

INDIVIDUAL DIFFERENCES IN CONCEPTUAL AND PROCEDURAL KNOWLEDGE OF FRACTIONS AND ALGEBRA: A RASCH-BASED ANALYSIS

Michael D'Erchie¹, Claire L. Forsmann², Michael Schneider², Andreas Obersteiner¹

¹Technical University of Munich, ²Trier University

Research on individual differences in mathematics learning often distinguishes between conceptual and procedural knowledge and has reported distinct learner profiles based on this distinction. However, most of these studies rely on sum scores that do not account for item difficulty. The present study applied latent profile analysis to ability measures derived from Rasch-scaling measures of conceptual and procedural knowledge in fractions and algebra among 357 seventh-grade students. Across both domains, profiles differed in overall proficiency but not in the relative levels of conceptual and procedural knowledge. No profiles showed high conceptual and low procedural knowledge, or vice versa. These findings support theoretical models that posit a mutually reinforcing development of both knowledge types.

INTRODUCTION

Research on mathematics education and educational psychology commonly distinguishes between conceptual and procedural knowledge (e.g., Hiebert & Lefevre, 1986; Rittle-Johnson, 2019). *Conceptual knowledge* is typically described as the knowledge and understanding of mathematical facts and ideas and the relations among them. A *procedure* is a “step-by-step instruction that prescribes how to complete a task” (Hiebert & Lefevre, 1986, p. 6), and *procedural knowledge* refers to the ability to carry out these mathematical procedures and algorithms.

For example, conceptual knowledge of fractions includes understanding that fractions represent part-whole relationships or how fractions and their properties differ from other number domains, while procedural knowledge includes skills such as adding two fractions by following the algorithm of first finding a common denominator, extending both fractions, and adding the resulting numerators (e.g., Lenz et al., 2019). In algebra, conceptual knowledge may include an understanding of the idea of equivalence, whereas an aspect of procedural knowledge is the ability to apply equivalence transformations to solve linear equations (e.g., Schneider et al., 2011).

The conceptual-procedural distinction has led researchers to assume that learners may differ not only in their overall levels of knowledge, but also in their levels of conceptual and procedural knowledge separately. Such reasoning has motivated person-centered approaches, particularly in the domain of fractions, that seek to identify qualitatively

distinct learner profiles, such as students who exhibit relatively strong procedural but weak conceptual knowledge, or vice versa (Bempeni et al., 2018; Hallett et al., 2010; Hallett et al., 2012; Lenz et al., 2024; Lenz & Wittmann, 2020). Empirical studies indeed report a range of distinct learner profiles, encompassing both students with similarly high or low levels of conceptual and procedural knowledge and students characterized by marked conceptual–procedural imbalances. Across studies, such cross-level profiles frequently accounted for substantial proportions of the sample (e.g., Bempeni et al., 2018; Lenz et al., 2024) and, in some cohorts, even represented the dominant pattern (e.g., Hallett et al., 2012).

These frequently observed cross-level profiles are interesting as they are theoretically unexpected. The most widely accepted model in the field of conceptual and procedural knowledge is the *iterative model* by Rittle-Johnson et al. (2001). The model posits that conceptual and procedural knowledge develop in a mutual way, such that gains in one form of knowledge support and promote gains in the other. Thus, from a theoretical perspective, one would not expect students to exhibit substantial imbalances in their levels of conceptual and procedural knowledge.

One possible explanation for the contradiction between empirical findings and theoretical predictions is methodological. Previous studies used sum scores or residualized scores for latent profile or cluster analyses. This is problematic because these scores do not account for item difficulty, although differences in average item difficulty between subscales can create apparent cross-level profiles. To illustrate this point, consider a student with average levels of conceptual and procedural knowledge who is administered, relative to his ability level, a set of difficult conceptual knowledge items and a set of easy procedural knowledge items. Such a student is likely to answer most procedural knowledge items correctly while failing many conceptual knowledge items, not because of true differences in his or her knowledge levels, but because of differential item difficulty. Consequently, previously observed cross-level profiles may in part be artifacts of measurement rather than reflections of actual differences between students' conceptual and procedural knowledge.

Another limitation of previous research on individual differences in conceptual and procedural knowledge is its focus on one single domain, namely fractions. Thus, it remains unclear whether profiles found in the domains of fractions generalize to other mathematical content domains.

The present study

This study addresses the potential confounding of sum scores with test difficulty by applying latent profile analysis to ability measures of conceptual and procedural fraction knowledge derived from Rasch-scaling, which specifically accounts for item difficulty (e.g., Boone, 2016).

Embedded within a larger project investigating fraction and algebra knowledge, a second aim was to examine whether similar individual profiles emerge across both domains or whether patterns of individual differences differ across both domains.

METHODS

Sample and procedure

A total of 357 seventh-grade students (178 girls, 179 boys; $M = 12.4$ years, $SD = 0.56$) from academic-track secondary schools in Germany participated in our study. Data were collected in whole-class settings using paper-pencil tests during three 45-minute lessons. The study received approval from an institutional review board and the local school authorities. Informed parental consent and student assent were obtained from all participants.

Measures

Conceptual fraction knowledge items assessed fraction magnitude understanding (reasoning about magnitudes without computation), the density property (understanding infinitely many fractions exist between any two fractions), and part-whole interpretations (e.g., shading given portions of polygons). Procedural fraction knowledge items focused on performing fraction arithmetic and extending or simplifying fractions to equivalent forms. Conceptual algebra knowledge items targeted central algebra concepts such as variables, equivalence, and equations. Procedural algebra knowledge items required students to calculate numerical values of algebraic expressions, simplify algebraic expressions, and solve linear equations. Example items are given in Table 1.

Weighted Likelihood Estimation (WLE) reliabilities ranged between .47 and .72. Slightly lower WLE reliabilities than the commonly used .70 benchmark are to be expected, as the measures of conceptual and procedural knowledge assessed different aspects of these constructs, resulting in heterogeneous scales that may increase standard errors and thus reduce WLE reliability.

Data analysis

Item-level analyses were conducted using Rasch modeling to examine the psychometric properties of the fraction and algebra measures. Rasch analyses were carried out in R using the *TAM* package. These analyses were used to obtain person ability estimates that served as input for subsequent person-centered analyses. None of the items showed Infit/Outfit values below 0.5 or above 1.5, respectively, suggesting that the items showed no evidence of misfit to the Rasch model.

Latent profile analyses were conducted separately for fractions and algebra using Mplus Version 8.11 (Muthén & Muthén, 1998-2017). For each domain, profile solutions ranging from two to eight profiles were estimated. To evaluate model fit and determine the optimal number of profiles, we focused on the BLRT p -value. The best profile solution was selected as the most parsimonious model whose BLRT was still

significant. Other statistical criteria, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), sample-size adjusted BIC, and entropy, were also taken into account when selecting the best profile solution.

Table 1: Example items for conceptual and procedural knowledge

Knowledge facet	Sample item
<i>Conceptual fraction knowledge</i>	How many fractions are between $3/8$ and $7/8$? a) Three b) Five c) A finite number, but more than five d) Infinitely many
<i>Procedural fraction knowledge</i>	Calculate: $11/12 - 1/3 + 1/2 = \dots$
<i>Conceptual algebra knowledge</i>	Lucy bought 3 ducks for 36 dollars. She has written the following equation: $3d = 36$. What does d stand for? a) Price of a duck b) A duck c) Ducks d) Dollars
<i>Procedural algebra knowledge</i>	Solve for x : $4 \cdot (2x - 3) = 2 \cdot (3x - 4)$

RESULTS

Discriminant validity of conceptual and procedural knowledge

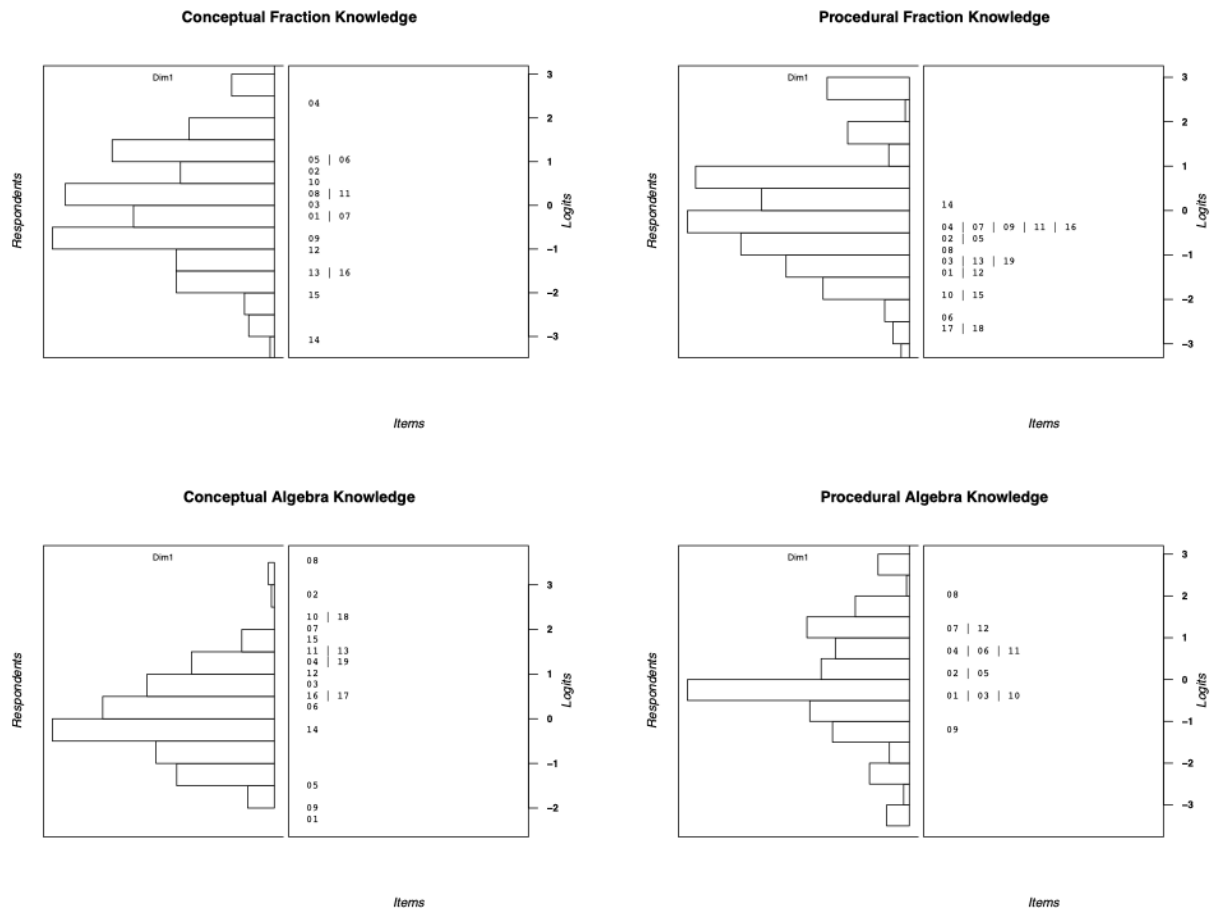
Across both fractions and algebra, information criteria consistently favored the two-dimensional Rasch model over the one-dimensional model. For fractions, the two-dimensional model improved fit by $\Delta\text{AIC} = 73$ and $\Delta\text{BIC} = 66$; for algebra, by $\Delta\text{AIC} = 57$ and $\Delta\text{BIC} = 50$. These results show that conceptual and procedural knowledge are better represented as separable latent dimensions.

Wright Maps

Figure 1 shows the Wright Maps for conceptual fraction, procedural fraction, conceptual algebra, and procedural algebra knowledge. Across all four maps, person abilities (left panels) span a broad range and generally overlap with the item difficulty distributions (right panels), indicating adequate test targeting. Conceptual knowledge items in both domains cover the full range of person abilities. Procedural fraction knowledge items are concentrated in the lower half of the ability distribution, limiting

their capacity to differentiate among high-achieving students. Procedural algebra knowledge items are slightly shifted toward the upper half of the ability range, reducing their sensitivity for distinguishing among lower-achieving students.

Figure 1: Wright Maps for the LPA variables



Profiles of individual differences in conceptual and procedural knowledge of fractions and algebra

For fractions, model fit indices improved monotonically from the 2- to the 6-profile solution, with substantial reductions in AIC ($\Delta\text{AIC} = 43$) and sample-size adjusted BIC ($\Delta\text{sample size adjusted BIC} = 37$), while BIC was relatively unchanged ($\Delta\text{BIC} = 1$). The 6-profile solution showed the highest entropy value (.87) and was supported by a significant BLRT ($p < .001$). Adding a seventh profile, for the first time, did not meaningfully improve model fit (BLRT p -value = .187). Thus, we considered the 6-profile solution as the best-fitting solution. Average posterior probabilities for profile membership ranged from .88 to 1.00, indicating high classification accuracy.

For algebra, information criteria favored a 3-profile solution, with further increases in the number of profiles yielding non-significant BLRTs ($ps \geq .237$) and negligible improvements in AIC ($\Delta\text{AIC} \leq 2$). The 3-profile solution showed the lowest BIC and sample-size adjusted BIC. Entropy for the 3-profile solution was acceptable (.70). The

3-profile solution demonstrated moderate to good classification accuracy, with average posterior probabilities ranging from .78 to .90.

The higher number of profiles in fractions suggests that students are more heterogeneous in their fraction knowledge than in their algebra knowledge.

Figure 2: Latent profiles for conceptual and procedural knowledge in fractions (left) and algebra (right)

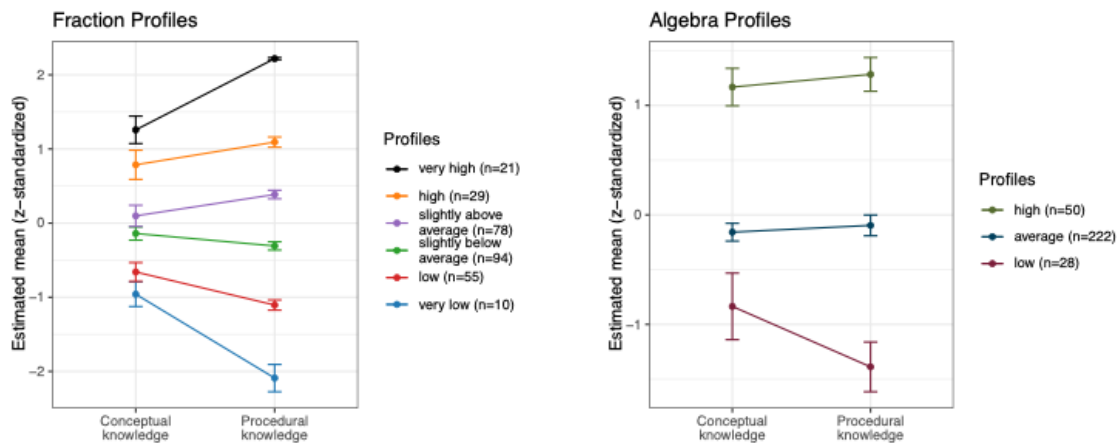


Figure 2 displays the levels of conceptual and procedural knowledge for each of the identified profiles in the domains of fractions (left) and algebra (right). In both domains, profiles are primarily ordered by overall level of performance: higher conceptual knowledge is accompanied by higher procedural knowledge, and vice versa. There was no evidence for profiles characterized by uneven levels of conceptual and procedural knowledge.

For both fractions and algebra, procedural knowledge shows larger variation than conceptual knowledge, indicating that procedural knowledge differentiates better between profiles. This pattern is particularly pronounced in the fraction domain, where procedural knowledge differs markedly between the lowest and highest profiles, whereas conceptual knowledge varies more moderately.

The distribution of students across profiles was uneven. In the fraction domain, very low ($n = 10$) and very high ($n = 21$) profiles were rare compared to intermediate profiles. A similar pattern emerged for algebra, with comparatively small low ($n = 28$) and high ($n = 50$) profiles. Overall, this distribution indicates that most students clustered around moderate levels of conceptual and procedural knowledge, with relatively few students at the extremes.

DISCUSSION

The present study followed a person-centered approach to examine individual differences in conceptual and procedural knowledge in the domains of fractions and

algebra. Unlike earlier research on individual differences in conceptual and procedural knowledge, however, our study's contribution lies in its methodological approach. We applied latent profile analysis to Rasch-scaled measures that account for item difficulty. This matters because systematic difficulty differences between conceptual and procedural items can artificially create apparent cross-level profiles even when true abilities are similar.

Using this approach, latent profile analyses identified profiles that differed primarily in overall level of proficiency rather than in the relative balance between conceptual and procedural knowledge across both the fraction and algebra domains. In other words, students' levels of conceptual and procedural knowledge tended to align (high-high, moderate-moderate, low-low). Importantly, we found no evidence of students with substantially varying levels of conceptual and procedural knowledge. This pattern held across both mathematical domains investigated, suggesting that it reflects a domain-general pattern of how these knowledge types relate rather than domain-specific effects.

The absence of such cross-level profiles aligns with theories emphasizing that conceptual and procedural knowledge develop iteratively and reinforce each other (e.g., Rittle-Johnson et al., 2001). Once formal instruction on a topic has ended, persistent imbalances are unlikely, because gains in one type of knowledge typically support gains in the other. Our person-centered evidence supports this theoretical assumption. As such, our findings challenge the idea of categorizing students as "conceptual learners" or "procedural learners." This has implications for practice. Students who struggle with procedures likely also struggle with concepts, so they need support with both, rather than instruction tailored to a supposed learner type.

Limitations and future directions

The present study captures individual differences only at a single time point and cannot address developmental dynamics. Although we found no evidence for cross-level profiles, temporary imbalances between conceptual and procedural knowledge may occur during learning and dissolve over time. Future longitudinal studies could uncover such short-term fluctuations. Moreover, procedural fraction items showed limited differentiation among high-achieving students, while procedural algebra items showed limited differentiation among low-achieving students (see section 'Wright Maps'). However, these students would fall into the highest or lowest procedural profiles, which consistently showed aligned conceptual knowledge, suggesting cross-level profiles are unlikely beyond our measurement range. Finally, the present study focused on fractions and algebra only. Future research could extend this approach to other mathematical domains to test the generalizability of our findings.

Conclusion

Using ability measures derived from Rasch-scaling, this study shows that individual differences in conceptual and procedural knowledge of fractions and algebra are best

characterized by differences in overall proficiency rather than by qualitatively distinct knowledge profiles. The absence of cross-level profiles suggests that conceptual and procedural knowledge develop in an intertwined manner, consistent with the iterative model of conceptual and procedural knowledge by Rittle-Johnson et al. (2001).

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