# MATHEMATICAL PROBLEM-SOLVING KNOWLEDGE FOR TEACHING: INSIGHTS FROM A SCOPING REVIEW

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*An important goal of mathematics education is for students to become competent in solving problems. Therefore, it is essential that research clarifies what is required of teachers to appropriately integrate Problem Solving into their mathematics teaching. For this reason, we conducted a Scoping Review to provide a broad and well-founded overview of research on teaching Problem Solving, resulting in the identification of 11 research foci. In this paper, we focus primarily on one of these foci, namely Teachers' Knowledge. By using connections to further research foci, we show how the overview allows us to complement and expand existing concepts developed by research. A key insight is that knowledge of classroom environments for Problem Solving is a category that could complement existing knowledge frameworks for teaching Problem Solving.*

## INTRODUCTION

If mathematical problems are defined as tasks for which individuals do not have readily available strategies for solving (e.g., Lester, 2013), then focussing on Problem Solving (PS) suits the theme of PME 49—*sensing and making sense of mathematics*—very well. PS greatly requires sense-making of mathematics, because problem solvers must be able to mobilise and apply their mathematical skills flexibly. There is a consensus in research that promoting PS skills should be a key objective in mathematics education. For students, however, gaining experience and acquiring skills in PS is not only relevant to achieving success in learning mathematics, but also to coping with future real-life challenges. In this respect, it is of great importance for teachers to foster their students' PS proficiency. Teachers play a crucial role in enabling students to gain PS experiences (Chapman, 2017; Lester & Cai, 2016).

For teachers to meet these requirements, it is necessary to clarify what teachers should be capable of in order to optimally integrate PS into their mathematics teaching. This is particularly important for teacher education, whether at the university level or for in-service teacher trainings. In this regard, it is advantageous that there is a long tradition of research on PS, which has yielded comprehensive insights into PS teaching. Although there are studies that examine developments (e.g., Lester & Cai, 2016) or the current state (e.g., Liljedahl & Cai, 2021) of research on (teaching) PS, no broad overview based on a systematically developed body of literature is available to date.

Consequently, this results in our research objective. Based on a comprehensive and systematically identified body of research literature on PS teaching, we strive to provide an overview of research on PS teaching by elaborating on key research foci.



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This approach may lead to implications for research in that it highlights important research findings that have hardly been considered for PS teaching in the past. Furthermore, the detailed overview we aim to generate could have implications for teacher education, e.g. by providing impulses for in-depth explorations with (prospective) teachers that target specific aspects of PS teaching. After presenting the research foci, the focus of this paper is on the knowledge teachers should possess for PS teaching. We illustrate the extent to which the above-described approach enriches research, particularly conceptual considerations research on PS teaching has generated.

## METHOD

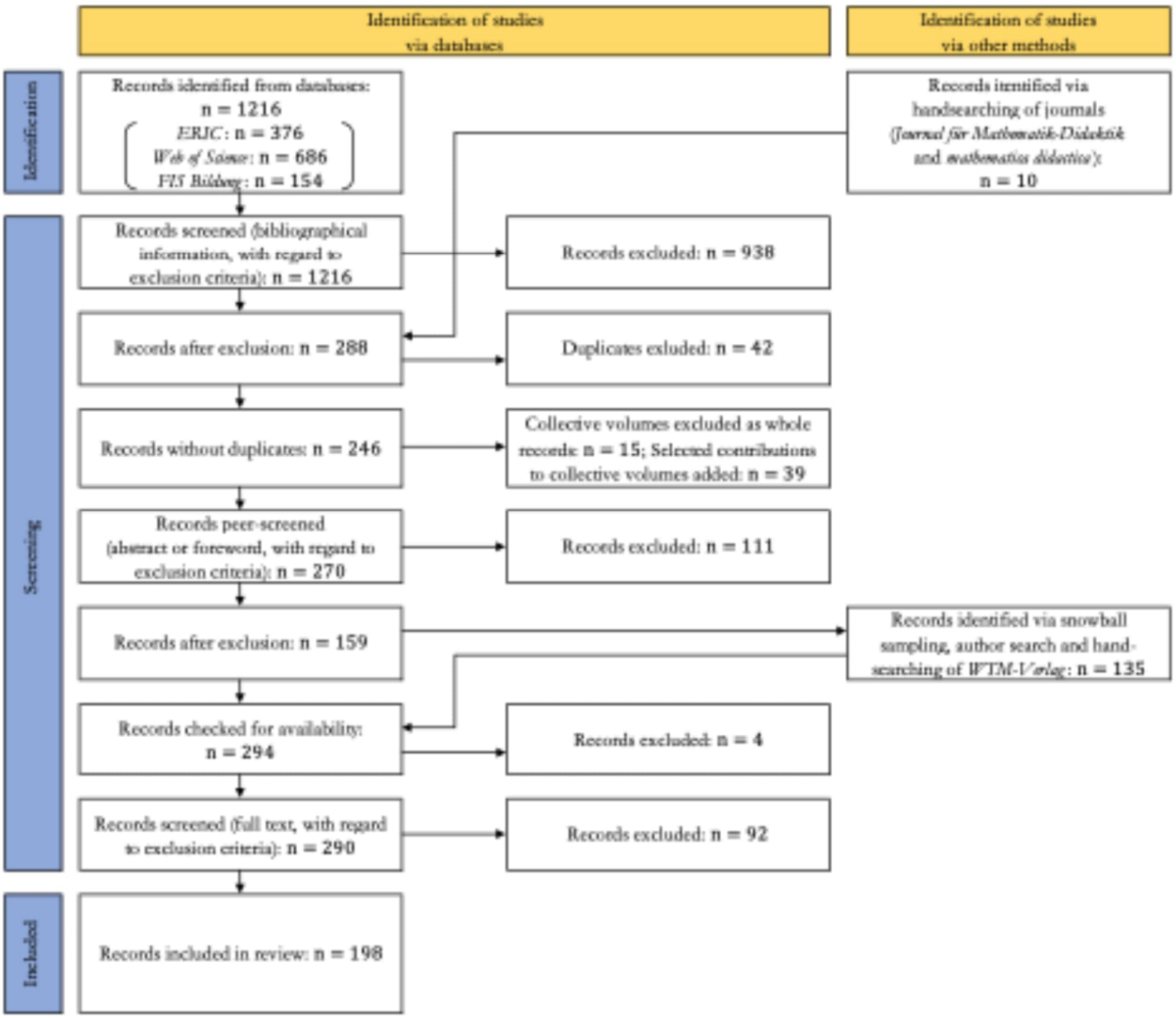
To generate broad yet well-founded insights into research on PS teaching, a systematic approach is required, including systematic database searches and the formulation of clearly defined inclusion and exclusion criteria. Moreover, the identification of suitable publications should not be limited to peer-reviewed articles. We assume that to grasp the wide range of research foci and perspectives, various publication formats such as monographs, contributions to collective volumes, practitioners' journals, and even grey literature should be considered. In addition to the systematic search, further methods for identifying publications should be used, e.g., hand-searching journals or snowball sampling. Finally, our approach should allow for adjustments to the process during implementation. If, for example, a publishing house appears to have a large number of suitable publications, we may take a closer look at the entire contributions to PS research published there. Taken together, the systematic literature search with the above-mentioned additions, the features of our approach described above, and the objective of providing an overview of research on PS teaching, constitute what is known as a *Scoping Review*. Our procedure for identifying literature is based on the PRISMA extension for Scoping Reviews (PRISMA-ScR; Tricco et al., 2018).

For the literature search, we did not set any restrictions on publication dates. However, restrictions apply to the language in which they are written. In addition to publications written in English, we decided to include publications written in German. The reason for this is the long-standing tradition of German research on PS teaching, which has produced findings that may enrich the research field.

The process of identifying publications began with database searches in *ERIC*, *Web of Science* and *FIS Bildung*, yielding a total of 1216 publications. Because of the vast number of articles, there was a need for exclusion criteria (Ex): Publications should be excluded if they are (Ex1) not exclusively concerned with mathematics education, (Ex2) not primarily focussed on PS (e.g., dealing with *word problems*), (Ex3) not dealing with either teachers or students at schools, (Ex4) not referring to PS concepts, but rather focus on how specific content can be implemented in a PS context, (Ex5) addressing specific learning environments, or (Ex6) dealing mainly with curricula or textbooks.

The application of these criteria resulted in the exclusion of 938 publications. However, the number of publications could also increase during the screening and identification process shown in Figure 1. The following explanations illustrate this: After the initial exclusion described above, as well as the addition of publications identified via hand-searching of journals and the exclusion of duplicates, 246 publications remained. These publications contained 15 collective volumes, which were removed. Simultaneously, we identified 39 contributions to these volumes that align with our interests. Combined, this increased the number of publications to 270 at this stage of the process. The subsequent screening reduced the number of publications to 159. These 159 publications were then used to identify further publications using additional methods (e.g., snowball sampling), which increased the total number to 294.

Figure 1: Review Flow Diagram based on PRISMA-ScR




After the entire process of identifying and screening publications, which included a final full-text screening, 198 publications remained. These publications were analysed to work towards our objective. For this purpose, while reading these publications, text passages were identified that provide insight into findings of research on PS teaching,

respectively into what should be considered when teaching PS. These passages were then assigned to inductively generated research foci (see Results). Two points are important to note here: Firstly, our knowledge of the research field has certainly influenced the generation of the research foci. We were aware that, for example, guiding students during PS processes is widely discussed in research on PS. Secondly, passages could be assigned to several research foci at once. For example, one of the seven teaching principles Lester (2013) formulated based on 40 years of research is the *multiple roles for the teacher principle*. He summarised this as follows:

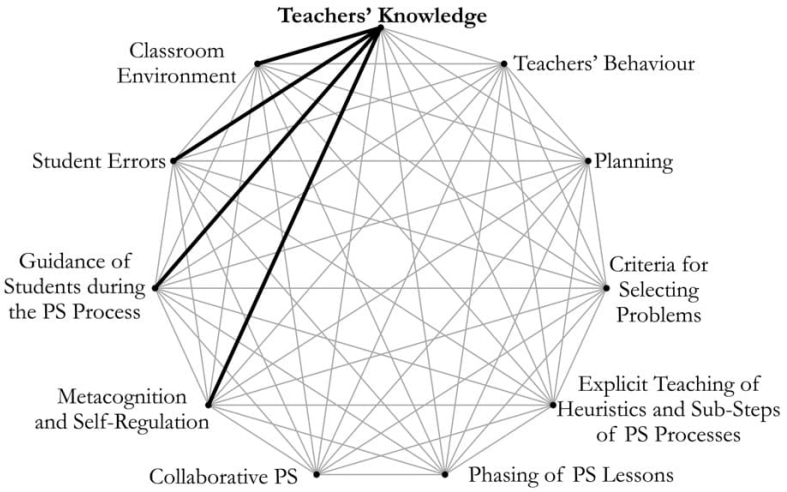
> Problem-solving instruction that emphasizes the development of metacognitive skills should involve the teacher in three different, but related, roles: (a) as an external monitor, (b) as a facilitator of students' metacognitive awareness, and (c) as a model of a metacognitively-adept problem solver (Lester, 2013, pp. 272–273).

In this passage, two research foci were identified: *Guidance of Students during the PS process* and *Metacognition and Self-Regulation*. This example shows why we opted against trying to construct a system of distinct research foci, since they are intertwined, if not interdependent. It is the connection between research foci that creates the depth of detail in the overview. Because research foci could also be further subdivided, this means that there can be no 'correct' number of them to be identified. Therefore, it should not be the objective to find a verified and exclusive system of research foci. Besides, achieving the objective of generating an overview of research on PS does not depend on such a system. It is rather important that the overview is extensible insofar as new research developments can be integrated whenever necessary.

## RESULTS

We identified 11 research foci in the literature on PS teaching. These foci are not presented hierarchically. However, they are interconnected (as illustrated in Figure 2) and may also depend on each other. An expert consultation within our working group indicated the validity of our approach and the results.

Figure 2: Research foci identified in the literature on PS teaching



Since a full discussion of all research foci exceeds the scope of a conference contribution, the following section primarily addresses *Teachers' Knowledge*, together with links to four other research foci (as depicted by dark lines in Figure 2).

## DEEPENING THE RESULTS: A FOCUS ON TEACHERS' KNOWLEDGE

Under the research focus *Teachers' Knowledge*, we collected considerations regarding the specific knowledge that teachers should possess for PS teaching. We identified Chapman's (2015) *mathematical problem-solving knowledge for teaching* (MPSKT) framework as central, as it is one of the most frequently referred-to and recent concepts. It consists of *PS content knowledge* (PSCK), *pedagogical PS knowledge* (PPSK) and knowledge of *affective factors and beliefs*. She emphasised that it is not just the knowledge categories themselves, but also the links between them that contribute to MPSKT as well. The framework appears to be "a complex network of interdependent knowledge" (p. 32).

In contrast to previously published knowledge frameworks for mathematics teaching in general, MPSKT specifically refers to PS. Such specification is crucial because relevant facets of PS may remain underrepresented in more general frameworks. For example, mathematical processes such as representing or communicating are of great importance for PS. In this context, Foster et al. (2014) noted that established frameworks such as the MKT (*mathematical knowledge for teaching*) framework by Hill et al. (2008)—consisting of *subject matter knowledge* and *pedagogical content knowledge*—omit the integration of mathematical and pedagogical process knowledge. Chapman (2015), however, addresses PS process perspectives. For instance, in the section on *knowledge of PS*, a component of PSCK (like *knowledge of problems* and *knowledge of problem posing*), she asserts that teachers must possess PS concept and process knowledge "in order to understand […] the thinking involved in the process of reaching a solution" (p. 26).

To identify facets of MPSKT, Chapman (2015) drew on a sample of studies published between 1922 and 2013. However, it was less her goal to base the framework on an "exhaustive review" (p. 20), but rather to provide a brief overview of each knowledge category. This approach entails that there may be some knowledge facets which are either underemphasised or not integrated in MPSKT. If this were the case, then it would also apply to follow-up research building on the framework.

It is at this point that our review yields new insights. Overall, it became evident that MPSKT already covers many research achievements that we have also identified as significant within the research foci. However, our insights into the research literature suggest that in the MPSKT, some knowledge categories lack details that would help to understand more precisely how knowledge can be applied. Moreover, we found that some research foci were not given sufficient consideration by Chapman (2015) but should be considered for identifying knowledge facets that complement the MPSKT

framework. Below, we demonstrate this by connecting *Teachers' Knowledge* with other research foci.

The research focus *Classroom Environment* emphasises primarily the question, which classroom conditions should prevail to optimally engage students in PS processes. In this regard, our literature analyses reveal two important questions to consider: First, how are interactions between students and with the teacher regulated? Second, what characterises the emotional climate in the classroom? In the MPSKT framework, knowledge about these two questions is given little consideration. To approach this, we initially take a look at interactions during PS processes. As PS in mathematics classrooms can be understood as a sociocultural process, in which individuals interact and work together toward a goal (Koichu, 2019), norms that regulate these interactions are essential. In this regard, Stephan and Whitenack (2005) consider social and sociomathematical norms that students should be involved in negotiating. In short, social norms include agreements about how to deal with other students' ideas at a communication level, and sociomathematical norms contain shared criteria for classifying these ideas mathematically. In PS processes, for example, it is important that students' ideas are dealt with respectfully, even if they are considered ineffective or erroneous. Furthermore, some researchers noted that errors in PS processes have inherent learning potential (e.g., McIntosh & Jarrett, 2000). Consequently, erroneous approaches or ineffective ideas may play a significant role in PS processes. On this basis, two conclusions can be drawn. First, students and teachers must agree on the learning potential of errors. Second, it is important that students find themselves in an environment where they do not fear negative reactions or harassment, which is where the connection to emotional atmosphere becomes apparent. For teachers, this means ensuring that shared agreements are adhered to and acting in accordance with them. Laine (2019) concludes that teachers play "a central role in constructing the emotional atmosphere" (p. 15). In summary, *knowledge of classroom environment for PS* is a category that could be integrated into frameworks of teacher knowledge for PS teaching.

At this point, further connections can be discussed, on the one hand, to the research focus *Student Errors*. Especially when the learning potential of erroneous approaches is to be utilised, teachers should, as Chapman (2015) demanded, possess knowledge of students' difficulties and should be able to interpret them from students' perspectives, as well as to support them on this basis. With regard to our analysis of the research literature, we suggest extending this knowledge facet. Teachers should also know what kind of errors in PS should be given particular attention to in PS processes. Although errors in PS are not uniformly defined in the research literature, some researchers emphasise *strategic errors* (i.e., students using inappropriate heuristic strategies) in particular. For example, research by Heinrich (2004) has shown that students usually do not recognise strategic errors themselves. Moreover, none of the PS processes he observed proceeded without the students replacing their initially chosen solution strategy. This is precisely why teachers should support their students when such errors

occur. Besides, teachers should pay particular attention to situations in which students aim to change their solution strategy.

On the other hand, connections to the research foci *Guidance of Students during the PS Process* and *Metacognition and Self-Regulation* emerge. If students get the opportunity to pursue inappropriate ideas or, more specifically, make strategic errors, they may show emotions such as anger or frustration. Building on the knowledge of *affective factors and beliefs* presented by Chapman (2015), we suggest that teachers should familiarise their students with strategies for self-regulating emotions. For example, a study by Tzohar-Rozen and Kramarski (2017) shows that students who received guidance for meta-affective self-regulation performed similarly to those who received support for metacognition. Their implication is that teachers should decide what guidance on self-regulation to provide based on each student's individual learning style. In this regard, we suggest supplementing the *knowledge of instructional practices for PS,* which is a component of PPSK (as is *knowledge of students as problem solvers*). Teachers should not only be familiar with instructional practices for metacognition (Chapman, 2015) but also should be aware of how to support students' meta-affective self-regulation.

## CONCLUSION

Our research aims to provide an overview of PS teaching research. Using a systematic approach, we identified 198 publications and elaborated 11 research foci based on them (see Figure 2). Since this is a large number of publications covering different types of research literature, the overview provides a deep insight into the field. Furthermore, the level of detail enables us to complement and expand upon existing concepts developed by researchers. In this paper, we illustrate this using Chapman's (2015) MPSKT framework as an example. Connecting the research focus *Teachers' Knowledge* with other foci led us, on the one hand, to suggest extensions to knowledge categories. On the other hand, considering the research focus *Classroom Environment* indicates that *knowledge of classroom environment for PS* is a category that could be incorporated into knowledge frameworks for PS teaching, especially MPSKT.

These insights demonstrate that our approach can enrich research on PS teaching. In other words, it allows us to draw broader pictures of existing concepts. However, such insights could remain theoretical in nature. Therefore, future research should utilise the richness of details inherent in the elaborated research foci for empirical explorations. For example, a comprehensive tool for class observations or for analysing lesson videos could be developed. This would then allow for a more explicit view of PS teaching, e.g., by enabling more accurate interpretations of teacher behaviour. The results of such analyses could, in turn, be significant for teacher education insofar as they provide impulses for teacher training.

## References

Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT, 3*(1), 19–36. https://doi.org/10.31129/lumat.v3i1.1049

Chapman, O. (2017). Mathematics Teachers' Ways of Supporting Students' Learning of Problem Solving. In M. Stein (Ed.), *A Life's Time for Mathematics Education and Problem Solving* (pp. 45–69). WTM. https://doi.org/10.37626/GA9783959870641.0.04

Foster, C., Wake, G. & Swan, M. (2014). Mathematical Knowledge for Teaching Problem Solving: Lessons from Lesson Study. In P. Liljedahl, C. Nicol, S. Oesterle & D. Allan (Eds.), *Proc. of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 3, pp. 97–104). PME.

Heinrich, F. (2004). *Strategische Flexibilität beim Lösen mathematischer Probleme. Theoretische Analysen und empirische Erkundungen über das Wechseln von Lösungsanläufen* [Strategic flexibility when solving mathematics problems. Theoretical analyses and empirical explorations of switching solution approaches]. Dr. Kovač.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400. https://doi.org/10.5951/jresematheduc.39.4.0372

Koichu, B. (2019). A Discursively Oriented Conceptualization of Mathematical Problem Solving. In P. Felmer, P. Liljedahl & B. Koichu (Eds.), *Problem Solving in Mathematics Instruction and Teacher Professional Development* (pp. 43–66). Springer.

Laine, A. (2019). How to promote learning in problem-solving? In A. Kuzle, I. Gebel & B. Rott (Eds.), *Implementation Research on Problem Solving in School Settings* (pp. 3–18). WTM. https://doi.org/10.37626/GA9783959871167.0.01

Lester, F. K., & Cai, J. (2016). Can Mathematical Problem Solving Be Taught? Preliminary Answers from 30 Years of Research. In P. Felmer, E. Pehkonen & J. Kilpatrick (Eds.), *Posing and Solving Mathematical Problems* (pp. 117–135). Springer.

Lester, F. K. (2013). Thoughts About Research On Mathematical Problem-Solving Instruction. *The Mathematics Enthusiast, 10*(1), 245–278.

Liljedahl, P., & Cai, J. (2021). Empirical research on problem solving and problem posing: a look at the state of the art. *ZDM, 53*(4), 723–735.

McIntosh, R., & Jarrett, J. (2000). *Teaching mathematical problem solving: Implementing the vision. A Literature Review*. NWREL.

Stephan, M., & Whitenack, J. (2005). Establishing Classroom Social and Sociomathematical Norms for Problem Solving. In F. K. Lester (Ed.), *Teaching mathematics through problem solving: Prekindergarten – Grade 6* (pp. 149–162). NCTM.

Tricco, A. C. et al. (2018). PRISMA Extension for Scoping Reviews (PRISMA-ScR): Checklist and Explanation. *Annals of Internal Medicine, 169*(7), 467–473.

Tzohar-Rozen, M., & Kramarski, B. (2017). Metacognition and Meta-Affect in Young Students: Does it Make a Difference in Mathematical Problem Solving? *Teacher College Record, 119*(13), Article 130314. https://doi.org/10.1177/016146811711901308