PROSPECTIVE SECONDARY TEACHERS' PERSONAL RELATIONSHIP WITH COMBINATORICS: IMPLICATIONS FOR THEIR DIDACTICAL AND MATHEMATICAL TRAINING

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This paper explores prospective secondary mathematics teachers' personal relationship with combinatorics (PR_C). Existing research on the teaching and learning of combinatorics calls for an expanded role of combinatorics in secondary mathematics curricula; however, there is limited understanding of how teachers are trained in this domain and their views of combinatorics content. This study's findings reveal a diverse range of PR_C, some of which may prevent prospective teachers from exploiting the recognised educational potential of combinatorics. Participants demonstrated difficulties applying techniques appropriate for secondary classrooms and justifying techniques learned at university. Results point to several promising avenues for strengthening mathematics teacher training related to combinatorics.

INTRODUCTION

The teaching and learning of combinatorics is a growing area of research in mathematics education. Although numerous studies have documented students' difficulties with combinatorics (e.g., Batanero et al., 1997), recent work has highlighted its potential for promoting mathematical reasoning and supporting the development of mathematical ideas such as generalisation and equivalence (Lockwood & Reed, 2020). Combinatorics also has important applications in domains such as graph theory, coding, game theory and cryptography. On this basis, researchers such as Lockwood et al. (2020) and Sandefur et al. (2022) have argued for a greater presence of combinatorics in curricula, including at the secondary level.

As a result, researchers have increasingly turned their attention to teachers, who play a vital role in the teaching and learning of combinatorics and whose involvement is essential for a more meaningful inclusion of combinatorics content in secondary mathematics: "research needs to include studies that provide information on teacher learning in this domain [and] studies on how teachers come to see this domain as connected to the curricula that they teach" (Lockwood et al., 2020, p. 9). The first author's master's study (Morcos, 2025) addresses this issue by comparing secondary and university combinatorics content to better understand current practices at these two levels of education, and by investigating how prospective teachers are affected by consistencies and inconsistencies between combinatorics content in secondary and university curricula. This paper specifically seeks to analyse: 1) prospective secondary mathematics teachers' views and perceptions of combinatorics content, and 2) possible

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links between these perceptions and prospective teachers' experience with combinatorics-related practices as secondary and university students. We believe that this type of research can contribute to ongoing discussions about the inclusion of combinatorics in secondary mathematics, with a view to adapting teacher training so as to make this inclusion more effective.

THEORETICAL FRAMEWORK

This paper's aim is twofold: 1) to analyse prospective teachers' perceptions of combinatorics, and 2) to explore possible links between these perceptions and current practices in secondary mathematics and in university mathematics courses in teacher training programmes. Our study employs tools from the Anthropological Theory of the Didactic (ATD—Chevallard, 1999). ATD posits that the teaching and learning of mathematics are human practices shaped by the institutions in which they occur. In ATD, an institution is a social system that defines, regulates, and legitimises specific practices with respect to a given body of knowledge. Within institutions, individuals occupy specific positions or roles that, in turn, orient their practices. Chevallard (2003) defines two theoretical constructs related to practices and positions. First, the *institutional relationship* (IR) with content refers to the way content is defined, organised and used within a given institution. In this paper, we focus on the IR with combinatorics content in two specific institutions: secondary mathematics ($IR_{SM\text{-}C}$) and university mathematics courses in teacher training programmes ($IR_{UM\text{-}C}$). Second, an individual's *personal relationship* (PR) with a given content refers to the system of interactions between the individual and that content: the way the individual uses, manipulates, talks about, describes and perceives the content. An individual's PR evolves over time and through experience, influenced by the multiple institutions in which they participate and by their practices related to specific content.

In ATD, relationships with content can be investigated using the analytical tool of the *praxeology*. Praxeologies comprise a practical block, which includes tasks and the techniques used to solve them, and a theoretical block, which consists of the rationales (technological discourses) that support, explain or justify the use of specific techniques, together with the theory that validates these rationales. Praxeologies related to specific content within an institution provide insight into the IR with that content, while praxeologies deployed by an individual shed light on that individual's PR with the content. This theoretical approach was used by González-Martín & Hernandes-Gomes (2020) to study engineering teachers' personal relationships with mathematics and to investigate how these teachers' backgrounds influence their PR and, consequently, their practices. In this paper, we use these constructs to address the following research question: *What are the characteristics of prospective teachers´ personal relationship with combinatorics (PR_C) and how does their PR_C relate to $IR_{SM\text{-}C}$ and $IR_{UM\text{-}C}$*? We have previously published our results of the analysis of $IR_{SM\text{-}C}$ and $IR_{UM\text{-}C}$ (Morcos, 2025; Morcos & González-Martín, 2025a, 2025b, 2025c). In this paper, we summarise

the main findings concerning IR_{SM-C} and IR_{UM-C}, then we present our results to the research question.

METHODOLOGY

This research is based at the Université de Montréal (UdeM), in Canada, and draws on two data sources: textbook analysis (references are provided here) and interviews with prospective teachers. The textbook analysis illuminates institutional praxeologies and thus provides a model of IR_{SM-C} and IR_{UM-C}. We analysed the praxeologies related to combinatorics identified in all three secondary mathematics textbooks (ST1, ST2, ST3) approved by the Québec Ministry of Education, and in the two university textbooks (UT1, UT2) used in the UdeM teacher training programme for its *Probability* and *Discrete Mathematics* courses. These courses are taken by a wide range of students and are the only two courses that address combinatorics in UdeM's training programme for prospective secondary teachers.

For this study, we conducted individual semi-structured interviews with four prospective teachers (T1, T2, T3, and T4) enrolled in the UdeM programme. All participants had successfully completed the two mathematics courses that address combinatorics. Participants were first asked to solve six combinatorics tasks from the analysed textbooks (four from the STs and two from the UTs), justifying their solutions and explaining their rationales. This revealed certain aspects of their personal relationship with combinatorics (PR_C). In the second part of the interviews, the participants answered questions about their views on combinatorics, allowing for a more detailed characterisation of their PR_C. The interview data were also used to triangulate the findings from the textbook analysis. The interviews were transcribed for analysis, with excerpts translated from French into English. The analysis was open coded (Saldaña, 2013), based on participants' responses concerning their views on combinatorics and the connections we established with their training and secondary mathematics. The initial data analysis was conducted by the first author and verified by the second author, with minor disagreements resolved when necessary.

RESULTS AND DISCUSSION

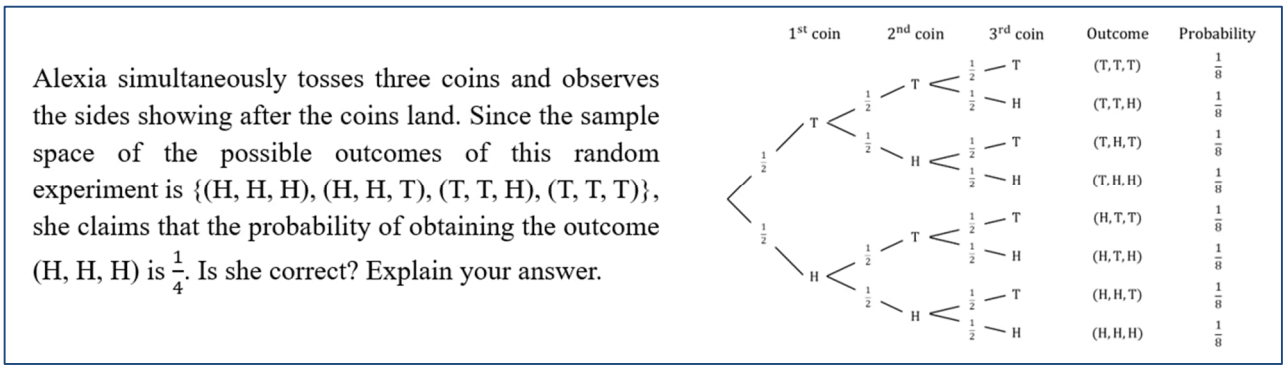
The institutional relationships with combinatorics (IR_{SM-C} and IR_{UM-C})

As indicated above, we summarise here results published in Morcos (2025) and in Morcos & González-Martín (2025a, 2025b, 2025c) concerning IR_{SM-C} and IR_{UM-C}, in order to then make connections between the participants' PR_C and the institutions in which they have studied this content.

Our textbook analysis reveals important differences between IR_{SM-C} and IR_{UM-C}. In the three secondary textbooks analysed, combinatorics content is not explicitly identified; rather, it is introduced in the probability chapter, which strongly influences the praxeologies related to combinatorics. Tasks involving combinatorics are framed as probability problems that require students to count favourable and possible outcomes of a random experiment. Various techniques are proposed to carry out these counts,

including systematic enumeration, tree diagrams, grids and tables, Venn diagrams, and combinatorial rules such as the multiplication principle[1]. This could, in principle, support connections within and across contents, and thus foster rich mathematical reasoning; however, these techniques, while grounded in combinatorics, are presented in all three textbooks as tools for probability tasks. The technological discourses that accompany the techniques are anchored in probability theory, and omit rationales related to combinatorial practices and reasoning. For example, ST1 uses tree diagrams to list outcomes and to determine the probability of an event, as illustrated in Figure 1, while masking important combinatorial notions such as equivalence and structured enumeration and providing no clear rationale to support the development of mathematical reasoning. Similarly, the multiplication principle, fundamental in combinatorics, is introduced specifically to solve probability tasks (for more details and for possible impacts on prospective teachers, see Morcos & González-Martín, 2025a). This suggests that, in secondary mathematics, the *raison d'être* of combinatorics is reduced to the counting of favourable and possible outcomes of probability experiments. Consequently, $IR_{SM\text{-}C}$ appears limited to the application of combinatorics content in probability, both practically (in terms of tasks and techniques) and theoretically (in terms of explanations and justifications). This limits the potential role and nature of combinatorics in secondary mathematics.

Figure 1: Example of a task solved with a tree diagram in ST1 (p. 347)



In each of the university textbooks analysed, combinatorics is addressed in a chapter devoted to combinatorial analysis. Thus, combinatorics is studied independently of probability, which positions it as a specific branch of mathematics. It is then used to solve various tasks for applications in fields such as programming (in addition to probability). This approach to organising content is a key characteristic of $IR_{UM\text{-}C}$ and differs qualitatively from $IR_{SM\text{-}C}$. Moreover, in the UTs, praxeologies are organised around specific counting formulae associated with corresponding sets of tasks (for instance, "permutations" or "combinations"). Solving techniques largely rely on ready-to-use formulae that are absent from the STs. In the UTs, techniques are

[1] "The total number of possible outcomes in a random multi-stage experiment is equal to the product of the number of possible results at each stage" (ST3, p. 371).

institutionalised as theorems, with only a few technological discourses provided to explain or justify the techniques and their use. Tasks are grouped according to the counting formula suggested for their solution, resulting in the compartmentalisation of some equivalent tasks (see, for example, Morcos & González-Martín, 2025c).

Prospective teachers' personal relationship with combinatorics (PR_C)

We observe a variety of PR_C among the four participants. First, T2's PR_C incorporates elements present in both $IR_{SM\text{-}C}$ and $IR_{UM\text{-}C}$. T2 defines combinatorics as "the branch of mathematics concerned with counting things" but also recognises its role in probability. When asked to comment on the statement, "In secondary mathematics, combinatorics content is clearly identified", they respond as follows:

> False. […] I had never heard the word "combinatorics" before my *Discrete Mathematics* course. The idea that it is a branch of mathematics in itself, ultimately, was completely new to me. So, I would say that, in my head, it was just a way to find the answer in probability, because that's how we saw it in secondary school. [At university], now we talk about it as a branch of mathematics. It's, like, we dedicate a chapter to it, not necessarily in the context of probability, but really in the context of counting, pure counting.

T2's views have been shaped by their experience both at secondary school ($IR_{SM\text{-}C}$) and at university ($IR_{UM\text{-}C}$), revealing an evolution in their PR_C. This suggests that university combinatorics courses in teacher training programmes ($IR_{UM\text{-}C}$) can contribute to the development of teachers' PR_C. When asked whether combinatorics can help foster deep mathematical thinking, T2 responds: "I think so. […] Because there are different ways to get to the same answer. Appealing contexts. And sometimes there are twists that, I don't know, that show mathematical thinking and mathematical richness, for example when considering the complementary [set or event]". This view echoes that of Lockwood et al. (2020) and Sandefur et al. (2022), who highlight the potential of combinatorics in secondary mathematics and advocate for its expanded presence in secondary curricula. However, this broader view does not necessarily enable teachers to mobilise praxeologies from both institutions and deepen their teaching practices. T2 correctly solves only half of the six interview tasks, drawing mostly on factorials and formulae learned at university. T2 also notes that techniques used in their secondary school classes, such as systematic enumeration and tree diagrams, were rarely seen in their university courses. This may give the perception that secondary school techniques are informal: "[Enumerating outcomes] is the equivalent of trial and error in my opinion, meaning, it's like a brute force method, really. […] It's not forbidden [at university], I think, but I think that it was never formally shown, that's for sure".

In T4's interview, $IR_{UM\text{-}C}$ seems to provide advantages and constraints. While T4 views combinatorics as "the art of counting" and as "one of the most useful things" in mathematics (with applications in game theory) they also raise concerns about the algorithmic approach to combinatorics in $IR_{UM\text{-}C}$, which led them to learn the university content by heart rather than develop praxeologies useful for secondary teaching. For example, when solving a task requiring the selection without order of three elements

from a set of six, T4 states: "That's how it was defined in the *Discrete Mathematics* course, [like a combination]. Because otherwise, to me, it didn't mean anything, the $\binom{6}{3}$, it doesn't mean anything to me". T4 is unable to produce an alternative technique for the same task without using the formula (which is absent from $IR_{SM\text{-}C}$), solving only one of the interview tasks correctly. This suggests that, although the two university courses have helped broaden T4's view on combinatorics, the $IR_{UM\text{-}C}$ praxeologies have constrained their PR_C by failing to provide techniques that align with $IR_{SM\text{-}C}$ or strategies that would enable them to develop more intuitive techniques on their own.

The PR_C of T1 and T3 display elements that appear more closely aligned with $IR_{SM\text{-}C}$, as both participants view combinatorics as a tool for probability. T1 suggests that combinatorics may be "the ability to express the probability of an event within a [set of possibilities]". When solving the interview tasks—four of the six correctly—T1 mobilises various techniques from secondary and university mathematics (systematic enumeration, tree diagrams, factorials and formulae, etc.) but struggles to justify these practices and explain their reasoning, which would be useful for teaching. These difficulties lead T1 to identify a gap in $IR_{UM\text{-}C}$ concerning rationales: "If only [university] teachers were able to not just be theoretical, but also to explain the why and the ins and outs, well, of course that would help. But I never had a teacher who taught like this, it was always theoretical, and, like, the formula". This formula-based approach, characteristic of $IR_{UM\text{-}C}$, seems to hinder T1's ability to perceive and exploit the full potential of combinatorics.

Like T1, T3 defines combinatorics as "the introduction to probability", although they give several examples of other fields that make use of combinatorics, such as computer science, whereas T1 mentions applications in probability only. T3 successfully solves three of the six tasks, using praxeologies observed in $IR_{SM\text{-}C}$ and $IR_{UM\text{-}C}$ (similarly to T1). However, T3 notes that their university training in combinatorics did not include discussions about "how to teach" combinatorics or about common student errors and difficulties. In fact, all four participants spontaneously indicate at the end of their interview that they appreciated our questions related to the teaching of combinatorics in secondary school and found them relevant to their future role as teachers. This suggests a possible avenue of research on teacher training in combinatorics; namely whether stronger links between university didactics courses and mathematics courses addressing combinatorics could foster a PR_C that bridges the gap between $IR_{SM\text{-}C}$ and $IR_{UM\text{-}C}$ and unlock the potential of combinatorics for secondary mathematics.

We also observe differing views among the participants regarding the potential of combinatorics to help secondary students develop deep mathematical thinking. While T3 recognises this potential, T1's view appears more limited:

> Yes. Yes, there is a mathematical richness behind combinatorics that is maybe more limited than the richness in calculus. Because in calculus […] you can calculate again and again, you could go very far. But in combinatorics, it's like a closed box that goes a little less towards infinity. We always go back to the same things. […] Is it really an infinite subject?

> Maybe, but like, from what I've learned, I feel like it is more a small box that contains a few ways to do things, and there are tools in that box that we work with. So, there isn't that expansion that we could find in other mathematical branches.

It appears that the combinatorics training received at university does not lead to a uniform PR_C among participants. While some participants see the many applications of combinatorics and its potential for secondary students, others view it merely as a toolbox for probability. We also observe varied repertoires of techniques in the participants' PR_C, with some unable to mobilise techniques aligned with $IR_{SM\text{-}C}$, relying instead on ready-to-use formulae from $IR_{UM\text{-}C}$. All participants struggled to justify the combinatorics techniques they employed.

FINAL REMARKS

The results point to diverse PR_C among the participants, influenced by the IR with combinatorics in secondary mathematics ($IR_{SM\text{-}C}$) and university mathematics courses in the teacher training programme ($IR_{UM\text{-}C}$). In $IR_{SM\text{-}C}$, combinatorics is primarily a tool for probability tasks, and both T1 and T3 share this limited view. Although secondary mathematics introduces various techniques, their uses and justifications remain confined to probability theory; moreover, these techniques are not revisited at university, and the participants who draw on these techniques struggle to justify them. In contrast, $IR_{UM\text{-}C}$, presents combinatorics as a specific branch of mathematics, which seems to have allowed T2 and T4 to develop a broader view of combinatorics, suggesting that their university training had a positive influence. However, $IR_{UM\text{-}C}$ tends to use ready-made formulae and lacks rationales to support mathematical reasoning, which limits its potential to be exploited by prospective teachers.

By identifying links between prospective teachers' PR_C and $IR_{SM\text{-}C}$ or $IR_{UM\text{-}C}$, this study identifies areas for improvement in combinatorics education. In particular, if the place and role of combinatorics in secondary mathematics are to be expanded (as recommended by researchers), university courses could be enhanced to support the development of meaningful practices among prospective teachers. This could be achieved by making more explicit connections with secondary practices and by incorporating a didactic dimension in the combinatorics training of prospective teachers. Overall, our results indicate that prospective teachers' PR_C may not currently support the optimal teaching and learning of combinatorics in secondary mathematics, which could hinder efforts to increase its presence in curricula. Further research is needed on the teaching and learning of combinatorics, particularly on ways to strengthen both the mathematical and didactical training of teachers.

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