SECONDARY TEACHERS' INTERPRETATIONS OF STUDENTS' GEOMETRIC ERRORS LINKED TO SPATIAL REASONING

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This study investigates how secondary school mathematics teachers identify and interpret students' errors in a geometry item, analysed through the theoretical lens of spatial reasoning. The item was first administered to a sample of 10th-grade students to examine whether students' errors were framed by spatial reasoning. Subsequently, a questionnaire was administered to 19 in-service secondary school teachers to collect information on their interpretations. Results indicate that while some teachers correctly identify spatial errors, those who do not adopt a spatial reasoning perspective tend to have greater difficulty in recognizing and interpreting students' errors. These findings highlight the need to strengthen teachers' hermeneutical competence through targeted professional development programs.

INTRODUCTION

Spatial reasoning has long been a subject of research, particularly due to its close connection with mathematics education (Bishop, 1980). Rather than a single, universally accepted definition, spatial reasoning can be understood as the set of skills required to mentally represent, manipulate, and to transition between two-dimensional and three-dimensional representations (Hydaiat et al., 2023). Promoting the development of these skills from an early age is crucial (Sinclair & Bruce, 2014), as they constitute a fundamental component of mathematical learning and are closely related to students' mathematics achievement (Hawes et al., 2015). Accordingly, various educational interventions have been studied and implemented with the aim of enhancing students' spatial reasoning to improve their mathematics achievement (Lowrie et al., 2017). At the same time, less is known about teachers' knowledge and perceptions in this area, particularly regarding their ability to recognize and interpret classroom situations in which errors related to spatial reasoning occur. In this regard, literature suggests that teachers often encounter challenges in identifying and interpreting the underlying causes of students' errors (Cibien, 2025). It is precisely in this context that the present study is situated: exploring if and how secondary school mathematics teachers identify and interpret students' geometric errors, which can be analysed through the theoretical lens of spatial reasoning. This study is part of the broader national project *Mathematics Standardized Assessment as a Tool for Teachers' Professional Development – MaSt*, aimed at investigating mathematics teachers' knowledge in relation to national large-scale assessments (LSAs), with particular focus to the interpretation of emerged errors, in order to develop an innovative model for

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teachers' professional development (Viola et al., 2024). This research aims not only to contribute to the *MaSt* project but also to strengthen the connection between research on spatial reasoning and teacher education, thereby supporting professional development and enhancing teachers' competencies in this domain (Burte et al., 2020).

THEORETICAL BACKGROUND

Spatial reasoning is a multidimensional construct encompassing a range of interconnected cognitive skills that are fundamental for mathematical learning and, more broadly, for STEM disciplines (Lowrie et al., 2020). Among these, *spatial visualization* refers to the ability to mentally manipulate complex objects and spatial configurations, requiring the management of multiple cognitive steps and the execution of sequences of mental transformations. *Mental rotation* involves the capacity to imagine the rotation of two- or three-dimensional figures, enabling the recognition of a figure in a different position following rotation. *Spatial orientation* pertains to the ability to adopt perspectives different from one's own, anticipating and mentally representing the position of objects in space. Finally, *spatial structuring* entails the mental decomposition of complex objects and the understanding of the relationships among their components.

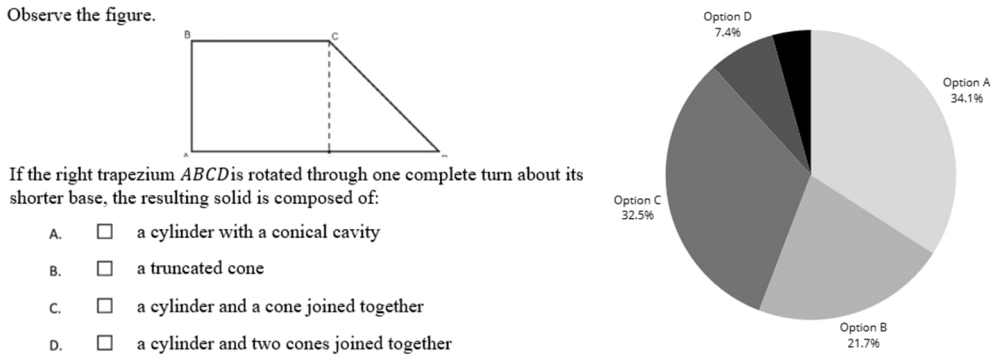
To explore how mathematics teachers reason about students' errors in terms of spatial reasoning, it is adopted the theoretical lens of hermeneutical competence (Cibien, 2026), defined as the practice-related competence that mathematics teachers mobilize to accurately identify and interpret students' errors based on the analysis of mathematical texts, such as items from national LSAs. This construct has been delineated by combining, according to the integrative networking approach (Prediger et al., 2008), components of interpretative knowledge (Ribeiro et al., 2013) and diagnostic competence in error situations (Heinrichs & Kaiser, 2018). Hermeneutical competence thus integrates elements from both theoretical frameworks: on the one hand, the knowledge necessary to make sense of students' responses, which is central to interpretative knowledge; on the other hand, the ability to infer the possible causes of errors without immediately providing corrective solutions, as emphasized in diagnostic competence in error situations. It is therefore a dynamic skill closely tied to classroom practice, requiring teachers to consider the multiple situations that may arise and the subjective interaction of the student with the mathematical content. Furthermore, it's practice-related nature implies that hermeneutical competence can be enriched through professional development activities, where the use of LSA evidence allows teachers to ground their inferences in consolidated research findings rather than in solely subjective experience. In this way, high hermeneutical competence requires teachers to systematically analyse an LSAs item, reflecting on the types of errors students are likely to make and reasoning through the processes that could lead to those mistakes. This involves identifying and interpreting students' errors in a manner consistent with the item's purpose and the mathematical knowledge and skills it is intended to assess.

This study thus addresses the following research question: How do mathematics teachers identify and interpret errors related to spatial reasoning in a geometry item?

METHODOLOGY

For the purposes of the present study, Item 23 from the 2017 Italian national LSAs INVALSI was selected (Figure 1a), administered nationwide to Grade 10 students. It is a closed-ended multiple-choice question and was selected because it is representative of a macro-phenomenon identified at the national level (Bolondi & Ferretti, 2021): specifically, it shows a high percentage of incorrect responses, and the incorrect options can be interpreted in light of well-established constructs in the mathematics education literature. For this specific item, students' errors may be attributed to difficulties related to spatial reasoning. The item requires students to identify the solid obtained by rotating a right trapezium through a full turn around its shorter base. National data, publicly available from the Gestinv database (www.gestinv.it), show that only 34.1% of students at the national level provided the correct answer (Figure 1b). The most frequently chosen incorrect option was C (32.5%), selected by a proportion of students almost equivalent to that of the correct answer, followed by option B (21.7%) and, finally, option D (7.4%). Missing or invalid responses accounted for 4.3% of the sample. It is plausible that students who selected incorrect options made errors related to the mental rotation of the two-dimensional figure involved (Lowrie et al., 2020). In particular, students who chose option C may have rotated the trapezium around its longer base; those who selected option B may have considered a rotation around the height; finally, students who chose option D appear to have rotated the trapezium around its longer base as well, but without taking into account the specific properties of a right trapezium (this may be due to their reliance on a more familiar "standard" representation of a scalene trapezium, which leads to imagining the generation of two cones rather than a single one as a result of the rotation).

Figure 1. (a) Left: Item 23 from the 2017 INVALSI LSAs for Grade 10 students (author's translation); (b) Right: response percentages of the national sample.



Since the item was administered nationwide in a closed-ended format, no written evidence of students' reasoning was available. Therefore, it was decided to administer the item in an open-ended format, requiring students to provide an explicit justification

for their selected answer. The administration took place in two different schools and involved six Grade 10 classes, for a total of 125 students. This allowed for the collection of authentic protocols to analyse strategies and errors through the lens of spatial reasoning. In some cases, semi-structured interviews (Patton, 1980) were conducted following the administration of the item to further explore the reasoning described by the students.

To investigate how mathematics teachers interpret this item and the errors associated with it, an online questionnaire was also administered via Google Forms to 19 in-service secondary school mathematics teachers from various Italian regions. To ensure a heterogeneous (though not statistically representative) sample and to avoid potential bias, participating teachers were selected from schools not involved in the student administration. The design of the questionnaire was inspired by the work of the SIRD research group (Arzarello & Ferretti, 2022): in particular, the results presented in this contribution refer to responses collected through questions validated within that research framework. In the questionnaire, teachers were shown the item in Figure 1a, without being provided with information about national response percentages, and were asked to answer two questions: (Q1) *In your opinion, what knowledge and competencies should a student possess in order to solve the item correctly?*; (Q2) *The correct answer is option A. What do you think has been the most frequent wrong option nationwide? Why?*. Q1 was intended to investigate how teachers framed the item in terms of its purpose and the competencies involved, while Q2 aimed to explore their ability to identify the most frequently selected incorrect option and to interpret the underlying error. Teachers' responses were classified using a thematic analysis (Cohen et al., 2018) and provided information on the respondents' hermeneutical competence, in terms of item analysis and the identification and interpretation of students' errors.

RESULTS

From the experimentation with students, a response pattern emerged that differs slightly from that observed in the national sample. In particular, a higher percentage of students selected the correct answer (44%), while the distribution of incorrect responses resembles the national pattern: option C was the most frequently chosen incorrect answer (18.4%), followed closely by option B (14.4%), and option D was selected by only 2.4% of students. Notably**,** the proportion of students who left the item blank was much higher than in the national results, with 20.8% providing no answer, and half of these students do not even provide a justification. It was therefore decided to conduct interviews with students in order to further investigate their reasoning (or their lack of response). The interviews were conducted in pairs, for time-related reasons, and a total of 20 students from the same class were interviewed. In general, both in the written justifications and during the interviews, students' performance on the item appeared to be related to their capacity to mentally represent the solid in rotation; this therefore suggests a specific difficulty with mental rotation. Particularly interesting is the case of an interview between two students, referred to by the

pseudonyms Lex and Bloom for privacy reasons, during which the need to use real objects to represent the required situation emerged. Lex had not selected any answer during the item administration and had not provided an explanation; Bloom, on the other hand, had chosen the correct answer and, during the interview, explained that she was able to imagine the rotation only after having "experienced" it with a sheet of paper: "I took the sheet and made it rotate, and it's a cylinder. If I also rotate the cut, this becomes a cone, but the fact that it goes downward and not upward creates a hole." [when she mentioned the "cut", she was referring to the oblique side of the trapezium]. During the interview, she mimed her reasoning using a cut-out sheet of paper and her hands: in particular, keeping her finger fixed on the oblique side of the trapezium, she attempted to mimic its "trajectory" to emphasize the resulting solid. This explanation allowed Lex to visualize the required situation, and she herself finally remarked that without a "concrete" aid she would never have been able to imagine the rotation. This interaction suggests that, in addition to strictly geometric knowledge and skills, the "concrete" experience of reproducing the situation may support item resolution and, more generally, help overcome difficulties related to spatial reasoning.

Regarding the teachers, who completed the questionnaire without any prior knowledge of the national or experimental data, their responses to Q1 are summarized in Table 1. All participants identified multiple types of knowledge necessary to solve the item; consequently, the percentages in Table 1 reflect the frequency of each domain mentioned rather than the proportion of individual teachers.

Table 1: Frequency of knowledge domains identified by teachers in response to Q1.

Knowledge domains identified by teachers as necessary to solve the item	Percentage of responses referring to each domain
Geometric transformations (rotation)	31.58%
Solid geometry	47.37%
Plan geometry	21.05%
Spatial geometry	31.58%
General geometric knowledge	21.05%
Knowledge of mathematical language	10.53%

Regarding the indicated competencies, Table 2 shows that many teachers referred to visuospatial skills related to spatial reasoning. As before, percentages reflect the frequency of mentions rather than the number of respondents, as multiple competencies could be identified in a single answer. Of particular relevance for the purposes of the study is the link between these responses and what emerges from the answers to Q2. The analysis shows that 11 teachers out of 19 (approximately 57.8% of the sample) correctly identify the most frequently chosen incorrect response option. Among the

remaining 8 participants, 5 identify option B as the most common incorrect answer, 1 indicates D, 1 states that he does not know which to choose, and 1 provides no justification.

Table 2: Frequency of competencies identified by teachers for Q2.

Competencies identified by teachers as necessary to solve the item	Percentage of responses referring to each competence
Visuospatial skills	57.89%
2D vs. 3D comparison	10.53%
Imagination/Abstraction	21.05%
Text comprehension	10.53%

Among the 11 teachers who correctly identify the most frequently chosen incorrect option, 7 (approximately 63.6%) refer to visuospatial competencies as necessary for solving the item, whereas among those who do not identify the most frequent incorrect option, 4 out of eight (50%) refer to such competencies. Analysing the justifications, teachers who correctly identify the most frequently chosen incorrect option tend to refer to difficulties related to visualization or imagination of the proposed situation (for example: “C, because they struggle to imagine the concavity.”), or to errors in the “manipulation” of the individual solids correctly identified (for instance: “C, because the error consists in failing to subtract the cone and instead adding it.”). Their responses therefore suggest that they locate the causes of error in issues closely related to spatial reasoning. Conversely, those who do not select option C as the most chosen incorrect option appear to focus primarily on the failure to recognize elements of the two-dimensional figure (for example: “Incorrect identification of the side around which the trapezoid is rotated.”), a difficulty that does not fall strictly within the components of spatial reasoning considered here.

These results therefore suggest the existence of a relationship between the way teachers interpret the item, in terms of the knowledge and competencies required for its solution, and their ability to identify and interpret potential errors. In particular, those who frame the item through the lens of spatial reasoning seem better able to interpret errors in a way that aligns with national trends. Interestingly, while some teachers highlighted the role of language in geometry as a key factor for solving the item correctly (Tables 1 and 2), none mentioned that a reformulation of the item or even alternative approaches, such as the physical strategy observed in the case of Bloom and Lex, could help avoid potential errors.

CONCLUSION

This study investigated how secondary school mathematics teachers identify and interpret student errors in a geometry item characterized by specific challenges related

to spatial reasoning (Hidayat et al., 2023). Preliminary analysis of a Grade 10 student sample confirmed that the item is particularly demanding due to the mental rotation required to visualize the solid (Lowrie et al., 2020). However, the data reveal a discrepancy among teachers: not all participants utilized spatial reasoning to frame the task or the associated errors. Specifically, those who did not adopt this interpretative lens appeared to face greater difficulty in recognizing the underlying causes of the most frequent student errors. These findings align with existing literature, which indicates that identifying and interpreting students' mistakes often represents a challenge for teachers (Cibien, 2025).

In summary, addressing the research question, the results show that while some teachers successfully identify and interpret errors related to spatial reasoning, another portion of the sample exhibits difficulties in this process. This suggests a need to strengthen this component of their hermeneutical competence (Cibien, 2026). Beyond spatial aspects, the data highlight the importance of students working concretely with the analysed situations (as seen in the cases of Lex and Bloom) and reveal a marginal, yet present, reflection by teachers on the linguistic component of mathematics. This latter aspect is already a focus of the *MaSt* project, which aims to design targeted professional development programs (Viola et al., 2024); the results of the present research thus offer a concrete contribution to promoting similar interventions within the context of geometry.

Despite the limitations of the small sample size, future research aims to extend the study to verify if these findings persist on a larger scale, incorporating further national macro-phenomena (Bolondi & Ferretti, 2021) observable through the lens of spatial reasoning. In conclusion, consistent with Burte et al. (2020), there is a clear need for specific teacher professional development courses on these topics to better support them in recognizing and facilitating spatial reasoning within the classroom.

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