CHILDREN'S PATTERN INTERPRETATIONS AT THE PROCESS AND CONTENT LEVEL IN THE EXPLORATION OF STRUCTURAL PROPERTIES OF OPERATIONS

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Mathematical structures are crucial for learning mathematics. Patterns can provide access to these. But studies show that recognizing patterns is not enough to understand the underlying mathematical structures. This paper presents a design research study with 42 elementary school learners who explore the mathematical structures of constancy properties of operations (e.g., 16 + 29 = 15 + 30). Based on reconstructed interpretations, the results provide a more detailed differentiation of the connection between recognizing the pattern at the process level and knowledge of structures at the content level.

THEORETICAL BACKGROUND

Mathematical patterns and structures

Structures are one of the big ideas of mathematics (Kieran, 2018; Mason et al., 2009). They are, therefore, also of fundamental importance for the learning of mathematics, especially for structural thinking, which involves seeing and expressing structures and properties of numbers, operations, and expressions – a central dimension of early algebraic thinking (Kieran, 2022). While structures in mathematics education were not defined for a long time (Kieran, 2018), two different perspectives on structures have emerged (Venkat et al., 2019). Although (emergent) structures refer to local relationships that are represented in specific cases (Venkat et al., 2019), this paper focuses on *mathematical structures*, which, in line with Akinwunmi and Steinweg (2022) and Mason et al. (2009), are defined as *general mathematical properties and relations* that are not directly visible. Various studies show that learners who perceive mathematical structures develop a deeper conceptual understanding (Mason et al., 2009; Mulligan & Mitchelmore, 2009).

Abstract mathematical structures must be made accessible to learners in representations that embody these relationships (Schifter, 2018). *Patterns*, understood as *perceptible regularities*, can represent structures and at the same time assist in becoming aware of them (Akinwunmi & Steinweg, 2022). However, simply seeing a pattern is not enough to perceive structures as general properties (Mason et al., 2009), as studies show that some learners make generalizations with a focus on patterns and develop no deeper understanding (Akinwunmi & Steinweg, 2022). Only when learners perceive a relationship as an instance of a general property they can appreciate the

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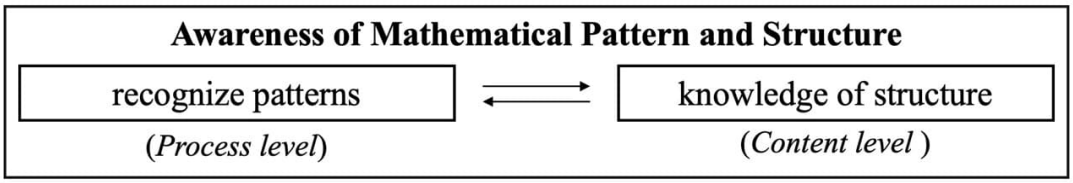
underlying structure (Mason et al., 2009). In order to *develop an understanding of the structures* behind the patterns, Mulligan and Mitchelmore (2009) assume an awareness of mathematical pattern and structure (AMPS) that has two interdependent components: "one cognitive (knowledge of structure) and one meta-cognitive (a tendency to seek and analyse patterns). Both are likely to be general features of how students perceive and react to their environment" (p. 38). In the associated interview study with 103 first-year students, Mulligan and Mitchelmore (2009) reconstructed five stages of structural development. Due to the generality of the construct of the AMPS for different contents, the description of the stages is very broad and no distinction between the two components, *recognition of patterns* at the *process level* and *knowledge of structures* at the *content level*, is made. While both components are assumed to co-exist, the connection between them has not yet been empirically investigated and differentiated, so that concrete research findings are still missing.

Figure 1: components of awareness of mathematical patterns and structures based on Mulligan & Mitchelmore (2009)



In order to support learners in understanding the mathematical structures behind the patterns, this present study therefore addressed the research gap by empirically investigating the connection between the two components for the content of structural properties of operations.

Structural properties of operations

One approach to understanding mathematical structures is to encourage learners to find calculation strategies that make it easier to solve tasks (Mason et al., 2009). Focusing on the behavior of operations in this way helps learners see operations not only as processes, but as objects with specific properties (Schifter, 2018). In accordance with Schifter (2018), this study understands the *structural properties of operations* to include not only commutativity, associativity, distributivity, the properties of the additive and multiplicative identities, and the understanding that addition and subtraction (as well as multiplication and division) are inverses of each other, but also as structures that can be derived from these laws. Even though learners often already use these properties implicitly in calculation strategies, not all learners understand them or have thought about when and why the strategies work, leading to many learners transferring the structures from one arithmetic operation to another (Schifter, 2018).

Currently, there are very few empirical studies on the so-called *constancy properties of operations* (in an addition expression, if one addend decreases by an amount and the

other increases by the same amount, the sum remains the same/ in a subtraction expression, if the minuend and the subtrahend decrease (or both increase) by the same amount, the difference remains the same) (Wilkie & Hopkins, 2024). Schifter's (2018) research on constancy properties showed that even after an intervention (lesson sequences exploring different sets of generalizations about operations), not all learners relate their findings to the operations. In their study, Wilkie and Hopkins (2024) showed that comparing the heights of vertical towers of blocks proved helpful for understanding the constancy of difference, but just over half of the learners developed an awareness of the generalization, and some learners had conceptual difficulties recognizing the directional aspect of the change. Both studies indicate that further research is needed on how learners can develop an understanding of the mathematical structures of constancy properties.

PRESENT STUDY AND RESEARCH QUESTIONS

In accordance with the research gap described above, a topic-specific design research study (Prediger et al., 2015) was initiated. This study aimed, on the one hand, at developing a teaching-learning arrangement to explore the mathematical structures of constancy properties and, on the other hand, at investigating the interpretations of learners in the pattern tasks developed. In order to address the importance of recognizing patterns at the process level and knowledge of mathematical relationships and structures at the content level, as well as the connections between the two levels, the study focuses on the following three research questions:

RQ 1: Which learners' interpretations can be reconstructed when recognizing patterns? (*Process level*)

RQ 2: Which learners' interpretations can be reconstructed with regard to mathematical relationships and structures? (*Content level*)

RQ 3: How are the interpretations at the process and content level connected, and how do the learners' interpretations relate to the mathematical structures of the constancy properties?

METHODOLOGY

Data collection and tasks

The newly developed teaching-learning arrangement on constancy properties in addition and subtraction expressions, consisting of five units (45 minutes each), was implemented in three iterative cycles in the form of design experiments. A total of 42 second and third graders (7-9-year-old) took part. The sessions were videotaped and transcribed. In the series of lessons, the learners first explored patterns in various sequences of related expressions and formulated rules that address for which changes in addition and subtraction tasks the results remain constant. The paper focuses on the task “verify patterns”, in which learners are asked to verify various sequences of related expressions (Fig. 2) for the presence of constancy properties (rule applies/rule does not

apply) based on the patterns. For this purpose, learners were given various examples and counterexamples of the constancy properties of addition and subtraction, which they are asked to sort without calculating and justify their classification (Tilke, 2026).

Figure 2: sequences of related expressions for the task “verify patterns”

199 + 743	50 + 37	453 + 287	53 + 28	17 + 59
200 + 742	52 + 35	454 + 285	54 + 26	18 + 60
201 + 741	54 + 33	455 + 283	55 + 24	19 + 61
202 + 740	56 + 31	456 + 281	56 + 22	20 + 62

Data analysis

A qualitative data analysis of 26 cases was conducted using a newly developed *analytical tool for reconstructing interpretations of mathematical patterns* (Tilke, 2026) in order to gain insights into learners' interpretations at the content and process levels. For this purpose, the *epistemologically oriented analysis of mathematical interactions* according to Steinbring (2005) was combined with the *category model of visual structuring ability* according to Söbbeke (2005). The combination of the two instruments made it possible, on the one hand, to focus on the type of relationship between the sign to be interpreted and the learners' reference context in order to explain how learners organize representations of mathematical patterns and which patterns they recognize in them (process level, RQ 1). On the other hand, the interaction between the sign and the reference context can be used to reconstruct which mathematical relationships and structures learners perceive in the representations (content level, RQ 2). The relationships are represented in a triangle (Figs. 3 & 5), which also makes it possible to examine the connections between the two levels (RQ 3).

RESULTS

Interpretations at the process level and at the content level

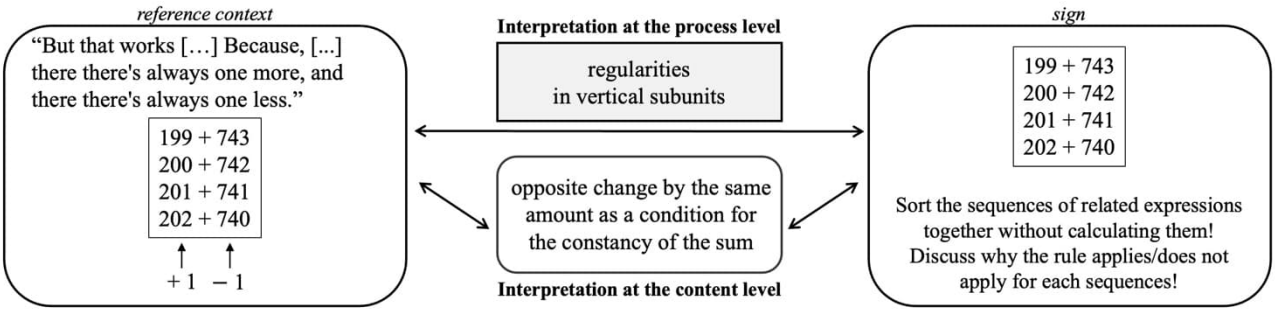
In the following, two cases, Mara and Leni, are used to illustrate selected interpretations at the process level and at the content level, before the connections between the two levels are explained using the results of the qualitative analyses of all learners. These two cases are examples of two central, different interpretations at the process level and at the content level in this activity.

In the task to verify various patterns for the mathematical structures of constancy properties, Mara selects the sequence of related expressions 199 + 743 (Fig. 3, sign) and explains:

Mara: But that works (*moves the sequences of related expressions 199 + 743 to ‘rule applies’*). […] Because, [...] there (*points to the first addends*) there's always one more, and there (*points to the second addends*) there's always one less.

In this interpretation, Mara, like other children in the study, groups the addends into vertical subunits ("there (points to the first addends) … and there (points to the second addends)"), establishes relationships between the elements of a subunit, and describes growing and decreasing patterns ("always one more/less"). This interpretation at the *process level* is referred to as *regularities in vertical subunits*. At the *content level*, Mara justifies the constancy of the sum ('rule applies') by means of an opposite change by the same amount. This interpretation is therefore referred to as an *opposite (same direction) change by the same amount as a condition for the constancy of the sum (constancy of the difference)*. Using the developed analysis instrument, Mara's interpretation can be illustrated as follows (Fig. 3).

Figure 3: Mara's interpretation of the sequence 199 + 743



In the second case, Leni categorizes the sequence of related expressions 453 + 287 (Fig. 2) as fulfilling the constancy properties, but her interpretation differs from Mara's:

Leni: That (*points to the sequence of related expressions 453 + 287*) is also big small (*moves the sequence of related expressions to 'rule applies'*).

Leni, similar to Mara, combines the two addends into vertical subunits and relates the elements of these to each other. Leni describes the direction of change ("big small") and thus recognizes partial aspects of the pattern. Like other children who use this interpretation, she does not explicitly state the amount of change. This interpretation at the *process level* is referred to as *partial aspects of regularities in vertical subunits*. Based on her explanation and the classification of the sequence with a change in the opposite direction by different amounts as fulfilling the constancy properties (incorrect classification), Leni interprets the *opposite direction of change as a condition for the constancy of the sum* at the *content level*.

Learners with this interpretation only focus on the same direction of change for subtraction. Furthermore, the same interpretation could be reconstructed at the *process level* among learners who focused exclusively on the *amount of change*. At the *content level*, another interpretation could be reconstructed, namely the *same amount of change as a condition for the constancy properties*.

Connections between the interpretations at the process and content level

Through analyses such as those described above, the following insights into the connections between the interpretations at both, the process and the content levels were gained: In this study, some learners, like Mara, describe the (complete) *regularities in vertical subunits* at the process level and, at the content level, interpret the *opposite (same direction) change by the same amount as a condition for the constancy properties*. These learners consider all relevant properties of the constancy of the sum or difference. Another group of learners, like Leni, interpret *aspects of regularity in vertical subunits* at the process level. Depending on which aspect (direction or amount) the learners focus on, they use the *opposite (same direction)* or *the same amount of change as a condition for the constancy properties* at the content level. With regard to the recognition of mathematical structures, these learners must be encouraged to perceive the complete regularities – including the recognition of the other partial aspect – in the vertical subunits, as otherwise they will also classify counterexamples, e.g., in addition sequences of related expressions with a change in the opposite direction by different amounts, as depicting the constancy properties. Additionally, the study reveals a new, particularly important connection between process and content level. This particular group of learners also interprets the complete *regularities in the vertical subunits* at the process level and, at the content level, identifies the *opposite (same direction)* or *the same amount of change as a condition for the constancy properties* (Tilke, 2026; Fig. 4).

Figure 4: Connections between the interpretations at the process and content level

Content level / Process level	Opposite (same direction) change by the same amount as a condition ...	Opposite (same direction) change as a condition ...	Same amount of change as a condition ...
Regularities in vertical subunits	X *(z. B. Mara)*	X *(z. B. Max)*	X
Partial aspects of regularities in vertical subunits		X *(z. B. Leni)*	X

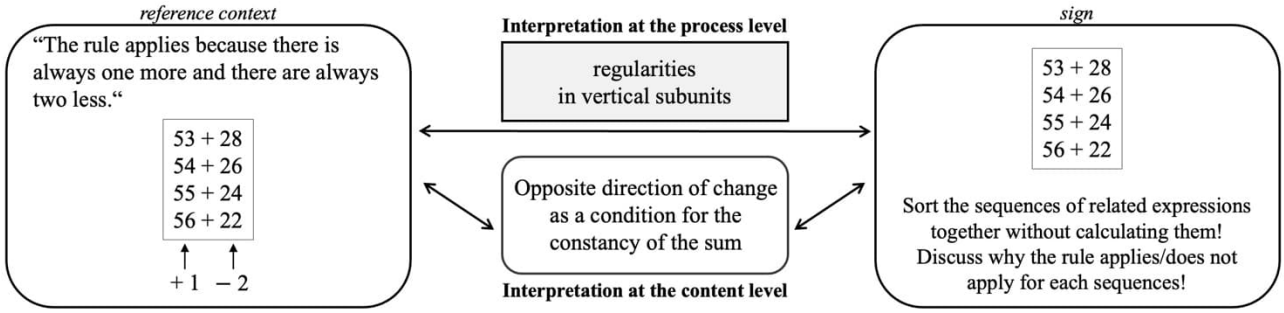
The specific nature of the last-mentioned connection will be explained based on Max's interpretation. When classifying the sequence of related expressions 53 + 28 (Fig. 5, sign), Max explains:

> Max: That (*moves the sequence of related expressions 53 + 28 to 'rule applies'*) is the most difficult. [...] The rule applies because there (*points to the first addends*) is always one more and there (*points to the second addends*) are always two less.

In this case, Max points out the complete *regularities in the vertical subunits* ("always one more/two less") at the *process level*. Due to the classification of the counterexample with a change in the opposite direction by different amounts as fulfilling the constancy properties (incorrect classification), it becomes obvious at the

content level that he only considers the *opposite direction of the change as a condition for the constancy of the sum* and does not take the different amounts into account (Fig. 5).

Figure 5: Max's interpretation of the sequence 53 + 28



At this point, the data clearly shows that simply recognizing the pattern is not enough; knowledge of the mathematical structure is also essential. Learners like Max therefore need support at the content level in order to develop a meaningful understanding of the constancy properties (Tilke, 2026).

DISCUSSION AND CONCLUSION

The study was able to reconstruct different interpretations at the process level and at the content level when exploring the constancy properties of addition and subtraction expressions and to demonstrate connections between the two levels. The results show that, on the one hand, it is essential to recognize the patterns completely, and on the other hand, knowledge of the mathematical structures is required in order to be able to draw conclusions regarding the mathematical structures based on the patterns. In regard to recognizing constancy properties, this means that learners must create vertical subunits at the process level and recognize in which direction and by what amount the addends (the minuend and the subtrahend) change. At the content level, learners must also know that the sum (the difference) only remains constant if the addends are changed in opposite directions (minuend and subtrahend in the same direction) by the same amount. Both levels are interdependent. The study thus further differentiates the connection between the two components assumed by Mulligan and Mitchelmore (2009) through empirical findings on constancy properties. Due to the topic-specific nature of the study, it is limited to first connections between the two levels, so that the transferability of the results to other mathematical structures (of operations) must be investigated in future studies.

With regard to the further development of mathematics teaching, the results indicate that learners need targeted support in exploring the mathematical structures of constancy properties at both the process and content levels. Furthermore, the study shows that considering counterexamples is particularly effective in identifying incomplete understanding of mathematical structures through incorrect classifications.

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