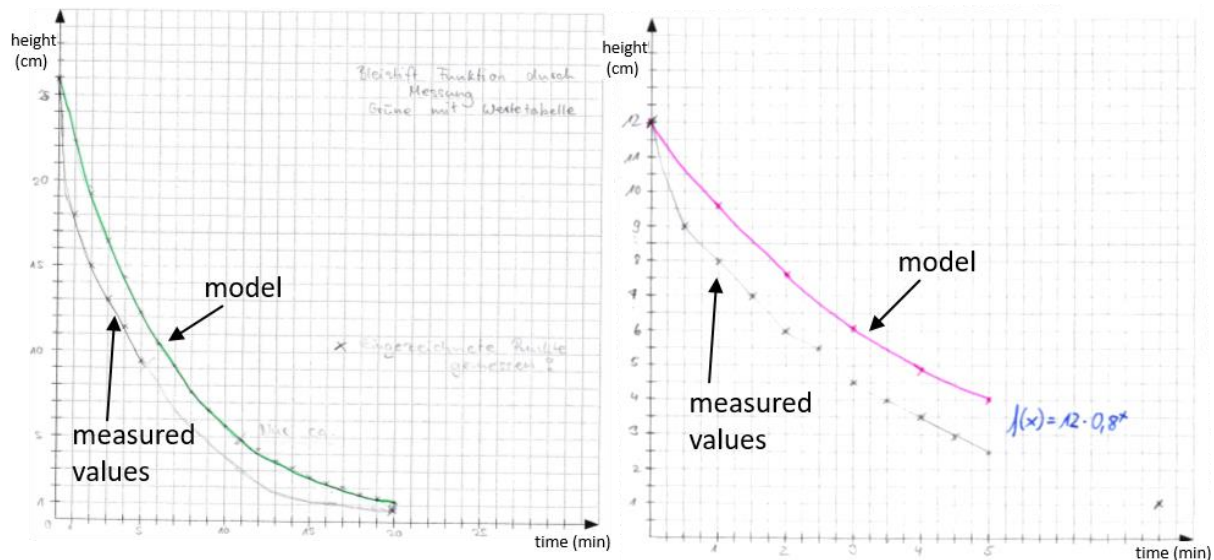


Figure 3: Model and data of a student who identified no significant deviations (left picture), and of a student who identified significant deviations (right picture)

Ten students indicated that they see significant deviations between their models and experimental data (e.g. “*The values differ greatly. The measured values are not described adequately by our function.*”). All of these students suggested to improve the fit between their models and the experimental data by increasing the number and precision of the measurements during the experiment (e.g. “*The model can definitely be improved by better measuring.*”). No student suggested to use more or even all measured values to estimate a .

It is noteworthy that students’ judgement, whether the deviations between the experimental data and the model are significant is very subjective. Comparing models from students who did not indicate relevant deviations and models from students who did, reveals that the models and deviations from the data are quite similar. For example, both models in Fig. 3 reveal similar systematic shortcomings.

Relevance of the Validation

Being asked, why the validation subtask was relevant, two students stated that it was not important for them (“*It was not relevant for us!*”).

16 students argued that comparing their model with their experimental data was relevant for their modelling-process. Three different argumentations could be identified in students' answers:

Eight students wrote that it was important in order to understand how fast the froth decreased and to determine the quality of the beer froth (e.g. *"It was important to notice the speed of the decay. With the speed you can recognize and explain the quality of the beer."*). However, these eight students did not explicitly explain how the validation contributed to the evaluation of the quality of beer froth.

Two students argued that validation was necessary in order to set up a model that can adequately predict the further decay of beer froth (*"It was important since – theoretically – the function can approximately predict the quality and the amount of froth that is still there after »x« time."*). These students saw their model not only as a description of their data but as a tool to predict the further decay.

Evaluating the fit of ones' model with the data, as a reason why validation is relevant, was mentioned by six students (e.g. *"Verification whether function and data can fit together."*). Two of these students explicitly linked the comparison of model and data with the evaluation of the froth quality (*"It was important to see that – in contrast to our function – the decay of froth was faster, indicating that the quality was not good."*).

DISCUSSION AND OUTLOOK

Mathematical modelling and in particular model validation are very important (Niss, 1994). In the context of modelling with functions, the results of our study show that many students struggle with validation. Some students did not recognize systematic shortcomings of their models (e.g. Fig. 3, left). Those students who noticed relevant deviations between their model and experimental data argued that these deviations are the result of imprecise measurements during the experiment. Systematic shortcomings of fit between model and data – as apparent in the right picture in Fig. 3 – are attributed to measurement errors (cf. Carrejo & Marshall, 2007). Consequently, students suggested to improve the data by more precise measurements instead of revising the model in order to increase the fit between data and model. It is surprising that even the three students who used more than two consecutive measured values to estimate a , did not suggest using the rest of their data in order to further improve their models.

It seems that the majority of students put more trust in their mathematical models than in their experimental data, with the result that they try to adjust their data to the model instead of the other way around. However, in the reflection statements of those two students who linked the comparison of model and data with the evaluation of beer froth quality some doubts about their

models came through (“*It was important for judging the beer quality, because – in contrast to our function – the beer froth decreased much faster.*”). But even these students wanted to improve the fit between model and data by more precise measurements during the experiment.

Since we worked with a small group of students from two classes using only one task and experiment, our results are rather explorative and should be confirmed by further studies with similar tasks and experiments as well as a larger sample of students.

The question remains, why students put more trust in their models than in their data and why they prefer to adjust the data instead of revising the model. A possible reason is that students are more familiar with tasks which use unrealistic data, already fitting well to the intended model (cf. Engel, 2010). Furthermore, it is possible that students’ hesitation to change the model is rooted in a belief that mathematics is always precise and that mathematical tasks have only one correct solution (cf. Schoenfeld, 1992). With this belief, students might assume that the first model they set up is “correct”. However, another reason could be that students simply find it easier to improve their measurement than to revise their model. Stimulated recall interviews (with students who worked on experiment-based modelling tasks) could be used to gather more insights into students’ reasons for putting more trust in their models than in their data. We will follow this strategy within the next DBR-cycle of the MaMEx-project.

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USING AN INQUIRY-ORIENTED CALCULUS TEXTBOOK TO PROMOTE INQUIRY: A CASE IN UNIVERSITY CALCULUS

Saba Gerami¹, Vilma Mesa¹, & Yannis Liakos¹

¹University of Michigan, United States of America

We report how an inquiry-oriented, open source, and open access calculus textbook shaped one university instructor's planning and enactment of his lessons. We use two analytical lenses—curricular noticing (Dietiker et al., 2018) and Inquiry in Mathematics Instruction (Gerami et al., 2020)—using various sources of data (surveys, bi-weekly logs, classroom observations, and instructor interviews). We found that the instructor relied heavily on the textbook to plan his lessons and that his enactment of the lesson plans resulted in meaningful interactions about calculus ideas in the spirit of inquiry.

INTRODUCTION AND BACKGROUND

University calculus instruction, specifically lecturing, has been blamed for the high proportion of students leaving science and engineering programs (Rasmussen & Ellis, 2013). Citing current evidence that ‘active learning’ is associated with higher student performance (Freeman et al., 2014; Laursen et al., 2014), the popularity of inquiry-oriented teaching and learning in university mathematics has increased, as has the interest in creating curriculum materials that can support inquiry (Haberler et al., 2018). Whereas there is empirical research on how inquiry-oriented research-based curriculum materials support teaching and learning (e.g., Rasmussen & Kwon, 2007), there is scant research on how non-research-based textbooks that are designed to engage students shape instructors’ teaching. An exception is Fukawa-Connelly’s (2016) study about an undergraduate abstract algebra instructor who used his own non-research-based inquiry-oriented curriculum. He showed that even though the instructor wanted students to engage in various defining and proving practices and some elements for inquiry were present in his curriculum, the enactment of the curriculum did not seem to support that goal because of the absence of design principles that would allow the instructor to “devolve much responsibility to [the students] in the defining and conjecturing phase” (p. 747).

Our study contributes to this body of research, as we seek to understand how a textbook designed to create inquiry opportunities in a first-year calculus course shaped two teaching processes of one university instructor (Casey, pseudonym): planning and instruction. We chose a case study approach (Yin, 2003) because it afforded us an in-depth analysis of the processes of planning and instruction

in a bounded system: the teaching of calculus by one instructor. We chose to work with Casey, because he mentioned wanting to implement inquiry into his calculus course, after having some success implementing inquiry in a few of his upper-division courses. Because we were interested in how the textbook entered the processes of planning and instruction, we thought that an instructor interested in incorporating inquiry but who lectured in his calculus course would be ideal to see whether the textbook was the catalyst to implement inquiry. We pose the following research questions: How does an inquiry-oriented calculus textbook shape lesson planning by a calculus instructor in inquiry-oriented ways? and, How does an inquiry-oriented calculus textbook shape one college instructor's interactions with students and the mathematics at stake in inquiry-oriented ways?

The inquiry-oriented textbook: Active Calculus

To teach his course, Casey used *Active Calculus* (Boelkins, 2019), an open source and open access textbook created in PreTeXt (<https://pretextbook.org>) in an HTML format. The book is designed to “actively engage students in learning the subject through an activity-driven approach in which the vast majority of the examples are generated by students” (Boelkins, 2019, *preface for instructors*). The textbook is conceived as a workbook with activities that have to be done with peers in class; students are expected to ask questions, make mistakes, and write about and talk through the concepts (see *preface for students*). The textbook includes three interactive features: links to GeoGebra animations (interactive applets), preview activities (students can submit their responses to questions about the upcoming material), and WeBWorK (<https://webwork.maa.org>) exercises that provide immediate feedback to students. Each section is designed to map class time in a five-phase instructional sequence: 1) students completing a preview activity prior to each class meeting, 2) short class discussion, 3) short lecture and discussion based on the preview activity, 4) students working on an activity while engaging with peers, 5) instructor wrapping ideas up. The materials include a YouTube channel with short videos for every section and a set of worksheets (prep assignments) that are given to the students to do as homework before class.

THEORETICAL FRAMEWORK

In this study, we see artefacts as an interdependent element that mediates and modifies the interactions between the teacher and his or her students and the mathematics at stake as they engage in activities that bring them together (Rezat, 2013). We attend to two activities, planning and enactment of a lesson, and seek to understand them from the perspective of the teacher.

We conceive of planning as the set of activities instructors engage in to generate a *plan* that outlines the goals, activities, times, roles, etc., of the teacher and the students in the classroom. We used Dietiker and colleagues' (2018) *curricular*

noticing framework to conceptualize this activity vis a vis the textbook. Curricular noticing refers to how teachers “take advantage of the opportunities in curricula in mathematically and pedagogically productive ways” to create such plan (p. 524). The framework consists of three phases: 1) curricular attending refers to the set of actions involved in viewing or visually taking in information within the curriculum; 2) curricular interpreting includes the actions teachers take to make sense, mathematically and pedagogically, of the information they have visually taken in during the attending phase; and 3) curricular responding describes the actions teachers use to make curricular decisions and the way in which those decisions are carried out in the classroom. The three phases follow one another consecutively, but teachers may refer to prior phases as needed (see Figure 1).

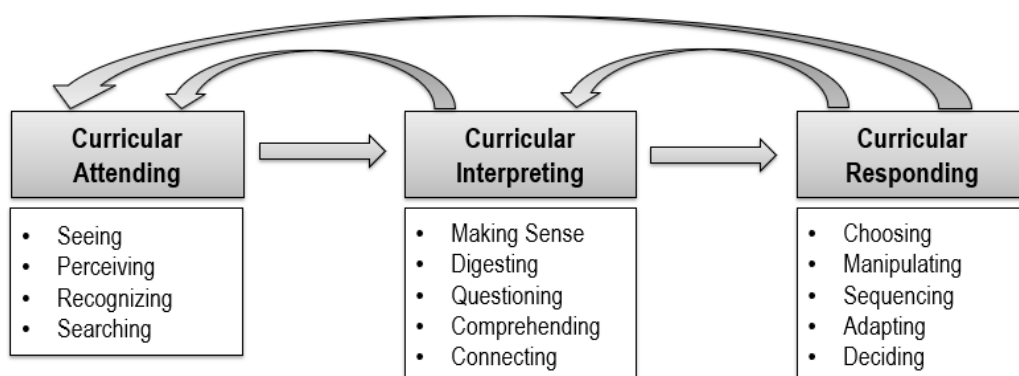


Figure 1: The curricular noticing framework (adapted from Dietiker et al., 2018)

Instruction is conceptualized here as the interactions between the teacher, the students, and the mathematics, bounded by particular contexts. Attending to the interactions, we identify various dimensions of inquiry (or pillars, Laursen & Rasmussen, 2019) to define inquiry instruction as instruction that supports inquiry in the classroom (see Figure 2).

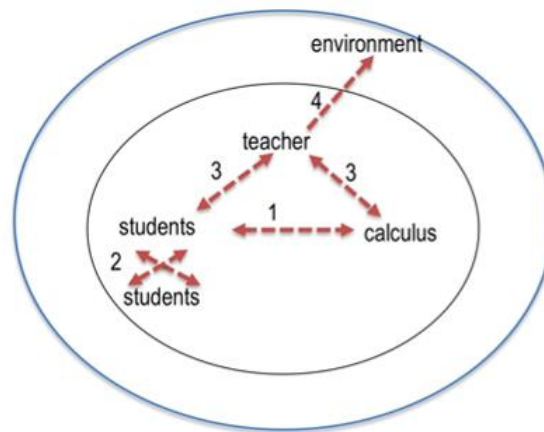


Figure 2: Dimensions of inquiry in the classroom (adapted from Cohen et al., 2003)

Inquiry in the classroom can refer to: 1. *Individual student inquiry*, that is, the extent to which students engage with “coherent and meaningful mathematical tasks”; 2. *collective student inquiry*, that is, the extent to which “students collaboratively process mathematical ideas”; 3. *instructor inquiry*, that is, the extent to which the instructors express interest and curiosity about how individual and groups of students are thinking about and processing mathematical ideas and use that information to guide the interactions; or 4. *environment that supports inquiry for all*, that is, an environment in which equitable participation is expected and supported through lesson design and facilitation choices (Laursen & Rasmussen, 2019, p. 138). To define the specific inquiry-oriented processes under each dimension, we identified actions (behaviours or items) from 15 extant observation instruments and questionnaires concerned with inquiry-oriented classroom practice (e.g., Cawley et al., 2018; Laursen et al., 2014; Shultz, 2020; full list of instruments and definitions provided upon request). We mapped them into the interactions outlined in Figure 2. This process resulted in The Inquiry in Mathematics Instruction framework, shown in Figure 3, with 28 behaviours (which we identify by a number representing a dimension and a letter, e.g., “3a: Interactive lecture” is a behaviour under the *third* dimension).

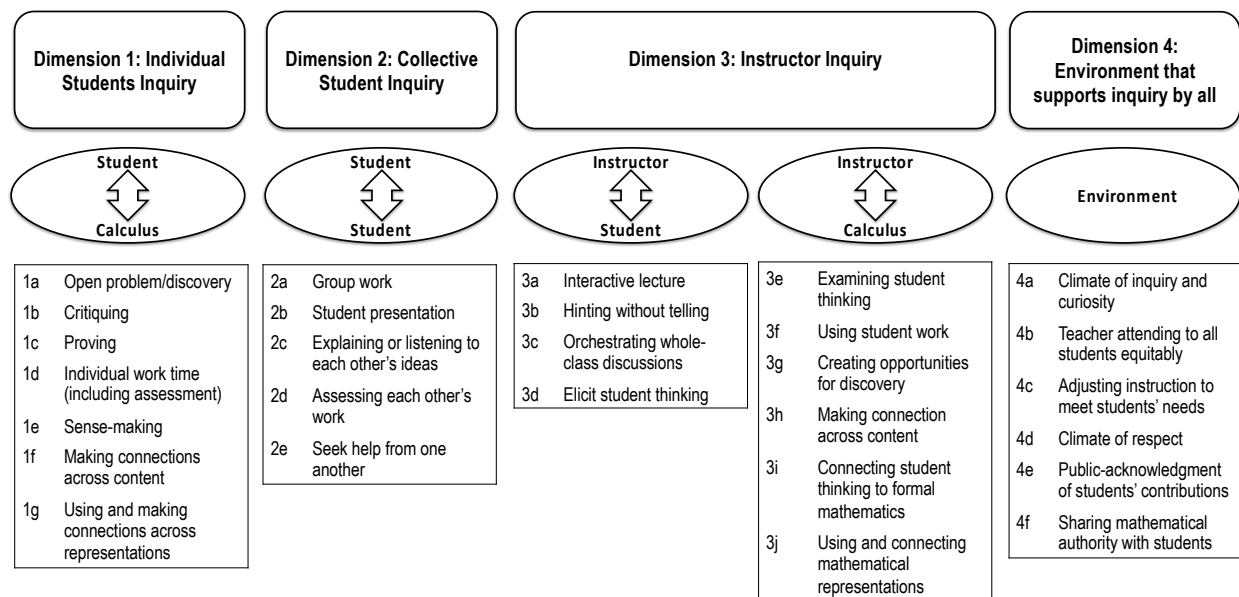


Figure 3: The Inquiry in Mathematics Instruction framework (Gerami et al., 2020)

METHODS

Data for this case study were collected over the Fall 2019 semester, as part of a large project (Beezer et al., 2018) that involved 18 instructors. Casey had nine years of experience in teaching at the university level and taught at a small private university in the Midwestern United States. He had eight students (three females) in his calculus course, all majoring in STEM fields. The data included a teacher survey (collected before teaching started), five teacher logs (short surveys collected through the semester), course syllabus, audio recordings and fieldnotes of three one-hour interviews with the instructor, and video-recordings and fieldnotes of four classroom observations (during the 10th week of the term: Monday, Wednesday, and Friday morning, and a lab on Friday afternoon). In the first interview, we video-recorded Casey to capture his lesson planning process. In the second interview, we discussed his enactment process using clips from classroom observations and lesson planning. In the third interview, we revisited some themes of planning and enactment.

To analyse Casey's planning, we focused on the set of actions that constituted Casey's curricular noticing using relevant pieces from his records: We listened to each interview, reviewed the field notes, and highlighted aspects that were directly relevant to planning or instruction with the textbook. To analyse Casey's instruction, we identified video segments that exemplified interactions between Casey, his students and the mathematics that were shaped by the textbook features and resulted in mathematically meaningful exchanges. Here we showcase the results of our analyses of two such segments, using the Inquiry in Mathematics Instruction framework. We shared a summary of our findings

and assertions with Casey to make sure that he agreed with the representation and interpretation of our findings.

FINDINGS

Using Active Calculus during Planning Inquiry Instruction

Casey's planning exhibited the three phases of curricular noticing as he produced two documents: the lesson plan (a to-do checklist to be used in class) and the prep assignment (a plan for student work to be completed in advance of the lessons). First, he searched within and reflected on the resources he had at hand—the textbook, Boelkins' and his own previous notes and prep assignments—searching for specific content and features; this gave evidence of the curricular attending phase. In the second phase, curricular interpreting, he made sense of the materials by interacting with them and by anticipating students' difficulties and the time needed to complete each activity. He recognized opportunities embedded in this particular set of curriculum materials, thinking they might “provoke discussion among his students” or be redundant because they repeat the content in the textbook. Casey questioned and made sense of the amount of class time that he would need, should he choose to include these textbook features. In the curricular responding phase, Casey made several decisions related to use of textbook features in both his lesson plan and his prep assignment. Casey's decisions were mainly about selecting and sequencing textbook features in both documents, and how he and his students were going to interact with them (e.g., in whole-class discussion or group work). Casey closely followed the textbook and its supplemental materials in designing these documents, consequently embedding the author's inquiry-oriented intentions for textbook use. For instance, following the author's suggestions, Casey expected his students to come to class after interacting with the content via the prep assignment, so that whole-class discussions could be generated at the beginning of the lessons.

Using Active Calculus during Inquiry Instruction

We documented several actions using the Inquiry in Mathematics Instruction framework in all of Casey's lessons that we observed. He started each lesson by collecting the prep assignment for the day and asking students if they had any questions. This opening generated whole-class discussions or interactive lectures (3a, 3c), with Casey inquiring into student thinking (3d), evaluating their thinking (3e), and making connections across content and representations (1f, 1g, 3h, 3j). Throughout the lesson, they followed the textbook and its activities, often referring to the mathematics as sections of the textbook. When working on problems at the board, Casey interacted with the students by asking them next-step questions, calling them by their names, and eliciting their thinking (3a, 3d, 3e, 3f). When students worked individually or in groups (1d, 2a), Casey spent time with all students (4b), going back and forth between

groups (4c), eliciting student thinking (3d), and checking their work and connecting it to formal mathematics (3i). During these interactions, Casey seemed comfortable not answering students' questions directly, allowing them to struggle with open problems (1a). Moreover, we analysed two segments of instruction and coded every utterance by Casey and his students. Although we cannot show our analyses of these segments here due to lack of space, in Figure 4 we present a summary of the inquiry actions observed in instruction and those supported by the textbook.

	Dimension 1: Individual Students Inquiry							Dimension 2: Collective Student Inquiry					Dimension 3: Instructor Inquiry										Dimension 4: Environment that supports inquiry by all					
	Student ↔ Calculus							Student ↔ Student					Instructor ↔ Student				Instructor ↔ Calculus						Environment					
	1a	1b	1c	1d	1e	1f	1g	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	3f	3g	3h	3i	3j	4a	4b	4c	4d	4e	4f
Episode 1					●	●	●						○			○	○	○	●	●		●	○		○	○	○	○
Episode 2			○		●	●					○		○	●			○	○		●	●	●	○			○	○	○

Figure 4: Actions enacting inquiry by dimension; a circle represents that the particular action was identified in the episode; a filled circle means that the textbook supported the interaction.

DISCUSSION

In this study, we perused two research questions: How does an inquiry-oriented calculus textbook shape lesson planning by a calculus instructor in inquiry-oriented ways? and, How does an inquiry-oriented calculus textbook shape one college instructor's interactions with students and the mathematics at stake in inquiry-oriented ways? We answer our first research question as follows: Casey designed his prep assignments and lesson plans carefully and efficiently because the textbook and its supplemental materials made it easy to do so. We think that the availability of the resources, especially the textbook with its interactive features and its ancillary materials, afforded Casey time to think through the available information, meticulously tweaking the details that were not aligned with his goals and visions. Because this was his first time teaching the course with *Active Calculus* or with inquiry, Casey might have relied on the textbook and the author's suggestions more, as he built up knowledge and experience with the textbook. It has been reported that instructors tend to be more attentive to new textbooks and that as they gain confidence, they tend to use it less (Mesa & Griffiths, 2012). To answer our second research question, we assert: Casey's instruction was inquiry-oriented because he closely followed his prep assignments and lesson plans which were based on a textbook and its supplemental materials that supported inquiry. Regarding this assertion, we believe that this was the case in part, because Casey had wanted to use inquiry

in calculus in the first place. From interviews we know that he had had some success doing inquiry in his upper division courses and was already predisposed to use it. That the students also did the work required of them ahead of time was important for the inquiry enactment to happen—and he had planned for it. Students came to class with questions that were freely asked and got into the relevant topics much quicker and with more focus. Casey might have created a classroom climate that supported students being comfortable with inquiry, included the appropriate incentives for students to do the work (homework credit), or students may have found the prep assignments (including the YouTube explanations and animations) interesting and the amount of work appropriate.

Thus, we believe that we have evidence demonstrating that this textbook facilitated a shift in Casey's way of teaching the course towards inquiry compared to his previous teaching of calculus. From our interviews with him, we know that Casey used to lecture before using *Active Calculus* and was dissatisfied with it but had not engaged in a change because his textbooks did not support his visions and goals. Casey had been slowly building more inquiry into his upper-division classes but found it very time consuming to design calculus activities that would enable students to explore the ideas in class. This hurdle was resolved when he had the opportunity to take part in the research project and teach with a textbook that supports inquiry in calculus.

Our study suggests that investigating how textbooks shape undergraduate mathematics education is an important area of research. Research should describe and analyse curriculum on aspects beyond their content to see how they afford opportunities for altering practice. Our study shows that a textbook that is oriented towards inquiry can support instructors in implementing it. Although supporters of inquiry have traditionally focused on shifting university instructors' practices via professional development opportunities or participation of faculty in research projects, our findings suggest that they could consider curriculum materials as a complementary resource and catalyst for instructional change (Laursen, 2019). Our findings are encouraging for advocates of reshaping teaching of university calculus by promoting inquiry using a resource that is readily and freely available to all.

Acknowledgements

Funding for this work has been provided by the National Science Foundation through Awards IUSE 1624634, 1821509. Carlos Quiroz, Yue Ma, Thomas Judson, and Lynn Chamberlain have provided assistance in the project. We thank Casey for participating in this study and for his suggestions and comments on early findings of this paper.

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“I DON’T NEED THIS” - UNDERSTANDING PRESERVICE TEACHERS DISAFFECTION IN MATHEMATICS

Lara Gildehaus¹ and Michael Liebendörfer¹

Paderborn University, Germany

In some countries, preservice mathematics teachers enrol in specific teaching degree programs but share some lectures with mathematics majors. In this setting, we analyse the phenomenon of preservice teachers’ deeming university mathematics as irrelevant through the lens of social psychology. Group interviews of in total 14 preservice teachers were analysed for students’ positioning of themselves and others within the mathematics community. The students experienced the two conflicting communities of mathematics and teacher education, which seem incompatible. Feeling excluded, they may project their negative experience against the community of mathematics. The proclaimed irrelevance should thus be seen as an expression of tensions in their identity and a reaction on their rejection by the mathematics community.

TEACHER TRAINING IN MATHEMATICS

In their training, secondary mathematics teachers often take courses together with mathematics majors. Yet, they may have separated programs. In Germany, preservice higher secondary teachers formally chose a different degree from the beginning and have different study schedules than mathematics majors. Yet, they often attend the first-year lectures on analysis and linear algebra together. Later, preservice teachers tend to have more and more specific courses on mathematics education and pedagogical contents (e.g., Bauer & Hefendehl-Hebeker, 2019).

While it is well known and discussed, that university mathematics can be challenging for all beginners, “it is striking, not to say frightening, how negatively preservice teachers assess their studies” (Mischau & Blunck, 2006, p. 49). While differences in learning prerequisites between mathematics majors and preservice teachers are rather small at the beginning of their studies (Bauer & Hefendehl-Hebeker, 2019), preservice teachers strongly lose their interest in university mathematics after the first weeks and some describe themselves as being excluded from the mathematics community (Ufer, Rach & Kosiol, 2017; Liebendörfer, 2014). They drop out of mathematics studies more often and are disaffected with their study content. Many students criticize university mathematics as being irrelevant for their future profession and demand more practice-related content (Bauer & Hefendehl-Hebeker, 2019; Liebendörfer,

2014). Accordingly, they report to copy homework and use surface learning strategies more often (Liebendörfer & Göller, 2016).

Current interventions therefore address this desire for more practical relevance (e.g. practice semesters and internships in schools) and relevance of content (e.g. specific tasks connecting university mathematics with school mathematics; Eichler & Isaev, 2017). Whereas such interventions are meant to comply with the demands of preservice teachers, their critique often seems to sustain (Bauer & Hefendehl-Hebeker, 2019) and may have further reasons. We therefore aim to understand preservice teachers' disaffection from mathematics from a social psychological perspective.

STUDENTS' IDENTITY AND PROJECTIONS

With their concept of "leading identity", Black et al. (2010) introduced a construct that may frame preservice teachers' concrete career aspirations. "Leading activities are those which are significant to the development of the individual's psyche through the emergence of new motives for engagement." (Black et al., 2010, p. 55). Alongside those motives comes a new understanding of self – the leading identity –, which reflects the hierarchy of motives. A leading identity focused on the later career aspiration would rather see the "use value" in studying mathematics to pursue the aspirations, in contrast to a leading identity that is focused on the activity of studying and becoming a university student, which values qualification one may obtain (Black et al., 2010). Accordingly, preservice teachers may rather search for a "use value" of their studies, than mathematics majors, who are more diverse in their career opportunities.

Similar to these results on individual level, Solomon (2007) found that mathematics undergraduates may find themselves within potentially conflicting communities during their studies. While one may expect first-year students, who managed to participate successfully in the mathematics community, will build a functional mathematics learner identity, this was not the case for all students. "On the contrary, students who describe identities of heavily alignment can appear unworried by their lack of participation in mathematics, successful as they are in the more dominant local communities" (Solomon, 2007, p. 79). Considering preservice teachers as a specific community with their own leading identities and ways of participation may thus explain the differing behaviour compared to mathematics majors. It remains unclear, however, if, and if so, how this community relates to the mathematics community.

The social psychological concept of projection (Baumeister et al., 1998, p. 1090) may clarify this relation, as it provides a frame to analyse preservice teachers' disaffection from mathematics along their positioning to the mathematics community. A projection is a psychological mechanism of attribution that is usually triggered by experiences that challenge one's own

identity, e.g. fear. It describes a defence mechanism to repress impulses that the individual does not allow for self-perception – because they contradict the self-image, for example. For this purpose, a projection screen is sought to transform fear, into hatred of something that is not supposed to belong to oneself, and to relieve the self of psychological strains. The projection screen is then “the foreign” – a construct that developed from one's own unconscious fantasies and affectation. Obviously, “the foreign” is constructed to take ambivalences out of the self and to clarify one's own identity. In contrast to “the foreign” one can easily determine oneself, by what one is not. A group-stabilizing mechanism of exclusion and inclusion succeeds: “The foreign” is generalized and perceived as fundamentally incompatible with the own identity. One valorises oneself as belonging to “one's own” by devaluing “the foreign”. A longing for a community of wholesome identity arises and legitimates defence against “the foreign” (Pohl, 2017).

The preservice teachers' challenge to mediate between one's leading identity on the individual level and to be torn between different communities might be seen as a base for projections. If students' disaffection from university mathematics could be seen as part of a projection, then the demand for more practice-oriented content could not easily be satisfied by including some practice-oriented elements in their studies.

Research questions

To explain preservice teachers' disaffection and demand for practice-oriented content from a social psychological perspective, we pose the following research questions:

RQ1: How do preservice teachers position themselves and others in relation to university mathematics?

RQ2: Which elements of projection (according to Pohl) can we find in the positioning?

METHODS AND DESIGN

To answer these questions, three semi-structured group interviews with four to five preservice teachers each (n=14, 8 female) were analysed. The interviews were conducted at the University of Paderborn two months after the first semester started. The interviewees participated voluntarily and they were guaranteed anonymity. German secondary teachers need to study two subjects and the participants represented diverse second subjects (e.g. sciences, foreign languages, or social science). In Paderborn, the first semester includes a regular lecture on linear algebra together with mathematics majors and a specific introductory course for preservice teachers only.

The interviews focussed on the students' experiences in the first semester and their identity. Questions in the guide included the following: “To what extent do

you identify yourself here at the university? How much do you feel yourself as mathematicians? How would you describe your parents what mathematicians are and do?”

In the data analysis, we first coded passages with statements about one's own position and descriptions of other persons within the study program (lecturers, fellow students, tutors). These passages were then examined and structured for specific references to the teaching profession. In a second pass, indicators of projections were sought in the identified passages (generalizations, revaluations/devaluations, longing for a wholesome identity), which were then interpreted based on theory (Pohl, 2017). All presented citations were translated from German by the first author.

RESULTS

Regarding RQ 1, all students said they had always been good in mathematics and mostly had enjoyed it in school. Facing cognitive struggle, especially in linear algebra, challenged their mathematical identity. They felt frustration and irritation (“Here you think to yourself: I work and work and still nothing comes of it”). Very quickly, they developed a leading identity of becoming a teacher. The profession was very dominant in students’ self-descriptions and teaching was labelled as “the dream-job”. For most students, teaching itself was described to be more important than the subject was:

Anna: So, maths is not my big love now either, so it's not that I'm happy like some mathematician there in my math world, but I just want to teach it to someone.

Hence, their new motives of engagement were quickly orientated towards formal requirements, particularly on the graded weekly homework:

Thomas: So at the beginning, it [the goal] was to understand that. And then, after the first two results or so, it was then: “we need points now and no matter how”.

Although they had been studying for only two months, all groups substantially questioned the relevance of the content:

Melissa: Well, I actually don't see the point of having our lectures with mathematics majors. Sure, maybe a certain part, but not as it is at the moment. And as abstract as it is at the moment, I don't think you'll ever need it in school, not even in a few years. I don't see that being purposeful.

Positioning along this content-relevance started from Melissa’s mostly individual statement, but also referred to generalizations that were legitimized with the school curricula (“You don’t need it. It is simply not part of the school curricula”). In contrast, the specific lecture for preservice teachers was widely

accepted. We should note, however, that students experienced less cognitive demand in this lecture.

Thus, for most of the students, the uncertainties experienced in the transition to university mathematics seemed to strengthen the focus on their actual goal of becoming a teacher. In addition to this inner struggle, the students also described external struggle, feeling degraded by mathematics major students and faculty members:

Marc: I think it's always a pity when people come and say, oh yeah, you're only doing a teaching degree.

They argued that mathematics majors as well as the professor in their linear algebra course see themselves as something better and therefore exclude preservice teachers. Part of this exclusion was also being recognized as less competent:

Melissa: They [mathematics majors] really think as if they are something better. And I think that preservice teachers are always a bit excluded.

Melissa: [...] But you always have the feeling that he [the professor] doesn't really think much of preservice teachers either. [...]

Miranda: So you always notice that you're talked down to.

Melissa: Like: "Yes, we do that so that the future teachers sitting here can teach their students that, so they can then solve the math problems of tomorrow." And then you think to yourself: "are we too stupid for that or what"?

Anna: [...] for them [mathematics major tutors] it's all logic and easy of course and in their point of view we're just stupid because we don't understand it.

In line with this degradation, the students positioned themselves as more distant from the professor and less part of the mathematical community than mathematics majors:

Melissa: Well, they're already into this abstract thinking, which we all don't seem to be yet.

Laura: They are more on the professor's level than ours.

Melissa: Well, they always get along really well with him and he also makes jokes with them. But we just don't understand it.

The two communities seemed rather incompatible:

Melissa: So, computer scientists and mathematicians always sit together and preservice teachers always sit together somehow. And I think that it just doesn't come together at all.

This perceived incompatibility made it difficult to participate in both communities or to negotiate ones positioning. The preservice teachers' community was formed based on their common career aspirations. Being positioned as preservice teacher, however, resulted in downgrading and separation from the mathematics community, e.g. being considered less competent. Some students quickly adapted to this position and described themselves as preservice teachers rather than mathematicians:

Marc: I would never call myself a mathematician, for that I also see more and more in the homework that it's just not like that.

Other students struggled with this positioning, trying to avoid the attribution as less competent. However, they failed negotiating this position and could not participate successfully in the mathematics community (e.g. they felt marginalized by their tutors and struggled with the weekly exercise). One female student reported participating mathematically as she successfully discussed her homework with tutors. Notably, she is the only student who did not show a strict leading identity of becoming a teacher. Her participation in the mathematics community lead to distance herself from the preservice teachers' community, but she tried to negotiate this incompatibility:

Lydia: That's why I don't know yet if I'll eventually be a computer scientist instead of a teacher, I don't know. I wouldn't either/ Why can you only be one thing, // why can't you be everything?

There is an ambivalence in students' self-positioning. Being positioned as a preservice teacher, who is generally less competent, was mostly denied and criticized. However, it was also accepted sometimes and used to legitimate lower effort and achievement, as the content was proclaimed not relevant:

Miranda: As if my students would ask me: "What is a group homomorphism?" [...] So, if I have basics with which I can understand that, then I think that's enough. Then I don't really need to do all the stuff in LinA [linear algebra].

Mathematically, a group homomorphism is a very basic concept, from which Miranda distances herself, claiming she only needs basics to become a teacher. Such ambivalences are now discussed in the framework of projection in RQ 2:

Projections related to mathematics majors could be found in all interviews. Notably, they varied in their extent, mostly linked to the perceived identity tensions. Those, who could not identify themselves as "just teachers" described most precisely their projection based on the nerd stereotype:

Anna Well, I don't know if they look different, but, I don't know, I often find that their behaviour is a bit/[...] They are a bit in their own world. For them, everything is always logic. They all understand it immediately. And I think that these mathematics majors, in part, that it is more difficult with them, for example, to form learning groups,

because I think they are often so on their own level. And they are not the kind of people who are capable of social interaction and empathy, who sit down with others and do that. But I think they are also often really loners, lone fighters.

This description used the stereotype that higher mathematical competence implies lower social competence. Major students were seen to live in their own world, which separates them and their active living from the preservice teachers. Projecting their label of being incompetent, preservice teachers described major students to always understand everything. Those interrelations can also be found in Miranda's quote. Additionally, she deemed the majors responsible for the separation of the two worlds:

Miranda: You just have the feeling that they don't have any other topics among each other. So I don't want to generalize that either, but there are so many: "Yes, that was quite simple, it was totally logical. That was it, I understood it all." And then you think to yourself, "Yes, that makes me happy for you, maybe you can explain that to someone." But then they all just hang out with each other. They also don't manage to break it down, to make it more understandable."

As Miranda directly answered to Anna, we see that they constructed a contradictory perspective: Majors were described to be "lone fighters", who "hang out with each other". This perspective includes generalizations that are negatively attributed for majors and positively attributed for the preservice teachers: Mathematics majors are conceived as a homogeneous group of nerds, with no other themes than mathematics:

Miranda: I just think that all the mathematics students really only, so from the feeling, they all just hang out at home after university and only deal with maths.

Preservice teachers instead are described to be "still all okay" and "more social". This separation associated with positive or negative attributions was not limited to fellow students, but also projected on tutors who are higher semester students:

Melissa: And I honestly have to say, that you can really recognize a difference between the tutors in EmDA [the specific lecture] and LinA [the linear algebra lecture] [...]. In EmDA, I think, there are only people who are real preservice teachers and where you realize, they are really nice.

In contrast to this, Lydia, who did not have the leading identity of becoming a teacher, described the least downgrading of mathematics majors and hence the lowest extent of projection: She only referred to the "cute nerds" she met at a first-semester party.

Following the framework of projection, we found that the interviewed students experienced tensions between their developing leading identity of becoming a teacher and their perceived position as downgraded, less competent preservice teachers with less access to the mathematics community. Students who struggled with this positioning distanced themselves from mathematics. This can be reconstructed as a projection: Based on their fear of being unsuccessful in their studies and being excluded from the mathematics community, preservice teachers constructed the image of mathematics majors being nerds who lack social skills and developed a negative identity towards these students. This, however, inevitably goes hand in hand with their own disaffection from the scientific content.

DISCUSSION

We analysed three group interviews investigating preservice teachers positioning and possible projections with regard to their disaffection. We firstly found that the interviewed students quickly defined themselves in terms of their career goal and only secondarily in terms of mathematics. This is consistent with previous research (Bauer & Hefendehl-Hebeker, 2019). The results indicated a situation of incompatible communities of preservice teachers and mathematics majors. The preservice teachers described their position as downgraded and perceived to be less competent. Hence, they experienced challenging identity work, positioning themselves to this attribution, which formed a base to build projections. Secondly, the analysis of this projection made visible, that the demand of preservice teachers for more relevant content is not (only) a rational fact, neither exclusively indicated by their leading identity. It is also part of the projection that occurred in the disaffection from university mathematics including the content and the major students. At the same time, this projection is caused by the ambivalent identity tensions that the preservice teachers experience in their attempt to find their way into two conflicting communities.

The split into two communities is ambiguous: On the one hand, the preservice teachers' community forms the basis for specific participation and identity formation. On the other hand, it reinforces the downgrading, which could legitimize actions such as copying homework and surface learning based on students' leading vocational identity. The resulting projections thereby exclude participation in the mathematics community.

Changes towards more and more equal participation of preservice teachers thus need to consider the outer circumstances, such as faculty members' implicit positioning, as well as preservice teachers' inner tensions along their leading identity. Specific preservice teachers' lectures provide a basis for experiencing the wholesome identity students seek in regard to their projection. Hence, such interventions may have mostly situated effects. As long as preservice teachers

share lectures with mathematics majors and experience inner tensions due to excessive cognitive demands, we may assume that they will position themselves as future teachers. They may use this position to participate less intensive and will represent their position publicly, e.g. articulate their demands. Specific interventions must thus not only serve students' longing for practical relevance, but also reduce their tensions by valuing the content they learn as relevant for their profession, as well as worth to participate mathematically (e.g., Eichler & Isaev, 2017). Our research highlights a need to examine not only student satisfaction with the interventions, but also whether they promote identification with mathematics.

Limitations & further indications

The results presented here are highly dependent on the study context and thus at most exemplarily for very similarly organized teacher education. Further research, for example in the context of pure teaching degree programs or entirely mixed degree programs, is desirable. Furthermore, the identified incompatibility of the two communities could not be fully reconstructed. Further research is needed here to more concretely understand the positioning of preservice teachers. In particular, future research should take into account the perspectives of major students and faculty members, to better reflect the social context of the communities. The “all knowing” mathematics majors are obviously the preservice teachers’ construct and usually face similar struggle, probably positioning themselves along a different leading identity.

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EXTENDING PISA'S MATHEMATICS SELF-EFFICACY SCALE TO A MULTIDIMENSIONAL MEASUREMENT MODEL: RESULTS OF A SWISS NATIONAL LARGE-SCALE ASSESSMENT

Boris Girnat

University of Hildesheim, Germany

This paper presents an extension of the unidimensional PISA 2003/2012 mathematics self-efficacy scale to a four-dimensional measuring model related to the mathematical subdomains algebra, applied mathematics, geometry, and probability theory. Its first application in a Swiss large-scale assessment shows the following results: 1) The four-dimensional model allows a more fine-grained analysis of group differences, illustrated here with respect to gender and schools levels. 2) The subdimensions of self-efficacy are good predictors for the mathematics test outcome, but work differently: algebra and applied mathematics are most important. 3) The explanatory value of the predictors is different in homogeneous and heterogeneous groups and can be supplemented by a scale on mathematics self-concept (only) in homogeneous groups.

INTRODUCTION: PISA'S MATHEMATICS SELF-ASSESSMENT

In 2016, the first national school assessment took place in Switzerland, focussed on mathematics in grade 9 (Konsortium ÜGK, 2019). Typical for a large-scale assessment, this survey consisted of two parts: a performance test and a context questionnaire gathering data that are suspected to allow a deeper insight in the outcome of the performance test (cf. Martin, Mullis, Arora, & Preuschoff, 2014). Some variables of a context questionnaire are often linked to self-assessment, following the idea that students' beliefs about their own abilities could be a good background variable to analyse and interpret their test scores. Mathematics self-assessment can be measured in different ways. Usually, two approaches are used: The first one is related to a person's so-called *mathematics self-concept*. It is measured by items based on general statements like "I have always believed that mathematics is one of my best subjects" (cf. Marsh, 1990). The second approach is called *mathematics self-efficacy* and is based on Bandura's theory of (academic) self-efficacy (cf. Bandura, 1977 & 1986). Bandura defined self-efficacy expectation as "people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performances" (Bandura, 1986, p. 391). Applied to mathematics, Bandura's theory implies the strategy to measure mathematics self-assessment by items that allow a person to express his or her level of

confidence about feeling able to solve specific problems that are relevant to mathematics in general or to a specific mathematical subdomain of interest. Research has shown that scales based on these two approaches are correlated, but empirically distinguishable (cf. Multon, Brown, & Lent, 1991). Both of them are good predictors of test performance (Hackett & Betz, 1989).

The supervising group of researcher that were responsible for the Swiss context questionnaire decided to measure both mathematics self-concept and mathematics self-efficacy (cf. Hascher, Brühwiler, & Girnat, 2019). They followed the PISA framework, adopting the items of PISA 2003 and 2012 (OECD, 2005, pp. 291–294, & OECD, 2014, pp. 322–323). However, a Swiss pre-study with about 2,000 participants in 2015 had shown that the PISA self-efficacy scale could not be regarded as unidimensional. A factor analysis led to the conclusion that the different subdomains of mathematics formed related, but empirically distinguishable factors that should not be mixed up in one unidimensional scale (cf. Girnat, 2018). An independent study came to the same result analysing the original PISA data of 2003 (Oberski, 2014, p. 13). Therefore, the supervisors of the Swiss test decided to use a multidimensional measuring model. As far as possible, the PISA items were reused. However, the PISA scales did not contain enough items to implement this idea: With regard to the Swiss curriculum, it was necessary to measure self-efficacy with respect to elementary geometry and probability theory – the PISA scale contains neither of them –; for statistical reasons, each scale should consist of at least four items to be a sufficient measurement tool (cf. Beaujean, 2014, pp. 145–152). Insofar, the supervising group decided to use a “4x4 arrangement” that relies on four scales, each of them containing four items to measure the following mathematics subdomains: applied mathematics (app), algebra (alg), elementary geometry (geo), and probability theory (prb), reusing as many PISA items as possible:

seff.app1) Calculating how much cheaper a TV would be after a 30% discount.
(PISA)

seff.app2) Calculating how many square metres of tiles you need to cover a floor.
(PISA)

seff.app3) Calculating the petrol consumption rate of a car. (PISA)

seff.app4) Finding the actual distance between two places on a map with a 1:10 000 scale that problem. (PISA)

seff.alg1) Solving an equation like $3x+5=17$. (PISA)

seff.alg2) Solving an equation like $2(x+3) = (x+3)(x-3)$. (PISA)

seff.alg3) Developing and simplifying an algebraic expression like $2a(5a-3b)^2$.

seff.alg4) Solving an equation like $2x-3=4x+5$.

seff.geo1) Applying the Pythagorean Theorem to calculate the length of one side of a tri-angle.

seff.geo2) Constructing a perpendicular bisector using compass and ruler.

seff.geo3) Calculating the area of a parallelogram.

seff.geo4) Constructing the focus of a triangle.

seff.prb1) Calculating the probability of throwing a dice twice in succession to achieve two sixes.

seff.prb2) Calculating the probability of getting the first prize in a lottery.

seff.prb3) Calculating how likely it is to take two sweets of the same colour from a sweet jar.

seff.prb4) Calculating how likely it is that two pupils in a class have the same birthday.

As mathematics self-efficacy is related to mathematics self-concept, a short scale to measure the latter (abbreviated as *matcon*) was included in the questionnaire, also based on items used by PISA (cf. OECD, 2013b, p. 95):

matcon1) I get good grades in mathematics. (PISA)

matcon2) Mathematics is one of my best subjects. (PISA, shortened)

matcon3) I have always been good at mathematics.

The research questions related to these scales are as follows: 1) Are the statistical properties of the four-dimensional model of self-efficacy sufficient? 2) How can this model be used to gain deeper insights into the test population? Following PISA, gender differences and differences concerning different school level are of a special interest (cf. OECD, 2013b, p. 91). 3) How are the four scales related to each other and to the mathematics self-concept scale? 4) Are the four scales good predictors for the test scores of the participants? After a short description of the Swiss test, these question will be answered in the following section.

CONTENT, SAMPLE, AND METHODS

A total of 22,423 students took part in the Swiss test in 2016. This population was a representative sample of Swiss students in class 9 (according to the Swiss numbering grade 11). Insofar, exactly the same grade was tested, which is also the basis of the PISA studies. The context questionnaire was available in two variants: The first variant was focussed on sociological issues. Only the second variant contained questions related to mathematics. This variant was worked on by 11,131 students and is the basis of the following analysis (Konsortium ÜGK, 2019). The performance test (cf. Girnat & Linneweber-Lammerskitten, 2019) consisted of 180 test items that were related to five subdomains of mathematics (data and probability, quantities and measurement, functional relationships, numbers and variables, space and shape) and five mathematical processes (reasoning and argument, representation and communication, concepts and knowledge, mathematisation, operations and calculations). This framework is quite similar to that one used in PISA (cf. OECD 2013a, p. 26), however, the items were designed in such a way that they meet the Swiss curriculum more precisely than a worldwide study like PISA can do. The sampling design and

the evaluation of the test followed the standards of PISA (cf. Angelone & Keller, 2019). The data from the test are linked to the questionnaire data. A Rasch model is used for the test, including 50 plausible values as technical tools (cf. Mislevy, 1991). The questionnaire data and the plausible values are evaluated using structural equation modelling (Loehlin & Beaujean, 2017, pp. 95–125). This is the same method that PISA applies (cf. OECD, 2005, p. 293). The calculations were done using the R packages TAM (Robitzsch, Kiefer, & Wu, 2019) and lavaan (Rosseel, 2012).

RESULTS

According to the research questions, the first step of the evaluation is to check the statistical properties of the scales. Cronbach's alpha is used as a measure of reliability (cf. Cronbach, 1951). Various quality criteria (fit indices) are known from the context of the structural equation modelling. CFI, SRMR and RMSEA are reported here (for the definition and interpretation of these values cf. Beaujean, 2014, pp. 153–166; a short summary: CFI should be greater than 0.95, but definitively not below 0.90, SRMR should be lower than 0.06 and RMSEA lower than 0.05 or 0.08 according to different sources).

Scale	Cronbach's alpha	CFI	SRMR	RMSEA
matcon	0.89	n.a.	n.a.	n.a.
seff.app	0.79	0.993	0.017	0.047
seff.alg	0.90	0.951	0.036	0.076
seff.geo	0.76	0.994	0.013	0.039
seff.prb	0.88	0.982	0.026	0.079

Table 1: Reliabilities and fit indices of the scales

The values reported in Table 1 indicate good reliabilities and fit indices (the latter are not available for matcon, since this scale consists of three items only). Hence, the scales are usable measuring instruments. Next, the correlations between the scales and the test outcome are reported. Since a structural equation model is used, the correlations reported here are latent correlations, i.e. these correlations are stripped from the measurement error observed variables are contaminated with and, hence, they reflect the relationship between the underlying latent concept more accurately than normal (Pearson) correlations between the row sums of the scales (cf. Beaujean, 2014). The asterisks here and in the following denote the usual significance levels.

	matcon	seff.app	seff.alg	seff.geo	seff.prb
test	0.36***	0.55***	0.53***	0.55***	0.30***
matcon		0.56***	0.38***	0.45***	0.41***

seff.app	0.66***	0.87***	0.69***
seff.alg		0.73***	0.39***
seff.geo			0.49***

Table 2: Latent correlations (SRMR 0.052, RMSEA 0.059)

The correlations reported in Table 2 are mostly moderate (maybe except for the pair seff.app and seff.geo), which indicates that the underlying concept are empirically distinguishable and do in fact measure different aspects of mathematics self-efficacy.

The next focus is set to group differences. Following the PISA framework, gender differences are regarded first, and different school levels are the second part of this examination. In Switzerland, mathematics is taught from grade 7 to 9 on three different levels: Level “a” is the lowest, “e” the middle, and “p” the highest.

In PISA 2012, there are some remarks on gender difference concerning mathematics self-efficacy: “No gender differences in confidence are observed when students are asked about doing tasks that are more abstract and clearly match classroom content, such as solving a linear or a quadratic equation. However, gender differences are striking when students are asked to report their ability to solve applied mathematical tasks.” (OECD, 2013, p. 91). This statement is based on analysing the single items of the PISA scale. By doing so, the authors implicitly admit that is questionable to combine these items to a unidimensional scale. Analysing single items is a questionable method, since single self-efficacy items are focussed on just one specific task and are much more affected by random measurement errors than a scale based on several items. Having the scales introduced here, this issue can now be answered on a profound basis.

Table 3 shows the mean differences between the relevant groups. The differences are reported in terms of Cohen’s d (Cohen, 1988), i.e. the mean of one group (the “reference group”) is set to zero (in Table 3 the group following “vs”, e.g. “male” in case of gender) and the mean of the other group is given as the difference to the reference group using the standard deviation as the measurement unit. Cohen’s d is usually interpreted as follows (Cohen, 1988): $d = 0.2$ indicates a small effect, $d = 0.5$ a medium effect, and $d = 0.8$ a strong effect.

scale	female vs male	e vs a	p vs a	p vs a
test	-0.14***	1.02***	0.96***	1.92***

matcon	-0.63***	0.06*	0.00	0.06*
seff.app	-0.60***	0.39***	0.40***	0.79***
seff.alg	0.00	0.56***	0.57***	1.13***
seff.geo	-0.23***	0.43***	0.57***	1.00***
seff.prb	-0.54***	0.14***	0.06*	0.20**

Table 3: Mean differences concerning gender and school levels

Table 3 shows some remarkable results: The mean difference between male and female students is relatively small ($d = -0.14$), however, the differences concerning self-concept is approximately four times as large ($d = -0.63$). In view of this discrepancy, one can speak of a considerable self-underestimation of female students, if they rated themselves on an abstract level of items related to their mathematics abilities. This observation does also hold with respect to the self-efficacy scale focussed on applied mathematics and probability theory, but not on the one related to algebra. Insofar, the conjecture stated in PISA 2012 can be verified: Female students report a lower ability to solve applied mathematical tasks, but this is not the case as far as algebraic topics are concerned. If this finding holds beyond algebra and can be extended to tasks “that are more abstract and clearly match classroom content” is an open question.

With regard to the different school level, one observation is most striking: There are (very) large differences concerning the test outcome, however, mathematics self-concept does not reflect these differences to the slightest extend. The self-efficacy scales reflect this difference at least about half. This suggests the hypothesis that self-concept is a measure that does not work across groups, if the groups differ substantially in their performance. The self-efficacy scales, however, – probably because they are linked to specific mathematical tasks – are able to determine cross-group differences.

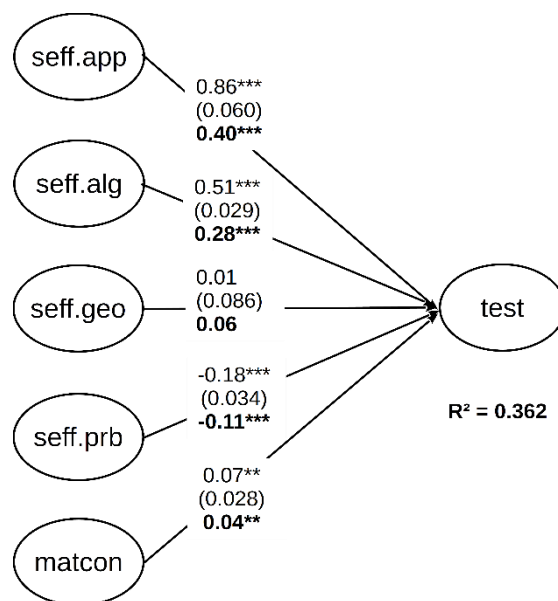


Figure 1: Linear model to predict test outcome (SRMR 0.037, RMSEA 0.052)

The last part of the analysis is dedicated to linear models (Searle & Gruber, 2016). In Figure 1, a path diagram is shown that uses all the scales as predictors of the test score (the value on top of each arrow expresses the unstandardised regression coefficient, the second value (in brackets) is the standard error of this coefficient, and the third value (bold) is the corresponding standardised regression coefficient; only the latter can be compared between different arrows and models). Overall, the variance explained by this model is considerable ($R^2 = 0.362$). However, the impact of the predictors is rather diverse: Geometry is not significant; and – very astonishing – probability theory has a negative impact. Only self-efficacy concerning algebra and applied mathematics are powerful predictors, whereas self-concept has just a very small impact.

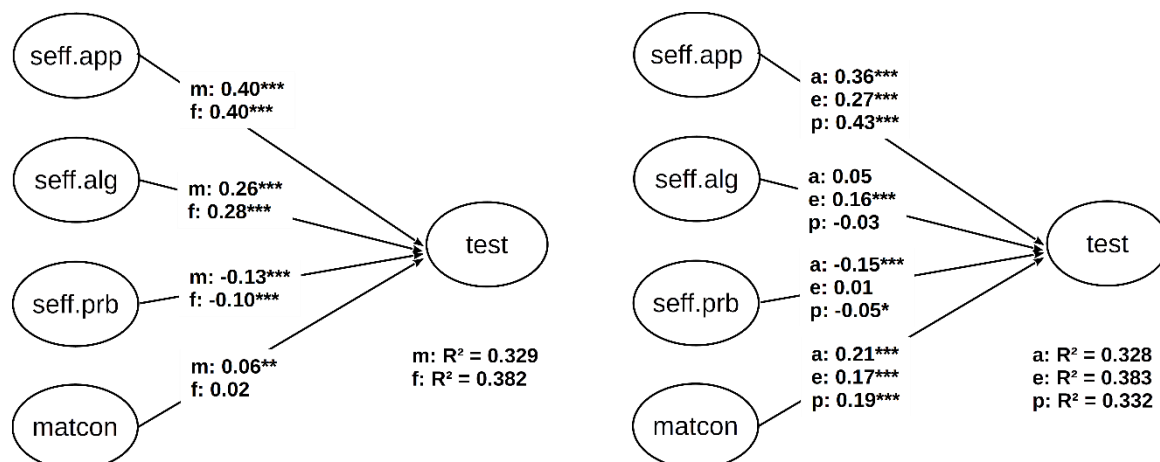


Figure 2: Linear models for gender and school levels (SRMR 0.039, RMSEA 0.053)

Figure 2 shows two models in which the regression was carried out differently for the grouping variables gender and school types (only the standardised regression coefficients are reported here, and the insignificant predictor related

to geometry is omitted). While the model on the left shows no remarkable gender differences, the model on the right reveals informative information about self-concept: If you divide the overall sample (Figure 1) into the more homogeneous subsamples related to school levels (Figure 2, right side), the self-concept will become a strong predictor, while the role of self-efficacy in algebra will decrease. This supports the hypothesis that was already expressed in the context of group differences (Table 3): The self-concept seems to be a scale that is only meaningful in relatively homogeneous groups, whereas the self-efficacy scales allow a comparison of students even in heterogeneous groups.

CONCLUSIONS

The four-dimensional extension of the PISA mathematics self-efficacy scale presented here allows deeper insights than the original unidimensional PISA model: It provides a valid statistical basis to examine group differences related to different subdomains of mathematics. It can be confirmed that gender difference do not appear with respect to algebra, while they are remarkable (and excessively high) concerning applied mathematics, geometry, and probability theory. Both, the group differences related to school levels and the linear models, reveal a fundamental difference in the nature of the self-concept and self-efficacy scales: The first do only work within relatively homogenous groups; the latter are able to determine cross-group differences also within a heterogeneous sample. The reason may be the fact that self-efficacy items a formulated on the basis of concrete mathematical tasks that seems to work as “objective anchors” across groups, whereas the abstractly worded self-efficacy items appears to be understood by students as being relative to their classmates and their average abilities. Overall and across groups, the two self-efficacy scales on applied mathematics and algebra are the most powerful predictors to test outcome.

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FRUSTRATED AND HELPLESS - SOURCES AND CONSEQUENCES OF STUDENTS' NEGATIVE DEACTIVATING EMOTIONS IN UNIVERSITY MATHEMATICS

Robin Göller¹ and Lara Gildehaus²

¹Leuphana University Lüneburg, Germany

Paderborn University, Germany

In this paper, we try to systematize and to explain different types and sources of frustrations which arise in the transition from school to university with Pekrun's control-value theory. Empirical basis are problem-centered interviews with 21 mathematics students during their first year of study at university. The results show that a low action-outcome control can be considered as the main source of frustrations, but also that the importance of certain values should not be underestimated. In particular, the consideration of consequences of frustrations allows new approaches to explain phenomena known in higher mathematics education, such as the copying of exercises or a devaluation of mathematical contents, which are discussed.

FRUSTRATION IN UNIVERSITY MATHEMATICS

Frustration is a commonly reported emotion of first-year university mathematics students (Göller, 2020), and yet, like emotions in general, is relatively rarely considered by research in undergraduate mathematics education. Theoretical and empirical evidence indicates that frustration as a deactivating negative emotion undermines motivation, reduces flow, and is related to task irrelevant thinking, shallow information processing, and weaker academic achievement (Boekaerts & Pekrun, 2015) and is therefore rather undesirable in academic contexts.

However, studies in undergraduate mathematics education report various sources of students' frustrations in the transition from school to university, such as being stuck during problem solving, perceiving limited autonomy, the fast pace of the courses, inefficient learning strategies, the need to change previously acquired ways of thinking, difficult rapport with truth and reasoning in mathematics, insufficient academic and moral support on the part of teachers, and poor achievement (Liebendörfer & Hochmuth, 2015; Martínez-Sierra & García-González, 2016; Sierpinska, 2006). Such frustrations may differ in their duration, intensity, type and genesis. In this paper, we try to systematize and to explain these types and sources of frustrations with Pekrun's (2006) control-value theory and investigate concrete consequences of such frustrations, which

highlight the importance of increased attention to frustration, or emotions in general, in teaching and research in undergraduate mathematics and teacher education.

THEORY: CONCEPTUALIZING EMOTIONS

Emotions, as we consider frustration, are understood as affective episodes which constantly mediate between changing events and social contexts and the reactions and experiences of the individual (Scherer & Moors, 2019). Such emotion episodes comprise various components which include appraisals of the situation, action preparation, physiological responses, expressive behavior, and subjective feelings (Scherer & Moors, 2019).

Emotions can be categorized by their valence (positive – negative) and degree of activation (Boekaerts & Pekrun, 2015): For example, joy and hope are activating positive emotions, whereas contentment and relief are deactivating positive emotions. Anger and fear are negative activating emotions, whereas hopelessness and frustration are deactivating negative emotions. As mentioned above, negative deactivating emotions (and frustration and helplessness in particular) negatively interfere with desirable learning processes and performance (Boekaerts & Pekrun, 2015). In the following, when we speak of frustration, we refer to these deactivating emotions described by words like “frustrating”, “helpless”, “despair”, and “depressed”.

In academic contexts, *achievement emotions* (e.g. pride, shame, hope, anger, anxiety) which refer to achievement activities (e.g. learning or studying) or achievement outcomes (e.g. grades), and *epistemic emotions* (e.g. curiosity, surprise, frustration at unsolved problems) which refer to the comprehension process of novel information or to problem-solving can be distinguished (Boekaerts & Pekrun, 2015). For example, in mathematics problem solving, frustration as an epistemic emotion (e. g. at not deriving a correct solution to a mathematics problem) can be considered less problematic, as the focus is on the cognitive incongruity that resulted from the unsolved problem, than frustration as an achievement emotion, where the focus is on personal failure and the inability to solve the problem (Muis, Psaradellis, Lajoie, Di Leo, & Chevrier, 2015).

Sources of frustration in Pekrun’s control-value theory

The control-value theory of achievement emotions (Pekrun, 2006) posits that achievement emotions are a multiplicative function two groups of appraisals: (1) The *subjectively perceived control* over achievement activities and their outcomes and (2) the *subjective values* of these activities and outcomes. Both, subjective control and subjective values can refer prospectively and retrospectively to outcomes as well as to activities, and accordingly result in *prospective outcome emotions* (e.g., anticipatory joy, when subjective control and value is high, and helplessness, when subjective control is low),

retrospective outcome emotions (e.g., pride, when subjective control and value is high, and anger, when subjective control is low), and *activity emotions* (e.g., enjoyment when subjective control and value is high, and frustration, when subjective control is low, Pekrun, 2006).

The (prospective) total outcome-control expectancy appraises the overall controllability and probability of an achievement outcome on the basis of situation-outcome, action-control and action-outcome expectancies. *Situation-outcome* expectancies are expectancies of (positive or negative) outcomes that the situation will produce without self-action, respectively if no countermeasures are taken. *Action control* expectancies are expectancies that actions can be initiated and executed autonomously. They are closely related to Bandura's self-efficacy (Bandura, 1997). *Action-outcome* expectancies are expectancies that one's own actions will produce a positive outcome, or prevent, reduce, or end negative outcomes. Retrospectively, perceived outcome control attributes the causes of success and failure to one's own actions, the self, external circumstances, or other people. External attributions are related to situation-outcome expectancies and internal attributions are related to action-control and action-outcome expectancies (Pekrun, 2006; Weiner, 1985). Values in control-value theory are distinguished in intrinsic and extrinsic values, whereby *intrinsic values* refer to the value of an activity or outcome per se, and *extrinsic values* refer to the instrumental utility of actions or outcomes for achieving other goals. Outcomes and activities in control-value theory can be negatively valued, e.g., in form of the subjective value (respectively cost) of an outcome that is appraised as failure, or when the effort required by an activity is experienced as unpleasant (Pekrun, 2006).

According to the control-value theory, frustration and helplessness can occur as prospective and retrospective outcome emotion, as well as activity emotion: As a prospective outcome emotion frustration occurs in terms of helplessness when outcome control expectancy is low. As a retrospective outcome emotion, frustration results from a negatively valued outcome „attribution-independent" (Boekaerts & Pekrun, 2015), but especially in the case of a low perceived outcome control. As described above, frustration as an activity emotion is aroused by low action control (Pekrun, 2006), especially when the subjective, intrinsic value of the learning activity is negative, or if a task is perceived as too demanding and effortful (Boekaerts & Pekrun, 2015). In sum, frustration is expected to appear in different forms of emotions, with low perceived control being the main source in each form. It should be noted that due to the theoretically assumed multiplicative structure in control-value theory, when the perceived control is low, the level of frustration is expected to increase with the value. Recapitulating the sources of frustrations of mathematics students mentioned at the beginning of this paper, it shows that these sources all refer to low perceived control, with different foci in terms of outcomes and activities.

Studying mathematics in Germany: institutionally predetermined situation-outcome expectancies and values

Perceived control and value, and thus emotions, are influenced by the academic environment. Some recent studies report frustrations of (German) mathematics students within the first year at university (Göller, 2020; Liebendörfer & Hochmuth, 2015). However, sources and consequences of these reported frustrations have not yet been systematically investigated in these studies.

Mathematics modules of the first semesters at German universities normally consist of lectures and related exercises, for several study programs. The lectures introduce mathematical theory, i.e., definitions, examples, theorems and their proofs are presented. The exercises are handed out weekly and have to be worked on by students in self-study. Students' solutions are submitted, corrected and discussed in a separate lesson. In order to pass such a module, usually a certain number of exercises (often 50 % of all exercises) has to be solved correctly and a written exam has to be passed.

This institutional setting predetermines some situation-outcome expectancies and values: If the exercises are not successfully completed and the written exam is not passed, the module will be failed. Hence it can be assumed that (1) students' situation-outcome expectancy is that they will fail these modules if no self-action or countermeasures are taken and that (2) the exercises and the written exam have a high extrinsic outcome value.

Research Questions

In the following, sources and consequences of frustration and helplessness in the academic environment just described shall be identified. For this purpose, the following research questions will be investigated:

RQ1: Which sources of frustrations with learning mathematics at university do students report? Or more precisely: In which contexts do students use words like “frustrating”, “helpless”, “despair”, or “depressing” in self-reports?

RQ2: Which consequences of frustrations with learning mathematics at university do students report?

METHODS AND DESIGN

The empirical basis for the results of the present study are problem-centered interviews on self-regulated learning (Göller, 2020) with a total of 21 students (14 of whom were female) at up to four interview times in their first year of study at university. Ten interviewees (9 female) were enrolled in the degree program for mathematics teachers at upper secondary level, seven (3 female) in the degree program Mathematics B.Sc., two (1 female) in the degree program Physics B.Sc., and two (1 female) in the degree program business education.

The respective interviews had a duration of about 45 minutes, were audio-recorded, and completely transcribed.

To investigate the questions listed above, these transcripts were searched for the word fragments “frust”, “hilfl” (German hilflos = helpless), “verzw” (German verzweifeln = (to) despair), and “depri” (German deprimiert/deprimierend = depressed/depressing), as operationalization of frustration. These words were not used by the interviewer, i.e., they were used by the interviewees on their own initiative. As context for identifying sources of frustration, the transcript segments (between two interview questions) that contained these words were analyzed using Grounded Theory coding methods. The Interview excerpts presented here were translated from German by the authors.

RESULTS

Of the 21 interviewed students, 17 used one of the words like “frustrating”, “depressing”, or “despair” at least once in at least one interview. Frustration is experienced in varying degrees of intensity and duration and can occur both situationally (as epistemic emotion) when working on a specific exercise

This exercise is just very frustrating, because you see: Fixed point, you recognize that, you’re happy and think, okay, then I can just take the lecture and apply it. And then you realize that it doesn’t work at all. That’s a mean exercise.

as well as in form of continuous, recurring frustration as achievement emotion that becomes a rather permanent condition:

It’s the same thing over and over again. You look at the exercise sheet, think you don’t understand anything. You’re depressed. You go for a drink. Still don’t understand anything. Saturday won’t be any better.

Table 1 lists the categories found with regard to control and value in the context of students’ reports of frustration as achievement emotion.

Code (N° codes/persons)	Example quotations
Low action-outcome expectancy understanding (38/12)	<p>... and then at some point you get frustrated, because it’s just stuff that you don’t understand anymore.</p> <p>It’s depressing when you’re about halfway somewhere in the middle of nowhere and you don’t know what to do.</p>
Low outcome control score (16/8)	<p>I expect to get it [the exercise sheet] back and then be depressed again.</p> <p>Exercise 2 was really frustrating. I really thought: “Cool. This is probably right. Definitely 1 and 2 is probably correct.” And then I only scored 30 percent on 1.</p>
Low intrinsic value (13/5)	<p>But in math, I have to say, it has frustrated me more than it has cheered me up. And a bit of the fun in mathematics is</p>

	already lost.
	... the fun factor is really nil at the moment.
Low extrinsic value profession (5/3)	I'd rather learn the content for school more intensively which I need, instead of learning so much background knowledge.
	I don't need it later in life [as a teacher].
(Too) high cost (22/9)	And just this inner voice that keeps saying: "Yes, you still have a whole math exercise sheet, which takes forever." That [...] somehow already completely destroys me psychically, knowing that I still have such a huge amount of work ahead of me that I somehow still have to do. So somehow that already makes me psychologically totally unstable, unhappy, I don't know. It frustrates me.
Social comparisons (12/7)	It's really frustrating sometimes when you talk to other people and they accomplish some exercises just totally easily.
	But of course, it's depressing when you see people who can do it easily.

Table 1: Codes and example quotations of sources of frustration. In parentheses, the number of units coded and persons who used statements assigned to these codes.

Accordingly, the most frequent source of student frustration is a low action-outcome expectancy in the sense that students try to solve mathematics exercises or understand lecture content but fail to do so. For some students, this is a rather permanent condition. However, in addition to this understanding-related outcome-control, frustrations resulting from low outcome-control of exercise scores are also reported.

In terms of values, a low intrinsic value in the form of not enjoying the engagement with the mathematical contents is reported in the context of frustrations. In addition, some student teachers also reported a low utility (extrinsic value) of university mathematics content with regard to school in the context of frustrations. As further sources of frustration, (too) high costs in terms of time, effort and also psychological cost, as well as social comparisons were also identified, which are regarded here as specific forms of values, or as influencing them (see discussion). In particular, the social context can both frustrate and inhibit frustration, depending on whether or not similar difficulties are attributed to others:

Well, because you're usually not the only one or the only one in math, who despairs, it is actually okay. So as long as you have some kind of contact with

your peers, it works. I mean, you're usually not alone. If an exercise is completely difficult, then there are at least five other people you know who can't do it either. And that's always a little comfort. But of course, it's depressing when you see people who can do it easily.

Consequences of frustrations (RQ2) can be assigned to these sources and may all be interpreted as attempts to avoid or reduce frustration. The most frequent consequences are less autonomous strategies (6 codes / 6 persons) and an adjustment of values (14/5). Less autonomous strategies, such as searching for exercise solutions on the internet or copying the solutions of others aim to reduce frustrations caused by low action-outcome expectancy or outcome control and may even be supported by exercise scores:

The second exercise sheet was the most depressing, because I did exercises 1, 2 and 4 myself. And I only had 30 to 50 percent everywhere. And the exercise I got from the internet or from someone else, of course I had 90 [percent], and then I thought to myself: Great! Well, hm.

An adjustment of values usually consists of devaluing activities that are perceived as frustrating (e.g., solving exercises autonomously, trying to understand lecture contents) or outcomes that are difficult to achieve (being good at mathematics, achieving good grades)

And then you say to yourself, okay, I want to be a teacher, I don't have to be such a superb mathematician.

and instead valorizing other activities (e.g., thinking along with the exercises, working on certain types of exercises autonomously) and outcomes (e.g., understanding exercise solutions of others, pass exam). The following quotation sums up all these considerations quite well:

I could kind of work more on the exercises or read the lecture notes. I do think it is helpful that you know roughly, where which definition is, but I don't know. Actually, I could manage that, but as I said, you are desperate after the two lectures where you sit there and take notes. If you then get out and then think, okay, let's sit down somewhere, to understand everything. I think that would just frustrate me for the whole semester that I don't understand it, that I am then completely in despair in the end. That's why, at the moment, I find that I could just do more thinking along with the exercises. I think that would be the most important thing. Because I really don't do that at the moment. Some exercises I just copy. Calculation exercises are fine, but some proofs or something like that I occasionally have the meaning explained to me and stuff like that. But that you really know what you're doing, like last semester, I don't know anymore.

DISCUSSION

Consistent with control-value theory, low subjectively perceived control over achievement outcomes was identified as the main source of students'

frustrations in the present study. In particular, low action-outcome expectancies (for understanding), but also low outcome control in terms of exercise scores were identified as problematic, but not action control. Obviously, this may relate to the fact that university students are confident to read lecture notes or books and perform outer actions to work on exercises (writing, looking things up, etc.). However, there is a high degree of uncertainty about what to do in order to achieve the desired outcomes (e.g., understand lecture content, develop correct exercise solutions autonomously, cf. Göller, 2020). Copying exercise solutions or searching for exercise solutions on the internet, and not re-reading lecture notes, which were reported here as consequences, can thus also be seen as strategies to avoid frustration by avoiding situations with low perceived control (such as working on the exercises autonomously).

The results also highlight the importance of values for the emergence of frustration and suggest a more detailed differentiation of values that includes costs and social comparisons (as done e.g. by Wigfield, Rosenzweig, & Eccles, 2017). It is not clear whether the low intrinsic and extrinsic values reported here should be considered a source or a consequence of frustration. On the one hand, frustration could emerge because students feel forced to engage with intrinsically or extrinsically low-valued mathematical content. On the other hand - and this is an advantage of Pekrun's (2006) approach over Wigfield et al.'s (2017) - due to the high extrinsic value of the exercises with simultaneous low outcome control, a devaluation of exercises or lecture contents can reduce the value of the exercises or lecture contents as a whole and thus also the frustration aroused. The consequences described here as adjustments of values also illustrate ways of valorizing or devaluing certain activities and outcomes in order to achieve a (subjectively) valuable and preferably frustration-free (because the valorized activities and outcomes are easier to control) participation in university mathematics.

Limitations and prospect

When interpreting the results, the institutional characteristics and the interview situation, in which students reported emotions and activities either retrospectively with knowledge about the outcomes or prospectively with regard to their outcome expectancies, must be taken into account. Different and possibly additional sources of frustration may occur in other environments. Due to the small qualitative sample and due to the rather superficial data analysis (keyword search and no systematic analysis of the complete material), the categories presented here are not necessarily exhaustive. In particular, the numbers given here for units coded and persons are only lower limits, since interviewees may refer to these codes in other parts of the interviews which were not analyzed for this study. However, the categories are suitable for qualitatively illustrating and further differentiating all sources of frustration described in Pekrun's (2006) control value theory. Overall, the approach taken

here for a systematic consideration of emotions provides relevant perspectives and new explanations for phenomena that are known from practice and research (e.g., copying, devaluing mathematical contents) but are not yet sufficiently understood. Further research is desirable at this point.

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PROSPECTIVE PRIMARY SCHOOL TEACHERS' AND SECONDARY SCHOOL MATH TEACHERS' MATHEMATICS KNOWLEDGE AND BELIEFS

David Maximiliano Gómez¹ and María Victoria Martínez Videla¹

¹Institute of Educational Sciences (ICEd), Universidad de O'Higgins, Rancagua, Chile

Teachers' mathematical knowledge and beliefs are important components of teachers' professional knowledge. In this work, we compared the primary school mathematical knowledge and beliefs of prospective primary school teachers and prospective secondary school mathematics teachers at the beginning of their first year of undergraduate studies. The school mathematical knowledge of prospective secondary math teachers was in general higher than that of prospective primary teachers, particularly in Algebra. It was also higher across the three TIMSS cognitive domains (Knowing, Applying, Reasoning). Prospective teachers' beliefs about the teaching and learning of mathematics were more similar across programs than beliefs about the nature of mathematics and about one's self-concept as learner of mathematics.

INTRODUCTION

Different models have established the domains of knowledge and practice that an individual should develop to become a mathematics teacher (Shulman, 1987; Ball, Thames, & Phelps, 2008). Other authors have enriched these models of knowledge by incorporating the individual's belief system (e.g. Beswick, Callingham, & Watson, 2012; Carrillo, Contreras, & Flores, 2013).

There are many studies aiming at measuring prospective teachers' and in-service teachers' different types of professional knowledge: content knowledge, pedagogical content knowledge, and/or beliefs (e.g. Beswick & Goos, 2012; Blömeke, Suhl, & Kaiser, 2011; Tatto & Senk, 2011). These studies, however, show great variability in the populations considered. For instance, Depaepe et al. (2015) contrasted mathematics knowledge in the domain of rational numbers between prospective primary school teachers (PST) and secondary school mathematics teachers (SSMT), showing that SSMT students have higher content knowledge than PST students. However, it should be noted that in some countries PST and SSMT programs are not directly comparable, as the former tend to be undergraduate programs or programs not offered by universities and therefore not leading to an academic degree, whereas the latter may take the form of undergraduate or graduate programs.

The Chilean Context

In Chile, legal regulations allow only universities to offer teacher training programs of any educational level and modality (early childhood education, primary school, special education, secondary school). Moreover, recent legal changes require these programs to follow specific guidelines such as the application of diagnostic assessments to all enrolled students at the beginning of the training programs, and the design and implementation of formative actions based on the results.

Martínez Videla et al. (2019) developed an assessment of mathematics content knowledge and beliefs for students enrolled in primary school teacher training programs. The knowledge section of the instrument focuses on school mathematical knowledge, that is to say mathematical content knowledge specific to the primary school levels, as these contents are those that the prospective teachers are expected to teach once working in the school system.

School Mathematical Knowledge and Beliefs

School Mathematical Knowledge (SMK) is considered as the contents and skills that a school curriculum defines in order to foster greater capacities in a country's citizens, such as the abilities to think abstractly and systematically, to experiment and learn to learn, to communicate and work collaboratively, to solve problems, to handle uncertainty, and to adapt to change (Kerr, 2002). This definition of SMK includes not only mathematical content, but also skills related to mathematical activities. Martínez Videla et al.'s (2019) instrument conceptualized mathematical skills using the TIMSS framework (Grønmo, Lindquist, Arora, & Mullis, 2013), which appears to be more consistent with instruments that seek to determine how much a person knows according to a prescribed curriculum.

The second element, beliefs, is understood not only as a verbalization of what is believed, but also as the willingness to act in a certain way (Wilson & Cooney, 2002). It is also considered that beliefs do not operate independently, but rather as a belief system that may be understood as a metaphor to represent a possible structure of the beliefs of an individual, considering them as understandings and premises about the world, perceived as true by who sustains them, that imply personal, cognitive, and affective codes and that predispose people towards certain forms of action (Lebrija, Flores, & Trejos, 2010; Lester, Garofalo, & Kroll, 1989). There are different ways of categorizing beliefs about mathematics education, describing different aspects of the mathematical activities and interactions that take place in the classroom. Martínez Videla et al. (2019) chose to use the following categories, based on the proposal by Op't Eynde, De Corte, and Verschaffel (2002): Beliefs about the nature of mathematics, Beliefs about the process of teaching and learning of mathematics, and Beliefs about one's self-concept.

The Present Study

At the beginning of the academic year 2019, our Institution applied the aforementioned instrument to all first-year students of its six undergraduate teacher training programs. This report focuses on the data collected in the primary school teacher (PST) and the secondary school mathematics teacher (SSMT) training programs, contrasting their mathematics knowledge and beliefs across programs. As a starting hypothesis, we expected SSMT students to obtain higher knowledge scores and to exhibit more positive appreciations of mathematics and of themselves as mathematics learners than PST students. The specific research questions that guided our analyses were the following: [RQ1] Do SSMT students obtain higher scores than PST students across all content and cognitive domains of mathematics knowledge, or this difference varies across domains? [RQ2] In what aspects do the mathematics beliefs of SSMT and PST differ the least and the most? [RQ3] Are these answers affected by gender composition differences between programs?

METHODS

Participants

The assessment was applied to all first-year students of the six undergraduate teacher training programs of Universidad de O'Higgins, before the beginning of the academic year. In this report, we analyzed the data from 47 students of the primary school teacher (PST) program [40 women, 7 men] and 41 students of the secondary school mathematics teacher (SSMT) program [17 women, 24 men]. Data from additional 5 students (2 from PST and 3 from SSMT) were excluded because they omitted more than 25% of the mathematical knowledge items. All students gave written consent to use their data for research purposes.

Instrument

We applied the Mathematics Knowledge and Beliefs Instrument developed by Martínez Videla et al. (2019). The knowledge section contains 40 multiple choice items organized into the five content categories of the Chilean primary education mathematics curriculum—Number, Geometry, Measurement, Data and Chance, and Patterns and Algebra—(MINEDUC, 2012) as well as the three cognitive domains of the TIMSS 2015 Mathematics Framework—Knowing, Applying, and Reasoning—(Grønmo et al., 2013). The beliefs section contains Likert scale items in which students indicate their degree of agreement with 47 statements about teaching and learning, about their self-concept as learners of mathematics, and about the nature of mathematics. The Likert scales had 4 levels: 1-strongly disagree, 2-somewhat disagree, 3-somewhat agree, 4-strongly agree.

Data Analysis

We analyzed students' knowledge by computing percentages of correct answers with respect to non-omitted items, first considering the full instrument and later separating items according to their content and cognitive domains. Students' beliefs were compared across programs by looking at the difference between the agreement scores indicated by students in each program for each statement.

RESULTS

Mathematics Knowledge

Figure 1 shows the distributions of overall scores in the mathematics knowledge section. Pre-service secondary math teachers showed significantly better overall knowledge scores than pre-service primary teachers (74% vs. 61% correct). Table 1 presents the results by content domains. Geometry was the domain with the highest scores, whereas Data exhibited the lowest ones. A direct comparison between PST and SSMT showed that, although the SSMT students obtained higher scores across all categories, the magnitude of the score differences was the smallest for geometry items and the largest for algebra items.

Beliefs

We analyzed prospective teachers' beliefs by contrasting their degrees of agreement to the presented statements across the two programs. Table 2 presents the statements that showed the smallest and largest magnitude differences in agreement between PST and SSMT. Interestingly, the statements showing the smallest differences between programs were all related to teaching and learning, whereas four out of the five items with the largest differences were about students' self-concept as mathematics learners and one was about mathematics.

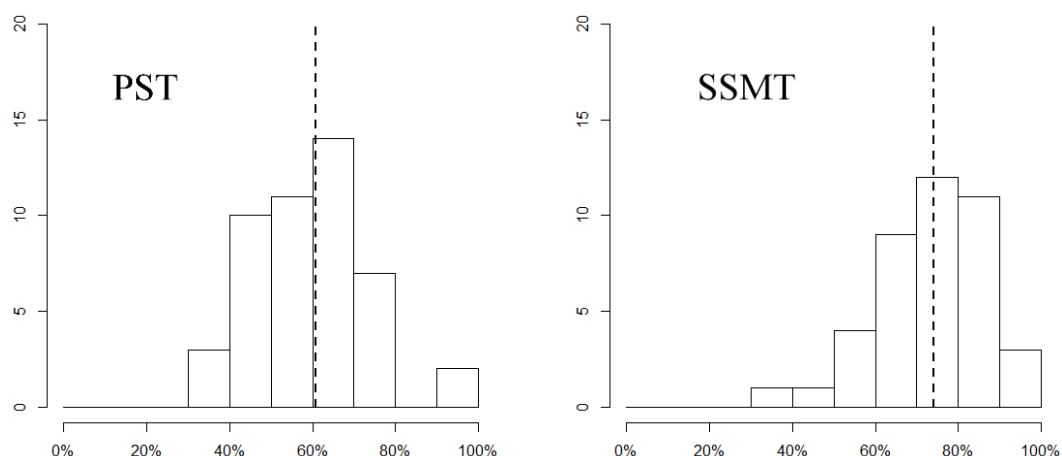


Figure 1: Histograms of overall mathematics knowledge scores.
Vertical dashed lines depict each group's average score.

		PST	SSMT	Overall	Difference
Content domains	Number	61%	73%	67%	12%
	Geometry	79%	83%	81%	4%
	Measurement	64%	79%	71%	15%
	Data	44%	60%	51%	16%
	Algebra	47%	73%	60%	26%
Cognitive domains	Knowing	63%	77%	69%	14%
	Applying	63%	75%	68%	12%
	Reasoning	49%	63%	55%	14%

Table 1. Knowledge scores by mathematics content and cognitive domains

	PST	SSMT	Diff.
Good math teachers are creative. ^{TL}	3.37	3.38	-0.01
Math teachers must know what contents their students know, to build their lessons from that information. ^{TL}	3.83	3.82	0.01
In a good math lesson, the teacher constantly asks their students to reflect about the newly acquired knowledge. ^{TL}	3.84	3.82	0.02
Good math teachers must propose clear and simple problems. ^{TL}	2.74	2.77	-0.03
Math teachers must adapt to their students' needs and work from the abilities of each of them. ^{TL}	3.76	3.73	0.03
Math is interesting for me. ^{SL}	2.87	3.91	-1.04
Math is mechanical and boring. ^{MA}	2.28	1.23	1.05
Learning math is difficult for me. ^{SL}	3.11	1.93	1.18
I struggle to understand math. ^{SL}	3.20	1.87	1.33
I enjoy doing math. ^{SL}	2.46	3.80	-1.34

Table 2: Statements that show the smallest (top) and largest (bottom) agreement differences between students in the PST and SSMT programs. Statement categories: TL: Teaching and learning, SL: Self-concept and learning, MA: Mathematics.

Gender Perspective

It is possible that our results are affected by gender differences, given the large difference in gender composition between both programs: 85% of PST students were women, as opposed to 41% of SSMT students. To explore this issue, we repeated our previous analyses considering only female students (40 in PST, 17 in SSMT). Given the smaller sample sizes in the analyses of this section, however, our results should be considered as preliminary and investigated in more depth in the future.

Looking at female students across both programs, we observed a similar pattern of results to that of the full sample (Table 1) across content domains and cognitive domains with one exception: the gap between programs in Reasoning grew to 23% mostly driven by a higher Reasoning score of 70% of women in SSMT. As for beliefs, the five statements with the largest difference across programs remain related to self-concept and mathematics, whereas only three of the five statements with the smallest differences across programs remain in the teaching and learning category. The two statements eliciting the most similar agreement scores were in this case about math (“mathematics establishes a single path to solve a given problem”) and about self-concept (“only the most capable math students can solve problems requiring multiple steps”), with students in both programs highly disagreeing with both statements (ratings of 1.83-1.82 and of 1.53-1.54 for PST and SSMT students, respectively).

DISCUSSION

We have presented the results of a mathematics knowledge and beliefs diagnostic assessment applied to prospective primary school and secondary school math teachers at the beginning of their undergraduate studies.

The knowledge section of the test focused on contents of the Chilean primary school curriculum, ensuring that the level of difficulty of the mathematics involved was appropriate for the students of both programs. Answering RQ1, we observed systematic differences in knowledge scores in favor of SSMT students with variations across content domains but not across cognitive domains. In terms of contents, a large difference was expected to emerge in algebraic items because algebra is a traditionally difficult domain for the general student population (Stacey & Chick, 2004). On the other hand, the content domain with the smallest score difference between PST and SSMT was geometry, which could reflect a less marked focus on geometry learning objectives in secondary school. Differences between both programs in scores across cognitive domains were quite similar.

Regarding RQ2, it was foreseeable that PST and SSMT students would differ importantly in their beliefs about mathematics and about themselves as learners of mathematics, but it was surprising that both groups largely agreed on how a good math teacher/lesson looks like.

We also explored gender differences (RQ3), observing that mathematics content knowledge scores were very similar in the full sample and the women subsample, with the exception of an increased Reasoning score exhibited by female students in the SSMT program that deserves to be further investigated.

It is relevant to note that results of our assessment are not directly comparable to some large-scale studies such as the Teacher Education and Development Study in Mathematics (TEDS-M), because this one focuses on the knowledge of PST students at the end of their program whereas the present research focuses at the beginning. Our results show that PST and SSMT students differ importantly in their mathematical knowledge and beliefs already at the beginning of their training, meaning that these differences are unlikely to be directly driven by the training, but rather indirectly through students' program selection preferences.

Further comparative research between PST and SSMT programs may have a relevant impact in the quality of school mathematics education. A better understanding of the initial state of mathematics knowledge and beliefs of PST and SSMT students is essential for institutions to design and implement their training programs, and a more comprehensive focus on PST and SSMT programs can also contribute in facilitating school students' transition process from primary to secondary school.

Acknowledgements

The authors thank the School of Education of Universidad de O'Higgins and its students for their support for this research.

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FROM DISCRETENESS TO INFINITY: STAGES IN STUDENTS' UNDERSTANDING OF THE RATIONAL NUMBER DENSITY

Juan Manuel González-Forte¹, Ceneida Fernández¹, Jo Van Hoof², & Wim Van Dooren²

¹University of Alicante (Spain)

²KU Leuven, Centre for Instructional Psychology and Technology, Belgium

This research investigated students' intermediate stages in understanding the dense structure of rational numbers. A cross-sectional study with 953 students from 5th to 10th grade was performed. After an inductive analysis coding the open answers to a question about how many numbers there are between two given rational numbers, a TwoStep Cluster Analysis was carried out revealing different student reasoning profiles. Results showed that the most naïve natural number bias did not disappear at the end of secondary school. Moreover, different intermediate stages in the understanding of density were found along grades. A characteristic of these stages is that the understanding of infinity was reached in decimal numbers earlier than in fractions.

THEORETICAL AND EMPIRICAL BACKGROUND

Recent research has focused on natural number knowledge interference as one of the main explanations of students' difficulties in understanding rational numbers - a phenomenon called *natural number bias* (Smith, Salomon, & Carey, 2005; Van Hoof, Verschaffel, & Van Dooren, 2015). This phenomenon has been studied in three domains: Rational numbers size, arithmetic operations, and density (Gómez & Dartnell, 2018; McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015).

In the present research, researchers focus on the domain of density. This has been considered the most difficult and natural number biased domain (McMullen et al., 2015; Smith et al., 2005). However, studies focusing on individual differences over age in this domain are scarce. The natural number set is discrete since between two numbers there is a finite (possibly zero) number of numbers (e.g., only the number 4 is between the numbers 3 and 5). However, the rational number set is dense since there is an infinite number of numbers between any two rational numbers (Smith et al., 2005).

The idea of discreteness, developed through experience with natural numbers is considered by Vamvakoussi and Vosniadou (2004) as a “fundamental presupposition which constrains students' understanding of the structure of the

set of rational numbers” (p. 457) and causing numerous conceptual difficulties in primary and secondary school students (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004) and even in undergraduates (Tirosh, Fischbein, Graeber, & Wilson, 1999). Students believe that between two rational numbers there are no other numbers or there is a finite number of numbers. For instance, students believe that between the “pseudo-consecutive” fractions $\frac{5}{7}$ and $\frac{6}{7}$ there are no numbers, or that between $\frac{1}{2}$ and $\frac{1}{4}$ is only the number $\frac{1}{3}$ (Merenluoto & Lehtinen, 2004; Tirosh et al., 1999). In decimal numbers, students think that between the “pseudo-consecutive” decimals 0.59 and 0.60, it is not possible to find other numbers, or that between 1.22 and 1.24 is only the number 1.23 (Moss & Case, 1999).

Rational numbers can be represented as both fractions or decimals numbers (e.g., $\frac{3}{4}$ and 0.75 are alternative representations of the same rational number) (Carpenter, Fennema, & Romberg, 1993). Previous research has shown that students sometimes treat fractions and decimals numbers as unrelated sets of numbers, rather than as interchangeable representations of the same number (Khouri & Zazkis, 1994). Furthermore, regarding the different representations, previous research has obtained opposite results. In some studies, students were better able to explain the dense nature of decimals numbers than the dense nature of fractions (McMullen & Van Hoof, 2019; Tirosh et al., 1999; Vamvakoussi & Vosniadou, 2010). Other studies (e.g., Vamvakoussi & Vosniadou, 2004) found an opposite result. Moreover, some students tend to believe that there are only decimals numbers between two decimals numbers and fractions between two fractions (Vamvakoussi & Vosniadou, 2010).

Vamvakoussi and Vosniadou (2004; 2010) found that understanding the density of rational numbers is not an all or nothing issue: They identified some intermediate stages in the understanding of density in secondary school students. They described several expected students’ answers patterns (hypothesised profiles) and then, with interviews (Vamvakoussi & Vosniadou, 2004) or a test (Vamvakoussi & Vosniadou, 2010), they found examples of students’ answers for these hypothesized profiles. The profiles were: Students who considered that there is a finite number of numbers between two pseudo-consecutive rational numbers; students who thought that decimals are dense, whereas fractions are discrete, and *vice versa*; students who were reluctant to accept that there may be decimals between two fractions, and *vice versa*; and finally, students who correctly considered that there is an infinite number of numbers between any two numbers regardless of their symbolic representation. However, no, or very few students could be fit in some of these profiles. Therefore, these profiles could not be representative. Furthermore, they obtained other students’ answer patterns that differed from the profiles hypothesized.

Researchers extend previous research by performing a cross-sectional research with a large sample of primary and secondary school students (from 5th to 10th grade) and by determining profiles after an inductive analysis of students' answers to an open question about how many numbers there are between two given rational numbers. Therefore, the aim of this research is to identify and characterise intermediate stages in primary and secondary school students' understanding of density. Furthermore, researchers examine the evolution of these stages over a large age range, from primary to secondary education.

METHOD

Participants were 953 Spanish primary and secondary school students from 5th grade ($n = 115$), 6th grade ($n = 139$), 7th grade ($n = 162$), 8th grade ($n = 173$), 9th grade ($n = 174$), and 10th grade ($n = 190$). There was approximately the same number of boys and girls in each age group. The participating schools were five primary schools and five secondary schools, and students were from mixed socio-economic backgrounds.

To design the instrument, researchers adapted the density items of the Rational Number Sense Test (RNST), developed and validated by Van Hoof et al. (2015). It is a paper-and-pencil test that contains six density items in which students had to answer how many numbers there are between two fractions or two decimal numbers given. There are three fraction items: $2/5$ and $3/5$ (pseudo-consecutive fractions); $2/5$ and $4/5$ (non-pseudo-consecutive fractions with the same denominator); $5/9$ and $5/6$ (non-pseudo-consecutive fractions with the same numerator). There are three decimal items: 1.42 and 1.43 (pseudo-consecutive decimals); 1.9 and 1.40 (non-pseudo-consecutive decimals); 2.3 and 2.6 (non-pseudo-consecutive decimals). Students were asked individually to solve the test during a mathematics lesson at school. The items were presented in random order in eight different versions. No time limit was used, as a time limitation could encourage natural number biased reasoning.

Four researchers inductively analyzed the students' answers to identify categories according to the nature of the answer. Seven categories were identified: i) *Infinite*: Students who answered that there is an infinite number of numbers between the two given ones; ii) *Difference*: Students who calculated and reported the difference between the two numbers given (e.g., 0.3 is between 2.3 and 2.6); iii) *Naïve consecutive*: Students who answered that there is no other number between two pseudo-consecutive numbers (e.g., between 1.42 and 1.43 or between $2/5$ and $3/5$, there are no numbers) and between two non-pseudo-consecutive numbers they gave a finite list of consecutive numbers (e.g., the numbers 2.4 and 2.5 are between 2.3 and 2.6 or $3/5$ is between $2/5$ and $4/5$) or the number of numbers of this list (e.g., there are 2 numbers between 2.3. and 2.6 or there is 1 number between $2/5$ and $4/5$); iv) *Finite consecutive*: Students who gave a finite list of consecutive numbers between the numbers

after adding a decimal and then counting on in decimal numbers (e.g., the numbers 1.421, 1.422, 1.423..., 1.429 are between 1.42 and 1.43) or after adding a decimal in the numerator in fractions (e.g., the numbers $2.1/5$, $2.2/5$, $2.3/5$..., $2.9/5$ are between $2/5$ and $3/5$) or they gave the corresponding number of numbers of these lists (e.g., there are 9 numbers between 1.42 and 1.43 or there are 9 numbers between $2/5$ and $3/5$); v) *Finite*: Students who gave other specific numbers included between the numbers given; vi) *Rest*: Students who gave specific numbers not included between the numbers given; vii) *Blank answers*.

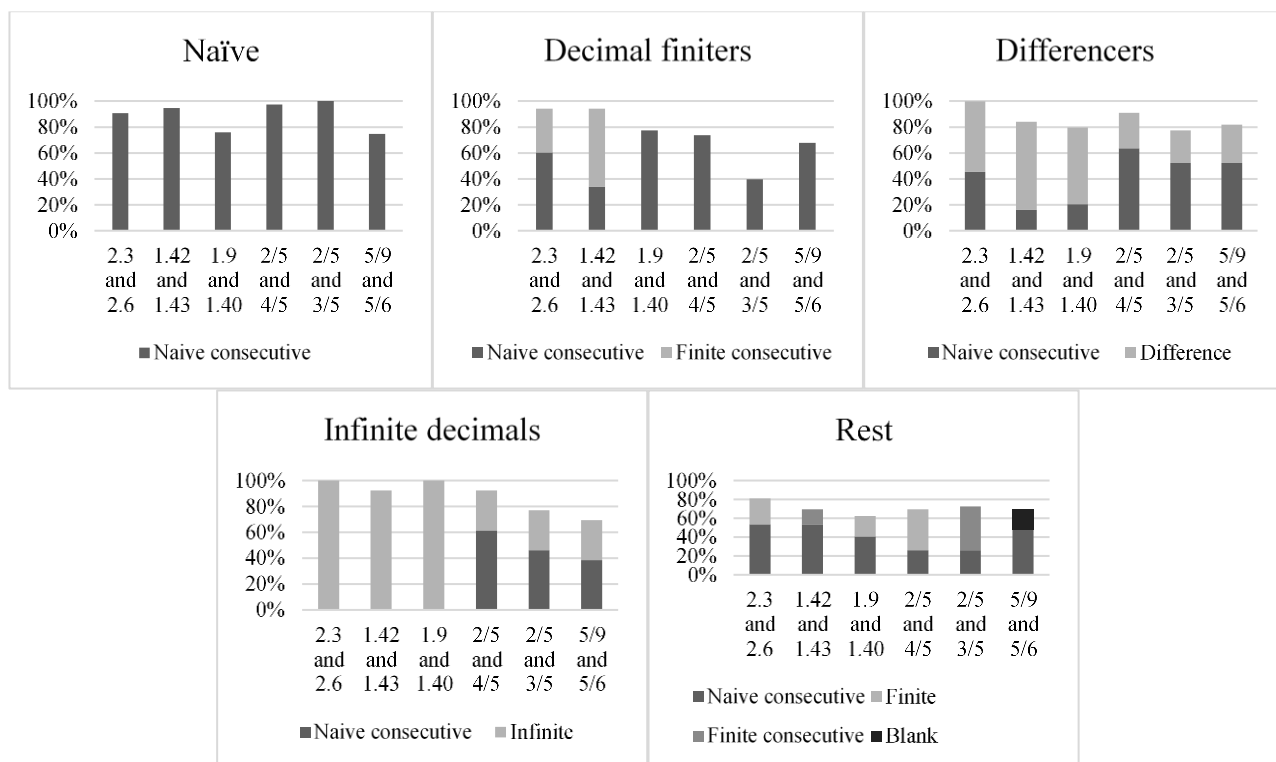
With these categories, a TwoStep Cluster Analysis with categorical data was performed to identify groups of students (profiles) with qualitatively similar answers patterns. Given the complexity of our coding scheme, many intermediate states of understanding could be expected. Therefore, we analyzed data separately for age groups, obtaining students' profiles in 5th and 6th grade, in 7th and 8th grade, and in 9th and 10th grade. The statistical software used was SPSS version 25.

RESULTS

In this section, firstly, researchers determine the number of profiles and describe them. Secondly, we show the evolution of these profiles from 5th to 10th grade.

Determining and Describing the Profiles

The number of profiles was determined according to a low BIC and from an interpretative viewpoint. In 5th and 6th grade, researchers chose the five students' profiles solution. Figure 1 shows the characteristics of the profiles identified in 5th and 6th grade. The X-axis consists of the six test items, and the Y-axis consists of the percentages of frequency of the largest group(s) (categories) identified in the inductive analysis.

Figure 1: Characteristics of students' profiles in 5th and 6th grade

- *Naïve*: Students who considered that there is no other number between two pseudo-consecutive numbers, and there is a finite number of numbers between two non-pseudo-consecutive numbers.
- *Decimal finiters*: Students who started to consider that there is a finite number of numbers between two pseudo and non-pseudo-consecutive decimals (there is a subgroup of students that still considered that between two pseudo-consecutive decimals there is no other number). However, they considered that there is no other number between two pseudo-consecutive fractions.
- *Differencers*: Students who calculated the difference between two decimals but considered that there is no other number between two pseudo-consecutive fractions, and there is a finite number of numbers between two non-pseudo-consecutive fractions. Although a subgroup of students also calculated the difference in fractions.
- *Infinite decimals*: Students who considered that there is an infinite number of numbers between two decimals, but there is no other number between two pseudo-consecutive fractions, and a finite number of numbers between two non-pseudo-consecutive fractions. However, there is a subgroup of students who started recognizing that there is an infinite number of numbers between fractions.
- *Rest*: Students with a low performance in general who solved the items without any recognizable pattern.

In 7th and 8th grade, we chose the six students' profiles solution. Figure 2 shows the characteristics of each profile identified.

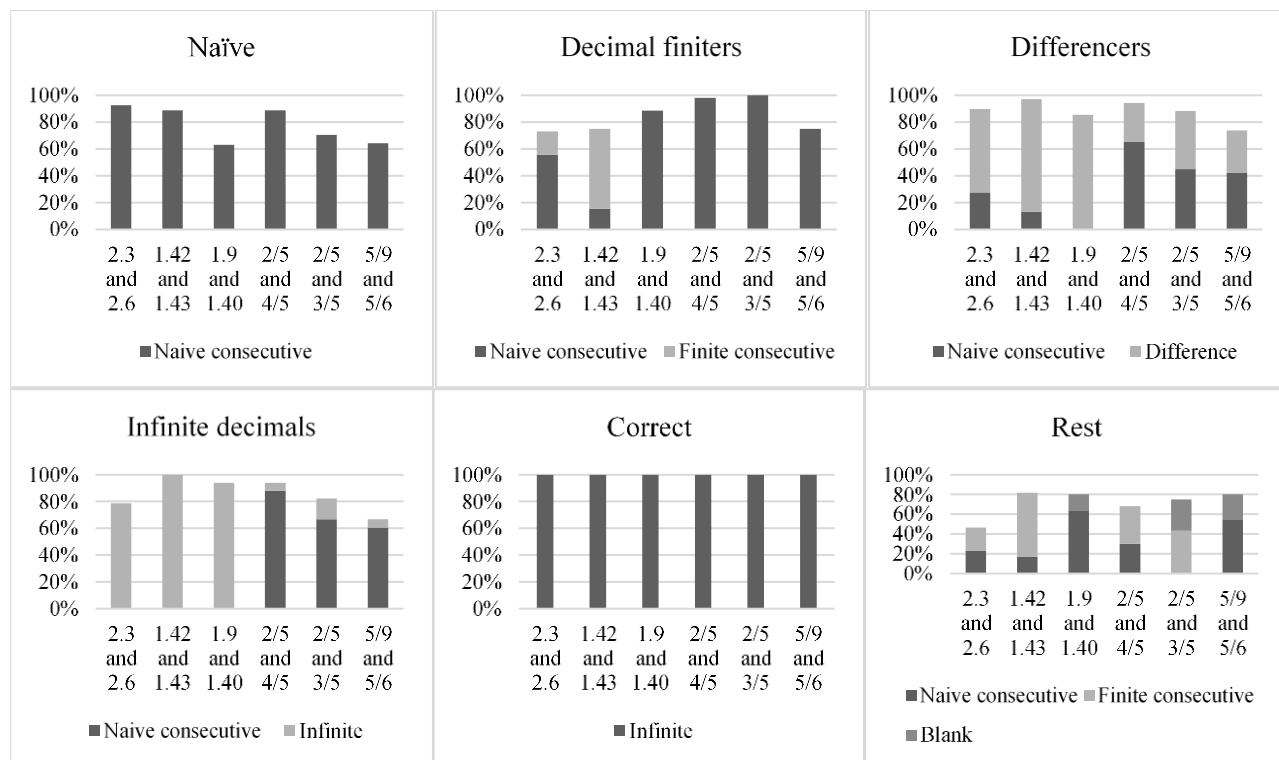


Figure 2: Characteristics of students' profiles in 7th and 8th grade

In these grades, the same profiles than in 5th and 6th grade were identified and we identified a new one:

- *Correct* profile: Students who considered that there is an infinite number of numbers between two fractions and two decimals.

In 9th and 10th grade, we chose the 6 students' profiles solution. Figure 3 shows the characteristics of each profile identified.

Figure 3: Characteristics of students' profiles in 9th and 10th grade

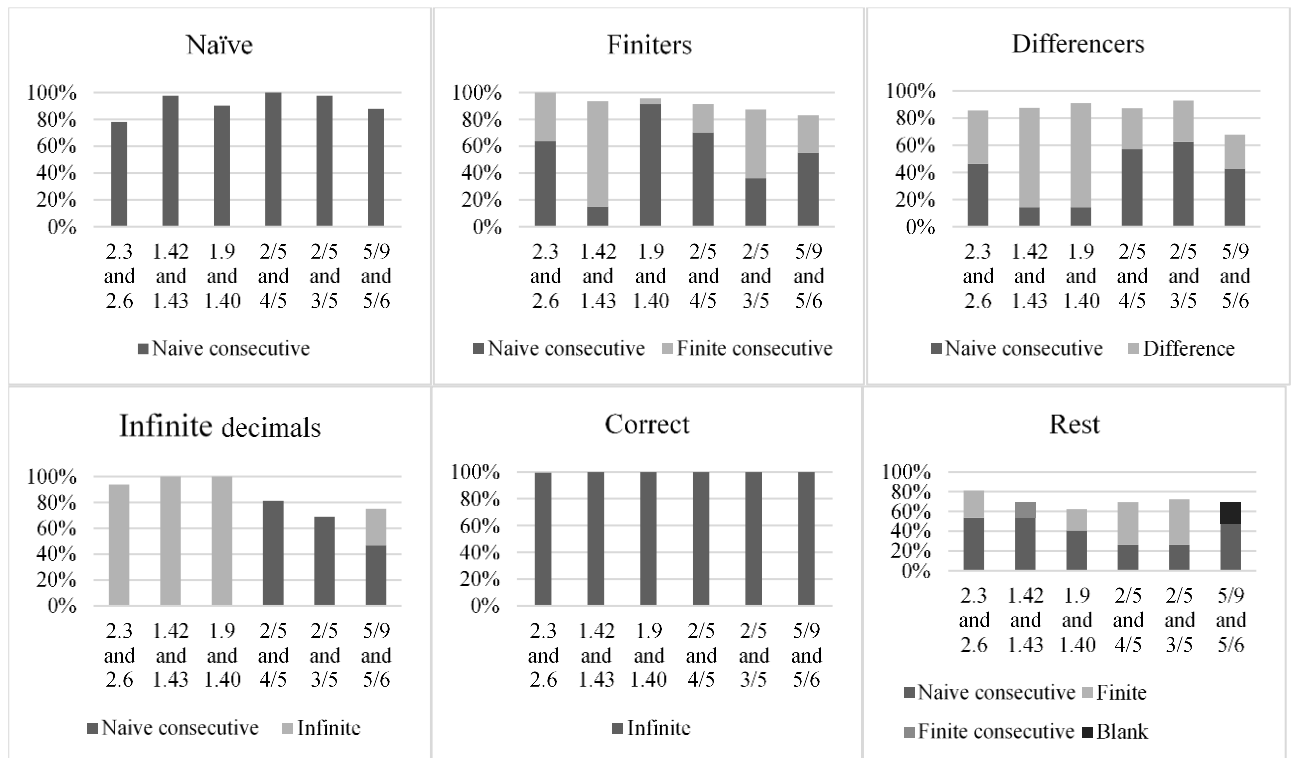


Figure 3: Characteristics of students' profiles in 9th and 10th grade

In these grades, the *Decimal finiters* profile was not identified, but researchers identified a new profile:

- *Finiters* profile: Students who started to consider that there is a finite number of numbers between two pseudo and non-pseudo-consecutive decimals and fractions.

Evolution of the Profiles

Figure 4 shows the evolution of each profile from 5th to 10th grade. The *Naïve* profile decreased as the grades advanced (29.50% in 5th and 6th grade, and 11.30% in 9th and 10th grade). However, this result indicates that the most naïve natural number bias seems not to disappear in the last grades of the secondary school, neither in fractions nor in decimal numbers.

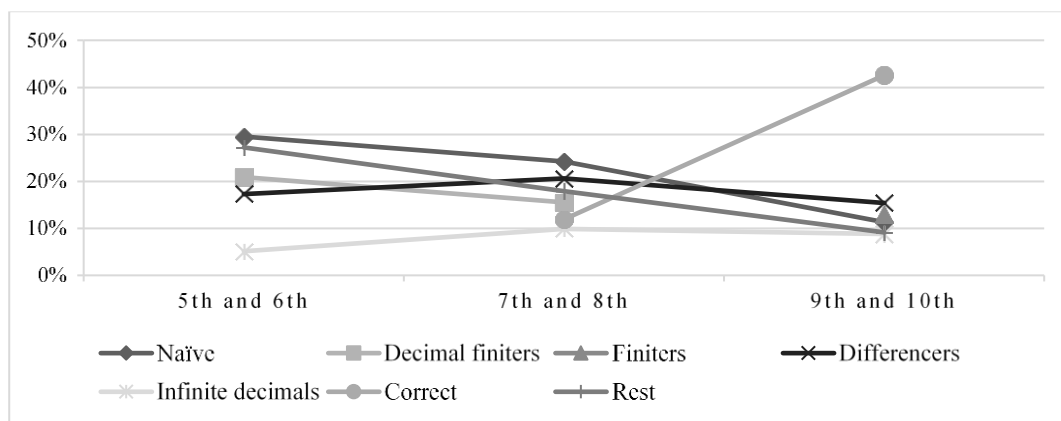


Figure 4: Evolution of the profiles from 5th to 10th grade

The *Decimal finiters* profile also decreased along grades, disappearing in 9th and 10th grade, where it got replaced by a *Finiters* profile. This result seems to show that students started to believe that there is a finite number of numbers between two decimals and then between two fractions.

The decrease of the *Naïve* and *Decimal Finiters* profiles corresponded to an increase of the *Correct* profile (0.0% in 5th and 6th grade, and 42.60% in 9th and 10th grade) and of the *Infinite decimals* profile (5.10% in 5th and 6th grade and 8.80% in 9th and 10th grade). This result shows that density is first understood with decimal numbers and later with fractions. Moreover, decimal infiniteness was reached even in some primary school students. Finally, the *Differencers* profile remained stable along grades.

DISCUSSION AND CONCLUSIONS

The aim of this research was to identify and characterize intermediate stages in students' density understanding and to examine the evolution of these stages from primary to secondary education. Through an inductive and a cluster analysis, different profiles were identified showing different stages in students' density understanding.

The clearest natural number bias, denoted as *Naïve* profile, was higher in 5th and 6th grade and decreased along grades, but it did not disappear towards the end of the secondary school (Vamvakoussi & Vosniadou, 2010). The following stage of discreteness corresponds to the *Decimal finiters* profile (it was identified from 5th to 8th grade). These students had overcome the naïve discreteness in decimal numbers. However, this profile was not identified in 9th and 10th grade, where the *Finiters* profile appeared. These last students showed to have overcome naïve discreteness both in fractions and decimal numbers. The *Differencers* profile evidenced a group of students who determined the number of numbers between the two given by subtracting both numbers.

The transition from discreteness to infiniteness in decimal numbers was shown by the presence of the *Infinite decimals* profile. In this profile, students considered that there is an infinite number of numbers between two pseudo and non-pseudo-consecutive decimals numbers. However, students of this profile were still reluctant to recognize the infiniteness in fractions. The last stage was reached by the *Correct* profile –not identified in 5th and 6th grade- and showed an understanding of the density concept both in fractions and decimal numbers. However, at the end of secondary school, still less than half of the students were in this profile.

Further research could focus on longitudinal designs to examine how learners' individual understanding of rational number density progresses over time. This could clarify possible transitions between profiles.

Acknowledgments

This research has been supported by Conselleria d'Educació, Investigació, Cultura i Esport (Generalitat Valenciana, Spain) (PROMETEO/2017/135) and by the University of Alicante (UAFPU2018-035).

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TEACHERS' PRACTICES AND RESOURCE USE IN TEACHING DERIVATIVES. A STUDY OF THE AFRICAN CONTEXT: THE CASE OF CAMEROON

Alejandro S. González-Martín¹ and Casimir Jojo Nseanpa¹

Département de didactique, Université de Montréal, Canada

Although there is abundant research literature on the difficulties, students face in learning derivatives, research on teaching practices is lacking. This paper proposes a study of teachers' practices and use of resources in teaching derivatives to better identify the teachers' decisions and their justifications. Our study focuses on Cameroon, a country with strong institutional constraints (a single textbook and a national examination). Our study of three teachers reveals that these constraints have a very strong influence on their activity, including their approach to teaching and their use of resources.

INTRODUCTION

Derivatives are one of the most important topics studied in high school (in many countries) and in postsecondary mathematics programs. For many students, this topic is a prerequisite to university studies and a gateway to other mathematical topics in various fields. Research in mathematics education has already reported many difficulties linked to the learning of derivatives (e.g., Hitt & González-Martín, 2016) and of its different aspects (e.g., Zandieh, 2000). This has led to experiments in attempting to improve the learning of derivatives (e.g., Giraldo, Tall, & Carvalho, 2003). While a number of studies have examined the learning of derivatives and have proposed interventions to facilitate this learning, the number of studies analyzing teaching practices, or how derivatives are presented in textbooks and other resources used by teachers, remains quite low.

One recent study on the teaching of derivatives is by Park (2015; 2016), who analyzed three teachers' approaches to defining the derivative at a point using limits and then transitioning to the derivative of a function on an interval (Park, 2015). Park (2016) also examined how derivatives are introduced in three manuals widely used in the United States. Park's work shows that both the teachers and the textbooks use symbolic notations and graphic illustrations without making explicit links between them. Moreover, the teachers used secant lines, tangents, and symbolic notation to explain the derivative at a point without making the links between these explicit. They also used the symbolic notation of the derivative at a point to shift to the derivative over an interval by simply changing the coordinates of the point by the variable. Moreover, the

teachers presented the properties of the derived function with only a few explanatory examples.

We emphasize again that there is not an abundance of literature on teaching practices related to derivatives, or on the introduction of the notion of derivative in resources used by teachers. Topics ripe for exploration include teachers' perspectives on their students' prior knowledge; teachers' visions of the important topics to cover when teaching derivatives (Park, 2015); and the analysis of resources other than textbooks. Numerous studies have highlighted the way in which teachers at different levels make use of resources in their teaching (e.g., González-Martín, Nardi, & Biza, 2018; Gueudet, 2017), as well as the constraints and opportunities provided by these resources (Gueudet & Trouche, 2009). National examinations also have a major impact on teachers' practices, influencing content and course planning (e.g., Rozenwajn & Dumay, 2014).

The research described in this paper seeks to contribute to the scant existing literature on teaching practices used to introduce derivatives. We seek to study the similarity of teachers' practices and the way derivatives are presented in teaching resources, while also considering the various constraints that may hinder teachers' work. In addition, we note that the existing literature on derivatives (which mostly focuses on how students learn them) and on mathematics teachers' practices and use of resources primarily concerns studies conducted in Europe and North America, with very little mathematics education literature reporting on studies conducted in developing countries. To help bridge this gap, our study focuses on the African context, and more specifically Cameroon, a context with which this paper's second author is very familiar. We hope this study may help to identify issues that may not always be present in developed countries, but that may have a significant impact on the teaching and learning of mathematics.

THEORETICAL FRAMEWORK

Since we are interested in teachers' practices and their use of resources in teaching derivatives, we have applied elements of the anthropological theory of the didactic (ATD – Chevallard, 1999) and of the documentational approach (DA – Gueudet & Trouche, 2009), following the work of González-Martín *et al.* (2018).

In analyzing practices, ATD proposes the useful tool of *praxeology*, which is composed of four elements: a task (or type of task) to solve, techniques used to carry out the tasks, technologies (or rationales) that justify and explain the techniques, and a theory that justifies the technologies. Chevallard (1999) distinguishes *didactic praxeologies* to describe the act of teaching.

ATD also suggests that teaching institutions, through their official documents and guidelines, establish an *institutional relationship* to the content being taught

and learned; in other words, institutions influence what individuals in a given *position* (e.g., teacher or student) can do, and how they relate to the content in question. Individuals who have belonged (or who belong simultaneously) to different institutions have their own *personal relationship* to this content. For instance, Bronner (1997) showed that some secondary teachers in France have ideas concerning irrational and real numbers that are not reflected in France's official national education program. Faced with this situation, some teachers restrict their teaching to the program's requirements (thus their teaching conforms to the institutional relationship with real numbers), whereas others supplement their teaching with additional details in the hope that their students will better grasp the content's subtleties (thereby making their personal relationship include in their teaching items not anticipated by the institutional relationship).

Finally, DA acknowledges that teachers use a variety of resources when preparing to teach. These can be either physical (the official program, textbooks, etc.) or intangible (a discussion with a colleague, their own training, etc.). The various resources used to teach content, together with these resources' schemes of use, are termed a *document* (Gueudet & Trouche, 2009). González-Martín *et al.* (2018) showed that many of these schemes of use are influenced by the teachers' own personal relationship with the content they teach.

With these tools, we can reformulate the aim of this paper. We wish to study the link between the institutional relationship with derivatives on the one hand and teachers' practices and use of various resources in teaching derivatives on the other. We also seek to identify specific elements in the African context that may be less present in existing literature.

THE CAMEROONIAN CONTEXT

In Cameroon, students attend secondary school between the ages of 12 and 18. Cameroon calls *secondary* that which in other countries may be considered as pre-university or college-level studies. Derivatives are introduced in the penultimate year of this cycle, called *première* (students are 17 years old), after the study of functions, limits, and continuity. The content on derivatives in *première* includes: differentiable function at a point; derivative of a function at a point (including left and right derivatives); geometric interpretation of the derivative at a point; equation of the tangent of a curve at a point; derivative function; derivative of the addition, the product, and the quotient of functions and of $f(ax + b)$, with f being differentiable; variation of a function in an interval depending on the sign of the derivative; extrema. The Ministère des enseignements secondaires [Ministry of Education] asks teachers to introduce derivatives at a point by calculating the limit of $(f(x) - f(a))/(x - a)$ when $x \rightarrow a$, but it does not provide any didactic suggestions on how to make this

introduction, or about making the shift from a derivative at a point to a derivative function.

The Ministry provides the public schools with a list of approved textbooks; however, as of the 2018-2019 school year, only one approved textbook has been available to teach each course, including mathematics (Tegninko, Sielenou, Bouda, Pokam, & Boudy, 2014). This means that all public schools use the same textbook chosen by the Ministry. Moreover, there are national examinations for students in May and June). For students in *première*, questions concerning functions, limits, continuity, derivatives, and sketching the graph of a function represent approximately 42% of the examination questions concerning derivatives usually concern the derivative of a function and studying the variation of a function using the sign of the derivative.

Finally, the training of secondary teachers falls under the auspices of the Ministère de l'enseignement supérieur [Ministry of Higher Education]. Teachers' pre-service training is provided by the *Écoles Normales Supérieures* (ÉNS). When students enter an ÉNS after finishing their secondary studies, they must complete a three-year Bachelor of Mathematics (first cycle), followed by two more years of training (second cycle). This second cycle includes additional courses in university mathematics (approximately 50% of the cycle) along with courses in education. Throughout this training, relatively little emphasis is placed on the specific aspects of mathematics education.

METHODS AND ANALYSES

The research presented in this paper is part of the second author's doctoral thesis, which, under the qualitative paradigm, is developed as a multi-case study. Semi-structured interviews, class observations and documents (textbook and other resources) were used to collect the data. All three teachers who volunteered to participate in the research (T1, T2, T3) are male; all are legally qualified, teach in *première* during the 2019-2020 school year, and work in three different schools in Yaoundé.

In this paper we focus on data from the participants. The data collection was structured in three stages: 1) preliminary interviews concerning the teachers' personal relationship to derivatives, their use of resources, and their lesson preparations; 2) observations of the teaching of derivatives in class; 3) post-teaching interviews to compare the teachers' planning with the actual teaching and discuss some episodes. The interviews and observations provided data to study the teachers' adherence to the institutional relationship with derivatives, as well as their documentation work in their specific context, with major institutional constraints (the imposition of an official textbook, an official exam). We note that, for this paper, we have mostly used data from the interviews, with some additional details culled from our observations.

The analysis of the interviews was performed using the theoretical tools provided by ATD and DA. We first examined the teachers' statements to identify elements associated with their personal relationship to derivatives (how they define derivatives, what they consider important about them, the exercises they value, etc.). We then identified the main techniques they used to teach the content (providing definitions, working on exercises, etc.) and their rationales (technologies). We also identified the resources they use, as well as their schemes of use (mainly, rationales concerning why they used the resources, their aims, etc.). The following section summarizes our main results.

DATA ANALYSIS

During the preliminary interviews, each of the three participants discussed his vision of derivatives – a vision that encompassed many of derivatives' mathematical meanings (Figure 1):

T1	T2	T3
The derivative has several meanings [...] mathematically, it is the slope of the tangent line to the curve of the function at a point. Practically, it represents a speed [...] in the Cameroonian context, children must master the derivative much more like the slope of the tangent line at a given point.	The derivative for me is a mathematical tool [...] which has many physical applications and, therefore, I associate it more with a speed.	For me, the derivative is first the result of a limit... and so, I see it as the limit of the rate of change of a function.

Figure 1: The participants' views on derivatives

Their responses show that their personal relationship to derivatives includes several aspects of this topic (slope, speed, limit, rate of change). However, despite the different elements present in their personal relationship, all three teachers follow a similar technique to introduce derivatives using rates of change (Figure 2):

T1	T2	T3
By calculating the limit of a certain rate of change to determine the derivative at a point.	First, I wanted to make them understand that the notion of derivative is linked to rate of change, which they have seen in previous years.	Because of the requirement of the program, I restricted myself to the limit of the rate of change.

Figure 2: The participants' choice to introduce derivatives

Three of them also stated, at various points in the interviews, that their choices are determined by the official guidelines: they follow the Ministry's instructions and begin with the limit of a rate of change because this is what the official program dictates. Therefore, their technique for introducing derivatives is mainly explained by the rationale (technology) that they had to do what the program requires. The teachers also noted that they followed the approach of the textbook. We see, therefore, that the introduction students receive to derivatives is restricted to an abstract, limit-focused approach, and that the teachers do not call for connections with speed or use other more intuitive

approaches. Moreover, we can see that the three participants are “good subjects of the institution,” doing what the institution expects individuals in their position to do (institutional relationship). This may explain why T1, who sees derivatives primarily as a slope, introduces them as the limit of a rate of change, or why associations with physical meaning (T1 and T2) are not present. Their statements are supported by classroom observations: all three teachers start introducing derivatives at a point, and they do so by calculating the limit of a rate of change.

Our classroom observations also confirm that the three teachers organize their introduction to derivatives by using the textbook as their main guide. In this sense, the passage from the derivative at a point to the derivative as a function is made in a very immediate way (between the first and second lessons, by simply replacing a generic “ x_0 ” with “ x ”), and the teachers move quickly to introduce techniques to solve tasks concerning derivatives. In this sense, their approach is like the participants in Park’s (2015) study. Our participants’ approaches prioritize computational aspects to provide students with a set of rules and formulae that will be later applied in the exercises. Despite this, the three stated during the interviews that they hoped students would be able to develop a better understanding of what they were doing (e.g., T1: “they discover by themselves”).

Our data also indicate that, despite occupying the same position and following the same program, the teachers exhibited some differences in their views concerning which elements to highlight in the chapter on derivatives (Figure 3):

T1	T2	T3
In the Cameroonian context, what matters more is the use of the derivative to construct functions [meaning studying functions to later sketch their graph] [...] to use the tangent [meaning calculating the equation of the tangent line at a point].	In the chapter about derivatives, what is more important to teach is how to find the derivative of polynomial and rational functions. I start with the aspects that are necessary for the exam [...] we must respect the institutional guidelines. The State limits the way in which we must teach the derivative, we do not have a choice to change [this approach].	What is more important in the teaching of the derivative are its applications [...] the search for extrema, the search for the direction of the variation of functions.

Figure 3: The participants’ views about the most important aspects of derivatives

As we can see, even if they hold the same position, the teachers may emphasize different aspects of the content in the classroom. This behavior is consistent with the results of González-Martín *et al.* (2018), where five participants using the same resource exhibited variations in their teaching. However, in opposition to González-Martín *et al.*’s (2018) results, where the participants had some conceptual objectives in their teaching, we may observe here that the three participants highlighted operational aspects concerning derivatives as the key aspects to be learned. We conjecture that this may be a consequence of the

strong influence that the program and the examination (which focus on operational aspects) exert on their practices. In the interviews, many of the teachers' rationales for the way in which they organize their teaching were reduced to the program and the exam. We also conjecture that their pre-service training, with its emphasis on mathematics to the detriment of didactic components, may influence their views.

We note that, in Figures 2 and 3, the program is seen as somehow restricting what teachers can do: T3 states that, because of the program, he limits himself to introducing derivatives in a certain way, while T2 claims that the State (through the program) limits the way in which teachers can teach derivatives. We emphasize that the program was mentioned as a factor at several points during the interviews.

We also note that all the elements concerning derivatives mentioned by the participants as important for students to learn are aligned with the objectives of the official mathematics program. These objectives are usually reflected in the questions on the national exams that take place at the end of the school year, and this examination has a strong influence on the teachers' practices and on their choice of resources in preparing their courses. In fact, the three participants stated that their main resource is the official textbook, since it allows them to cover the required content that will appear on the national examinations. We see how institutional constraints limit the teachers' documentation work. They also discussed their use of the Internet, particularly their participation in online forums for teachers, but they insisted that they use this resource minimally, as a complement to the activities provided by their main resource. Figure 4 presents some interview excerpts concerning the influence of the national examination on the teachers' practices:

T1	T2	T3
The goal of the students is to pass their end-of-year exam. So, everything related to the derivative must be what often comes up in the examination or what the official examination requires [...] If you look in the official textbook it is the type of activity like that [...] It is clear, that is not open to debate. The textbook is much more interested in this [type of activity].	What seems more important to me is firstly their examination because you know that if a student does not pass his examination, the parent will say that the teacher did a poor job. So, what is important for me now is to use the derivative as a mathematical tool for the study of functions... For now, derivatives [are taught] to help students pass their examination and move on to higher studies in [the last year of secondary] and later at university. [...] I try to stick to [the content and activities of] the official textbook. When we do problem-solving lessons [<i>travaux dirigés</i>], we first do the exercises that are in the official textbook.	Keep in mind that these students must prepare for the year-end examination; this is our main objective at the moment.

Figure 4: The role of the national examination in the participants' practices

In this case, we can clearly see how the three participants keep the national examination in mind, and how they see their role as preparing their students to pass this examination. Here, the three teachers are explicit about the constraints

to which teachers are subject in the Cameroonian context, namely the ministerial evaluations that generally take place at the end of the school year. We can see how this examination, together with the official textbook provided by the Ministry, strongly directs their documentation work.

FINAL REMARKS

Our preliminary results are consistent with previous research: teachers prioritize algebraic aspects to the detriment of all other aspects such as graphic tools for introducing the derivative. They follow the institutional relationship and emphasize the derivative as the limit of a rate of change, which limits their use, for example, of the notion of tangent to introduce the derivative. As in Park's (2015) study, they favor symbolic notation and shift quickly from the derivative at a point to the derivative function.

Our data show that the participants' activity is strongly conditioned by the injunctions of the official program, by the official textbook, and by the Ministry-imposed evaluations. Although their personal relationship includes several aspects of derivatives, these are not present in the teachers' introduction of this content to their students. In this sense, their practice reflects the content of the main resource (the textbook), with some possible variations and additional exercises. We also note a lack of agency: most of the rationales (technologies) they use to justify their teaching techniques are reduced to their need to follow the dictates of the program. Another important element influencing their resource use and their practices stems from institutional constraints: the teachers believe their main goal is to prepare students to pass the national examination. They use expressions such as "the goal of the students is to pass," "derivatives [are taught] to help students pass their examination and to move on to higher studies," or "these students must prepare for the year-end examination." Note that these elements are very strong in the Cameroonian context. This, in addition to the weaker didactic component in the teachers' pre-service training, may lead the teachers to not question the official guidelines.

Regarding the use of ATD and DA to study these issues, we believe that they allow a better understanding of some of the teachers' practices, both in terms of their planning and their use of resources. Even though their personal relationship with derivatives encompasses several different aspects, the latter are not mobilized. Given the institutional constraints, the main rationales seem to be "to follow the program" and "to prepare for the national examination." In this context, we can clearly see how these constraints influence the participants' choices in introducing derivatives, as well as their use of resources, which is mostly reduced to a single textbook.

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APPLICATIONS IN CALCULUS FOR ENGINEERING. THE CASES OF FIVE TEACHERS WITH DIFFERENT BACKGROUNDS

Alejandro S. González-Martín¹ and Gisela Hernandez-Gomes¹

¹Département de Didactique, Université de Montréal, Canada

Recent literature and engineering reports suggest that the mathematical training of engineers should include more applications to help students connect mathematical content with professional engineering practices. In this paper, using tools from the anthropological theory of the didactic (ATD), we analyse data from interviews conducted with five calculus teachers in engineering programs, all of whom possess different academic and professional backgrounds. Our data suggest that, while they all seek to make their course more “engineering-oriented,” the teachers’ practices seem to be quite different. Only those teachers with extensive professional engineering experience provide realistic applications.

INTRODUCTION

The training of engineers is putting increased emphasis on the development of mathematical competence to meet industry needs in the 21st century. For instance, a document discussed at and drafted by the SEFI Mathematics Working Group in 2013 (Alpers, 2013) highlights and exemplifies eight mathematical competences required by students: thinking mathematically; reasoning mathematically; posing and solving mathematical problems; modelling mathematically; representing mathematical entities; handling mathematical symbols and formalism; communicating in, with, and about mathematics; and making use of aids and tools. More recently, van der Wal, Bakker, and Drijvers (2017) also proposed a set of skills (what they call techno-mathematical literacy) to highlight the fact that 21st-century engineers need to go beyond the ability to calculate and estimate, which is now insufficient. In addition, Beswick and Fraser (2019) state the following:

“For mathematics teachers to contribute to STEM and 21st-century competence agendas they need knowledge of their discipline and how to teach it as a foundation upon which to build their capacity to integrate mathematics with other disciplines and to teach 21st century skills beyond discipline knowledge.” (p.963)

The above authors call for a shift from merely learning mathematical content to developing mathematical competence specifically for the workplace. In engineering programs, mathematics is usually taught separately, in isolation from engineering courses (González-Martín, Gueudet, Barquero, & Romo-

Vázquez, in press), which may reduce the likelihood that students will develop the above competences. Specifically, mathematics courses in engineering tend toward a significant level of abstraction, without explicit connections to engineering practices (Christensen, 2008). This may result in students failing to develop crucial mathematical competences, and is one reason why mathematics courses in engineering programs often have high failure and dropout rates (González-Martín et al., in press). Indeed, the relationship between calculus and engineering's client disciplines is ripe for research (Rasmussen, Marrongelle, & Borba, 2014).

Modelling activities are usually recommended to bridge the gap between mathematical content and engineering practices (e.g., González-Martín et al., in press), as well as the use of Project-Based Learning; however, certain modelling activities can be unrealistic or artificial. For instance, Alves et al. (2016) identified that some calculus teachers in engineering programs believe they should focus on simply teaching the course content, without incorporating it into “fake” (p.137) projects. One may presume that teachers who have used mathematics in non-academic contexts (engineering or other) are more likely—and better equipped—to offer concrete applications of mathematical content than teachers who lack this experience (e.g., Nathan, Tran, Atwood, Prevost, & Phelps, 2010; Nicol, 2002), thereby avoiding unrealistic or “fake” modelling activities. For instance, Nicol (2002) pointed out that when a teacher has experience using mathematics in a non-academic context, this could “[help] students connect mathematics, to real life and work” (p.291). This echoes the results of Nathan et al. (2010), who argue that “practicing engineers present a more nuanced picture of the relationship between mathematics knowledge and engineering practice” (p.420). The results of our recent study (González-Martín & Hernandez-Gomes, 2020) on two teachers with different backgrounds suggest that engineering teachers lean on their professional experience to justify some of their teaching practices. It is still uncertain how teachers with different backgrounds are likely to tackle the same mathematical content, and what kind of applications to engineering they may provide their students.

To help close this research gap, this paper examines how calculus instructors in engineering programs use applications, paying special attention to the connections between the teachers' background and these applications. Despite the recommendations to integrate mathematical content with modelling activities, the structure of many engineering programs still separates mathematics from engineering courses, imposing institutional constraints that may hinder the use of modelling. Our research question can be formulated as follows: What type of applications do calculus teachers in engineering programs provide, and how do these applications relate to their professional and academic backgrounds?

THEORETICAL FRAMEWORK

We seek to study one aspect of calculus teachers' practices (their use of applications) in a specific institutional context and relate these practices to their background (acquired in other institutions). We therefore have adopted a framework that provides tools to study institutionally situated practices: the anthropological theory of the didactic (ATD) (Chevallard, 1999). ATD uses the construct of "praxeology," which considers the following: types of tasks (what to do); techniques (ways of performing tasks of a given type); technologies (rationales that describe, explain and justify techniques); and theories (which function as a basis of and support for the rationales). ATD also acknowledges that individuals are influenced by their belonging (or having belonged) to different institutions. Consequently, they may use techniques and rationales acquired in one institution to solve the same task in another institution. For more details, see González-Martín & Hernandez-Gomes (2020).

Our research focuses on calculus teachers in engineering programs. Although these individuals occupy the same institution and perform the same general task (*teach a calculus course*), their experience in other institutions may affect the way they accomplish this task. For instance, they may choose to engage in the sub-task, *provide applications of the content*, and, in that case, use different techniques and rationales to support their techniques. Our aim is to identify these techniques and rationales and observe how they relate to the teachers' professional and academic backgrounds.

METHODOLOGY

For this paper, we use data from interviews conducted in September 2015. To investigate the practices of calculus teachers with different backgrounds, we interviewed five university teachers with extensive experience teaching mathematics courses in engineering programs (see Hernandez-Gomes & González-Martín, 2016). They all had been teaching in São Paulo, Brazil, for at least 10 years. At the time of the interviews, Teachers A, C, and D were teaching calculus at the same engineering school, Teacher E was teaching at another school and Teacher B was teaching at both schools. They each received a preliminary survey on their academic and professional backgrounds, which allowed us to categorize their profiles (Figure 1).

A: Female	B: Female	C: Male	D: Female	E: Male
<ul style="list-style-type: none"> • Bach. in Math. • Master in Math. • No PhD. 	<ul style="list-style-type: none"> • Bach. in Math. • Master in Space Eng. • PhD. in Mech. Eng. 	<ul style="list-style-type: none"> • Bach. in Math. • Master in Theology • No PhD. 	<ul style="list-style-type: none"> • Bach. in Elec. Eng. • Master in Elec. Eng. • PhD. in Elec. Eng. 	<ul style="list-style-type: none"> • Bach. in Elec. Eng. • Master in Math. Ed. • PhD. in Math. Ed.
<ul style="list-style-type: none"> • High School Math. Teacher • University Math. Teacher 	<ul style="list-style-type: none"> • High School Math. Teacher • University Math. Teacher • Trainee in an electrical company • Researcher in Eng. Education. and Optimization 	<ul style="list-style-type: none"> • High School Math. Teacher • University Math. Teacher • Director of the Faculty of Sciences 	<ul style="list-style-type: none"> • University Math. Teacher • Telephone and Telecomm. Engineer • Researcher in Eng. Education and Telecomm 	<ul style="list-style-type: none"> • University Math. Teacher • Trainee in engineering • Coordinator in an environmental pollution control company

Figure 1: Academic (blue) and professional (green) profile of five participants

The five semi-structured interviews were conducted and audio recorded in each participant's office (with an average duration of one hour each), and then transcribed and coded. For each teacher, we identified the main praxeologies (tasks, techniques and rationales) they use to teach calculus by identifying points in the interviews where they described specific sub-tasks. For instance, for the sub-task involving the presentation of properties and results, a statement such as “to understand this or that result, we end up doing some proofs” (Teacher A) was seen as evidence of the teacher using the technique “do some proofs” to accomplish this sub-task. We then used thematic analysis to describe each participant's praxeology. The transcriptions were analysed separately by each researcher and differences were then resolved to arrive at a consensus. We created tables for each teacher, identifying the main sub-tasks (e.g., providing examples, creating tables, choosing appropriate exercises, etc.), their techniques and the rationales given for these techniques.

In this paper, we focus on interview excerpts in which participants discuss their use of applications. We examine their techniques (e.g., the type of applications they consider), their justifications for providing these applications, and the connection with the participants' backgrounds and experience.

RESULTS

At the beginning of the interviews, the participants were asked to reflect on the types of exercises they propose most often in their praxeologies in teaching calculus courses. Figure 2 synthesizes the main elements of the participants' practices:

Teacher	Types of exercises
A	<ul style="list-style-type: none"> • “we work more than just practice... Since it's calculus [we do] exercises concerning graphs, those things... [...] We don't do many proofs. But in order to understand this or that result, we end up doing some proofs too. So, it's a mix of everything.”
B	<ul style="list-style-type: none"> • “[The exercises I provide] are more practical. [...] Some problems, and, when I have problems applicable to engineering, I think it's quite interesting, this type of exercise, which can exemplify the application of this concept in [the student's] field.”
C	<ul style="list-style-type: none"> • “[The exercises are] more practical, since we are in engineering. For instance,

	yesterday [in class], I wanted to justify the first fundamental limit, and I said, ‘now, I’m going to provide a justification for engineers.’ [...] I made a table, inserted some values [...] It’s more practical; I believe that in engineering, theory must be minimized as much as possible.”
D	<ul style="list-style-type: none"> “As any teacher, you have a preference for some type [of exercise], some type of function, and obviously I end up using things that have more applications for electrical engineering.”
E	<ul style="list-style-type: none"> “The part regarding functions and limits is quite theoretical. So, it goes: calculate this limit, find the inverse function [...], sketch the graph [...]. Later, in derivatives [...], we explore the determination of maxima and minima, problems concerning rate of change. Then, problems of maxima and minima and rate of change — they are more practical, they have a practical application. Before that, problems are more conceptual. Do this, do that. Later, rate of change, we have... ah... you can have an inverted cone with water escaping... it’s being filled at a given rate of cubic meters per minute, what is the rate of change of the height in relation to the time [...]?”

Figure 2: The participants’ views on the exercises they propose

We may observe a variety of approaches. Although the five teachers use the rationale that engineering students need more “practice,” we see that their techniques vary. Whereas Teachers B and D appear to consider the professional profile of their students more directly (“I have problems applicable to engineering” or “applications for electrical engineering”), Teachers A, C, and E appear to propose more “classic” activities, typical of a mathematics course. They see “doing exercises concerning graphs,” creating tables or proposing (classic) exercises about maxima and minima as ways to make their course more “applied” and better attuned to their students’ profile.

We then asked participants for concrete examples of applications they provide to get a clearer idea of the activities they develop in their praxeologies. Figure 3 summarizes their answers.

Teacher	Applications provided during the interview
A	<ul style="list-style-type: none"> Exercises about graphs; “we are actually missing some applications”; “[students being in their first year is a constraint], since they still haven’t seen anything about engineering. They are actually seeing basic mathematics.”
B	<ul style="list-style-type: none"> “Let’s consider a course on maxima and minima, with a two-variable function. So, you can [...] calculate the tangent plane to a given point, you can exemplify your surface with a spherical surface, calculating the minimal distance. You can provide an example of a satellite in orbit, and there you want to calculate the minimal distance [...] to the position of the antennae. These are examples that you give them [...] Then, you obviously get to connect theory and some applications. It’s obvious that you make some approximations [...] since you are talking about satellites, and you cannot consider all those [...] parameters [...] in a simulation that you should obviously consider in a real job, but there you provide a practical example about that concept.” “We are going to talk about definite integrals. [Students] will use that a lot. Because [engineers] want numbers. [...] He will have a variation from a value a to a value b, then he wants to calculate something. And that thing comes from applications.”
C	<ul style="list-style-type: none"> “I always found [providing applications] difficult. First, I’m not an engineer, and I don’t have the experience of an engineer. And then, students do not yet have that experience, since I start with [students in their] first and second semesters, so they haven’t acquired concepts from

	<p>engineering. So, I always found that very difficult, but when I talk to other engineer colleagues, I always try to find out, ‘do you use this?’ [...] Then, I say that [in my course]: ‘this thing here, there is somewhere in engineering, in the professional courses, where you are going to use it.’ [...] But I think it’s not very helpful for the student.”</p> <ul style="list-style-type: none"> • “It’s easier with physics, since physics and calculus go together [...] But I think that, to do that type of work, the teacher needs to be a mathematician and a physicist. If not, he won’t do it well [...]”
D	<ul style="list-style-type: none"> • “I have a look at the exercise and see where one can apply it. I then give a contextualization in addition to the exercise.” • “[...] I’m going to do, for instance, an integrator circuit; I need to know what an integrator circuit does; I need to know the integral, how I throw a pulse and it starts integrating, the curve goes up. Then, I have my differentiating circuit, we model circuits with mathematics. With an equation... a second order filter is a polynomial equation of [...] second grade. Then, I cannot dissociate one thing from the other.” • “I think that engineering is equations, graphs, and tables. If the engineer cannot interpret that, he doesn’t know anything. [...] So, when I’m teaching to production engineering [students], for instance, I provide analogies with examples from the stock market, since they work a lot with that. And I ask them how you create a function for interest rates...”
E	<ul style="list-style-type: none"> • “Since [the course] is in the basic cycle [the first years of basic courses], you have diverse applications. You have something from production engineering, something more from mechanical engineering, something from electric engineering. So, it’s generic. It’s not specific, I won’t ... give specific applications for a field. They are more generic, more towards physics. I’d say that it’s because it’s the first semester, and the student is not yet taking professional courses. So, [the activities] are more generic.”

Figure 3: Applications provided by the participants

We observe three distinct positions. The first technique does not provide specific applications, or provides only classic applications when teaching optimization and calculation of areas. This is the case with Teacher A, who uses the rationale that first-year students lack experience. A second technique provides some applications (mainly using physics or contextualized exercises), while also possibly informing students that they will use this content elsewhere; we observe this in Teachers C and E, who state the common rationale that students are in their first-year (interpreted as “students do not know applications yet” or as “there are many different profiles”). A third technique provides more realistic applications, directly related to engineering practices. We see this with Teachers B and D, who, in justifying their use of this technique, occasionally fall back on their experience and their vision of engineers’ preferences (e.g., “because [engineers] want numbers”) and professional needs (e.g., “engineering is equations, graphs, and tables”).

First, we note that the only teachers who use realistic applications are Teachers B and D. They are also the only ones with experience working as engineers. This is consistent with Nicol’s (2002) and Nathan et al.’s (2010) observations. Second, we note that Teachers A, C, and D use the fact that their students are in their first year as a rationale to justify the teachers’ difficulty in finding suitable applications. This correlates with the study conducted by Alves et al. (2016), in which only calculus teachers stated that including mathematical content in applications in the first year of engineering could be “useless” or “fake.” This

perspective could stem from a lack of knowledge of engineering practices. The participants also reflected on how their own training and experience influence their teaching practices with respect to the kind of applications they offer as examples to their students (Figure 4).

Teacher	Types of exercises
A	<ul style="list-style-type: none"> • “since my training is in mathematics, we want to do many proofs, which I think are not always suitable for an engineering course [...] So, [my training] has an influence in that sense. But the good thing is, there’s always a curious student, and they pose the question, and then you must do a proof, even if it’s as an aside...” • “I think I limited much of my content to mathematics, due to my training. Because we hear [...] that [students] will need [this content] later.”
B	<ul style="list-style-type: none"> • “[in my training] I had teachers who were mathematicians, also mathematicians with an engineering perspective, and also engineers. And I believe that influenced my training a lot. I believe that a [...] pure mathematician has a different perspective on mathematics, on this differential and integral calculus. [...] I don’t think you need that much rigour. A mathematician teaching calculus, he doesn’t think about application. He thinks of mathematics as mathematics. How and where it comes from, how I prove it [...]—not how the temperature will go up or down, or how you apply air conditioning ... an engineer is more preoccupied with the latter.” • “[my training] contributed a lot, having this perspective for applications. An engineering student won’t want much theory; he wants to know how he’ll use these concepts in his practical life. [...] Therefore, this heterogeneous training that is not just linked to thinking and proofs, is very beneficial in an engineering course, I have no doubt. Since, when a student asks me, ‘how could I use that?’ [...], even the book I use can have no practical examples, but [...] it takes me five minutes to think and be able to tell him: ‘look, in that course you’re going to use this.’”

C	<ul style="list-style-type: none"> • “First, I’m not an engineer, and I don’t have the experience of an engineer.” • “In the beginning, I had a lot of difficulty, since most of the time I had to teach a topic that I myself didn’t know too well. So, I had to study a lot; I used different books; I was always self-taught [...] And so books became essential. I always tell [my students]: “you cannot be an engineer if you don’t have a book at home; you need to have a library.””
D	<ul style="list-style-type: none"> • “[my training influences me] in how I contextualize, in the application of exercises; it influences quite a bit. [...] And us, in electrical engineering, let’s say we model engineering, circuits, components, through mathematics. So, one thing becomes another. [...] I have my differentiating circuit; we model circuits with mathematics. With an equation ... a second order filter is a polynomial equation of [...] second grade. Then, I cannot dissociate one thing from the other. For me, it’s just one thing.” • “My vision of calculus for engineering is that it’s modelling. I see our oscilloscope and I see a function, in the same way that I see a function, and I think of the electrical signal associated with that function.”
E	<ul style="list-style-type: none"> • “[my training] influences, yes. For instance, when we’re solving a problem later in the semester. How do I explain to my students, “you’ll be engineers, what does an engineer do?” [...] How can we think when analyzing? What are the data of the problem? What is being asked? And then I’ll think of a strategy. [...] What knowledge will I gather and how do I articulate it to get there? Here, my training as an engineer is very influential at that point.” • “Look, I don’t know the day-to-day practice [of engineers] [...], since although I’m an engineer, I didn’t work for long in the field; I spent very little time there.”

Figure 4: The participants’ reflections on their training

We observe that all participants see their training as an important source of justification (rationale) for some practices. Teachers A and C state clearly that their training is not in engineering, and that this fact influences some of their practices in teaching calculus. We also observe a possible justification for Teacher E’s lack of realistic applications: he does not have knowledge of the day-to-day practice of engineers. However, he believes his experience allows him to come up with a strategy for solving problems.

FINAL CONSIDERATIONS

We believe our results provide further insight into Nicol’s (2002) and Nathan et al.’s (2010) observations. As we noted in our previous study concerning two teachers (González-Martín & Hernandez-Gomes, 2020), teachers’ backgrounds and experience may provide important rationales for their teaching practices. In the case of engineering programs, it appears that experience in the professional workplace may provide teachers with a different way of looking at mathematics and of connecting with applications, as seen with Teachers B and D. Professional engineering experience also appears to provide teachers with a wider repertoire of applications for their calculus courses. In our study, the teachers who believe first-year engineering students do not have a solid enough background to connect calculus with realistic applications had no experience working as professional engineers.

As stated above, the data presented in this paper come from interviews conducted in 2015 for a more general purpose. However, we believe the interviews provide sufficient data concerning the participants' knowledge of realistic applications, and their ability to include this knowledge in their techniques. These five teachers, with their different backgrounds and different teaching practices, make up an interesting pool of participants. We intend to conduct another set of interviews to delve deeper into the participants' practices and gather more information about their praxeologies in their calculus courses. This will provide material for future publications.

Acknowledgments

The authors wish to thank the five teachers for their participation and for sharing their experience. This research was funded by grant 435-2016-0526 of the Social Sciences and Humanities Research Council (SSHRC) through Canada's Insight program.

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CONCEPTUALIZING EXPERTISE FOR TEACHER PROFESSIONAL DEVELOPMENT IN TEACHING STOCHASTICS

Birgit Griesel¹

¹Paderborn University, Germany

In order to address the intended aims of a professional development (PD) intervention adequately, the purported expertise must be conceptualized in detail, including a specification of the learning at classroom level. This paper utilizes an expertise framework (Prediger, 2019a, following Bromme, 1992) that distinguishes between jobs, pedagogical tools, thinking categories, and orientations to illustrate the concept-ualization of a PD unit that focuses on explanations, interpretation, and understanding when deploying digital tools for the teaching of stochastics at secondary level.

INTRODUCTION: CHALLENGES IN TEACHING STOCHASTICS

Teaching stochastics (statistics and probability calculation) presents a challenge for many teachers, for various reasons (Batanero, Burrill, & Reading, 2011): Some have little or no personal experience of being taught stochastics themselves, others feel insecure because of the uncertain nature of statements referring to probability, and others again deem themselves unprepared for the technological demands. All these issues have led to a high demand for PD courses covering stochastics in Germany.

The designers of PD courses on stochastics at the German Center for Mathematics Teacher Education, DZLM, intend to address understanding, and not (only) procedure (Barzel & Biehler, 2016), at the classroom level as well as at the teacher PD level. This is in keeping with the standards of higher education, but it presents the challenge for PD designers and facilitators to attend to content *and* didactics. (For other more popular areas of mathematics, like calculus or geometry, PD courses can concentrate on didactical and methodical consideration alone.) For example, in a DZLM PD course on stochastics developed at Paderborn University, only 20% (9 out of 44 participants) found they had fully attained the goal of learning about the didactic value of simulations, although this was an explicit focus. One participant stated “sometimes simulations are more confusing than helpful”, another wrote that it was “not clear in which situations simulations make sense”, and more than one complained that it would not be worth the lesson time to teach students how to code the simulations. Consequently, a sound theoretical basis for the expertise

in teaching stochastics is indispensable when aiming at a systematic re-design of the course.

THEORETICAL BACKGROUND

Theory elements for Design Research (van den Akker, 2013) in teacher PD can be distinguished as categorical, descriptive, normative, explanatory, or predictive, each with different functions and structures (Prediger, 2019b). These elements are closely related and help to answer *what* should be addressed and *how* this can be orchestrated (like in Griese, Rösken-Winter, & Binner, 2020), thus offering a perspective of how to re-conceptualize PD design. The idea of this paper is to explore a framework (Prediger, 2019a) in regard to describing the expertise necessary for teaching stochastics and for conducting PD courses. Thus, insights into the structure and interdependencies of teacher expertise (and how to promote it) can be obtained. The objective is to use the framework for both the re-design of PD courses and the qualification of facilitators. The idea behind this model differs from the approach to describe teacher competence and their impact on teaching quality (Kunter, Klusmann, Baumert, Richter, Voss, & Hachfeld, 2013) insofar as it focuses on the development of teaching skills and thus seems better suited for the PD perspective.

Following Bromme (1992), Prediger (2019a) developed a framework for conceptualizing content-specific teacher expertise which describes *jobs* as “typical, often complex situational demands of subject-matter teaching that are most relevant to the PD content in view” (p. 369) and how teachers cope with them. This situated approach allows to disentangle teacher practices by also describing their *categories* for thinking and noticing, the *pedagogical tools* they employ, and their underlying beliefs (*orientations*) that in their complex interplay influence the effect of a PD intervention. The specifications naturally relate to the teaching content that is the focus of the PD intervention. By utilizing these categorical and descriptive constructs, normative elements can be phrased in detail – and explanatory and maybe even predictive statements are to be gained. This is worthwhile, so the central question of this paper is:

Can a framework for content-specific teacher expertise that describes expertise in jobs, pedagogical tools, thinking categories, and orientations help to improve a PD course on stochastics by offering answers to *what* and *how* questions?

CONTEXT OF THE PD COURSE

This paper is based on the design of and research around a five-day PD course for stochastics at upper secondary level (Oesterhaus & Biehler, 2014) that specifically addresses the use of digital tools. In particular, we focus on that part of the first day of the course where suggestions are presented on how to promote a deeper understanding of distributions, and participants experience and discuss various tasks and activities. The evaluation of the PD course showed that the intended aim of displaying the advantages of employing digital

simulations fell short of expectations. When considering reasons for this feedback and pondering options for improvement of the PD course, the question how to proceed emerged. One answer was to use a framework for teacher expertise in order to comprehend the learning obstacles in this context and find approaches to overcome them.

UTILIZING THE FRAMEWORK FOR CONCEPTUALIZING TEACHING EXPERTISE IN STOCHASTICS

Classroom Level

Following Prediger (2019a), we start with compiling a learning map for stochastics for the classroom level (summarized in Table 1). In analogy to her structure of the categories, we describe the content goals, the activities and linguistic practices, and the lexical means and resources.

	Content goals	Activity and linguistic practice	Lexical means and resources
Procedural / local	<ul style="list-style-type: none"> - probabilities - characteristic variables - accumulation limits - ... 	calculate	apply formulae (e.g. binomial distribution, for mean or variance)
Conceptual / global	<ul style="list-style-type: none"> - interpret probabilities - make evidence statements - vary scenario (e.g. increase n in a binomial distribution) 	use tables, calculator, software etc.	elaborate on if-then scenarios; employ dynamic software; conduct repeated simulations

Table 1: Learning map for stochastics at classroom level

Content goals cover what the students are expected to master, in different levels of complexity. The basic level comprises goals for stochastics that can be classified as procedural or local (middle row in Table 1), where the view is quantitative on one specific characteristic or value of a distribution. For students to show they have reached these goals, they will apply formulae or rules, and use calculators, software, or tables (which we sum up under *resources*) and do calculations (which we term an *activity*). At this level, no verbal utterances are necessary, and therefore no lexical means are employed. Another level of content goals refers to conceptual or global aspects (bottom row in Table 1), where the view is on a distribution as a whole, on understanding its characteristic features, thus on qualitative aspects. Learners are here expected to vary scenarios, e.g. change a parameter in a distribution, and come to the conclusion that if you increase n in a binomial distribution, then the histograms of absolute frequencies will become flatter and wider, but the histograms

showing relative frequencies will be narrower. In order to show that learners have reached these goals, more complex activities are needed which include language and will therefore be termed *activities and linguistic practice*. At this goal level, learners will interpret, conclude, explain, and give reasons. The resources employed to realize these activities are more complex and are therefore extended to *lexical means and resources*. Learners elaborate on conditional deliberations and consider if-then scenarios (like the conclusion described above in this paragraph).

Apart from lexical means to explain the scenarios, learners can also resort to technical resources. Dynamic software seems well-suited to represent varying scenarios, as it allows to vary parameters, e.g. via sliders. Obviously, in a changing scenario, the focus is not on certain values or numbers (as in the procedural / local level), as they do not carry potential for explanations in themselves but only in comparison to other values or numbers. The same is true for simulations, where the specific value of a relative frequency is not very informative, but repeating the simulation can reveal both what is characteristic of a distribution and what is random. These resources (dynamic software and repeated simulations) can support an understanding which is then operationalized with the help of the lexical means described above.

Teacher PD Level

The learning map described in Table 1 is an essential resource for teachers to identify which *category*, thought pattern, or reference frame is addressed by a learning activity. The map supports teachers to make informed decisions when planning and performing teaching sequences. It is crucial that students experience teaching that covers not only the procedural / local content goals, but also the conceptual / global ones (this refers to *what* should be covered). The learning map offers activities and linguistic practices that attend to them (suggesting to *how* to accomplish the intended goals).

The *jobs* for teaching stochastics (Table 2) are phrased in analogy to Prediger (2019a) as demanding, noticing, developing, and supporting resp. explanations, interpretations, and reasoning. They each address different aspects of teacher behavior. *Demanding* explanations, interpretations, and reasoning means teachers set tasks and activities that refer to the activities and linguistic practices of that level. *Noticing* stresses that teachers need to have a reference system to diagnose which goal is being addressed at a certain moment. *Developing* and *supporting* emphasize the fact that teachers address the learning process. The job of *identifying* the language and resources relevant for stochastics is another basic *job* which is crucial for all others. This equally means identifying what is *not* relevant – as it is quite easy to get sidetracked by the technical specificities of software coding, or by inconsequential language issues like declension.

The *pedagogical tools* for stochastics, that support teachers in *how* to address certain learning goals, comprise, among others, motivating and cognitively activating tasks with authentic background, experiments (e.g. throwing coins or dice), in particular utilizing equivalent random experiments for authentic situations, which is an essential element of stochastics modeling, interactive visualization (e.g. with sliders for the parameters of a distribution), pre-coded simulations, and language scaffolding.

Jobs	demanding explanations / interpretations / reasoning	noticing ...	developing ...	supporting ...	
	identifying language and resources (not) relevant for stochastics				
Pedagogical tools	cognitively activating tasks	interactive visualizations	pre-coded simulations	equivalent random experiments	language scaffolding
Categories	learning map for stochastics at classroom level (Table 1)				
Orientations	addressing understanding and procedure	considering digital tools as means to a goal	regarding certain language issues as relevant		

Table 2: Specification of the framework for teaching stochastics, classroom level

All these aspects refer to what can be observed in a classroom situation. They are, however, influenced by the *orientations* of the teacher who orchestrates the classroom activities. The orientations relevant for teaching stochastics are an awareness that both procedure and understanding are to be addressed in stochastics (as in any other mathematical content area), regarding certain language issues as relevant for stochastics (which involves the *job* of identifying these issues), and considering digital tools as means to reach a goal, and not as a goal in itself.

Table 2 summarizes the framework for stochastics-specific teacher expertise. Various entries there will help to avoid unsatisfactory feedback on the didactic value of simulations, e.g. recommending pre-coded simulations (a *tool*) or considering digital tools as means to a goal (an *orientation*). The matrix in Table 2, however, is not to be read vertically, but horizontally, meaning that the *jobs* are to be understood as one unit, and the *pedagogical tools*, the *categories*, and the *orientations* reflect the complex practice. Moreover, the *tools* and *orientations* are not to be understood as a closed list. The relations between the rows in Table 2 are points of interest and further exploration.

Exemplification: Teaching Sequence for Addressing Distributions

We will look into these interrelations in more detail, exemplified by a teaching sequence for addressing distributions. The sequence starts with the 10/20-test

problem (Figure 1), which represents a pedagogical *tool*, and sets the goal of exploring what lies behind it (demanding reasoning, a *job*). To incorporate students' intuitive thinking, their ideas as to which test is easier to pass and why are collected and discussed, without yet revealing if their reasoning is correct because the learning goal is to allow them to reflect upon the problem in detail, and not to memorize the correct answer. This serves to develop reasoning skills, and its realization is based on the teacher's *orientation* that understanding is a worthwhile learning goal.

One test has ten questions with two answers each, one correct and one false. Another test has twenty questions with two answers each, one correct and one false. You pass each test if you have 60 % or more correct answers.

If you are merely guessing, which test is easier to pass: the test with ten, or the test with twenty questions – or are both equally difficult?

Figure 1: The 10/20-test problem, for addressing distributions

With experiments of tests with 10 or 20 questions, a first exploration of the phenomenon can be created, and students can utter their considerations as to the reasons, which relates to the teacher's *job* of noticing. In order to systemize the observations (i.e. support students' reasoning, a *job*), the teacher can introduce an equivalent random experiment, e.g. throwing ten or twenty coins and counting how often "tail" appears, as a representation of a correct answer. These hands-on experiments may lead to an interpretation which test is easier to pass. Students' reasoning can be supported (a *job*) by guiding and categorizing their arguments.

To get a more reliable basis for conclusions, it seems natural to gather more data, i.e. to simulate the experiment with the help of software. The software used should allow easy handling and visualization, and some steps of the simulation can be prepared in advance, which can be viewed as a *pedagogical tool*, based on the *orientation* that digital tools (here, at least) are a means to reach a goal, and not a learning goal in itself.

Spreadsheets or other software can be used to create visualizations (a *pedagogical tool*) in the form of histograms that show the distribution of the percentages of correct answers for the test with 10 respectively 20 questions. An important feature (*pedagogical tool*) is Excel's F9 key (or a similar key): pressing it results in a new simulation, which the dependent visualizations follow. This feature in particular allows the learner to observe what is a characteristic of the distribution, and what is random. It helps to focus the attention on the characteristics of the distribution (conceptual content goal) and not on particular values (procedural content goal).

It can be observed in the visualizations displaying the relative frequencies of correct answers that the "pass" areas for the test with 10 questions is a bigger

percentage of the total area of the histogram than the “pass” area for the test with 20 questions (certainly students will need help with phrasing a statement like this, a *job* the teacher can fulfil by offering language scaffolding, a *pedagogical tool*), thus yielding the insight that it is easier to pass the test with 10 questions by merely guessing. How probable it is to pass the tests is not relevant at the moment (although the software would easily produce approximations for these probabilities), and the teacher identifies this as not relevant here, which is another *job*.

The general rule that lies behind this phenomenon is that percentages vary less when the number of repetitions is increased (n.b. the absolute numbers vary more), and therefore are more likely to pass a threshold (in our example, 60%) that is not the expected value (here, 50%). At this stage, this generality cannot be phrased, but the example of the 10/20-test problem serves as a milestone task that is remembered and can be referred to when analogous phenomena are considered. Besides, later, when the binomial distribution has been covered, the phenomenon observed in this context can be connected to the values of the mean and variances of the numbers and percentages of correct answers in a test with 10 respectively 20 questions.

Facilitator PD Level: A Further Perspective

Conceptualizing the expertise of facilitators, who present PD courses, is also possible with an analogous framework that follows the principle of nesting teacher expertise in the *categories*, similarly to the nesting of the learning map in the categories of the framework at classroom level. It can be used to specify facilitators’ *jobs* (e.g. attending to practices or discussions), *pedagogical tools* (e.g. the move of pointing to elements of the learning map), *categories*, and *orientations* (e.g. an appreciation for the PD course participants). It becomes apparent that, apart from the *categories*, these specifications are not content-specific, but describe more general characteristics.

CONCLUSION

Our analyses show that the first step, specifying the content goals, as well as what activities and linguistic practices students follow and which lexical means and resources they employ to do so, helps to structure *what* is to be taught and *how* these teaching goals can be attained. The specifications of *how* to employ software and simulations and why this can promote the conceptual learning goals is a major result unveiled by Prediger’s framework. The next step, elaborating on the *jobs*, *pedagogical tools*, and *orientations*, shows that these are connected in various ways and influence teachers’ classroom actions that aim at reaching the learning goals they have decided upon. The *pedagogical tools* are closely related to the lexical means and resources and support teachers in attending to their *jobs*. The *orientations* represent guidelines that influence or even determine which *pedagogical tools* are chosen, and which content goals

are being addressed. They also help to polish the *pedagogical tool*, in the above example in the details to first use hands-on experiments, followed by pre-coded simulations that can be repeated, and visualizations showing the distribution and not single values. Teachers need not necessarily be aware of their *orientations*, but if they are, this serves to rationalize their choice of tool or move.

Thus, the framework permits explanations and even predictions, and helps to shape the PD intervention by suggesting specific uses of digital tools (*repeated* simulations, *dynamic* software) and avoiding sidetracks (e.g. the coding of simulations). These insights prompt *what* should be addressed in teacher PD and *how* this can be orchestrated. In sum, the framework for conceptualizing content-specific teacher expertise has proved useful for stochastics, as it suggests various ways to convince teachers to develop their teaching and offers concrete tools and resources for support.

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MENTAL SIMULATION OF AN ENCODED REPRESENTATION ON ARITHMETIC WORD PROBLEM SOLVING

Katarina Gvozdic¹ and Emmanuel Sander¹

¹IDEA Lab, Faculty of Psychology and Educational Sciences, Université de Genève, Switzerland

Studies showed that even before formal instruction, children can solve some arithmetic word problems by using informal solving strategies. The current study investigates the processes underlying the use of these informal strategies. We propose that the efficiency of the mental simulation of the encoded representation influences the difficulty of word problems and the use of formal solving strategies that reflect principle-based knowledge. Three experiments, two collective classroom experiments and a collection of verbal protocols, with 383 2nd grade students, revealed that the cost of the mental simulation influenced the performance and solving strategies. Principle-based strategies are dominant only on high cost mental simulation problems, reflecting a re-representation process. Theoretical and pedagogical implications are discussed.

INTRODUCTION

Arithmetic word problems are an important part of mathematics instruction all over the world (Verschaffel, Schukajlow, Star, & Van Dooren, 2020). Repeated findings have shown that problems which belong to different semantic categories but share the same underlying arithmetic structure yield different degrees of difficulty and are solved using different solving strategies (De Corte & Verschaffel, 1987; Gros, Thibaut, & Sander, 2020; Riley, Greeno, & Heller, 1983). Numerous research has therefore investigated the processes involved in solving such problems and most would agree that an arithmetic word problem leads to the construction of a situation model due to its semantic characteristics (Reusser, 1990; Verschaffel, Greer, & De Corte, 2000).

One current approach, the Situation Strategy First framework (Brissiaud & Sander, 2010), proposes that the situation depicted in the word problem provides the solver with situation-based solving strategies, which will be preferentially used when it is efficient. This would be the case on the Change 2 problem “Luc is playing with 22 marbles at recess. During recess, he loses 4 marbles. How many marbles does Luc have now?”, since the informal situation-based solving strategy efficiently leads to the solution. Yet, the numerical magnitudes within the word problem can influence the difficulty of the informal

situation-based solving strategy. For example, when the problem is turned into “Luc is playing with 22 marbles at recess. During recess, he loses 18 marbles. How many marbles does Luc have now?”, the informal situation-based solving strategy is computationally inefficient. When solvers succeed to find the answer, they no longer solve it by relying on the initial representation of the problem which led them to use *direct subtraction*. They actually find the answer to this problem by re-representing the problem and using *indirect addition*.

This kind of strategy switch is also described in the literature on non-word problems (Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2013). In fact, both of these strategies correspond to the arithmetic operation of subtraction, yet the arithmetic format is different (Campbell, 2008). In the current study we propose that solvers encode the initial representation of a problem by relying on a conception of arithmetic that is activated by the problem statement. Two encodings compete: the wide-spread conception of subtraction as *taking away* (Fischbein, 1987; Lakoff & Núñez, 2000), aligned with the use of *direct subtraction* and the *determining the difference* conception (van den Heuvel-Panhuizen & Treffers, 2009), aligned with the use of *indirect addition*. Second, we propose that the process leading to the use of informal strategies is the mental simulation of the initial encoded representation of the problem. According to Barsalou (1999, p. 586), mental simulation consists of the construction of “specific images of entities and events that go beyond particular entities and events experienced in the past”. The involvement of dynamic and perceptual simulations in text comprehension and the processing of abstract concepts also gives great importance to the process of mental simulation in contexts where there are no actions involved, such as it would be the case on static word problems (Hostetter & Alibali, 2018; Zwaan, Madden, Yaxley, & Aveyard, 2004). We propose that in order to go beyond the mental simulation of the encoded representation and use a solving strategy based on arithmetic principles, a solver needs to recode the initial representation by relying on a different arithmetic conception.

In the current study we created arithmetic word problems whose mental simulation of the initial encoded representation would have either high or low cost. We predicted that different performance rates and different solving strategies will be observed as a function of the cost of the mental simulation of the encoded representation. We chose static problems, that do not allow straightforward mental simulation because of the absence of actions involved in the described situation. Thus, the mental simulation is only made possible due to the arithmetic conception evoked, entailing an encoding that triggers either a mental *direct subtraction* (*taking away* conception) or a mental *indirect addition* (*determining the difference* conception). On problems where the mental simulation of the initial encoding is inefficient and bears a high cost, we first expect to find lower performance rates and less formal solving strategies – those

that do not reflect the initial encoding. Second, we expected that the process of mental simulation will remain prevalent throughout the school year and that we will replicate the findings six months later. Third, we expected to observe informal solving strategies that reflect the mental simulation of the initial encoding of the problem on low cost mental simulation problems, and formal strategies reflecting a recoded representation mainly on high cost mental simulation problems.

METHOD

Participants

341 second-grade students from 16 classes coming from 11 elementary schools from working-class neighbourhoods in France participated in a collective classroom study at the beginning of the year. The average age of the children was 7.60 years ($SD = 0.33$, 177 girls). At the second time of testing, six months later there were 269 second-grade students from the initial cohort who participated (mean age = 8.02 years, $SD = 0.33$, 138 girls). Lastly, verbal protocols were conducted with 42 second grade students who were not part of the initial cohorts (mean age = 7.93 years, $SD = 0.26$, 23 girls).

Materials

There were 8 types of subtraction and addition problems belonging to 3 major categories corresponding to Compare problems 1, 2, 3, 4, Combine problems 1, 2, and Equalizing problems 1 and 2 from Riley et al.'s (1983) classification (Table 1). The number triplets involved in the data and the solution are (31, 27, 4), (33, 29, 4), (41, 38, 3), and (42, 39, 3). The number triplets were combined in order to create high and low cost mental simulation versions of each problem category in the same way as it was done in previous studies (Brissiaud & Sander, 2010; Gvozdic & Sander, 2020). To control for the impact of position, numerical sets and context, 8 different problem sets were created. Another 8 problem sets were 'mirror' sets in which the low cost version of one problem would be presented in its high cost counterpart, while the high cost problem would be presented in its low cost counterpart.

Problem category	Low cost mental simulation	High cost mental simulation
Compare 1	“There are 27 roses and 31 daisies in the bouquet. How many daisies are there more than roses in the bouquet?”	“There are 4 roses and 31 daisies in the bouquet. How many daisies are there more than roses in the bouquet?”
Compare 4	“Anna has 31 euros. Susan has 4 euros less than Anna. How many euros does Susan have?”	“Anna has 31 euros. Susan has 27 euros less than Anna. How many euros does Susan have?”
Equalizing 1	“There are 27 oranges and 31 pears in the basket. How many oranges should we add to have as many oranges as we do pears?”	“There are 4 oranges and 31 pears in the basket. How many oranges should we add to have as many oranges as we do pears?”
Combine 1	“There are 27 blue marbles and 4 red marbles in Marc's bag. How many marbles are there in Marc's bag?”	“There are 4 blue marbles and 27 red marbles in Marc's bag. How many marbles are there in Marc's bag?”

Table 1: Examples of arithmetic word problems in their low and high cost mental simulation version for the number triplet (31, 27, 4)

Problem category	Cost of mental simulation	Informal strategies		Formal strategies	
Combine 1	Low	$27 + 4 = \square$		$4 + 27 = \square$	
	High	$4 + 27 = \square$		$27 + 4 = \square$	
Combine 2	Low	$27 + \square = 31$	$31 - \square = 27$	$31 - 27 = \square$	$\square + 27 = 31$
	High	$4 + \square = 31$	$31 - \square = 4$	$31 - 4 = \square$	$\square + 4 = 31$
Compare 1	Low	$27 + \square = 31$	$31 - \square = 27$	$31 - 27 = \square$	$\square + 27 = 31$
	High	$4 + \square = 31$	$31 - \square = 4$	$31 - 4 = \square$	$\square + 4 = 31$
Compare 2	Low	$31 - \square = 27$	$27 + \square = 31$	$31 - 27 = \square$	$\square + 27 = 31$
	High	$31 - \square = 4$	$4 + \square = 31$	$31 - 4 = \square$	$\square + 4 = 31$
Compare 3	Low	$27 + 4 = \square$		$4 + 27 = \square$	
	High	$4 + 27 = \square$		$27 + 4 = \square$	
Compare 4	Low	$31 - 4 = \square$		$4 + \square = 31$	$31 - \square = 4$
	High	$31 - 27 = \square$		$27 + \square = 31$	$31 - \square = 27$
Equalizing 1	Low	$27 + \square = 31$	$31 - \square = 27$	$31 - 27 = \square$	$\square + 27 = 31$
	High	$4 + \square = 31$		$31 - 4 = \square$	$\square + 4 = 31$
Equalizing 2	Low	$31 - \square = 27$	$27 + \square = 31$	$31 - 27 = \square$	$\square + 27 = 31$
	High	$31 - \square = 4$	$4 + \square = 31$	$31 - 4 = \square$	$\square + 4 = 31$

Table 2: Classification of solving strategies for different problem categories from Riley, Greeno and Heller's (1983) classification

Procedure

Each student solved 4 low cost mental simulation problems and 4 high cost ones. In the collective classroom part of the study, each student received an 8

page booklet. Each problem was read aloud twice, and students then had one minute to write the number that was the solution. Regarding the collection of verbal protocols, students were asked to give an oral explanation of how they found the solution after writing down the answers. Their responses were recorded and transcribed.

Scoring

The solutions noted by the children were scored with 1 point when the numerical answer was exact or, in order to allow for mistakes in counting procedures, within the range of plus or minus one of the exact values. Any other answers received 0 points. On the verbal protocols, the strategy students described were also assessed. Two coders evaluated the solving strategies of 10 students by writing down the number sentence they considered corresponds to the descriptions children gave. The initially obtained inter-rater reliability was 98.75% with the Cohen's kappa score of 0.982, providing an almost perfect level of agreement. The informal strategies corresponded to the mental simulation of the initial encoding, and the formal strategies corresponded to strategies that did not correspond to the initial encoding (Table 2). Two separate codings were done, one for the informal strategy and one for the formal strategy. When the student provided an informal strategy, this was scored as 1 point for the informal strategy. When a student described a formal solving strategy, this was scored as 1 for the formal strategy.

RESULTS

Performance

We compared students' success rates on low and high cost mental simulation problems. Since the data points for the responses were binary and recorded in a repeated design (with low and high cost mental simulation problems), we conducted random-effects logistic regressions. We constructed a generalized linear mixed model (GLMM) with a binary distribution with the cost of mental simulation (low vs. high) as the fixed factors, while participants and problem categories were included as the random effects. In accordance with our hypotheses, at the first time of testing, the analyses showed a highly significant main effect of the cost of mental simulation on performance ($\beta = 1.05$, $z = 11.12$, $p < .001$). The low cost mental simulation problems had a 1.69 times higher success rate than high cost mental simulation problems (Figure 1A). At the second time of testing in the collective classroom study, the effect was replicated the analyses revealed a highly significant main effect of the cost of mental simulation on performance ($\beta = 1.33$, $z = 12.22$, $p < .001$). The low cost mental simulation problems had a 1.59 times higher success rate than high cost ones (Figure 1B). Lastly, the gap in performance on low and high cost mental simulation problems was also replicated in the verbal protocols, confirming that

low cost mental simulation problems are significantly easier ($\beta = 1.4$, $z = 789.6$, $p < .001$) (Figure 1C).

Solving Strategies

Further on, we analysed the strategies used by the students. We aimed to show that re-representation leading to the use of arithmetic principles was predominant only when mental simulation was inefficient. We conducted two GLMMs, one with the informal strategies and one with the formal strategies. Both were GLMMs with a binary distribution and the cost of mental simulation as the fixed factor and participants and the problem categories as the random effects. As predicted, both differences were significant. Informal strategies were used significantly more on low cost mental simulation problems than on high cost mental simulation problems ($\beta = 3.12$, $z = 8.04$, $p < .001$). Formal strategies were used significantly more on high cost mental simulation problems than on low cost ones ($\beta = -3.95$, $z = -4.99$, $p < .001$) (Table 3).

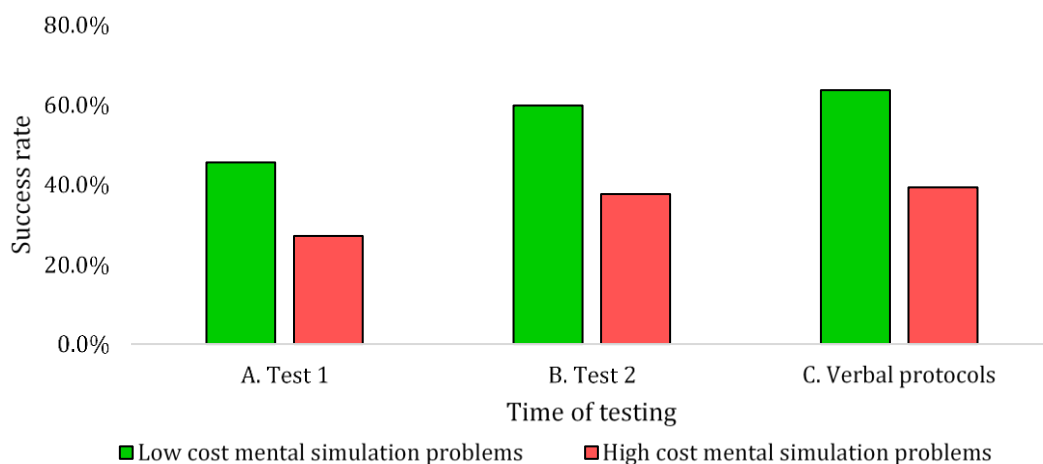


Figure 1: Success rates at the three different times

	Problem types	
	Low cost mental simulation	High cost mental simulation
Informal strategy	97%	23%
Formal strategy	3%	77%

Table 3: Rate of use of informal and formal solving strategies

DISCUSSION

In the current study we proposed that the cost of the mental simulation of the encoded representation would constrain the difficulty of different word problems as well as the solving strategies. Problems that do not depict actions in their wording were chosen in order to show that mental simulation operates on

the encoded representation rather than the depicted situation. By looking at the performance rates, we demonstrated that the problems hypothesized to involve a low cost for the mental simulation of the encoded representation are and remain easier for students throughout the school year. We also provided evidence that formal solving strategies that do not reflect the initial encoding are dominant only among high cost mental simulation problems, whereas informal solving strategies are almost systematic on low cost mental simulation problems. Overall, our study provides evidence that behind student's use of informal solving strategies is a non-mathematical mental simulation of the encoded representation, while the use of formal arithmetic strategies is dependent on the recoding of the initial representation. The processes we propose take place in arithmetic problem are illustrated in Figure 2. As this figure displays, the mental simulation does not operate directly on the situation described by the problem, but on the encoding of this situation, constrained by the arithmetic conception evoked by the problem statement.

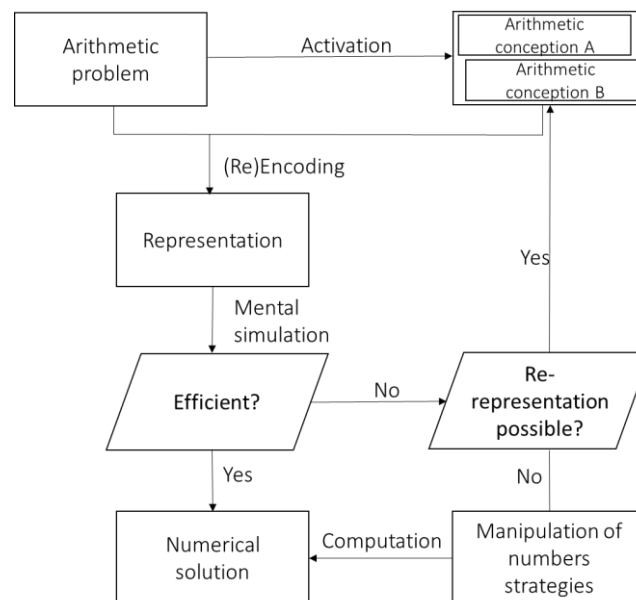


Figure 2: Modelling the encoding and recoding processes in word problem solving

An essential objective in mathematics education is to provide students with the necessary knowledge to select the most appropriate strategy for finding the solution to a problem (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). Our findings provide insights for evaluating the acquisition of such knowledge. Indeed, if problems which can be easily simulated mentally are used in school evaluations, we are not actually evaluating an adaptive strategy choice but merely informal knowledge with which students already come to class. Our findings provide insight into what kind of content is better suited for evaluating actual learning objectives: it is only when the initial encoding of the content makes informal solving strategies difficult to use, that students are actually given the opportunity to put their arithmetic knowledge to the test. Furthermore,

it has been demonstrated that when teachers are faced with problems that are intuition-consistent, they overlook the difficulties that such content poses for students (Gvozdic & Sander, 2018). Our current findings therefore also bear high pedagogical relevance not only evaluating adaptive expertise, but for teaching it, since it could be useful to work on comparing different kinds of low and high cost mental simulation problems in order to overcome intuitive conceptions and favour the acquisition and use of formal solving strategies.

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978-616-93830-1-7