BEGINNING THIRD-GRADE STUDENTS' MANIPULATIVE-BASED UNDERSTANDING OF THE PLACE-VALUE POSITION PRINCIPLE FOR TWO-DIGIT AND THREE-DIGIT NUMBERS

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One important of primary school mathematics education is understanding of the base-ten number system, encompassing the regrouping and the place-value position principle. When extending the number space, students need to extend their understanding of these principles. Converting between manipulatives and symbolic number representations is one indicator of such emerging understanding. We investigate whether beginning third-grade students can transfer their understanding of the place-value position principle from two-digit to three-digit numbers. Students show substantial, but not perfect manipulative-based understanding for two-digit numbers, and that they can transfer it to three-digit numbers to some extent. We find no evidence that this transfer is supported by the salience of the principle in the manipulative.

INTRODUCTION

Understanding the place-value system is the basis for learning arithmetic skills. It includes the knowledge of the decimal structure of our number system that forms the conceptual foundation of multi-digit number representations and operations. Moeller et al. (2011) showed that first-grade place-value understanding predicts third-grade arithmetic performance. Understanding this symbolic notation entails two fundamental principles (Jensen et al., 2021): The *regrouping principle* posits that an initially unstructured quantity is structured by sequentially bundling ten units or bundles each into one larger bundle, until no further collection of ten or more equal units or bundles is available. For example, 123 initial units can be grouped into one hundred, two tens, and three units in the decimal system. The *place-value position principle* comprises that the symbolic number representation may be obtained by noting down the number of bundles as a digit for each bundle size, starting with the smallest unit (units) on the rightmost position and proceeding with increasing bundle size. Thus, the position of each digit in the symbolic number representation indicates which bundle size it belongs to, and its value indicates how often the bundle occurs in the structured quantity. In Germany, place-value notation of two-digit numbers (up to 99) is introduced during grade 2 and extended to three-digit numbers during grade 3 (Staatsinstitut für Schulqualität und Bildungsforschung, 2014).

Several studies have demonstrated that students struggle to comprehend the symbolic place-value notation (e.g., Wittmann & Lenz, 2025; Jensen et al., 2021; Herzog & Fritz, 2022; Fromme, 2017). Herzog and Fritz (2022) report that about 63% of the

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students at the start of grade 3 can master two-digit decimal number representations, including both principles, primarily reporting "syntactical errors" (such as noting 90051 for 951), which strongly relate to not understanding the place-value position principle. Only two-thirds of their third-grade students in Jensen et al.'s (2021) study showed understanding of the place-value position principle. While many of these studies have focused on the understanding of primarily symbolic representations *after* completion of instruction on a new number range, there is only few evidence on how students can make sense of decimal number notations at points of their learning trajectory, while the number system is extended, for example, from two- to three-digit numbers. Such evidence is, however, crucial for an evidence-based design of instruction on these extensions of the number range.

In this report, we investigate how students can transfer their acquired understanding of the place-value position principle from two-digit numbers to three-digit numbers, when representing and reading off numbers from two different (digital) manipulatives.

THEORETICAL BACKGROUND

In his Triple-Code-Model Dehaene (1992) describes how (adult) individuals deal with numbers and their representations. He focuses on mental transcoding between verbal, symbolic, and approximative, linear, and mostly unstructured quantity representations. Also in mathematics education, the ability to transcode between differently structured representations of the same mathematical object is considered a key aspect of conceptual understanding (Duval, 2006). Thus, models of place-value understanding adopted this perspective (e.g., Van de Walle et al., 2023; Fromme, 2017; Herzog & Fritz, 2019). Often, manipulatives are introduced as additional representations to support acquisition and application of mathematical concepts such as place-value system (Donovan & Fyfe, 2022). Manipulatives are concrete or digital objects that allow students to access and interact with to mathematical structures and principles in a concrete, embodied or visual way. Base-ten blocks and place-value charts are frequently used manipulatives for place-value concepts (Wittmann & Lenz, 2025). From the representational transfer perspective, the regrouping principle deals mostly with re-structuring a quantity within a given representation (*treatment* in Duval, 2006), for example, by exchanging 10 ten rods for one hundred flat, or by moving a counter from the tens to the hundreds column. The place-value position principle indicates how the symbolic decimal number representation relates to other representations of the quantity (*conversion* in Duval, 2006). We define *manipulative-based understanding of the place-value position principle* as the ability to convert numbers from manipulatives into symbolic number notation and—extending the definition by Wittmann and Lenz (2025)—also vice versa.

Different manipulatives may differ in the mathematical structures and principles to which they provide access. Base-ten blocks (fig. 1) provide very salient access to the decimal bundling structure, because the equivalence of 10 ten rods with one hundred flat is directly observable. The place-value position principle is, however, less salient,

since the same number can be represented, independently of the position of the different blocks. Contrary, a place-value chart with counters (fig. 1) does make place-value position principle more salient, since the value of a counter for the whole number obviously depends on its position in the chart. In this representation, however, the regrouping principle is only encoded in the title row of the chart, without visible access to the bundling structure. Instructional strategies such as concreteness fading (Fyfe et al., 2015) are based on the idea that students should be provided with manipulatives that make the targeted structure very salient at the start of concept acquisition, but that this salience should be reduced later (cf. Resnick et al., 1998).

Based on an initial, manipulative-based understanding, a more abstract understanding of place-value may be targeted with concreteness fading strategies (Fyfe et al., 2015). Moreover, when introducing new operations with decimal numbers, tasks may be translated back to the manipulative-based representation to allow meaningful treatment (Uttal et al., 2013). This, however, requires the ability to convert between symbolic and manipulative-based representations of the same number, which means it requires manipulative-based understanding of the place-value position principle. When it comes to extensions of the number range, such as from two-digit to three-digit numbers in grade 3 in Germany, manipulatives may serve as a tool to connect to acquired manipulative-based understanding of place-value principles and to extend this understanding to the new number range. It is unclear, however, to which extent students can informally succeed in such a transfer to the new number range based on the mathematical structures encoded in the manipulatives without further support, and whether the manipulative of the direction of the conversion (to vs. from the symbolic representation) affects this transfer.

GOALS OF THE PRESENT STUDY

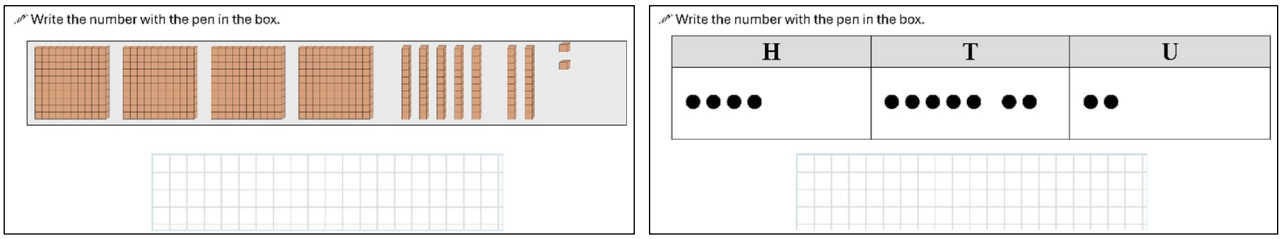
Based on the reported state of research, we were interested whether third-grade students (before introducing to three-digit numbers) could represent and read off symbolic numbers when using digital versions of the place-value chart—with a quite salient place-value position structure—and the base-ten blocks—with a less salient place-value position structure (Q1). We further expected that students would transfer their acquired knowledge of the place-value position principle on two-digit numbers to three-digit numbers to a limited extent, leading to lower but not negligible solution rates for three-digit numbers compared to two-digit numbers (Q2). We expected that this transfer would be easier (reflected by a weaker disadvantage for tasks with three-digit number compared to two-digit numbers) for the place-value chart due to the more salient place-value position structure (Q3). Finally, we explored whether the transfer to three-digit numbers would pose particular problems for representing or for reading off symbolic numbers from the two manipulatives (Q4).

METHODS

From September to November 2024, we conducted a cross-sectional study with 742 students from 41 third-grade classes in Germany (N_{female} = 354, $N_{genderMissing}$ = 6; M_{age} = 8.32 years, range = *6–12*). The precondition was that numbers up to 1000 had not yet been introduced at that time. One class had already received the introduction, which is why it was excluded from the analysis.

36 tasks on the place-value position principle were presented on tablets. The students had 15 minutes to answer the tasks. The manipulatives (base-ten blocks, place-value chart with counters) were available as interactive tools on tablets; symbolic number notations were printed on the screen or could be written with a pen on the tablet. The task was to transfer the number from the initial number representation into the specified target number representation. There were four sets of tasks, which differed in the initial number representation (Base-ten blocks, place-value chart with counters, symbolic number notation). Seven tasks each required the transfer from the symbolic number notation to the place-value chart or to the base-ten blocks. There were also 7 tasks involving transferring from the place-value chart to the symbolic number notation. Because the base-ten blocks can be presented in an unstructured way, there were more tasks to do during the transfer from the base-ten blocks to the symbolic number notation (15 tasks). Figure 1 shows examples in which the symbolic number notation is to be written from the base-ten blocks or the place-value chart with counters. Within these sets, the tasks were systematically varied.

Figure 1: Examples for place-value tasks with transfer from the base-ten blocks or place-value chart with counters into the symbolic number notation (own illustration).



The tasks also differed according to the number of digits in the presented number, the number of positions with digit 0, and for the base-ten blocks in whether the sequence of blocks matched the symbolic number notations (as in fig. 1). There were 4 tasks with one-digit, 10 tasks with two-digit, and 22 tasks with three-digit numbers.

The students' answers were recorded per task and coded using a coding manual ($\kappa > 0.80$). We analysed our data descriptively, and with a generalized linear mixed model (GLMM), a generalization of logistic regression and ANOVA. We used *digit number* (two vs. three-digit), *manipulative* (place-value chart vs. base-ten blocks), and conversion *direction* (*represent* a given symbolic number on the manipulative vs. *read off* a number from the manipulative and represent it symbolically) as fixed factors, and

included random factors for the persons and the items. We report Chi-Square Statistics for omnibus tests and odds ratios as effect sizes for differences between conditions.

RESULTS

Table 1 shows the solution rates by digit number, manipulative, and transfer direction. The average solution rate across all tasks was 80.9 %. This shows high, but not perfect overall performance. Just under 12 % of the students made no errors in the tasks they completed. Solution rates for single-digit items were very high, so we decided to include only two- and three-digit number tasks in the GLMM analysis.

Table 1: Solution rates depending on the number of digits of the number and the transfer between the representations in the tasks

	One-digit numbers	Two-digit numbers	Three-digit numbers	Total
Symbolic number notation to base-ten blocks	95.2 %	92.3 %	83.7 %	87.7 %
Base-ten blocks to symbolic number notation	97.0 %	86.7 %	77.6 %	81.5 %
Symbolic number notation to place-value chart	89.9 %	76.0 %	72.6 %	76.0 %
Place-value chart to symbolic number notation	95.3 %	82.7 %	71.4 %	78.1 %
Total	94.2 %	84.6 %	76.5 %	80.9 %

The GLMM analysis for two- and three-digit number tasks showed a significant main effect of digit number ($\chi^2(1) = 11.30$, $p < .001$) and transfer direction ($\chi^2(1) = 6.90$, $p < .01$). Beyond their main effects, there is only one significant interaction of manipulative and transfer direction ($\chi^2(1) = 8.01$, $p < .01$). Analysing this interaction (Q1) showed that performance did not differ significantly between the two manipulatives when reading off numbers from the manipulatives into the symbolic form ($OR = 1.22$, $p = 1.00$). When representing numbers on the manipulatives, however, students showed significantly lower performance on the place-value chart than on the base-ten blocks with a large effect size ($OR = 4.07$, $p < .0001$). The same descriptive pattern can be observed in the solution rates for one-digit number tasks.

Regarding Q2, we found significantly lower performance on three-digit (76.5 %) than on two-digit number tasks (84.6 %, $OR = 4.51$, $p < .0001$) with a large effect size. Yet, solution rates for three-digit numbers were above 65 % in each condition, indicating that students could cope with the yet unknown three-digit numbers to some extent.

For Q3 and Q4, we found small effect sizes for the insignificant interactions of the number of digits with either manipulative ($OR = 0.65$, $p = .23$) or transfer direction ($OR = 0.76$, $p = .43$), indicating that the transfer of acquired knowledge did not observably depend on these task characteristics. Also, the three-way interaction was insignificant (p = .21).

DISCUSSION

Starting from students' problems with the main principles of the decimal place-value system (Herzog & Fritz, 2022; Jensen et al., 2021; Wittmann & Lenz, 2025), and the role of manipulatives for mathematics learning (Carbonneau et al., 2013), we investigated to which extent students can represent and read off symbolic numbers from versions of two manipulatives—base-ten blocks and place-value chart with counters. Such tasks dig into the sound understanding of the place-value position principle. Given the role of manipulative-based actions for initial concept acquisition and extension, we were particularly interested in the extent to which students could transfer their acquired knowledge for two-digit numbers to yet unknown three-digit numbers. We draw the following main conclusions from our study:

We had expected that, due to the more salient structure of the place-value position principle, performance on the place-value chart would be higher than with the base-ten blocks. This is even more noteworthy, since we could include number representations, in which the sequence of object frequencies did not match the symbolic number notation (e.g., 3T 2H 1U or 1T 2H 2T 1U for 231), in tasks requiring to read off numbers from the base-ten block manipulatives, but not from the place-value chart. However, reading off numbers turned out equally hard for both manipulatives, and representing numbers was even harder on the place-value chart than with the base-ten blocks. Given the unique potential of this manipulative to highlight that the value of each counter (or digit) depends on its position—a major idea of place-value notation—it might be promising to investigate whether the observed difficulties are really due to an intrinsic complexity of the representation, or if they merely arise, because the place-value chart—particularly with counters—is less frequently used in classroom instruction in Germany than the base-ten blocks. Another reason might be that the matching of columns to bundle sizes is not given by visual quantity cues (as in the base-ten blocks) in the place-value chart, but only by the bundle size symbols H, T, or U. However, Herzog and Fritz (2022), however, report that identifying bundle names for digit positions is very easy for participants of their study. More research on this manipulative may investigate which task characteristics are specifically challenging for students, and also consider their thinking processes while transcoding the numbers.

As expected, we find that students show lower performance for the unknown three-digit numbers than for the two-digit numbers, but performance was still substantial (76.5%) and compares to solution rates observed with other notations, such as 1H 2T 3U for 123 *after* introducing three-digit numbers (68.3%, Jensen et al., 2021). This indicates that an initial transfer of acquired knowledge on two-digit numbers to

corresponding tasks on three-digit numbers with manipulatives prior to explicitly introducing three-digit numbers may be challenging, but not impossible. Given results on students' problems with symbolic multi-digit numbers (e.g., Fromme, 2017), this initial and informal transfer could serve as a springboard to establish meaningful, manipulative-based conceptions of how symbolic three-digit numbers represent decimally structured quantities.

Finally, we found no evidence that transferring knowledge from two-digit to three-digit numbers would be affected by either the manipulative, or by whether numbers need to be converted from the manipulative to the symbolical notation, or vice versa. The different salience of the place-value position principle in the manipulatives or other factors influencing the overall solution rates therefore do not appear to strongly impact transfer from two- to three-digit numbers. Also, neither using acquired two-digit knowledge to represent nor to read off three-digit numbers at the manipulative seems to be observably harder than the other. This indicates that this transfer may be a quite general process, which can be mastered (if it can) largely independent from the concrete task.

Our results need to be seen considering a range of limitations. We focused on conversions between the manipulatives and the symbolic number representation (Duval, 2006), which primarily requires understanding of the place-value position principle. Thus, we cannot provide any insights on the regrouping principle or the combined application of both principles. Furthermore, we focused on two very specific and digital manipulatives. Usability and familiarity may have affected overall performance. However, the transfer to three-digit numbers did not seem to be affected by the differences between the manipulatives. Finally, we investigated students at a very specific point of their learning trajectory—just before introducing three-digit numbers. While our findings are surely restricted to this context, we emphasize that they are of specific importance for the design of mathematical learning opportunities at this point. For example, discussing typical errors (e.g., representing 107 as 1T 7U) in the place-value chart may help to discuss the place-value position principle explicitly, due to its increased salience at this manipulative.

In summary, our study indicates that students can use manipulatives to represent and thus make sense of symbolic two-digit numbers quite well at this point of their learning trajectory. Informal transfer to three-digit numbers poses problems but does not seem out of reach. Even though students seem to have generally fewer problems when working with the base-ten blocks than with the place-value chart with counters, we did not find evidence that the transfer to three-digit numbers is particularly harder with any of the two manipulatives. As long as place-value charts do not get more attention in instruction, this may speak for using base-ten blocks in a first step of the transition to three-digit numbers. However, the specific potential of place-value charts for making the place-value position principle accessible should not be neglected.

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