MATHEMATICS CONTENT AND MATHEMATICS PEDAGOGY KNOWLEDGE: HOW DO THEY INTERACT?

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This paper raises questions about interactions between mathematics content and mathematics pedagogy knowledge. We draw on data from a larger study on examining secondary preservice mathematics teachers' proficiency in algebra and its teaching. These data were collected in an Australian initial teacher education course, where a set of algebraic word problems were implemented, and were analysed by using sample indicators of school algebra (SA) and algebra teaching (AT) knowledge, drawn from the Teachers' Knowledge of Algebra for Teaching framework (Reckase et al., 2015). The findings highlight complex interactions between the two knowledge categories, where SA knowledge provides a foundation for AT yet does not guarantee it.

INTRODUCTION

The fields of teacher education and professional development largely recognise the importance for teachers to have profound content knowledge and its pedagogy. Much of the beginnings of knowledge for teaching are rooted in Shulman's (1986) research that highlights and distinguishes between content knowledge (CK) and Pedagogical Content Knowledge (PCK). Knowledge of mathematics for teaching, widely known as Mathematical Knowledge for Teaching (MKT; Ball et al., 2008), originated from the notion of PCK. MKT has two broad components: Subject Matter Knowledge (SMK, or CK) and PCK. It is defined as the content knowledge used in recognising, understanding, and responding to the mathematical situations encountered in the classroom (Ball et al., 2008). Informed by earlier works, Baumert et al. (2010) and Blömeke et al. (2016) continued to study teacher knowledge in mathematics. Their conceptualisation similarly included CK and PCK as core aspects of the mathematical knowledge that is specialised, and specific for teachers to have.

Whilst these conceptual frameworks have been widely used in the research on teacher knowledge of mathematics, they are general. Teaching mathematics, especially in secondary schools, would require knowledge in specific content areas such as algebra, geometry, and trigonometry. There is still much to be known about secondary teachers' topic-specific knowledge and understanding. Furthermore, while it is typically acknowledged that the knowledge and understanding concerning a specific content domain, for instance algebra, geometry, or calculus (i.e. CK) would form a basis for its teaching (PCK), much remains unknown about the interactions between teachers' mathematics and mathematics pedagogy knowledge. Addressing the mentioned gaps, the current paper aimed to examine these interactions, in the context of algebra. The

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investigation presented here contributes to the ongoing discussions on mathematics teacher preparation and professional development. Additionally, it offers an analytical approach for analysing teachers' knowledge specific to algebra and its teaching that can inform future research and practice.

CONCEPTUAL FRAMEWORK

The conceptual framework used in this paper, titled Teachers' Knowledge of Algebra for Teaching (Reckase et al., 2015), is derived from that of McCrory et al. (2012) who categorised the knowledge required to teach algebra as: School Algebra, Advanced Mathematics, and Algebra for Teaching. School Algebra (SA) refers to the algebraic concepts that teachers would be expected to teach in secondary school classrooms. Advanced Mathematics refers to university level mathematics concepts that provide the teacher with an appreciation for the development of mathematical ideas beyond that which it taught in secondary school. Algebra for Teaching includes pedagogical knowledge such as strategies, common student misconceptions, and alternative definitions and affordances. The knowledge in this category "may fall into the category of mathematical pedagogical content knowledge, or it may be pure mathematical content applied in teaching" (Reckase et al., 2015, p. 252). In the current paper, we focused on algebra-specific knowledge (hereafter *SA knowledge*) and Algebra for Teaching (hereafter *AT knowledge*). We drew on data from a larger research study where it was not the intention to assess advanced mathematics knowledge, and therefore, the data available in this paper does not lend itself to an assessment of the category of Advanced Mathematics.

This framework has been applied in previous studies to assess teacher knowledge for teaching algebra. Reckase et al. (2015) developed a test of teacher knowledge of algebra spanning all three knowledge categories. Osei and Agyei (2023) applied the expanded Knowledge of Algebra for Teaching (or KAT) framework, based on the framework of McCroy et al (2012), to explore the knowledge for teaching algebra of in-field and out-of-field junior high school teachers in Ghana. They found that *SA knowledge* for both groups was the strongest knowledge domain, followed by *AT knowledge* for in-field teachers. They also found that *AT knowledge* initially improves with teaching experience by comparing teachers with 5-10 years' experience to those with less than 5 years' experience. The difference was not pronounced for teachers with over 10 years' experience compared to those with 5-10 years' experience. While not using the KAT framework, Lowrie and Jorgensen (2016), and Hatisaru et al. (2024) examined preservice secondary mathematics teachers' demonstration of algebraic skills (*SA knowledge*) in Australia. The authors suggested that a firm algebra knowledge base is necessary to ensure effective teaching (see also Baumert et al., 2010), and eventually to ensure students' learning of algebra.

METHODOLOGICAL APPROACH

The larger study used an exploratory approach to examine secondary preservice mathematics teachers' (PMTs') content-related knowledge and understanding for teaching algebra associated with solving algebraic word problems. In this paper, we focused on their *SA knowledge* and *AT knowledge* (Reckase et al., 2015). We applied these knowledge categories to the solutions and explanations that PMTs provided in response to a problem involving simultaneous equations. The following research question guided the paper: What are the qualities of *SA knowledge* and *AT knowledge* visible in PMTs' responses to the given problem, and how do these knowledge categories interact?

Data Gathering and Instrument

Data for the larger study were gathered from fifteen PMTs enrolled in a mathematics pedagogy unit at a metropolitan university in Australia. These PMTs would teach mathematics to students in years 7 to 12 (ages 12 to 18) after graduation. Within this unit, PMTs undertook problem solving activities, as part of their learning activities, for five weeks during Semester 2, 2023 where the focus was on Number and Algebra. Participation in the research was purely voluntarily, and PMTs' informed consent was obtained. At the beginning of each of these five classes, PMTs were asked to complete a reflection form with a purposefully selected algebraic word problem recorded on it. In total, five reflection forms and five problems were presented to the participants; each reflection form was completed in 20-25 minutes.

Table 1: The knowledge categories and reflection form items

Knowledge category	Item (#)
SA knowledge: implement different strategies to solve problems	*Item #1*: Think of and explain as many different possible solutions to the problem as you can. Name them as Solution A, Solution B, Solution C, so on.
AT knowledge: understand the strengths or weaknesses of solution strategies in teaching	*Item #2*: Assume you are solving this problem in the classroom. Please identify: (a) the solution(s) that you would use to solve it and why; (b) the solution(s) you hope your students would use, why.
AT knowledge: understand the potential of a problem in teaching mathematics	*Item #3*: Would this problem be useful for your students? Would it resemble problems you use?

SA: School Algebra; AT: Algebra Teaching

The reflection forms consisted of two pages. The first page presented a problem and Item #1 that required PMTs to solve the problem in as many different ways as possible. The second page had Item #2 and Item #3 (see Table 1). These items were extensions of Item #1 and situated in the context of teaching. These three items were generated

because we were interested in assessing PMTs' content-related understanding of school algebra (Item #1) and their understanding of the strengths and weaknesses of particular solution strategies to an algebraic problem (Item #2), as well as their capability to assess the potential of a given problem for teaching the relevant content (Item #3).

Participants and Task

To answer the research question in this paper, we analysed the responses of two PMTs, pseudonyms Logan and Noah. This was because, based on their performance in solving all five problems, overall, Logan and Noah were typical examples representing the PMTs whose *SA knowledge* was weaker (Logan) or stronger (Noah), and who responded to all *AT* items. This enabled us to describe how *SA knowledge* and *AT knowledge* interact, or how *SA knowledge* could influence the *AT knowledge*. Logan was doing a mathematics minor while Noah was doing a mathematics major. At the time of the study, they had taught school students in the capacity of a teacher assistant as either a preservice teacher on placement or as a private mathematics tutor. Due to page restriction, among the five algebraic word problems, we picked the Any Two Numbers (below) problem. The other good reason was we have reported the data associated to the other problems elsewhere (Hatisaru et al., 2024).

> If you are given the sum and difference of any two numbers, show that you can always find out what the numbers are. (Kieran, 1992, cited in Hatisaru et al., 2024)

Data Analysis

We analysed the data deductively using the two knowledge categories mentioned earlier: *SA knowledge* and *AT knowledge*. We independently examined the participants' solutions to the problem (Item #1) and read their responses to reflective questions (Item #2 and Item #3) seeking instances where these knowledge categories were demonstrated. To analyse their *SA knowledge*, we examined how the participants solved the problem; what strategy or strategies they used (algebraic: symbolic / graphical, and numerical: systematic / unsystematic). To analyse their *AT knowledge*, we examined the participants' understanding of the strength or weaknesses of these strategies in teaching algebra. Additionally, we assessed the participants' understanding of the usefulness of this problem for teaching simultaneous equations. The following sections present illustrations of algebraic understanding that reveal the two selected participants' *SA knowledge* and *AT knowledge*, and how these knowledge categories interact.

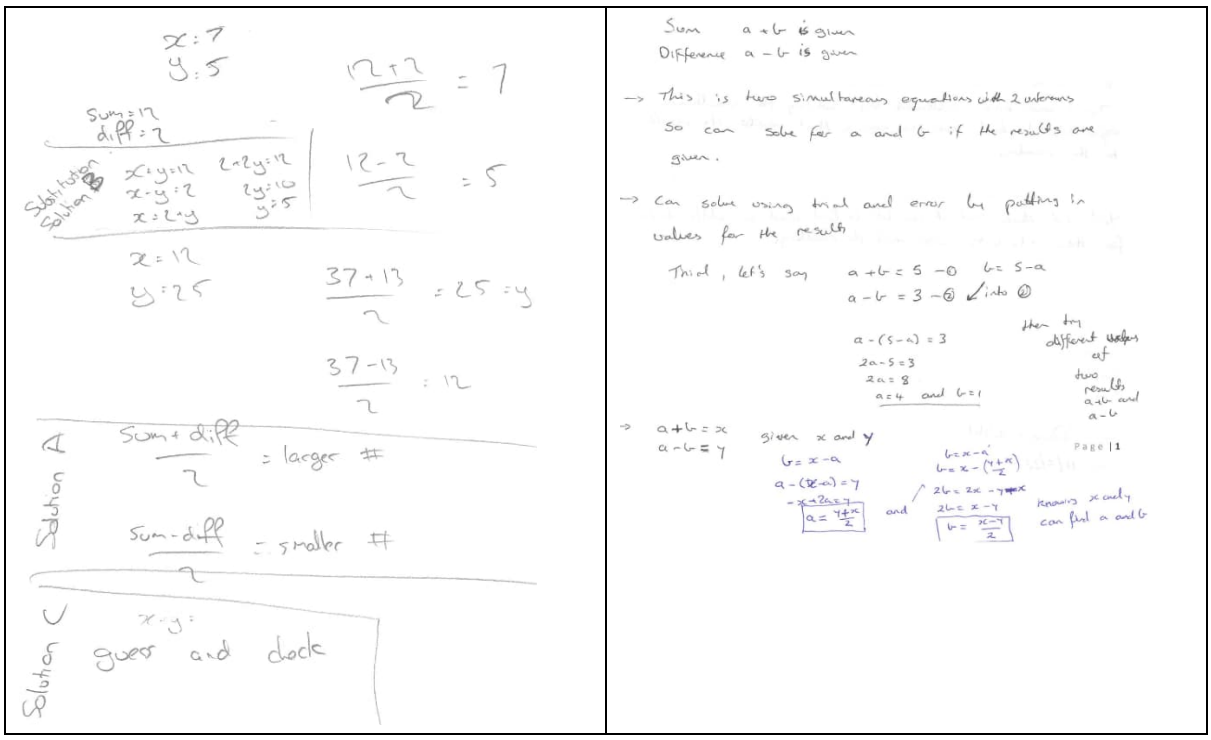
FINDINGS

SA and *AT Knowledge* of Logan

Overall, Logan's *SA knowledge* in terms of solving the Any Two Numbers problem and generating viable strategies to the problem was poor. As presented in Figure 1, Logan suggested three strategies to solve the problem. Solution A presented a correct semi-general result, although there was a complete absence of working or reasoning as

to how the result was derived (i.e. they did not show the results / address the question). The result was not fully general, or perhaps incomplete, in that the difference was not defined. Specifically, Logan likely assumed the difference to be defined as the larger number subtract the smaller number, but that was not made clear. Under that assumption Logan's result is correct; however, if the difference is defined more generally, then it is incorrect. Solution B provided a partial illustration of the steps for solving the simultaneous equations for a specific choice of sum and difference values, but Logan stopped short of solving for both numbers (just solved for one). Solution C simply indicated that the solution could be obtained by "guess and check". Simply saying guess-and-check without specifying any strategy to guide the iterations of the process does not constitute a valid mathematical solution.

Figure 1: Solutions of Logan (left) and Noah (right) to the given problem



Likewise, Logan's performance in *AT knowledge* items was unsatisfactory. Among the three solutions, Logan indicated that they would use "Solution B – substitution method [in teaching] as this is a valid method when progressing through mathematics in school" (Item #2a) and would hope to see this solution from students (Item #2b). Logan seemed to be unable to judge the correctness of the strategies they used (i.e. that Solution B did not show the result for a general sum and difference). Their assessment of the usefulness of the problem (Item #3) was that "it would force them to think outside the box". This assessment did not include any specific reference to the underlying mathematics. All that could be inferred from this is that Logan recognised that this was not a basic/standard problem, and that it would serve to extend students beyond those basic/standard problems.

SA and *AT Knowledge* of Noah

Noah gave a correct answer to the Any Two Numbers problem and generated viable strategies (*SA knowledge*). As shown in Figure 1, Noah presented two solutions. The first solution involved the specification of particular values for the sum and difference and the use of the substitution method to solve the associated simultaneous equations. This was then generalised to derive formulae for the two numbers given arbitrary sum and difference values. These formulae were supported by a full derivation which included a clear definition of the difference.

Noah's responses to the *AT* items were equally satisfactory. In response to Item #2a, Noah stated that they:

> Realised that it was a simultaneous equations problem and then thought that can apply test cases. Then went on to try to find a general solution if given the result of $a + b$ and $a - b$ as x and y.

This demonstrates an awareness of good problem-solving strategies, and how to scaffold the consideration of a more difficult problem by considering easier versions (i.e. test cases) initially to build understanding. In their response to Item #2b, Noah indicated that they would like their students to use:

> Trial and error followed by solving the simultaneous equations to find two expressions that relate the results to the numbers.

It is important to note that Noah's use of the term "trial and error" is misleading, as what they mean is the use of test cases and the solution of the resulting simultaneous equations. The key point is that Noah hoped that their students would approach the problem in the same way that they did, by starting with some specific examples to build understanding, and then generalise.

Like Logan, Noah did not refer to any mathematical specifics when reflecting on the usefulness of the problem (Item #3) and simply stated that: "It would be best used for differentiation for those students who need the challenge." This again reflects the recognition that this is not a basic/standard simultaneous equations problem, and its utility lies in extending students beyond a basic understanding/application.

DISCUSSION

Secondary preservice mathematics teachers (PMTs) solved the Any Two Numbers problem presented above. Their *SA knowledge* was captured by Item #1: ability to solve the problem using different strategies and explain them. *AT knowledge* is captured by Item #2: choosing teaching strategies and anticipating students' approaches, and Item #3: judging the problem's usefulness/potential for teaching (see Table 1). The analysis of responses of two PMTs (Logan and Noah) to the problem has revealed both strengths and weaknesses in their *SA knowledge* and *AT knowledge*. The provided elicitations about the given problem can inform mathematics teacher educators in enacting a vision

of teacher education and/or professional development and plausibly can improve the design of relevant professional learning programs.

The analysis has highlighted important interactions between the two knowledge categories (Figure 2). Firstly, Noah who demonstrated stronger *SA knowledge*, as evidenced by their ability to solve the given problem by using viable strategies, also tended to show a better understanding of how to apply these strategies in teaching contexts (*AT knowledge*). Conversely, Logan whose own capability to solve the problem was poor showed limited scope to use viable solution strategies in assumed teaching situations. This finding supports the notion that mathematics content and mathematics pedagogy knowledge are closely related and that a solid foundation in mathematical content is essential for conducting effective teaching (Blömeke et al., 2016). Next, while overall it has appeared to be that *SA knowledge* and *AT knowledge* were associated, that association appeared to be nonlinear. More specifically, even Noah who gave firm solutions to the problem (*SA knowledge*), showed instances where their *AT knowledge* was lacking, particularly in their ability to reflect on the mathematical base of the given problem. This suggests that while a strong content knowledge base is necessary, it is not sufficient on its own to ensure effective teaching. Having viable strategies at hand and analysing the suitability of particular problems for teaching particular content are essential parts of *AT knowledge* (Reckase et al., 2015).

Mathematics content and mathematics pedagogy knowledge, which are defined as two distinct categories of discipline-specific knowledge, typically associate with one another, and share common elements (Blömeke et al., 2016). The observed interactions between *SA knowledge* and *AT knowledge* aligns with previous research suggesting complex interactions between these knowledge domains (Lowrie & Jorgensen, 2016; McCrory et al., 2012; Osei & Agyei, 2023). While a strong *SA knowledge* does not guarantee well-developed *AT knowledge*, it appears to provide a foundation upon which *AT knowledge* can be built. Conversely, limitations in *SA knowledge* could constrain demonstration of *AT knowledge*. This implies that teachers lacking firm algebra understanding may struggle to unpack algebraic ideas for teaching effectively.

Figure 2: The SA-and-AT knowledge interaction pathway

Logan

Negative *SA knowledge → AT knowledge interaction*: poor *SA knowledge*, concerning solving the given problem in different ways, lead to *AT* choices that risk demonstrating incomplete reasoning to students and offer thin pedagogical rationales.

Noah

Positive *SA knowledge → AT knowledge interaction*: a firmer *SA knowledge* lead to coherent *AT* that scaffolded teaching and aligned student expectations, though Item #3 commentary could be enriched with a mathematical base.

CONCLUSION AND FUTURE RESEARCH

Based on the responses of Logan and Noah, in Figure 2, we present a model of *SA-and-AT knowledge interaction pathway*. According to this model, *SA knowledge* can hinder *AT knowledge*, as in Logan's responses, or can cultivate it, as in the responses of Noah. This model, however, is hypothetical and generated using data from only two PMTs. While it illustrates possible pathways between content-specific knowledge and the teaching of this content, the model should be treated as tentative. Future research with a larger and more diverse sample is needed to test, refine, and validate these interactions. Furthermore, this work could be extended in several ways. Firstly, data could be gathered from university mathematics students who do not have a pedagogical/teaching background. This may provide insight into the extent to which *AT knowledge* follows, or does not follow, naturally from a *SA knowledge* base. Secondly, the reflection forms can be extended to allow insights to be drawn regarding Advanced Mathematics knowledge domain and its interaction with *AT knowledge*. It would seem reasonable to expect that the Advanced Mathematics knowledge would enhance the depth and breadth of *SA knowledge*, thus providing a superior base upon which *AT knowledge* can be built.

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