DISCOVERING "WHYS" BEHIND THE CHANGE SIDE–CHANGE SIGN RULE IN ALGEBRA

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The change side–change sign procedure (CS procedure) for equation solving is often used by students as a memorized rule without justification. This paper reports on findings from the DynaMat project that sees remediation not as repetition of procedures, but as access to mathematical meaning through Digital Exploratory Objects (DEOs). Grounded in Commognition, the study analyzes a case of two grade-10 students to investigate whether and how the remedial activities supported the substantiation of the CS procedure. The findings show that the intervention supported students through a guided process of discourse expansion with the DEOs, followed by a compression in which the CS procedure came to be seen as a substantiated shortcut.

KNOWING "WHY" BEHIND RULES AND LEARNING DIFFICULTIES IN MATHEMATICS

The "change side–change sign" procedure (hereon CS procedure) for equation solving, also known in Italian as the "transportation rule", is widely taught in terms of moving numbers and letters across the equals sign of the equation and inverting their sign in the process. The CS procedure is mathematically valid, and justifiable by the equivalence principles–specifically by adding or subtracting the same value to both sides of an equation. However, research shows that students often use it as a memorized rule, applying it syntactically without understanding, controlling its meaning or knowing how to justify it (Kieran, 1992). More in general, the literature suggests that students who experience low achievement in mathematics due to learning difficulties are more often offered below-grade-level content focused on memorized routines, rather than opportunities for inquiry, reasoning, and problem-solving (e.g., Kurz et al., 2014; Lambert, 2018). Instead, if procedures such as CS procedure are learned meaningfully, they can be compressed and unraveled at will, as Thurston (1990) states:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. (Thurston, 1990, p. 846).

The DynaMat project (2022–2025) has employed a design-based research approach that sees remediation not as repetition of procedures, but as access to mathematical meaning, through sequences of activities leveraging Digital Exploratory Objects (DEOs). The project has designed and trialed two sequences of activities with 28 grade-10 students (15-16 years old). In this paper the protagonists are Alice and Rina

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(pseudonyms), whose data allowed us to gain insight into how our remedial intervention sequence benefitted their meaning making of the CS procedure.

THEORETICAL FRAME OF THIS STUDY

We adopt Commognition (Sfard, 2008) as it enables us to move beyond deficit discourse by framing low achievement as a difficulty in deritualizing participation and by operationalizing "compression" in observable discursive terms, without recourse to unobservable internal mental processes. Within this framework, doing mathematics is seen as participating in a special discourse, characterized by its specific *words, visual mediators, narratives* and *routines* and studying students' mathematical learning means to analyze the development of their mathematical discourse. This development can be described in terms of alternating *expansions* and *compressions* (Sfard, 2008, pp. 119–123). The increase in the amount and complexity of discursive elements leads to a discursive expansion, while a compression involves a reduction of the complexity of the discourse through the rise to a *metalevel*. For instance, a potentially infinite series of different arithmetic narratives as "5=3+2 therefore 5+4=3+2+4" can be compressed into an algebraic, or meta-arithmetic, narrative as "if a=b then a±c=b±c".

In this study we operationalize *discursive compression* through the following commognitive constructs: objectification (in particular saming), substantiability and discourse transitions to metalevel. Objectification leads to the emergence of new objects within a discourse, and it includes the key process of *saming*, that is "giving a common name to things that, although seemingly unrelated, can be seen in certain contexts as equivalent" (Sfard, 2008, p. 96). Among the processes of deritualization (Lavie et al., 2019), we focus on *substantiability*, which we relate to discursive compression by viewing the substantiation of a procedure as a shift to a metalevel, that is, to a discourse about the procedure itself (Sfard, 2008). This shift occurs when students use the result of a procedure to address the task situation and they justify why the procedure is appropriate, thereby authoring meaningful narratives about its justification. Finally, we want to consider other forms of *metalevel discourse*, specifically focusing on students' metadiscursive rules underpinning their narratives about "knowing why" in mathematics (e.g., why a procedure works, how to justify its steps, whether it is important to "know why").

RESEARCH QUESTIONS

In this paper we address the following research questions:

- Do the proposed remedial activities with digital artefacts support struggling students to substantiate the CS procedure? And if so, how?
- Does the intervention influence students' metadiscursive rules concerning the management of "know why" in mathematics discourse? And if so, how?

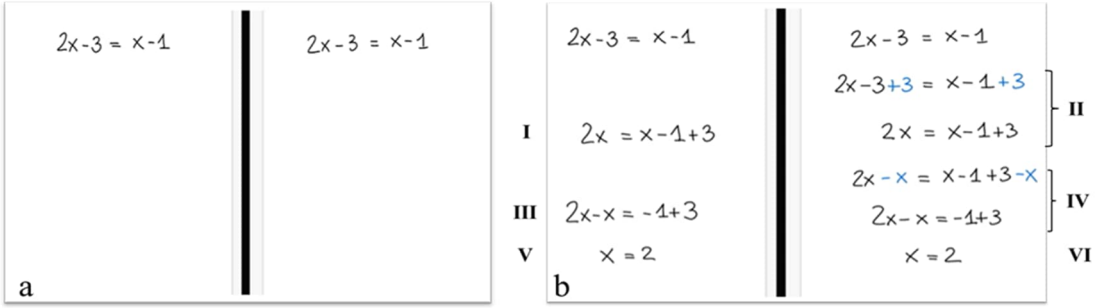
METHODS

The DynaMat project aimed to overcome the limited impact of traditional remedial mathematics interventions through iterative cycles of design, implementation, and analysis. It involved 15–16-year-old students with persistent mathematical difficulties (identified by themselves and by their teachers) in after-school tutoring sessions at a research center, with activities guided by a researcher-tutor. Each participant undertook an individual incoming interview during the first session, and then participated, alone or in a dyad, to five sessions lasting approximately 2 hours each. Qualitative data (videos, transcripts, and notes) were collected and analyzed.

The design of the remedial intervention is framed in terms of remediation through phases of expansion of students' discourse (Antonini et al., 2023) in the context of various DEOs based on the metaphor of balance for equality (two-pan scales and hanging mobiles). These are used to promote discourse expansion, including the oral and written expression of "scale balancing" principles such as those shown in Table 1, that can eventually be seen as corresponding to equation equivalence principles. In the newly developed discourse, the principles can be used to justify actions on the DEOs or constructing meaningful and endorsable narratives, and such actions can be represented on symbolic realizations of the equations in focus. At this phase the activities aim to promote saming between the realizations with the DEOs and in algebraic symbols (Weingarden et al., 2024) and between "scale balancing" principles and equation principles, recognizing them as the *same* equivalence principles.

Then the remedial intervention is designed to explicitly relate the CS procedure learned in school with the justified steps of the procedures discovered by the students in the context of the DEOs. We will focus in particular on one activity based on a silent video (SV) (Kristinsdóttir, 2022) in which the screen is split in half (Figure 1a): the CS procedure is enacted on the left; while on the right-hand side of the screen the mysterious solver proceeds by transforming the original equation into equivalent ones adding or subtracting equal terms from both sides of the equation (Figure 1b). The video begins with a voice saying: "In this video we want to justify all the steps in the solving procedure of an equation" and it continues silently.

Figure 1: Screenshots of the video involved in the SV activity, showing its beginning (a) and its end (b); numerals indicate the order of appearance during the SV.



Students can control the video playback, watching it multiple times, pausing it, and fast-forwarding or rewinding. The task situation asks them to watch the video and then to "try to understand and make sense of what happens". This SV activity is aimed to foster students' discursive substantiation of the CS procedure (instantiated twice on the left), recognizing it as a shorter version of the meaningful manipulations on the right, justified by the principles discovered in the previous activities with DEOs. It was proposed to the dyad Alice and Rina after four hours (two sessions) on DEOs (two-pan scales and hanging mobiles). In addressing our research questions, we selected excerpts from: (1) the students' incoming interview; (2) their work on the activities with DEOs during which they elaborated the written equivalence principles; (3) the SV activity.

ANALYSES OF SELECTED EXCERPTS

(1) From students' incoming interview

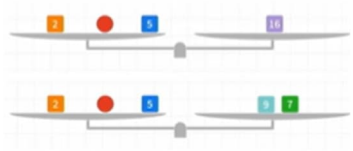
For space limitations we provide here only a brief summary of what emerged in Alice and Rina's incoming interview. Some task situations explicitly involved finding the solution(s) of a linear equation (e.g. "13 – a = 13 + 11" or "3 + a = a + 3"). Alice and Rina's discourse about equations almost uniquely depended on their remembering of the CS procedure (e.g., A: "I remember that all the letters have to be moved to one side and the numbers to the other"; R: "The letters always stay on the left, while the numbers on the right, […] change the sign when you take it over"). Both students always attempted to instantiate the recalled procedure, but often they did it in a non-canonical way (e.g., not changing a sign or dividing instead of subtracting) and without alternative procedures to use when being stuck. Moreover, neither student ever justified why the activated procedure works and when explicitly asked, both responded that they did not know or could not remember it (A: "[...] I don't remember why but I remember that you do like this"). Overall, based on what emerged from the interviews, the metadiscursive rules that seem underpin Alice and Rina's discourse about equations could be expressed as "Solving an equation means to apply the CS procedure", "It is enough to remember what to do" and "it is not important to remember the reasons for the rules applied".

(2) Discourse expansion through the activities with DEOs

The first activities presented to Alice and Rina involved scales with various weights, all with known value (e.g., rows 1 and 2 of Table 1). The typical task situation from which each activity started was: "The scale that you see is balanced. If possible, put on [or take off] the pans 5 objects [or other number] so that the scale at the end will still be balanced". As the students engaged with these activities, they authored narratives about when the scales are balanced (e.g., A: "[A scale is balanced] when there is the same value on one [plate] as on the other", R: "We can add the same value on both sides", R: "[In a balanced scale] if we remove the same value, it remains balanced"). In these cases, the tutor intervened by asking the students to try to write down what they had just said, leading them to produce a series of written endorsed narratives that

were collected on their notes (right column of Table 1). The task situations also involved unknown weights (e.g., row 3 of Table 1). In the students' narratives the "scale balancing" principles became the justifications for answering "why" questions about their actions (physical or envisioned) on the DEOs. We also note that Alice and Rina had become completely comfortable transforming "scale situations" (column 1 in Table 1) into symbolic notation (column 2 in Table 1), suggesting that saming between realizations of the principles had occurred.

Table 1: Graphical and symbolic realizations of A&R's "scale balancing" principles.

Graphical realizations	Symbolic realizations	A & R's "scale balancing" principles
	$(4+5+7+6+3)=(3+5+8+2+2+5)$	The scale is balanced when the sum of the weights on both pans is equal
	$\cancel{4}+5+7+6+3 = 3+5+8+\cancel{2}+\cancel{2}+5$	If a scale is balanced and from both sides we add or remove the same collective weight it remains balanced
	$2+0+5=16$ $2+0+5=9+7$	If a scale is balanced and you substitute various weights on a pan with an overall equivalent weight, the scale stays balanced

(3) The silent video (SV) activity

This activity, including the interactions with the tutor, takes Alice and Rina about 18 minutes (and 201 lines of transcript) to complete. We selected some lines from the transcript that show how the students come to substantiate, for the first time, the CS procedure that they remember from school. They talk about the two halves of the screen: the left hand-side shows the "normal" [60] way learned in school with the "x" "carried over" [15]; and the right hand side shows a "strange" [60] way which initially they state not to fully capture ("I got this part, but this one [no]" [12]) even though they connect it to scales ("like on a scale, if you add or remove the same weight from both plates, it will continue to be balanced" [12]). At the end, however, they conclude that what happens on the right is the justification of "why when you move a number to the other side, the sign changes" [138].

31 A: Like, in the first equation you move […] the x's to the left and the numbers to the right.

35 A: So, the x is moved to the left, and the numbers to the right, and the first one [the three] is moved, and when you do this, the signs change.

40 R: Now, he has to explain what he did here [right-hand side of the screen].

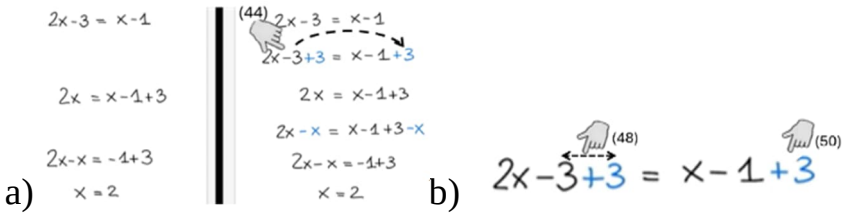
41 A: Yes, but why does he do it? […] in my opinion it's to prove that it doesn't change.

44 R: Because… Basically, where was the three? Here? Minus three has to be moved here and it becomes plus three [pointing as in Figure 2a] … no wait, I have it in my head but I don't know how to say it.

46 R: If you add plus three to both sides…

47 A: Yeah, the result doesn't change, I mean it keeps on being an equivalence.

Figure 2: Pointing gestures at lines 44 (a), 48 and 50 (b)



48 R: It doesn't change and so consequently here, since they're opposites, it cancels out [pointing as in Figure 2b].

50 R: And here [pointing as in Figure 2b], plus three remains, which would be with the sign changed.

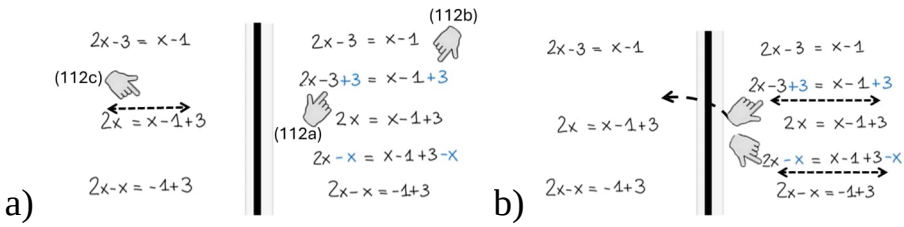
The students describe similarly the following steps that appear on the two sides of the screen. Then the tutor asks them to re-explain what they observed. They describe the left-hand side as the "normal" procedure [101] consisting of "Putting the numbers on one side and the x's on the other" [103]. Then they start talking about the other side:

108 R: Since the first principle [see row 2 in Table 1] says that if you add or subtract the same value, etc., then if you add plus three here, you also have to add it on the other side.

110 R: Here it cancels out, while on the other side plus three remains, so with the sign changed.

112 R: Here there was minus and here it became plus. Which is, let's say, what we usually do [pointing as in Figure 3a].

Figure 3: Pointing gestures at lines 112 (a) and 175 (b)



122 A: Because here you directly change the sign, I mean, if I want to move minus three, minus three has to go over there and it directly changes to plus three. […] Instead, maybe here he justifies the reason why I change the sign.

132 A: And doing this is kind of the justification for moving the minus three to the other side, which then becomes plus three. [...] For this sign change.

138 R: That is why when you move a number to the other side, the sign changes.

171 A: So this [left-hand side] is the normal equation, that is, the one you do without steps. Instead, this one [right-hand side] motivates.

175 A: Because here these, this and this could very well also be over here because they're meant to motivate, they're like intermediate steps that we do automatically [pointing as in Figure 3b].

The episode testifies the presence of different saming processes taking place. The visual mediators on the right-hand side of the screen are samed with the scale DEO ([12]) and some syntactic actions on those mediators are recognized as being justified by the same equivalence principles they wrote during the previous activities ([46-47], [108]). Moreover, during the excerpt the two students come to recognize that the actions on the left- and right-hand side of the screen can be seen as being the same (first implicitly in [50], then clearer in [112] and [175]).

Concerning substantiability of the CS procedure, we notice that the saming processes provide Alice and Rina with a justification of why the procedure works. Specifically, at the end of the excerpt, they recognize the right-hand side as the justification of "why when you move a number to the other side, the sign changes" [138]. Since they justify what happens on the right-hand side using the "scale balancing" principles (which they now apply in a context that no longer refers to scales), such principles now also justify the CS procedure.

As a further element, we observe the students' use of "why" and "because" throughout the episode. The words appear 25 times, both in the first part as they solve the task (e.g., [41], [44]), and in the second part of the episode when they re-explain to the tutor what observed (e.g., [122], [138], [175]). This suggests emerging independence of the two students in authoring endorsable narratives about "knowing why" in mathematics, and it highlights a possible shift in their metadiscursive rules: at least in this activity, discovering why the CS procedure works and how to justify its steps.

Overall, the analyses suggest that a discursive compression is taking place in Alice and Rina's discourse: the CS procedure has become a shortcut for the symbolic manipulations of an equation, justified by the "scale balancing" principles.

DISCUSSION AND CONCLUSION

In this case study we addressed whether, and how, the proposed remedial activities with digital artefacts support struggling students in substantiating the CS procedure, and whether this process influences students' metadiscursive rules concerning the management of "why" in mathematical discourse. Our analyses show that the intervention supported Alice and Rina through a guided process of expansion, through guided interaction with the DEOs, followed by compression of their mathematical discourse. Through the SV activity they came to speak of the CS procedure–the "normal" way of solving a linear equation–as a shortcut, substantiated by principles they had discursively constructed and endorsed. In this sense, the intervention enabled a (re)compression that had not occurred in their prior learning. This result is significant because there is evidence that this compression entailed further objectification of the

signifiers involved in linear equations. The terms appearing in the two procedures (e.g., "x", "3") are not simply "carried over" from one side of the equation to the other, but arise as new objects that appear and cancel out (thus disappearing) as a consequence of justified actions. The overall process of "saming" the CS procedure with the equivalence-based procedure thus requires overcoming this ontological difference, a move that appears to be supported by prior discourse expansion in the context of DEOs. More broadly, the findings are consistent with Lambert's (2018) plea for all students to engage in meaning making. Moreover, it provides an example of how a remedial intervention can be successful in a situation known to be problematic in the literature (Kieran, 1992). From a theoretical point of view, the study operationalizes the construct of *compression* (Thurston, 1990; Sfard, 2008) offering a new tool, within the theory of Commognition, to study a fundamental process in mathematics learning.

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