FLEXIBILITY (BEYOND PROCEDURAL) IN PROBLEM SOLVING

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In solving mathematics problems of the type that are non-routine and where the solver encounters blockages to overcome, the problem solving process is likely to require flexibility that is beyond merely alternative ways of tackling routine tasks. We thus make a distinction between procedural flexibility and heuristical flexibility in the context of problem solving. By analysing a case of a student teacher solving the "nice number problem", we found that the insights from his procedural flexibilities can potentially be reasonably translated as heuristical flexibility in a productive way. That he did not do so led to reflections that can open up new areas of research in mathematics problem solving.

INTRODUCTION

For research in mathematics education, flexibility is defined as knowledge of multiple strategies and the ability to use these strategies in innovative ways in different mathematical problem solving situations (Baroody & Dowker, 2003; Rittle-Johnson & Star, 2007; Schneider et al., 2011). There are more fine-grained categorisations of flexibility in this field: potential flexibility (defined as knowledge of multiple strategies for a given mathematical task), practical flexibility (defined as the ability to select the most appropriate strategy for a given mathematical task). The term procedural flexibility encapsulates both the features of potential flexibility and practical flexibility. Although procedural flexibility as defined this way broadly includes flexibility-use in "problem solving contexts", work by the researchers in this field tend to focus on tasks that can be completed by well-known standard procedures, such as solving linear equations (Star et al., 2022; Wang et al., 2019; Xu et al., 2017). This is perhaps unsurprising as "procedural" in procedural flexibility points to research directed at studying contexts where solutions to task can be compared in terms of specific "procedures", and hence the choice of tasks as such. For this reason, and for the purposes of our study as reported in this paper, we take the liberty to narrow the use of procedural flexibility to knowledge of multiple procedures and the ability to select a suitable procedure to complete a specific familiar mathematical task.

BEYOND PROCEDURAL FLEXIBILITY

But our interest in this study includes flexibility as used within the broader context of mathematics problem solving. [By "problem", we mean it in the sense as Schoenfeld (1985) defined it: Not routine exercises that evoke application of standard procedures, but tasks that create mental roadblocks for one who engages in it and where exploration

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, pp. 931-938. PME.

of strategies is involved to tackle the roadblocks]. Our premise is that for problem solving, the kinds of flexibility required is beyond procedural flexibility as we defined it above. To illustrate this point, take this problem: An integer is removed from a set of positive integers {1, 2, …, n}. The average of the remaining set of numbers is $\frac{163}{4}$. What is the removed number? There is certainly scope in the process of solving the problem for procedural flexibility. For instance, a student may want to understand the problem by substituting n = 10 to find out the effect of one-number-removal on the average. Suppose the student wants to find the average after "1" is removed. There is thus a specific task in which he needs to enact a procedure: Find the average of 2, 3, …, 10. To do so, he may apply procedural flexibility in that more than one standard procedure can be used to complete this specific task. But to make progress in this problem, more is needed. Beyond the heuristic of substituting random values for n and k (the integer to be removed from the set), at some point, there is a need to be flexible also in heuristic use – that is, a shift, say, to another heuristic of fixing n at around 80 (since the removal of an integer will not vary the average by much) and then solving for k using an equation. This latter form of flexibility we provisionally label as "heuristical flexibility" and we define it as flexible ways of advancing towards a goal within the context of problem solving, and the advancement is novel in the sense that the solver has not confronted the same goal-within-problem context previously.

The purpose of making the distinction between procedural flexibility and heuristical flexibility is so that we can study their interactions within the context of problem solving. The assumption is that though they can be distinguished, they can affect each other significantly in problem solving. And in the study reported here, we focus on how procedural flexibility can potentially lead to heuristical flexibility. In particular, we report a case study of how a student exhibited procedural flexibility that can potentially be translated into helpful heuristical flexibility. That the connection did not actually materialise in the actual problem solving process of the student leads to reflections about the possible productive interactions between the two types of flexibility.

To be clear, we are aware that the existing literature accounts for the broadness of flexibility in the use of the term "strategic/strategy flexibility" (e.g., Bolat & Arslan, 2024; Verschaffel, 2024). But currently, this term is not sufficiently well-defined in that it can refer to both our constructs of procedural flexibility and heuristical flexibility. For our purpose – as we want to distinguish them to see their interactions – we retain our terminology.

METHOD

The first step was the selection of a suitable problem for the purpose of the study. We think these are necessary characteristics of the problem: (a) As we were looking for sustained problem solving so that there would be opportunities to trace pockets of procedural and heuristical flexibilities, and their interactions, the problem needed to be one whose solution is not so obvious; it should be of sufficient depth for students to

maintain sustained deepening of probes and hence motivation to be flexible; (b) and yet it should not be so challenging as to discourage the need for flexibility altogether. It should allow one to access the problem without the need of specialised techniques which only select people may possess; (c) the components of the solutions to the problem ought to be more obviously connected so that the solver would be naturally drawn to see potential of how procedural flexibility in one small part of the problem can be extended for heuristical flexibility across a larger part of the problem.

We decided on the "Nice Number Problem": A 'nice' number is a number that can be expressed as the sum of a string of two or more consecutive positive integers. Determine which of the numbers from 50 to 70 inclusive are nice.

The Nice Number Problem was given to three student teachers in China to attempt. They were asked to devote up to 30 minutes to attempt the solution on their own. While solving the problem, they were to articulate as much as they can their own thinking processes. This was similar to the "think aloud protocol" used in Schoenfeld's (1985) research. In addition, each of them video-recorded their concurrent writings and articulated thoughts throughout the problem solving process. The video was directed at their writings on paper as they solved the problem. For the purpose of this study, we identified Zhi Wei (pseudonym) for in-depth analysis for the case as his procedural flexibility moments were most relevant to the purpose of our study.

We borrowed heavily from Schoenfeld (2000) goal-based methodology for the parsing of the problem solving process. In particular, the accounting of how lower-tier goals can be nested within higher-tier goals in that methodology parallels well our differentiation of procedural flexibility (which can be viewed as ways of fulfilling a lower-tier goal) in support of heuristical flexibility (to fulfill a higher-tier goal that leads directly to solving a part of the problem).

For analysis, we first parsed the data using as fine-grained a lens as we can to capture all the details related to his goals and actions with respect to his problem solving process. The goals were differentiated into 3 nested levels: L1 refers to the overarching goal; L2 refers to the heuristical goal – including the use of different heuristics, if any – in support of the L1 Goal, and L3 are the execution goals – including the use of different techniques flexibly in support of the L2 Goal. As illustration, the relationship between these goal levels in relation to the actual work of problem solving by Zhi Wei is then put together as shown in Figure 1 for the first part when he worked on odd nice numbers.

As shown in Figure 1, the case for odd nice numbers was straightforward for Zhi Wei and hence there was likely no motivation for flexibility – whether procedural or heuristical. The flexible moments occurred predominantly in the next section of his problem solving effort when he turned in search of nice numbers among the even numbers in the 50-70 range.

Figure 1: An example of a L1 goal frame-size that relates the goals at different levels

L1: Conjecture and prove all odd numbers (3 or more) are nice				
L2: Confirm conjecture with more cases		L2: re-conjecture to 2-consecutive	L2: prove using algebra	
L3: Try 7	L3: re-check definition		L3: Break down an odd number into two consecutive integers	L3: check addends are integers
Verified: Write "7=3+4"	"confirmed"	Insight drawn from previous cases	writes "$n = \frac{n-1}{2} + \frac{n+1}{2}$"	"Yes, it works"

1.57 2.05 2.24 2.28 2.50

ZHI WEI'S PROBLEM SOLVING: LACK OF TRANSFERENCE FROM PROCEDURAL FLEXIBILITY TO HEURISTICAL FLEXIBILITY

In our analysis, we identified two distinct occurrences of procedural flexibility moments in Zhi Wei's problem solving work. They were both located in the broader L1 goal context of "Find all even nice numbers in the 50-70 range". To support this goal, Zhi Wei set a L2-level heuristical goal of "Use the AP S_n formula by incrementally changing r in "r + …" strings, starting from r = 1". Task A as listed in Table 1 shows the execution of the "1 + …" string in support of the L1- and L2-goals. Task A involved finding the sum 1 + … + 11. Zhi Wei demonstrated the knowledge and ability to use two different procedures: Procedure A1 and Procedure A2 as shown in Table 1. The ability to apply the alternative Procedure A2 – which we call simply as "add term" – is further verified when he directly wrote the sum $S_{12} = 78$. This is an evidence of procedural flexibility.

Table 1: Procedural flexibility in finding the sum 1 + … + 11

Task A: Find the sum of 1 + … + 11			
7.37	He wrote "$S_{11} = \frac{(1+11)\times 11}{2} = 66$" Said "66 is nice" Said "55 + 11 is more efficient"	Procedure A1 Directly using AP formula to find nice numbers $S_{11} = \frac{(1 + 11) \times 11}{2} = 66$	Procedure A2 Based on the former conclusion to find $S_{11} = S_{10} + 11 = 55 + 11 = 66$
7.50	Directly wrote "$S_{12} = 78$"		Clearly not using the AP formula anymore, but just adds 12 to 66 to obtain 78 $S_{12} = S_{11} + 12 = 78$

Similarly, procedural flexibility was again demonstrated in the context of computing "2 + …" strings with Task B: He used both Procedure B1 and Procedure B2 to find the sum 2 + … + 10, as shown in Table 2. For simplicity, we label Procedure B2 as "subtract terms".

Table 2: Procedural flexibility in finding the sum 2 + … + 10

Task B: Find the sum of 2 + … + 10			
12.22	He wrote, "$\frac{(2+10)\times 9}{2} = 54$". Said "54 is also a nice number, looks like no problem in my calculation, 1 to 10 is 55 and 2 to 10 is 54"	Procedure B1	Procedure B2
		Using AP sum formula with the first term 2 to find nice numbers $\frac{(2+10)\times 9}{2} = 54$	*Based on the former conclusion of S_{10} with the first term 1 to check* $\frac{(2+10)\times 9}{2} = S_{10} - 1 = 55 - 1 = 54$

Surprisingly, the insights displayed in Procedure A2 (add term) and Procedure B2 (subtract term) were not subsequently employed by Zhi Wei heuristically. In fact, for the rest of his problem solving endeavour to fulfill the L1 goal of "find all even nice numbers in the 50-70 range", he stuck to the same heuristic of "use AP S_n formula for "r + …" strings all the way to r = 11" before he gave up altogether. There are evidences from the data that he gradually felt that the heuristic did not work sufficiently efficiently for him and he thus gave up after persevering up to the "11 + …" case. Hypothetically, the flexible insights obtained from Procedure A2 and Procedure B2 can potentially be translated into a heuristical alternative that could potentially fulfill the L1 goals more efficiently. When the "add term" and "subtract term" heuristic is combined with the "application of AP S_n formula" heuristic, the way to find (and maintain parity of) even nice numbers in the range may proceed as shown in Figure 2.

Figure 2: Hypothetical use of heuristical flexibility in listing all even nice numbers

55 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 [Apply AP formula]
54 = 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 [Subtract term]
52 = 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 [Subtract term]
60 = 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 [Subtract term, Add term]
66 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 [Add terms, or Apply AP Formula]
56 = 5 + 6 + 7 + 8 + 9 + 10 + 11 [Subtract terms]
68 = 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 [Add terms]
50 = 8 + 9 + 10 + 11 + 12 [Subtract terms]
60 = 10 + 11 + 12 + 13 + 14 [Subtract terms, Add terms]
50 = 11 + 12 + 13 + 14 [Subtract term]
54 = 12 + 13 + 14 + 15 [Subtract term, Add term]
70 = 12 + 13 + 14 + 15 + 16 [Add term]
58 = 13 + 14 + 15 + 16 [Subtract term]
62 = 14 + 15 + 16 + 17 [Subtract term, Add term]

This combination of heuristic choices – hence heuristical flexibility derived from the insights of procedural flexibilities – is more directly oriented towards the L1 goal and thus more efficient compared to the strict AP Formula-application heuristic used by Zhi Wei. It is consequential in terms of problem solving experiences for students as an appropriate use of such heuristical flexibility – which was overlooked by Zhi Wei - could have potentially changed one's experience of problem solving from a sense of unproductiveness to one of success.

DISCUSSION AND CONCLUSION

We presented a case of how insights obtained from procedural flexibilities can be reasonably translated to heuristical flexibility to more efficiently solve the same problem, but yet was unemployed by the student. He used these insights at the procedural level (at the L3-goal level), but was unable to take a step (back) to apply it at the heuristical level (at the L2-goal level) to fulfil the same L1 goal.

First, that procedural flexibilities can be 'elevated' quite obviously to a heuristical level is a potential domain of interesting research in mathematics problem solving that is erstwhile yet unexplored. This is different framing from the more traditional stages approach (such as Polya's model) or process approach (such as focusing on learning the actual heuristics as 'new' processes) of teaching about problem solving. It is a shift towards using what students already displayed at the procedural flexibility level for heuristical purposes, and hence a shift from students' acquiring altogether 'new' heuristics to one of building upon their own working procedures towards productive heuristics. In so doing, there is a likelihood of greater acceptance and ownership of these heuristics for themselves.

Humbly, we also think that this is a useful extension to Schoenfeld's (1985) work on mathematics problem solving. In his work, cognitive resources (which procedural flexibility is a part of) and heuristics (which heuristical flexibility is a part of) were presented as essential components of problem solving but their interactions were not explicated at this level of detail. In this study, we specify aspects of these components that are closely related and which teachers can practically make links in actual instruction about problem solving, such as, "this add-subtract term technique can become another way of generating other even nice numbers?" Of course, this requires knowledge and readiness of teachers in taking advantage of such procedural flexibilities in students' work as insights for productive heuristical adjustments. We think that such sensitivities in students' personal awareness of how their procedural flexibilities can be translated into heuristical flexibility as a habit can be an area of more intentional future research in problem solving. Examples of such research direction may include: Is this type of transference applicable to other mathematics problems? Would a re-framing of problem solving instruction into such a transference really helpful for teachers and students?

Second, it is worthwhile to reflect more deeply into why Zhi Wei did not make the 'heuristical leap' even when he was already able to apply the add-subtract technique

for certain specific tasks. In the case of Zhi Wei, his heuristical inflexibility can also be interpreted as the flip side of his persistence to stick to the "apply AP formula" heuristic – as seen from he persevered all the way to the "11 + …" case. In other words, being heuristical flexibility was pushed back from the foreground of his problem solving concentration because what was central to his pursuit was to gain increasing fluency with the heuristic that he had chosen – and provided him with successes (in terms of find even nice numbers) initially. This tension between flexibility and fluency at the heuristical level reminded us of our earlier work which discussed the need to hold both procedural fluency and procedural flexibility among Chinese students (Li & Leong, 2026). From the findings of this study, it seems there is a case for the construct of "heuristical fluency" which is held in tension against heuristical flexibility too. Reinterpreted into Zhi Wei's case, his procedural flexibility at specific tasks with the potential of rendering it into efficient heuristical flexibility was jettisoned not due to procedural fluency, but more of the drive towards attaining heuristical fluency – to become more and more proficient with the use of the same "apply AP" heuristic.

Just like the juxtaposition of procedural fluency and procedural flexibility, the argument is not for one over the other, but the appropriate use of one (or both) in relation to the task at hand. Similarly, there is much to be said about the need to persevere for heuristical fluency in problem solving; but at the same time, there is a need to develop the ability to draw insights from one's own problem solving process for alternative efficient heuristics (that is, heuristical flexibility) to complement heuristical fluency. This area of developing in students the right 'division of labour' between heuristical fluency and heuristical flexibility in problem solving is yet another direction of research worth exploring in future.

References

Baroody, A. J., & Dowker, A. (Eds.). (2003). *The development of arithmetic concepts and skills: Constructing adaptive expertise*. Mahwah, NJ: Lawrence Erlbaum.

Bolat, R.C., & Arslan, C. (2024). Examination of mathematics teachers' strategic flexibility in solving mathematical problems. *Thinking Skills and Creativity, 54*, 1-23.

Li, X., & Leong, Y.H. (2026). Solving fractional equations: (in)flexibility and its roots. *International Journal of Mathematical Education in Science and Technology, 57*(1), 153-172.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561–574. https://doi.org/10.1037/0022-0663.99.3.561

Schoenfeld, A. (1985). *Mathematical problem solving.* Academic Press.

Schoenfeld, A. H., (2000). Models of the teaching process. *Journal of Mathematical Behavior, 18*(3), 243-261.

Schneider, M., Rittle-Johnson, B., & Star, J. R. (2011). Relations among conceptual knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Developmental Psychology, 47*(6), 1525–1538. https://doi.org/10.1037/a0024997

Star, J., Tuomela, D., Joglar Prieto, N. et al (2022). Exploring students' procedural flexibility in three countries. *International Journal of STEM Education, 9*(4). http://dx.doi.org/10.1186/s40594-021-00322-y.

Verschaffel, L. (2024). Strategy flexibility in mathematics. *ZDM, 56*, 115-126.

Xu, L., Liu, R. -D., Star, J. R., Wang, J., Liu, Y., & Zhen, R. (2017). Measures of Potential Flexibility and Practical Flexibility in Equation Solving. *Frontiers in psychology, 8*, 1368. https://doi.org/10.3389/fpsyg.2017.01368

Wang, J., Liu, R. D., Star, J. R., Liu, Y., & Zhen, R. (2019). The moderating effect of regulatory focus in the relationship between potential flexibility and practical flexibility. *Contemporary educational psychology, 56*, 218-227. https://doi.org/10.1016/j.cedpsych.2019.01.013