GRADE 6 STUDENTS' EQUATION SOLVING AFTER DIRECT INSTRUCTION AND GUIDED CONSTRUCTION

Anna Ida Säfström[1,2], Björn Palmberg[1,2], Johan Sidenvall[1,3]

[1]Umeå Mathematics Education Research Centre, Umeå university; [2]Department of Science and Mathematics Education, Umeå university; [3]Municipality of Hudiksvall

Learning algebra is difficult for many students. Recent reviews highlight the need for more research on challenges students encounter when learning about equations and studies of learning in real classrooms. This paper reports on a small-scale experiment aimed at investigating how teaching design affect Grade 6 students' equation solving, with a focus on the comparison between direct instruction and guided construction. Our results indicate that teaching design affect what approaches students learn, but to a lesser extent their proficiency in solving equations, in line with similar studies investigating other mathematical topics. Based on these findings, we provide suggestions for development of guided construction designs that support student's equation learning.

INTRODUCTION

Learning algebra is difficult for many students (Xie & Cai, 2022) and declines in algebra knowledge have been identified (Hodgen et al., 2024). Recent reviews highlight the need for more research on challenges encountered when students move from pre-algebra to the letter/symbol form of equations (Kieran, 2022), as well as more classroom studies focusing how teaching affect student learning (Ellis & Özgür, 2024).

Learning linear equation solving

Two broad types of obstacles to overcome in learning algebra, and especially equation solving, concern understanding equations and what their solutions are, and becoming proficient in simplifying equations by transforming them through a number of steps until the value of the unknown can be easily determined (Jankvist & Niss; 2015). Relating to the former, Xie and Cai (2022) found that Chinese 5th-grade students struggled more with equations using letters rather than other forms of representing the unknown. Regarding transformations of equations, the influential study by Filloy and Rojano (1989) argued that solving some types of equations (e.g., $Ax + B = C$) only requires use of inverse operations, and that actual operation on the unknowns only is required for equations with at least two appearances of the unknown, exemplified by $Ax + B = Cx + D$. Although many studies since have focused on this specific type of equation, Elkjaer and Kieran (2025) recently argued that operating on the unknown is also required when solving equations with several unknowns on one side only, and that this has been mostly overlooked (for exceptions, see e.g., Vlassis, 2002). Furthermore, they identified specific common errors with regards to such equations. Studies also

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show that students' who seemingly have successfully overcome these two obstacles, can revert to previously erroneous ideas in the face of new equations (e.g., Filloy & Rojano, 1989).

Teaching linear equation solving

While it has been argued that the existent research on classroom algebra teaching is insufficient (Ellis & Özgür, 2024), there are some studies testing different teaching designs empirically. Booth et al. (2015) found that providing worked examples benefit students' learning, especially if they have low prior knowledge. While worked examples are often characterised as a type of direct instruction contrasted to approaches aiming to guide students' own construction of methods (Zhang et al., 2023), it is important to note that Booth et al. (2015) compared regular teaching only—which is likely to have included demonstrations of equation solving by means of worked examples—to regular teaching together with additional worked examples prompting students to explain why the solution was correct or incorrect. It is thus not clear how this study weighs in on the debate regarding the relative merits of direct instruction and guided construction (see e.g., de Jong et al., 2023; Zhang et al., 2023). Others have found that sequential worked examples benefit students with low prior knowledge more, while the opposite was true for students with higher prior knowledge (Rittle-Johnson et al., 2009). Star and Rittle-Johnson (2008) focused multiple and efficient ways of simplifying equations, comparing prompts to construct different ways of simplifying the same equation and direct instruction on such different ways. They found no significant differences in accuracy between the groups one day after training ended but that the prompt condition led to greater use of multiple strategies whereas direct instruction rendered greater use of efficient strategies. In sum, studies have shown that teaching design matters and provided valuable insights into relative merits of different teaching designs. However, detailed information on how different designs work for learning to solve equations is still scarce, in particular regarding how learning from direct instruction compares to learning from guided construction, and how guided construction can be designed to support students' construction of the simplification approach to equation solving.

THEORETICAL FRAMEWORK

In this study, we focus equations requiring operating on the unknown, that is equations including at least two occurrences of the unknown (Filloy & Rojano, 1989), with whole-number coefficients. In previous research, we identified two main approaches that can be used to solve such equations generally and correctly: *simplification*, defined as transforming the equation through a number of steps until the value of the unknown can be easily determined, and *substitution*, defined as repeatedly replacing the unknowns with a selected number and evaluating both sides of the equation, until a number that makes the two sides equal is found.

In line with Elkjaer and Kieran (2025), we classify equations requiring operating on the unknown into two types: (a) equations with multiple occurrences of the unknown on one side of the equal sign and no unknowns on the other, and (b) equations with one or more unknowns on both sides of the equal sign. The rationale for these types is that they require different types of simplification steps: type (a) only require *gathering of unknowns*, while type (b) require *removal of equal unknown terms* from each side of the equal sign, sometimes combined with gathering of unknowns.

The difficulty of using these steps may depend on the terms involved (Filloy & Rojano, 1989) and a reasonable prerequisite is that one can use the step for constants. For gathering of unknowns, we hypothesise the progression: x-terms with coefficient 1, x-terms with coefficients ≥ 1, x-terms with coefficient -1, x-terms with coefficients ≤ -1. For removal of equal unknown terms, we hypothesise the progression: visibly equal terms with coefficients ≥ 1, such as $5x$ in the equation $3 + 5x = x + 5x$; visibly equal terms with coefficients ≤ -1; non-visible equal terms with coefficients ≥ 1, such as in $5x$ in the equation $3 + 5x = 6x$; non-visible terms with coefficients ≤ -1.

AIM AND RESEARCH QUESTIONS

The aim of this study is to investigate how teaching design affect students' early development of equation solving approaches, with a focus on the comparison between two groups: students learning by direct instruction (DI) and students learning by guided construction (GC). Our aim is specified in two research questions: To what extent do students in the two groups attempt and solve equations of type (a) and (b)? Do students in the two groups differ in terms of their use and success of equation solving approaches and simplification steps?

METHOD

The study was conducted within a design research project, aiming to develop insights about how tasks and teacher support can be designed to promote students' learning of mathematical methods in Grade 4–6. It was preceded by development and testing of tentative teaching designs in a series of small-scale trials together with participating teachers. The data used in this study entails a pretest one week before training and a posttest one week after training. The training consisted of three lessons.

The pretest was designed to test the use of properties of operations and equalities prerequisite for equation solving: relationships between basic operations; additive inverse; the distributive law and removal of equal constant terms on both sides of the equal sign. The last task was an equation with several unknowns on both sides of the equal sign (the same as the last equation on the posttest), included to identify students who already knew a general method for solving equations.

One Swedish Grade 6 class (n=31) participated in the study. The textbook series used in this class included multiple opportunities to solve equations with one unknown and one operation (such as $A + X = B$ and $A \cdot X = B$), but the class had not received any formal instruction on equation solving or simplification of algebraic expressions prior

to the study. Half of the class (n=15) was taught using the direct instruction approach and the other half (n=16) was taught using the guided construction approach. The groups were formed using stratified randomisation based on the teachers' assessment and the number of correctly solved items on the pretest. The number of properties used were similar in the two groups (ID: M = 3.1, SD = 1.4; GC: M = 2.8, SD = 1.7). No student solved or operated on the unknowns on the last item of the pretest.

The students' regular teacher, who also participated in the development of the designs, taught both groups on the same days distributed over one week. Lesson 1–2 focussed on equations of type (a) and Lesson 3 focussed on equations of type (b). In the DI group, each lesson started with the teacher showing and explaining the method for a few examples, followed by students individually using the method. If students needed help, the teacher described the method again. In the GC group, the full lesson time was devoted to students' individual work on a sequence of equations which followed the progressions described above to support students' development of the simplification method. If students needed help, the teacher supported the students' own construction of solution approaches by asking for students' reasoning so far, supporting connections to previously solved tasks and providing occasional hints. All lessons were observed and audio recorded, and the data showed that the two approaches were implemented according to plan.

The posttest consisted of 8 items. Items 1–5 consisted of equations of type (a) of increasing difficulty: $x + x + x + x + x = 15$, $24 = x + 2x + 3x$, $5y + 3y + 4 = 20$, $16 = 8x – x – x – x – x$, and $5x + 10x – 3x + 2 = 26$. Thus, Item 1 required gathering of *x*-terms with coefficient 1, Items 2–3 gathering of *x*-terms with coefficients ≥1; Item 4 gathering of *x*-terms with coefficients ≥1 or -1; and Item 5 gathering of *x*-terms with any whole number coefficients. Items 6–8 consisted of equations of type (b) of increasing difficulty: $15 + 4z = 5z + 4z$, $10x – x = 7x – 3x + 30$ and $15 + 4x + 4 + 2x – 21 = 6 + x + 4 + 2x + 9$. Item 6 can thus be solved by removal of visible *x*-terms with coefficients ≥1, while Items 7–8 require gathering of terms followed by removal of non-visible *x*-terms with coefficients ≥1.

Analysis

Analysis focussed three aspects, based on the theoretical framework: general solution approach, correct gathering of terms and correct removal of equal terms. Each aspect was assigned a code accompanied by an argument, explaining the empirical basis for the code. For all aspects, it was possible to assign the codes *None* and *Cannot be determined*. General approach was coded as either *Simplification*, if the student tried to gather and/or remove equal terms to simplify the equation to the form AX=B; *Substitution*, if the student tried to substitute the *x*:es for a number and evaluate whether that made the equation correct; *Guessing*, if the student only guessed an answer without evaluating the equation; *Multiple*, if the student combined at least two of the previously described approaches, *Other*, if the student neither simplified, substituted or guessed. Correct gathering of terms was coded as either: positive constants, negative constants,

x-terms with coefficient 1, *x*-terms with coefficient -1, *x*-terms with coefficients >1, or *x*-terms with coefficients <-1. The codes were defined accumulatively meaning, e.g., that the code "*x*-terms with coefficient -1" included cases where the student also gathered constants and/or *x*-terms with coefficients ≥1. Removal of equal terms on each side of the equal sign correctly, and if so what kinds of terms: positive constants, negative constants, visible *x*-terms with coefficients ≥1, or non-visible *x*-terms with coefficients ≥1. These codes were also defined accumulatively.

All data was analysed independently by two researchers, with each author coding approximately two thirds of the data. This was done in three phases, with discussion leading to clarification and revision of codes in between. The first phase included 6% of the data and the second phase included 47% of the data. In the last phase, including 48% of the data, agreement ranged from 75–98% with an average of 91%, and Cohen's kappa ranged from 0.64 to 0.94, with an average of 0.82. All disagreements were reviewed by one researcher who, based on the coders' arguments, settled on one code. For 91% of the disagreements, this could be done using the existing code book, which meant that the disagreements could be seen as mistakes. For general approach, the codes guessing, multiple, other and cannot be determined were collapsed into the code *Unclear/other*. These codes where generally used for solutions that were unclear or sparsely documented, and there were very few cases of guessing and multiple.

Due to the small scale, our study is underpowered to, through hypothesis testing, identify anything other than very large effects. Therefore, we refrain from reporting p-values accompanying statistical tests to avoid interpretations of large p-values as evidence of no effect. We do however report on point estimates of sizes of effects to further describe parts of our results.

RESULTS

In this section, we report on the results of the analysis of the posttest, answering our research questions.

Research Question 1: Success rates for the equation types (a) and (b)

The DI group attempted a total of 85 equations (M = 5.7, SD = 2.1) and the GC group attempted a total of 113 items (M = 7.1, SD = 1.7). The additional equations attempted by the GC group was mainly of type (b). As shown in Table 1, the DI group solved more equations overall (Hedges' g = .26) and with multiple unknowns on one side of the equal sign (type (a), Hedges' g = .55), while the GC group solved more equations with unknowns on both sides (type (b), Hedges' g = .29).

Table 1: Average number of correctly solved equations with SD in parenthesis

Equation type	(a)	(b)	Total
DI	3.8 (1.3)	0.5 (0.9)	4.3 (1.9)
GC	2.9 (1.8)	0.8 (1.0)	3.7 (2.6)

Research Question 2: Solution approaches and success rates

In total, simplification was used in 57% of the solutions, and substitution in 20% of the solutions. The relation between group and approaches used was very strong (Cramer's v = .72). The DI group almost exclusively used simplification, while the approaches used in the GC group were more evenly distributed across simplification, substitution and unclear/other (Table 2). Both groups where most successful when using simplification on type (a) equations, but the GC group was also highly successful when using substitution on type (a) equations and simplification on type (b) equations (Table 3). The overall lower success rate of the GC group on type (a) equations (Table 1) can be explained by the frequent use of unclear/other approaches, which were very seldom successful.

Table 2: Frequency and percentage use of approaches in the two groups

	Simplification	Substitution	Unclear/other
DI	83 (98%)	2 (2%)	0 (0%)
GC	29 (26%)	38 (34%)	46 (41%)

Table 3: Success rates for different approaches and equation types

	Type (a)		Type (b)	
	DI	GC	DI	GC
Simplification	88%	92%	37%	100%
Substitution	50%	84%	–	54%
Unclear/other	–	9%	–	–

On the other hand, the high success rates for simplification in the GC group can be explained by most students in this group only using this approach on Item 1 and the consistent and successful use by one student on all items. In contrast, DI students used simplification steps for equations including more difficult terms. In the DI group, 93% of students gathered x-terms with coefficients ≥ 1 and 67% gathered x-terms with coefficients ≤ -1, while only 31% and 6% of GC students did so, respectively. The two groups were similarly successful in removing visible x-terms, (DI: 20%, GC: 19%) but while 20% of students in the DI group removed non-visible x-terms correctly, only 6% of GC students did so.

CONCLUSION AND DISCUSSION

The results of this study revealed only a small difference in overall success rate but large differences in approaches used, when comparing the two groups. DI students relied mainly on simplification, whereas GC students employed a wider range of

approaches, including substitution and other less systematic methods. A similar pattern was discovered by Heinze et al. (2015) for multi-digit addition and subtraction. They neither found differences regarding overall success, but that students receiving more direct instruction on advanced approaches used them more frequently, while students learning solely through own construction of approaches used their own, but sometimes less advanced, approaches more often. However, Heinze et al. (2015) found that the difference in use of advanced approaches decreased over time, possibly due to these methods being learnt superficially. Complementing the present study with a delayed posttest could reveal if this may also be the case for the simplification approach to equation solving.

In both groups, students were more successful in solving equations with multiple unknowns on only one side of the equal sign (type (a)) than with unknowns on both sides of the equal sign (type (b)). In relation to the lesson time devoted to each type—two lessons for type (a), one lesson for type (b)—this is not surprising. However, it contrasts previous findings that equations with several unknowns on one side are more challenging for students than equations with unknowns on both sides (Elkjær & Kieran, 2025). Our study indicates that students can learn to solve type (a) equations even in rather short interventions, if given the opportunity.

The results highlight two aspects that may provide guidance on how to improve the GC teaching design. First, the GC group's frequent use of unclear/other approaches led to incorrect answers, while the GC group seem to be at least as successful as the DI group when it comes to simplification and substitution. A possible adjustment of the GC teaching design could therefore be to identify students who already have, or are at risk of developing, incorrect strategies, and to find ways to support these students in developing correct methods. One possibility would be to blend GC and ID designs (de Jong et al., 2023), for example by providing more direct instruction after initial attempts at constructing a method, or to target such instruction at students who are at risk of developing ineffective strategies. Second, the results show that the GC group consistently used simplification on the first task, but that most students then switched to substitution on subsequent tasks. This suggests that the GC design did support the development of gathering x-terms with coefficient 1, but not the continued development of this step to include x-terms with other coefficients. A more detailed analysis of the students' solutions during the training could provide further insights into how the design can be refined to better support the development of simplification.

Lastly, we note some limitations of this study: the small sample size made it underpowered for hypothesis testing and the analysis only included written responses and did not consider students' real-time reasoning neither in the test nor during training (e.g. Ellis & Özgür, 2024). In addition, the short duration of the intervention may not have been sufficient for all students to learn effective equation solving approaches.

Additional information

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