DESIGN PRINCIPLES FOR SUPPORTING KNOWLEDGE INTEGRATION IN A DIGITAL GAME-BASED LEARNING ENVIRONMENT

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Guided by theories of knowledge integration and adaptive expertise, this paper describes the design of number line estimation tasks that target adaptive rational number knowledge. We propose three design principles: (1) eliciting learners' current knowledge, (2) introducing linking ideas and increasing element interactivity, and (3) embedding multiple solution routes with on-the-spot decisions. The design aims to stimulate students to coordinate five knowledge components embedded in number line estimation tasks—magnitude mapping, relational reasoning, cross-notation conversion, within-notation transformation, and arithmetic operations—and the manipulation of number line features (including non-integer-unbounded number lines) and target formats as two central mechanisms to support knowledge integration.

INTRODUCTION

The development of mathematical knowledge that students can use in novel situations is a main goal of mathematics education. However, in the domain of rational numbers, learning fractions and decimals and their connections can be challenging. Traditional curricula typically introduce fractions and decimals in separate units and emphasize rote procedural knowledge (Hurst & Cordes, 2018). This approach may prepare students with basic rational number knowledge but also may limit the development of knowledge that can be easily applied in future problem solving.

One of the consequences of teaching decimals and fractions separately is the lack of cross-notation knowledge with rational numbers, the ability to understand and translate between different forms of the same magnitude, e.g., decimal and fraction notations (Braithwaite et al., 2022). Students who are proficient with within-notation contexts (e.g., ¾ + ¼) may struggle to adapt their knowledge to cross-notation contexts (e.g., 0.75 + ¼), in which efficient problem solving may require them to recognize that 0.75 and ¾ refer to the same value (McMullen et al., 2020). This may also lead to a lack of conceptual meaning of procedural knowledge, where students may be familiar with the arithmetic procedures of adding fractions or decimals but not connect these to their magnitude (Siegler & Lortie-Forgues, 2015).

The process of knowledge integration, as may be needed for rational number knowledge, is a progressive process of differentiating, connecting, and reorganizing ideas into increasingly coherent understanding (Linn, 2005). The process isn't automatic and requires targeted support that explicitly bridges different representations

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and the conceptual and procedural knowledge (Schiller et al., 2024). Linn (2005) describes four interrelated processes that support knowledge integration: (1) eliciting learners' current ideas, (2) adding normative ideas that reorganize and connect existing ideas, (3) developing criteria for evaluating ideas and their integration, and (4) sorting out ideas by resolving contradictions, identifying overlaps and gaps, and seeking connections through reflection.

Adaptive expertise, the ability to apply meaningful conceptual and procedural knowledge flexibly to solve problems (Baroody, 2003), extends ideas of knowledge integration by focusing on when and why knowledge is applied and how learners select or generate strategies under novel or demanding conditions. Tasks that promote adaptive expertise should elicit on-the-spot decisions, strategy comparison, and revision where no single rehearsed procedure suffices (Feltovich et al., 1997). In the present paper, we focus on adaptive rational number knowledge (ARNK), a behavioural manifestation of adaptive expertise in the rational number domain, defined as well-integrated rational number knowledge that can be flexibly applied in novel problems and contexts (McMullen et al., 2020). Building on theories of knowledge integration (Linn, 2005) and adaptive expertise (Feltovich et al., 1997), we propose three design principles to support the development of ARNK and examine how to apply these principles in designing number line estimation tasks.

Principle 1: The tasks should activate learners' existing knowledge at the beginning of the learning process, making it usable in action (Linn, 2005). In this way, knowledge integration begins with existing concepts and procedures, which subsequently become objects of coordination.

Principle 2: The tasks should introduce linking ideas that connect knowledge elements and increase element interactivity, which require learners to coordinate multiple (procedural and/or conceptual) knowledge components (Feltovich et al., 1997; Linn, 2005). In this way, knowledge integration may develop as an outcome of repeated connection and coordination of knowledge components within and across task items.

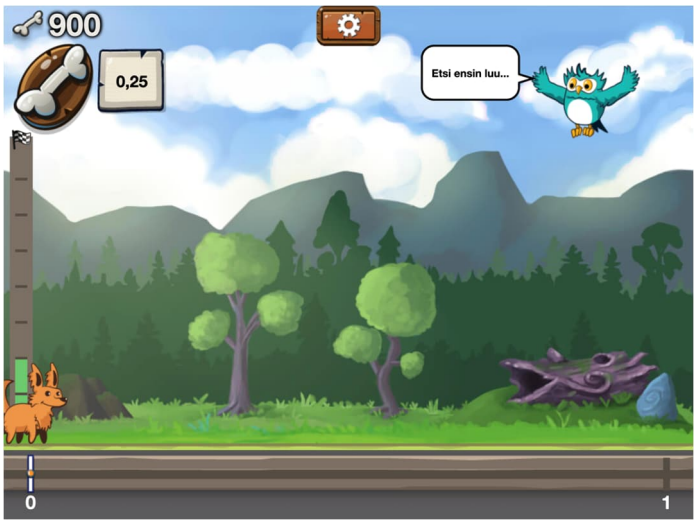
Principle 3: The tasks should embed opportunities for multiple legitimate solutions that require on-the-spot strategic decisions (Feltovich et al., 1997). This requires learners to select and evaluate strategies rather than following familiar or prescribed routes. The on-the-spot strategic decisions should be consequential in the sense that more than one route can lead to success, even if the routes differ in efficiency or requisite knowledge.

A NUMBER LINE GAME AS A TESTBED OF DEVELOPING ARNK

Number Trace is configurable math-game research and learning environment developed for rational number instruction (see Figure 1) that uses the visual representation of the number line to link rational-number magnitude with the concepts of unit, scale, and spatial distance (Kiili et al., 2021). In the game, learners help a dog locate a bone by placing the target number on the number line. Number line tasks also

allow systematic manipulation of targets and scale cues (e.g., endpoints, reference intervals, and tick marks), making them a flexible setting for our goal of designing tasks that require coordination of multiple aspects of rational-number knowledge (McMullen et al., 2024).

Figure 1 Number Trace game environment



In the number line task design, we attempt to include five knowledge components for developing adaptive rational number knowledge: (1) magnitude mapping (MM), interpreting a rational number as a magnitude and mapping it to a spatial location on the number line (2) cross-notation conversion (CC), converting between fractions and decimals; (3) relational reasoning (RR), reasoning about relations among rational numbers (e.g., 1/2 is twice as large as 0.25); (4) within-notation transformation (WT), converting within a notation to create a more workable form (e.g., simplifying or expanding fractions); and (5) arithmetic operations (AO), using arithmetic operations with rational numbers (Braithwaite et al., 2022; McMullen et al., 2020; Siegler & Lortie-Forgues, 2015).

OPERATIONALIZING THE DESIGN PRINCIPLES IN NUMBER LINE ESTIMATION: SIMULTANEOUS TASK DESIGN

We refer to the task mechanism type that embeds the previously mentioned design principles as simultaneous tasks, which require learners to coordinate multiple knowledge components and make on-the-spot decisions. Figure 2 shows how we designed these simultaneous tasks with number line estimation using six exemplar items. In each row, the blue boxes present the target to be estimated, and the columns present the two number line types: 1) an integer-bounded number line, and 2) a non-integer unbounded number line. We consider the typical number estimation on the 0-1 number line (Figure 2, Task 1) as the baseline task, whereas the remaining tasks function as simultaneous tasks. Importantly, in simultaneous tasks, different decisions during problem solving can lead to different combinations of knowledge components being required. Accordingly, for each task, we report (Figure 2): (1) the baseline

knowledge demands that apply regardless of the decision route (Baseline KD), (2) the on-the-spot decisions learners make during problem solving (On-the-spot decision), and (3) additional knowledge demands that become necessary under particular decision routes (KD in different decision routes).

Figure 2 Simultaneous number line tasks: exemplar items, decision points, and knowledge demands

Integer-bounded number line			Non-integer-unbounded number line		
Task 1 : 0.33			**Task 2: 0.33**		
Baseline KD	On-the-spot decision	KD in different decision routes	Baseline KD	On-the-spot decision	KD in different decision routes
MM	-	-	MM, CC, RR	$CC^{*} \times WT^{*} + NE^{*}$	CC^{*d}: Baseline CC^{*f}: Baseline + WT
Task 3 : $\frac{2}{3}$ – 0.33			**Task 4: $\frac{2}{3}$ – 0.33**		
Baseline KD	On-the-spot decision	KD in different decision routes	Baseline KD	On-the-spot decision	KD in different decision routes
MM	$AO^{*} \times CC^{*} + NE^{*}$	AO^{*n}: Baseline $AO^{*y} \times CC^{*d}$ or $AO^{*y} \times CC^{*f}$: Baseline + CC + AO	MM, CC, RR	$AO^{*} \times CC^{*} + NE^{*}$	$AO^{*n} \times CC^{*d}$ or $AO^{*n} \times CC^{*f}$: Baseline + WT $AO^{*y} \times CC^{*d}$ or $AO^{*y} \times CC^{*f}$: Baseline+ AO + WT
Task 5: $\frac{6}{5}$ + 3.4			**Task 6: $\frac{6}{5}$ + 3.4**		
Baseline KD	On-the-spot decision	KD in different decision routes	Baseline KD	On-the-spot decision	KD in different decision routes
MM, CC, AO, WT	$AO^{*} \times CC^{*} + WT^{*}$	$AO^{*n} \times CC^{*d}$ or $AO^{*n} \times CC^{*f}$ or $AO^{*y} \times CC^{*d}$ or $AO^{*y} \times CC^{*f}$: Baseline	MM, CC, WT, RR	$AO^{*} \times CC^{*} + WT^{*}$	$AO^{*n} \times CC^{*d}$ or $AO^{*n} \times CC^{*f}$: Baseline $AO^{*y} \times CC^{*d}$ or $AO^{*y} \times CC^{*f}$: Baseline + AO

MM = Decimal/fraction magnitude mapping; CC = Cross-notation conversion; WT = Within-notation transformation; AO = Arithmetic operations; RR= Relational reasoning; AO^{*} = Arithmetic operations decision; CC^{*} = Cross-notation converting direction decision; WT^{*} = Within-notation transforming direction decision; NE^{*}= numeric estimation

Design principle 1 involves designing tasks that elicit students' prior knowledge across contexts, which is done through the basic number line game environment that uses a

supportive visual representation of rational numbers' magnitude. They also make visible that different representations of the same value map to the same location (Schiller et al., 2024). To this end, baseline tasks (see Figure 2, Task 1) are introduced at the beginning and interleaved across decimal and fraction forms. This gameplay activates magnitude reasoning for solving number line estimation tasks and for being an object coordinated in subsequent simultaneous tasks.

Design Principle 2 emphasizes introducing linking ideas that connect previously separated knowledge elements and increasing element interactivity by requiring coordination of multiple knowledge components within a single task (Feltovich et al., 1997; Linn, 2005;). We operationalized this principle by embedding cross-notation conversion (CC) and relational reasoning (RR) as linking components in simultaneous tasks. CC links decimal and fraction knowledge by requiring conversion procedures and makes their equivalent magnitudes explicit. RR links procedures and concepts by requiring relational reasoning about different rational number magnitudes and notational knowledge (e.g., CC or within-notation transformation (WT)) to create comparable notations (e.g., converting 1/3 to 3/9 to compare with 1/9).

As shown in Figure 2, CC is incorporated in the tasks by using a non-integer unbounded number line or an arithmetic sentence target that involves numbers in different notations (Task 2, 4, 5 and 6). However, CC is not inevitable in tasks with cross-notation arithmetic sentence as target if the number line structure supports a solution through direct magnitude mapping, such as estimating 2/3 − 0.33 on a 0–1 number line (Task 3) by sequentially estimating the spatial distances of 2/3 and 0.33 and mapping them on the line. In this case, the baseline KD only involves magnitude mapping.

Relational reasoning (RR) is embedded by shifting the number line from an integer-based bounded to a non-integer unbounded number line (Task 2, 4, and 6). For example, placing 0.33 with a 1/6 marker (Task 2) requires reasoning the relation between 0.33 and 1/6 rather than direct magnitude mapping, as in mapping 0.33 on 0-1 number line (Task 1). The same logic applies to task with arithmetic sentence targets (Task 4 and 6). For example, placing 2/3 − 0.33 with a 1/9 marker (Task 4) requires inferring relations among 2/3, 0.33, and 1/9 or between the arithmetic result 2/3 − 0.33 ≈ 1/3 and 1/9.

Arithmetic operations (AO) and within-notation transformation (WT) can also be introduced explicitly by using an arithmetic sentence as the target or by designing target–marker pairs that are not directly comparable within a notation, including after cross-notation conversion. For instance, 2/3 − 0.33 may be converted to 1/3, but comparing 1/3 to 1/9 requires WT by expressing 1/3 as 3/9, enabling proportional reasoning between the two magnitudes. The designed simultaneous tasks required the coordination of these above-mentioned knowledge components in one task, requiring learners to connect different concepts and treat procedures (e.g., WT) as meaning-preserving ways to compare and locate the underlying magnitude on the number line, rather than as isolated steps.

Design Principle 3 focuses on embedding on-the-spot decision-making to help learners learn to apply knowledge in ways that can strategically redirect their solution paths (Feltovich et al., 1997). This process can activate different knowledge components and affect the efficiency of coordinating these knowledge components. This principle builds on the knowledge integration demands introduced in Principle 2. What distinguishes Principle 3 is that the design of targets and number line structure allows for in-play decisions to lead to different solutions strategies.

One decision type is conversion direction (CC^{*}) where learners decide whether to follow a fraction-route (CC^{*f}) or a decimal-route (CC^{*d}). For example, in Task 2, a CC^{*d} route may convert 1/6 to 0.166… and then manage repeating decimals through within-notation transformation (WT) before relational reasoning to locate 0.33. A CC*f route may treat 0.33 as approximately 1/3 with numeric estimation (NE^{*}) and construct comparable fraction forms (WT) to compare it to 1/6. Both routes coordinate WT with MM and RR, but the fraction-oriented route may be more efficient for identifying the relation.

A second type of decision is arithmetic strategy (AO^{*}). For example, in Task 4 (Figure 2), learners may avoid explicit computation (AO^{*n}) by inferring magnitudes in relation to 1/9 (e.g., comparing 2/3 with 1/9 and 0.33 with 1/9), which can also require WT. Alternatively, learners may compute first (AO^{*y}): $2/3 - 0.33 \approx 1/3$, which requires CC before mapping the result and using WT ($1/3 = 3/9$) to make it comparable with 1/9. In both routes, numeric estimation (NE^{*}) is required for $0.33 \approx 1/3$. Although both routes require coordination of multiple knowledge components, the AO^{*y} route may support more efficient relational inferences in Task 4.

A third type of decision involves within-notation transformation (WT^{*}). For example, with the target 6/5 + 3.4 (Figure 2, Task 5), the 3–5 bound makes transformations such as rewriting 6/5 in mixed-number form easier to align with integer bounds, whereas a non-integer reference marker, such as 9/5 (Task 6), transformations with improper fractions can make direct relational comparison based on numerator more transparent.

These on-the-spot decisions allow for multiple procedural variations within a single task and set of tasks. Nonetheless, successful paths converge on the same core knowledge components because the number line and cue structure make certain components unavoidable. Consequently, effective routes can differ across tasks and repeatedly applying the same on-the-spot decisions may be ultimately inefficient.

DISCUSSION

This paper contributes a theory-aligned blueprint for simultaneous task design intended to support adaptive rational number knowledge (ARNK) within a game-based learning environment. Across tasks, the design combines (a) introducing linking knowledge and increased element interactivity, requiring coordination among magnitude mapping, cross- and within-notation conversion, arithmetic, and relational reasoning, and (b) decision-rich routes that invite learners to select strategies rather than rely on a single

routine. In this sense, the number line game is positioned as a site for both applying familiar ideas and exploring alternative strategies under changing task features (McMullen, 2024).

Thus, we argue the simultaneous task requires learners to coordinate multiple knowledge elements and compare alternative solution routes. Even if the overall performance of students on the task may be low, the process of trying ideas, testing routes, and evaluating strategies under uncertainty may be productive, including cases where learners end with "I don't know" (Linn, 2005). However, for designers and teachers, "simultaneous" should not be equated with simply adding more steps or more representations. Greater simultaneity does not necessarily imply multiple viable and efficient strategy possibilities if on-the-spot decisions collapse into a single viable route with high coordination demands that leave no space for route comparison (Feltovich et al., 1997). Rather, the design value lies in how task features (e.g., targets and marker/range structures) are configured to require coordination and to create consequential decision points and multiple strategies (Van Dooren, 2023). At the same time, the design highlights an important limitation: high simultaneity demands may presuppose a certain level of prior knowledge, and learners with weaker component knowledge may require additional scaffolding or direct support (Linn, 2005).

Future work can use this design to examine learners' strategy construction in simultaneous tasks. Multi-modal or think-aloud approaches may be useful to trace how learners respond to decision points and how they compare and revise routes during the problem-solving process. Research can also explore how simultaneous tasks function in socio-cultural learning contexts, where learner interactions shape sense-making and strategy construction, or add reflection activities based on Linn's (2005) knowledge integration principle to investigate its contribution in knowledge integration. Overall, this paper provides a theoretical justification and blueprint for the design of simultaneous tasks and the theoretical lens to analyze tasks that require coordination of knowledge components and consequential decision-making.

Additional information

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