EMBODIED INSTRUCTIONAL INTERVENTIONS FOR THE EMERGENCE OF FORMAL GEOMETRY REASONING

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This design-based research (DBR) tests the self–friend–skeptic framework as embodied moves for the teacher to shift spatial-geometry reasoning from measurement to proof. In five teaching experiments (TE) with 16 non-STEM education undergraduates, an instructor-guided body-sized-cube activity was iteratively refined. Naming a 'skeptic' often failed, whereas justifying to an absent friend without a cube reliably prompted general, non-demonstrative warrants. Progressively constraining affordances recruited an inner skeptic and semiotic resources. We propose a classroom-ready principle: escalate audiences-of-conviction framing while withholding manipulatives.

INTRODUCTION

Over the past two decades, mathematics education has increasingly foregrounded embodiment, showing that gesture, movement, and material engagement are constitutive of mathematical thinking and learning (e.g., de Freitas & Sinclair, 2013). In line with the PME49 theme, Sensing and Making Sense of Mathematics, the present study views mathematical sense-making as emerging through bodily experience and interaction with the material world. We focus on the role of the teacher in embodied learning environments—a dimension that, despite growing attention (e.g., Abrahamson et al., 2022; Boonstra et al., 2025; Way & Ginns, 2024), remains underexplored. In particular, advanced forms of high-school reasoning and proving, especially in geometry, have only recently begun to be investigated from embodied perspectives (Marco & Shvarts, 2025). Addressing this gap, the present design-based research (DBR) study articulates empirically grounded design principles for teacher interventions that scaffold students' transition from 'naive' measurement-based reasoning to formal proving in spatial geometry.

THEORETICAL BACKGROUND

Embodied design and the teacher's role

A recent PME Research Forum synthesized intertwined principles of embodied design for mathematics: involve students' bodies in activity; enable immediate sensorimotor coupling with artifacts so that their affordances—perceived possibilities for action—can be discovered and support learning; design with semiotic sensitivity; orchestrate multiple semiotic registers to foster multimodal engagement and "languaging" (Palatnik et al., 2023). Recent studies highlight the teacher's embodied role: how

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instructors shape learners' perceptual fields, direct attention to relevant cues, and revoice gestures to support conceptual reorganization (Abrahamson et al., 2022; Boonstra et al., 2025; Way & Ginns, 2024). Yet, the field still lacks empirically grounded design principles for teacher facilitation in embodied contexts that involve advanced reasoning and proof (Marco & Shvarts, 2025; Palatnik & Marco, 2026).

Mathematical conviction and proving

Proving serves multiple functions in mathematics education. Stylianides et al. (2017) identified three complementary perspectives: proving as problem solving, as convincing, and as a socially embedded activity. This study focuses on the convincing perspective, viewing proof as a process through which learners establish and communicate mathematical conviction in relation to disciplinary norms. To conceptualize this process, we adopt Mason et al.'s (2010) self–friend–skeptic framework, which distinguishes between convincing different audiences. According to Mason et al., convincing oneself involves intuitive reasoning that builds personal confidence; convincing a friend requires articulating reasoning so others can follow; and convincing a skeptic demands more rigorous justification. These researchers also emphasize the role of an internal skeptic, an inner voice that subjects one's own reasoning to critical scrutiny, while stressing proof as an iterative process between personal insight and social validation via culturally accepted mathematical argumentation.

From embodied argumentation to the current study

This study builds on our PME48 work (Marco & Palatnik, 2025), which introduced embodied argumentation—the use of bodily movement and interaction with physical 3D models to construct and communicate convincing proofs in spatial geometry. That study demonstrated how embodied arguments served to convince different audiences: oneself through perceptual engagement, a friend through shared visualization, and a skeptic through verbalized descriptions of actions. Accordingly, this DBR investigates how to implement the self–friend–skeptic framework to support a shift from measurement-based reasoning to formal proving in spatial geometry. Our RQ: *In spatial geometry, how can the self–friend–skeptic framework be enacted to foster formal proof?*

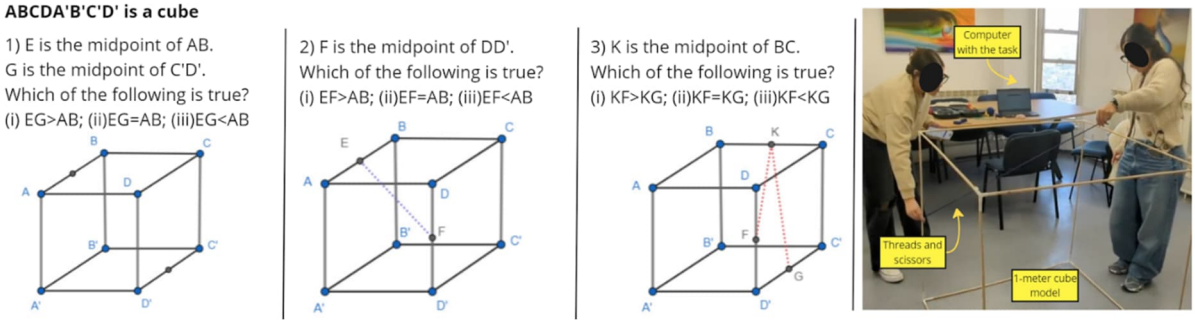
METHOD

This study employed a design-based research (DBR) methodology (Bakker, 2018) to develop and refine principles that support students' proving in spatial geometry. Each iteration was guided by a hypothesis about how learning might unfold within the designed environment. Following Bell et al. (2004), design principles were treated as intermediate constructs—inductively derived from successful cases and revised when they proved unproductive.

Research settings

Across six teaching experiments (TEs), we iteratively refined an instructor-guided spatial-geometry activity using a body-sized cube (Figure 1). TE1 was a preliminary try-out with Edi, an advanced mathematics bachelor who rejected measurement and therefore remains outside the current analytic scope. From TE2 onward, we adjusted task affordances, instructional interventions (prompts), specifically the audiences-of-conviction framing, to examine how self–friend–skeptic interventions may shift reasoning to formal argumentation.

Figure 1: The Midsegments in a Cube Task and research settings.



Sessions were conducted in a lab with a tangible cube and materials for auxiliary constructions, and a computer-based task. After IRB approval and consent, we recruited 16 education undergraduates with non-STEM specializations to work in pairs (TE1 was a single participant). Sessions lasted 23–54 min and were video-recorded (side camera; eye-tracking glasses). Eye-tracking was essential in this 3D, multi-artifact setting, as it revealed each participant's exact viewpoint and captured moment-to-moment shifts between artifacts, thereby reducing inferential ambiguity about what drew attention. We operationalized the self–friend–skeptic framework by asking, for each item, what would convince oneself, a friend, and a skeptic; participants entered skeptic-facing arguments in the digital form. Table 1 summarizes the TEs and instructor's interventions.

Data collection and analysis

Video data were annotated in ELAN 6.4. We marked all instructional interventions and the immediate participant reactions that followed them, including verbal responses and embodied actions. Each intervention-reaction episode was treated as a unit of instructional interaction. All interventions were then submitted to qualitative content analysis (Mayring, 2021), yielding five categories of instructional interventions: (1) Prompting Student Action on the Environment, (2) Clarifying and Elaborating Reasoning; (3) Instructor Action in the Environment; (4) Eliciting Prior Mathematical Knowledge; (5) Framing Audiences of Conviction. The present paper focuses on episodes triggered by Framing Audiences of Conviction and immediate participants' responses. Although later developments may occur, we delimit claims to short-term

effects because subsequent reasoning often reflects additional interventions and contextual factors, reducing the confidence of causal inference.

Table 1: Overview of relevant DBR cycles, participants, and interventions.

TE#	Participants	High school (HS) Math level	Session duration	# instructional interventions (total time)	# instructional interventions on audience framing
TE2	Li & Nina	Intermediate	23:22	9 (1:27)	1
TE3	Mika & Alia	Intermediate	48:24	35 (6:08)	7
TE4.1	Su & Ali	High	49:23	31 (4:07)	6
TE4.2	Sivan & Yael	Intermediate	51:55	47 (9:49)	8
TE5.1	Or & Din	High	40:42	34 (6:12)	1
TE5.2	Rim & Suha	High	1:03:42	49 (6:18)	11
TE6.1	Ina & Lara	Intermediate	53:51	51 (6:51)	5
TE6.2	Sara & Hila	Intermediate	40:56	51 (6:05)	12
Total:			6:19:15	307 (46:57)	51

FINDINGS

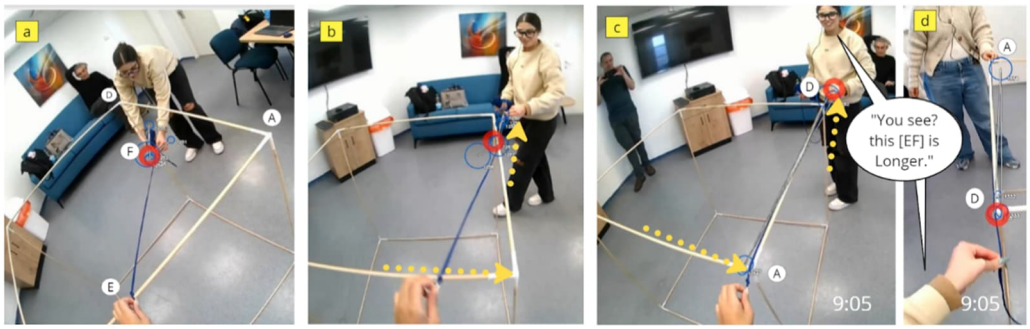
We report how Mason et al.'s (2010) self–friend–skeptic framework evolved across five teaching experiments (TE2, 3, 4.1, 4.2, 6.1). Rather than treating the framework as a fixed scaffold, each iteration probed its pedagogical potential: when does invoking the skeptic advance students' reasoning, and when does it fail to do so. The throughline of the findings is a shift from naming a social foreign persona ("the skeptic") to shaping communicative conditions ("a friend who is absent and lacks the cube"). We call this design move *distancing the friend*. Preserving Mason's progression in audiences-of-conviction orientation but operationalizing it as a constraint on available affordances and modalities—pushing students from visual demonstrations toward verbal arguments that go beyond concrete physical interactions with the environment.

TE2: Baseline: measurement persists, formalization is fragile

TE2 intentionally omitted the self–friend–skeptic interventions to observe participants' native trajectory. Across items, the participants, Li and Nina, used measurement systematically (Figure 2). Yet they inconsistently treated them as sufficient: went

beyond measurement in the second item but relied solely on it in the third. Two observations followed. First, measurements are attractive and easy to stage with the cube model; second, in the absence of explicit audience constraints, formalization that emerges is not always sustained. This led us to speculate that making audience expectations more explicit with Mason et al.'s framework might stabilize more rigorous justification across items.

Figure 2: A measurement for showing that EF>AD. (c) and (d) occurred. Yellow arrows indicate hand movements.



TE3: A 'social' skeptic fails, yet an inner skeptic emerges

In TE3, we introduced the self–friend–skeptic progression, but its uptake was surprising. Mika and Alia did not differentiate between a "friend" and a 'skeptic.' For example, when asked whether a particular argument would convince themselves, a friend, or a skeptic, Mika responded (9:00), "People. If I show that this is the measurement…" When pressed to imagine "a tough skeptic… a math teacher or lecturer," Mika replied she would use: "Measurements. Same same." Naming a socially authoritative skeptic did not nudge away from measurement.

Even so, later in the session, we observed several indications of their effort to construct more mathematically rigorous arguments and to refrain from relying on measurements. In Item 3, for instance, they abandoned measurement altogether, even as Mika asked Alia ironically, "(26:41) Should we compare threads again like Neanderthals?" This self-aware humor reflected an emerging sensitivity to mathematical standards of justification, a nascent inner skeptic whose origin was unclear but evidently not an immediate reaction to the introduced figure of the skeptic. Thus, while Mason's framework remained conceptually relevant, the social specification of the skeptic ("a tough teacher or lecturer") did not reconfigure their reasoning.

TE4.1: Absent friend elicits formal reasoning

In TE4.1, we again introduced the skeptic as a demanding mathematics teacher. Su and Ali (High HS level) began with a quick measurement in Item 1. When the instructor asked if it was convincing for them:

Su: "Basically yes, in a visual manner…"

Ali: "If we show it to him, yes. But can we show it to the friend? Or do we need to explain?"

Instructor: "Say, if you needed to say something to them or write…"

This one clarification, that the friend cannot be shown, only told, was enough to pivot them from empirical measurements to constructing a proof: they built a right triangle with EG as hypotenuse, established that it equals a face diagonal, and used the Pythagorean theorem to conclude it was longer than the cube's side. Notably, the 'skeptic' was not reintroduced. The modal constraint, 'cannot be shown,' did what the social persona had failed to do in TE3. This was the seed of distancing the friend: keep the audience socially close, but remove access to embodied demonstration and require verbal warranting modalities. However, at this stage, we could not determine if an absent friend sufficed because Su and Ali had a strong inner skeptic from their high HS background.

TE4.2: Skeptic induces defense; cube removal elicits abstraction

TE4.2 involved two participants (intermediate HS level) who first established EG longer than a cube edge by measuring. We then posed the Mason triad explicitly ("...what convinced you, what would convince a friend who is not here, and how you would convince a tough skeptic…"):

Sivan: "I would show it to a person physically. In the physical model, it's easier."

Instructor(s): "Suppose he [the skeptic] says: yes, for this cube it works… it's not exactly straight… not exactly the midpoint…"

Sivan: "I would tell him to bring his own cube… do it in several ways until you understand you are wrong."

Instructor: "Suppose we did not have a cube…"

Yael: "To be completely honest? I would not make the effort [laughs]."

Instructor: "If you had only the diagram…"

Yael: "Maybe using numbers?… in a cube all the edges are equal, EG is exactly the midpoint… maybe triangles… I don't remember the math…"

Two contrasts are striking. First, naming a "tough skeptic" elicited affective resistance and an adversarial tone ("bring your own cube," "until you understand you are wrong"). The skeptic became a social rival rather than an epistemic prompt. Second, removing the cube—a deprivation of affordances—quickly steered Yael toward abstraction: recalling givens, proposing numerical scaffolds, and cautiously mentioning imaginary triangle structure.

Similar patterns appeared in earlier TEs: in TE1, Edi called the skeptic "the devil," and in TE4.1, proof-oriented reasoning emerged only after clarifying the absent friend. The design lesson was that embodied distancing—removing the cube and requiring verbal or written explanations—better supports formalization than invoking a socially distant skeptic. Drawing on TE4.2 and prior TEs, we therefore dropped the skeptic persona and institutionalized distancing the friend: students first state what convinces them,

then convince a friend, and, if still using measurements, an absent friend without a cube.

TE6: Distancing the friend reliably shifts measurements to general reasons

Under the revised protocol, TE5 and TE6 showed that distancing the friend was highly productive in stirring reasoning to more mathematically formal argumentation—the emergence of an inner skeptic. For instance, one example from TE6.1, Item 1. After participants applied relative measuring to EG, the Instructor asked if an argument could convince an absent friend:

Ina:	"If he believes me, then yes."
Lara:	"I would tell him that I measured the diagonal and it came out longer than the edge."
Instructor:	"But what if he doesn't have a cube or a 3D pen?"
Ina:	"Then I do need to explain that it's always like that."
Lara:	"You'd need some mathematical background, I assume there's some rule..."

The absent-friend constraint triggered two shifts: a generality talk ("always like that") and a normative warrant search ("some rule"). No adversarial framing was required; the communicative condition was sufficient to recruit an inner skeptic.

DISCUSSION

Across five TEs, our progressively refined interventions embodied Mason et al.'s (2010) self–friend–skeptic framework: convincing was materially orchestrated by, first, permitting manipulation, then restricting physical interaction, so sensorimotor traces were verbalized as generalized warrants. In particular, the imaginary absent-friend condition (without a cube) propelled students to reorganize reasoning to be communicable without demonstration. This resonates with Boonstra et al. (2025), who show how teachers transform an initially non-mathematized action space into a geometrical one through reflection and verbalization ('embodied abstraction'). Yet their eight categories of teacher interventions do not address conviction or audiences; our audience-framed communicative constraint (self–friend–absent friend–skeptic) expands their account by curtailing measurement-laden affordances and steering learners toward shareable, general, culturally acceptable arguments.

Abrahamson et al. (2022, p. 169) suggest that to "transition students from movement to mathematics," one should "introduce symbolic artifacts into the activity space such as a grid of lines, numerals, or rhythmic structures." Our distancing-the-friend move complements this approach: we restrict the affordances of physical artifacts, prompting students to introduce semiotic artifacts on their own (e.g., Yael: "maybe using numbers?… maybe triangles").

To conclude, distancing the friend contributes a design principle for high-school geometry while progressively withholding manipulatives. This constraint may recruit

an inner skeptic, shift reasoning from measurement to general properties, and elicit student-generated semiotic resources (e.g., mathematical concepts), advancing abstraction and deductive reasoning in geometry.

Acknowledgments

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