STRATEGIES USED BY HIGHER-PERFORMING STUDENTS IN SOLVING COMBINATORIAL PROBLEMS

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This study investigates the strategies that students employ when solving combinatorial problems. Therefore, a written test with 26 items was administered to 282 secondary students at the end of their school career. This paper focuses on higher-performing students who solved at least 75% of the tasks correctly, as their strategy use provides insights into successful approaches. The application of strategies was (overall and correctly applied) as well as the frequency and variance of strategies per person were analyzed. The findings indicate that higher-performing students predominantly rely on formulas, arithmetic rules, and systematic enumeration compared to other strategies and highlight the importance of fostering different combinatorial strategies in instruction and provide implications for the design of teaching learning environments.

INTRODUCTION

Combinatorics is an important topic in discrete mathematics and is covered in the central written examinations at the end of school in Germany (e.g., MBJS, 2022). Kapur (1970) highlighted its relevance based on several aspects still valid today: Combinatorial problems are independent of calculus, making them accessible across different grade levels while offering opportunities to engage with challenging problems that highlight the need of advanced mathematical ideas. Moreover, they foster systematic thinking, generalisation, and counting, while also supporting the development of mathematical concepts, such as equivalence relations and order relations, functions, and sampling (e.g., Kapur, 1970). Despite its importance in mathematics, combinatorics is considered a challenging topic, as numerous studies show that secondary school students (e.g., Eizenberg & Zaslavsky, 2003) and university students (Hadar & Hadass, 1981; Lockwood, 2013) have difficulties solving combinatorial problems.

A deeper understanding of these challenges requires analysing the solution strategies students use when solving combinatorial problems, as well as the factors determining the choice and appropriateness of these strategies (Batanero et al., 1997). To gain insights into productive and successful approaches to combinatorial problem solving, this paper focuses on higher-performing students and investigates the solution strategies they use. By identifying patterns in successful strategy use, including the co-occurrence of strategies within individual students' solutions and the diversity of strategies employed across individuals, the study aims to derive implications for promoting combinatorial thinking.

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THEORETICAL BACKGROUND AND RESEARCH QUESTION

Learners often struggle with combinatorial tasks (Batanero et al., 1997; English, 2005; Lamanna et al., 2022; Roa, 2000). To support learning in combinatorics, it is necessary to gain an understanding of learners' solutions to identify (less) viable approaches, strategies and concepts. This article therefore examines the strategies learners use when working on specific tasks, since different strategies influence task success (e.g. Roa, 2000).

Research on solving combinatorial problems has identified a range of strategies that students (secondary school students aged 11–14 (Fischbein & Gazit, 1988) and 14–18 (Batanero et al., 1997; Lamanna et al., 2022), as well as elementary school students (English, 2005) or university students (Roa, 2000; Godino et al., 2005) draw on when solving such tasks. The following section outlines central solution strategies discussed in the literature, with a particular focus on their characteristics, typical patterns of use, and associated difficulties.

One fundamental strategy for solving combinatorial problems is **enumeration**, that is, the listing of all possible configurations (English, 2005). When enumeration is carried out systematically, elements are combined according to a fixed scheme, which typically results in complete and correct solutions. In contrast, a-systematic enumeration often follows a trial-and-error approach and tends to produce incomplete or erroneous listings (Fischbein & Gazit, 1988; Lamanna et al., 2022). As a visual support for enumeration, **tree diagrams** can be used to facilitate the structured identification of all possible cases (Fischbein, 1975; Roa, 2000). Despite their didactic potential, however, empirical studies indicate that tree diagrams are rarely used by students and, when applied, are often implemented inaccurately or incompletely (Fischbein & Gazit, 1988; Lamanna et al., 2022).

Beyond enumeration, students may draw on more general problem-solving strategies, such as **the reference to another problem** or **decomposing a task into sub-problems** (e.g., Pólya, 1995; Rott et al., 2023). When using analogies, new combinatorial problems are mapped onto previously encountered or well-known situations, a process that has been observed particularly among higher-performing students (Godino et al., 2005). Decomposition into sub-problems allows complex tasks to be broken down into smaller, more manageable parts, whose partial results can subsequently be integrated into an overall solution (Roa, 2000). These strategies support a structured, step-by-step approach to problem solving and are often combined with other strategies.

At the symbolic-mathematical level, combinatorial problems can be solved using **arithmetic rules**, such as the product and sum rule, as well as **combinatorial formulas**. Empirical findings show that students frequently use these strategies in combination with (partial) enumeration, for example by first listing selected cases to identify structural regularities before applying arithmetic rules to determine the total number of possible outcomes (Roa, 2000). Successful application of arithmetic rules and formulas requires students to correctly identify the underlying combinatorial

operation (permutation, arrangement, or combination, with or without repetition) and to apply the relevant parameters appropriately. However, research consistently indicates that this requirement poses substantial difficulties for many learners, particularly in problems involving multiple parameters, which are especially prone to confusion and error (Roa, 2000; Lamanna et al., 2022).

Regarding students' performance levels, higher-performing students (as measured by the number of correctly solved problems) tend to employ strategies such as systematic enumeration, decomposition into sub-problems, reference to another problem, and the product rule in a targeted and precise manner (Godino et al., 2005; Roa, 2000). Inaccurate or incomplete solutions are often observed in lower-performing students, as they apply inappropriate formulas or misinterpret problems (Batanero et al., 1997).

Previous studies analyzing learners' use of strategies have focused either on younger students (e.g., Batanero et al., 1997) or on university students (e.g., Roa, 2000). In comparison, relatively little is known about the strategic behavior of students at the end of upper secondary school (aged 17 to 18). To date, this age group has been explicitly studied primarily in the study by Lamanna et al. (2022).

Given the current state of research, it is essential to investigate more comprehensively how successful students solve combinatorial problems. They could provide insights into how and which different strategies contribute to success, thus enabling a nuanced understanding of combinatorial thinking in the school context. To address this gap, the present study analyzes the strategies used to solve a broader and more varied set of combinatorial problems compared to previous work by Lamanna et al. (2022). Expanding their study, which focused on four tasks (involving permutation and combination), we investigate 26 tasks that cover all combinatorial operations taught in German schoolbooks (Thomas & Pöhler, submitted): permutation and arrangement with and without repetition, as well as combination without repetition. The analysis investigates the strategies used by higher-performing students and identified which of these strategies led to correct solutions. This allowed us to analyse both students' strategy preferences and the effectiveness of different strategies. Against this theoretical background, the present study addresses the following research question:

> *What strategies do higher-performing students use when solving combinatorial problems, and how do these strategies differ in their success rates when working on these problems?*

METHODICAL APPROACH

To address the research question, a total of 282 students from 21 upper secondary school classes in Brandenburg, Germany, participated in the study. A performance-based sub-sample was created for the analysis. Each of the 26 tasks was scored dichotomously (1 = correct, 0 = incorrect), yielding a maximum score of 26 points. Students scoring between 19 and 26 points were classified as higher-performing and therefore represented the top 25% of the score distribution. This resulted in a sub-

sample of 14 higher-performing students. The participants were between 17 and 18 years old and attended different types of schools and courses. Since combinatorics are part of the central focus of the final examinations (e.g., MBJS, 2022), all students had received instruction in this topic (according to teacher reports, an average of four 90-minute sessions).

The survey was administered as a paper-and-pencil test during a regular mathematics lesson. Students were given 45 minutes to complete 26 open-ended combinatorial word problems individually, without tools. This time limit was based on two previous pilot studies ($N = 141$) that confirmed its practicality. All answers were provided in written form.

The tasks were designed to systematically represent various combinatorial operations in school (permutation and arrangement with and without repetition, as well as combination without repetition), combinatorial models (selection, partition, and distribution, according to Dubois, 1984), different type of contexts (abstract (using numbers or letters) and concrete (using objects or people) according to Batanero et al., 1997), and the number of configurations (low and high according to Fischbein and Gazit, 1988) (see figure 1).

Figure 1: Exemplary item (arrangement with repetition, selection, abstract, and low)

> There are two labelled marbles (A and K) in a bag. We select a marble, note its letter, put it back, and then repeat the process.
> How many different letter combinations can be formed in this way?

The strategies were analyzed using qualitative content analysis according to Mayring (2015). The deductive category system was based on the strategies described in earlier studies (as noted above) (e.g., Godino et al., 2005; Roa, 2000). These strategies include:

- **Enumeration** (systematic and a-systematic), **tree diagrams**, **reference to another problem**, **decomposing a task into sub-problems**, **arithmetic rules**, and **combinatorial formulas**.

During coding, the system was inductively expanded to include the following:

- **Mixed strategies** (a combination of several strategies); **Others** (stating only the result without an identifiable strategy, random calculations, naming the operation without justification or using solution methods that do not correspond to any known strategies); **No answer**.

An independent second coder evaluated 25% of the dataset. The intercoder reliability κ was calculated in MAXQDA 2024 and resulted in a value of $\kappa = .97$. According to the classification of Landis and Koch (1977), this value indicates almost perfect agreement. Consensus was subsequently found for the disjunctive cases.

RESULTS

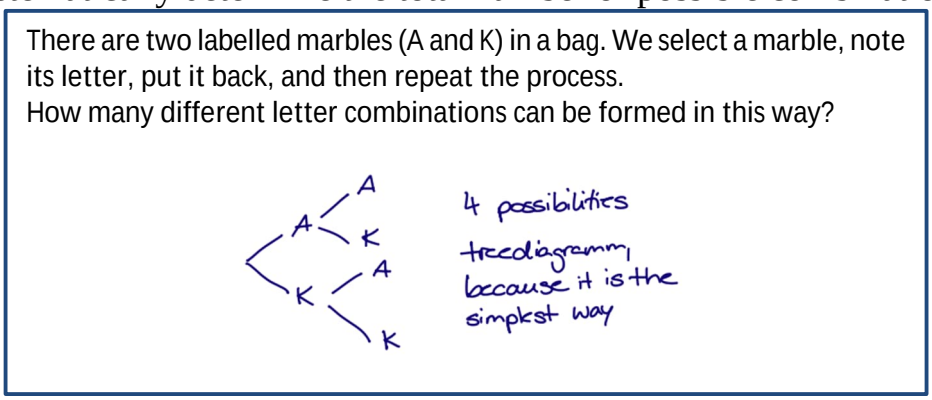
To address the research question of which solution strategies higher-performing students use when solving combinatorial problems and how these strategies differ in terms of their success rate, the use of different strategies was analyzed among students who correctly answered at least 75% of the 26 items (see Table 1). Fourteen students met this criterion, resulting in a total of 364 responses (14 students x 26 items) that were included in the analysis.

Table 1: Strategy use and successful application of the strategy use among higher-performing students (N = 14; 364 responses) in relative frequencies

Strategy	Strategy use	Successful application
Systematic enumeration	16.3%	90.9%
A-systematic enumeration	0.3%	0%
Tree diagram	0.9%	100%
Reference to another problem	0%	0%
Sub-problem decomposition	2.7%	88.9%
Arithmetic rules	13.0%	65.9%
Formula	41.7%	75.9%
Mixed strategies	2.1%	57.1%
Other	29.3%	83.8%
No answer	0.9%	0%

The descriptive analysis shows that higher-performing students predominantly rely on formulas (41.7% strategy use). The formula strategy exhibited a comparatively high success rate, with 107 correct solutions (75.9%). Systematic enumeration was also one of the most used strategy (16.3% strategy use) and proved to be very successful (96.4%). Mixed strategies combining different approaches were documented in seven responses (2.1% strategy use), of which five resulted in correct solutions (57.1%). When mixed strategies were used, the combination most frequently involved systematic enumeration and the application of arithmetic rules. The responses in the category other among high-performing students with 29.3% strategy use mainly comprised the pure statement of results (e.g., 24 possibilities). The tree diagram strategy appeared only in three cases (0.8%), but each time resulted in a correct solution (100%). In figure 2, for example, a complete tree diagram was used to determine all possible letter combinations, resulting in the correct solution.

Figure 2: Illustration of the solution strategy: a complete tree diagram used to systematically determine the total number of possible combinations



Overall, the descriptive analysis shows that higher-performing students draw on a wide range of solution strategies, and that both the frequency of use and the success rates of these strategies vary considerably. This broad spectrum is also evident at the individual level: many students employ more than one strategy in their written test. While two students relied exclusively on a single strategy (relied on formulas), the others applied to five different strategies. The majority (9 out of 14 students) employed four or five different strategies.

When considering each student separately, formula use emerged as the most frequently adopted strategy for most students: eight out of fourteen relied primarily on formulas. The remaining six students use systematic enumeration or arithmetic rules that represent their dominant strategy. This distribution indicates notable variation in dominant strategy preferences among higher-performing students, despite comparable overall performance levels.

DISCUSSION AND CONCLUSION

This study provides initial insights into the strategic approaches of higher-performing students when solving combinatorial problems. A central finding is their frequent and successful application of formulas, which aligns with previous research showing increased reliance on formal strategies following instruction in combinatorics (Lamanna et al., 2022). Higher-performing students appear to possess sufficient knowledge to identify and correctly apply the appropriate formula, which Roa (2000) frequently identified as a source of error – especially if understanding is not well established (Batanero et al., 1997). The relatively frequent use of systematic enumeration also confirms existing findings (English, 2005; Fischbein & Gazit, 1988; Lamanna et al., 2022). The frequent and correct implementation of systematic enumeration once again underscores the importance of rule-based listing strategies as a reliable approach to solving combinatorial problems. The very low adoption of tree diagrams is striking. This is consistent with prior research highlighting their didactic potential, while noting that they are rarely taught systematically in the classroom and are therefore rarely established as a problem-solving tool (Fischbein, 1975).

Furthermore, several strategies were rarely or never observed in the available data. Whether this absence is due to limited exposure in class or to conscious or unconscious decisions made by students when choosing strategies cannot be determined. Against this background, the results seem to underscore the importance of providing students with access to a wide range of approaches to solving combinatorial problems in order to support flexible and adaptive strategy use.

Regarding the number of strategies employed, it was found that most high-performing students used several different strategies within a test. Taken together, the findings indicate that higher-performing students draw on flexible and heterogeneous strategy repertoires rather than relying on a single dominant approach. This suggests that successful combinatorial problem solving is associated not only with the development of specific strategies, such as formulas, but also with the ability to coordinate and switch between multiple strategies at the individual level (Godino et al., 2005). In summary, the wide range of strategies employed by higher-performing students demonstrates that effective processing is not limited to a single strategy. Therefore, future teaching and learning concepts should present multiple strategies for combinatorics and encourage students to their flexible application.

Due to the small sub-sample (14 out of 282 students), the findings should be interpreted as exploratory insights into patterns of successful strategy use and can only be generalized to a limited extent. Moreover, the written data allow conclusions only about strategies that were explicitly documented, not about those potentially available but unused, as their absence may reflect limited instructional exposure, the time frame of the test, or strategic choice. This limitation is particularly relevant given the rare adoption of tree diagrams, which may be attributable to instructional practices rather than students' strategic competence (Fischbein, 1975). Using qualitative methods such as thinking-aloud or interviews could therefore reveal whether unused strategies were available, and whether the selection was influenced by limited teaching experience, task characteristics or personal preferences. More broadly, as students' strategy use is shaped by prior instructional experiences, differences in strategy frequency and success cannot be interpreted independently of teaching contexts. Future research should therefore examine students' strategy decisions more directly, include different performance groups, and investigate how targeted instructional approaches may support more flexible combinatorial problem solving (Roa, 2000; Godino et al., 2005).

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