PITS AND PILES, DIRECTIONS, AND DEBT: PRE-SERVICE TEACHERS' CONCEPTIONS OF NEGATIVE NUMBERS

Emily Macmillan, Jenni Ingram

University of Oxford

Children naturally use fingers or objects to count whole numbers, but there is no similarly intuitive way of representing negative numbers. Using scripting as a proxy for practice, we investigated the conceptions pre-service teachers drew on through the representations they used and the structures they highlighted when explaining how to calculate with negative numbers. We categorised the conceptions into quantity (debt, counterbalance, and relativity), movement (translation and rotation), position (order and symmetry), and symbolic (removed from context). The pre-service teachers used multiple conceptions and representations between and within explanations, rarely overtly drawing attention to mathematical structures.

INTRODUCTION

Research into students' understanding of negative numbers is in its infancy (Bofferding, 2019). Most research in this field reports on children's conceptions of integers, showing they use unconventional methods in an effort to make sense of negative numbers, whilst drawing on a range of representations (Wessman-Enzinger, 2019). Little research has been conducted into teachers' conceptions of negative numbers. Literature reports on teachers' and researchers' recommendations for which representations to use in the classroom to help students to "gradually refine and modify their understandings toward … those the mathematics community at large accepts as correct and useful in solving mathematical and scientific problems" (Harel, 2019, p. 338). However, there is not one "correct" conception of negative numbers, but a range of conceptions and representations that foreground different underlying structures (Tsang et al., 2015). The aim of this study is to identify the conceptions of negative numbers pre-service teachers (PSTs) draw on through the representations they use and the structures they emphasise. Using scripting as an approximation for practice (Zazkis & Herbst, 2018) we investigated PSTs' explanations in three whole-class scenarios involving negative arithmetic. We asked the research questions: 1) What conceptions of negative numbers do student teachers draw on? 2) What representations do they use to convey these conceptions? 3) What connections do they make between these representations and mathematical structures? By answering these questions, we present a classification of conceptions of negative numbers, linking these conceptions to representations and descriptions of underlying mathematical structures they can highlight. The results of this study will inform the development of instructional practices for supporting the development of PSTs engaged in Initial Teacher Education (ITE) and form the basis for further research into the teaching of negative numbers.

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LITERATURE REVIEW

Whilst whole numbers can be represented by physical objects, negative numbers are less perceptually obvious (Alibali & Norton, 2019). Negative numbers can be traced back to 250 BCE in the context of debt (Varma et al., 2019). Debt is one of many *conceptions* of negative numbers which have been written about extensively and without consensus. Bofferding (2019) presents the "multiple meanings" of negative numbers in four categories: *bookkeeping* (gaining and losing money or points), *counterbalance* (cancelling positive with negative values, for example net worth or "floats and anchors"), *relativity* (number represented as an action rather than a quantity), and *translation* (changes in elevation, temperature, or movement). Varma et al. (2019) add to this list with a focus on *symmetry* by conceptualising the number line folding at zero (Tsang et al., 2015; Varma et al., 2019). Wessman-Enzinger (2019) considers *debt/credit* and *net worth* as "instructional models" rather than conceptions, and promotes conceptualising integers as *directed quantities*. Sarma et al. (2024), in their "historical and scholarly context", categorise conceptualisation of negative numbers in a different way again, as *spatial*, *temporal* and *possessional*. These attempts to connect negative numbers to observable contexts all break down at some point (Wessman-Enzinger, 2019), because negative numbers do not "exist as tangible entities in the same way that positive number do" (Bofferding, 2019, p. 252).

Negative numbers can be represented in a variety of ways including words, symbols, images, and physical objects (Jitendra et al., 2016). Common representations of negative numbers include: number lines; double-sided counters; symbolic notation such as $-1 + -4$; and analogies such as pits and piles (Macmillan, Forthcoming). Different representations lend themselves to different conceptions of negative numbers (Bofferding, 2019). For example, the term 'negative numbers' is used throughout this report instead of the term 'directed numbers' which suggests a 'movement' conception. Mathematicians need to be able to move fluently between representations (Otte, 2006) and between conceptions of negative numbers (Harel, 2019), acting "on the thing itself rather than merely upon the description of it" (Otte, 2006, p. 17).

Different conceptions and representations of negative numbers can foreground different mathematical *structures.* The term *structure* is used variably across the literature to describe properties, relationships, and generalisations (Venkat et al., 2019). Within the literature, authors similarly use a range of meanings for *structure*. For example, Tsang et al. (2015) use *structure* to refer to properties of *order* and *magnitude* (p. 159) and the *additive inverse law*: $x + -x = 0$ (p. 155). It is used here to describe mathematical generalities that underpin negative number arithmetic including properties such as commutativity of addition, and laws such as the additive inverse.

METHODS

Twenty PSTs learning to teach mathematics to 11–18-year-olds were asked to script three imaginary whole class conversations in pairs, two about negative number

addition and subtraction and the third about multiplication. Scripting, the act of creating a dialogue between a teacher and their students (Zazkis & Herbst, 2018), is an approximation of practice (Campbell & Baldinger, 2022) and can be used to "bridge pedagogies of investigation and enactments and to support teacher learning in ways that are more sustaining and generative" (Crespo, 2018, p.255). Pedagogies such as approximations of practice are commonly used in ITE to support PSTs to tackle the theory/practice divide (Grossman et al., 2009). When scripting, the scripter is asked to plan what they might say as the teacher, anticipate how students might react, and how they might respond, writing an explanation for their choices (Crespo, 2018). Scripting tasks pay attention to the complex and situational nature of teaching as a social activity, whilst reducing its complexity (Campbell & Baldinger, 2022), allowing PSTs to focus on a particular element of practice identified by the instructor (Grossman et al., 2009). Analysis of scripts created by PSTs provides insight into their mathematical conceptions and their pedagogical choices (Zazkis & Herbst, 2018).

Scripting Task 1 (ST1) (Figure 1) was designed to investigate the PSTs' existing conceptions of negative numbers (RQ1), their choice of representations (including language) (RQ2) and the structures they highlighted (RQ3). Scripting Task 2 (ST2) (Figure 2) encouraged the PSTs to focus on two specific representations (double-sided counters and number lines), to investigate the connections they made between the representations and mathematical structures (RQ3). Scripting Task 3 (ST3) (Figure 3) was intentionally less structured, requiring PSTs to select their own conceptions and representations (RQ1, 2 and 3). ST1 was used before, and ST2 and ST3 after, a university session titled 'Building a Sense of Structure' during which different conceptions and representations for negative numbers were investigated and the structures they can reveal were discussed.

RESULTS

What conceptions of negative numbers do PSTs draw on?

PSTs described negative numbers in four broad categories as: quantities, movements, positions, or symbols. The descriptions used by the PSTs within these categories were varied and attended to different features which allowed further classification (Table 1). Some PSTs used a particular conception consistently, whereas others used multiple conceptions within the same explanations. For example, in ST1, one PST wrote:

> Teacher: Then, if I subtract the negative number, I'm undoing the negative direction and adding instead

Figure 1: Scripting Task 1

Imagine you give the following task to a Year 7 mixed prior attainment group.

1. $-5 + 3$
2. $-5 + -3$
3. $-5 - -3$
4. $-5 - 3$

As you walk around the classroom you notice that the first two students, Alpha and Beta, have given an answer of -8 to question 3. You decide to stop the class and start a whole-class discussion. What can you imagine saying and doing next? What do you imagine the students will say and do in return? Continue the imaginary conversation below.

Class exchanges	Explanation
Teacher: Hands-up if you think the answer to question 3 is negative eight. (nearly half the students raise their hands)	To check to see whether this is a mistake that more than 2 students have made.
Teacher: What answer did you get to question 2? Alpha? Alpha: Minus eight	To draw students' attention to the answer to Q2 being -8
Teacher: Negative eight for question 2 as well. You got negative eight for question 2 and also for question 3. Do you think the answers to question 2 and question 3 can be the same?	To correct language use (minus → negative). To draw attention to the similarities and differences between the questions.

Figure 2: Scripting Task 2

Two students give different explanations for question 2.

Continue the conversation.

Class exchanges	Explanation
Student 1: I started with five reds and added three more. Student 2: I did it a different way. I went down five and then down another three. Teacher...	

Figure 3: Scripting Task 3

The following lesson the students are considering this question:

$$-3 \times -2$$

Class exchanges	Explanation
Student: Why does a minus times a minus equal a plus? Teacher...	

The use of the term ‘undoing’ could be categorised as a *relativity* conception, treating the ‘negative number’ as an action. However, the reference to ‘negative direction’ suggests a *translation* conception, where the number is conceived as a movement. The representations PSTs used and the explanations they gave in their scripts helped to clarify their conceptions, but it was often not possible to assign one category, demonstrating the complexity of the topic and the challenges of teaching it.

Table 1: Classification of negative number conceptions

Conception		Example
Quantity:	Debt	Gaining and losing money/points on a game
	Counterbalance	Net worth, pits and piles
	Relativity	Doing/undoing – number as an action
Movement:	Translation	Vectors on a number line, magnitude and direction
	Rotation	Movement on a number line, rotating 180° when minus symbol is seen
Position:	Order	Relative position on the number line
	Symmetry	Folding or ‘flipping’ the number line at zero
Symbolic:	Abstract	Referring to ‘rules’ without context

What representations do PSTs use to convey these conceptions?

Seven types of representations were revealed in the scripts: written (using symbols), verbal, analogies, double-sided counters, points on a number line, arrows on a number line, and reflecting or ‘flipping’ on a number line. All scripts used some written and verbal representations, but despite ST1 prompting PSTs to consider language use, some PSTs did not convert or connect written representations (e.g. $2 + -3$) to verbal ones (e.g. “two subtract negative three”). There are many ways of reading ‘$-$’ including negative, minus, subtract, take away, and difference between. The specific choice of language can emphasise different conceptions and underlying structure (Bofferding, 2019; Macmillan & Ingram, Forthcoming). For example, Bofferding and Hoffman noted a correlation between students’ performance on questions asking them which of a pair of integers was ‘higher’ and ‘more’, suggesting that students interpreted ‘higher’ in terms of magnitude (Bofferding, 2019). This was reflected in the research reported here, where PSTs regularly paired a movement to the left on a horizontal number line with the phrase ‘gets lower’.

PSTs who explained using a quantity conception used double sided counters or analogies of debt or pits and piles. Counters were used differently in ST3 where they

were used in an abstract procedural way which introduced the rule of 'flipping', drawing on a symbolic conception. This change aligns with two proposed different conceptions of the minus sign, as a difference, or a multiplication by negative one (Bofferding, 2019). Those who approached their explanations from a movement conception used number lines, but number lines were not on their own an indicator of a movement conception. PSTs used multiple conceptions with the same representation, particularly with number lines. For example, when representing $-5 + -3$ in ST1, in three scripts -5 was represented as a point on a line (position, order conception), and -3 as an arrow (movement, translation conception). In one of these scripts this was combined with a north/south analogy describing 'starting at -5' (position, order conception) and rotating (movement, rotation conception). Similarly, in one script in ST3, the analogy of pits and piles (quantity, counterbalance conception) was used alongside a number line (movement, translation conception). In ST3, multiplication by -1 was represented by 'flipping' or reflecting the number line about zero (position, symmetry conception), where the number was represented by arrows (movement, translation conception) or a point (position, order conception). In all three scripting tasks, when using number lines or counters, PSTs used representations which were visual (drawn), physical (describing students walking forwards and backwards, or instructions to use counters on a visualiser) and imagined (talking about 'going up' and 'down' an imaginary number line, or imagining counters).

What connections do PSTs make between these representations and mathematical structures?

Representations present opportunities to foreground mathematical structures. For example, using 'zero pairs' with double-sided counters presents an opportunity to emphasise the additive inverse property ($a + -a = 0$) which is "key in the conceptualization of negative numbers" (Harel, 2019, p. 335), whereas representing a number as an arrow on a number line highlights structures of magnitude and direction. In three scripts in ST1, PSTs focused on the structure of negative arithmetic, attempting to generalise to 'subtracting a negative number is the same as adding a positive number'. In two of these scripts, PSTs attempted to connect this structure to a number line representation which drew on position and movement conceptions, whilst the third presented it as a rule (symbolic conception). In two other scripts in ST1, PSTs focused on addressing the conception that holds for whole number but not integers, that 'addition makes larger', using number lines to demonstrate this. One of these representations used a symmetry conception, with numbers represented as arrows (movement conception) symmetrically about -5. In contrast, in ST3, where the task was focused on a generality, "Why does a minus times a minus equal a plus?", five of the nine scripts explicitly addressed the additive inverse ($-1 \times a = -a$) using a range of representations including counters, number lines, algebra and analogies.

CONCLUSIONS

We identified and categorised pre-service teachers' conceptions of negative numbers as: quantity (debt, counterbalance, and relativity), movement (translation and rotation), position (order and symmetry), and symbolic (removed from context). The representations they used to convey these conceptions were: written, verbal, analogies, double-sided counters, position on a number line, arrows on a number line, and reflecting the number line. Number lines were more likely to be used with movement conceptions and counters with quantity conceptions, but representations were more likely to draw on a range of conceptions simultaneously. The multiplicity of conceptions being drawn on within a single explanation demonstrates the complexity of teaching and learning about negative numbers and may contribute to the PSTs making few explicit connections between representations and structures in their scripts.

Whilst the results of this study reflect and build on prior research into students' negative number conceptions, the small scale and the pool of respondents being from the same ITE institution may have limited the variety in the responses we received. Furthermore, whilst scripting provided insight into the scripter's thinking, we cannot conclude that PSTs would choose to draw on the same conceptions or representations in a real lesson. This research should be built on by collecting responses from PSTs in a variety of ITE situations (for example School Centred Initial Teacher Training) and by extending the data collection techniques to classroom observations.

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