BASIC ARITHMETIC SKILLS OF ELEMENTARY SCHOOL TEACHER STUDENTS IN DIFFERENT NUMBER RANGES

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Arithmetical skills are essential for informed participation in society and are cumulatively developed through mathematics education up to higher education. This study examines the relationship between elementary school teacher students' skills with natural numbers, integers and rational numbers. Data from 1,711 elementary school teacher students were analyzed using item response theory and structural equation modelling. Results revealed high skills with natural numbers and rational numbers, but notably lower performance with integers. Structural equation modelling showed exceptionally strong relationships between natural number and integer skills, with both direct and indirect pathways from natural numbers to rational numbers, indicating partial mediation.

INTRODUCTION

The acquisition of mathematical skills is not viewed as a series of isolated skills, but as a cumulative process of conceptual restructuring and expansion. In arithmetic, the focus is particularly on developing an understanding of numbers and operations across the number ranges, which is also reflected in the curricular design of arithmetic content (In Germany, this is generally organized in the order of natural numbers ($\mathbb{N}_0$), followed by integers ($\mathbb{Z}$), and rational numbers ($\mathbb{Q}$), KMK, 2022a, 2022b). In this context, the ‘mental number line’ is a dynamic structure due to curricular arrangements generally starts its development with small natural numbers, then expands rightward to larger natural numbers, leftward to negative integers, and "interstitially" to include fractions and decimals (Siegler & Lortie-Forgues, 2014). A fundamental question in mathematics education research concerns the extent to which skills acquired in working with natural numbers predict and support skills in subsequent number ranges.

Of course, the goal of mathematics education is for learners to have an integrated and differentiated understanding of numbers and operations in various number ranges by the end of their schooling. Elementary school teacher students are once again completing a course of study that activates their own skills in dealing with numbers and arithmetic operations, enabling them to use the teaching methods they have learned to prepare and support children as they transition to new number ranges. Prospective teachers should have at least basic arithmetic skills in the $\mathbb{N}_0$ which is the prime focus in elementary school teaching, but also basic arithmetic skills in the number ranges that follow on from elementary school ($\mathbb{Z}$ and $\mathbb{Q}$). In this study, we explore the question of

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what basic arithmetic skills prospective elementary school teachers have at the beginning of their studies and how these skills are interrelated.

THEORETICAL BACKGROUND

Empirical evidence supports this cumulative development. Early natural number arithmetic skills form a foundational core that longitudinally predicts later skills with rational numbers, algebra, and overall mathematics achievement (Bailey et al., 2014). In the primary and lower secondary school curriculum, mathematics education in the field of number and operations generally starts with $\mathbb{N}_0$ before expanding the number range to include $\mathbb{Z}$ and $\mathbb{Q}$.

The transition from $\mathbb{N}_0$ to $\mathbb{Z}$ presents distinct challenges as negative numbers violate assumptions students form with $\mathbb{N}_0$ – that zero means nothing or that numbers always increase with larger numerals. It shows that children initially treat the minus sign purely as subtraction rather than recognizing its unary function (Bofferding, 2014). Vlassis (2004) argues that achieving flexibility in negativity—recognizing the minus sign's multiple meanings – requires robust natural number understanding as a foundation.

The transition to $\mathbb{Q}$ brings further advantages and challenges. While skills with $\mathbb{N}_0$ support the learning of $\mathbb{Q}$ in many ways (Bailey et al., 2014), it can also lead to systematic errors known as the natural number bias (e.g., thinking $\frac{1}{5} > \frac{1}{3}$ because $5 > 3$; Ni & Zhou, 2005). Overcoming this bias requires conceptual change, a radical restructuring of previous understanding (Vamvakoussi & Vosniadou, 2004). Overall, transfer across number ranges is not uniform: properties of $\mathbb{N}_0$such as discreteness and successor-based reasoning can interfere with learning $\mathbb{Z}$ and $\mathbb{Q}$, giving rise to systematic misconceptions and a natural number bias that constrains understanding of density and relative magnitude (Kainulainen et al., 2017; Vamvakoussi & Vosniadou, 2004). Research shows that even experts and adults still show traces of the natural number bias in terms of slower reaction times when tasks are incongruent with natural number properties (e.g., comparing $\frac{11}{15}$ and $\frac{2}{3}$), suggesting that the initial $\mathbb{N}$ schema is never fully deleted but must be actively inhibited (Vamvakoussi et al., 2012).

Development in these number ranges typically involves gradual conceptual change through intermediate conceptions rather than abrupt restructuring (Kainulainen et al., 2017). Nevertheless, rational number arithmetic poses additional challenges, contributing to persistent difficulties that can extend into adulthood and even affect prospective teachers (Vamvakoussi et al., 2012). Hence, it is of particular interest to examine the arithmetic skills of prospective elementary school mathematics teachers and how their skills in the different number ranges are interconnected.

For prospective elementary school teachers, it is additionally relevant to understand the developmental relationships of number skills in different number ranges. When entering university studies, these students should have successfully navigated number

range transitions that elementary students find challenging. International comparative studies suggest that German primary school mathematics teachers tend to have average mathematical skills (as proposed by the TEDS-M study; Tatto et al., 2012). This raises the question of the extent to which prospective primary mathematics teachers have basic arithmetic skills in dealing with $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$ (as the number ranges relevant to primary education or those directly following them), and how these skills are interconnected across the three number ranges and to what extent the misconceptions and difficulties mentioned above also exist in this cohort.

We conceptualize basic arithmetic skills through three facets (1- number sense, 2- operations, 3- skillful operations) across each number range, informed by curricular frameworks and research (KMK, 2022b; a detailed description of the conceptualization is given in Holland et al., 2025). *Number sense* encompasses understanding numbers and their structural properties—recognizing magnitudes, ordinality, and relationships within and between number systems. *Operations* refers to skills in executing basic arithmetic procedures and representing them mentally, semi-written, and in written forms. The facet *skillful operations* captures coordinated, flexible use of arithmetic in complex contexts, including expressions, sequences, and application of arithmetic laws. This framework structures our investigation of prospective teachers' arithmetic skills across $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$, allowing examination of both: specific skills in one of the number ranges and cross-range relationships.

In this contribution, we therefore want to address the following three research questions:

(1) What basic arithmetic skills do prospective elementary school mathematics teachers have at the beginning of their studies in the fields of $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{Q}$?

(2) How are the skills in dealing with $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$ interrelated among prospective mathematics teachers at the beginning of their studies?

(3) What are the direct and indirect effects of pre-service mathematics teachers' skills with $\mathbb{N}_0$ and $\mathbb{Z}$ on their skills with $\mathbb{Q}$?

METHODS

A test was developed to assess the basic arithmetic skills of prospective elementary school mathematics teachers. The content and structure of the test were based on the conceptualization presented in Table 1. A total of 228 items were developed, which are distributed across the conceptualized areas as shown in Table 1.

The assessment was implemented using the online survey platform SoSci. From the full item pool, each participant was presented with an individually randomized subset of 60 items. The sampling process was structured to ensure that the distribution of items across the domains of $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$, as well as across the facets of number sense, operations, and skilful operations, reflected the proportional representation in the

overall item pool. The testing time was limited to 60 minutes. Participants were not allowed to use any aids during the assessment, except for pen and paper.

Table 1: Number of items in the different test facets

	$\mathbb{N}_0$	$\mathbb{Z}$	$\mathbb{Q}$	Total
Number sense	18	12	42	72
Operations	24	18	30	72
Skilful Operations	24	24	36	84
Total	66	54	108	228

Data collection occurred at the beginning of each semester (winter 2024/25, summer 2025, and winter 2025/26) at two different universities from two federal states in Germany. In total, N = 1,711 elementary school teacher students participated in the study. The sample consisted of 1,497 women (87.5%) and 205 men (12%). Four participants reported their gender as diverse and five participants did not provide any information about their gender. The average age of the participants was 21.53 years (SD = 3.16).

All answers were scored dichotomously as correct (1) or incorrect (0). If items were skipped, this was considered a lack of skill and scored as incorrect (0). If items were not reached or were not available for answering due to random selection, this was treated as a missing response. To answer research question 1, the average percentage score achieved is reported for each recorded numerical range. To answer research questions 2 and 3, a three-dimensional ($\mathbb{N}_0, \mathbb{Z}, \mathbb{Q}$) Item Response Theory model was applied to scale the responses using the software Conquest. Difficulty parameters were estimated for each item based on its proportion of correct responses, while an ability parameter was assigned to each participant according to their performance on the administered items. To evaluate the quality of the scaling, several scale indices were reported, including the EAP reliability (analogous to Cronbach's alpha), as well as detailed item statistics (see Table 2). The item statistics for the test scaling across the full sample were considered satisfactory. The weighted mean square fit values fell within the recommended range (0.80 – 1.20; see Bond & Fox, 2015), and the average item discrimination was above 0.30. The scales' reliability (i.e., 0.73, 0.75 and 0.82) could likewise be regarded as satisfactory.

The person ability estimates revealed varying distributions across the three dimensions. Dimension 1 ($\mathbb{N}_0$) showed the narrowest range of person abilities, spanning from -2.23 to +1.80 logits ($SD \approx 0.77$). Dimension 2 ($\mathbb{Z}$) demonstrated a moderate distribution ranging from -2.44 to +2.00 logits ($SD \approx 0.85$). Dimension 3 ($\mathbb{Q}$) exhibited the widest distribution of person abilities, ranging from - 2.93 to +2.38 logits ($SD \approx 0.97$).

Table 2: Statistical parameters from the scaling analyses of the test assessing basic arithmetic skills

Dim.	EAP reliability	WMNSQ Min	WMNSQ Max	r_{it} Mean
$\mathbb{N}_0$	0.73	0.91	1.07	0.30
$\mathbb{Z}$	0.75	0.92	1.08	0.33
$\mathbb{Q}$	0.82	0.86	1.17	0.37

In order to analyze the skills of prospective mathematics teachers in the three focused number ranges, the EAP values for each person and each of the three number ranges were taken from the scaling and linearly transformed so that the mean value was 50 and the standard deviation was 10 in each case. To answer research question 2, the manifest and latent correlations between the personal ability scores of the three dimensions are considered, and finally, to answer the third research question, a mediation model is analyzed using MPlus software, which includes the personal ability scores of the students from the three number ranges.

RESULTS

The descriptive findings on the percentage of total points achieved by prospective mathematics teachers in the three number ranges (RQ1) are shown in Table 3. It appears that students at the beginning of their studies are most confident in the area of $\mathbb{N}_0$, as they achieve the highest percentage of total points in this area (81.3% items answered correctly). Interestingly, students achieve higher average scores in the test with $\mathbb{Q}$ (77.7% correct) than in the test with $\mathbb{Z}$ (68.1%).

The manifest and latent correlations between the students' personality trait scores (RQ2) are reported in Table 4. The error-corrected correlations are relatively high (0.82–0.88). As expected, the correlations of the four test dimensions at the manifest level are lower (0.54–0.74), but still within an acceptable range.

Table 3: Mean Percentage Scores Across the Three Test Sections

	N	Min	Max	Mean	SD
$\mathbb{N}_0$	1,711	18.2	100	81.3	12.4
$\mathbb{Z}$	1,711	12.5	100	68.1	16.6
$\mathbb{Q}$	1,711	6.7	100	77.7	13.1

The three-dimensional Rasch scaling revealed varying levels of variance across dimensions. Dimension 1 ($\mathbb{N}_0$) showed the lowest variance (0.59), while Dimension 2 ($\mathbb{Z}$) demonstrated moderate variance (0.72), and Dimension 3 ($\mathbb{Q}$) exhibited the highest

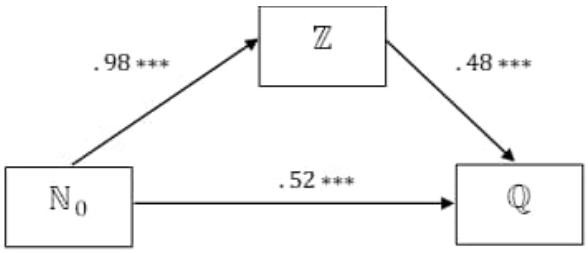
variance (0.93). All variances remained below 1.0, suggesting a relatively homogeneous sample distribution across all three dimensions.

Table 4: Latent & manifest correlation of students' skills in the three number ranges

	$\mathbb{N}_0$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{N}_0$		0.54	0.65
$\mathbb{Z}$	0.82		0.72
$\mathbb{Q}$	0.87	0.88	

Finally, Figure 1 shows the results of the mediation analysis, controlled for duration of studies and high school graduation grade. The path analysis revealed significant relationships among the three skills in the different number ranges. Skills in operating in $\mathbb{N}_0$ showed a very strong positive effect on skills in $\mathbb{Z}$ ($\beta = .98$, $p < .001$). In turn, skills in $\mathbb{Z}$ had a significant positive effect on skills in $\mathbb{Q}$ ($\beta = .48$, $p < .001$). In addition, skills in $\mathbb{N}_0$ exerted a direct effect on skills in $\mathbb{Q}$ ($\beta = .52$, $p < .001$). Thus, both the direct path from $\mathbb{N}_0$ to rational numbers and the indirect path via $\mathbb{Z}$ were statistically significant, indicating a partial mediation effect.

Figure 1: Mediation model between skills in number ranges



The variance explanation of the dependent variables is $R^2 = 96.1\%$ for integer skills and $R^2 = 98.2\%$ for rational number skills.

DISCUSSION

This study examined the structure of 1,711 prospective elementary school teachers' basic arithmetic skills in different number ranges resulting in three key findings: First, students' skills in the different number ranges – strongest with natural numbers and rational numbers but weaker with integers – is surprising given integers' simpler structure. This finding raises concerns for teacher preparation since elementary school teachers must prepare students for this challenging number range extension. Second, there is high latent correlations between students' skills in the three number ranges indicating that students' understanding of numbers and operations in the three numerical areas is closely linked. In a path model, we even find an extremely high regression coefficient between $\mathbb{N}_0$ and $\mathbb{Z}$, which suggests that the prospective teachers'

understanding of numbers and operations is already very strongly interconnected and that their skills in dealing with $\mathbb{N}_0$ almost completely predict their skills in the area of $\mathbb{Z}$. Thirdly, it shows that skills in dealing with $\mathbb{N}_0$ largely predict both skills with $\mathbb{Z}$ and skills with $\mathbb{Q}$. The mediation model shows that the relationship between $\mathbb{N}_0$ and $\mathbb{Q}$ cannot be fully explained by $\mathbb{Z}$ as mediator between the two number ranges. A significant direct effect remains, indicating that foundational concepts in natural number skills directly support rational number understanding without requiring complete integer mastery. This could indicate that the frequently chosen curricular arrangement of number ranges ($\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$) does not necessarily have to follow this order, as it is not reflected in the structure of the skills of prospective teachers. In addition, the prospective mathematics teachers may have had several opportunities to learn about $\mathbb{Q}$ before starting their studies, so that no natural number bias occurred.

Several limitations should be taken into account. Our sample of beginning elementary teacher students likely possesses profound arithmetical skills. High solution rates and small variance may reflect ceiling effects. Of course, this is a cross-sectional assessment through which no causal relationships can be described. In addition, the observed relationships might reflect instruction history rather than necessary developmental sequences. Future longitudinal research following students through active learning of these domains would provide stronger evidence about developmental mechanisms. Future research should also examine how relationships between number range skills evolve during teacher preparation and investigate instructional approaches leveraging the dual-pathway structure. However, the results indicate that if prospective elementary school teachers develop rational number skills primarily through natural number connections while possessing weaker $\mathbb{Z}$ understanding, they may struggle to prepare the number range that follows in the next grades and to anticipate and address problems children may have in dealing with negative numbers.

In conclusion, the results show that prospective teachers' mathematical skills across the number ranges from $\mathbb{N}_0$ to $\mathbb{Q}$ are strongly interrelated. The near-perfect $\mathbb{N}_0$ - $\mathbb{Z}$ relationship suggests successful foundational learning, while dual pathways to $\mathbb{Q}$ skills indicate multiple conceptual routes rather than a strict hierarchy. Teacher education must develop an integrated understanding across domains, with particular attention to integer skills and conceptual change among number systems.

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