FROM HALF OF A HALF TO A THIRD ON THE NUMBER LINE: INTERPRETING AND RESPONDING TO FRACTION THINKING

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Fractions are challenging, requiring teachers to interpret and respond effectively to students' reasoning. This study examines how 12 experienced teachers (ET) and 10 pre-service teachers (PST) interpret and respond to problematic situations involving fractional knowledge on number lines. Using professional noticing as a framework, we analyzed two dimensions: interpreting and responding. Fraction sub-constructs and units coordination guided analysis. Participants viewed fraction-related video clips and answered open-ended questions probing noticing skills. Findings show ET more often demonstrated advanced interpretations and responses involving units coordination, while PST relied on quotient subconstruct. Results underscore the need for targeted teacher education on content-specific noticing.

RATIONALE AND RESEARCH QUESTION

Fractions are widely recognized as one of the most challenging topics in school mathematics (Siegler et al., 2013), making it essential to give special attention to how this content is learned and taught. It is essential for teachers to interpret and make sense of what they observe in children's mathematical activities for effective teaching. Recent research emphasizes the need for content-specific noticing to better understand what teachers notice and how they respond in different mathematical domains (Amador et al., 2025). Furthermore, teachers' prior experiences influence their noticing skills, suggesting that examining these differences in the context of fractions can provide valuable insights into what is needed for effective professional development and teacher education programs at the universities. Our interest lies in understanding what noticing looks like in challenging school topics such as fractions and what we can learn from both experienced and inexperienced teachers when addressing this question. Specifically, we aim to define and examine the processes of "interpreting" and "responding" to students' fractional knowledge on the number line.

Research Question. How do pre- and in-service teachers interpret and respond to situations of learning and teaching that require children's fractional knowledge?

THEORETICAL FRAMEWORK AND RELATED LITERATURE

Interpreting and Responding

This study is grounded in the concept of professional noticing of children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010; Jacobs & Empson, 2021),

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which is central to responsive teaching. Professional noticing is defined as a set of three interrelated skills that are temporally and conceptually linked in teachers' daily work:

1. **Attending** to the details in children's strategies,
2. **Interpreting** children's understandings as reflected in those strategies, and
3. **Deciding how to respond** based on those understandings.

Prior research suggests that teacher questioning and professional development experience positively influence students' achievement in fractions (Empson & Jacobs, 2021). However, less is known about how teachers—both pre-service and in-service—interpret and respond to problematic situations that require children's fractional knowledge, particularly in representations such as the number line. This study adopts professional noticing as its theoretical lens to examine what it means to be skillful in interpreting and responding to children's fractional knowledge on number line. By focusing on a content-specific domain (fractions), the study addresses calls for research on whether noticing expertise is general or varies across mathematical topics (Jacobs, Lamb, & Philipp, 2010; Amador et al., 2025).

Fraction Knowledge

The sub-constructs and units coordination promise to be as useful frameworks to conceptualize "fraction knowledge", which are essential for understanding both the students as well as how teachers interpret and respond to children's fractional reasoning. Fractions are conceptually complex and have multiple interpretations as sub-constructs in the literature (Kieren,1993): (a) Part-whole sub-construct (e.g., 3/4 as three parts out of four equal parts), (b) Quotient sub-construct (e.g., 3/4 as 3 divided by 4), (c) Operator sub-construct (e.g., 3/4as three times one-fourth of a whole), (d) Ratio sub-construct (e.g., 3/4 as a ratio of three to four), and (e) Measure sub-construct (e.g., 3/4 as a quantity measured using one-fourth as a unit). These sub-constructs provide a framework for analyzing both children's reasoning and teachers' responses. In the context of number lines, successful reasoning often requires unit coordination—the ability to partition a whole into equal units and coordinate these units across other units (Hackenberg & Sevinc, 2024; Steffe & Olive, 2010). For example, solving a problem that requires identifying $\frac{1}{4}$on a number line involves coordinating two levels of units: 0-1 interval needs to be equally partitioned, where 4 smaller (equal)units connected it will be at the same location as "1" .

Teachers' interpretations and responses to students' fractional reasoning are influenced by their understanding of these sub-constructs and unit coordination. When children misinterpret fractions—such as labeling an unequal partitioned interval as $\frac{1}{3}$ on the numberline—teachers must recognize the underlying reasoning (often they might base on part-whole sub construct) and respond in ways that guide students toward more sophisticated understandings, such as measurement and operator sub-constructs. This requires teachers to integrate interpretations of multiple fraction sub-constructs and

emphasize the importance of equal partitioning, as well as how to construct these parts using reference points and intervals—particularly in number line contexts, therefore coordinating units.

By combining fraction sub-constructs with Hackenberg's unit coordination framework, this study conceptualizes interpreting and responding as content-specific noticing skills. These skills involve identifying which sub-construct(s) a child is using, recognizing gaps in reasoning, and responding with strategies that promote coordination of units and deeper fractional understanding.

METHODOLOGY

The use of video for teacher learning and researching has become a common practice in recent years (e.g., König et al., 2022, for a review). Video-based approaches offer significant advantages, such as the ability to re-watch classroom interactions, analyze them in detail, and engage in collaborative discussions about teaching and learning. In this study, video clips of clinical interviews involving fractions were used as stimuli for interviews. Participants were asked to interpret the child's reasoning and explain how they would respond, allowing researchers to capture teachers' noticing skills—specifically their ability to attend, interpret, and respond to students' mathematical thinking. Four video clips related to fractions were purposefully selected as tools for this study (with durations of approximately 3 minutes, 6 minutes, 3 minutes, and 1 minute). Each clip featured one student and a researcher engaged in solving a fraction problem involving number lines. All clips presented a problematic situation where the researcher was expected to ask questions to probe the student's thinking; however, the purpose of these interactions was not to teach but to reveal reasoning processes.

After viewing each clip, participants responded to seven open-ended questions designed to elicit their professional noticing in a semi-structured interview setting. These questions explored how participants interpreted students' thinking, identified misconceptions, evaluated the researcher's actions, and described how they would respond if they were the teacher. Interviews provided insight into participants' pedagogical reasoning and decision-making processes.

Each interview lasted approximately 30–45 minutes, was video recorded, transcribed, and analyzed using a thematic coding process. This approach allowed for systematic examination of teachers' interpretations and responses to students' fractional reasoning.

One example is a three-minute videoclip of the interaction between Researcher A and Student A (see Figure 1). The researcher initially asks the student to draw a number line marking 0 and 1, and then to place and label the midpoint of 0-1 and expected answer was 1/2. The task is extended to marking half of the interval 0-1/2, where the student after marking the half, incorrectly labels it as 1/3, highlighting a misunderstanding in fractional concepts and unequal partitioning.

Student A: This one (for the midpoint between 0 and ½) would be one-half too. Do we include this one? (The student asks this while pointing to 1. see Figure 1)

Researcher A: Yes, you should think about the whole number line.

Student A: (The student thinks for a moment.) One-third.

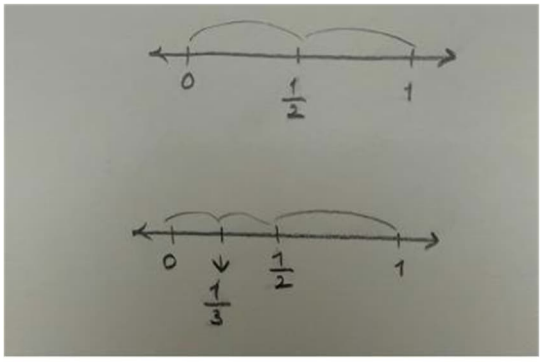
Researcher A: If you want, you can write that under the point you marked. So you think it's one-third, right? Why?

Student A: Because I started counting from 0 (The student begins counting the interval starting from 0). I counted the interval 3 times and got one-third (The student shows that he divided the interval between 0 and 1 into 3 parts).

Researcher A: Alright, I see. Can you explain again how you figured out one-third?

Student A: Here, we're counting again—one, two, three. This point right here is one. Since we counted all of them, we write 3 underneath.

Figure 1: Reconstructed drawing of half of half as one third



Our Interpretation: Student A did not pay attention to the equal lengths when labelling half of half as one third. This suggests that the child was focusing on counting intervals and operating primarily within the part-whole sub-construct. The child may not yet coordinate two levels of units: the first level being one of the four equal lengths within the 0–1 interval, where the ½–1 interval should also be hypothetically partitioned into two equal mini-units; and the second level being the entire 0–1 interval on the number line.

Participants, Data Collection and Analysis

The study involved 12 experienced teachers (ET) with 6 to 25 years of teaching experience, and 10 pre-service teachers (PST) enrolled in the third or fourth year of a four-year teacher education program

All interviews were video recorded, transcribed, and analyzed using a thematic coding process. Responses were categorized into two main dimensions: interpreting (the situation, why the child had the answer of "1/3"), and responding (if you were the teacher of this child, what to do next). Comparative analysis was conducted to identify patterns and differences between the two groups: experienced and pre-service teachers.

We highlighted four categories with color coded related to **"interpreting"**: The **Red group** includes responses that focused primarily on counting and transferring the part-whole subconstruct without considering "equal parts." These responses lacked attention and interpretation in relation to the fundamental concept of "equal" in partitioning. The **Green group** represents responses that highlighted an important construct: the child "does not seem to realize that the intervals need to be equal." These responses emphasized that the idea of "equal parts" within the number line

representation is important (and missing in the child's actions) but did not go beyond this basic observation and interpretation. The **Yellow group** reflects more advanced interpretations. Transcripts in this group indicate that teachers were aware of what the child understood—partitioning a whole—but also recognized that the child ignored the fact that the interval from ½ to 1 was larger than the first two small intervals counted. These responses could also encourage students to think about creating equal parts within the undivided ½–1 interval and connecting this back to 0–1 as the whole unit on the number line. This demonstrates a more sophisticated understanding of units coordination which teachers use in their interpretation of child's actions. The **Blue group** includes responses—primarily from PSTs—that focused on symbolic interpretation of 1/3(quotient sub-construct) and counting with whole numbers, without reference to fractions or the number line context. These responses emphasized procedural meaning, such as "one divided by three," rather than what is missing conceptually in the videoclip. Here are some transcript examples for the categories:

1. No mention of "equal parts or equal interval" (Red)

   "The reason the student gave one-third as an answer here is because Student A ordered the numbers. He is not thinking in terms of fractions—he is just ordering the numbers directly. He is not paying attention to the intervals." (AK, Experienced Teacher)

2. Mention of "equal parts and equal interval in 0-1 interval" (Green)

   "Actually, on the second number line, the student doesn't divide it into 3 equal parts correctly—the intervals aren't equal. I think he found one-third using the same logic he used to find one-half, without looking at the intervals. He does not seem to realize that the intervals need to be equal."(SESE, PST)

3. Advanced interpretations: "how to get equal parts in 0-1 interval" (Yellow)

   "The student knows how to divide a whole into parts. But he doesn't have the concept of whether those parts are equal or not. Why? Because the interval between 0 and one-half is a half, and he divided that into two, so now he has a quarter. But there's also a half between one-half and 1—so the idea of dividing into equal parts isn't there." (NNKG, Experienced Teacher)

4. Interpreting "1/3 as a symbol" (Blue)

   "… The key issue is that they didn't think about needing three intervals. I think they just counted one, two, three to come up with an answer—just to give a response."(SUSÖ, PST)

We highlighted four categories with color coded related to **"responding"**: We revisited teachers' responses to the question of how they would respond to the student's thinking and what actions they would take. Units coordination emerged as a key concept, particularly when teachers used ideas such as dividing "1" into four equal parts and recognizing that four of these parts make up the unit/whole on the number line. This reflects an advanced responding skill. Our color-coded levels for responding might align with the analysis and colors used in the "interpreting" findings. The **Red group** includes responses that were either too advanced for the student's actions (e.g.,

focusing on equivalent fractions) or relied on general representations within the part-whole subconstruct, such as guiding students to count within parts. The **Green group** represents responses that focused on "equal parts" within the number line representation—related but not detailed. These are acceptable but less sophisticated. The **Yellow group** includes responses that not only directed students toward equal parts but also encouraged them to think about how to create equal parts within the undivided interval from ½ to 1 and connect this back to 0–1 as the whole unit on the number line. This demonstrates a more advanced responding skill. The **Blue group** reflects responses—primarily from PSTs—that guided students to interpret fractions as the division of whole numbers (e.g., "one divided by three"), stripping the concept from the number line context and focusing solely on procedural meaning. In addition to these groups, there was a separate category for "no responding" as "I would do nothing".

1.Surface level responding such as "I would give daily life examples" (Red)

What would I do? "The child does not know what they represent. First, I would have them draw a fraction, for example. Draw a circle and show what 1/2 means. Then show what it would be for 1/4. After that, move on to the number line..."(AG, Experienced Teacher).

2. Equal parts but no units coordination (Green**)**

What would I do? "The student could be asked whether the intervals on the number line are equal by showing them. Instead of going directly in a procedural way, the teacher could question the student's answers. I would try to question the student's way of thinking. I would want to understand why the student gave that answer" (AA, Preservice Teacher).

3. Half of the other half: units coordination (Yellow)

What would I do? Can we talk about a fraction when the parts are not equal? Because if the intervals are not equal to each other, we cannot talk about a fraction anyway. So, I would first make the student question that and then ask: how can we make them equal? The student needed to divide the other half into equal parts. In that case, I think the student could have found one-fourth more easily. Also, maybe half of a half—between 0 and 1/2—now, half of a whole can be drawn separately from the number line. Yes, exactly, a symbol, a rectangle representing a whole can be drawn, and then we divide it in half—now this is equal to one-half. Then, if we divide this again in half, and guide the student by asking: what fraction of the whole is this?—I think the student could understand better (ZS, Experienced Teacher).

4. Fractions as division: Quotient sub-construct (Blue)

What would I do? The child wrote 1/3 here (referring to the place where the student put ⅓). Why did they do that? They did it by counting. I would say something like: "If you took half of one using one-half and one-half, what would you get?" If the child still says one-third, I would do this: "What would happen if we divided this? …do the operation. Most likely, the child would not be able to do this, but I would try (EM, Preservice Teacher).

RESULTS

Interpreting (see Table 1): Most ET and PST recognized the student's misconception but varied in depth. About half of PSTs and slightly more than half of ETs noted the need for "equal parts" (Green). A few ETs provided advanced interpretations involving units coordination (Yellow), while PSTs rarely did so, often viewing fractions as "division of two numbers" (Blue). Responding(see Table 2): ETs more frequently suggested strategies emphasizing equal partitioning and units coordination, while PSTs relied on general or procedural approaches. Advanced responses (Yellow) included guiding students to divide intervals and use related representations. Only PSTs focused on quotient sub-construct (Blue).

Table 1. Interpreting why half of half is a third

Group	**Red** No Mention of equal parts or equal interval	**Green** equal parts or equal interval	**Yellow** Advanced interpretation	**Blue** Symbolic meaning of 1/3
Experienced teachers (12)	AK, AG(2)	EA, ES, ESA, GY, NK, TK, HÇ (7)	HY, NNKG ZS(3)	-
Pre-service teachers (10)	FŞ, MEE, YK (3)	CÇ, EM, FY, SESE, SNS (5)	-	AA, SUSÖ (2)

Table 2: Responding to the child's mistake

Group	Red Part-whole subconstruct	Green Equal distance or Equal pieces within NL rep	Yellow Advanced Responding (Units coordination)	Blue Symbolic meaning of 1/3 (Quotient sub-construct)	No Answer
Experienced teachers (12)	AG AK TK(Change: Green to Red) (3)	NNKG (Change: Y⟶ G), EA, ES, GY HY (Change: Y⟶ G) NK (6)	ZS, ESA (Change: G ⟶ Y) (2)		HÇ (Change from G) (1)
Pre-service teachers (10)	FY, SESE (Change: Green to Red) (2)	AA(Change: B⟶ G), CC, SNS, SUSO (Change: B ⟶G) (4)	FŞ (Change: R⟶ Y) (1)	EM (Change: G⟶ B), MEE (Change: R⟶ B), YK (Change: R⟶ B) (3)	

DISCUSSIONS

When we examined the results of both interpreting and responding, and how certain participants moved between groups, we can confidently state that it is more predictable for ETs to move from a more advanced group in "interpreting" to a lower-level group in "responding" within the context of fractions. Approximately one-third of the teachers (4 teachers: TK, NNKG, HY, HC) made this move, while only one teacher

progressed to a more advanced group in "responding" after interpreting the student's thinking. Out of 12 ETs, more than half remained within the same color groups.

Moving from blue to green in responding is also a strong improvement, achieved by two PSTs. However, interestingly, three PSTs shifted from red or green interpreting actions to blue responding. Overall, out of 10 PSTs, only three remained within the same color groups for both interpreting and responding. This indicates that PSTs' responses are less predictable; even when they successfully interpret children's errors in fractions, the quality of their subsequent responses may not align with expectations. Therefore, this aspect requires targeted support in teacher education, as it appears to be particularly challenging. Recent studies suggest that when teacher education emphasizes providing opportunities related to "fractions", PSTs demonstrate higher levels of responding (Authors, 2026) skills for the video cases used in the research. The findings of this study further highlight the need for such approaches.

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