PARTITIONING OF DISCRETE AND CONTINUOUS QUANTITIES AT KINDERGARTEN

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This paper reports findings from a design research study aimed at fostering early multiplicative reasoning. Kindergarten children were introduced to an estimate-and measure procedure (indirect partitioning) for identifying submultiples of discrete and continuous quantities. Pre- and post-interviews examined children's performance, appropriation of the taught procedure, and the emergence of their own strategies. Results showed improvement across quantity and submultiple types for all children who completed the posttest. Analysis of video-recorded solution processes revealed a range of indirect and direct partitioning strategies. The findings suggest that parallel work with discrete and continuous quantities is feasible for young children and can substantially support early multiplicative reasoning.

THEORETICAL FRAMEWORK

Research indicates that, from an early age, children can discern simple multiplicative and proportional relationships, particularly for continuous quantities and especially for the relations of half and double (Fitzpatrick & Hallett, 2019). However, when children are tasked with finding a fractional part of a quantity—such as in fair-sharing problems—they tend to perform better with discrete quantities than with continuous ones. This advantage arises because discrete contexts allow children to use strategies such as one-to-one or one-to-many correspondence, continuing until the quantity is fully allocated (Park & Nunes, 2001). In contrast, strategies for continuous quantities, such as repeated halving (Hunting & Davis, 1991), are mainly based on the children's ability to estimate halves—which constrains the range of fractional parts they can produce. Importantly, successfully finding a fractional part does not guarantee understanding of the underlying mathematical principles. Nevertheless, research suggests that such principled understanding develops in parallel for both discrete and continuous quantities (Kornilaki & Nunes, 2005).

Procedural challenges associated with partitioning continuous quantities may be one reason why mathematics curricula typically introduce multiplicative reasoning learning objectives first in the context of discrete quantities (e.g., equal-groups situations), and only subsequently in the context of continuous quantities (e.g., the introduction of fractions via area models) (Sophian, 2008; Vamvakoussi & Kaldrimidou, 2018). A second, arguably deeper, reason lies in the historically entrenched distinction between discrete and continuous quantities, which has influenced mathematical thought for millennia (Lucas, 2000). Nevertheless, some

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researchers emphasize the similarities between discrete and continuous quantities and how the human mind processes them (Sophian, 2008; Steffe, 2013). Furthermore, it has been argued that an overemphasis on discrete quantities, natural numbers as cardinal numbers, and additive relationships in early mathematics education may not adequately support children in building the conceptual foundation necessary for later learning about multiplicative notions, particularly fractions (Hunting & Davis, 1991; Sophian, 2008; Vamvakoussi & Kaldrimidou, 2018). The question arises as to whether fostering multiplicative reasoning for discrete and continuous quantities in parallel at an earlier stage of instruction is a feasible and fruitful goal.

We have been conducting a topic-specific design research study (Gravemeijer & Prediger, 2019) to address this question. We adopted a relational approach (Polotskaia & Savard, 2021) to multiplication, according to which the expression $A = n \times b$ captures a single multiplicative relationship between A and b that can be interpreted from two complementary perspectives: $A = n \times b$ and $b = 1/n \times b$ (e.g., six is the triple of two / two is one third of six). Note that A and b may be numerical or physical quantities, and n represents the ratio A/b, which can itself be either an intensive quantity (e.g., three candies per child) or a bare number.

We designed an instructional program aimed at supporting children to a) identify multiplicative relationships across various types of multiplicative situations for both discrete and continuous quantities, specifically, length; b) express multiplicative relationships from two perspectives; and c) find multiples and submultiples of given quantities.

The core of the program consists of three activities addressing three types of multiplicative situations (Greer, 1992): multiplicative comparison, multiplicative change, and proportional problems (for a more detailed description of the activities, see Pitta & Vamvakoussi, 2025). A key element of the program is the introduction of vocabulary pertinent to multiplicative notions, as vocabulary has been found to play an important role in the development of multiplicative reasoning (De Keersmaeker et al., 2024), possibly because it helps children recognize the same relationships across different contexts (Hunting & Davis, 1991). Specifically, we introduced terms for multiples and submultiples (i.e., fraction names) as well as terms for composite units to support children in unitizing (i.e., structuring a quantity into units), a foundational multiplicative reasoning competence that is relevant in both discrete and continuous contexts (Sophian, 2008; Steffe, 2013). We should note that in Greek there is a salient regularity in the formation of all the introduced terms, as they are derived from the words for cardinal and ordinal numbers through a standard suffix. For example, the term triad—*triada* in Greek—is formed from the word *tri*a (three).

In addition, we have devised a meaningful way to circumvent the procedural challenges associated with directly partitioning continuous quantities. We designed an indirect partitioning procedure, an estimate-and-measure one, employing the inverse process of partitioning: Suppose that the unknown quantity b is one third of a quantity A. First,

b is estimated and selected from a set of options; then, one checks whether three *b's* measure *A* exactly. This procedure relates to the measurement aspect of fraction (Wilkins & Norton, 2018), urges children to (mentally) engage in unitizing (Steffe, 2013), but also allows for elementary strategies like "trial and error". Further, the initial quantity remains intact, while there is the possibility to see a partitioned version of it.

The instructional program embedding this procedure was revised and redesigned over the course of four cycles of the study, along with its linguistic and other representational tools (Pitta & Vamvakoussi, 2025). In this paper, we present findings regarding submultiples from the latest enactment with kindergarten children. We examined (a) whether children's ability to identify submultiples of quantities improved across types of quantities and relationships, (b) if and how they appropriated the taught procedure, and (c) whether they developed their own strategies.

METHOD

Design elements: Indirect partitioning and representational tools

Manipulatives made of cardboard were used to represent the quantities—dots for discrete, and bars for continuous quantities. For discrete quantities, the manipulatives were available in two forms: A collection of dots that could be freely manipulated, and copies of groups of dots glued onto strips of paper, representing composite units.

We designed a concrete representation of the ratio *n* between the known (*A*) and the unknown quantity (*b*) to help children keep track of important elements of the problem situation, notably *n*. This tool (hereafter, ratio tool) consisted of arrows with a common starting point. On the left side there was always one arrow, while on the right there were *n* arrows representing the numerical value of *n*. The ratio tool was linked to equipartitioning of *A* from right to left, and to iteration of *b* from left to right. Figure 1 presents the ratio tool and its use in the taught procedure for *n*=2, for discrete and continuous quantities. Specifically:

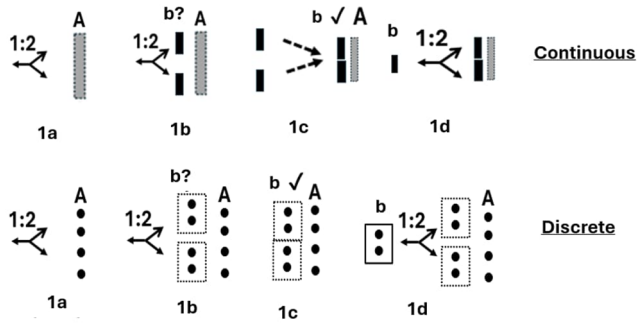
Children first select the appropriate tool for the given problem (Fig. 1a), estimate the unknown quantity *b*, and select it from five available options. They then map *n* copies of *b* onto the *n* arrows (Fig. 1b); and check whether these copies measure *A* exactly (Fig. 1c). If so, one copy of *b* is placed on the left arrow (Fig. 1d); otherwise, the estimate is revised.

Mapping copies of *b* onto the *n* arrows of the ratio tool constitutes an intermediate step (IS) in this estimate-and-measure procedure, which we term *Indirect Partitioning-IS*. This step enables children to keep track of *n*, while the whole setting supports children to coordinate the number of iterations of *b* with the partitioning of *A*.

During the intervention, the researcher-teacher (first author) posed the problems and gave children time to process them. She introduced the procedures and modelled them again when necessary. Children's own strategies, as well as variations of the taught procedures, were acknowledged as correct if they were valid. The relationship between

A and *b* was described from two perspectives (e.g., *A* is the triple of *b*/ *b* is one third of A). In addition, *b* was described both as a unit of measurement and as part of a whole (e.g., *b* fits three times into *A; b* is one of the three equal parts of *A*).

Figure 1: Illustration of the procedure for Indirect Partitioning with the ratio tool.



Participants

An entire mixed-age class from an urban private kindergarten in Greece participated in this intervention. Thirteen children were at their second (Y2) and nine at their first (Y1) Kindergarten year (mean age 5 years 7 months, ranging from 5 years and 7 months to 6 years and 4 months). They were all native Greek speakers and participated voluntarily with their parents' informed consent.

Procedure and Research Tools

The children were pre- and post-tested via individual interviews. The interviews were conducted in two parts and were video recorded. The students participated in the intervention in two groups of 11 mixed-age children; each group participated in nine sessions (approximately 45 minutes each) that took place in their classroom. During the intervention, each problem was elaborated first for continuous, and immediately after for discrete quantities, to highlight the underlying similarities.

There were two common tasks at pre- and post-test (hereafter, Task 1 and Task 2) which were not addressed during the intervention. Each task had 3 trials for *n=2, 3, 4*, a total of 12 problems with *b* as the unknown quantity. Task 1 included partitive division problems posed as follows "Three friends will share *these candies/ this chocolate bar* fairly. Can you show me what each friend will get?"

In Task 2, the children were given a quantity and were explicitly asked to find a submultiple of this quantity. The problems were posed as follows: "I have *these candies/ this chocolate bar*. I will give George one-third of the chocolate. What do you think George will get?" In Task 2, recognizing and interpreting the terms was a prerequisite for solving the problems.

All quantities were represented as in the intervention, the ratio tool was available, and there were representations of the "recipients" in Task 1, serving as another representation of the numerical value of *n*. Children were given 5 options for each trial and were asked to explain their answers to ensure that their choices were not random.

RESULTS

Seven out of 9 Y1 children did not provide any correct answers at pretest and either refused to participate or quit at posttest. Table 1 presents the frequency of correct answers for the remaining 15 children, by quantity type, task, and trial *(n = 2, 3, 4)*, at both test moments. At pretest, very few correct responses were observed, and only for *n = 2*; these responses came from 4 children. At posttest, all children answered correctly for *n = 2*; and 11 children (including a Y1 child) for *n = 3* and *n = 4* as well (hereafter, Group A). We note that errors were produced by 4 children (hereafter, Group B), who all succeeded twice for *n=3*; three succeeded at least once for *n*=4, whereas the fourth, a Y1 child, failed all trials for *n=4*. Overall, correct answers in the discrete and continuous case were similar in frequency, with a slight difference in favour of the latter for *n=4*; and more correct answers were observed in Task1.

Table 1. Correct-response frequencies by quantity type, task, and trial, at pre/posttest

n	Type of quantity							
	Discrete				Continuous			
	Task 1[a]		Task 2[a]		Task 1[a]		Task 2[a]	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
2	3	15	0	15	0	15	2	15
3	0	15	0	11	0	15	0	11
4	0	12	0	11	0	14	0	11
Total	3	42	0	37	0	44	0	37

[a] Maximum number of correct responses per test moment: 15

Based on data from Table 1, correct answers at posttest were 160 (88.9 % in a total of *12 x 15 =180*). Among those only 5 (3.1 %) were not adequately justified, neither via a solution process, nor verbally; all corresponded to Group B. In 15 cases (9.4%) only verbal explanations were provided.

We focus on the 140 cases (87.5%) in which a solution process was applied (with or without verbal explanation). The applied solution processes were slightly more in the discrete case (76 instances, 54.3% of 240). Based on the video recordings, children's solution processes were described in detail in written form and iteratively compared by the two authors. The following categories were agreed upon:

Indirect Partitioning-IS (N=8, 5.7%). The taught procedure was scarce, with 5 instances in the continuous case.

Indirect Partitioning (N=40, 28.6%). This is a variation of the taught procedure in which the intermediate step was omitted (Example 1). It occurred predominantly in the continuous case (34 instances).

Example 1. The child seeks to find 1/3 of a chocolate bar (*A*). He selects a *b*, takes 3 copies of it, aligns them alongside the given bar, verifies that they together match the same length as *A*, takes one of them and presents it as his response.

Indirect Partitioning –unit iteration (N=11, 7.9%). This is an advanced variation of the taught procedure, in which *A* is directly measured by one copy of *b* (Example 2). It occurred predominantly in the continuous case, but there were 3 instances in the discrete case as well (Example 2).

Example 2. The child seeks to share 9 "candies" fairly among 3 "kids". He selects a group of three from the available options and measures *A* by moving the strip alongside A, while reciting "one, two, three".

Direct Partitioning/ 1-1 (N=17, 12.1%). This process corresponds to the exhaustive one-to-one correspondence, where a representation of n (typically, the "recipients") was employed.

Direct Partitioning/ many-to-one (N=25, 17.9%). In this process, a composite unit was selected in advance as *b* and many-to-one correspondence was carried out, mapping *b* onto a representation of *n* (Example 3). In some cases, the child's first selection (typically, a group of two) needed to be revised.

Example 3. The child seeks to share 9 "candies" fairly among 3 "kids". She selects a group of three from the available options and gives it to the first "kid"; she selects two more copies and gives them to the remaining "kids". She says, "Each kid gets a triad".

Direct Partitioning (N=39, 27.9%). In this process, the child operated on *A*, and decomposed it in *n* parts without using any concrete representation of *n*. In the discrete case (22 instances), it consisted of partitioning *A* into equal groups (Example 4). In some cases, the first attempt (typically, by groups of two) failed and the child retried.

Example 4. The child seeks to find 1/3 of 9 "candies". She looks at the candies for a few moments, separates one group of three and rearranges the remaining "candies" in two groups of three. She says, "Here it is, the one third. I have three triads and I take out one".

In the continuous case (17 instances), Direct Partitioning occurred predominantly for *n=2* where children estimated the midpoint of the bar, marking it with their hand. A similar strategy was applied three times for *n =3* (Example 5).

Example 5. The child seeks to share a "chocolate bar" fairly among 3 "kids". He says, "There are three kids, I need to cut the chocolate into three pieces". He places his hand where he estimates the first piece ends. He looks at the options without removing his hand. Then he selects the correct option and says, "This is one third of your chocolate".

To summarize, two variations of the taught procedure Indirect Partitioning-IS were identified in children's solution processes; in both, children did not use any concrete representation to keep track of the number of multiple units, or iterations of a single unit. Three solution strategies involving direct partitioning of A occurred via one-to-one and many-to-one correspondence; and partitioning *A* without using a concrete representation to keep track of the number of parts. Indirect partitioning was employed

mainly, but not exclusively, in the continuous case; and vice versa for Direct Partitioning. We note that all children used both indirect and direct partitioning processes but only Group A children used the ones that we consider more advanced—Indirect Partitioning with unit iteration, and Direct Partitioning.

DISCUSSION-CONCLUSIONS

The findings of this study indicate that prior to the intervention, successful solving of the tasks was scarce and confined to "halving" in fair-sharing contexts, After the intervention, all Y2 and few Y1 children completed all tasks, across quantity types and relationships. A smaller group succeeded consistently for "half" but inconsistently for "thirds" and "fourths"; they did better in the fair-sharing task; and slightly better in the continuous, than in the discrete case. These findings agree with literature highlighting greater facility with "one half" and "fair sharing" situations (Fitzpatrick & Hallett, 2019); and are consistent with findings indicating that it is procedural, rather than conceptual difficulties that young children face with division tasks in continuous contexts (Kornillaki & Nunes, 2005).

Our findings also indicate that the children appropriated and varied the taught procedure in productive ways; and were not discouraged from developing and advancing their own strategies. We note that none of the variations of the taught procedure or any of the direct partitioning strategies were introduced by the researcher; rather, the children contributed, shared and developed these strategies, starting with exhaustive one-to-one correspondence, the most elementary one (Park & Nunes, 2001). A common feature of more advanced solution processes is less reliance on concrete representations of the number of parts; in addition, there appear to emerge important components of multiplicative reasoning such as identifying and using composite units and measuring with unit iteration (Sophian, 2008; Steffe, 2013).

Overall, the findings suggest that pre-primary children can engage meaningfully with multiplicative relationships for both discrete and continuous quantities when instruction foregrounds relational meanings, and key multiplicative notions such as measurement. At the same time, the limited participation of most Year 1 children in the posttest raises certain concerns. These children were willing to participate in classroom activities but needed peer support to complete the submultiples task. Apparently, they were unwilling to engage in an one-to-one testing situation, probably because they perceived these tasks as overly challenging. Further design iterations are needed to explore whether and how such instructional approaches can be adapted to better support younger children's engagement with multiplicative notions.

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