HIGH SCHOOLERS' SPONTANEOUS USE OF CROSS-NOTATION KNOWLEDGE OF RATIONAL NUMBERS

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Many students face difficulties in acquiring the skills needed for such adaptive expertise with rational numbers, a key aim of education systems worldwide. This study investigates the possibility that cross-notation knowledge, understanding the relationships between different notations of rational numbers, should be considered a feature of adaptive expertise with rational numbers. 447 high school students completed measures of adaptive expertise, including the Arithmetic Sentence Production Task. Results indicate that students who employed cross-notation solutions on this task performed well on other measures of adaptive expertise. These findings underscore the importance of cross-notation knowledge for adaptive expertise with rational numbers.

INTRODUCTION

Rational number knowledge is crucial for both future STEM achievement and success outside formal learning contexts (D'Erchie et al., 2025; Siegler et al., 2012). However, many students struggle to apply their rational number knowledge in unfamiliar settings (Schiller et al., 2024). One specific feature of rational number knowledge that students struggle with is cross-notation knowledge, or knowledge of how the different notations of rational numbers are related to each other (Braithwaite et al., 2022). Despite the importance of cross-notation knowledge, prior examinations of this type of knowledge have focused on student performance only on a narrow range of tasks (Braithwaite et al., 2022). Less is known about how students use cross-notation knowledge in novel tasks, nor how this knowledge relates to other features of adaptive expertise with school mathematics. The present study aims to examine this gap to better understand the holistic nature of adaptive expertise with rational numbers in high school students.

Adaptive expertise requires malleable, fluid knowledge and skills that are readily applicable in novel situations, such as in future learning or everyday life (Baroody, 2003; Van Dooren, 2023). This contrasts with routine expertise, which is typified by static, inflexible knowledge that is only easily applicable in well-practiced situations. The differences between routine and adaptive expertise can be described as qualitative in nature. For instance, a student with routine expertise may be able to accurately solve standard fraction and decimal arithmetic problems but may not be able to use fractions

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, pp. 1011-1018. PME.

and decimals interchangeably in solving an unfamiliar algebraic equation involving rational numbers. In contrast, a student with adaptive expertise would more readily recognize connections between the fraction and decimal properties, allowing them to flexibly apply their rational number knowledge in a way to solve the algebraic task effectively.

Adaptive expertise is defined by well-connected knowledge networks that allow for reasoning, as well as actions, that use multiple bits of knowledge and skills both between and within different topics (Van Dooren, 2023). Well-connected knowledge may help support transfer by providing more contextual diversity where specific knowledge can be relevant and thus may be critical for use in situations that are not well-rehearsed (Schwartz et al., 2005). However, the structure of mathematics instruction may exacerbate the disconnected view that fractions and decimals are distinct numbers and not interchangeable notations of the same magnitudes.

Substantial individual differences in cross-notational knowledge, that is the knowledge about the relation between fractions and decimals, appear even among students with many years of experience with both notation types (Schiller & Siegler, 2023). Individual differences in cross-notation knowledge appear important for mathematical development, even beyond within-notation knowledge (Braithwaite et al., 2022; Schiller & Siegler, 2023). Cross-notation knowledge also appears to be supportive of higher performance on the arithmetic sentence production task, which measures a behavioral manifestation of adaptive expertise, adaptive rational number knowledge (McMullen et al., 2020). In this task, participants are asked to produce as many arithmetic sentences as possible that equal a target number (e.g., 1) using a set of given numbers (e.g., ½; 0.5; 0.25; ¼; 4) within a limited period of time. In the present study we examine if a crucial feature of having high levels of adaptive rational number knowledge is the level of cross-notational knowledge, as an indicator of the level of knowledge integration about different rational number concepts.

In order to investigate more closely the role of the use of cross-notation knowledge in adaptive expertise, we examine a group of high school students' performance across a number of tasks aimed at measuring adaptive expertise with school mathematics. We therefore ask:

1. How do high school students use cross-notation knowledge on the arithmetic sentence production task?

2. How does cross-notation use relate to other measures of adaptive expertise with whole and rational number arithmetic?

METHODS

Setting and participants

A total of 447 students from a single suburban high school in the southeastern United States participated in this study. The sample comprised 45.2% female, 52.8% male,

and 0.2% non-binary individuals. Participants were enrolled in grades 9 through 12, with the large majority (78%) of participants in either grade 9 or grade 10. The study received ethical approval from the ethical review board of the University of Turku, along with approval from the local school board, principal, and teachers; guardians were informed of the study with the opportunity to opt out of participating. Students were also informed about the purposes of the study and allowed to not participate without negative repercussions before and during testing.

Instruments

Arithmetic Sentence Production Task (McMullen et al., 2020)*:* This task included four items that required students to produce as many different arithmetic sentences as they could that equaled a target solution, using a set of given numbers, in 90 seconds per item (e.g., using the numbers .25, 1/4, .5, 1/2, and 4 to arrive at a total of 1). Two of the items included only whole numbers in the set of given numbers (and prompted students to produce arithmetic sentences that totaled to an integer), whereas the other two items included non-integer rational numbers (that prompted students to produce arithmetic sequences that totaled to a rational number; 1/2 or 3). The number of correct arithmetic sentences generated by each student for each of the four items was coded as the total number of correct solutions. For items with rational numbers, the number of times a participant used both a fraction and decimal in a single solution (e.g., 1/2+.5=1) was coded as the total number of cross-notational solutions.

After completing all items, students were asked to evaluate sample responses for the arithmetic sentence production task from two hypothetical students five-point Likert scales (e.g., How much would your teacher like Stephan's way? ("Not much"<->"A lot").

Fill-in-the-Blank Task: This task included two pages, each with eight items. Each item displayed an arithmetic sentence with missing number(s) and/or operation(s) (e.g., 2 * [blank] + 2 [blank] 9 = 38; 1/4 ÷ (3/4 [blank] 1/4) = 1/2), where the aim was to complete the sentence so that it was true. One page included items with only whole numbers, while the other page included items with non-integer rational numbers. Participants had three minutes to complete each page. The outcome measure was the total number of arithmetic sentences that were true on each page.

Rapid Verification Task: Based on the number sets task used in previous studies (e.g., Mazzocco & Hanich, 2010), this task presented participants with a target value (e.g., 59 or 1/2) and a set of arithmetic expressions. Participants were asked to circle all expressions equal to the target solution as quickly as possible within a time limit of one minute per item. Two items involved solely whole numbers and two items included rational numbers. The outcome measure was the total number of correctly circled responses on each page.

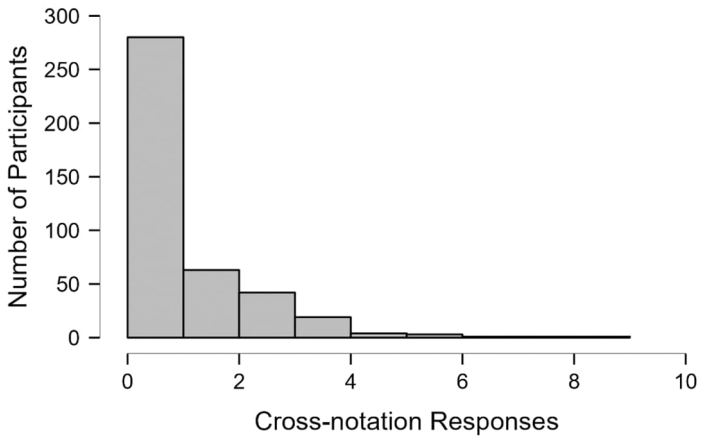
Strategic Flexibility: Participants completed a three-part assessment of their strategic flexibility with arithmetic and algebra problems (Coppersmith Garcia & Star, 2022).

Only the first two parts are used in the present study. In Part 1, students were asked to complete five problems, each of which had multiple potential solution strategies. In Part 2, students were asked to complete the same problems from Part 1 using a different solution strategy than the one they used before, without looking back at Part 1. Two independent coders classified all solutions by strategy type (standard, better-than-standard, or below standard) and subsequently resolved all disagreements. Students were given a score for being flexible in their use of problem-solving strategies if they correctly used a different strategy to solve a task in both parts of the test. Thus, the maximum possible score for the strategic flexibility measure was 5.

RESULTS

Figure 1 displays the distribution of cross-notation solutions provided by participants. As can be seen, a large proportion (46%) of participants did not provide a single correct cross-notation solution. Despite this somewhat non-normal distribution (Skewness = 1.60, SE = 0.12; Kurtosis = 3.47, SE = 0.24), cross-notation solutions were still strongly associated with overall performance on the rational number items of the arithmetic sentence production task (r = .61, p < .001).

Figure 1: Histogram of the number of correct cross-notation solutions



In order to avoid confounding overall performance and cross-notation use and to take into account the long-tailed distribution of cross-notation solutions, we split students in four groups based on whether they were above or below the median values for overall performance (6 correct answers) and cross-notation use (0 cross-notation solutions) on the arithmetic sentence production task, rational number items. The four groups were: (1) Low performance, no cross-notation (*Low, No Cross-Note*; n=126), (2) Low performance, at least one cross-notation (*Low, Cross-Note*; n=53), (3) High performance, no cross-notation (*High, No Cross-Note*; n=66), and (4) High performance, at least one cross-notation (*High, Cross-Note*; n=169).

One-way ANOVAs with group membership as the fixed factor and Likert scale responses to the two prompts as dependent variables revealed no statistically significant group differences for the interpretation scales, F (3, 396) = 0.33, p = .81, $\eta 2$ = .002, and fourth items, F (3, 396) = 0.80, p = .49, $\eta 2$ = .006. We interpret this finding

to mean that students' use of cross-notation solutions was not impacted by differing interpretations of the task.

Next, we compare the groups' performance on the fill-in-the-blank and rapid verification tasks separated by whole and rational number items, in order to explore whether cross-notation use was related to performance on these tasks. One-way ANOVAs with group membership as the fixed factor and performance as dependent measures revealed statistically significant group differences for all tasks (Table 1).

As shown in Table 1, all ANOVAs revealed differences in performance between the groups. As expected, Bonferroni corrected post-hoc comparisons revealed significant differences between the low and high performance groups for all measures. There were significant differences between the *Low, no Cross-Note* and the *Low, Cross-Note* groups on Fill-in-the-blank Whole number (mean difference = 0.78, SE = 0.25, Cohen's d = 0.52, p_{bonf} = .011) and Rapid Verification Rational (mean difference = 0.97, SE = 0.35, Cohen's d = 0.46, p_{bonf} = .033), but not on Fill-in-the-blank Rational number (mean difference = 0.60, SE = 0.26, Cohen's d = 0.37, p_{bonf} = .15) or Rapid Verification Whole (mean difference=0.52, SE=0.35, Cohen's d=0.24, p_{bonf}=.85) tasks.

Table 1: Performance on fill-in-the-blank and rapid verification tasks by arithmetic sentence production task groupings.

	Fill-in-the-Blank Whole	Fill-in-the-Blank Rational	Rapid Verification Whole	Rapid Verification Rational
	$F(3,410)=41.77$, $p<.001$, $\eta^2=.23$	$F(3,410)=47.26$, $p<.001$, $\eta^2=.26$	$F(3,410)=39.05$, $p<.001$, $\eta^2=.22$	$F(3,410)=60.41$, $p<.001$, $\eta^2=.31$
Low, No Cross-Note	4.65 (0.16) [a]	1.86 (0.13) [a]	17.97 (0.21) [a]	9.84 (0.21) [a]
Low, Cross-Note	5.43 (0.22) [b]	2.45 (0.19) [a]	18.49 (0.31) [a]	10.81 (0.27) [b]
High, No Cross-Note	6.21 (0.17) [c]	3.54 (0.23) [b]	19.86 (0.24) [b]	12.18 (0.26) [c]
High, Cross-Note	6.59 (0.10) [c]	4.01 (0.13) [b]	20.57 (0.16) [b]	13.09 (0.16) [d]

Note: Cells with different letters differ significantly based on Bonferroni post-hoc comparisons (within columns)

There were significant differences between the *High, No Cross-Note* and the *High, Cross-Note* groups only on the Rapid Verification Rational (mean difference = 0.91, SE = 0.31, Cohen's d = 0.43, p_{bonf} = .021), but not on Fill-in-the-blank Whole number

(mean difference = 0.37, SE = 0.22, Cohen's d = 0.25, p_{bonf} = .54), Fill-in-the-blank Rational number (mean difference = 0.47, SE = 0.24, Cohen's d = 0.29, p_{bonf} = .28) or Rapid Verification Whole (mean difference = 0.71, SE = 0.32, Cohen's d = 0.33, p_{bonf} = .15) tasks. These results suggest overall performance on the arithmetic sentence production task and the use of cross-notation solutions are both positively related to performance on the fill-in-the-blank and rapid verification tasks, beyond overall performance on the rational number arithmetic sentence production task.

Due to the low number of students who displayed strategic flexibility across the whole measure, arithmetic sentence production task groups were cross-tabulated with the dichotomous variable of strategic flexibility (defined above as the use of multiple strategies to solve a single task at least once). Table 2 presents the cross-tabulations, which had a significant chi-squared value χ^2 (3) = 29.70, $p < .001$, indicating a relation between arithmetic sentence production groupings and strategic flexibility. Standardized residual values per cell indicated that there were fewer students than expected in the Low, No Cross-Note group who had at least one item in which they received a flexibility score, and more students than expected in the High, Cross-Note group who had at least one flexibility score.

Table 2: Cross-tabulations of arithmetic sentence production task groupings and strategic flexibility groups.

		Flexibility Grouping		
		None	At least one	Total
Low, No Cross-Note	Count	97	29	126
	St. res.	4.75	-4.75	
Low, Cross-Note	Count	36	17	53
	St. res.	1.31	-1.31	
High, No Cross-Note	Count	34	32	66
	St. res.	-1.47	1.47	
High, Cross-Note	Count	80	89	169
	St res.	-4.24	4.24	
Total	Count	247	167	

DISCUSSION

These results provide novel insights into the role of cross-notation knowledge in relation to adaptive expertise in school mathematics. Overall, we find the use of cross-notation solutions to be fairly rare and unevenly distributed in our sample of high

school students. Our results show that those students who used cross-notational solutions on the arithmetic sentence production task had higher performance on other measures of adaptive expertise. This suggests that those with high cross-notational knowledge may have an overall higher level of knowledge integration across topics in school mathematics. Importantly, those students who did not use cross-notation solutions had lower levels of adaptive expertise irrespective of their overall performance on the arithmetic sentence production task. Importantly, we find that these differences did not appear to be explained by differences in students' interpretation of the task requirements.

The results of the study expand upon recent research emphasizing the importance of cross-notational knowledge and expanding the conceptualization of the role of knowledge integration in mathematics development. Our results highlight that it is not only declarative knowledge of the relation between fraction and decimal notations that may be relevant for mathematical development (cf. Braithwaite et al., 2022), but it is also the spontaneous use of this knowledge in novel contexts. This may be important for future learning, for instance, being able to recognize when a particular notation is more suitable for a calculation process in equation solving.

This result also furthers understanding about the role of knowledge integration in mathematical development itself. Prior studies on adaptive number knowledge, both with whole and rational number arithmetic, have proposed that high performance on the arithmetic sentence production task requires well-integrated knowledge of numerical characteristics and arithmetic relations (McMullen et al., 2020). The present study is the first to examine how that might manifest with a particular aspect of numerical knowledge, in this case, cross-notational knowledge. The distinct contribution of cross-notational knowledge to differentiating performers on other measures of adaptive expertise, beyond overall performance on the arithmetic sentence production task, suggests that at least in the case of notations with fractions and decimals, knowledge integration may be a relevant feature of adaptive expertise.

Several limitations limit the potential for interpreting our findings more broadly. First, using the total number of cross-notation solutions as the main measure fails to consider the quality of the produced sentences beyond their correctness. As well, the causal mechanism that would explain the relation between cross-notation knowledge and other features of adaptive expertise, or a common cause, is unexamined in the present study. Nonetheless, this study provides greater clarity on how cross-notation knowledge involving rational numbers may fit into educators' and researchers' understanding of what adaptive expertise with rational numbers entails. The ability to recognize opportunities to apply their cross-notational knowledge in novel contexts is also a valuable trait for students when engaging with algebra or other mathematical problem-solving activities, both in and out of the classroom.

Acknowledgments: This study is funded by the Research Council of Finland (EDUCA Flagship #358924, #358947, SCAFFLEX #363159).

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