DEVELOPMENT AND VALIDATION OF A TEST INSTRUMENT FOR ASSESSING FUNCTIONAL THINKING IN SECONDARY EDUCATION

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Functional thinking is considered a fundamental prerequisite for understanding key mathematical concepts. This study examines the structure of functional thinking based on the facets of input-output assignment, covariation and objects. To this end, an economic testing instrument was developed and evaluated with N = 153 Year 11 students. Confirmatory factor analyses confirm that the theoretically postulated three-dimensional model represents the data significantly better than a one-dimensional model. The instrument enables a differentiated diagnosis of individual thinking patterns at the transition to upper secondary school and provides a valid basis for research and practice in mathematics education.

INTRODUCTION

Functional thinking refers to the ability to recognise, generalise, describe and represent patterns and relationships. These relationships are understood as independent mathematical objects. The development of functional thinking is fundamental to learners' ability to model real-life situations mathematically and solve complex problems (Blanton & Kaput, 2011). Moreover, functional thinking is a prerequisite for understanding concepts such as slope or derivative (Thompson & Carlson, 2017; Greefrath et al., 2023). It is therefore important that students develop a deep understanding of functional patterns and relationships at lower secondary level. Understanding a function primarily occurs via the “concept image”, that is the totality of cognitive structures that learners individually associate with the term (Vinner, 1983). For functional thinking, the concept image includes different facets, namely input-output assignment, covariation and object. To our knowledge, however, no validated test instrument for secondary education exists to date that captures these different cognitive structures as self-reported information and thus allows a diagnosis of functional thinking. Existing testing instruments typically rely on performance testing, which tends to focus on solutions rather than thought processes (Pittalis et al., 2025). Due to the importance of functional thinking for further learning, in this study, a theoretically based short test instrument is to be developed and evaluated.

FUNCTIONAL THINKING

The understanding of functions in mathematics education is based on various cognitive approaches that complement each other in their complexity. For this reason,

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correspondence relationships and covariational thinking are described as two fundamental approaches to introducing functions (Smith, 2008). Building on this, mathematics education distinguishes between three essential facets of functional thinking that help learners to gain a deep understanding of functional relationships (Doorman et al., 2012; Pittalis et al., 2020). These facets – input-output assignment, covariation, and object – enable the development of a holistic understanding of functional relationships (Günster & Weigand, 2020; Wei et al., 2025). For primary school students, this structure of functional thinking can be verified, whereby recursive patterns were also examined in greater detail (Pittalis et al., 2020).

Input-output assignment

From this perspective, functions can be understood as mappings between input and output. This understanding is characterised by its representation in the form of a chain of effects that illustrates the sequence from input to calculation to output. While the computational aspect is often perceived more strongly at the beginning of the learning process, this initially vague idea changes into a deeper understanding of the dependencies between input and output as the learning process continues (Doorman et al., 2012). Accordingly, the correspondence relationship focuses on the construction of function equations that articulate a direct relationship between the variables. In this context, the function is understood as an explicit rule that directly assigns a value of the dependent variable to a value of the independent variable (Smith, 2008). The correspondence relationship can help to understand the underlying structure of the function, as it sharpens the perception of the relational structure between the variables (Pittalis et al., 2025). A sound understanding of function is particularly evident in the ability to recognise and cognitively process these correspondence relationships within different forms of representation such as graphs, equations or tables (Günster & Weigand, 2020).

Covariation

In terms of covariation, functions are conceptualised as dynamic processes, focusing on the simultaneous change of two variables (Doorman et al., 2012). The underlying principle is that the independent variable passes through an interval, inducing a corresponding change in the dependent variable within its range of values. Thus, this facet focuses on the investigation of the interdependence and dynamics of this joint change, with tables and graphs serving as preferred forms of representation to gradually explain the simultaneous variation (Doorman et al., 2012). Unlike variational reasoning, which considers only the continuous change of a single variable (e.g. time), covariational reasoning requires the mental coordination of two such processes. This ability to perform smooth continuous covariation is a necessary condition for a sound understanding of calculus. Without it, concepts such as slope or derivative remain at a purely calculation-oriented, formulaic level (Thompson & Carlson, 2017). Typical learning activities to support covariational thinking address the planning and evaluation of variations and the resulting covariations (Günster & Weigand, 2020).

Object

In another view, the function is regarded as an independent mathematical object. This means that attributes can be used to describe a function as a whole (e.g. periodicity, monotonicity, linearity). Furthermore, functions in terms of independent mathematical objects can be manipulated (DeMarois & Tall, 1999). Within this notion, one and the same object can be represented in different ways, for example by means of tables, graphs, formulas or verbal descriptions (Doorman et al., 2012). This structural perspective is particularly promoted by tasks at a global level, such as comparing families of functions or by advanced mathematical operations such as differentiation or integration (Doorman et al., 2012).

In connection with functional thinking, the importance of different representations (e.g. graphs, tables, terms) is emphasised. Learners should not only be able to work with a suitable representation but also be flexible in selecting one (Acevedo Nistal et al., 2012). Studies show that learning with multiple forms of representation promotes qualitative functional thinking (Rolfes et al., 2022). However, deficits often arise when changing between different forms of representation (Dubinski & Wilson, 2013).

PRESENT STUDY

The central facets of functional thinking are input-output assignment, covariation and object (Doormann et al., 2012), each of which can be represented in various ways (Acevedo Nistal et al., 2012). Understanding a function often does not come from its formal definition, but rather from cognitive schemata. Therefore, visual representations, examples and experiences are relevant for the concept image (Vinner, 1983). However, to our knowledge, there is a lack of test instruments that assesses these cognitive schemata and provide information on how the three facets of functional thinking are developed in secondary students. Moreover, from a theoretical perspective, the question arises as to whether the three-dimensional structure of functional thinking as described in the theoretical section can be confirmed. This leads to the first question:

Q1. To what extent can functional thinking be empirically described as a latent construct with the facets input-output assignment, covariation and object?

Due to its importance for further learning (Thompson & Carlson, 2017), the ability to easily assess functional thinking is an important goal. In particular, the individual thoughts of learners and the different facets of functional thinking should be assessed individually. We therefore ask the question:

Q2. How can the functional thinking of secondary school learners be assessed in detail using a written test instrument with the facets input-output assignment, covariation and object?

METHOD

Sample

Data was collected at three grammar schools and one comprehensive school in Germany. Of the 153 participants, 72 were female (47.1%), 79 were male (51.6%) and one was diverse (0.7%; one missing value). The sample consists of Year 11 students who were taught in eleven different courses. No information on age was collected.

Test instrument

For this study, we developed a new test instrument to assess learners' functional thinking. In particular, the test measures the extent to which students' thinking corresponds to given mathematical ways of reasoning. This allows agreement with the facets of functional thinking to be specifically collected and objectively evaluated. For each item, the participants were presented with a mathematical situation and a correct way of reasoning that specifically addressed one facet of functional thinking. They then used a five-point Likert scale to rate the extent to which the given statement corresponds to their own thinking (from 'does not apply at all' to 'applies completely').

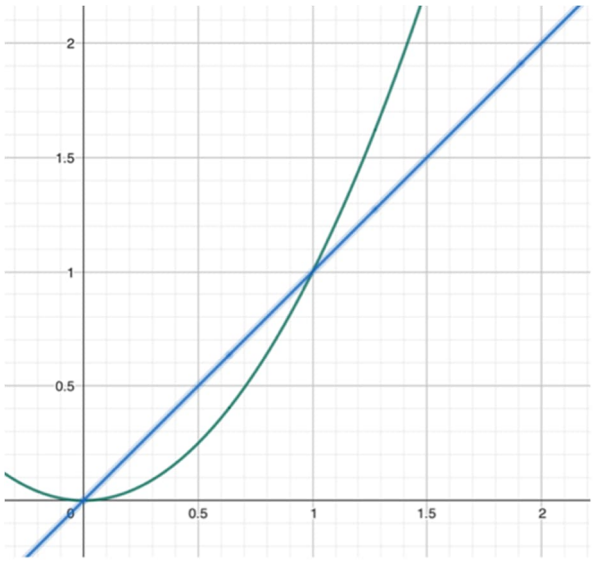
A total of nine items were used, three of which addressed the facet of input-output assignment, three addressed the covariation facet, and three addressed the function-as-object facet. The representations graph, table and term occurred across all items. Figure 1 shows an example of a covariation item and Figure 2 provides an example of a function-as-object item.

Figure 1: Sample item for covariation

Given are $f(x) = x$ and $g(x) = x^2$.

How much does this statement apply to what you think about this function?

I compare how the function values change: For x > 1, the values of g(x) increase faster than those of f(x).



☐ does not apply at all ☐ does not really apply ☐ partly applies, partly does not apply ☐ applies somewhat ☐ applies completely

There is a risk that statements may be misinterpreted or that participants may tend to overestimate or underestimate the degree of agreement. The authors assume that these errors do not produce systematic bias but should be regarded as statistical noise.

Study design

This is a one-off cross-sectional study. The dependent variable is the level of self-reported functional thinking facets measured using the test instrument. The study was conducted in July 2025 as part of the regular mathematics lessons at the end of the 11th school year and was carried out using the digital survey tool SoSci Survey.

To ensure standardisation and comparability of the results, the test was conducted individually and without the use of tools (e.g. calculators or formula collections). Before the test began, uniform instructions were given, emphasising the importance of individual work for research purposes. The test participants completed the test in less than 10 minutes after careful consideration.

Figure 2: Sample item for object

Given is the function: $f(x) = x^3 - 3x$

How much does this statement apply to what you think about this function?

I see that the functional values do not behave constantly in relation to x.

☐ does not apply at all	☐ does not really apply	☐ partly applies, partly does not apply	☐ applies somewhat	☐ applies completely

Data analysis

Confirmatory factor analyses (CFA) were performed using the JASP software (version 0.95) and the R package `lavaan` to examine the factorial structure of the scales. The models examined specified one or three latent factors, which were operationalised by a total of nine items. Codes from 1 to 5 were assigned to each item. The codes were added to obtain the scale values. Of the N = 153 data sets, 10 data sets were excluded because in these cases students quit the survey early. The full information maximum likelihood (FIML) method in `lavaan` was used to handle missing data, as it can be assumed that the data are not systematically missing.

RESULTS

Functional thinking empirically described with three facets (Q1)

Two competing models of factor structure were examined: a one-dimensional model (Model 1), in which all items load on a single latent factor, and a three-dimensional model (Model 2), which is based on theoretical assumptions about functional thinking. Model fit was assessed using various fit indices. A detailed comparison of the fit indices of both models is presented in Table 1.

The three-dimensional model (Model 2) showed a better fit to the data than the one-dimensional model (Model 1). In particular, the values for the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) were lower for the three-dimensional model.

Table 1: Fit indices for the 1-dimensional and 3-dimensional model structure

Model	χ^2	df	p	CFI	TLI	RMSEA [90% CI]	SRMR	AIC	BIC
1	37.56	27	.085	.94	.92	.05 [.00, .09]	.05	3,876.65	3,957.21
2	29.63	24	.197	.97	.95	.04 [.00, .08]	.05	3,874.72	3,869.30

Functional thinking assessed using a test instrument with three facets (Q2)

The model quality of the three-dimensional model 2 was checked using the chi-square test and common fit indices. The chi-square test was not significant, $\chi^2(24) = 29.63$, p=.197, indicating a good fit between the theoretical model and the observed data. Further fit indices confirmed this very good model fit (see Table 1). All indicator variables showed significant loadings on their respective factors ($p < .001$, except for the marker variables fixed for scaling and one item with $p = .045$). The latent factors showed significant variances: Factor 1 'Input-output assignment' ($Var = 0.239, p = .024$), Factor 2 'Covariation' ($Var = 0.237, p = .053$) and Factor 3 'Object' ($Var = 0.264, p = .005$). The correlations between the latent factors were low but reached the level of significance. There was a correlation of $r = .014$ ($p = .024$) between Factor 1 and Factor 2, $r = .019$ ($p < .001$) between Factor 1 and Factor 3, and $r = .022$ ($p = .002$) between Factor 2 and Factor 3.

In the sample examined, the mean values of Factor 1 'input-output assignment' ($M = 12.47, SD = 2.38$) and Factor 3 'object' ($M = 13.28, SD = 2.17$) were higher than those of Factor 2 'covariation' ($M = 11.07, SD = 2.88$).

DISCUSSION

The results show that the three-dimensional model fits significantly better than the one-dimensional model. The empirical findings thus support the theoretically postulated structure of functional thinking (Doorman et al., 2012; Wei et al., 2025) and are consistent with previous studies (Pittalis et al., 2020). The low correlations between the three dimensions also indicate that input-output assignment, covariation and object represent largely independent facets, at least at lower secondary education.

The results also show that it is possible to develop a short but valid test instrument to assess students functional thinking at the end of lower secondary education. In our test

instrument, the three central facets of functional thinking (Doorman et al., 2012) are taken into account and can be evaluated and compared separately. In this sample, for example, we analysed the functional thinking of participants at the transition from lower secondary to upper secondary education. The results indicate that the covariation facet was less pronounced than the other two facets. As covariational thinking is an important prerequisite for developing meaningful concepts in calculus at upper secondary level, the results can be used to specifically promote individual facets of functional thinking.

The choice of items that ask learners to self-report on their thinking about functions allows for more individualised diagnoses regardless of performance (Vinner, 1983). Alternatively, tasks with a right-wrong evaluation could be used. However, these offer less diagnostic precision between the three components at the margins of the performance spectrum. Furthermore, with such formats, it remains unclear whether the solution is actually based on the intended aspect of functional thinking.

Despite the insightful results, the present study is subject to certain limitations. First, the construct of functional thinking was not represented in its entire theoretical broadness; in particular, the restriction to sample tasks from lower secondary education limits the generalisability of the findings to other areas of mathematics. In order to ensure a high degree of test economy and to keep the burden on the test participants low, a reduced number of items was chosen. Finally, it should be noted that the covariation factor is less robustly supported empirically than the other dimensions, which is why the relevant results should be interpreted with caution.

CONCLUSION

The study provides theoretical insights into the structure of functional thinking and, as a result for research and practice, a test instrument for assessing the functional thinking of learners from the end of lower secondary school onwards. This test instrument is suitable both for a rapid but differentiated assessment of the three facets of functional thinking and for assessment in combination with other test instruments, e.g. for basic mental models of the derivative (Greefrath et al., 2023). In the future, it can also be used to investigate the extent to which functional thinking depends on a particular lesson or rather on the learners' prerequisites (cf. Harel et al., 2006, p. 163). Moreover, the test instrument could be used to further investigate the potential of computer tools in supporting the development of covariational thinking (Kaput & Schorr, 2007).

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