DESIGNING ONLINE ASSESSMENT TO STUDY CONCEPTIONS OF EQUIVALENT LINEAR EQUATIONS IN AN MLR ENVIRONMENT

Julia Bagdadi and Michal Yerushalmy

University of Haifa

This paper reports on the design of an automated assessment for exploring students' conceptions of equivalence of linear equations in a multiple-linked representation (MLR) environment. Students' approaches to equivalence are examined in a function-based context, where an equation is represented as a comparison of two functions. Common solutions are visually represented by a shared X value of intersection points, derived by representation of a sequence of solution-preserving transformations. Implemented in the STEP platform, answers to Example Eliciting Tasks (EETs) – including triangulation of choice of examples from lists of equations, learner-generated examples and choice items are automatically analysed.

INTRODUCTION

This paper reports on the design and use of an online activity for assessing students' approaches when identifying equivalent linear equations in an MLR algebra context. The assessment was implemented using the STEP platform, an online environment designed to support work on EETs (Olsher, 2019; Olsher & Yerushalmy, 2020). STEP makes it possible for task designers to have rich data on student work and descriptive analytics that identify valuable aspects in students' work which educators can use for higher-inference judgments like student conceptions and reasoning processes. Here, STEP supports triangulation across three coordinated data sources.

THEORETICAL BACKGROUND

Equivalent equations over the real numbers are often defined through a *common* approach, according to which two equations are equivalent if they share the domain and the same solution set (Gelfand & Shen, 1993). Solving equations in school through writing a set of equivalent equations, however, relies on a *transformational* approach, whereby equivalence is established through a finite sequence of solution-preserving transformations. Cook, Richardson, Reed & Lockwood (2023) conceptual analysis assumes that in school algebra students need to understand that *transformations* preserve the *common* characteristic. Yet, research on symbolic manipulations shows that algebra students' attention is frequently drawn to surface features of symbolic transformations, not to the underlying characteristic (Larkin, 1989). For example, Kieran and Martínez-Hernández (2022) studied students' challenge of attending to invariant solution sets when symbolic forms change. Noster and Siller (2025) empirically distinguished between recognizing, producing, and justifying equivalence.

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, pp. 123-130. PME.

Attempting to provide greater conceptual understanding of solving equations procedures, functions'-based algebra in the last decades looked at linear equations represented as comparison of functions. When a linear equation is represented as two linear functions whose graphs intersect, algebraically equivalent equations typically produce different pairs of lines, while preserving only the x-value of the intersection point. Equivalence of equations produced through legitimate transformations and keeping a common solution no longer corresponds to the sameness of graphs and same intersection point (as for example in the solving of systems of equations). Thus, this representational system may have the undesired effect of covering over the commonality that students are supposed to focus on. Yerushalmy & Chazan (2002) exemplify this challenge when an invariant solution becomes visually different. Thus, our goal is to study whether students attend to this representational gap and to qualitatively address the following research questions: 1. What can online MLR EETs assessments automatically reveal about how students identify equivalent linear equations? 2. What information can analysis of multiple inputs (learner-choices examples, learner-generated examples LGE, and forced choices) provide that may support formative assessment of conceptions of equivalence of equations?

METHOD

The assessment: EETs have been shown to be particularly effective for revealing learners' concept images, because they require learners to articulate boundaries of a concept through construction of examples and non-examples (Watson & Mason, 2005). However, such tasks pose challenges for assessment due to the open variability of LGEs. The STEP platform was developed to address this challenge. Within this framework, online MLRs play a dual role. They provide learners with immediate feedback on the consequences of their actions, and they provide the assessment system with structured data that can be analysed automatically.

The participants were 21 ninth-grade students studying in advanced mathematics classes. The experiment was conducted as part of an algebra class where students individually worked with their laptops. The students had previous experience solving linear equations symbolically and interpreting linear functions graphically. As an introduction to the experiment, the students worked on a preparatory STEP task in which they were presented with an applet of a linear equation and identified equations with one solution (intersecting lines), no solution (two parallel lines), and an equation with a solution for all x (two common lines).

The experimental task involved an applet that includes a given linear equation $-2x-10=2x+6$ represented symbolically in the form $f(x)=g(x)$ and graphically by an ID of two intersecting linear functions in a Cartesian coordinate system. The solution of the equation was represented by a vertical dashed line that highlighted the x-value of the point of intersection. The core of the task is a dynamic list of 8 linear equations (see Table 1 below); 5 equivalent and 3 non-equivalent equations. The graph window showed the given equation $-2x-10=2x+6$ (in red and blue colours) and one of the 8

equations that students could choose from (in green colour). Students may generate multiple lists of eight equations that preserve the same structural categories of equivalence and non-equivalence while varying numerical parameters. Thus, example elicitation operates not only through LGEs but also through students' choices from dynamically changing given example spaces. The task consisted of three parts. Part A focused on equivalent equations; students selected from the list of 8 equations an equation equivalent to the given equation (LST) and generated additional equivalent equations of their own (INP). Part B focused on non-equivalent equations; students selected from the same or similar list and also generated their own non-equivalent equations. In Part C, students responded to forced-choice claims about the invariance of the solution's graphic representation. They had to decide whether the two equations share an intersection point (x and y values) or an x-value of the intersection.

METHOD OF DATA ANALYSIS

The analysis focused on identifying patterns in the students' submissions using the categories defined in Table 1. The table is organized along the algebraic structures repeated in each generated list. The STEP platform automatically recorded the equation choices from the list, as well as the equivalent and non-equivalent equations that the student generated. The records are dynamic, allowing observation of the work. STEP coded the submissions according to the predefined equations' categories (Table 1) and presented graphs and tables of comparative analysis within and across the three parts.

FINDINGS

Findings in Part A show that all equations that students chose (LST) and all generated examples (INP) were indeed equivalent to the given equation, indicating that students were generally able to identify equivalence. Most submissions clustered around three structural categories: explicit solution format (EX), operations on both sides (CHBS), and restructuring of one side (CH1S-IDN).

Table 2 presents the distribution of students' submissions by types of equivalent equations. Almost half of the LST and more than half of the INP equations belonged to the EX type.

Less than a third resulted from transformation on one side (CH1S). The list included two equivalent equations representing transformations on one side. One equation represented restructuring of an expression on one side, based on the distributive law, resulting in an identical function (CH1S-IDN). The other transformation preserved equivalence because the expression added to one side evaluated to zero at the solution of the original equation (CH1S). In both cases, the graphical representations of the given and the transformed equations shared an intersection point (x and y values).

Table 1: Structural categories and coding of equations used in the task.

Symbolic structural description	Representative equation	Graphical invariant (MLR)	Status	Type's Code
Explicit solution format	-3x = 12	Same x-value of intersection; one-line horizontal	Equivalent	EX
Same operation on both sides (add/subtract)	-2x-10-10 = 2x+6-10	Same x-value; different intersection	Equivalent	CHBS-A
Same operation on both sides (multiplication)	8x+40 = -8x-24	Same x-value; intersection may be outside window	Equivalent	CHBS-M
Reduction to zero (all terms to one side) with an error	-2x+2x+6-10 = 0	One line is the x-axis, and one line horizontal	Non-Equivalent	CHBS-Z
Shifting with error a term from one side to other side	-2x = 2x+6-10	Different x-value of intersection	Non-Equivalent	CHBS
One side changed, neutral at solution	-2x-10=-3x-14	Same intersection; one line coincides	Equivalent	CH1S
One side changed, non-neutral at solution	-2x-10 = 2x+6+ +x-4	Different x-value of intersection	Non-equivalent	CH1S
Restructuring one side (identity)	5(-0.4x-2)=2x+6	Same pair of graphs	Equivalent	CH1S-IDN

Table 2: Distribution of examples submitted by types of equivalent equations (Part A)

Examples	EX	CHBS-A&M	CH1S	CH1S- IDN
Chosen LST	45% (16/36)	11% (4/36)	22% (8/36)	22% (8/36)
Generated INP	65% (24/37)	16% (6/37)	3% (1/37)	16% (6/37)

Approximately one tenth of the choices were equations involving the same operations applied to both sides (CHBS). This result was surprising, given that symmetrical manipulations are central to the balance model of equations and are commonly emphasized in school instruction. Moreover, in Part B, three submissions mistakenly identified an equivalent equation of the form M*f(x)=M*g(x) (M=7) as non-equivalent.

It might be that the reference was graphical in these cases and the point of intersection of two lines was not even visible in the applet window and therefore was mistakenly classified as a non-equivalent equation (Figure 1).

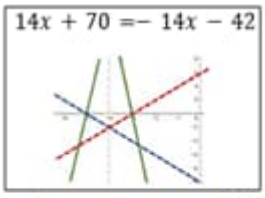


Figure 1: Graphical representations CHBS

In Part B, similarly to Part A, more than half of the INP non-equivalent equations were in EX structure. There were students (such as S16 in Figure 2), who consistently generated equations of the EX type in all their submissions. But, as the given lists did not include EX for non-equivalent equations S16 chose CHBS operations on both sides and then generated EX examples of non-equivalent equations (Figure 2 B).

In Part C, 18 out of 21 students marked the claim that equivalent equations share an x-value intersection. Three students marked the claim that there must be a common point that shares (x,y) values of intersection for a pair of equivalent equations. One of them is S21 who selected the CH1S equation from the list and also generated an equivalent and a non-equivalent equation type CH1S. S21 seemed to consistently pay attention to the graphical feature of a common point of intersection when identifying equivalence (Figure 3 A). To generate possible non-equivalent examples S21 chose to change one side in a way that “moves” one line away from the common intersection (Figure 3 B).

Figure 2: Student S16 submissions of equivalent (A); non-equivalent equations (B)

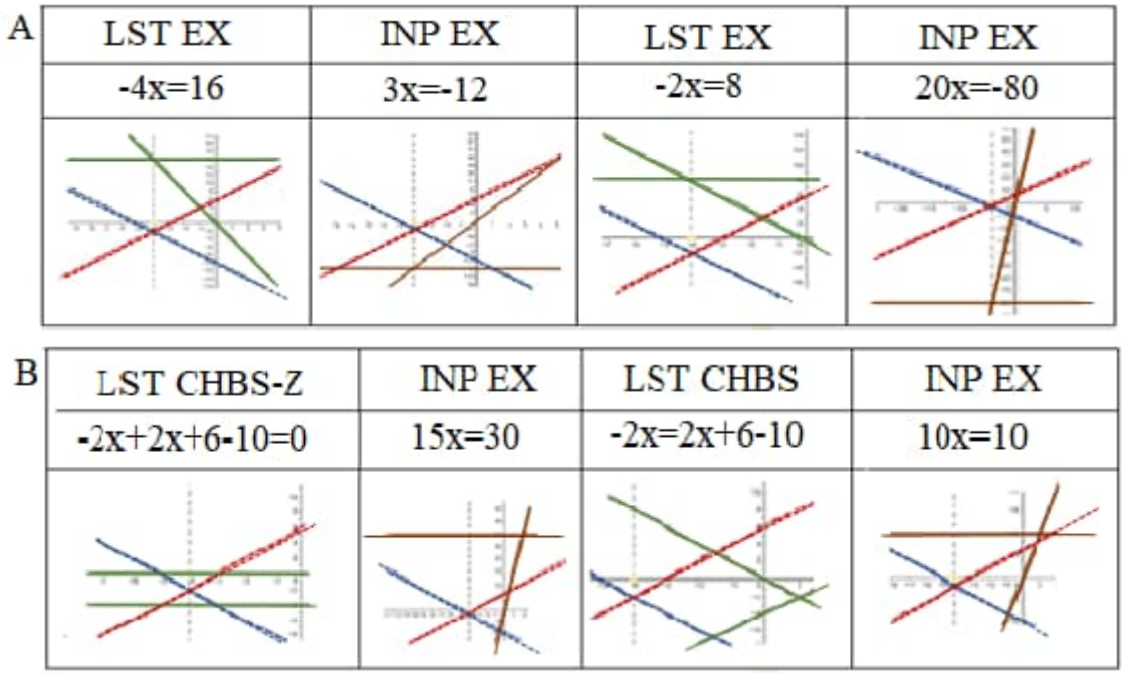
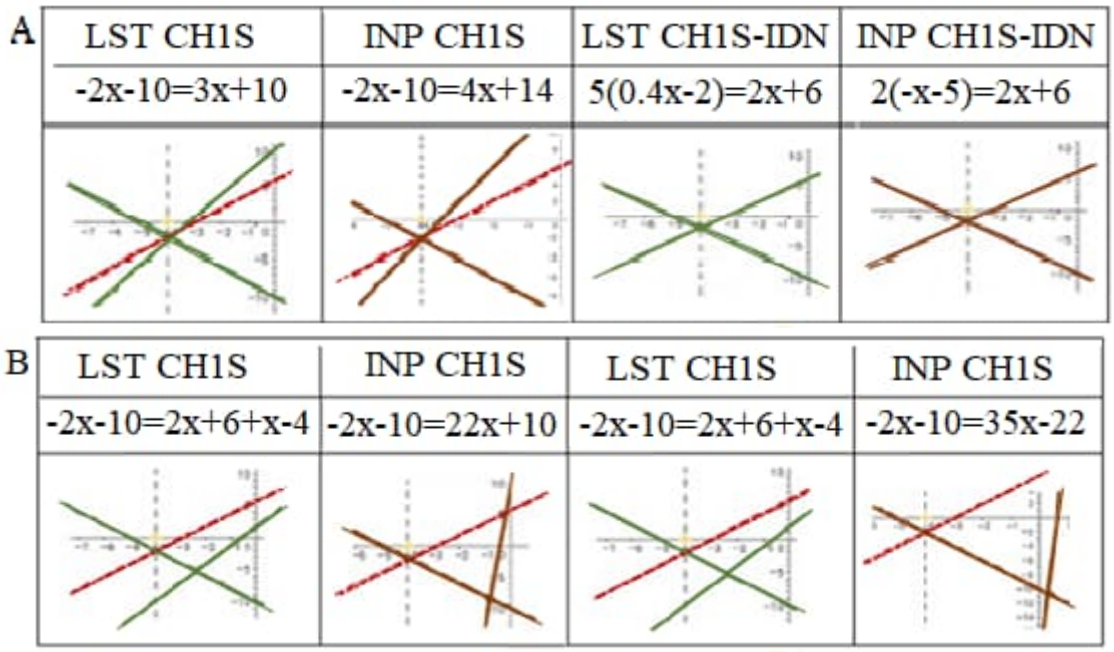


Figure 3: Student S21 submissions of equivalent (A); non-equivalent equations (B)



DISCUSSION AND CONCLUSION

This study examined students' reasoning about equivalent linear equations through an automated example-eliciting assessment embedded in a multiple-linked representation (MLR) environment. The findings highlight how students' sense making of equivalence is deeply intertwined with what they can perceive, attend to, and

coordinate across symbolic and graphical representations. A prominent pattern in the data is students' strong reliance on *common solution* equivalence. For most students, equivalence was associated with "having the same answer," expressed symbolically or claimed as a shared x-value of intersection. However, another mode of reasoning derived from the work showing consistent attention to a common point of intersection when identifying equivalence. This highlights a tension introduced by function-based setting, where valid algebraic transformations do not necessarily preserve visually expected features. The MLR environment further revealed tensions between algebraic legitimacy and perceptual accessibility. Symmetrical transformations applied to both sides of an equation were sometimes rejected as non-equivalent when the resulting intersection point was outside the visible window.

From an embodied perspective, this suggests that when invariance cannot be *sensed* in the representation, its mathematical legitimacy may not be recognized.

From a formative assessment perspective, the study demonstrates the value of the patterns we were seeing to create a conceptions' scheme of coding that might support teachers with data that would help them think about how their students are thinking about the notion of equivalence. By combining analysis of selections from structured lists, learner-generated examples, and responses to conceptual claims together with natural-language trained models, the assessment would potentially have the power to further identify, explain and support teachers in relating to distinctions in students' conceptions of equivalence that would be difficult to detect otherwise.

References

Cook, J. P., Richardson, A., Reed, Z., & Lockwood, E. (2023). Using conceptual analyses to resolve the tension between advanced and secondary mathematics: the cases of equivalence and inverse. *ZDM–Mathematics Education*, *55*(4), 753-766.

Gelfand, I. M., & Shen, A. (1993). *Algebra*. Birkhäuser.

Kieran, C., & Martínez-Hernández, C. (2022). Coordinating visible and invisible sameness in equivalence transformations. *ZDM–Mathematics Education*, *54*, 1215–1227. https://doi.org/10.1007/s11858-022-01374-9

Larkin, J. H. (1989). Robust performance in algebra: The role of the problem representation. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*, Vol. 4, pp. 120–134). Lawrence Erlbaum Associates, Inc; National Council of Teachers of Mathematics.

Noster, N., & Siller, H.-S. (2025). Transforming equations into equivalent equations: An empirical study of equation transformation capability as a multidimensional construct. *Educational Studies in Mathematics*, *120*, 249–267. https://doi.org/10.1007/s10649-024-10287-3

Olsher, S. (2019). Making good practice common using computer aided formative assessment. In G. Aldon & J. Trgalová (Eds.), *Technology in Mathematics Teaching (pp. 31–47).* Springer. https://doi.org/10.1007/978-3-030-19741-4_2

Watson, A., & Mason, J. (2005). Mathematics as a constructive activity: Learners generating examples. Lawrence Erlbaum.

Yerushalmy, M. & Chazan D. (2002). Flux in school algebra: Curricular change, graphing technology, and research on student learning and teacher knowledge. In L. English et al. (Eds.) *Handbook of International Research in Mathematics Education*. pp. 725-756. Hillsdale, NJ: Erlbaum.

Yerushalmy, M., Olsher, S. (2020). Online assessment of students' reasoning when solving example-eliciting tasks: using conjunction and disjunction to increase the power of examples. *ZDM–Mathematics Education, 52*, 1033–1049. https://doi.org/10.1007/s11858-020-01134-0