INSTRUCTIONAL LANGUAGE IN SCHOOL TEXTBOOK PROOFS: A CROSS-CULTURAL STUDY

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Proofs are often conceptualised as sequences of deductively connected assertions, yet in practice they frequently include instructions that guide the reader's actions. Little is known about how culturally or linguistically dependent such proof language may be. This study examines how instructional language is used to structure written proofs in two school textbook traditions, by analysing multiple versions of the same theorem. In doing so, we refine an analytical framework for proof discourse. In one tradition, proofs are densely expressed through symbolic statements with few instructions, whereas in another they rely on frequent verbal instructions that guide the reader step by step. These contrasts identify culturally contingent features of proof language and their potential implications for students' engagement with proving.

INTRODUCTION

Proving lies at the core of mathematical activity (Skovsmose, 2024), and many school curricula require students to engage in related processes (*e.g.*, Brazil: Simas & Amaral, 2025; Japan: Fujita & Jones, 2014; Italy, Norway, and Slovakia: Kohanová et al., 2025). This foregrounds philosophical and epistemological questions, such as what a proof is and how it is communicated in basic education, as key concerns for research on proof. Recent studies suggest that characterising proofs as mere sequences of logical assertions does not adequately capture actual proof practice (Tanswell & Inglis, 2024; Weber & Tanswell, 2022). In practice, many proofs appear at least partly as lists of instructions, so-called recipe proofs, that guide readers through actions leading to the desired result (Weber & Tanswell, 2022).

The prominence of such instructional forms has long been noted. Pimm (1987) and Ernest (1998) observed that imperatives are a recurring feature of mathematical writing and a "topic worthy of considerable attention" (Pimm, 1987, p. 72), although neither examined them in depth. More recently, Weber and Tanswell (2022) extend this line of inquiry by analysing imperatives systematically and by proposing a model of what it means to understand a recipe proof. They argue that such proofs require a different kind of comprehension than models that treat them as mere sequences of assertions, since readers must track and justify a construction process specified by instructions. Complementing this, Tanswell and Inglis (2024) draw on a large corpus of research papers to show that imperatives and other instructional forms are

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widespread in contemporary mathematical proofs, indicating that recipe-like features are a pervasive aspect of modern proof writing rather than a marginal phenomenon.

These developments align with perspectives that view proofs as a specialised textual genre. From this standpoint, linguistic choices play an active role in how mathematical arguments are communicated and interpreted (Dawkins & Vroom, 2025). Proving can therefore be understood as a culturally situated practice, shaped by mathematical and educational traditions that establish distinct norms for argumentation and proof presentation (Jahnke et al., 2023). This suggests that the linguistic realisation of proofs, including the use of instructions, may vary systematically across cultures.

Although argumentation and proving practices differ across educational traditions (Knipping & Reid, 2019), much less is known about how such differences appear in *written* proofs, particularly in school textbooks. Empirical work on the linguistic structure of textbook proofs is limited, as most analyses have focused on classroom discourse or on conceptual classifications of imperatives rather than on how textbooks organise instructional language across countries. This gap and the relevance of the theme motivate our research question: *How is instructional language used to structure written proofs in Austrian and Brazilian lower-secondary mathematics textbooks, and how do these patterns differ across the two contexts?*

To address this question, we analyse proofs of the triangle area formula drawn from seven Austrian and seven Brazilian lower-secondary textbooks, which allows us to keep the mathematical content fixed and to conduct a focused cross-cultural comparison. The analysis examines how proofs are linguistically organised and the role of instructions in structuring the proving process in each context. In what follows, we outline the theoretical perspectives that inform our analysis and the framework used to classify the linguistic units within each proof.

THEORETICAL FRAMEWORK

For the purpose of this study, we conceptualise a proof as an explanation accepted by a specific mathematical community (Balacheff, 1987; Jahnke et al., 2023) and regard its written form as a specialised genre of mathematical text rather than a purely logical object (Dawkins & Vroom, 2025). As a genre, proof is characterised by recurrent ways of organising the activity of proving, including how assumptions, intermediate steps and conclusions are introduced, connected and warranted in local and global argumentation structures (Knipping & Reid, 2019). These organisational patterns are realised through a specialised linguistic and semiotic register, whose choices not only express logical relations but also orient the reader’s actions and attention within the argument (Dawkins & Vroom, 2025; Jahnke et al., 2023). Understanding proof as text therefore entails examining how linguistic forms structure the reader’s epistemic engagement in the proving process, a perspective that provides the basis for analysing the role of instructional language in written proofs.

In practice, particularly in school geometry, proofs are rarely presented solely as static sequences of deductively linked assertions. Instead, they frequently include explicit instructions that guide the reader through constructions or transformations (Wittmann, 2014). Such proofs have been discussed under the label recipe proofs, emphasising their procedural organisation and reliance on action-guiding language (Weber & Tanswell, 2022; Fischer & Oldenburg, submitted for publication). In these cases, the progression of the argument is intertwined with operations on diagrams or objects, and the execution of these actions serves as the justification for the claim (Weber & Tanswell, 2022). This operative dimension has long been recognised in mathematics education, particularly in studies of geometric reasoning that describe proofs as construction-based or quasi-empirical (Herbst, 2002; Wittmann, 2014). Viewing proofs in this way foregrounds the role of action-guiding language in written proofs, raising questions about how such instructions contribute to the organisation of proving.

To analyse these forms of action-guiding language more precisely, drawing on Rotman's (2006) semiotic account and the analyses of Weber and Tanswell (2022), instructions are understood as linguistic moves that manage the reader's engagement with a proof. These moves typically appear as imperatives or in first-person plural constructions, and what matters is their functional role within the unfolding argument rather than their grammatical surface form. Rotman distinguishes between *inclusive* instructions, which establish a shared frame of reference or "world" (e.g., assumptions, scenarios, definitions, perspectives), and *exclusive* instructions, which dictate concrete operations within an already established frame. This provides an initial functional division of instructional language. Weber (2023) refines this view by identifying additional categories, *inference*, *observational* and *reference* instructions, that capture further ways in which proofs direct readers to conclude a fact, attend to a feature or locate a prior result. Together this distinction offers a structured lens for examining how textbooks organise the instructional components of proving.

Beyond these functional distinctions, research on proving has emphasised that mathematical argumentation is shaped by culturally embedded norms. Here, culture refers to shared norms and conventions that shape how proofs are produced, communicated and interpreted within mathematical communities. These norms form what some authors describe as argumentation cultures, which provide shared expectations about what counts as an acceptable justification and how it should be presented (Jahnke et al., 2023). Comparative studies illustrate this diversity in concrete practices of argumentation. Knipping and Reid's analyses of French and German geometry lessons, for instance, revealed contrasting approaches to constructing and validating arguments, with French classrooms focusing more on explicit verbal justification and German classrooms relying more heavily on diagram-based reasoning (Knipping & Reid, 2019). More recent work comparing German and

Japanese contexts has shown further variation in how examples, diagrams and inferential steps are integrated into proving (Knipping et al., 2025).

These differences suggest that cultural traditions may influence not only how proving is enacted in classrooms but also how proofs are linguistically realised in written materials. Viewing proofs as culturally situated practices thus makes it meaningful to examine whether, and in what ways, textbooks from different countries organise instructional language differently when presenting geometric proofs. This perspective frames our comparative analysis of Austrian and Brazilian textbook series, focusing on how instructions are used to structure the proving process.

METHODOLOGY

To ground our comparison in a shared mathematical context, we selected the proof of the triangle area formula. This theorem is well suited for comparative analysis because it appears in all of the textbooks examined and admits several different derivations. These include presentations based on a specific numerical triangle, constructive dissection arguments, generic proofs for arbitrary triangles, and derivations from the parallelogram area formula. This diversity of legitimate methods creates variation in how the proof is written, making the theorem a robust and comparable context across the two textbook traditions.

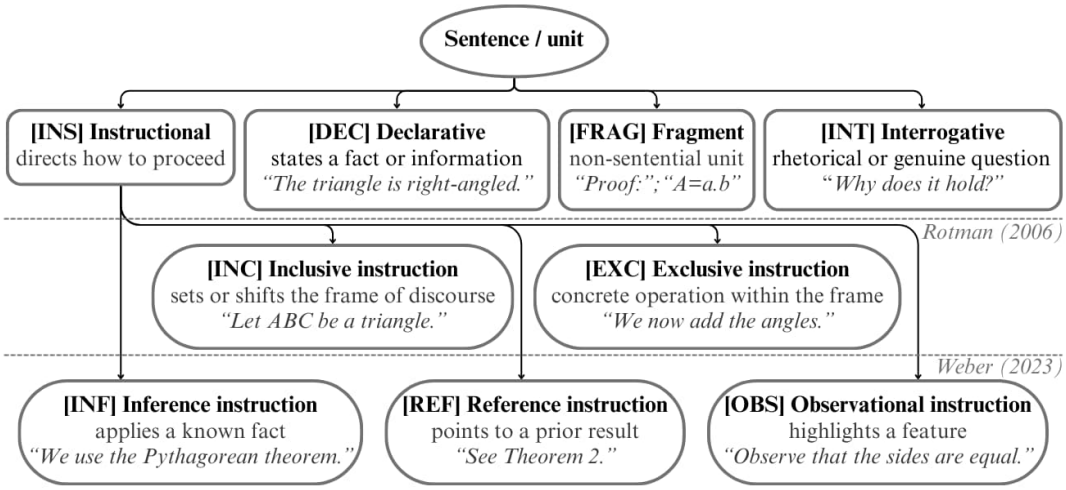
To examine how this proof is linguistically realised across contexts, we analysed fourteen lower-secondary mathematics textbooks, seven Brazilian and seven Austrian, containing fifteen proofs of the triangle area formula (one Austrian book presents two proofs). The Brazilian textbooks, written in Portuguese, are all intended for Grade 7, while the Austrian textbooks, written in German, are intended for Grade 6. In both countries, textbooks were selected from the most recent official lists of approved materials for lower-secondary education. In Brazil, they correspond to the most widely adopted titles among the 13 approved collections for this grade band; in Austria, they were chosen from the 19 officially approved titles for the common lower-secondary curriculum. This sampling strategy ensures representativeness while keeping the mathematical content fixed, enabling a focused comparison of the linguistic organisation of the proofs.

A systematic examination of the linguistic structure of the proofs required a clear unit of analysis. We considered every sentence or other standalone textual element within the proof as such a unit. Each proof was first segmented into syntactically complete units and then coded in a two-layer process following a framework developed on the theoretical background outlined before, see Figure 1. At the first level of coding, each unit was identified as an instructional unit or assigned to one of three categories defined for this study: declarative statements, for mathematical facts; interrogatives, which pose genuine or rhetorical questions; and textual fragments, which encompass structural textual elements such as headings or standalone expressions, especially formulas. At the second level, units coded as instructions were then classified into the

instructional types described in the theoretical framework (Weber, 2023): inclusive, exclusive, inference, reference and observational.

To ensure the reliability of this coding process, each proof was coded in its original language by a native-speaker researcher, and the English translation was coded independently by another researcher to ensure consistency across languages. We then compared codings and resolved discrepancies through discussion. All coding procedures were carried out using MaxQDA, a qualitative data analysis software.

Figure 1: Overview of the coding scheme for sentences/units in proofs.



Building on this coded dataset, we calculated the frequencies and proportions of each code and catalogued the verbs used in instructional units. We then examined the coded units qualitatively to identify the roles played by instructional language. This dual approach supports a focused comparison of linguistic structures across the two textbook traditions.

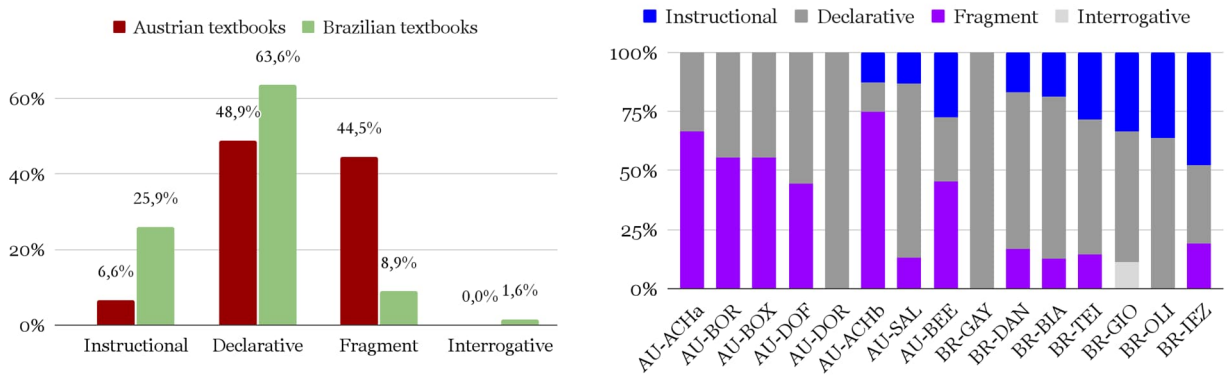
RESULTS AND DISCUSSION

Across the fifteen textbook proofs analysed, drawn from seven Austrian and seven Brazilian textbooks, clear cross-contextual contrasts emerged in how linguistic units structure the proof. A first difference occurs already in the first layer of our framework (Figure 1) and concerns the presence of instructional language: only three of the eight Austrian proofs include instructions (Figure 2, right), two of which occur in guided proof tasks on a specific example (e.g., Figure 3) rather than in stand-alone proofs, whereas six of the seven Brazilian proofs contain them. When aggregating all coded units, instructions account for less than 7% of the Austrian material and almost 26% of the Brazilian one. A further contrast concerns the markedly higher proportion of fragments in the Austrian corpus, which appear systematically at the conclusion of many proofs (Figure 3).

These patterns point to contrasting ways of organising support for the reading of proofs. In the Austrian corpus, fragments, typically titles or formulas restating the

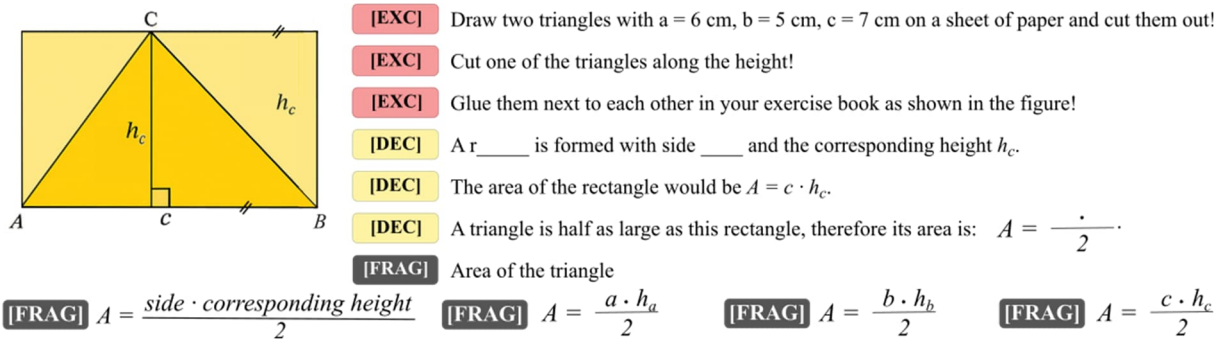
theorem, appear mainly as final summaries, foregrounding symbolic representation over discursive closure (Dawkins & Vroom, 2025). By contrast, in the Brazilian corpus the conclusion is usually integrated into the argumentative flow and frequently accompanied by explicit instructions. While understanding a proof requires readers to reconstruct inferential steps, the markedly higher proportion of instructional language in the Brazilian proofs provides more explicit guidance about how this reconstruction is to be carried out, whereas the predominantly declarative and fragment-based presentations in the Austrian corpus place greater responsibility on readers to supply these connections on their own.

Figure 2: Distribution of sentence types across the Austrian (AU) and Brazilian (BR) proofs, with aggregated proportions and individual textbook values.



Following our framework (Fig. 1), differences also appear in the types of instructions used across the two sets of proofs. *Inclusive instructions* occur in only one of the Austrian proofs, but in four of the Brazilian ones. *Exclusive instructions*, which appear in both contexts, display a similar surface profile: many of them involve action verbs such as “cut”, “copy”, “join” or “draw”, inviting the reader to carry out (or mentally simulate) materially grounded operations. In the Brazilian proofs, this category also includes imperatives such as “calculate”, “justify” and “follow”.

Figure 3: Example of proof adapted from AU-BEE (i.e., Beer et al., 2024, p. 128).



Compared with the construction-oriented exclusive instructions examined by Weber (2023), which coordinate abstract operations through “manipulative imagination”, the

exclusive instructions in our corpus serve a different epistemic purpose at school level, anchoring the proof in physically enactable actions. *Inference instructions* appeared in five Brazilian proofs only and mostly with the "we" form of instruction with the verbs "determine", "explain", "notice", "state", "verify" and "write". *Reference* and *observational instructions* are marginal in our corpus.

CONCLUDING REMARKS AND OUTLOOK

In response to our research question, this study shows that instructional language is used to organise textbook proofs in systematically different ways across the Austrian and Brazilian corpora. By examining how proofs are linguistically realised in these two textbook traditions, this study offers a methodological tool for analysing instructional language and proof structure. It also provides empirical evidence of cross-cultural patterns in these features. The findings contribute to current discussions on the nature of proof and the notion of rigour in school mathematics.

By showing that instructional language is not confined to advanced mathematical practice (Tanswell & Inglis, 2024) but is also a recurrent and culturally variable feature of proofs in textbooks, this study raises questions about its implications for students' learning of proof. An important direction for future research is to investigate whether, and in what ways, different linguistic organisations of proofs shape students' opportunities to engage with, interpret and make sense of proving.

One limitation of this study is its focus on a single proof of the triangle area formula, since discursive patterns may differ in other proofs or in domains outside geometry. The contrasts identified here may reflect idiomatic features of German and Portuguese or broader cultural conventions in Austria and Brazil; examining materials from additional German- and Portuguese-speaking countries will help clarify the origin of these differences. A further possible line of inquiry concerns the internal ordering of coded units within proofs: preliminary evidence from this corpus indicates systematic contrasts in how proofs are structured across the two contexts.

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