COUPLING REPRESENTATIONS FOR SENSE-MAKING OF SINUSOIDAL FUNCTION PARAMETERS: CONSTRAINING AS A RESOURCE?

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We report a design-based case study on grade 11 students' sense-making of sinusoidal parameters in a GeoGebra environment that dynamically couples the unit circle and the Cartesian graph. Drawing on Ainsworth's DeFT framework, the coupling was designed to serve a constraining function while students predicted and justified graphs associated with transformed motions (e.g., phase shift, reversed rotation). Analyses of small-group discourse and interviews reveal that the constraint can serve as a resource, supporting students' monitoring and self-correction; however, it can also be interpreted in locally plausible ways that hinder the intended coordination across representations.

INTRODUCTION

Trigonometric functions are a core topic in secondary and tertiary mathematics, yet students often struggle to construct coherent meanings that integrate unit-circle/triangle interpretations, symbolic expressions, and Cartesian graphs (e.g., Weber, 2005; Presmeg, 2008). Such difficulties are frequently associated with limited fluency in coordinating representations, which can keep students' work at an operational level and hinder sense-making of periodicity, symmetry, zeros, and extrema (e.g., Kidron, 2020). In classroom work on sinusoidal functions, this challenge is particularly visible in students' fragmented interpretations of parameters in expressions such as $y=a\sin(bx+c)+d$, where parameters are treated formally rather than connected to amplitude, frequency/period, phase shift, and vertical translation (e.g., Carlsen, 2018; Bornstein, 2020). To address not only *which* representations are introduced (and *in what order*) but *how* they are coordinated in activity, we draw on Ainsworth's DeFT framework (2006), which conceptualises learning with multiple representations in terms of design parameters, of the functions representations may serve (e.g., constraining), and of the tasks learners must accomplish to benefit from them. Prior research highlights the potential of dynamic multi-representational environments to support coordination of trigonometric meanings, while also documenting design tensions: linking supports may become "attention magnets", redirecting students' focus away from the intended invariant relationships (Palatnik et al., 2023). Building on these insights, we designed a GeoGebra environment that dynamically couples the unit circle and the Cartesian graph through a shared parameter, aiming to use this coupling as a constraining resource for students' sense-making of

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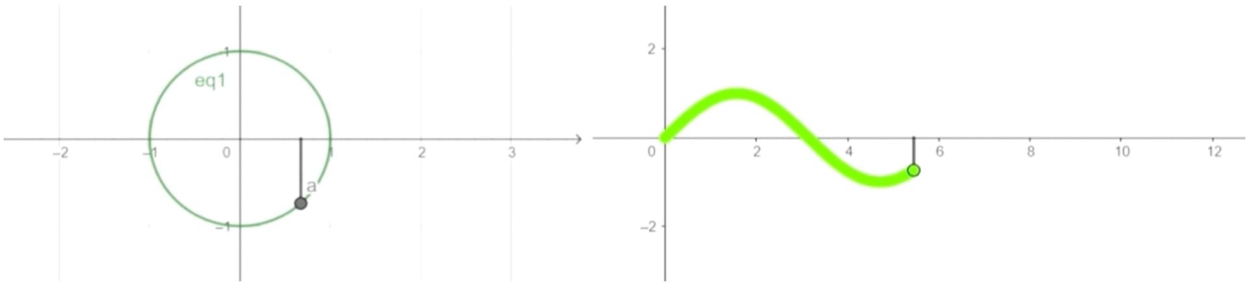
parameter changes in sinusoidal functions. Unlike studies addressing the initial emergence of the sine graph (Shvarts & Alberto, 2021; Shvarts & van Helden, 2023), our participants had already coordinated unit-circle and Cartesian meanings for *y=sin(x)*. The instructional goal was therefore to support an extension of this coordination to more general sinusoids through a prediction task in which students reconstruct transformed graphs from the coupled displays. Accordingly, we investigate whether and how the coupling is used in students' reasoning when they are asked to predict and justify graphs associated with transformed sinusoidal motions.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

Our study is grounded in the view that perceptuo-motor and symbolic learning constitute two inextricable components of mathematical practice. As highlighted by Robutti (2009), this synthesis is captured by the *mathematics-laboratory*, a methodology based on structured activities in which learners build meanings through practice—doing, observing, imitating, and communicating—and where meaning construction is strictly bound, on the one hand, to the use of artefacts and, on the other hand, to interactions among people working together (teacher and students). Within this way of conceiving mathematics teaching, we sought to identify the most appropriate activity for student interaction in order to foster the coordination of multiple representations. To do this, we draw on Ainsworth's DeFT framework (Design, Functions, Tasks) (Ainsworth, 2006). DeFT distinguishes three dimensions that matter when we design a learning activity with multiple representations: (i) design parameters of a representational system (e.g., number of representations involved, information represented, etc.), (ii) the pedagogical functions representations may serve, and (iii) the cognitive tasks requested to learners when interacting with them. Considering the second dimension, multiple representations may serve: *a complementary* function when different representations provide different information or support different cognitive processes; a *constraining* function when one of the representations is used to constrain the interpretation of another one; a *constructive* function when coordinating representations promotes abstraction, generalisation, and explicit awareness of relationships between representations (Ainsworth, 2006).

In our setting, we operationalise constraining through a dynamic coupling between unit circle and the Cartesian graph of sine function in GeoGebra environment (Fig.1).

Figure1: Coupling of representations in GeoGebra environment



A shared parameter t links a point $A=(cos(t), sin(t))$ moving on the circle to a corresponding point $(t, sin(t))$ moving on the Cartesian plane with Tracing activated that progressively generates the sine graph. The two displays, therefore, are coordinated by a structural invariant: at any moment, the ordinate of point A coincides with the ordinate of the graph-tracing point. We refer to this coordinated setup as *coupling of representations*, to t as the *coupling parameter* which is also the variable on the horizontal axis in the Cartesian graph; the *coupling constraint* is the ordinate correspondence of the moving points across displays. Building on students' prior familiarity with the basic sine coupling, we extend the coupling to explore families of sinusoids by introducing a second point B in the unit circle display whose motion is specified relative to A (e.g., changes in radius, angular velocity, or starting position of B with respect to A). The examples provided are summarized in Table 1. Students are asked to predict and justify the corresponding transformed graph, thereby making the coupling potentially available as a reasoning resource.

Table 1. The transformed sinusoid representation as the motion of an additional point.

Sinusoid	Motion of the additional point B (relative to the base point A)
$y = a\ sin(x), a > 0$	Same angular velocity as point A, but on a circle of radius a
$y = sin(bx), b \neq 0$	Angular velocity equal to b times the angular velocity of point A (reversing rotation direction, if $b < 0$)
$y = sin(x + c)$	Same angular velocity as point A, but starting from the position corresponding to angle c instead of 0

In line with DeFT, we examine the *functions* the coupling comes to serve in students' activity, by tracing how students use the coupling to construct, test, or revise predictions. We also attend to how students construe the coupling parameter on the Cartesian axis, as this shapes what the coupling can constrain. Against this background, we examine whether and how students mobilise the coupling as a constraining resource during prediction of the Cartesian graph for generic sinusoidal functions: do signs of attention to the structural constraint between the two representations emerge in students' discourse? If so, what role does this constraint play in predictive processes?

METHOD

The study involved a grade 11 class in an Italian vocational upper secondary school specialising in electronics. Within this context, teaching trigonometric functions is particularly meaningful when it can be connected to the experience students may have in an electronics laboratory, where students engage with instruments (e.g., interferometers) and where graphical representations of wave phenomena are part of their familiar experience. On this basis, we designed an instructional sequence aimed at exploring how changes in the parameters of sinusoidal functions affect their graphs, by integrating dynamic visualisation and coordination across representations. The class

consisted of nine students (below the Italian average). During the activity, students worked in three groups of three, which allowed the researchers to monitor group work closely. Trigonometric functions are covered in this track's mathematics curriculum. Prior to the instructional sequence reported here, students were familiar with the general notion of function and with elementary families (linear, quadratic, exponential). They had previously encountered the definitions of sine and cosine, respectively, as the projection onto *y*-axis and onto *x*-axis of the point on the unit circle determining a given angle. They had also worked with Cartesian graph of sine function and with trigonometric expressions, including relations among associated angles.

We designed and implemented a 13-lesson instructional sequence (mid-March to mid-May 2025) focused on exploring sinusoidal functions and their key features as amplitude, phase shift, and frequency vary, and on introducing the addition of sinusoidal functions, also in connection with the acoustic phenomenon of beats. The sequence was designed to integrate multiple representations (algebraic, Cartesian graph, unit-circle/trigonometric representation, and the auditory phenomenon), with GeoGebra used to support dynamic visualisations. The first activity of the sequence served to introduce the coupling unit circle-Cartesian graph as previously described. As discussed in our theoretical framework, this coupling was designed to allow the unit-circle representation to exert a constraining function over the Cartesian graph. However, since the function that representations serve depends on learners' goals and interpretations (Ainsworth, 2006), the coupling was introduced through a *silent video* task (Kristinsdóttir et al., 2020) to elicit students' initial readings of it (Fig. 1 shows a screenshot of the video). Students worked in pairs to produce a commentary, which was then discussed in plenary. The video introduced the coupled setup progressively (moving point on the circle, projection segment, then the graph-generating point on the Cartesian plane) without making the dependence on the coupling parameter explicit, thereby inviting students to reconstruct the coordination based on prior knowledge.

In the focal activity analysed in this paper, each group was shown another silent video structurally similar to the introductory one. Each video began by re-establishing the coupling for sine, and then introduced an additional point *B* whose motion instantiated one of the transformations listed in Table 1 (e.g., change in radius length – video 1, reversal of rotation direction – video 2, shift in starting position – video 3). Unlike the introductory video, graph-generating point corresponding to *B* was not displayed. After viewing the video, students entered a prediction phase in which they were asked to anticipate the Cartesian graph that would be generated by the (hidden) point associated with *B* and to justify their prediction. In line with the idea of "melting" target artefacts (Shvarts & Alberto, 2021), this choice was meant to position reinvention as the central task: rather than interpreting a ready-made transformed graph, students were invited to reconstruct it by coordinating information available across the coupled displays.

Collected data include audio recordings of students' small-group work during the task activity, audio-recorded post-activity group interviews, and students' written

protocols. Transcriptions and protocols were analysed for signs that students attended to the coupling between representations and used it to explain, justify, test, or revise their prediction of the Cartesian graph associated with the movement of point *B*. Among the three groups, the case $y=2sin(x)$ unfolded comparatively smoothly; therefore, we focus on the other two cases, $y=sin(x+c)$ and $y=-sin(x)$, where interpretive tensions made students' use of the constraint particularly visible.

REPRESENTATIONS IN USE

Example 1 – Phase shift: from 'parallel graphs' to translation

We report the transcript of an excerpt of the group engaged in predicting the Cartesian graph corresponding to the movement of a point *B*, whose angular velocity was the same as *A*, but started from an initial phase of 45° instead of 0°.

1 S6: There is a point *B* that starts 45° ahead of point *A*, which starts at 0°. And they follow each other

2 S4: Ok, I've got it. It means you get two sine functions that are parallel, separated by some amount

[…]

3 S4: Like, these two pens are parallel. If you put them like this, the pens aren't parallel anymore. But if they are parallel, regardless of which point you take, they will always have the same distance

[…]

4 S4: It's not parallel, it's translated

5 S4: It's translated. If it were parallel, there shouldn't be a point where they have the same sine value. I mean, if it were parallel, there shouldn't be a point where they have the same or opposite sine values... Because they should never meet. But if it's like this, they do meet

[students discuss about the phase value]

6 S4: If they have the same the same sine value, it means that they intersect

7 S6: Exactly

8 S4: So you have this thing here [presumably indicating the intersections between the two graphs]. So you have this thing here and this point here, and they intersect because they have the same sine value. Then there is a point where they will have opposite sine values…

[…]

9 S5: There is a certain distance, do they always have it?

10 S6: Yes

11 S4: No, they don't always have it. Otherwise, they would be parallel. If they always had the same distance, they would be parallel and would never meet

S4's initial prediction is that the graph corresponding to *B* is a curve "parallel" [2] to sine graph. The student extends a Euclidean notion of parallelism to function graphs, interpreting "parallel" as implying non-intersection ([5], [11]) and constant distance

([3], [11]). This reading is grounded in S4's attention to the relative motion of *A* and *B*: in [1], the expression "follow each other" suggests, in formal terms, an invariant arc length between *A* and *B*, which S4 appears to translate into an invariant distance between the corresponding graph-generating points [3]. Accordingly, S4 may be imagining a vertical translation of the sine graph by a vector $(0, n)$, an interpretation echoed by S5 and S6 ([9], [10]). However, the coupling constraint—requiring corresponding points to share the same ordinate—provides a criterion to test this prediction. Noticing that ordinates of *A* and *B* occasionally coincide ([6], [8]), S4 concludes that corresponding graphs must intersect and then rejects the "parallelism" hypothesis [5], as it contradicts the supposed non-intersection of parallel curves ([5], [11]). Guided by this internal logic of the constraint, S4 revises the prediction and identifies the transformation as a horizontal translation of the Cartesian graph [4].

Example 2 – Reversed rotation: tracing sine graph in the "opposite" direction

The following is an excerpt from a retrospective interview with another group, that explored the case where the point *B* on the unit circle moved at the same angular velocity as point *A*, but in a clockwise rather than counterclockwise direction.

1 S9: My first impression was that they were rotating at the same speed. And then there was the issue of direction—one counterclockwise and one clockwise. So, I also thought the sinusoids were more or less opposite. Then the radius; we checked if it was always one.

2 S8: I had the same impressions. One went counterclockwise, the other clockwise; the same thing in the graph of the sinusoid—they were opposites in terms of sign. In the drawing, I made the one going counterclockwise moving normally toward the right, while the one going clockwise went toward the left. Although, we then saw that it was in the same part [i.e. the Cartesian plane] but opposite.

In this excerpt, "opposite" sinusoids [1] is the key expression used to relate Cartesian graphs corresponding to *A* and *B*. S9 foregrounds the invariance of angular speed, combined with a reversal of direction, as the salient feature of the motion of *B* on the unit circle. In [2], the meaning of "opposite" becomes more specific: this "opposition" is translated into an inversion of the motion of the *A*-graph-generating point, initially imagining the abscissa of the *B*-graph-generating point progressing into the negative semi-axis (from the origin to the left). Accordingly, "opposite" is interpreted as a sign change in both coordinates of the *B*-graph-generating point ("opposites in terms of sign" [2]). In the next excerpt, the rationale for this prediction is further elicited by the interviewer, who asks the group to account for the discrepancy between their predicted graph (coincident with sine graph) generated by moving left from the origin and the graph shown in a second video, $y=-sin(x)$, generated by moving right from the origin.

4 S8: Reflecting on it, we thought it made more sense to do it this way, even though our version was still fairly accurate. That is, we felt this way was more precise.

5 Int: What was your original line of reasoning?

6	S9:	More than anything, the thing that made us think they were heading in two opposite directions was the fact that one was moving counterclockwise and the other clockwise.
7	S8:	In our view, the values were opposite, so it went in the other direction instead of following the same side of the graph.
8	S9:	But again, mainly the fact that one went clockwise and the other counter-clockwise. Yes, exactly. Because, let's say, the fact that the point moved in this direction [gesturing] led us to consider negative angles.

In [7], S8 makes explicit the reading of the coupling that led to the prediction above: to identify *B*-graph-generating point coordinates, the students take as the independent variable the angle subtended at point *B*. This interpretation is so dominant that it seems to override what is conveyed by the solution graph [4], namely that the two graphs should share the same coupling parameter. As a result, it is this (mis)interpretation of the coupling constraint that guides their construction of the new graph: the angle–ordinate relationship identified for point *A* is transferred to point *B*, without attending to the invariance of the coupling parameter across the two representations. However, this (mis)interpretation does not support the intended exploration of parameter variation: by reinstating a generic constraining relationship angle–ordinate for point *B*, the outcome is—except for change in radius length—again the same Cartesian graph of *sin(x)*, parameterized with respect to the angle *x* subtended at point *B*.

DISCUSSION AND CONCLUSION

In our instructional sequence, the unit circle and Cartesian graph are dynamically coupled so that corresponding points share the same ordinate, while the interpretation of the coupling parameter remains open. Introduced through a silent video, the coupling invites students to reconstruct a coordination "script". From a DeFT perspective, it is intended to serve a constraining function, limiting plausible readings of the graph to those consistent with the motion on the circle. Our data show, however, that a constraining link does not automatically become a constraining resource for exploring sinusoidal function families: students may recognise the coupling yet mobilise it in a way which do not allow generation of graphs of sinusoids with altered parameters, depending on how they construe the coupling parameter and use it as a resource. Across the two analysed episodes, the constraint played two roles. In the phase-shift episode, it functioned mainly as a monitoring criterion: students' initial "parallelism" prediction (perhaps constant vertical distance between the graphs) became untenable when coinciding ordinates implied intersections, supporting revision toward horizontal translation. In the reversed rotation episode, instead, students effectively reconstructed a generic constraining relationship, interpreting reversal as continuation into negative values of the coupling parameter; here the coupling still supported a prediction, but not in a way aligned with the design intention. These contrasts foreground a design tension: the openness that supports interpretive elicitation in the introductory silent video becomes harder to sustain when the task shifts to predicting transformed graphs. Here, the coupling built on the case *y=sinx* is

meant to remain the shared reference—i.e., the same coupling parameter should govern the generation of both the base and the transformed sinusoid—yet students may adopt locally coherent but incompatible readings of the independent variable and, consequently, of what the Cartesian trace represents. Overall, our findings sharpen the DeFT claim that multiple representations support learning only under specific conditions: learners must attend to relevant features, coordinate them dynamically, and use inconsistencies as triggers for reinterpretation. At the same time, the frictions observed have pedagogical potential, opening discussions about Cartesian graphs—e.g., what "parallel" could mean for curves, and how "generating a graph" depends on treating the graph as a dynamic trajectory rather than a static correspondence. From a design perspective, strengthening constraining as *power-in-use* in students' reasoning may require supporting students in actively constructing and validating the coupling (Shvarts & Alberto, 2021), potentially enhancing transfer to other sinusoids (Shvarts & van Helden, 2023).

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