TOWARDS THEORETICAL VALIDATION WITHIN COGNITIVE ROLE-PLAYING: A HIGH SCHOOL CASE STUDY

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This study frames theorem learning as a process of scientific "enculturation" fostering rational behaviour. We present a case study of a teaching setting that supports 10th graders' development of theoretical validation within Euclidean geometry. The analysis of the evolution of shared geometrical arguments along the didactical path shows a movement from naïve justification to theoretical validation within a geometrical system aligned with the standards of the mathematical community.

INTRODUCTION AND CONCEPTUAL BACKGROUND

This study is framed within the "approach to theorems" as a process of scientific "enculturation" (Morselli and Boero, 2009), consisting in the development of a form of rational behaviour, according to Habermas' (1998) perspective. We present a case study about a didactical setting contributing to the development of such enculturation.

In line with Mariotti's theoretical construct of Theorem (2006), we consider a theorem as a triple consisting of a Statement, a Proof, and a Theory, where theory is conceived both as knowledge (axioms, definitions, and theorems) and as meta-knowledge of the proving process (types of proofs and rules of inference). Statements and proofs gain meaning within a reference theory that ensures epistemic control. Proofs derive their meaning and rigor from their reference theory rather than from logical structure alone. From this perspective, proof construction is a dynamic and communicative process that only later leads to forms accepted by the mathematical community. Proof fulfills a central communicative function, into the need of a systematic reorganization of discourse aimed at making thought arguments more explicit and understandable both to oneself and to others (Zaslavsky et al., 2021). Proof can be seen as a socially situated practice in which meanings and acceptability criteria are negotiated within a community (de Villiers, 1990). Accordingly, we adopt a cultural perspective that acknowledges classroom context and allows, at least temporarily, proofs that are valid within the classroom community (Stylianides, 2007). Therefore, neither mathematical proof as a product nor the process through which it is constructed should be conceived as fixed or immutable; rather both are to be understood as culturally and socially situated activities, shaped by the level of awareness of the individual engaged in proving, whether a mathematician or a student. In this process the teacher plays a crucial role both as Agent of the Culture of Mathematics (Zaslavsky et al., 2021), leading students to explore mathematical knowledge, and as a cultural mediator to

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move students towards standards of theoretical validation, acting as guarantor for mathematical community and classroom community (Mariotti, 2006).

According to Morselli and Boero (2009), the process of scientific "enculturation" involves the intentional search for proving strategies, the control of epistemic aspects, and the communication of results through rules shared by the mathematical community. In mathematical proof, communication develops from self-communication to intra-group and then external communication addressed to the mathematical community, requiring explicit didactical attention (Ferrari, 2017). In line with this view, the Inside-Out Model includes the role of Blogger, responsible for communicating problem-solving processes and results according to mathematical norms (Albano et al., 2021a). For the Blogger role we distinguish two kinds of communication: inner communication -seen as an inner dialogue- and external communication -aimed at understanding- according to Habermas' perspective (1998). Moreover, the construction of a proof takes place as a social act, particularly for the Blogger role, thus requiring an intermediate phase within a restricted community whose shared language is still in the process of development (Albano et al., 2024). This process of social dimension of proof (Balacheff, 2010) is concerned with the Blogger role and characterized in a rationality perspective in Albano et al. (2024). The cognitive role-playing of Inside Out model enables the immersive engagement of participants in collaborative problem-solving activities through their identification with the characters of a story, experiencing the various cognitive roles that come into play in a mathematician's mind during the proving process, conceived as a particular form of problem solving. In this paper we focus on Blogger, but for sake of completeness, let us briefly recall the other roles: Promoter is in charge of advancing the exploration to move the problem-solving process forward, drawing on available resources or expert's advices; Boss is responsible for overseeing the problem-solving process, aligning prior results, goals, group engagement, and collaboration; Pest, representing critical thinking, seeks to ensure problem-solving validity by challenging conjectures and checking results against theory; Guru symbolizes wisdom and access to external resources for overcoming impasses, and acts as a cultural mediator between the group and the mathematical community. The actions of students engaged in immersive role-playing are guided by the aims and by the cognitive role of the character they play. Students can play as Actors -that are required to solve the proposed problems through direct engagement- or as Onlookers -that are required to actively observe the behavior of the corresponding Actor, to monitor the engagement with the problem, the actions, and the contributions, and to record all observations-. The roles of Actor and Onlooker, as well as the roles associated with the individual characters, rotate from one episode to the next in order to allow all students to experience the different roles involved and the varying levels of engagement in cognitive role-playing.

In this study, we aim to investigate how playing the role of Blogger, understood in a collective sense through the Actor/Outlooker dynamic, can support the emergence of

rationality in 10th-grade students' collective arguments involved in geometrical proving processes. To this aim we formulate the following research question:

RQ: How the above role-playing setting supports the development of modes of theoretical validation?

Specifically, we analyse how collectively shared arguments evolve along the didactical path, looking at the evolution of Blogger work/interventions.

METHODS

Context of research

In this paper, we present a teaching experiment carried out between March and May 2025 in 10th grade classes, involving 37 students of two Liceo Scientifico in South Italy. The intervention consisted of a total of 20 hours for each class, during which students engaged in problem-solving activities in Euclidean geometry, with a specific focus on Books I and III of Euclid's *Elements*.

Design and task

We conduct a qualitative case study (Cohen et al., 2002) in which we observe the evolution of the Blogger's role within a group engaged in conjecturing and proving activities. We chose to focus on Blogger as a specific instance, as this character acts as an intermediary between the group and mathematical culture, taking responsibility for reorganising and reformulating the group's work for external communication, while all actions remain collectively shared and negotiated within the group. The activities are developed through collaborative interactions in digital chats in which the students, playing the characters immersing in Moodle-platform, interact and collaborate in geometric problem-solving activities, communicating among themselves. The didactical path consists of various open-ended tasks presented in narrative form through a sequence of comic-strip episodes set in a dystopian world and requiring students to engage in conjecturing processes and, subsequently, in the construction of a proof aimed at the theoretical validation of the formulated statements.

Figure 1. Tasks and their translations

Task Ep.1.1 Task Ep.1.3 Task Ep.1.4 Task Ep.2 Task Ep.3

1.1: How many circles pass through a point P?
1.3: Let's see... we are therefore asking ourselves how many circles pass through A and B and where their centres are located.
1.4: Is it possible to draw a circle passing through A and B, with a centre that is not on the perpendicular bisector of AB?
2: Let us consider three non-collinear points... how many circles could you draw for the three points?
3: Look closely at the figure. What relationship do you think exists between angles alpha and beta?

The five tasks discussed in this paper are shown in Fig. 1 and correspond to distinct episodes (named Ep. 1.1. etc.) of the story, but between one episode and the next the students are asked to work on the task under the guidance of the Guru, steering by Teacher Rational Questioning Framework (Zhuang & Conner, 2022). It is worthy to note that the requests evolve along the progression of the students' work, giving rise to further, and sometimes more specific, tasks.

During the experiment, students are provided with on-demand access to the Euclidean Theory Bank (ETB), through a tool inspired by the Digital Toolkit for Proof (DTP) (Albano et al., 2021b). The ETB is a DTP, allowing students to reorganize and reformulate, through a logical deductive process, the theoretical justification of the statements they produce as conjectures. It contains axioms and a selected set of definitions and theorems drawn from Euclid's *Elements* and considered necessary for the resolution of the proposed tasks. Differing from the DTP, during the activities the ETB is enriched with new propositions conjectured and then proved by the students themselves, becoming in this way a dynamic and continuously evolving tool which the students feel as their own.

Data Collection and Tool of Analysis

The broader research project involves the analysis of multiple data sources, including synchronous discursive interactions in group chats, systematic observations by the teacher–researcher, and/or paper-based protocols. The present study focuses only on the emails sent to Euclid (Guru) at the end of each episode or sub-episode of the comic-strip narrative. These emails represent the final outcome of a process of collective negotiation and rewriting within the group, carried out by the Blogger for external communication with Euclid, representing the Agent of Mathematical Culture. To observe the development of modes of theoretical validation (Mariotti, 2006) looking at the evolution of Blogger final interventions, we chose to analyse the emails sent by Blogger to Euclid, using the analytical tool of Rational Mathematical Template of proof (Boero & Azrou, 2023). Thus, within collectively shared written texts, we look for the three components of Habermas' rationality (teleological, epistemic and communicative) search statements describing actions highlighting intentional proving strategies search, opinions revealing epistemic control and communication of results according to norms shared within the mathematical community (Morselli & Boero, 2009). From Habermas' point of view (1998) an action is rational when it is aim-oriented and is achieved by means that are intentionally chosen and implemented, an opinion is rational when it is possible to motivate it in a given context, a communication is rational when the use of language is oriented towards understanding.

RESULTS

Across the episodes, the role of Blogger was enacted by three different students. The role of Blogger was played in the initial episodes (1.1, 1.3, 1.4, and 1.5) by a student hereinafter named Eva, in episode 2 by a student named Tina, and in episode 3 named

Estelle. The three students were all characterised by a high level of accuracy and attentiveness and by active participation in the proposed activities. According to the teacher, all the three students show a school-related profile marked by commitment, rigorous study habits, and careful completion of assigned tasks. Consequently, they can be considered as sharing a largely comparable academic profile. Nevertheless some differences emerged: Eva was distinguished by strong participatory intuition and rigour, Tina by greater shyness, counterbalanced by a high level of preparation and Estelle by strong determination and a tendency towards perfectionism, despite being perceived by the teacher as slightly less strong than the other two. In the following, the transcripts of the emails in response to Tasks of Episodes 1.1 and 1.3.

1.1 - Eva's email to Euclid: Infinite circles pass through a point p because, since there are infinite points on the plane, it is possible to draw infinite circles passing through [P], all with different radii and centres. Did we manage it?!

1.3 - Eva's email to Euclid: Infinite circles pass through two points because it is possible to have infinite centres lying on the line perpendicular to the chord connecting the two points and passing through its midpoint, i.e. the perpendicular bisector.

In the statement "it is possible *to draw* infinite circles passing through [P]"(1.1. email) or "it is possible *to have* infinite centres *lying on* the line perpendicular to the chord" (1.3 email) we acknowledge a teleological dimension, as an action aim-oriented. The epistemic dimension emerges from the recognition of the infinitely many points in the plane (1.1) or on the perpendicular bisector (1.3) as potential centres of circles through P or through A and B. The communication is yet simple, the arguments are mathematically correct, but no explicit theoretical references are present. In both emails indeed we note the *implicit* use of the Postulate 3 of Euclid's Elements, but at this stage, the students had not yet requested access to the Euclidian Theory Bank. Moreover, the students' texts still reveal a search for confirmation from Euclid, who is recognised as the expert.

In solving Tasks of Episode 1.4, students requested access to the Theory Bank. In the following the email text produced:

1.4 - Eva's email to Euclid: It is not possible to draw a circle passing through A and B that does not have its centre on the perpendicular bisector of the segment due to property 2 of the perpendicular bisector of the segment, which tells us that if c is not on the perp. bisec. of AB, then AC and CB are unequal, and this cannot be possible because a circle is a set of points that are equidistant from the centre.

In the claim "It is not possible to draw a circle passing through A and B that does not have its centre on the perpendicular bisector" we observe the strategic effort made by the group in searching for possible representations that could satisfy the request. The reference to property 2 of the perpendicular bisector of a segment, which they had

previously demonstrated and that is been included in the Theory Bank, shows an epistemic control expressed through a negative conditional form that shifts the communication from an implicit level to a language closer to the norms shared by the mathematical community. Finally, there is another implicit reference to the definition of a circumference as a set of points equidistant from a centre.

We now report the transcript of the email texts produced in response to Tasks of Episode 2.

> 2 - Tina's email to Euclid: Through three non-collinear points A,B and C exactly one circle can be drawn, because the perpendicular bisectors of segments AB and BC intersect at a single point O, which is the centre of the circle. The perpendicular bisectors intersect at only one point because the points are not collinear. The segments OA, OB, and OC are in pair congruent: if O lies on the perpendicular bisector of AB, then OA and OB are congruent; if O lies on the perpendicular bisector of BC, then OB and OC are congruent. According to the common notion 1, the three segments are equal, that is, OA=OB=OC. According to Postulate 3, for every centre and every distance, there exists exactly one circle.

In the first sentence of the email, we recognise a teleological dimension in the awareness of being able to trace a unique circle that has as its centre exactly the unique meeting point of the two perpendicular bisectors indicated. From an epistemic point of view, there are theoretical justifications guided by visual evidence ('the perpendicular bisectors intersect at only one point because the points are not collinear'), even if in this case there is no explicit reference to Postulate 5 of Euclid's Elements. However, we observe greater attention towards the theoretical aspects involved in the construction of the proof. Tina's text shows indeed several propositions presented in conditional form, even if invoked only implicitly to justify some steps (e.g., "if O lies on the perpendicular bisector of AB, then OA and OB are congruent") with the exception of Common Notion 1 and Postulate 3 of Euclid's Elements. The communication shows a greater effort to align with the standards of the mathematical community, through a structuring of the proof that is increasingly oriented towards the theoretical validation of the statement.

We present now the transcript of the email texts produced in response to Tasks of Episode 3.

> 3 - Estella's email to Euclid: In the circle, segment AB is the diameter, since, according to Definition 17, "a diameter of a circle is a straight line drawn through the centre and terminated in both directions by the circumference of the circle, and it also bisects the circle," whereas segment AC is a chord. Triangle AOC is isosceles because sides AO and CO are congruent; therefore, according to Proposition 5, which claims that "the angles at the base of an isosceles triangle are equal to one another," the sum of these two angles

> (angle α and the angle at C) together with the angle at O of triangle AOC is equal to the sum of the interior angles of a triangle, that is, 180°. According to Proposition 13, which states that "if a straight line standing on another straight line makes angles, it will make either two right angles or angles equal to two right angles," the sum of angle β and the angle at O is also equal to 180°, since they are supplementary. From this, it follows that angle β is equal to 2α, because β+∠O=180° and 2α+∠O=180°. Finally, according to Common Notion 3, which states that "if equals are subtracted from equals, the remainders are equal," it can be deduced that angle α is half of angle β.

Estella's email shows a further change in the care taken in communication and in the conscious use of theoretical references involved in the proof, more adherent to mathematical norms. Propositions, common notions and definitions are no longer simply used, but are explicitly referred to from the Euclidean Theory Bank and quoted textually to support the entire argument. The final paragraph ("From this, it follows that") shows that Estella is exploiting the previous remarks to make new inferences aimed at proving the conjecture, towards a theoretical validation.

DISCUSSION AND CONCLUSION

The case study documented an evolution in the Blogger's arguments, moving from naïve justification to modes of theoretical validation within a geometrical system aligned with the standards of the mathematical community (Mariotti, 2006). We note that the student playing Blogger in Episode 3 is very rigorous and precise; she is keen to do well, and this has certainly influenced her performance in the role. In addition, here we can hypothesise that her approach was also shaped by having previously experienced the roles of Promoter, Pest, and Blogger in sequence, the first two as an outlooker and the third as an actor. We suppose that the choice of role assignments undoubtedly influences the inquiry process, and the results are certainly affected by such choices. Nevertheless, at the level of the class as a whole, an increasingly greater precision was observed in the writing of the emails to Euclid. This led us to reflect on the different emails that were gradually sent to Euclid, which reveal a growing awareness both with regard to epistemic aspects of theoretical control and to the communicative component, which became increasingly careful and clear, with far fewer implicit elements than in the initial submissions. The analysis of the emails demonstrates a growing expository clarity in communication with Euclid, in line with the standards required by the mathematical community, together with an increasingly frequent reference to theoretical aspects to support the validity of the claims made. Based on the further analysis of all the chats, looking at the Actor/Outlooker levels and at the rotation of the students playing the role of Blogger, we can hypothesize that this evolution is a process involving the entire class group. Finally, we underline that, although not included in this paper, the complete analysis of all the chats shows the key role of the guidance and the mediation of Euclid, as rational mediator, to allow the students to be successful in their moving towards forms of theoretical validation.

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