PROSPECTIVE TEACHERS' MEANINGS FOR FRACTION DIVISION: DEVELOPING AND APPLYING A MEASUREMENT INTERPRETATION

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This paper reports findings from an ongoing design-based research study aimed at enhancing primary teachers' conceptual understanding of fraction multiplication and division, focusing on the latter. An instructional sequence was designed to support the development of a measurement interpretation of division, transferable from natural to rational numbers. Data from 36 prospective teachers were collected through a pretest, a group task, and individual assignments involving interpretation of fraction division and task design. Analysis indicated that although many participants employed a measurement interpretation after the intervention, most did so inconsistently, often exhibiting misinterpretations of what constitutes a measurement situation. The results highlight specific challenges with division to be considered in teacher education.

INTRODUCTION AND THEORETICAL FRAMEWORK

Fraction division is widely recognized as one of the most demanding topics in fraction learning and teaching. Numerous studies over time have consistently indicated that pre- as well as in-service teachers have limited conceptual resources to assign meaning to this operation and its algorithms and to recognize the situations in which it applies (Copur-Gencturk, 2021; Ma, 1999; Olanoff et al., 2014; Tirosh, 2000). Such difficulties are likely to have adverse effects on students' learning (Tchoshanov et al., 2017).

Arithmetical operations derive their meaning either from the structure of the arithmetical system—through their relations to other operations—or from the situations they model (Greer, 1992; Nunes & Bryant, 2022). Typically, the transition from natural to rational numbers entails changes in both types of meaning. For example, within the set of natural numbers, division and multiplication can be conceptualized as distinct inverse operations. Following the introduction of reciprocals, however, division can be replaced by multiplication by the reciprocal of the divisor. This important change is reflected in the "invert and multiply" algorithm for fraction division, which teachers typically find challenging to explain (Copur-Gencturk, 2021). On the other hand, the meaning of division derived from the situation predominantly associated with this operation in the context of natural numbers (i.e., "fair sharing") does not generalize to rational numbers. Indeed, interpreting *D:d* as "*D* is partitioned into *d* equal parts" requires the divisor to be a natural number. This interpretation strongly influences how division is conceptualized and this is reflected

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in a range of well-documented difficulties, such as the expectation that "division always makes smaller". This expectation appears both in purely numerical contexts and in problem situations and has been observed among students as well as prospective and in-service teachers (Fischbein et al., 1985; Olanoff et al., 2014; Tirosh, 2000).

Given the profound changes in meaning involved in the shift from natural to rational numbers, developing conceptual understanding of fraction division entails conceptual change learning (Vosniadou et al., 2008). This process requires learners to be aware of their prior conceptualizations of this operation, to contextualize them when necessary, and to build the conceptual background needed to construct new meanings. This is evidently a complex and demanding process, typically time consuming, during which learners may misinterpret, and even distort new information to fit ideas that are more familiar to them; and exhibit inconsistencies in their thinking.

Building on this theoretical perspective, we have been conducting a topic-specific design-based research study (Gravemeijer & Prediger, 2019), aimed at developing resources for pre- and in-service teachers to support the development of conceptual understandings for multiplication and division. The key ideas underlying the design are: (a) raising teachers' awareness of the changes that occur in the shift from natural numbers to rational numbers with respect to the meanings of these operations, their interrelationship, and the implications thereof; and (b) supporting teachers in constructing productive meanings for these operations—i.e., meanings that are valid for both natural and rational numbers. In this paper we focus on division.

One meaning of division that is transferable from natural to rational numbers is measurement (or quotitive) division, according to which *D:d* means "how many times *d* fits into *D*." Fischbein et al. (1985) characterized measurement division as an "intuitive model," alongside partitive division; however, they noted that this model typically emerges in instructional contexts. Initially, this interpretation of division is associated with counting equal groups in a discrete quantity. Its primary association, however, lies in the measurement of continuous quantities, through which it can be extended to the domain of rational numbers. This model also requires adaptation when the divisor is larger than the dividend, in which case the question "how many times does *d* fit into *D*" must be reformulated as "what part of *d* fits into *D*".

Within the framework of our design-based research study, we organized the content pertaining to measurement division into a structured instructional sequence. First, partitive and measurement division were explicitly distinguished. Second, measurement division was introduced in the context of natural numbers, initially represented as discrete quantities (i.e., "how many groups of *d* in *D*?") and subsequently re-represented as continuous quantities, specifically lengths (i.e., "how many times does *d* fit into *D*?"). Third, fraction division was introduced by first considering unit-fraction divisors and then non-unit proper fraction divisors, while maintaining the same representational context of numbers as lengths. Finally, the

meaning of measurement division was extended to cases in which the divisor is larger than the dividend (i.e., "what part of *d* fits into *D*").

We implemented this sequence with a group of prospective teachers (hereafter, PTs) and examined whether they (a) assigned a measurement meaning to fraction division and (b) were able to design student tasks to support the construction of this meaning. We were particularly interested in whether, and how consistently, the PTs capitalized on the key ideas of the intervention in their task design, as well as whether misinterpretations or partial appropriations of these key ideas would emerge.

METHOD

Participants

The participants were 36 PTs, in the 7th semester of a 4-year study program at a Department of Primary Education in Greece. They participated in an elective course focusing on mathematics teaching and learning.

Procedure

The participants were pretested with a paper-and-pencil test targeting various aspects of their understanding of fraction multiplication and division. One week later, they participated in a three-hour class session, the first of a series of six. During this session, the PTs worked in small groups on a task (see below) designed to elicit their interpretations of fraction division. This was followed by a whole-class discussion. Subsequently, an interactive lecture was delivered by the last author, during which participants were actively engaged through questions, prompts to explore alternative examples, and opportunities to raise their own questions. The lecture materials, in the form of slides, were grounded in the instructional sequence described above and were made available to the PTs after the session via the course's digital platform. The PTs were given one individual assignment to be completed prior to the next class session.

Research tasks

The research tasks analyzed for this paper are the following:

a) One task from the pretest (hereafter, PreT); the PTs were asked to pose a problem pertaining to the operation 4:2/3.

b) Part of the group task that the PTs worked on during the course session (hereafter, GroupT). GroupT was presented as classroom scenario: A hypothetical student is asked to explain what 6:3 means to a younger student and responds with a fair-sharing problem (i.e., six candies shared equally by three children). Then he is asked the same for 6:1/3, to which he responds: "It's like before, I have six candies and…. Wait! This doesn't work; I can't have one third of a child. I can't do it". The PTs were asked to provide feedback in written, explaining what 6:1/3 means.

c) Two tasks from the individual assignment. In the first (T1), the PTs were asked to interpret 3:1/4 as measurement; use this interpretation to compute the quotient and

explain their reasoning. In the second (T2), the PTs were asked to design a series of at most three student tasks (T2.1, T2.2, T2.3), aiming at facilitating students to interpret 3:1/4 as measurement and use this interpretation to compute the quotient.

RESULTS

Two authors independently categorized PTs' responses as valid, partially valid, invalid, or blank, collaboratively identified meaningful subcategories, and allocated the responses to these subcategories, with a third author reviewing all categories; disagreements were resolved through discussion until consensus.

The first category (A) included valid responses to the tasks. It included three sub-categories (A1, A2, A3). A1 comprised problems of measurement division, either word problems (Example 1), or problems embedded in actual measurement situations (Example 2).

> **Example 1:** To make an orange cake, we need 3 litres of orange juice. We have a cup that holds 1/4 of a litre. How many cups will we need for the 3 litres of juice? (T2)

> **Example 2:** Three strings are provided. Students are divided into three groups, and each group is given one string. Each group is asked to make bracelets for Mother's Day, using 1/4 of the string for each bracelet so that it will not be too large for the wrist. How many bracelets will all three groups make in total? (T2)

A2 included verbal descriptions of the measurement meaning of division in numerical context (Example 3). Finally, A3 included problems of the type known part – unknown whole (Example 4).

> **Example 3:** We would tell him that he needs to find how many times 1/3 fits into 6 (GroupT).

> **Example 4:** A gift package contains 6 chocolates, and 2/3 of the package costs €4. How much does the whole package with the 6 chocolates cost?" (PreT)

Category B comprised partially valid responses, in the sense that a mathematically correct problem was posed that resulted in the intended outcome, albeit via the inverse operation. Category B had two sub-categories (B1, B2). B1 included equal-groups problems where the denominator of the divisor functioned as a multiplicative operand (Example 5). B2 included multiplication problems in which each unit of the dividend was represented by an area which was already partitioned, or children were immediately guided to partition it into four equal parts, with the aim of determining the total number of "pieces" (Example 6). We note in many B category responses, the PTs used "measurement language" like the PT in Example 5.

> **Example 5:** Giannis bought three boxes, each with four compartments, to store his toys. He wants to place one toy in each compartment. How many toys will he store in total? Each student will have one toy, will place it in each compartment in turn, and count how many times it fits in the boxes. (T2)

Example 6: The children are asked to cut a sheet of paper into three equal parts. Then, each of the three parts is further divided into four equal parts. As the children observe the resulting pieces, they visualize the quotient of 3 : 1/4. (T2)

Category C included invalid responses of various types, placed in three sub-categories. C1 included multiplication problems in which the divisor functioned as the multiplier (Example 7). C2 included responses which referred merely to procedural methods to calculate the result (e.g., the "invert and multiply algorithm"). C3 included unsolvable or non-sensical problems. Interestingly, in two of them, the divisor and the dividend had exchanged roles (Example 8).

Example 7: I have 6 pizzas and I want to eat 1/3 of the pizzas. How many pizzas will I eat? (GroupT)

Example 8: Four crackers cover two thirds of the surface of a small baking tray. How many square centimeters does one cracker cover? (PreT)

Finally, there was a category that included blank responses ("no answer" in the case of PreT; and "blank" in the case of T2, because 7 PTs and 19 PTs had provided only one and only two student tasks, respectively). Table 1 presents the frequency of response categories across tasks.

Table 1: Distribution of response categories by research task (Frequencies)

Category		Tasks						Total
		PreT	GroupT	T1	T2.1	T2.2	T2.3	
A	A1	2	2	21	13	11	7	56
	A2	0	4	0	0	0	0	4
	A3	2	0	0	0	0	0	2
B	B1	0	0	6	11	9	1	27
	B2	0	0	0	3	3	0	6
C	C1	6	2	0	0	0	0	8
	C2	0	5	9	9	1	0	24
	C3	10	3	0	0	5	2	20
No answer/Blank		16	0	0	0 [a]	7[a]	26[a]	49
	Total	36	16	36	36	36	36	196

[a] All PTs provided one student task, 29 provided a second one, 10 provided a third one.

Table 1 shows that at PreT, the dominant category was the "no answer" one, followed by C3 (unsolvable or non-sensical problems), while there is also a considerable number of C1 responses corresponding to problems that conflated division with multiplication

by the divisor. Only four valid responses were provided, corresponding to two word-problems of measurement division (A1), and two of the "known part-unknown whole" type (A3) that did not appear in subsequent tasks.

In GroupT, there was a slight increase in valid responses, because few groups evoked the "how many times 1/3 fits into 6" interpretation of the particular division as an explanation addressed to the hypothetical student. A2 did not appear in subsequent tasks. Five groups presented some procedural method as an explanation (C2), while C3 responses also appeared at group level.

In T1, there was a substantial increase in A1 responses, with relative frequency 0.58 in the total of 36 responses. We should note that in this task, PTs largely drew on the rationale and the representations ("numbers as lengths") of the intervention. There were still procedural responses (C2) and B1 responses—corresponding to equal-groups multiplication problems—appear for the first time.

In T2, most PTs provided less than three student tasks. We should note that the PTs did not use the representation context of the intervention (i.e., numbers as lengths); only one student task referred to length measurement (Example 2 above). Also, no PT introduced the measurement meaning of division in the context of natural numbers.

The relative frequency of A1 responses in the total of student tasks provided in T2.1, T.2.2, T2.3 was 0.36, 0.38 and 0.7, respectively, indicating that PTs who provided more student tasks were more prone to provide A1 responses. On the other hand, B1 responses rose in T2, and B2 responses appeared for the first time, while invalid responses C2 and C3 were still present in this task.

Overall, valid responses (Category A) were scarce before the intervention but increased substantially afterward, due to the rise in A1 responses employing a measurement interpretation of division. Invalid responses (Category C) were dominant prior to the intervention; together with "no responses" in the pretest (PreT), they accounted for the vast majority of answers. Following the intervention, Category C was represented mainly by procedure-oriented responses (C2). Responses in Category B appeared only after the intervention, indicating a conflation of multiplication problem situations with measurement division situations, given that evoking the measurement meaning of division was an explicit requirement in Tasks T1 and T2.

We next examined the individual PTs' response patterns in tasks T1 and T2, with respect to the number of A1 responses. Because the PTs provided different numbers of student tasks in T2, we stress that the number of A1 responses was considered relatively to the number of student tasks provided by the PT. For instance, a PT who provided 3 student tasks categorized as A1 was attributed to the same response pattern with a PT who provided 2 student tasks categorized as A1.

We identified 5 response patterns. The first (All A1) corresponds to PTs who consistently provided A1 responses in T1 and T2. The second (All A1 only in T2) corresponds to PTs that consistently gave A1 responses only to the student tasks they

provided. The third (A1 only in T1) corresponds to PTs that gave A1 responses only to T1. The fourth (No A1) corresponds to PTs who provided no A1 response to T1 or T2. Finally, the fifth response pattern corresponds to PTs who provided a mixture of A, B, and C responses. Table 2 presents the frequency of these response patterns. It shows that the dominant response pattern was A1 responses only to T1 task, that was the one closely related to the intervention. The next more frequent response pattern was "No A1". Only 8 PTs consistently provided A1 responses in T1 and T2.

Table 2. Frequency of response patterns across T1 and T2.

Response Pattern	All A1	All A1 only in T2	A1 only in T1	No A1	Other	Total
N	8	3	11	10	4	36

DISCUSSION - CONCLUSIONS

Fraction division is notoriously difficult, not only for students, but also for pre- and in-service teachers (Copur-Gencturk, 2021; Ma, 1999; Olanoff et al., 2014; Tirosh, 2000). We argued that developing conceptual understanding of fraction division requires conceptual change learning (Vosniadou et al., 2008) that is challenging, time consuming and characterized by inconsistencies that reflect partial understanding or distortion of information coming from instruction.

We designed an instructional sequence aimed at supporting PTs to grasp the measurement meaning of division, first within the context of natural numbers, and then extending it to fractions. We presented findings before and after a group of PTs' first encounter with the notion of measurement division in the context of the intervention. Our findings indicate that, prior to the intervention, very few PTs employed appropriate meanings for fraction division. The majority exhibited difficulties in posing problems pertaining to fraction division, and errors such as conflating fraction division with multiplication by the divisor. Collaborative attempts to explain what fraction division means elicited only few more sophisticated responses.

After the intervention session, most PTs did employ the measurement interpretation of division, albeit inconsistently. Moreover, a new category of responses emerged, where natural number multiplication problems (by the denominator of the divisor) were presented as measurement division situations. This could reflect a belief that a problem situation is characterized by the outcome, or a distortion of a key element of the intervention, namely the introduction of measurement division in the familiar context of natural numbers. Key elements of the intervention were identified only when the PTs had to explain fraction division themselves, but not in student tasks. Finally, the use of procedures to explain the meaning of division persisted after the intervention.

Overall, the findings of this study highlight the difficulties PTs face with fraction division, indicate that a first, short intervention caused phenomena associated with the

challenges of conceptual change learning, and point to the need for a long-term, systematic approach to support PTs in teaching division.

The present study was conducted within the framework of the research project titled: "Start early, take a different perspective": developing a learning trajectory to tackle the problems of conceptual change in the transition from natural to rational numbers", which is implemented under the H.F.R.I. (ELIDEK) action: "3rd Call for H.F.R.I. Research Projects for Faculty Members and Researchers" (H.F.R.I. Project No. 26442).

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