A MATHEMATICS TEACHER'S FUNCTIONAL APPROACHES AND LEVELS OF DECENTERING WHILE INTRODUCING QUADRATIC FUNCTIONS

Mervenur Belin[1], Işıl İşler-Baykal[2] & Gülseren Karagöz Akar[1]

[1]Boğaziçi University, Türkiye

[2]Middle East Technical University, Türkiye

As part of an intensive year-long multi-case study, in this paper, we investigated a mathematics teacher's functional approaches and levels of decentering while teaching quadratic functions. We specifically report from the data of three hours of teaching on the introduction to quadratic functions. Results showed that the teacher used correspondence and covariation approaches at four levels of decentering. Importantly, at decentering levels 2 and 3, the covariation approach was the dominant approach in the discourse. In this regard, a relationship was observed between the teacher's emphasis on functional approaches and the quality of interaction with students. Importantly, this study provided empirical evidence that covariational reasoning is a significant component of teachers' quality of teaching quadratic functions.

INTRODUCTION

Two conceptions of functions are emphasized in the literature: static (correspondence) and dynamic (covariation) (Oehrtman et al., 2008). Thompson and Carlson (2017) argued that "although the correspondence meaning of function solved problems that arose for mathematicians, introducing it in school mathematics made it nearly impossible for school students to see any intellectual need for it" (p. 422). However, Ellis (2011) argued that even though teachers might provide quantitatively rich tasks while teaching functions, students still might have difficulties in exploring and developing relationships between quantities. Therefore, teachers' tendencies to create a classroom environment for students to share and discuss their mathematical ideas, and also teachers' questions and their responses to students' questions, are critical to increase the quality of students' reasoning covariationally while learning functions (Ellis, 2011).

In this study, we specifically focused on one teacher's introduction to quadratic functions. There are several reasons for considering quadratic functions as the concept of interest. Quadratic functions represent the first non-linear function family students learn after linear functions. Therefore, it is important to emphasize how the nature of quadratic functions differs from that of linear functions. Importantly, Hohensee et al. (2021) reported that although there were opportunities for emphasizing covariational reasoning during the instruction on quadratic functions, there was no evidence about

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, pp. 163-170. PME.

teachers' support for such reasoning. Also, Yoon (2019) investigated the relationship between teachers' meanings of quadratic functions and their teaching practices, and also the ideas students constructed. The results suggested that teachers' having robust meanings about quadratic functions is powerful but not an adequate criterion to support students in constructing robust meanings. Therefore, examining teachers' teaching practices on functions with a focus on teacher-student interactions is an essential endeavor. This study particularly might provide an answer to the question of Baş-Ader and Carlson (2022) as "how a teacher's meaning for a specific mathematical idea influences her decentering actions" (p. 116) in the context of quadratic functions. Building on the aforementioned studies, we scrutinized the following research question: To what extent does a teacher's quality of interaction with students relate with emphasis on functional approaches to introduce quadratic functions conceptually?

CONCEPTUAL FRAMEWORK

This section focuses on two conceptual frameworks guiding the study: Functional approaches and the decentering framework. We used the decentering framework to explain how a teacher having different functional approaches interacts with her students to introduce quadratic functions.

Functional Approaches

Oehrtman et al. (2008) pointed to the indicators of the correspondence approach in someone's mathematical activity. First, a person might consider that functions are related to a rule. This also leads to considering a single value at a time as input or output by doing calculations (Oehrtman et al., 2008). Similarly, the person might tend to relate the inverse of a function to the algebraic and geometric properties. This consideration might stem from the idea that the graph provides a geometric figure of a function rather than representing a simultaneous relationship between variables (Moore, 2016).

Oehrtman et al. (2008) also explicated the indicators of someone's actions operating from the covariation approach. Specifically, considering the definition of a function, someone might see a function as a process and a dynamic relationship between input and output values. In relation to this, the person might explain the process without needing a formula. This further leads to imagining and explaining the process and relationship between input and output values without calculating single values of the function at a time. Since the person does not need to focus on discrete values, s/he can imagine how inputs and outputs change together and continuously. Additionally, the person might consider the inverse of a function as a relationship between input and output where the process is reversed. Therefore, functions are considered to be dynamic entities. Moreover, a person reasoning with the covariation approach might consider that the graph of a function represents the traces of the points, which are multiplicative objects that refer to the dynamic relationship between variables (Thompson & Carlson, 2017).

Decentering Framework

Carlson et al. (2007) proposed a five-level model to analyze teachers' interactions in a professional learning community, and recently, Baş-Ader and Carlson (2022) considered and reported on the decentering levels and elaborated on each level with teachers' behaviors and mental actions (see Table 1). In this study, we used levels of decentering to determine teachers' interactions with students in the context of quadratic functions.

Table 1. Decentering levels (adopted from Baş-Ader and Carlson [2022, p. 106])

Decentering Levels	Explanations
Level 0: No Interest	Shows no interest in the student's thinking, but shows interest in the student's answer. Makes no attempt to make sense of the student's thinking, but takes actions to get the student to say the correct answer
Level 1: Interest	Shows interest in the student's thinking, but makes no attempt to make sense of the student's thinking. Attempts to move the student to her way of thinking without trying to understand or build on the expressed thinking and perspective of the student
Level 2: Make Sense	Makes an effort to make sense of the student's thinking and perspective, but does not use this knowledge in communication
Level 3: Use	Makes sense of the student's thinking and perspective, and makes general moves to use the student's thinking when interacting with the student
Level 4: Use and Adjust	Constructs an image of the student's thinking and perspective, and then adjusts her action to take into account both the student's thinking and how the student might be interpreting her utterance

METHOD

This study is part of an extensive one-year-long multi-case study investigating how two teachers' covariational reasoning is related with their knowledge of mathematics and their teaching practices in quadratic functions. In this paper, we reported from one teacher's teaching practices and her focus on functional approaches while introducing quadratic functions. Miray was a mathematics teacher with five years of experience at an international private high school. She graduated from a mathematics teaching program at a public university and holds a master's degree in mathematics education. She was also a participant in a project about the inclusion of quantitative reasoning in a mathematics teaching methods course.

Data Collection and Analysis

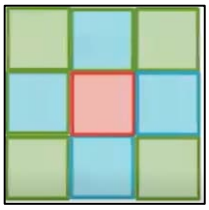
In order to investigate Miray's teaching actions and behaviors, and functional approaches during the instruction on quadratic functions, classroom observations were conducted, and video records of the instructions were obtained. Additionally, interviews were conducted before and after observations. Interviews before observations included questions about Miray's background information, experiences, and focus on content in teaching quadratic functions. Interview questions after observations included a discussion on prominent cases from her teaching. While the main data source was the video-recorded teachings, interviews were used as the supplementary data source. Class discussions on each task/example/exercise were considered as the unit of analysis to determine each excerpt for the analysis of teaching actions and behaviors according to the Decentering Framework (Baş-Ader & Carlson, 2022). Following, in each excerpt, her approaches to functions were analyzed through the indicators of correspondence and covariation approaches explained in the conceptual framework.

FINDINGS

In the first three hours of teaching, Miray focused on the rate of change and introduced quadratic relationships by implementing *The Squares from the Tiles Task*:

> A computer animation program shows a square (called the main square) that grows into larger squares by increasing the length of each side by one unit at each step of the animation. The main square is made up of smaller congruent squares that stay the same size. The main square is created by green squares (each of which has two edges on the perimeter), blue squares (each of which has one edge on the perimeter), red squares (with no edges on the perimeter). Every main square must have at least one red square in it. Write rules that determine how many green, blue, and red squares are needed to make a main square with any given side length.

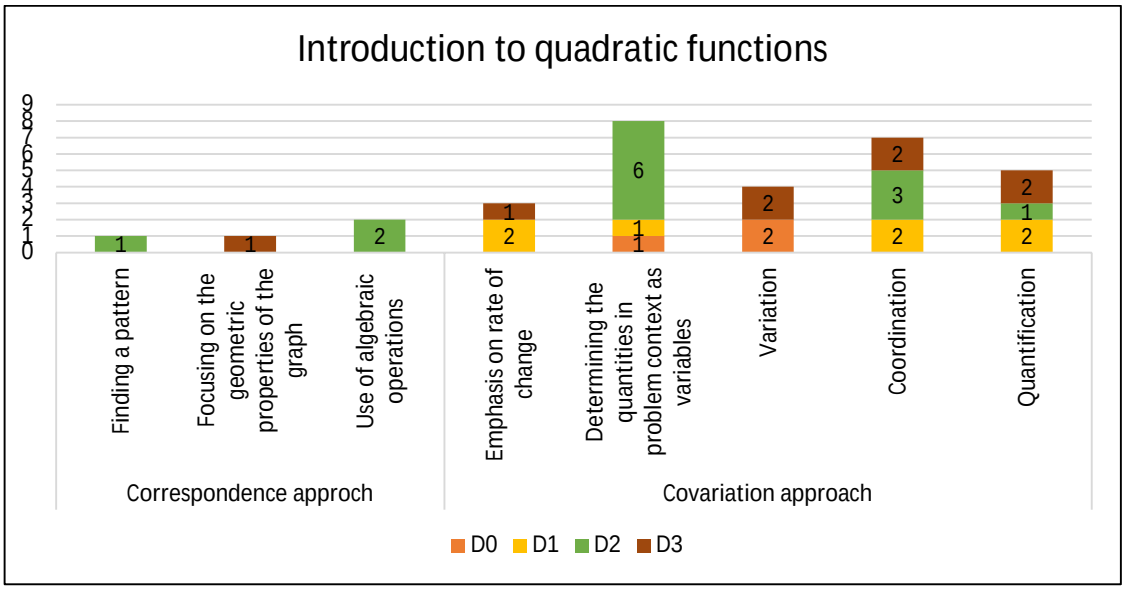
Figure 1: Initial main square for 'The Squares from the Tiles task'



She directly gave the task to students to work in small groups to determine the initial main square (see Figure 1), the main squares at each step for the changing side length and the numbers of red, blue and green squares. Then, they conducted the whole class discussion. She did not mention quadratic relationships at the beginning. The overall picture for Miray's consideration of functional approaches, along with her level of interactions with students, is shared in Figure 2. While Miray showed decentering *level 3-use*, the covariation approach was dominant, with her emphasis on the rate of change, variation, coordination, and quantification of the relationship between the quantities of interest. However, she also referred to the geometric properties of the graph during the

discussion. Similarly, while Miray interacted with students at decentering *level 2-make sense*, the dominant function approach was again the covariation approach with determining quantities in given problem contexts, variation of the values of quantities, and coordination of at least two quantities with an emphasis on the changes in the quantities additively and grossly. Rarely did she use a correspondence approach, such as finding a pattern for tabular values and using algebraic operations. While Miray interacted with students at the decentering level 1-interest, she focused on determining the quantities in the problem context, coordination and quantification of quantities. Similarly, at the decentering level 0-no interest, Miray's approach depicted determining the quantities in the problem context and variation of quantities. It is important to mention that the functional approaches occurred simultaneously as Miray and the students shifted between them easily. For this reason, while sharing sample excerpts, the observed approaches are explained together.

Figure 2: Miray's functional approaches while introducing quadratic functions



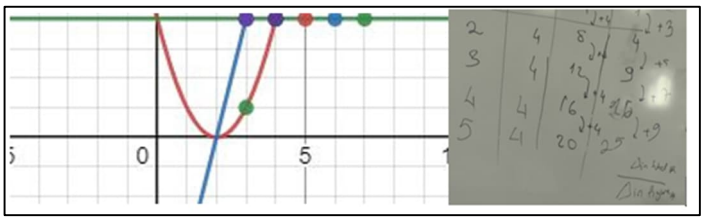
Miray importantly referred to students' thinking to enhance classroom discussion to develop specific mathematical ideas, such as the difference between linear and quadratic relationships. The following excerpt illustrates the idea that 'Functions can be characterized by their rates of change':

Miray: How would you S11 compare the parabola and the blue line?... How are they similar, and how are they different? The blue graph and the red graph?

S11: Red graph is 'steeper.' Steeper than the blue one.

Miray: If you take a look at this region (see Figure 3), it is not? How do you compare the steepness? What are you looking at exactly?

Figure 3: The graphs of constant, linear, and quadratic relationships and tabular values



S11: I am looking at the ‘slope’

Miray: Hmm. S4, do you agree with him?

S4: I mean, I do.

Miray: Eee. S3?

S3: Like the blue has the same rate of change, while the red has an increasing rate of change.

Miray: How can I show on the graph that the rate of change is increasing? (Teacher wants someone to come to the board)

S6: I can. Should I come to the board?

Miray: I mean. How you would show is up to you.

S6: We need to find the slope of these points (referring to two points on the red graph) and then compare the slopes of these points (referring to the other two points on the red graph). And then we will see that this is steeper. That means that for each figure we have, we have gone up, so the number is increased much more.

Miray: Maybe we can check (referring to table of values), like have one unit change here (change in the side length), we have a three-unit difference, here (change in the side length) we have one-unit change, we have a four-unit difference. No, it is not four. Was it three? Five. Five.

Data illustrated that Miray’s interaction with students was at decentering *level 3-use* as she operated from the student’s idea to determine what to ask further in the continuum of the discussion and gave explanations to enrich student thinking. First, she asked students to explain the nature of the difference between the graphs of the linear function (for the case of blue tiles) and the parabola (for the case of red tiles). This case represents Miray’s initiation of a discussion to advance students’ thinking. S11 considered steepness to be the property of differentiating the line and parabola, which might indicate reliance on *the geometric properties of the graph*. Then, Miray challenged students by showing a portion of the graph for which the line is steeper than the parabola to explain how they compare steepness. The students had different mathematical language, such as when S11 offered finding slope, S3 mentioned that the quadratic function graph had an increasing rate of change, while the line had a constant rate of change. Importantly, Miray continued with S3’s idea by asking how to show on the graph that the rate of change is increasing. That is, Miray used ‘rate of change’ rather than ‘slope.’ It might be possible that students were only thinking of the formula. This might be inferred from S6’s explanations for finding slopes of the points for which

S6 coordinated the step number and number of tiles and stated that at every step, the increase in the number of tiles was much greater. Therefore, in S6's explanations, there was an emphasis on directly coordinating the variables and grossly explaining how they are changing together. Miray further explained, "we can check, like have one unit change here, we have a three-unit difference, here we have one-unit change, we have a five-unit difference," was an indicator of *direct and systematic coordination and extensive quantification* since she coordinated how step number and number of tiles change additively. In the overall interaction, she aimed to advance the students' thinking by asking further questions. Specifically, while students used the word 'slope,' she continued with 'the rate of change.' In the interview, she explained her reasoning as follows:

Miray: He (S11) didn't satisfy me here in the first place. He said something steeper, something more visual... It doesn't say it increases faster. It says steeper. If there is an example like this, it is also steeper. This y equals x, and this y equals two x. The second one is steeper than the first one. Steepness is more, but it does not mean that the rate of change is different. It is a different property that makes it a parabola. I expect him to say that the rate of change is not constant.

R: You are not using the same language with students. When the student says slope, you insist on the rate of change.

Miray: We look at the instantaneous rate of change rather than the slope, in quadratic.... It seems to me that the slope will be more confusing for them and will lead to a wrong point. And they learned previously linear functions and said slope. They are trying to apply the same contextual concept to the parabola. They are looking at the change here. But actually, it is changing every moment. I don't want them to say slope when expressing it. I want them to go through the rate of change, how the variables change.

The interview data was important in the sense that it gave information about Miray's knowledge of student thinking. First of all, Miray considered that steepness gave information about a visual property of a graph. Therefore, in quadratic functions, making inferences about the changing nature of the rate of change might not be possible by only talking about the steepness of a graph. Second, Miray had several inferences and awareness of possible student misconceptions and possibilities of students' transfer of their knowledge of linear functions to quadratic functions. Specifically, Miray was aware that slope was a term to be used in linear functions illustrating the constant rate of change. Also, for Miray, in the case of a non-linear function, rather than considering the slope, using the rate of change as a term was mathematically correct.

CONCLUSION AND DISCUSSION

The overall findings indicated that the teacher's quality of interactions with students differed with a focus on different functional approaches. Specifically, Miray mostly emphasized the covariation approach; she could make sense of students' thinking and

further enrich students' ideas with additional questions based on students' answers, which was evidence for depicting decentering level 3-Use while interacting with students (Baş-Ader & Carlson, 2022). She mostly explained to or encouraged them by drawing their attention to the changes in the values of independent and dependent variables simultaneously, rather than directly giving function types. Miray's approach was an example of what Carlson and Moore (2015) suggested, that rather than giving a known function and its graph to expect students to explore how the variables are changing together, there is a need to provide students a space to create their own graph from a problem situation. In that sense, the findings provide empirical support for the relationship between embedding covariational reasoning in teaching and the quality of teacher-student interaction.

References

Baş-Ader, S., & Carlson, M. P. (2022). Decentering framework: A characterization of graduate student instructors' actions to understand and act on student thinking. *Mathematical Thinking and Learning, 24*(2), 99– 122.

Carlson, M. P., & Moore, K. C. (2015). The role of covariational reasoning in understanding and using the function concept. In E. A. Silver & P. A. Kenney (Eds.), *Lessons learned from research: Useful and useable research related to core mathematical practices* (Vol. 1) (pp. 279– 291). Reston, VA: National Council of Teachers of Mathematics.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 215– 238). New York: Springer.

Ellis, A.B., Özgür, Z., Doğan, M.F. (2023). A Conceptual Analysis of Early Function Through Quantitative Reasoning. In G. Karagöz Akar, İ.Ö. Zembat, S. Arslan, P.W. Thompson, (Eds.) *Quantitative reasoning in mathematics and science education. Mathematics education in the digital era*, (Vol 21., pp. 169 – 197). Springer, Cham.

Hohensee, C., Gartland, S., Willoughby, L., & Melville, M. (2021). Backward transfer influences from quadratic functions instruction on students' prior ways of covariational reasoning about linear functions. *The Journal of Mathematical Behavior, 61*, 100834.

Moore, K. C. (2016). Graphing as figurative and operative thought. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proc. of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 323– 330). Szeged, Hungary: PME.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics*. (Vol. 73, pp. 27–42). Mathematical Association of America.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Yoon, H. (2019). *The relationships between meanings teachers hold and meanings their students construct* [Unpublished Doctoral Dissertation]. Arizona State University.