ADVANCED MATHEMATICAL KNOWLEDGE FROM THE EYE OF EXPERTS: THE CASE OF LIMIT

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This study provides a detailed account of advanced mathematical knowledge specifically for mathematics teachers, focusing on the concept of limits. Structured interviews were conducted with five experts who have various expertise in the fields of mathematics and mathematics education. The results demonstrate how the concept of equality is interpreted differently in the context of limits, as opposed to other mathematical contexts. The role of covariational reasoning in the context of limits is also elucidated with empirical evidence, and the significance of topology in comprehending the concept of the limit is emphasized. Results point to some important implications for future research and teacher education in the learning of limits from an advanced mathematical point of view.

INTRODUCTION

A thorough comprehension of subject matter knowledge is essential for teachers (Shulman, 1986). Regardless of whether they are to teach in middle or secondary school, mathematics teachers usually take content courses as part of teacher education programs. Researchers have debated several key issues in this area. One major debate concerns the amount and type of subject matter knowledge needed to ensure successful mathematics instruction (e.g., Ball et al., 2005). On the other hand, Klein (1932) drew attention to the discrepancy between the mathematical knowledge that teachers acquire at the university level and the knowledge they actually need in classrooms. Mathematics teachers must have a profound understanding of the mathematical knowledge presented at the school level (Conference Board of the Mathematical Sciences [CBMS], 2012). Yet research has revealed a disparity between mathematics teachers' content knowledge and their use of this knowledge in classroom settings (e.g., Ball et al., 2005; Zazkis & Leikin, 2010).

Advanced mathematical knowledge refers to knowledge gained through undergraduate mathematics courses (Zazkis & Leikin, 2010). Despite the ongoing debate in the field, advanced mathematical knowledge is increasingly regarded as essential for mathematics teachers, as it shapes their comprehension and interpretation of the content they teach and, consequently, their instructional practices (Wasserman, 2016). Recent studies have examined the perceived value of advanced mathematics courses for teachers from the perspective of mathematicians (e.g., Hoffman & Even, 2018; Yan et al., 2022) and have highlighted distinct perspectives. For example, Ziegler and Loos

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(2017) focused on content-related issues, whereas Hoffman and Even (2018) emphasized the nature of mathematics itself. Similarly, Yan et al. (2022) identified advanced mathematics courses as crucial for improving problem-solving abilities, building cross-domain connections, and fostering deeper epistemological awareness.

In this regard, we pursued a similar objective of exploring the advanced mathematical knowledge that is essential for mathematics teachers, with a particular emphasis on the concept of limit, which is significant in the field of calculus and other mathematical fields (Tall, 2002). The formal definition signifies the transition to a more structured and rigorous mathematical reasoning (Swinyard & Larsen, 2012), serving as a reference point for subsequent concepts in calculus, including convergence, derivatives, and integrals (Cornu, 2002). We contend that re-capsulizing limits from the perspective of the lenses on advanced mathematical knowledge is essential. Albeit its importance, previous research has pointed to the difficulties of learners and possible reasons behind these difficulties. For example, teaching limits is primarily through graphs, tables, and symbolic representations, relying on informal notions such that "as x approaches a value, $f(x)$ approaches its limit" (Tall, 2002) not connecting to different concepts such as equality (Páez & Hitt, 2004) and not focusing on reasoning about the relationship between quantifiers as covarying quantities (Carlson et al, 2001) as well as epistemological obstacles related with infinity. Recently, Zazkis and Mamolo (2025) further reported on preservice teachers' ways of thinking about 0.999….=1 that showed that participants' main difficulty in reasoning about the equality sign stemmed from their conceptualization of potential infinity rather than actual infinity and their thinking about limits as approximation.

In this regard, we report an analysis of essential advanced mathematical knowledge regarding the concept of limits for mathematics teachers from experts, including mathematicians and mathematics educators. Yan et al.'s (2022) framework seemed appropriate for this purpose. This framework emphasizes a three-dimensional lens on the relevance of advanced mathematics studies for prospective secondary mathematics teachers. Accordingly, we posed the following research question for this study: "What advanced mathematical knowledge of the limit concept do experts identify as essential for prospective mathematics teachers?"

THEORETICAL FRAMEWORK

Yan et al. (2022) identified three essential elements of advanced mathematics studies for prospective mathematics teachers: (a) drawing connections across mathematical domains, (b) gaining mathematical experience for developing problem-solving abilities, and (c) increasing epistemological awareness of the subject. The first aspect is that advanced mathematics enables secondary mathematics teachers to gain a broadened perspective on mathematical topics. In the context of limits, it emphasizes the connections of limits to other mathematical concepts, including formal definitions, symbols, and representations—central topics in calculus and related domains (Cornu, 2002; Tall, 2002)—as well as concepts such as distance and neighborhoods. As the

second component of their framework, Yan et al. (2022) analyzed how advanced mathematics can provide teachers with diverse experiences, enabling them to develop mathematical flexibility and creative approaches to problem solving. Accordingly, the present study intended to identify specific mathematical approaches that could help teachers address limit-related problems more effectively. Yan et al. (2022) characterize epistemological awareness as teachers' recognition of the strengths and constraints of their mathematical knowledge as a third aspect. In this regard, this study examined the fundamental challenges encountered by teachers and students (Cornu, 2002; Moru, 2009) from the perspectives of mathematicians and mathematics educators.

METHOD

This study forms part of a broader project exploring how mathematics teachers develop their use of advanced mathematical knowledge in teaching the concept of limits. In this context, we performed structured interviews with experts to gain insights into the advanced mathematical knowledge that are essential for mathematics teachers (Clement, 2000). This study involved five experts (E1–E5), purposefully selected (Merriam & Tisdell, 2015) to represent a range of academic backgrounds. The sample of experts consisted of one mathematics educator (E2) and two mathematicians (E1, E4) affiliated with mathematics education departments; and, two mathematicians from mathematics departments (E3, E5). Each had over twenty years of academic experience with studies on advanced mathematics in mathematics and mathematics education, including analysis, algebra, number theory, geometry, and topology.

Structured interviews lasting 40 to 70 minutes were administered to experts through the Zoom platform. The interview questions were prepared with consideration of the framework components proposed by Yan et al. (2022) and the relevant literature on the limit concept and advanced mathematical knowledge. Initially, how experts connect the concept of limit across various mathematical domains, encompassing quantifiers, inequalities, absolute value, visualization, formal definitions, and related proofs or theorems. The second element of the framework, which concerns the acquisition of mathematical experience for the purpose of developing problem-solving abilities, entailed an investigation into mathematical problems that can be solved with greater efficacy by consideration of the underlying concepts that underpin the limit. Specifically, how experts comprehend the limit processes in the derivation of formulas for the area of a circle. Finally, we asked questions prompting the experts to consider the role of infinitesimals and the value of historical knowledge in the teaching of limits. Additionally, the role of limits in comprehending a frequently misconceived issue, the equality of 0.999... and 1, was posited to the participants.

A constant comparison approach was employed to analyze the data (Clement, 2000). Following verbatim transcription and the composition of memos, the first author performed open coding to analytically identify fundamental ideas, including mathematical connections, problem-solving procedures, and epistemological awareness. The codes were subsequently organized in an Excel sheet for comparative

analysis among experts and associated with the components of Yan et al.'s (2022) framework. Next, the research team collaborated to refine the categorization, settling disagreements through discussions. For example, the team reclassified covariational reasoning from 'connections' to 'problem solving' due to its relation to the ways of thinking and mathematical problem solving. Following this process providing clarity on the analysis, we wrote the narratives on the advanced mathematical knowledge that experts regard as essential for teachers in the context of limits.

FINDINGS

The findings are presented under three subheadings in accordance with the components of the framework guiding the study.

Drawing connections of limit across mathematical domains

Two issues were identified under this section: the relationship of limit with topology and different meanings of some concepts such as equality.

Although the limit is mainly known as a calculus term, Experts 1, 2, and 4 adopted some topological terms in their statements. For example, E1 made a distinction between the limit and accumulation points in topology and noted that a sequence may have some accumulation points but not a limit. Besides, E4 acknowledged that a teacher must be aware that “for a sequence to have a limit, it must first have an accumulation point.” Lastly, Expert 2 employed the term circle as a topological term, stating that “having a limit is akin to remaining wholly within a circle after a certain point.” Expert 2 also associated this notion with the quantifiers in the formal definition of limit, as outlined below:

> The part we call 'for every' is actually the radius of the circle. You can get it as small or as big as you want. However, once you set the radius, after a certain point, which is the ‘there exists’ part, all those values have to stay inside the circle; they cannot exceed its boundaries.

The epsilon–delta definition of limit necessarily requires the use of absolute value, which establishes the standard metric on the real line, together with the order properties of $\mathbb{R}$ underpinning the inequalities in the definition. However, as Expert 2's statements indicate, the term 'circle' was articulated without any apparent need for emphasis in the context of two-dimensional spaces. This flexible approach suggests that experts typically consider both one- and two-dimensional spaces collectively, without making a distinction between them for pedagogical purposes within the context of limits.

Furthermore, the meaning of mathematical symbols like equality, addition, and division was defined in relation to limits and other contexts within mathematics (E1, E2, E5). E1 and E2 exemplified how the meaning of equality in limit is changing as compared to its usual meanings. E1 claimed that a teacher must be able to think limit out of the ordinary arithmetic. Otherwise, he or she may encounter paradoxical situations. Furthermore, he presented an example of how the notion of equality might

be perceived variably within the framework of limit, employing the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ and paid attention to this expression noting that:

The equality in this expression is not an absolute 1 in the sense of exactly 1, neither more nor less. But it says that it cannot exceed 1. In fact, this sum is less than 1. But we take it as 1 in the sense of limit. I have taken infinite terms and written them one under the other. It does not mean that it made exactly 1. The sum of these terms does not mean that it made exactly 1 in the sense of nothing more, nothing less, but exactly 1. But it says that it cannot exceed 1. In the context of a limit, the value is 1; however, the actual sum is less than 1.

Expert 1 differentiated between equality to 1 as a numerical value and a limiting value, as is evident from the excerpts.

Gaining mathematical experience for developing problem-solving abilities in the limit

The second component of Yan et al.'s framework (2022) revealed two themes: the issues teachers face and the types of reasoning they need to engage in.

First and foremost, understanding the definition of limit conceptually and exploring the elements within the formal definition critically were emphasized by the experts. Experts E3, E4, and E5 noted that although the limit is defined at a point, examining the behavior of the function near that point is necessary. Secondly, some issues about the role of quantifiers were emphasized, as the definition of limit stated a claim for every epsilon, but it should be checked if it works sufficiently for small values of epsilon (E2, E3, E4).

Additionally, one expert (E1) strongly emphasized the importance of covariational reasoning in understanding the formal definition of limit. He highlighted its significance in three separate dimensions specifically: Firstly, he proposed an interpretation of the relationship between $x - a$ and $f(x) - L$ in the limit as follows: "In the case of $x - a$ remains less than δ, if $f(x) - L$ can be shown to be less than ε, then we deduce that the limit at that point is L." Secondly, he articulated the process of identifying a delta based on an assigned epsilon and stressed the importance of establishing a way of thinking about how δ is expected to change in response to changes in ε values. Finally, Expert 1 emphasized how changes in independent variables correlate to variations in dependent ones by highlighting the relationship between x and $f(x)$, and relating this lens to the dependence of epsilon and delta on one another.

Increasing epistemological awareness of the limit

The findings revealed four elements emphasized by experts about epistemological awareness within the context of limits. These issues arose from the differing meanings of terms in everyday life versus in limits, confusion faced during the limiting process, the misinterpretation of the symbols ε and δ, and the ineffective usage of infinitesimals in the context of limits.

Specifically, E3 addressed two points regarding the first issue. He noted that the everyday meaning of limit can be misleading, as it often conveys a sense of a boundary value. In addition, he remarked that the quantifier "there exists" is sometimes expressed verbally as "some," whereas interpreting it as "at least one" would be more appropriate to avoid misunderstandings. Additionally, the experts noted that the endless sequence of nines in 0.999... often hinders teachers from accepting its equality with 1. They therefore emphasized the importance of addressing such controversial issues with teachers by drawing on different branches of mathematics. In this regard, beyond clarifying the limiting process through series, equality should also be justified through the construction of real numbers. Thirdly, E4 addressed the interpretation of the symbols ε and δ. He emphasized that teachers should regard them as variables rather than fixed constants; otherwise, they might develop misconceptions, such as viewing 0.99… as the final number before 1. Lastly, experts 3, 4, and 5 expressed the view that the use of infinitesimals was not essential for introducing the limit and that it was challenging to comprehend even in itself.

DISCUSSION

This study investigated the advanced mathematical knowledge that teachers should possess in the context of limits, as viewed from experts' perspectives. We specifically drew on Yan et al. (2022) three-lenses framework narrowly focusing on identifying the scope of advanced mathematical knowledge with particular emphasis on the concept of limits. Experts' viewpoints referred to important notions including topology, equality, and covariational reasoning for teachers' understanding of limits at advanced mathematical knowledge.

First of all, drawing connections between limit and other mathematical domains, experts identified the denotations of symbols and signs, establishing their relationship to topology and the concept of equality. Experts have elaborated on certain topological terms in relation to the inequalities and the quantifiers "for every" and "there exists" in the formal definition of limit. In addition, the findings indicated that, similar to Zazkis and Mamolo (2025) study, according to the experts, the equality sign serves both as a fundamental connection linking the concept of limit to other mathematical domains and as an epistemological obstacle for teachers, particularly in the case of 0.999…=1. From the experts' perspective, the equality sign did not signify 'sameness' or 'do something to get the answer' (Usiskin et al., 2003); rather, it represented a state of equality attained at the conclusion of a process. According to the experts in our study, the construction of real numbers can provide a reasonable explanation of the equality 0.999…= 1 in its standard meaning in arithmetic, (Belin & Karagöz Akar, 2020a,b). In line with the experts' views, Choi and Do (2005) compared the various meanings of the equal sign in relation to 0.999…= 1. The study findings also revealed that one expert (Expert 1) offered a detailed analysis of the importance of covariational reasoning in the setting of limits. Although previous studies recognize covariational reasoning as central to other calculus concepts (Harel, 2008; Thompson & Carlson,

2017), its significance in limits has received little empirical attention (Carlson et al., 2001). This study provides an empirical account of how educators can intentionally engage learners in covariational reasoning related to limits—an issue that may be implicitly included in calculus but is rarely made explicit for mathematics majors and teachers. Although topology courses are not mandatory in many teacher education programs, the findings highlight that experts' emphasis calls on including a topological perspective in the instruction of limits. In addition, there is a need for more empirical research with teachers including the meaning of equality from the perspective of topological terms such as neighborhood or clustering with a focus on reasoning covariationally on quantifiers to improve mathematics teachers' understanding of and professional practice with limits.

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