PRESERVICE TEACHERS' UNDERSTANDING OF MULTIPLICATION: INSIGHTS VIA REPRESENTATIONS?

Daniela Assmus[1], Nina Bohlmann[4], Kerstin Bräuning[1], Caren Feskorn[1], Torsten Fritzlar[1], Heike Hahn[3], Simone Reinhold[2], Denise Resche[2]

[1]University of Halle-Wittenberg, [2]Leipzig University, [3]University of Erfurt, [4]Freie Universität Berlin

Teachers' specialized content knowledge on arithmetic operations is considered to play a meaningful role for teaching as learners' development depends on their teachers' understanding of relevant concepts, especially concerning the variety of strategies and representations for multiplication. This paper reports on a study with 863 preservice primary teachers from three different universities at the beginning of their university studies for teaching mathematics, using a written survey. Results of the study which included drawing tasks and tasks requiring preservice teachers' comments, illustrate a surprisingly wide variety of iconical representations, and moreover reveal the novices' challenges in finding corresponding comments to their drawings. Implications and obstructive details from the results are discussed.

INTRODUCTION

Developing appropriate concepts in basic numeracy is a major challenge in primary mathematics. Accordingly, this topic is also of great importance for teacher education. However, work with preservice teachers (PSTs) and related research put the appropriateness of prospective primary teachers' multiplicative understanding into question – which is likely to hinder teaching multiplication in primary. In addition to a wide variety of studies on primary students' multiplicative understanding (e.g. Lamprecht, 2020; Mulligan & Mitchelmore, 1997), there are various studies that address PSTs' understanding of multiplication – reaching from questionnaires about (prospective) primary teachers' beliefs to studies including iconical representations in various ways (e.g. Barmby & Milinković, 2011; Novikasari & Dede, 2021). However, there is little evidence on PSTs' multiplicative understanding at the very beginning of their university studies, and about iconical representations these PSTs design to encompass multiplicative structures. Therefore, the aim of the ongoing project *SaaLe-Multi* (Aßmus et al., 2025) is to reconstruct elements of PSTs' understanding of multiplication, and this paper addresses the questions: Which variety of drawings is shown by the PSTs? Do their comments correspond to the drawings?

THEORETICAL FRAMEWORK

Grundvorstellungen of multiplication

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, pp. 1211-1218. PME.

Multiplication is a mathematical key concept, and closely related to the understanding of division, fractions, or exponentiation. Fetzer and Tiedemann (2022) distinguish between *Grundvorstellungen* (*GV*), strategies and representations of multiplication. *Grundvorstellungen* (*basic concepts*; vom Hofe, 1995) can be understood as a prerequisite for translations between different forms of representation. On one hand, this means that basic concepts (*GV*) enable us to interpret a mathematical term appropriately and to find illustrative examples for this term – for example, a story, a picture, or an activity in the sense of Bruner's intermodal transfers (Bruner & Kenney, 1965, Bruner 1966). On the other hand, (everyday) situations, images, and actions can be interpreted mathematically and noted using a mathematical-symbolic expression. (*GV*; vom Hofe, 1995; vom Hofe & Roth, 2023)

Developing *GV* of multiplication is closely linked to repeated addition (Fischbein et al., 1985). Following this notion, the amount of equivalent sets (one factor) acts as the multiplier (or 'operator'), whereas the magnitude of each set is interpreted as the multiplicand (or 'operand'), with the union of all equivalent sets leading to the product. However, this understanding (e.g. in word problems or in fractions) is subject to limitations (Tirosh & Graber, 1989). Therefore, Greer (1992) complementary suggests *multiplicative comparison* (e. g. '4 times as many boys as girls), *rectangular arrays* (e. g. '4 rows with 5 counters in each row'), and the *Cartesian product* (e. g. 'number of possible girl-boy-pairs in a group of children'). Another notion which is internationally common for introductions to multiplication in textbooks includes *jumps on a number line* (Harries & Sutherland, 2000) which corresponds to the idea of 'jumping several times', with 'jumps' cover equal distances on a standardised scale. Usually, this also corresponds with the idea of repeated addition.

Representations of multiplication: Children's approaches and PSTs' knowledge

Understanding the variety of approaches to multiplication challenges teachers as they have to make important decisions, as for example:

> From an expert's point of view, which everyday actions have an inherent multiplicative structure and can be, for this reason, a suitable starting point for individual constitution of meaning? (Fetzer & Tiedemann, 2022, p. 2)

Regarding representations for multiplication, Mulligan and Mitchelmore (1997) encountered 'intuitive models' among children in grades 2 and 3 which referred to various calculation strategies for one-step, whole-number, multiplicative word problems (e.g. *direct counting*, *rhythmic counting*, *skip counting*, *additive calculation*). Against this background, (prospective) teachers are immensely challenged to foster young children's understanding of multiplication, and to support this by the use of iconical representations in visualizations and drawings (Cheeseman et al., 2023; Fetzer & Tiedemann, 2022; Harries & Barmby, 2007; Mulligan, 1992).

Referring to the work of Shulman (1986), the framework of MKT (*Mathematical Knowledge for Teaching*) introduced by Ball et al. (2008) distinguishes between

s*ubject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK). Teachers' ability to visually represent (mathematical) content and procedures and finding suitable variations of representations is regarded to be an important element of PCK as well as SMK (Ball et al., 2008). Focusing on SMK, *horizon content knowledge* (HCK), *common content knowledge* (CCK), and *specialized content knowledge* (SCK) have been identified as subdomains of SMK. As stated by Ball et al. (2008, p. 400), "(…) teaching involves making features of particular content visible to and learnable by students." Therefore, the elaboration of SCK is a principal aspect of teacher education – including knowledge of how to articulate mathematical explanations, and to find various representations of a mathematical content. In this sense, Milinković (2012) distinguishes four levels of "pedagogical knowledge of representations":

> (1) knowing different representations of concepts and procedures, (2) knowing how to use different representations in problem solving (3) knowing how to use different representations in problem posing (4) understanding representations as a tool in teaching. (Milinković, 2012, p. 210)

Subject matter knowledge: Limitations in teacher education for multiplication?

Concerning the use of representations for multiplication and division in the mathematics classroom, research has revealed PSTs' severe difficulties. For example, Barmby and Milinković (2011) asked PSTs to decide upon the appropriateness of presented iconical representations as the PSTs should consider addressing various aspects of multiplication in teaching situations (such as demonstrating the commutative or distributive law, or illustrating multiplication as repeated addition). They found that

> […] some teachers did not understand what was meant by these laws. The analysis also highlighted some limitations in teachers' use of representations, for example with the commutative law […] where a simplistic argument was provided for using the groups of strawberries representation (Barmby & Milinković, 2011, p. 110)

In a similar study with PSTs, Milinković (2012) found specific preferences in using representations which were not necessarily matching the characteristics of a given problem. Here, the choice for appropriate representations seemed to be linked to the level of the problems' abstractness. Novikasaki and Dede (2021) developed a questionnaire to explore the beliefs of Turkish teacher training students, and found that these struggled in identifying aspects of implementing multiplication in real life. The authors also stated that PSTs' mastery of multiplication was limited, and that many of them considered that only familiar strategies should be used in multiplication. Yet, very little is known about PSTs' understanding at the very beginning of their mathematics education university studies. These results and considerations underpin the need of a substantial analyses of first-year university PSTs' initial concepts and their emerging understanding of multiplication. As suggested in related research, PSTs' drawings and corresponding comments may provide insights into their initial understanding as these activities comprise and articulate facets of PSTs' *GV*. Later on, this may enrich university courses as well as (formative) assessments of PSTs' conceptual

development of SCK concerning *GV*, strategies and representations for multiplication, and offer substantial background to design adaptive teaching activities for PSTs.

METHODOLOGY

Research questions, data collection and analysis

Against the given theoretical background, the objective of the framing project *SaaLe-Multi* is to explore first-year university PSTs' initial concepts and their understanding of multiplication, following a wider range of research questions. The study presented in this paper refers to the following subordinated research questions:

- Which variety of drawings (iconical representations) is shown by the PSTs?
- (How) Do their explanatory comments correspond to the drawings?

Data collection focused on a written survey with open and closed items which was used in October and November 2024. The survey (Aßmus, 2025) was conducted during the first (arithmetic) mathematics methods courses for PSTs (N = 863) in three German universities (Erfurt n = 319, Halle/Saale n = 270, and Leipzig n = 274, in the German Federal States of Thuringia, Saxony Anhalt, and Saxony), including almost all primary mathematics PSTs who were first-year students at these universities, at that time.

Referring to the levels of "pedagogical knowledge of representations" (Milinković, 2012; see above), we focused on *level 1* (*knowing different representations of concepts and procedures*) with the first task of the questionnaire, asking all PSTs to draw a representation for the task 5 × 4, and to explain how this drawing relates to the task. Additionally, we asked the PSTs to "draw another drawing for the same task that differs significantly from the first representation", to explain all considerations and to describe how the two images differ (*knowing different iconical representations*).

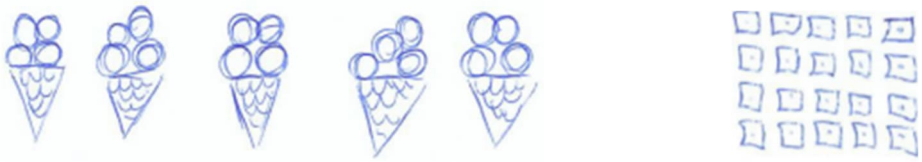
Following *Qualitative Content Analysis* Methodology (Mayring, 2015), data (drawings and comments) of all 863 PSTs was initially coded using inductive categorization. To ensure the reliability of the coding process, all data was independently analyzed by research scholar groups from the three universities. Identified discrepancies were discussed until consensus was reached, contributing to the refinement of the categories and the interpretative consistency of the analysis (following *consensual validation* in a qualitative research paradigm; see Maier, 1991).

RESULTS

Variety of drawings

To illustrate the task 5 × 4, the first-year PSTs drew representations which lead to the categories and refinements, namely sub-categories which were detected in the data, which are shown in table 1 (incl. a selection of frequently coded sub-categories).

Figure 1: PSTs' representations with equal sets (left) and rectangular array (right)



The majority of first-year PSTs (61 %) offered a drawing which included bundles with equal sets of the same cardinality (Fig. 1, left). The second most common drawings were rectangular arrays (resembling rectangulars), accounting for 18% (Fig. 1, right). Linear representations were rare. Similarly, obstructive illustrations (e.g. those with the factors represented separately using mixtures of icons and symbols) occurred only in few cases (4 %).

Table 1: Variety of iconical representations of first-year PSTs for the task 5×4

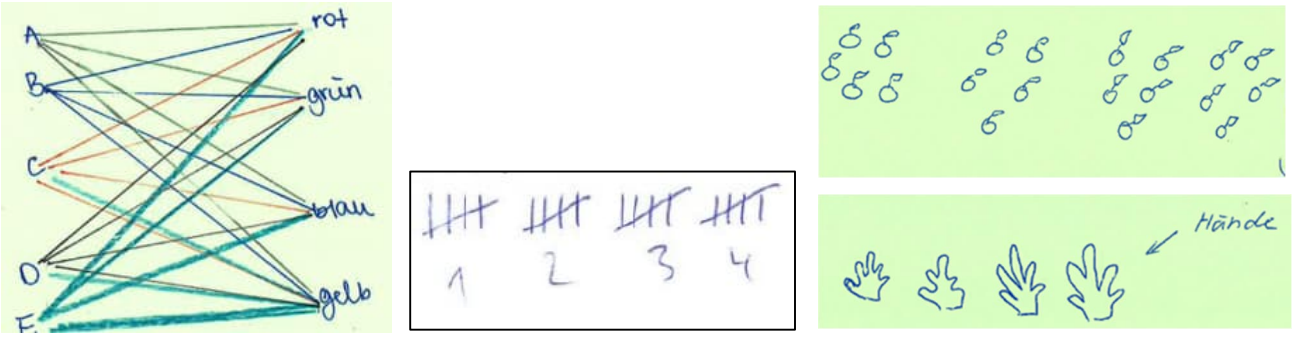
Categories for iconical representations	Description or examples	Refinement (frequently occurring sub-categories)
Bundles	Equal sets of items (see Fig. 1 left)	+ contextually embedded + part-whole relations
Rectangular array	Resembling rectangulars, number of rows and columns matches factors (see Fig. 1 right)	+ bundles marked + highlighting factors + part of bigger set + intersections of lines + unfinished "rectangle"
Number line	Equal jumps on line/ray	
Measurement	Units of length, money, area measurement, or time	+ context (e.g. market) + symbolic notation
Mapping	Mapping between elements of two sets	+ indicated by lines + using context
Combinations	Elements of two sets in Cartesian product	+ lines for all relations + additional matrix
Iconic representation of one factor	One factor is represented cardinally, the other symbolically.	
Both factors separately	AAAAA × BBBB	+ mathematical symbols + other items (and "×")

Correspondence between drawings and PSTs' comments

The explanations for bundles with equal sets of the same cardinality, and those who referred to rectangular arrays basically included highly corresponding one-on-one descriptions of the drawings, but on different levels of abstractness, e.g. "The illustration shows five bundles of four." (all quotes translated by the authors).

However, the analyses also encountered constraints, indicated exemplarily by the following excerpts from the data. For example, when *combinations* were drawn, descriptions sometimes lacked full information as in Figure 2 (left), where the second factor is missing in the comment (here: explicitely mentioning the second set of four colours for the candies): "There are five people (A, B, C, D, E) and each person eats four candies, in the end, they have eaten a total of 20 candies." The second example (Figure 2, middle) also reveals challenges in explaining the illustration using *Bundles*. If a bundle consisting of five lines was copied four times, as described in the explanation ("The number five is shown as lines and copied four times so that you can count the result."), there would have been five bundles, each including five lines at the end of this process.

Figure 2: Cartesian product and bundle representations



Furthermore, comments on the bundle representation helped to differentiate between dynamic resp. sequential, and static interpretations. For example, in Figure 2 (upper right) the bundles stand for the result of a process (PST's comment: "Clara is to buy five apples in each of four fruit shops. How many apples has she bought in the end?"), whereas the second picture of four "hands" (on the right in Figure 2) focuses on displaying a static situation (comment: "How many fingers do four hands have?").

DISCUSSION

Following our objective to explore first-year university PSTs' initial concepts and their understanding of multiplication (as part of *Grundvorstellungen*, *GV*), we saw a surprisingly wide variety of drawings (iconical representations). Overall, the drawings indicated that most PSTs started their studies in mathematics education with SCK that allowed them to offer at least one representation which comprised basic aspects of the concept of multiplication. However, there was at least a small group of PSTs who

obviously struggled while offering representations such as "*Both factors separately*" or "*Iconic representation of one factor*".

Categories which were identified in the first-year PSTs' drawings in this study resemble findings reported by Dabić and Milinković (2015) who describe how Serbian PSTs used iconical representations for multiplication (commutative law, in particular). They point at six different categories (*equal groups*, *array representation*, *number line*, *sets*, *area model* and *combined area/array model*), basically referring back to the representations introduced by Greer (1992), for example. This and the fairly similar results of our study are not surprising, since PSTs in these samplings are expected to have experienced these types of representations in their own learning of multiplication from grade 2 on. Yet, our results highlight specific findings concerning the sub-categories, offering a wider variety for subsequent in-depth-analysis of similar data.

However, PSTs positive efforts and constraints need to be addressed (anonymously) in prospective university studies, and the given examples (both for obstructive representations and mismatching descriptions) or further excerpts from the study's data have the potential to enrich university courses and assessments, as they may inspire discussions among PSTs about aspects of appropriateness, restrictions, or consequences for teaching primary children. Developing this kind of learning activities for university studies (and school), is one of the next activities in the project *SaaLe-Multi*. Furthermore, our results are limited as they do not cover all levels of "pedagogical knowledge of representations" (Milinković, 2012, p. 2010), entirely. Thus, the studies only offer a fairly small glimpse at PSTs' *GV* on multiplication, and further research should additionally enlighten the other three levels, in more detail.

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