FUNCTIONAL THINKING: THE INTERRELATION BETWEEN STUDENTS' FUNCTION CONCEPTIONS AND THEIR REPRESENTATIONAL COMPETENCIES

Susanne Digel[1], Ute Sproesser[2]

[1]RPTU University of Kaiserslautern-Landau
[2]Ludwigsburg University of Education

This study examines the structure and relationships of functional thinking facets, i. e., conceptions of functions and representational competencies. A sample of 678 students completed assessments on both facets. Rasch model comparisons identified all seven theory-based dimensions and showed varied correlations between them. The strongest correlations were found between changes involving situational representation and covariation/object and the weakest between changes with algebraic expressions and correspondence. This suggests that changes with situational representations rely on function conceptions, whereas changes with algebraic expressions often are performed algorithmically. Hence, instruction should cultivate function conceptions and comprehensible representational changes, while deemphasising fragile algorithms.

THEORETICAL BACKGROUND

Functional thinking describes the ability to think in relationships, dependencies and changes (Vollrath, 1989). This description makes clear, why functional thinking permeates various mathematical domains and is considered as an overarching idea, relevant in mathematics education and beyond (Krüger, 2019). One central facet of developing functional thinking is to grasp the different function aspects, i. e., the *correspondence* of an element of the definition set to exactly one element of the set of values; the *covariation* of the dependent variable when the independent variable is varied and the final aspect, in which the function is considered as an *object*. In order to develop functional thinking, these function aspects need to be internalized as basic mental models bridging the mathematical concept with the learner's individual concept formation (vom Hofe & Blum, 2016). In this sense, these basic mental models are indispensable for interpretating functions in real-world contexts and for their application. We will refer to them as *function conceptions* in accordance with the work by Thompson and Carlson (2017) on conceptualizing functions.

Another relevant facet of functional thinking are *representational competencies*, i.e. to adequately deal with and change between function representations (Vollrath, 1989). As mathematical functions are abstract, they can only be accessed by function representations such as function equations, graphs, tables or situational representations (e. g., descriptions or images) (Duval, 2006). Beyond giving access to the mathematical object of function, learning about different representations is also important for concept formation (Niss, 2020): Learners need to recognize that a function is not identical with

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, pp. 1411-1418. PME.

its representations and cannot be reduced to only (one) particular representation(s) as the distinct representations emphasize different function properties that are all relevant for the concept itself. When working with the distinct function representations, learners can experience and recognize how these representations are interconnected – an important prerequisite for changing between function representations. Such representational changes are often required for flexibly and adaptively solving various problems and tasks (Duval, 2006). In this respect, it should be noted that reasonably changing between function representations does not mean to algorithmically transfer source into target representation, but to understand and mentally represent these representations through function conceptions (vom Hofe et al., 2015), to recognize and connect their central properties relevant for the intended representational change (such as the parameters rate of change/slope and y-intercept in function equation and graph) as a base for converting from source to target representation (Adu-Gyamfi et al., 2012).On the one hand, representations remain incomprehensible schemata and isolated knowledge for the learner, without practical or cognitive value, if they cannot be mentally structured using function conceptions (vom Hofe et al., 2015). For instance, one uses the object aspect when identifying central properties of the graph of a linear function (i. e., slope and y-intercept) to convert them into the representation of a function equation. On the other hand, the abstract character of the function concept entails that the function conceptions can only be developed and observed via function representations (cf. vom Hofe & Blum, 2016). For instance, correspondence manifests in mapping x- and y-value of a function in a table of paired values, and covariation becomes apparent when observing how the y-value changes when the x-value traverses an interval on the x-axis (vom Hofe et al., 2015). Moreover, dealing with and changing between function representations might stimulate the learners to take different perspectives on functions and, hence, to develop their function conceptions. Thus, representational competencies and function conceptions are interrelated, but it remains unclear, how they interact.

Bossé et al. (2011) describe representational changes between situational and mathematical representations of functions as generally more difficult for learners than changes between mathematical representations – what is in line with teachers' preference of algebraic-graphical changes and neglection of situational representations in the function classroom (Sproesser et al., 2020) and what might be partly explained by the large differences in the abstraction level between situational and mathematical representations (Nitsch, 2015). However, prior research also indicates a particularly large heterogeneity in empirical difficulty for situational-graphical changes of representation (Sproesser et al., 2022). This finding might be explained by the fact that there is a large variety of possible tasks for changes between situation and function graph, whereas, e. g., tasks requiring changes between function equation and graph are relatively similar in structure and often solved rather algorithmically by learners (Sproesser et al., 2020). As such algorithms are less available for representational changes between situation and graph (Zindel, 2018), activating underlying function

conceptions could become more relevant than for, e. g., algebraic-graphical representational changes. Thus, these changes can pose learning opportunities to develop learners' function conceptions and, hence, to foster functional thinking.

The previous paragraphs indicate the relevance of function conceptions as well as representational competencies for functional thinking. Although from a theoretical perspective, at least several of their sub-facets (i. e., the distinct function conceptions and competencies related to concrete representations) might be closely connected, there is no empirical base for this assumption. This leads to the research questions (RQs) of the present study: RQ 1. Can functional thinking be structured according to the facets *function conceptions* and *representational competencies*, respectively their subfacets? RQ 2. How are these facets and subfacets related to each other? RQ 3. Which strengths and weaknesses can be observed with regard to students' function conceptions and their representational competencies?

METHODS

The study sample consists of 678 students (264 female, 288 male) from cooperation schools of Ludwigsburg University of Education and RPTU University of Kaiserslautern-Landau. The students came from 32 classes of academic track schools (ATS: 9 classes, 185 students, grade 7), medium track schools (MTS: 13 classes, 201 students, grade 8) and comprehensive schools (CS: 10 classes, 292 students, grade 8). As in Germany, the function concept is introduced in grade 7 in ATS and in grade 8 in MTS/CS, both grades were included, leading to a range of age between 12 and 16 years (M=13.5, SD=0.84). For this study, the participants worked on two paper-pencil-tests in temporal distance between a few days and some weeks after the teaching unit of functions depending on the mathematics teachers' possibilities. One ordinary lesson of 45 minutes was allocated for each test, whereas the order of the tests was alternated to control for possible learning and motivational effects by passing the tests.

Students' function conceptions were assessed with a short test on functional thinking (FT-short, 27 items, see Digel & Roth, 2020). More precisely, the subfacets were evaluated with seven items on correspondence (C), nine on covariation (V) and eleven on function as object (O). The test was already used in prior research and has proven to reliably measure students' function conceptions (EAP/PV reliabilities as joint dimension pre/post 0.80/0.86, Digel & Roth, 2022). The test instrument to measure students' competencies of dealing with and changing between function representations contains a total of 21 items (see Sproesser et al., 2022). The following subfacets were assessed: eight items required to change between situation and graph (SG), eight between algebraic expression and graph (AG), three between situation and algebraic expression (SA) and another three items only focused on knowledge about graph or equation without a change between representations (NC). This test on representational competencies was already used in prior research and has proven to reliably measure students' corresponding competencies (EAP/PV reliabilities for the dimensions SG, AG, SA and NC between 0.69 and 0.83).

Data was analyzed in IRT using R software. To determine the best representation of data, several Rasch models (in-between item design) were estimated. Models were compared by container ANOVA to perform a chi-squared test on significance and by evaluating information criteria. Based on the best fitting model, correlations between its dimensions were assessed, in order to evaluate the relationship between the function conceptions and representational competencies. Furthermore, mean values and standard deviations of the estimated person abilities for all dimensions were determined for the total sample and grouped by school form, in order to investigate students' potential strengths and weaknesses of these facets of functional thinking.

RESULTS

In order to answer RQ 1, we specified and compared different Rasch models for students' function conceptions and representational competencies (see Table 1 for an overview). The estimation of a two-dimensional Rasch-model (dimension 1: all items of the function conceptions test, dimension 2: all test items of the representational competencies test) showed good reliabilities $EAP\text{-}Rel_{Conc} = .79$ and $EAP\text{-}Rel_{Repr} = .77$. The two dimensions were linked through a mediocre correlation of r=.46.

To analyse correlations between the subfacets of function conceptions and of representational competencies, a second model with all subfacets of the two tests (see above: C, V, O, SG, AG, SA, NC) as separate dimensions was estimated, hence, a seven-dimensional Rasch-model. The estimation showed good to acceptable reliabilities for the seven dimensions: $EAP\text{-}Rel_C = .70$, $EAP\text{-}Rel_V = .70$, $EAP\text{-}Rel_O = .77$, $EAP\text{-}Rel_{SG} = .72$, $EAP\text{-}Rel_{AG} = .76$, $EAP\text{-}Rel_{SA} = .63$ and $EAP\text{-}Rel_{NC} = .60$. The correlation matrix will be presented after the model selection results.

Since the subfacets covariation and object did not form separable dimensions in some previous studies (Lichti & Roth, 2019; Digel & Roth, 2020), a six-dimensional Rasch-model was estimated (C, V/O, SG, AG, SA, NC). This model showed similarly good to acceptable reliabilities for the six dimensions: $EAP\text{-}Rel_C = .71$, $EAP\text{-}Rel_{VO} = .79$, $EAP\text{-}Rel_{SG} = .68$, $EAP\text{-}Rel_{AG} = .75$, $EAP\text{-}Rel_{SA} = .63$ and $EAP\text{-}Rel_{NC} = .61$.

The pairwise chi-squared tests using container ANOVA indicated significant differences between the two-, six- and seven-dimensional models, where the latter fitted the data significantly better ($p<.001$) than the other two did. Furthermore, most information criteria gave favour to the seven-dimensional model (see Table 1).

Table 1: Overview of information criteria for the compared models

	Deviance	AIC	AIC^3	BIC	aBIC	CAIC	AICC
2-dim. Model	29529.78	29694	29776	30064	29803	30146	29717
6-dim. Model	28873.93	29074	29174	29526	29208	29626	29109
7-dim. Model	28845.86	29060	29167	29543	29203	29650	29100

Regarding RQ 2 and based on the seven-dimensional model, we investigated the correlations between the subfacets (see Table 2). Altogether, the correlations showed large heterogeneity between .146 and .911 (all p-values <.001). The highest correlations were found between the three function conceptions, in particular between the C and O. For the dimensions of representational competencies, there were larger differences in the correlations' magnitude with the highest values for SA and SG as well as for NC and AG. Similarly, the correlations between the function conceptions and the representational competencies varied a lot with the highest correlations between SG and V respectively O and the smallest correlation for AG and C.

Table 2: Correlation matrix for the 7-dimensional Rasch-model (p<.001)

	C	V	O	SG	AG	SA
V	0.738					
O	0.855	0.911				
SG	0.361	0.705	0.663			
AG	0.146	0.352	0.333	0.537		
SA	0.346	0.384	0.422	0.730	0.570	
NC	0.314	0.219	0.379	0.488	0.743	0.541

With regard to RQ 3, Table 3 presents the estimated person abilities with mean values and standard deviations on the seven dimensions. Referring to the function conceptions, the learners showed the highest abilities for C and the lowest for the O. Referring to the representational competencies, the students had most difficulty with AG and were most successful with SA.

Table 3: Mean and standard deviation of person abilities in the seven dimensions

		C	V	O	SG	AG	SA	NC
Total number	M	1.91	-0.31	-1.15	-0.75	-1.49	0.34	-0.71
(n=678)	(SD)	(1.20)	(1.04)	(0.82)	(0.99)	(2.05)	(1.36)	(1.46)
ATS	M	2.57	0.27	-0.68	-0.27	-1.07	0.89	-0.38
(n=185)	(SD)	(1.03)	(1.01)	(0.76)	(0.97)	(1.95)	(1.22)	(1.38)
CS	M	1.60	-0.53	-1.33	-0.71	-1.38	0.55	-0.58
(n=201)	(SD)	(1.36)	(1.10)	(0.88)	(1.00)	(2.10)	(1.37)	(1.48)
MTS	M	1.67	-0.57	-1.37	-1.11	-1.85	-0.18	-1.02
(n=292)	(SD)	(1.01)	(0.86)	(0.68)	(0.85)	(2.02)	(1.25)	(1.45)

Note. ATS=academic track schools, CS=comprehensive schools, MTS=medium track school; M=mean value, SD=standard deviation, both on a logit scale.

Altogether, the mean values for the function conceptions were slightly distributed above the mean values for the representational competencies. The pattern found in the total sample was reproduced in all subsamples with the highest mean values for ATS and mostly the lowest for MTS.

DISCUSSION

With regard to RQ 1, this study indicates that students' function conceptions and representational competencies are separable since these two facets of functional thinking show a weak correlation in the two-dimensional model. Moreover, the best model fit for the seven-dimensional model and the mostly mediocre intercorrelations between its dimensions indicate that the subfacets, namely the three function conceptions and the four representational competency subfacets are empirically separable and, hence, can be considered as distinct (sub)facets of functional thinking.

Investigating RQ 2 and despite the separability of the subfacets, the high correlation between the conceptions covariation and object shows their interrelatedness in this sample. This might be partly explained by the way German lower secondary school students access the object aspect, less in the sense of concatenating or differentiating the mathematical object of function but rather from the perspective of evaluating the global form of a graph which overlaps with the covariation aspect. The relatively large differences in the correlations between the dimensions for the representational competency subfacets indicate that there is no overall competency of *dealing with and changing between representations* but that there are different competency subfacets depending mainly on the specific representations involved. In this sense, it appears plausible that SG and SA are relatively closely related, as the involved situational representation needs to be understood and connected to a mathematical representation. Similarly, the high correlation between NC and AG can be explained by the fact that learners only need to deal with the mathematical representations graph and equation.

With regard to the relatedness between the function conceptions and the representational competencies, the rather high correlations between SG and particularly V and O respectively imply that these conceptions are prerequisites for this representational change – presumably as there are no algorithmic procedures for this change and thus conceptions help to understand the underlying situation and bridge it to the graphical representation. In comparison, the rather low correlations between C, V, O and AG and NC indicate that dealing with and changing between these representations can be performed without drawing on these conceptions – presumably by using algorithmic procedures (cf. Sproesser et al., 2020).

The fact that C is by far the most pronounced of all subfacets in our sample might also explain why all correlations between this function conception and the representational competency subfacets were rather low: Presumably, most students disposed of the minimum level of C necessary to perform the representational changes and, hence, this conception (alone) did not enable or hinder a certain representational change.

With regard to RQ 3 and in line with prior research, the mean values of person abilities show a relative strong conception of C and difficulty with O (cf. Lichti & Roth, 2019; Thompson & Carlson, 2017). As expected, the highest values were found for ATS students, but the patterns were similar in all subsamples. Opposed to research by Bossé et al. (2011), students' competencies in SG and SA were more pronounced than in AG and mostly NC – although teachers often give priority to mathematical representations compared to situational ones in their function classroom (Sproesser et al., 2020). Moreover, the largest standard deviation was found for AG. This might be explained by forgetting effects in cases of longer temporal distances between the testing and the teaching unit on functions. Together with the rather low correlations between the function conceptions and AG, it can be hypothesized that the students perform this representational change rather algorithmically and without understanding, which is prone to error and forgetfulness. In contrast, the representational changes SA and SG were higher related to the function conceptions and disposed of higher mean values and lower standard deviations indicating better and more sustainable competency. In this sense, our research implies to foster the function conceptions in the mathematical classroom in conjunction with understanding-oriented representational changes. At the same time, our study confirms the recommendation to reduce algorithmic procedures that might be forgotten quickly and do not express any understanding or conceptions.

To end up, we want to touch upon one limitation of our study. Although one test focuses on function conceptions and the other on representational competencies, there is an overlap between both tests that should be taken into account, when interpreting the results: All items contain function representations and function conceptions are relevant when dealing with representations. However, the focus of the items could be allocated to (the subfacets of) function conceptions or representational competencies, which is currently substantiated by an expert rating.

References

Adu-Gyamfi, K., Stiff, L. V., & Bossé, M. J. (2012). Lost in translation: examining translation errors associated with mathematical representations. *School Science and Mathematics, 112*(3), 159–170. https://doi.org/10.1111/j.1949-8594.2011.00129.x

Bossé, M. J., Adu-Gyamfi, K., & Cheetham, M. R. (2011). Assessing the difficulty of mathematical translations: Synthesizing the literature and novel findings. International *Electronic Journal of Mathematics Education, 6*(3), 113–133. https://doi.org/10.29333/iejme/264

Digel, S., & Roth, J. (2020). A qualitative-experimental approach to functional thinking with a focus on covariation. In A. Donevska-Todorova, E. Faggiano, J. Trgalova, Z. Lavicza, R. Weinhandl, A. Clark-Wilson, & H.-G. Weigand (Eds.), *Proceedings of the 10th ERME Topic Conference on Mathematics Education in the Digital Age (MEDA) 2020* (pp. 167–174). Johannes Kepler University.

Digel, S., & Roth, J. (2022). Fostering a concept of function with combined experiments in distance and in-class learning. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas

(Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 227–234). PME.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Krüger, K. (2019). Functional thinking: The history of a didactical principle. In H.-G. Weigand, W. McCallum, M. Menghini, M. Neubrand, & G. Schubring (Eds.*), The legacy of Felix Klein* (pp. 35–54). Springer International Publishing.

Lichti, M., & Roth, J. (2019). Functional thinking − A three-dimensional construct? *Journal für Mathematik Didaktik, 40*(2), 169–195, DOI: 10.1007/s13138-019-00141-3

Niss, M. (2020). Functions Learning and Teaching. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 303–306). Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0_96

Nitsch, R (2015). *Diagnose von Lernschwierigkeiten im Bereich funktionaler Zusammenhänge*. [Diagnosing learning difficulties in the area of functional relationships] Springer.

Sproesser, U., Vogel, M., Dörfler, T., & Eichler, A. (2020). Typische Lernschwierigkeiten mit Darstellungswechseln bei elementaren Funktionen – Welche Schwierigkeiten kennen Lehrkräfte und wie schätzen sie Aufgabenbearbeitungen ihrer Klassen ein? [Typical learning difficulties with representational changes in elementary functions – What difficulties are teachers aware of and how do they judge their classes' performance?] *Mathematica Didactica, 43*(2020)2, 175–198. https://doi.org/10.18716/ojs/md/2020.1154

Sproesser, U., Vogel, M., Dörfler, T., & Eichler, A. (2022). Changing between representations of elementary functions: students' competencies and differences with a specific perspective on school track and gender. *International Journal for STEM Education, 9*(33). https://doi.org/10.1186/s40594-022-00350-2

Thompson, P.W., & Carlson, M.P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Vollrath, H.-J. (1989). Funktionales Denken. [Functional Thinking]. *Journal für Mathematik Didaktik, 10*, 3–37.

vom Hofe, R., Lotz, J., & Salle, A. (2015). Analysis: Leitidee Zuordnung und Veränderung. [Calculus: Guiding principle Correspondence and Change] In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Hrsg.), *Handbuch der Mathematikdidaktik* [Handbook of Mathematics Education] (pp. 149–184). Springer

vom Hofe, R., & Blum, W. (2016). "Grundvorstellungen" as a Category of Subject-Matter Didactics. *Journal für Mathematik Didaktik 37*, 225–254. https://doi.org/10.1007/s13138-016-0107-3

Zindel, C. (2018). *Den Kern des Funktionsbegriffs verstehen*. [Understanding the core of the functional concept]. Springer Spektrum.