CONCEPTUAL DIVERSITY OF LOGARITHMS: RESEARCH MATHEMATICIANS' VIEWS ON RELATIONS BETWEEN CONCEPT DEFINITIONS AND INTERPRETATIONS

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The concept of logarithms can be introduced through multiple mathematically equivalent yet conceptually different definitions, which may foreground different properties of the concept. This study investigates, using interviews, how research mathematicians relate such conceptualisations of logarithms. The result is a two-level model of meaning for the concept, distinguishing formal definition variants from facets of meaning and clustering definitions according to the mathematics to which they refer. The proposed model contributes a structure for the conceptual diversity of logarithms and offers a theoretical tool for research on the teaching and learning of logarithms.

INTRODUCTION

Few mathematical advancements have influenced science as profoundly as logarithms. Introduced in the 17th century, simplifying astronomical calculations, logarithms arose from the relation between arithmetic and geometric sequences by which multiplication in the latter sequence corresponds to addition in the former. Although rooted in basic arithmetic, logarithms were later formalised as inverses of exponentials and defined using limits or integrals (Ricardo, 2022), approaches that appear distant from the original one. Difficult to represent concretely, the meaning of logarithms is conveyed through abstract definitions, interpretations, numerical examples, and applications.

As mathematical concepts are interconnected, different definitions and interpretations of logarithms may activate different forms of prior knowledge among learners and foreground different conceptual properties. Lampinen and McClelland (2018) showed that different presentations of a new concept in abstract algebra supported different types of understanding. Furthermore, using multiple concept interpretations and clearly articulating their connections was shown to support understanding for some learners (Lampinen & McClelland, 2018). Understanding how presentations foreground different properties is vital in order to design instruction that supports conceptual understanding. In my PhD project, I seek to provide teachers with tools to introduce logarithms in ways that connect to relevant prior and subsequent mathematical knowledge. While the full project also draws on literature, textbooks, and workshops, the present paper focuses on research mathematicians' views on logarithms. The aim is to propose a model of meaning for the concept of logarithms based on mathematicians' associations elicited by different definitions and interpretations. The

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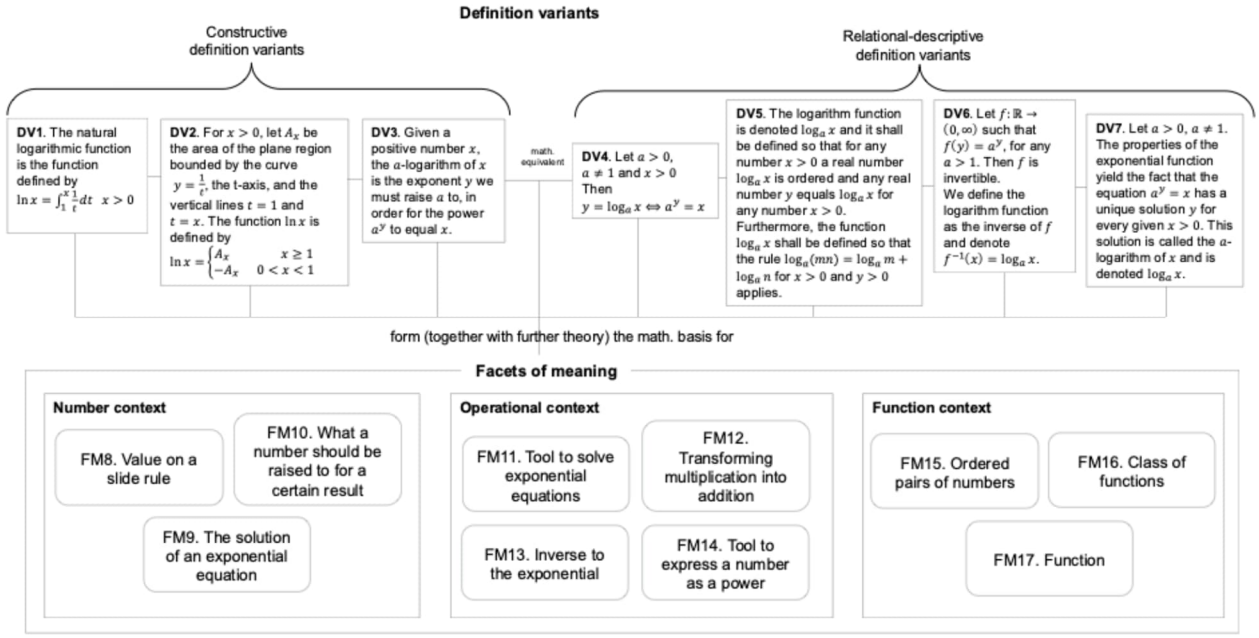
guiding research question is: *Which relations do research mathematicians suggest between different definitions and interpretations of the concept of logarithms?*

BACKGROUND

Previous research has identified multiple ways of defining and interpreting logarithms. Ricardo (2022) outlined four mathematically equivalent definitions of the natural logarithm, using the concepts of inverse, limit, integral, and a functional equation. Weber (2016) presents four 'basic models' of logarithms, including inverse exponents and multiplicative reasoning. He suggests that the inverse exponent definition indicates how to use logarithms to solve exponential equations (Weber, 2016), which provides an example of a proposed relation between conceptualisations of logarithms emphasising different but related properties. In contrast, Berezovski and Zazkis (2006) distinguished between interpreting logarithms as numbers, operations, or functions. The present study seeks to systematise definitions and interpretations, collectively called conceptualisations, through experts' meaning-making and evoked associations.

Subject matter analysis is a methodological approach to prepare mathematical content for instruction by preserving rigour while enhancing learner accessibility (Hußmann et al., 2016). This study extends work by Julsgård (accepted) which analysed logarithm introductions in university textbooks and, informed by Lankeit and Biehler (2024), proposed an initial model of meaning (Figure 1). This epistemological model includes mathematically equivalent yet conceptually diverse definition variants (DVs) of logarithms and facets of meaning (FMs) semantically highlighting specific properties.

Figure 1. Model of meaning for the concept of logarithms, from Julsgård (accepted).



According to the framework of Lankeit and Biehler (2024), DVs are conceptually distinct as they evoke different associations. This study explores such associations and their links to the concept's FMs through interviews with research mathematicians.

METHOD

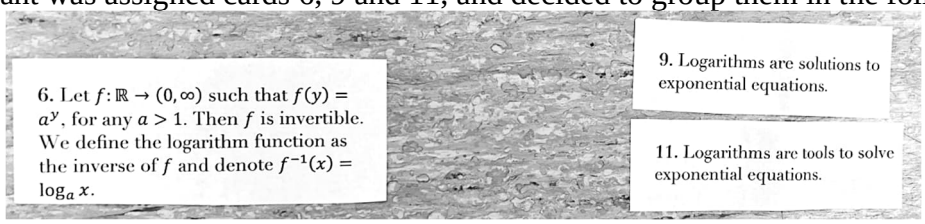
Invitations were sent to 81 PhD-qualified mathematicians at two Swedish universities. Twenty-one participated, of which 17 form the current sample. The participants represented a range of research areas, including pure mathematics (algebra, analysis, geometry) and applied fields (optimisation, statistics). The study followed ethical guidelines of the Swedish Research Council (2024) and all names are pseudonyms.

Data collection

Semi-structured interviews were conducted, allowing questions to be adapted to participants' experience, interests, and talkativeness. Prior to data collection, pilot interviews ($n = 7$) with PhD students in mathematical sciences led to reformulation, reordering, and addition of questions. The pilots also confirmed the value of including participants from diverse research areas, eliciting richer collective responses. The interviews were conducted individually and in person at the participants' workplaces, in Swedish or English, lasted 30–60 minutes, and were audio-recorded with occasional photographs of whiteboard work. The current study covers one section of the interview, aimed at facilitating reasoning about relations between logarithm conceptualisations. Inspired by the repertory grid technique (RGT, cf. Rozenszajn et al., 2021), the 17 DVs and FMs identified in the textbook analysis (Julsgård, accepted) were presented on separate cards. The participants partitioned randomly selected trios of cards into groups of 1 and 2 by identifying shared and distinguishing features, justifying each choice (example in Figure 2). This process was repeated on average five times per interview.

Figure 2. Example of interview procedure and direct coding process.

The participant was assigned cards 6, 9 and 11, and decided to group them in the following way.



The audio recording captured the following conversation.

Ove: Then I'll do this way: I'll put number six on its own and numbers nine and eleven on their own.

Interviewer: Why?

Ove: Well, because these are informal statements, nine and eleven, they're not really mathematically rigorous. The difference is quite significant between these and statement number six here, which is a more mathematically precise statement.

Direct coding produced the following table.

Pole A	Card(s)	Pole B	Card(s)	
Precise	6	Imprecise	9	11

Data analysis

To begin with, I conducted selected transcription and direct coding (cf. Point & Baruch, 2023). Using audio and visual data, I identified groupings made by the participants and labelled two poles representing the 'extremes' by which the partition was justified (see Figure 2). I also included relevant quotes to contextualise participants' justifications.

After familiarising myself with the data through selected transcription and direct coding, I constructed categories for types of partition justifications (Table 1). The mathematical content category reflects a higher order than the others, as it included various "this versus that" justifications (e.g., refers to functions versus refers to equations, or refers to the exponential function versus does not refer to the exponential function). Each partition was categorised, and in a few cases its two poles were coded to different categories. The first two categories address the content of the cards, while the remaining three mainly address the syntax.

Table 1. Categories and number of partitions encoded to each category.

Category			Total 96
Mathematical content			54
Specific to logarithms	vs	Not specific to logarithms	14
Precise	vs	Imprecise	6
Definition	vs	Not definition	20
Symbolic language	vs	Natural language	2

Next, I examined relations among the cards in two complementary ways. First, within the category mathematical content, I identified pairs that participants described as expressing mathematically equivalent statements. The pairs and their interconnection as well as lack of interconnection to other pairs were used to ascertain how conceptualisations could be meaningfully clustered in the resulting model.

Second, for the remaining categories, I analysed how individual cards were positioned across participants' partitions. This phase of analysis is analogous to the rating step in RGT (Rozenszajn et al., 2021), however aggregating data from all participants. Rather than focusing on single comparisons, this analysis identified systematic tendencies in how cards were characterised, for example as definitions rather than non-definitions, or as logarithm-specific rather than not specific. For each category, I examined whether particular cards were consistently associated with one pole or the other across interviews. This made it possible to distinguish cards that participants treated as formally defining the concept from those understood as a more informal description of properties. The resulting pattern informed the horizontal as well as the vertical arrangement of conceptualisations in the resulting model, presented in the next section.

RESULTS

Figure 3 presents the resulting model of meaning for the concept of logarithms, developed by integrating a prior textbook analysis (Julsgård, accepted) with research mathematicians' views on relations between different conceptualisations of logarithms. Vertically, the model distinguishes between a formal level consisting of definition variants (DVs) and a semantic level comprising facets of meaning (FMs). The cards belonging to the formal level (DVs) were described by the participants as precise and including a high density of mathematical symbols. The remaining were described as containing fewer mathematical symbols, to a greater extent imprecisely expressed, and not necessarily describing features specific to logarithms. For instance, the participant Lev argued that FM16 is not specific to logarithms because "sure it is good information, but the concept can still be anything". Gabriele similarly justified FM17 by noting that "a lot of things are functions". Rather than defining the concept, the FMs were understood as describing properties or applications of logarithms.

Horizontally, the DVs and FMs are clustered based on the participants' pairings and partitions of cards with associated reasonings. The DVs were commonly partitioned based on whether or not the definition referred to exponentials. DV3, DV4, DV6 and DV7 are based on the existence of exponential functions or exponential equations, while DV1, DV2 and DV5 introduce logarithms without mentioning exponentials. Additionally, the DVs not referring to exponentials are clustered into geometrical ones, defining the natural logarithm through area and integral, versus DV5 which uses the logarithmic property of converting products into sums to define the concept.

Dashed lines represent suggested interrelations between DVs and FMs. The analysis of pairings within the category mathematical content showed clearly that DV3, DV4, DV6 and DV7 were associated with FM9, FM10, FM11, FM13 and FM14. Additionally, DV1 and DV2 were associated with each other but not with any FM, and similarly for FM15, FM16 and FM17. Instances from participants' reasoning can further describe these clusters. Hani had a hard time partitioning DV1, DV7 and FM12, as he claimed them all to be completely different from each other. See below.

> If you see one [card], and then perhaps you don't really know the other, then it doesn't feel like it's super clear with the connection between, for example, this 1 over t [DV1] and this a to the power of y equals x [DV7], and even their connections to these calculation rules [FM12] are, well, it's not super clear.

Eddie concluded that there are essentially three clusters of conceptualisations, namely one expressing the geometrical definition (DV1, DV2), one introducing the logarithm as a function with certain structure-preserving properties (DV5, FM12), and a third speaking of the inverse relation to exponential functions (DV6, FM9 and others). Mattias arranged all the 17 conceptualisations into three branches: one for referrals to exponentials, one for the converting of products into sums, and one for the geometrical definitions. He called the branches "specifications in different directions". In this

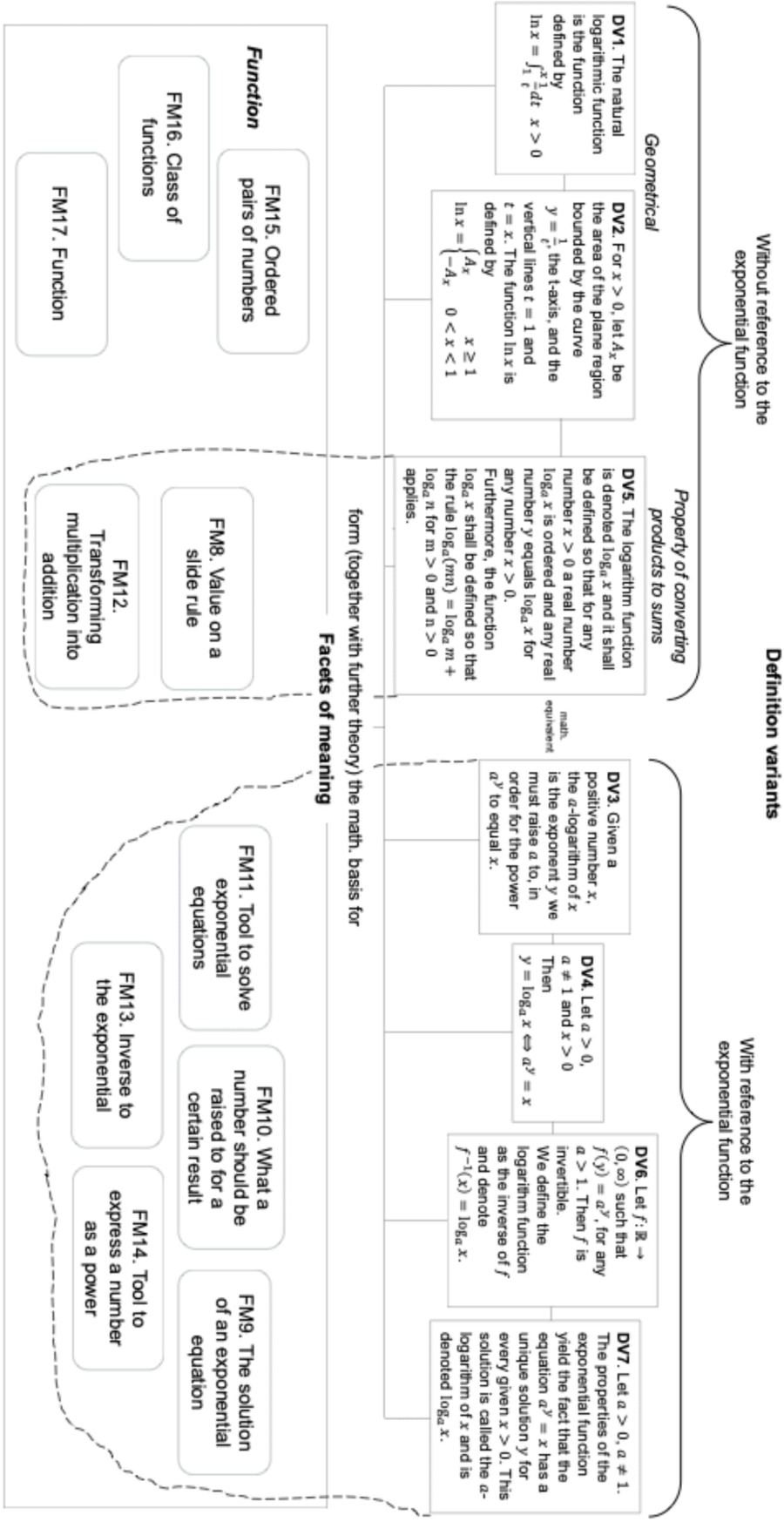


Figure 3. Model of meaning for the concept of logarithms, with definition variants and facets of meaning clustered according to associations regarding the mathematical content

arrangement, he put FM15–17 as the root of the three branches, reflecting true statements about logarithms without specifying the concept. Apart from this, FM15–17 were rarely associated with the DVs concerning their mathematical content.

DISCUSSION

This study examined the conceptual diversity of logarithms through research mathematicians' views on relations between different definitions and interpretations. Based on the relations they suggested, a model of meaning for the concept of logarithms was proposed (Figure 3). The paper's research question is answered through the relations represented in the vertical and horizontal dimensions of the model. Vertically, there is a distinction between DVs, sharing characteristics being formal and rigorous, and FMs describing properties and applications in less formal language. Horizontally, the DVs form three main groups, some with associated FMs, reflecting relations collectively suggested by the participating mathematicians.

The findings align with Weber's (2016) acknowledgement that the inverse exponent definition can be associated with the solving of exponential equations. In contrast, the interpretative frameworks proposed by Berezovski and Zazkis (2006) are not clearly reflected in the resulting model. Function-related FMs are rarely associated with DVs, while operational and number-based interpretations are distributed across clusters. The constructive versus relational-descriptive distinction of DVs in the prior textbook study (Figure 1) was not supported by interview data and was therefore omitted. However, the discrimination of cards 1–7 as DVs and cards 8–17 as FMs was reinforced by the participants' partitions of cards along the two poles definition and not definition. Additionally, this study adds characterisations of the levels based on the participants' justifications. As the interviews drew on definitions and interpretations from the textbook study, the current analysis does not address whether mathematicians would propose additional such. Further analysis of data is needed to extend the model.

The main contribution of this paper is a model of meaning for logarithms that can inform future research, teaching, and professional development. Similar to how fraction learning researchers make use of the idea that the concept of fractions comprises multiple sub-constructs (e.g., part-whole and ratio, cf. Charalambous & Pitta-Pantazi, 2007), research on the teaching and learning of logarithms may benefit from a structured conceptual framework. Empirically, the study contributes research mathematicians' perspectives on relations between logarithm definitions and interpretations, extending both individual (Weber, 2016) and theory-based (Berezovski & Zazkis, 2006) accounts. Regarding implications for teaching, the clusters of DVs and FMs in the proposed model indicate that different definitions foreground different properties of logarithms. Whether different DVs also support different types of understanding for learners, as found by Lampinen and McClelland (2018) in abstract algebra, remains an open question. The model organises a conceptual diversity that should be considered when teaching logarithms and in studying learners' conceptual and procedural development.

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