HOW DO FIRST-YEAR TEACHER STUDENTS EVALUATE VISUAL REPRESENTATIONS OF THE COMMUTATIVITY OF MULTIPLICATION?

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Visual representations are essential for teaching fundamental properties of multiplication such as commutativity. However, little is known about how beginning preservice teachers (PSTs) assess such representations. This study investigates how first-year PSTs at three German universities (N = 863) evaluate visual representations of the commutative law of multiplication and which criteria underlie their judgments. Results indicate that many students relied primarily on surface-related features (e.g., clarity or familiarity) rather than content-related aspects for evaluation. Even conceptually problematic representations were often rated positively. The findings point to a general surface orientation of some PSTs and highlight the need for teacher education to promote content-focused and comparative analyses of representations.

INTRODUCTION

Due to the abstract nature of mathematics, the importance of external representations, and thus also of visual representations, for mathematical activity and learning mathematics is repeatedly emphasized internationally (e.g., Arcavi, 2003). A meta-study that included 41 intervention studies conducted between 2010 and 2023 with a total of 10,562 learners demonstrated a significant positive impact of using diverse visualizations across different mathematical topics (Schoenherr et al., 2024). The use of appropriate representations is therefore considered a decisive factor for the quality of mathematics teaching (Dreher et al., 2024) and, accordingly, an important component of pedagogical content knowledge (Shulman, 1986), special content knowledge (Ball et al., 2008) or more recent models of professional competence for mathematics teachers (e.g., Lindmeier, 2011).

Multiplication and its properties are important topics in primary school mathematics, and visualizations are often used to teach them. The use of appropriate representations is essential, particularly for developing students' understanding of fundamental (and at the same time abstract) properties such as commutativity. However, little is known about how teachers select and use representations in their classroom and how they can acquire the relevant competences during their university education. The study presented here aims to contribute to filling this research gap by investigating how and according to which criteria first-year preservice teachers (PSTs) evaluate visual

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representations of the commutative law of multiplication. The results may provide insights into which learning opportunities are important for their mathematics education training.

REPRESENTATIONS FOR TEACHING MULTIPLICATION

It is often argued in the literature that multiplication represents a qualitative change in children's thinking compared to addition or subtraction because (in the most important understanding for primary school) the two factors have different meanings (Harries & Barmby, 2007). Therefore, the use of external, including visual representations, is very important for teaching multiplication in mathematics classes. They can be used not only to illustrate multiplicative situations or symbolic representations, but also to develop students' conceptual understanding. Frequently used types of representations are: set models (equal groups, arrays), length models (number line, line segments), area models (rectangle) or tree diagrams (Kosko, 2019). In particular, arrays are considered useful for developing multiplicative thinking and for studying the properties of multiplication because they visualize the three quantities involved at the same time and allow flexible partitioning (Hurst, 2015). This makes arrays particularly valuable as representations, since a thorough understanding of the basic properties of arithmetic operations is important not only for arithmetic but also for integrating arithmetic and algebra.

However, selecting, creating, evaluating, and using appropriate representations in the context of multiplication is challenging for (preservice) teachers. For example, in a study involving 56 PSTs who registered for their math semester at a U.S. university, only one participant was able to visualize the associative law of multiplication appropriately using a picture (Ding et al., 2013). Milinković (2012) examined preferences of 121 Serbian PSTs for primary school during their fourth year in using representations in the context of multiplication. Among other things, they were requested to select suitable representations for "$3 \times 4 = 4 \times 3$" and "$a \times b = b \times a$" from various given options. While 69% of the students considered an array as the best choice for visualizing the concrete example, only 6% used this representation for the abstract identity with symbols. In a larger study involving 445 PSTs at various stages of their studies from England and Serbia, Barmby and Milinković (2011) investigated which representations were selected for different content in the context of multiplication. Selecting a suitable representation for the commutative law was one of the most difficult tasks.

AN EMPIRICAL INVESTIGATION

Research interest

Based on the difficulties and uncertainties identified, it seems important that PSTs are able to acquire competencies in dealing with representations (including for multiplication) as part of their education. To this end, it would be useful to know what

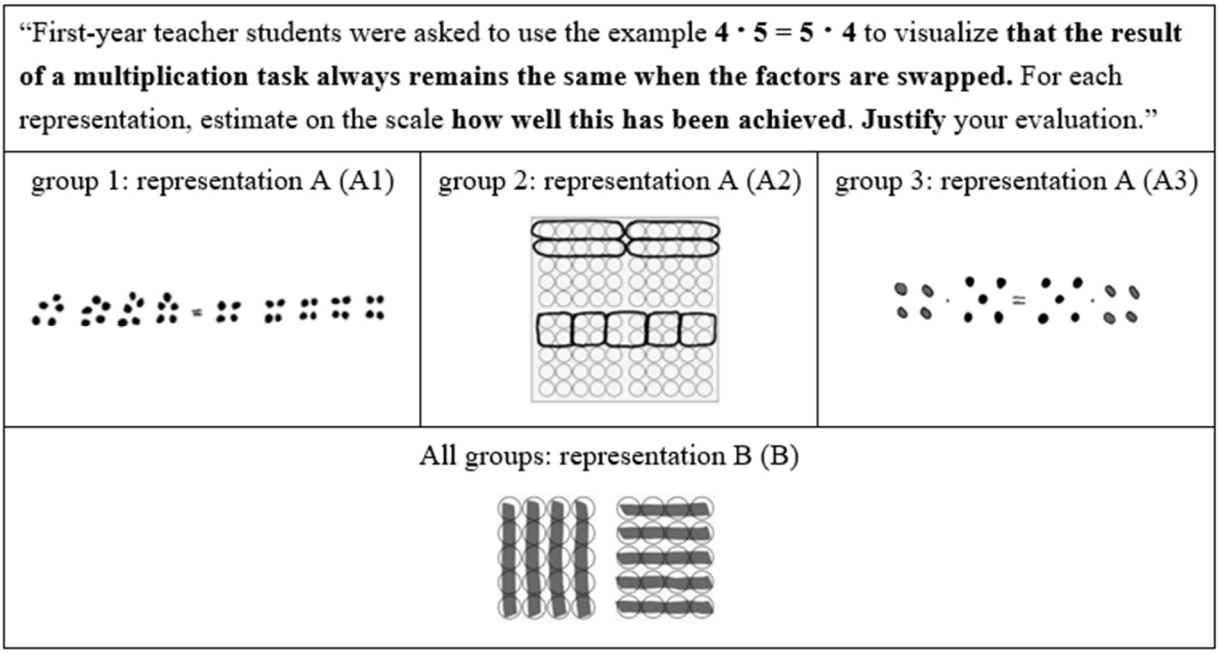
intuitive conceptions and evaluation criteria for representations first-year PSTs bring with them to university. This raises the following research questions:

1. How do PSTs evaluate representations of the commutativity of multiplication?
2. What criteria for evaluating the visual representations can be reconstructed?

Methodical approach and sampling

The study was conducted as a written survey of 863 first-year PSTs, mainly for primary schools, at three different German universities at the start of the semester. In addition to other items on multiplicative understanding, participants were asked to evaluate the quality of given visualizations of the commutative law of multiplication on a four-point scale (++, +, -, -- ; ++ standing for "best/most appropriate") and to justify their evaluations in free text. Three subgroups were formed for this purpose (group 1: N = 285, group 2: N = 284, group 3: N = 277; only complete and unambiguous evaluations were included). Each group was given two different representations, A and B, with representation B being the same in all groups (Fig. 1).

Figure 1: Item on commutativity (translated by the authors) and representations used (In Germany, "·" is used as symbol for multiplication)



The representations A1, A2, and B have a multiplicative structure by combining equal sets and the result can be captured by the number of dots. In A2 and B, the equality of the results of both tasks can be seen immediately, as both marked parts occupy the same area with dots arranged in the same way. The swapping of factors is only visualized in representation B. Since B can additionally be created for each product $a \times b$, it is the only one of the representations that is a generic example. Although A3 presents the factors in an easily perceivable form, which also makes it easy to visually

grasp the swapping of the factors, it does not visualize the commutative law due to its similarity to the symbolic equation and the lack of a multiplicative structure.

To analyze the data with regard to the types of arguments used, the students' written justifications were analyzed independently by two authors using deductive-inductive category formation (Mayring, 2015). The analysis was based on the PSTs' explicit written statements; results were validated by consensus.

RESULTS

The following table shows the evaluations of all representations; for easier comparison, the arithmetic mean was also calculated.

Table 1: Frequencies and arithmetic mean of students' evaluations
(Mean is calculated based on: ++: 1, +: 2, -: 3, --: 4.)

	++	+	-	--	Mean
group 1 A1	32.7%	47.5%	17.6%	2.1%	1.89
group 1 B	45.1%	28.9%	22.5%	3.5%	1.85
group 2 A2	31.4%	45.2%	20.8%	2.5%	1.94
group 2 B	52.3%	30.0%	14.8%	2.8%	1.68
group 3 A3	28.9%	31.8%	27.1%	12.3%	2.23
group 3 B	38.6%	35.0%	20.6%	5.8%	1.94

In all groups, both representations were rated positively by the majority of students. It seems particularly noteworthy that even the problematic representation A3 was rated positively by over 60% of students, with slightly less than 30% giving it the rating ++. However, inappropriate ideas about multiplication cannot explain these positive evaluations, as all students demonstrated an appropriate understanding of multiplication in other tasks in the survey. Therefore, a qualitative analysis will first be conducted for group 3 to work out which criteria the students used when evaluating representations A3 and B.

Criteria for evaluating the representations in group 3

Following Skemp (1982), surface-related arguments, which focus on the visual level of pictorial representations, can be distinguished from content-related arguments, which address the meaning of the representations. In addition, reference is often made to the assumed target group (primary school students) to reinforce or contextualize the evaluation. The following categories and subcategories are distinguished (specifications refer to representation A3):

- Surface-related arguments: related to layout (e.g., clarity of presentation, visibility of factors due to dice images, countability of individual elements),

related to given tasks (visibility of task elements (factors, terms, equation)), related to factor swapping (visibility of swapped factors)

- Content-related arguments: related to operation (visualization/recognizability of multiplication), related to results (visualization/recognizability of the result or equality of results), related to insufficient visualization of the statement
- Target group-related arguments (e.g., dots that are easy for children to count)

It is particularly interesting to compare the group of students who gave representation A3 a rating of ++ ("A3++") with the other groups (Table 2).

Table 2: Proportion of students using content-related arguments concerning A3

	A3++	A3+	A3-	A3--
Content-related arguments:	**35%**	**53.4%**	**73.3%**	**88.2%**
Related to operation	0%	5.7%	14.7%	14.7%
Related to results (positive)	35%	19.4%	0%	0%
Related to results (negative)	0%	29.5%	58.7%	58.8%
Related to insuff. visualiz. of the statem.	0%	1.1%	8%	20.6%

Their content-related arguments were exclusively related to results, and like the other arguments of this group, entirely positive. Thus, 35% of group A3++ judged that the equality of results (or, very rarely, the result itself) was represented or recognizable. In some cases, this was justified by the equal sign included in the representation. In 15.5% of cases, it could be concluded from the explanations that the students identified the result with the number of dots on the dice or concluded that the results were equal because the total number of dots was the same on both sides. Positive arguments related to results were less common in group A3+ and did not occur at all in the other groups. In contrast, the proportion of students with negative arguments related to results (result or equality of results not visualized/recognizable) increased toward a negative evaluation of the representation. Arguments related to operation were not mentioned at all in A3++ and were rarely mentioned in the other groups. They occurred more frequently for representation B (23% of the total group 3), probably because here the arrays with their markings must first be interpreted as a representation of multiplication, whereas in representation A3 the arithmetic operation seems to be given by the multiplication sign.

Overall, it can be seen that in group A3++, which comprises 28.9% of the 277 PSTs of group 3, content-related arguments played a significantly smaller role than in the other groups and that these were consistently based on a problematic or incorrect interpretation concerning the representation of the result. Instead, surface-related arguments (sometimes in combination with explanations concerning the target group)

were used. The (superficial) analogy to the given symbolic identity and the easy recognizability of the individual task elements seem to be central to these students' very positive evaluation of representation A3.

Considering the justifications for representation B, it becomes apparent that approximately one-third (32.9%) of the entire group 3 used content-related arguments (excluding the problematic interpretations concerning the (equality of) results) for both representations and, with a few exceptions, evaluated representation B as better than representation A3. Another third (30.3%) did not use content-related arguments for either representation, and 84.5% of these students ranked A3 higher than B. The consistent orientation toward deep vs. surface structures in both subgroups thus seems to be accompanied by a consistent preference for representation B vs. A3.

Comparison of the groups concerning criteria for evaluating

PSTs who justified their evaluations without referring to content-related arguments were not only found in group 3, but also in similar proportions in the other two groups. Based on this, it appears that this proportion is hardly dependent on the extent of the content-related differences between the two representations considered. Instead, these PSTs largely based their explanations on surface-related arguments. Even if it cannot be concluded from the written explanations that the PSTs used only superficial arguments in their evaluations, these were nevertheless primarily noteworthy from their point of view.

Comparing the arguments used, one striking commonality is that the representation of the swap of factors between the two tasks 4×5 and 5×4 is virtually not mentioned. Furthermore, the justifications do not address the possible generalizability of the representation in the sense of a generic example.

Differences can be seen in the group-specific characteristics of the arguments regarding representation B. In group 1, it was strikingly often emphasized positively that the equality of results is apparent at first glance, which corresponded with statements that this is not possible in representation A1. In group 2, it was noted that the total number of dots corresponds to the product, while representation A2 contains unused dots. In addition, the similarity of the factor representations in representation B (always rows) was mentioned in contrast to the different forms used in representation A2 (rows and "squares"). In group 3, in comparison to the other groups, representation B was noticeably often described as confusing and difficult to understand. In contrast, representation A3 was repeatedly mentioned as very clear.

Possibly, these different focuses, which are determined by the characteristics of representation A1, A2 and A3, indicate reasons for the different evaluations of representation B in the three groups (see Table 1). Thus, representation B in group 3 received the lowest average rating. Perhaps the immediate visibility of the multiplication task in A3 led to identifying the factors in B being viewed as a major obstacle to understand this representation. In group 2, representation B was rated

significantly higher than in the other groups. More than half of the students rated it ++. Although representations A2 and B differ in content primarily in their visualization of the interchangeability of factors and the possibility of generalization, this was not mentioned by the students. So, both representations may appear similar to them. There is no difference in the difficulty of understanding A2 and B, as the factors must first be identified in both representations. Due to this perceived similarity of the representations, the differences mentioned at the surface level (visualization of the factors by similar dot sets and no unnecessary dots in B) are particularly significant for the students. Also here, it is criteria on the surface level that have a strong influence on the evaluation, which, unlike in group 3, is in favor of presentation B.

DISCUSSION

When evaluating visual representations of the commutative law of multiplication, a significant proportion of first-year PSTs consider only surface-related characteristics. This is the case even when they are given two representations that are very different from a mathematics education perspective in terms of generalizability and potentials for reasoning (i.e., in terms of content-related characteristics). The results of the other tasks in the survey indicate that a lack of subject knowledge or inappropriate conceptions of multiplication cannot be causes for this. Rather, the results as a whole suggest that this group of students may have a general “surface orientation” with regard to teaching materials. A study by Battey et al. (2005) on PSTs’ criteria for evaluation software for mathematics teaching also revealed a focus more on surface features than on mathematics content or potentials for supporting students’ learning. Since there are no larger studies on a possible surface orientation of PSTs with regard to teaching materials, this seems to us a desideratum for research. At the same time, reducing this orientation would be an important challenge for teacher education.

The varying frequency of arguments used to justify the evaluations in the three groups suggests that the students took a comparative approach even without being explicitly asked to do so. In fact, we would argue that a comparative examination of representations, manipulatives, worksheets, textbooks, etc. in didactic courses could be an important way of encouraging PSTs to go beyond merely surface-related evaluations. By varying the differences between teaching materials related either to surface-related or content-related aspects, it might be possible to focus students’ attention on didactically relevant aspects for evaluation.

The partially differing evaluations of representation B by the three groups of PSTs (cf. Table 1) and the varying frequency of arguments used to justify the evaluations exemplify the context dependency of empirical results. On the one hand, this is obvious, and on the other hand, it reminds us not to neglect the potential influences of contextual characteristics when discussing the results of scientific studies.

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