SENSING AND MAKING SENSE OF DEFORMATIONS: EMBODIED PATHWAYS TO ADVANCED MATHEMATICAL THINKING AND EARLY TEACHER IDENTITY

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This study examines how undergraduate mathematics students developed advanced mathematical thinking and early teacher identity through embodied exploration of topological deformations. Students progressed through pre-rigorous, rigorous, and post-rigorous phases while designing teaching resources. Analysis through noticing and agency frameworks reveals how pedagogical awareness emerges when abstract mathematics is approached through embodied, multi-modal pathways. Findings show that mathematical noticing becomes pedagogical noticing when students exercise design agency, offering new perspectives on addressing Klein's second discontinuity through epistemological rather than content-based connections.

Klein's (2016) "second discontinuity" describes how prospective teachers struggle to connect abstract university mathematics to secondary school teaching. Despite decades of concern, empirical studies reveal teachers perceive minimal value in their university mathematics courses for classroom practice (Wasserman et al., 2023). This study explores an alternative approach grounded in embodied cognition (Palatnik et al., 2023; Palatnik & Cooper, 2025) by deliberately engaging students' sensory-motor experience with topological concepts to create pathways toward both abstract understanding and pedagogical awareness (Miranda, 2023). We investigate how third-year mathematics students developed understanding of deformations, conceived simultaneously as embodied transformations and formal homotopic equivalences, while recognising teaching potentials within this advanced topic. Three questions guide our investigation: *How do students make sense of abstract topological concepts through embodied activity? What teaching potentials do they notice when advanced mathematics is approached through multi-modal pathways? How does exercising agency in designing teaching resources shape their emerging identities as mathematics educators?*

THEORETICAL GROUNDING

Embodiment and Advanced Mathematical Thinking

Embodied cognition positions mathematical thinking as emerging through bodily interaction with our environment rather than purely abstract mental manipulation (Palatnik & Cooper, 2025). Mathematical activity is rooted in sensory-motor experience, even when dealing with abstract concepts. This perspective challenges traditional mind-body dualism, suggesting that perception itself is "perceptually guided

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, pp. 1027-1034. PME.

action" (Varela et al., 1991). On this strand, Advanced Mathematical Thinking (AMT) involves moving from intuitive, experience-based reasoning toward formal, definition-based mathematical work (Tall, 1991). In this regard, Tao's (2007) framework refines the path into three stages: *pre-rigorous* (informal, intuitive), *rigorous* (precise, formal), and *post-rigorous* (returning to intuition supported by rigorous foundations). This view accommodates embodied perspectives: the pre-rigorous stage involves sensory-motor exploration, while the post-rigorous stage requires integrating formal understanding back into embodied intuition. We trace them by using *noticing* and *agency* concepts.

Teacher noticing as an embodied, multi-layered practice

Teacher noticing, identifying mathematically significant moments and pedagogical opportunities, has become central to understanding teaching expertise (Sherin et al., 2011). Recent reconceptualisation emphasises noticing as embodied, layered, mutual, and meta (Coles & Helliwell, 2025), challenging traditional cognitive models of attending-interpreting-deciding. From an enactivist perspective, noticing emerges through engagement, carries affective colouring, and occurs at multiple levels simultaneously (Mason, 2002). Mason (2025) distinguishes a *detailed layer* (specific observations), a *middle layer* (recognising patterns), and an *abstract layer* (value judgements and purposes). For prospective teachers, noticing operates in dual registers: *mathematical noticing* (recognising structures, relationships, and reasoning patterns) and *pedagogical noticing* (identifying teaching potentials, anticipating difficulties, and recognising opportunities for accessibility). While Coles and Helliwell (2025) examine noticing in practising teachers during instruction, we extend their framework to prospective teachers learning advanced mathematics. We investigate whether and how pedagogical noticing emerges during content learning itself, particularly when that learning is structured through embodied, multi-modal pathways.

Agency in learning and teaching mathematics

Agency, the capacity to act independently and make meaningful choices, emerges through social interaction rather than as an individual possession (Pickering, 1995). Andersson and Norén (2011) identify multiple dimensions: students exercise *mathematical agency* through choices about which ideas to pursue and how to represent concepts, while *pedagogical agency* extends to decisions about sequencing activities and creating learning opportunities. Agency proves particularly significant for teacher education because it connects to identity formation (Holland et al., 2001): when students make consequential choices rather than implementing prescribed procedures, they develop identities as mathematical thinkers and educators. We conceptualise agency as operating across three interconnected dimensions in our context: mathematical agency, pedagogical agency, and epistemic agency.

RESEARCH DESIGN

This study emerged from a year-long algebraic topology course (2024/2025) for 30 third-year undergraduate mathematics students at an Italian university. It is part of a

research conducted over the last three years investigating the opportunities to recognise teaching potentials within advanced contents while learning. This course deliberately foregrounded connections between embodied intuition and formal reasoning.

Three-task learning sequence. We focus on the homotopic equivalences subject. Students engaged with deformations, working in groups through interconnected tasks:

Task 1 (ATEW—Algebraic Topology Embodied Working) guided students through three phases: pre-rigorous (imagining and describing what happens when "squashing" geometric objects: cylinders, cones, spheres); rigorous (translating intuitions into formal mathematics: parametric representations, retraction maps, continuity proofs); and post-rigorous (determining whether various objects could be obtained as deformation retracts, validating intuitions against formal definitions).

Task 2 (Resource Development) asked student groups to create resources on the topology topics for peers or second-year students, with complete designing freedom.

Task 3 (Implementation and Reflection) required groups to organise a lesson implementing their interactive resources and then, in a written individual interview, reflect on what they learnt mathematically and pedagogically.

Participants and data collection

Students worked organised in small groups. This paper focuses on one group (Ben, Mia, and Leo) selected through purposive sampling: (1) their resource most directly built on the deformations concept from Task 1, allowing us to trace development from learner to designer; (2) they created a comprehensive multi-modal resource integrating embodied argumentations, GeoGebra explorations, video explanations, formal mathematical development, and interactive Kahoot/Genially verification; and (3) their reflective responses showed the richest articulation of both mathematical and pedagogical insights. Data sources included Task 1 written responses showing sense-making through the three-phase progression; the group's GeoGebra Book and accompanying lesson notes revealing pedagogical choices; video recording of their one-hour lesson implementation; and individual written interview responses regarding mathematical learning, pedagogical insights, challenges faced, and identity shifts.

Analysis approach

We conducted qualitative content analysis using noticing and agency as analytical lenses. Our coding scheme distinguished six dimensions. ***Noticing dimensions***: *mathematical noticing* (*mN*): recognising mathematical structures, patterns, relationships, and reasoning strategies; *pedagogical noticing* (*pN*): identifying teaching potentials, anticipating learner difficulties, recognising opportunities for accessibility; *meta-noticing* (*metaN*): awareness of one's own learning processes, recognising how understanding develops. ***Agency dimensions***: *mathematical agency* (*mA*): choices about which mathematical ideas to emphasise, which theorems to prove versus illustrate, and which examples to develop; *pedagogical agency* (*pA*): decisions about

sequencing activities, choosing representational tools, and balancing abstraction, and concreteness; *epistemic agency* (*eA*): questioning assumptions, making conceptual distinctions visible, challenging common conflations. Analysis proceeded iteratively. All data were coded for evidence of each dimension, and their interactions examined.

FINDINGS AND ANALYSIS

Making sense through sensation: the pre-rigorous phase.

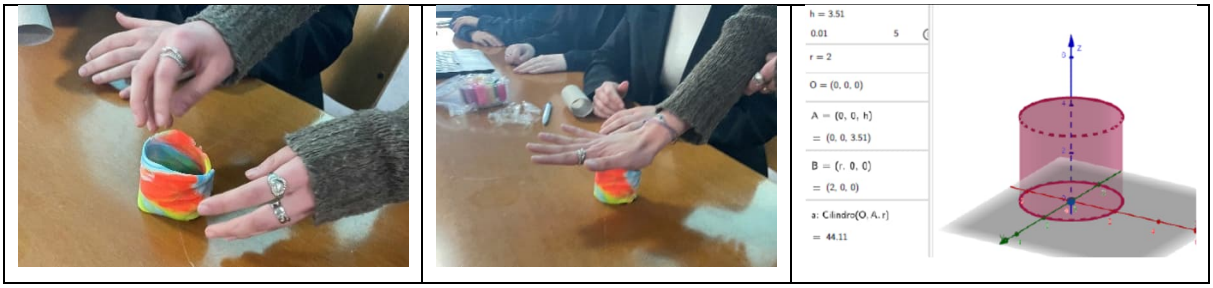
Students' initial responses revealed rich embodied reasoning. Mia wrote: "Deforming a cylinder by moving in the direction of its axis without tears (squashing it), we obtain a circle, while deforming a filled ball moving radially, we obtain a point." The language "squashing", and "moving" directly invokes bodily action, thinking through imagined physical manipulation (embodied *mN*). Ben made this explicit: "In this phase, we base ourselves on perceptual-motor intuitions. Physically manipulating models (clay, paper) helps visualise changes. Body movement and gestures become a bridge between visual intuition and topological understanding". This reveals both embodied reasoning (*mN*) and meta-awareness of it (*metaN*), recognising bodily engagement's epistemological function. Students' sense-making also captured essential topological ideas informally (*mN*): noting that transformations preserved connectedness ("not creating disjoint pieces") and maintained certain structures while eliminating others ("reducing thickness, making volume collapse"). When Mia hesitated about deforming a cone: "I obtain a disc ... it depends on direction and continuity". This revealed genuine mathematical thinking (*mN*), anticipating distinctions that would become formally significant. Importantly, students already exercised mathematical agency (*mA*) in this phase by choosing which features to emphasise in their descriptions. Leo focused on dimensional reduction ("3D objects become 2D or 1D"), while Mia emphasised preservation properties ("the boundary remains connected"). Students also demonstrated nascent pedagogical noticing (*pN*) even as learners. "The physical manipulation really helps seeing how this would work with students who struggle with abstraction" (Ben).

Formalising embodied intuition: the rigorous phase.

Translating embodied intuitions into mathematics, students moved from "squashing vertically" to defining specific retraction maps. Leo worked through carefully: "When 'squashing the cylinder vertically' (Figure 1), we apply a function compressing the *z*-coordinate. The retraction could be $(\cos 2\pi t, \sin 2\pi t, t') \rightarrow (\cos 2\pi t, \sin 2\pi t, 0)$. This takes each cylinder point and projects it on the *xy*-plane, fixing *z* at 0". The formal mathematics preserved geometric intuition: students didn't abandon embodied understanding but found mathematical structures capturing it (*mN*). Even verifying formal requirements, they maintained geometric language: "The cylinder image is a circle in plane z=0... Pre-images of open arcs are 'open strips' on the cylinder. Therefore, the map is continuous" (*mN*). This phase surfaced important distinctions requiring epistemic agency (*eA*). Ben noted "Even being continuous, they do not

always preserve properties! In the cylinder, yes. But…Möbius". This reveals epistemic agency (*eA*): questioning whether deformations guarantee formal properties (orientability). Students made this distinction visible rather than conflating deformations with homeomorphisms. Students also exercised pedagogical agency (*pA*) in how they would sequence their designing process. They choose to present geometric intuition first, then formal definitions, then verification, just the sequence they were experiencing, a pedagogical decision in ordering for understanding. Mia demonstrated meta-noticing (*metaN*) about the formalisation process: "Writing the formal proof forced me to articulate exactly what I was imagining. I realised my intuition was sometimes vague; the formalism made it precise". This awareness that formalisation clarifies intuition represents a significant epistemological understanding (*metaN*).

Figure 1: Exploring the cylinder deformations from the plasticine to GeoGebra.



Post-rigorous validation: returning to intuition.

Armed with formal definitions, students assessed initial intuitions. About the Möbius strip: "Yes, intuitively it can retract into the circle" (Mia). But 'intuitively' now meant something different than in the pre-rigorous phase. Ben clarified (*metaN*): "Now 'intuitive' means I can envision the formal construction, not just that it looks right. My intuition has been educated by the rigorous work". This distinction between 'naïve' and 'educated' intuition represents powerful meta-noticing (*metaN*), the awareness of how one's own thinking has been transformed by formal work. Leo refined his thinking beyond the original task, generating new examples (*mN*, *mA*). He noted about the filled torus: "It can retract to its central circle by collapsing each circular cross-section radially". This shows how the formal framework enabled generating new examples, enacting post-rigorous thinking where intuition and formalism work in concert. Students exercised mathematical agency (*mA*) in deciding which validation examples to pursue in depth. The group chose to focus on distinguishing subtle cases (Möbius strip vs. cylinder) rather than obvious ones, reflecting judgement about mathematical interest and difficulty. Ben confirms his epistemic agency (*eA*) by recalling the importance of questioning assumptions: "I thought any continuous deformation would preserve topological properties, but a retraction does not preserve orientability". This distinction matters mathematically. Making subtle distinctions visible rather than accepting surface-level understanding reflects epistemic agency (*eA*).

Noticing teaching potentials: from learner to designer.

When designing teaching resources, students shifted from noticing only mathematical structures to seeing teaching potentials in them. The group's deformations resource reveals *pN* through design choices: beginning with embodied, manipulable examples (recognising physical engagement creates accessible entry); including multiple representations (physical, digital, formal, interactive); and balancing abstraction with concreteness. Ben reflected, "I wanted to balance abstraction and concreteness. For instance, how much formalism to include in a visualisation, or intuition in a proof? I solved these through iterative creation, feedback, and revision". This represents strong *pN*, recognising different learners need different pathways. Mia articulated emerging *metaN*: "The process of selecting content, asking questions, and constructing structured explanations strengthened my understanding and improved my ability to communicate mathematical concepts. When you explain to others, you realise if a concept has been internalised". Mia noticed that teaching reveals understanding, a meta-pedagogical insight (*metaN*). Leo remarked on multiple representations' value: "Creating multiple representations showed me different 'faces' of the same object. Written equations, videos, and interactive GeoGebra offer complementary perspectives" (*pN*, *metaN*). Ben also noticed potential difficulties (*pN*): "Translating formal rigour into intuitive visualisation was challenging! Visually showing without losing mathematical accuracy proved difficult", an awareness that accessibility and rigour can be in tension (*pN*).

Exercising agency: mathematical and pedagogical choice-making.

The open-ended task created space for genuine agency. The deformations group exercised mathematical agency (*mA*) in selecting which theorems to prove versus illustrate. They proved the cylinder retraction rigorously but used visualisation for the torus retraction, reflecting judgements about pedagogical payoff versus mathematical depth. More significantly, they chose to emphasise the distinction between homeomorphism and deformation retraction, a subtle conceptual difference. Leo explained, "These are independent concepts but often confused. We wanted to make this distinction explicit because it's mathematically important but pedagogically tricky". This reflects both *mA* and *eA*. The choice was consequential; it shaped the entire resource structure. The group also exercised *pA* in choosing representational tools. Their decision to use GeoGebra for dynamic visualisation, Genially for interactive comparisons, and Kahoot for formative assessment reflected deliberate thinking about tool affordances. Ben explained (*pA*): "We created a GeoGebra book for formal aspects and exploration. We used Genially to explain through everyday comparisons, showing mathematical concepts are part of life, making them feel accessible. And we added quizzes to verify understanding while keeping engagement high". These choices reveal awareness that learning involves multiple processes (understanding, connecting, practising, assessing) and different tools support different processes (*pA*). This represents significant *pA*: not just choosing tools but understanding why particular tools serve pedagogical purposes.

Identity Transformation

Students experienced identity shifts. Mia: “This strengthened my interest in teaching as a career. I was satisfied with my work and enjoyed it!”. Leo: “My mathematical confidence grew enormously through solving visualisation problems and receiving feedback”. Ben: “Taking on the teacher role, I balanced theoretical with practical activities. Seeing the audience understand my explanations was extremely gratifying”. They recognised teaching deepened mathematical understanding: “When you explain to others, you really realise if a concept has been internalised” (Mia). “Translating formal rigour into intuitive visualisation forced me to think deeply about what these concepts really mean” (Leo). The recognition that teaching deepens learning represents epistemological awareness. The connection between agency and identity emerged clearly. Students who made choices experienced themselves differently than students who followed procedures. Ben articulated, “This wasn’t like assignments where we follow steps. We decided everything: what to teach, how to teach it, and what tools to use. Those decisions made it feel real, like actual teaching”. This agentic experience, making decisions that shaped others’ learning, proved identity-shaping. Students became people who make pedagogical choices, not just execute predetermined plans.

DISCUSSION AND CONCLUSION

Our findings reveal how embodied approaches to advanced mathematics create pathways toward pedagogical awareness. When students engage topological concepts through physical manipulation, digital exploration, and formal reasoning in concert, they don’t simply learn topology; they learn about learning topology. This meta-level learning differs fundamentally from traditional approaches where mathematics and pedagogy occupy separate courses. Our findings extend Palatnik et al.’s (2023) work by showing that students who experience embodied approaches as learners develop awareness of embodiment’s pedagogical value, making visible aspects of mathematical learning that remain invisible in purely formal instruction. Tao’s (2007) three-phase progression serves double duty: creating robust mathematical understanding while providing first-hand experience of learning progressions applicable at all educational levels. Students don’t merely hear about building on intuition before formalising; they live it, experiencing how physical engagement enables rather than opposes abstraction. Our analysis reveals specific dynamics: mathematical noticing (*mN*) in embodied contexts becomes pedagogical noticing (*pN*) when students exercise pedagogical agency (*pA*) in design tasks, mediated by meta-noticing (*metaN*). Agency serves as the activating condition for transforming implicit pedagogical insights into explicit pedagogical reasoning. Coles and Helliwell’s (2025) framework helps explain how students’ noticing evolved from embodied to layered (moving between details, patterns, and purposes) to mutual (shaped by audience responses). Having genuine choices, students experienced themselves as designers making consequential pedagogical decisions rather than following procedures. Students learnt that approaching abstract mathematics through embodied reasoning represents a powerful

pedagogical approach. This honours advanced mathematics while connecting it to teaching through how one approaches mathematical learning and supports others' learning, addressing Wasserman et al.'s (2023) ideas. When students sense and make sense of mathematical concepts through multi-modal pathways, they develop dual awareness: becoming both mathematicians who understand abstract structures and educators who understand how those structures might be made accessible to diverse learners. This dual development represents a promising direction for teacher training. Among the limitations, generalisability and transferability issues (case, context, subject) warrant acknowledgement. Future research must address the gaps.

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