DISCERNED ASPECTS OF MULTIPLICATIVE STRUCTURES EXPLAINING DIFFERENCES IN 6-YEAROLDS' ARITHMETIC SKILLS

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Critical for developing multiplicative reasoning skills is recognizing the multiplicative structure in the problem. In this study, we investigate which discerned and undiscerned aspects explain differences in 6-yearolds' ways of solving a task with a multiplicative structure. Drawing on variation theory of learning, qualitative analyses were conducted of interview data from 199 students solving an arithmetic task with a multiplicative structure. Four critical aspects were identified: the size of the unit, its iteration, the number of iterations, and the relation between iterations and the whole. Results show that learning to handle multiplicative structures requires discernment and coordination of all four aspects simultaneously.

INTRODUCTION

Students encounter arithmetic problem solving in their early school years and are expected to learn to operate with numbers in both additive and multiplicative tasks. Multiplicative reasoning is more complex than additive reasoning, as it requires keeping track of the iterations of a composite unit (Ulrich 2015; van Dooren et al., 2010; van den Heuvel-Panhuizen & Elia, 2020). This entails a challenge to many students because of the added complexity when the question is to find a number of iterations rather than a number of objects. The notion of units and how students experience numbers as a unit or composed by smaller units that can be iterated within a whole set, is thereby essential. How students learn to experience these relations is knowledge that would help understanding what teaching should attend to in order to support their arithmetic development, as well as why some students struggle with learning to reason multiplicatively. In this study, we address the following research question:

> What discerned and undiscerned aspects are critical for explaining the difference between 6-yearolds' ways of solving a task with a multiplicative structure?

The research question will be answered by a qualitative analysis of students' ways of solving a multiplicative task. The analysis is informed by the variation theory of learning (Marton, 2015), which goes beyond observed strategies used by the students by also interpreting the students' actions as a function of what aspects are discerned and undiscerned by them.

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, pp. 411-418. PME.

BACKGROUND

Learning to reason multiplicatively is more than learning certain strategies, it requires seeing numbers structured in a qualitatively advanced way that goes beyond for example handling multiplication as repeated addition (Nuñes & Bryant, 1996). Our interest lies in how a problem that includes a multiplicative structure can be interpreted, and what students find possible to do with the numbers given in the problem.

Multiplicative structures can be found in situations where two variables are linked by operations of multiplication or division, e.g., "one sticker costs 4 crowns, how many stickers can you get for 20 crowns?" where the price for one sticker is invariant and the number of stickers and total amount of crowns are proportionally related. Research has shown that learning the multiplicative structure of numbers is difficult (van Dooren et al., 2010; van den Heuvel-Panhuizen & Elia, 2020) because it differs from additive structures, which students more often encounter in their early years and daily life.

Based on Steffe's (1994) description of multiplicative development, students develop their multiplicative reasoning skills through formation of units and the organization or coordination of these units where a critical step is to perceive units of more than one as countable units. As empirically shown (e.g., Zwanch & Wilkins, 2021), being able to construct units is related to developing multiplicative reasoning skills. Thus, unitizing becomes a stepping stone for being able to reason about number of units, rather than number of single elements or the total number of elements in a set. Multiplicative reasoning where focus is directed towards such iterations of units, has in empirical studies been observed to be related to students' abilities to partition and compose units and see numbers as composed of units within units (Tzur et al., 2021). In tasks that require multiplicative reasoning, remaining to count in ones rather than coordinating units, is thereby a severe limitation (Ulrich, 2015). Critical for developing multiplicative reasoning skills, seems to be how students interpret the mathematical structure of a problem, usually demanding a shift from using (repeated) addition to recognizing the multiplicative structure in the problem (Askew et al., 2019). Nevertheless, even though students may learn to compose, decompose and identify units within units, studies have shown challenges transferring this ability to arithmetic problem solving situations (Björklund et al., 2024).

Steffe and scholars following his theoretical conjecture of multiplicative development, highlight unitizing and, in particular, coordinating units as one-to-many (in contrast to reasoning about additive relations which is about seeing two or more quantities as related to one another, see e.g., van den Heuvel-Panhuizen & Elia, 2020). What is lacking in the field of research on multiplicative reasoning is a detailed insight into what it means to unitize and handle multiplicative structures in an arithmetic task, taking the learner's perspective. A suggested classification by Cheeseman et al. (2020) focuses on young children's multiplicative thinking as visualization of equal group

structures. This is in line with the theoretical conjecture outlined above, where unitizing is key, and seems to focus on the learners encountering relevant tasks. However, Cheeseman et al. (2020) describe the development path in terms of emergent, partial or evident awareness as observed in children's ways of handling multiplicative tasks. This makes the classification a rather blunt tool, both to identify students' multiplicative reasoning skills and for teachers to know not only how a student experiences or understands a multiplicative task, but also how to support students developing understanding.

THEORETICAL FRAMEWORK

The theoretical framework in this study is the Variation theory of learning (Marton, 2015), with origins in phenomenological ontology. This means our research interest is to explore the world as it is experienced by the students. Mathematical skills, such as multiplicative reasoning, is considered a function of discerned aspects of numbers, which impacts on what the learner is able to do when encountering, for example an arithmetic task. Differences between students' ways of solving a task can then be described in terms of what aspects they have (and have not yet) discerned. By observing students' actions and ways of reasoning when solving a problem, this reveals the critical aspects to be discerned. Analyses informed by variation theory thereby may lead to insights into what is to be learnt to advance the student's knowledge and skills.

METHODS

Data for the present study is part of a larger research study of young students' arithmetic development conducted in Sweden. Of particular interest in this research is students' ways of experiencing numbers and number relations, why individual task-based interviews were conducted with students attending Swedish preschool class. To deepen the understanding of multiplicative reasoning, the present study focuses specifically on which aspects are critical for 6-year-olds to discern when solving a task with a multiplicative structure. In total, 344 students attending regular Swedish education were interviewed, randomly selected from 38 classes, 10 in each class. Written consent for participation was given by the students' legal guardians.

The interviews were conducted at the schools by researchers in mathematics education trained in interviewing young students. The task-based interviews took place at the start of the school year, when the students' mean age was six years. One interview item comprised three tasks assessing multiplicative reasoning, inspired by van den Heuvel-Panhuizen and Elia (2020). Students saw a picture of two children and five options (2, 3, 4, 6, 8 mittens). The interviewer asked: *How many mittens do these children need?* (Task 1) and: *If another child joins, how many mittens are needed?* (Task 2). Students selected from the given options. Those students answering Task 2 correctly received Task 3 orally (without options and picture): *If you have ten mittens, how many children is it enough for?* (Task 3). For this analysis, we included 199 students who completed

Task 3. Students' solutions were documented in a protocol and complemented by field notes capturing their actions and verbal expressions.

Analysis

The analysis, guided by the variation theory of learning (Marton, 2015), proceeded in several steps. Field notes were reviewed to gain an overview of the data, documenting each student's approach and answers in a table. The focus was on similarities and differences in how students handled numbers and number relations in the task, including reasoning, use of fingers or other actions, and their solutions. Different strategies were compared, advanced solutions were analyzed in detail and contrasted with the less advanced to identify nuances. Following variation theory, students' approaches reflect variations in discerned and undiscerned aspects of numbers. Through repeated readings and comparisons, we identified aspects critical for discerning multiplicative structures. Both discerned and undiscerned aspects were considered, based on the assumption that the more aspects a student discerns, the more advanced their understanding becomes. This enabled us to determine which aspects are critical for successfully handling multiplicative structures in arithmetic tasks.

RESULTS

In this section, we present the four critical aspects (A–D) identified in the analysis. These aspects are considered inclusive, meaning that all four must be discerned to successfully solve the multiplicative task. Consequently, the more aspects a student discerns, the deeper their understanding of the multiplicative structure. Empirical examples illustrate how the multiplicative structure is experienced differently depending on which aspects are discerned or remain undiscerned. Finally, we summarize the number of students discerning A, AB, ABC, and ABCD aspects, or none of these. Table 1 presents a brief description of the four critical aspects.

Table 1: Descriptions of aspects (A-D) critical to be discerned.

Aspect	
A	The size of the unit [2]
B	The unit [2] is iterated
C	A number of iterations of the composite unit [2's]
D	The composite units [2's] in relation to the whole [10] simultaneously

Discerning aspect A

A critical aspect for solving a task with a multiplicative structure is discerning the size of the unit. In our task (If you have ten mittens, how many children is it enough for?), the unit is 2—two mittens for each child. An example where this is discerned is when students say: “Each child has two” or “Children have two hands”. Such responses show that students have discerned that *the unit is 2*, that is, not the same as one child, and that each child needs two mittens. However, they do not necessarily relate this unit of 2 to the total number of mittens or discern that the unit needs to be iterated. If the student makes no further reasoning about how to solve the task, s/he has *only* discerned aspect A, and aspects B, C, and D remain undiscerned.

Discerning aspects A and B

When both aspects A and B are discerned in a task with a multiplicative structure, the size of the composite unit (2) and that the unit is iterated are also discerned. For example, one student said: “Eight or nine persons… (hesitates) two and two and two (raising two fingers at a time)”. Another student responded: “Two mittens, two mittens, two mittens… (pauses) I don’t know.” Compared to only discerning aspect A, discerning both A and B means that the composite unit is identified and understood as being repeated several times. However, it is *the equal number of elements or singles that constitute the unit* that is the focus of the students’ attention and the students do not recognize that the composed units are to be enumerated (aspect C) and do not simultaneously discern the number of iterations in relation to the whole (aspect D). Thus, both aspects C and D remain undiscerned.

Discerning aspects A, B, and C

In a more advanced way of handling the multiplicative task, aspects A, B, and C are discerned. This means that the composite unit (2) is recognized as being repeated several times, and that there are a number of iterations of this composed unit that are to be enumerated. In one case, a student places ten fingers on the table, arranges them into pairs (little finger close to ring finger, long finger close to index finger, the two thumbs, and so forth), leaving a gap between each pair, and responds: “Ten children, ten mittens.” Another student also uses fingers, grouping them by placing the two thumbs together and saying “two” then the index fingers, “two” followed by the middle fingers, “two” ring fingers, “two” and finally the little fingers, “two” before responding “six”. After grouping the units into five pairs, common answers are “ten” and “six”, respectively. When discerning A–C, the students discern the size of the composite unit, that the composite unit is iterated and *a number of iterations are to be enumerated*, but they do not yet discern how the number of iterations is related to the whole (10), why one student answers with the number of elements (10) not the number of units (5). Thus, aspect D (simultaneously discerning the unit of 2 in relation to the whole) may remain undiscerned.

Discerning aspects A, B, C, and D

When all critical aspects are discerned in a task with a multiplicative structure, the size of the composite unit (2), the fact that this unit is iterated, and the number of iterations (5) in relation to the whole (10) are simultaneously discerned. One student handles the task by raising two fingers at a time while saying: "One, two to one child; three, four to two children; five, six to three children; seven, eight to four children; nine, ten to five children." This is an example of a way of solving the task when all four aspects (A–D) are discerned. The way the students act in solving the task is quite similar to those in which aspects A–C are discerned. When discerning aspects A–C, students are able to discern the size of the composite unit and that there are a number of composite units to be enumerated. However, they do not discern the exact number of iterations in relation to the total number of elements. In contrast, discerning aspects A–D requires students to discern and *simultaneously coordinate the size of the composite unit, the number of its iterations, and the total number of elements*, a one-to-many relationship.

Arriving at the correct answer "five" is necessary for determining whether aspect D has been discerned and whether all four aspects have been identified in the analysis. In the dataset, we observed different solutions in which aspects A–D were discerned. Some students started with the whole (10) and split or subtracted in 2s, while others added five 2s, arriving at the answer "ten". Other students focused explicitly on the relationship between the number of composite units and the given whole number, for example saying: "Five—half of ten, because each one has two" or "Five, two mittens to each child".

Summary of discerned aspects

As presented in Table 2, 124 of 199 interviewed students did not discern any of the aspects identified as critical for solving a task with a multiplicative structure. A common answer was "*ten*" without making any kind of unitizing actions or utterances. These students most often approached the problem by guessing, counting from one to ten, or not giving any answer (see Table 2). To successfully solve the multiplicative task, the student needs to discern all four aspects (A–D). The results reveal that, as six-year-olds at the beginning of formal schooling in Sweden, one quarter of the students were able to discern all four aspects. Ten students were aware that the size of the composite unit was not the same as that of one child, but they were not able to reason further. In total, 15 students discerned aspects A–B or A–C, indicating that they recognized that the composite unit of 2 is iterated (aspects A and B) and that there are strategies for keeping track of the composite units (aspects A–C).

Table 2: Summary of students discerning A, AB, ABC, ABCD, or none of these aspects (N=199). Percentages do not add up to 100% due to rounding error.

Aspects discerned	Frequence	Percent
No aspects	124	62,3
A	10	5,0
A and B	6	3,0
A, B and C	9	4,5
A, B, C and D	50	25,1
Total	199	100

DISCUSSION

What does it mean to be able to reason multiplicatively when solving a task such as the mitten task, where the size of the repeated unit is not explicitly stated? To solve such a task, students must determine how many units constitute the whole. Although the total number of elements (10 mittens) is given, students must discern the relationship between the whole and the equal grouping of two mittens per child. Students at the beginning of formal schooling are likely unfamiliar with this type of problem, which makes the task challenging. The difficulty appears to lie in coordinating iterations of a composite unit within the total number of elements, that is, operating with numbers on different “levels” (Ulrich, 2015). While 25.1 % of the 6-yearolds in this study handled the multiplicative structure before formal schooling, 62.3 % had not yet moved from additive to multiplicative reasoning, that is, not recognizing the one-to-many relation in the task (Askew et al., 2019).

The task allows for a detailed analysis of the complexity involved in developing an understanding of multiplicative structures. Our analysis can be related to Cheeseman et al.’s (2020) categories of evident, partial, and emerged awareness. However, whereas partial awareness refers to recognizing equal groups without enumerating them, our analysis distinguishes between discerning that a composite unit is iterated (aspects A–B) and keeping track of the number of iterations (aspect A–C). This distinction offers a more nuanced understanding of the differences in 6-year-olds' approaches to solving multiplicative tasks. The critical aspects are inclusive, meaning that all four must be discerned to successfully solve the task. Although the difference between discerning aspects A–C and A–D is subtle, the analysis suggests that multiplicative reasoning requires the simultaneous coordination of the composite unit, the number of its iterations, and the whole.

The ability to reason multiplicatively is crucial for students’ mathematical learning in the middle grades (Zwanch & Wilkins, 2021). As multiplicative reasoning is more

complex than additive reasoning (Askew et al., 2019; van den Heuvel-Panhuizen & Elia, 2020), it is important to provide opportunities for multiplicative reasoning already in the lower grades.

Acknowledgement

The project is funded by the National Research Council, grant no. 2020-03712.

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