IDIOSYNCRATIC METAPHORS IN A MATHEMATICAL INVESTIGATION: THE SIEVE OF ERATOSTHENES

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We engaged undergraduate liberal arts students in investigating patterns within the Sieve of Eratosthenes, a historical method for finding prime numbers in a given interval by crossing out multiples of primes. The investigation used a digital application in which numbers appeared in a grid, and changes in grid width produced different visual displays of prime and composite numbers. The participants examined how the grid width related to the appearance of primes in specific columns and described their approaches and conclusions in the form of a dialogue between interlocutors, referred to as "scripting journeys." The reported research examines how participants use metaphors to articulate and develop their reasoning within a mathematical investigation.

INTRODUCTION, CONTEXT AND PURPOSE OF THE STUDY

Research in mathematics education has consistently shown that metaphors play a central role in how learners make sense of abstract ideas. Past contributions in mathematics education have considered metaphors, analogies, and images as conceptual tools that enable learners to organize initial ideas before these ideas become formal structures (English, 1997). Drawing from Lakoff and Núñez (2000), metaphors serve not just as images but as critical cognitive tools for constructing abstract mathematical knowledge. They enable students to scaffold their understanding by transforming abstract concepts into more familiar, concrete ideas. Chapman (1997) extended this view by showing that metaphors can also shape learners' approach to problem solving by drawing attention to particular aspects of a situation while leaving others aside.

Sfard (2015) argued that metaphors should not be viewed as final or fixed representations of students' thinking. Instead, they function as working ideas that help learners move toward more stable forms of reasoning. In this sense, changes in the metaphors learners use may indicate developments in their mathematical activity. This aligns with the assertion that "metaphors serve as 'transplants' from familiar discourses into mathematical discourse, enabling new conceptual systems and ways of seeing what would otherwise remain abstract" (Sfard, 2009, p. 50). Thus, metaphors enable students to bring familiar ideas into abstract spaces, opening new ways of thinking and understanding mathematical concepts.

Metaphors play a central role in classroom participation and mathematical communication. Teachers and students often rely on familiar frames such as "learning

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mathematics is a journey," "doing mathematics is work," and "presenting mathematics is a story" (Yee, 2017, p. 211) to make sense of abstract processes. These shared metaphors shape classroom participation and influence how learners position themselves in relation to mathematics (Kaphesi, 2014). At the university level, research indicates that students continue to depend on metaphors for mathematical sense-making. Osman et al. (2022) showed that undergraduates in science, engineering, and mathematics programs describe mathematics through metaphorical images that guide how they understand the discipline. In advanced courses, lecturers also rely on metaphors to communicate abstract ideas, and students adopt these expressions as part of their own reasoning (Olsen et al., 2020). Similar patterns appear in teacher education, where Scheiner et al. (2022) showed that prospective teachers use metaphorical structures when organizing content and anticipating student thinking. In these settings, metaphors function as shared tools that support mathematical sense-making.

Research in specific mathematical domains also highlights how metaphors help learners connect their everyday experiences with formal concepts. In number theory, Pradhan (2019) showed that students make sense of prime and composite numbers through familiar actions and cultural artifacts. He noted that "the abstract concepts can be made more meaningful by connecting learners' real-life situations… [and] metaphors provide a powerful tool to understand one domain of knowledge in terms of another" (p. 78). At a broader curricular level, Dietiker (2015) described curriculum as a "mathematical story" (p. 246), illustrating how learners build coherence by linking tasks and ideas across time.

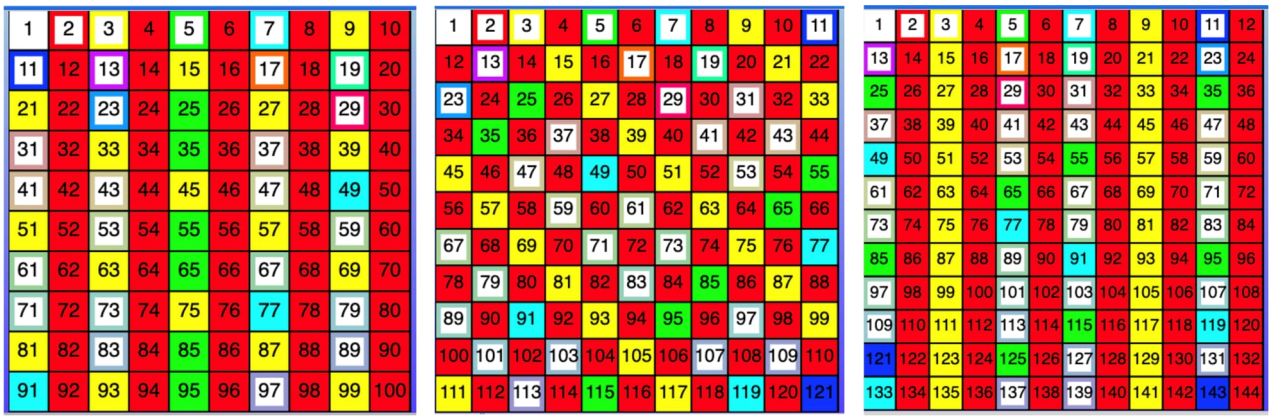
A common theme emerged across these studies: metaphors help learners move from what they notice to what they can explain. They tend to surface when students attempt to express causal relationships, test emerging ideas, or generalize structures. Pugalee and Schinck (2014) further emphasized this, stating that "students' metaphorical descriptions of mathematics as journeys, tools, or interconnected structures reflect their beliefs and influence how they make sense of mathematical tasks" (p. 115). This idea highlights how metaphors not only guide the understanding of abstract concepts but also shape how students approach mathematical challenges. The present study builds on this line of work by addressing the following research question: *How do undergraduate students use metaphors to articulate and develop their reasoning within mathematical investigations?*

METHODOLOGY

The participants in this study were undergraduate liberal arts students (n = 22) enrolled in a course titled *Numbers and Beyond*, an elective mathematics course used to fulfill a "course in quantitative reasoning" requirement for their degrees. The course addressed topics related to elementary number theory, such as prime and composite numbers, factors and multiples, divisibility, and prime decomposition. Participants were introduced to the Sieve of Eratosthenes, a historical method for finding prime

numbers in a given interval by crossing out the multiples of primes, through a digital application (www.visnos.com/demos/sieve-of-eratosthenes). In this application, numbers are displayed in a grid, multiples of primes can be crossed out manually or automatically, and the grid width can be changed from 4 to 30 (see Figure 1).

Figure 1: Sieve of Eratosthenes, grid widths of 10, 11, and 12.



Following the initial exploration of the Sieve with a grid width of 10, the participants completed a Sieve of Eratosthenes Task, in which they were invited to use the digital application and investigate in what columns prime numbers appear and in what columns there are no primes for different grid widths. Participants were required to present their investigation in the form of a dialogue, recording their approaches and explaining their findings. These approaches could include questions, conjectures (which were tested and either rejected or confirmed), and conclusions. This method of participants' reporting on how investigation was conducted is referred to in prior research as a "scripting journey" (Kercher et al., 2025), which is a novel data collection technique where "participants construct a dialogue that represents their experiences during a mathematical investigation" (p. 61), and what they choose to report represents what they found most relevant in their investigation.

The participants' scripting journeys, which provided an account of their exploration, comprised the data analyzed in this study. All scripts were imported into NVivo. We conducted line-by-line coding to identify metaphorical expressions, shifts in metaphor use, references to structural relationships, and moments of inquiry or verification, focusing on language drawn from non-mathematical domains. This initial phase generated a set of categories, including spatial metaphors, investigative metaphors, elimination metaphors, and narrative markers, which captured recurring patterns across the dialogues. The codes were then compared within and across student scripts to examine how metaphorical expressions aligned with or enabled the movement from noticing numerical regularities to articulating structural explanations.

Following Braun and Clarke's (2006) approach, the development of the themes was iterative and interpretive rather than a mechanical combination of codes. After initial coding, the categories were reorganized and refined into themes that captured how

students use metaphors to articulate and develop their reasoning within mathematical investigations. This process resulted in two analytical themes: (1) metaphors that describe mathematical structure, and (2) metaphors that frame the task through an investigative lens. These themes are presented in the following section.

FINDINGS

In this report, we focus on the scripting journeys of two pairs of students, Jay-Tim and Ken-Pam (pseudonyms), as their use of metaphors was most explicit and purposeful. Focusing first on the metaphors that describe a mathematical structure, we note that the Sieve of Eratosthenes itself is a metaphor, describing how primes are sieved out from the lure of numbers, similar to how a kitchen utensil is used to separate solids from liquids. In describing the process involved in the sieve, the usual reference in various learning resources is that multiples are "crossed out" or "shaded," so the uncrossed or unshaded numbers at the end of the process are prime.

However, in most participants' scripts, metaphorical language developed beyond these standard terms. The multiples of primes were described as being "eliminated," "killed," or "wiped out." Consequently, the remaining primes were seen as "survivors" in a process of "natural selection:"

Jay: It kind of feels like natural selection but for numbers.

Tim: Yeah, some get eliminated right away.

These metaphorical expressions then extended from elimination to reflect the survival of numbers that remain, highlighting the competition and weeding-out process. These metaphorical expressions continued to evolve as students articulated and developed their investigation and described the grid more systematically. In the following excerpt, Tim and Jay consider a grid with a width of 12:

Tim: Some columns are automatically doomed. They only ever hold multiples of 2 or multiples of 3 or whatever.

Jay: So if a column only spits out even numbers, then it is dead for primes.

Tim: Except maybe in the very first row, like 2 itself.

Tim: So those four columns are basically "composite factories." They never give us primes past the first row.

The reference to "doomed" columns and "composite factories" reframes columns not as organizational units but as sites in which primes cannot survive. Similarly, in the script of Pam and Ken, we observe this expanded perspective to describe broader regions of the grid. The following excerpt refers to a grid width of 30:

Pam: Whole neighborhoods of empty houses. Columns 2, 3, 4, 5, and 6 have no primes at all.

Ken: Columns that line up with multiples of base primes get wiped out. They overlap and form a desert of primes.

Pam: The smaller prime factors in the grid width, the more likely we are to see consecutive barren columns.

Ken: If the grid width K is highly composite, then many consecutive columns correspond to numbers divisible by small primes. That creates runs of empty columns.

Here, columns without primes are described as "neighborhoods of empty houses," where multiples are "wiped out." Eventually, noting "runs of empty columns," the students managed to formulate their conclusion: prime numbers can appear only in the columns in which the column number is coprime to the column width:

Ken: With width 10, primes beyond the first row only appear in columns 1, 3, 7, and 9. That's because the other columns all line up with multiples of 2 or 5. So the general rule here is: if a column number shares a factor with the grid width, it won't have primes—except maybe the very first row.

[…]

Ken: That generalization ties everything together. The prime-containing columns are those where the column number and the grid width have no common factors.

Next, in their scripting journeys, students used metaphors that frame a mathematical task through an investigative lens, positioning themselves as "mathematical detectives" who looked carefully at what appeared on the grid and checked their assumptions. They used this framing to organize their steps, explain their choices, and develop their line of inquiry. This is reflected in Jay's statement:

Jay: Yes, but let's act like detectives. You have to always check and confirm yourself. It will save you from future mistakes.

The notion of checking and confirming as a way of avoiding mistakes is seen by Jay as part of the practice of verification and systematics testing, employed by detectives. For Pam and Ken, the detective positioning adopted by the students guided their curiosity toward additional discoveries by considering different cases:

Pam: Absolutely. Let's keep going and discovering more, like detectives.

Ken: I am curious about different cases here. Why don't we try to find something where the primes appear in every column.

Further, Tim described their mathematical investigation as "narrowing down suspects in a mystery case."

Tim: Exactly. They're just the only candidates. It's like narrowing down suspects in a mystery case, you still have to test which ones are guilty of being prime.

Jay: I like that prime suspects.

The term "prime suspects," using homonymy of the term 'prime', shows a link between the terminology of detective investigation and that of mathematics. Tim's description of the activity as "narrowing down suspects in a mystery case" connects his reasoning

to an investigative process. In this frame of reference, the columns coprime with the grid width became "candidates" in mathematical terms or "suspects" in the detective sense, each needing further checking. Their phrase "guilty of being prime" continued this connection. Jay's response, "I like that prime suspects," showed interest in the idea and how the detective's frame shaped their talk.

This metaphor of "prime suspects" outlines further steps in their work. As investigators rule out suspects based on clear evidence, in a search for primes, the students removed columns from their search in which the column number shared a factor with the grid width. Consequently, focusing only on columns whose number was coprime to the grid width narrowed the search for those "guilty of being prime."

As they reflected on their work, Ken and Pam described the experience of finding primes as uncovering "hidden treasures:"

Pam: The more I hear about this, the more I think primes are like hidden treasures.

Ken: That's exactly how mathematicians describe them: beautiful, mysterious, almost like art.

Pam: But isn't it frustrating that we'll never find the "last" prime.

Ken: Maybe. But it's also comforting. It means there's always more to discover.

This moment in the dialogue shows how students began to see their mathematical work as an ongoing search that only pauses when the "hidden treasures" are discovered, leading to the beginning of another idea where there is "more to discover." Also, their comment about "never finding the last prime" captures the mix of curiosity and surprise they experience during the task. The open-ended nature of the task appeared to keep them engaged, and the treasure metaphor aligned with the way they worked, as each discovery encouraged them to seek out what might be hidden next.

DISCUSSION

This study explores how undergraduate students use metaphors to articulate and develop their reasoning while working with the Sieve of Eratosthenes. The findings highlight the central role metaphors played in shaping students' reasoning. Students used metaphors in two interconnected ways, which serve as the themes of this analysis: to describe the mathematical structure of divisibility and appearance of prime numbers within a grid, and to frame the investigative task, positioning students as active participants in the process of mathematical discovery.

This study addressed the research question: How do students use metaphors to articulate and develop their reasoning within mathematical investigations, by demonstrating that metaphors not only support understanding of abstract mathematical ideas but also organize students' investigative activity, positioning them as active participants in mathematical inquiry. By adopting the role of mathematical detectives, students engage more deeply with the task, transforming observations into structural explanations. This dual function of metaphors, describing structure and framing the

investigation, encourages students to take ownership of their learning, guiding them through more reflective and thorough work. Additionally, combining this task with the scripting journey approach provided a unique advantage, allowing students to reflect on and document their investigation, thereby enhancing their ability to track and refine their reasoning.

Students used metaphors to make abstract concepts tangible. Metaphors like “natural selection” and “neighborhoods of empty houses” helped students focus on divisibility relationships, describing how primes “survived” in certain columns while other numbers were “eliminated.” Beyond describing mathematical structures, metaphors shaped how students approached the task. By positioning themselves as “mathematical detectives,” students engaged in systematic checking and verification, narrowing “down suspects” to identify those “guilty of being prime.” This framing helped students justify their reasoning, reconsider assumptions, and develop generalizations, a process Grootenboer & Jorgensen (2009) and Hansen et al. (2022) describe as mathematical agency. The detective metaphor guided students’ reasoning and deepened their engagement.

This study contributes to research on metaphors in mathematics education by highlighting the unprompted use of idiosyncratic metaphors that helps students articulate their observations and describe their investigation. While previous studies focus on how metaphors provide a cognitive tool in facilitating understanding of abstract concepts (e.g., Falani et al., 2023), this study shows how metaphors shape the investigative process. Students, for example, use metaphors like “mathematical detectives” to guide their verification and exploration. It offers a unique contribution by showing how metaphors do not merely represent understanding but actively organize observations, conjectures, and generalizations during mathematical investigations.

In acknowledging the limitations of this study, we note that the sample size was small, and the grid’s maximal size of 30x30 in the digital application may have limited what students could explore. A larger grid could reveal additional patterns or push students to investigate higher numbers.

Future research could explore how metaphors evolve during mathematical activity and whether some are more effective than others in promoting conceptual breakthroughs. Further, future research may focus on the role the digital application played in students’ use of metaphors in their investigation. Extrapolating from Lakoff’s well-known title *Metaphors We Live By*, we highlight the importance of extending research on the “metaphors students move by” in their mathematical investigations.

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