SUPPORT IN THE USE OF THE NUMBER LINE: A CASE STUDY

Anna Lisa Simon[1], Florian Schindler[2], & Maike Schindler[1]

[1]University of Cologne, [2]Heidelberg University of Education

The number line (NL) is an important tool in mathematics education for developing students' understanding of numbers, yet its use is often not immediately accessible to students. To date, little is known about how students can be supported in learning to use the NL and how their related learning processes unfold. In this paper, we trace the learning process of Emir, a fourth-grade student with difficulties using the NL. Based on Emir's verbalizations, eye movements, gestures, and actions, we qualitatively follow his learning process over the course of a school support study, focusing on reference-point use and distance interpretations. We use Emir's case to showcase the volatile learning process of coming to use the NL and its reference points, and identify prominent challenges and turning points in supporting students' use of the NL.

INTRODUCTION

The number line (NL) is an essential representation in mathematics education (Gaidoschik, 2024). Beyond its role in representing the ordinal aspect of numbers, the NL is particularly useful for building and understanding numerical relationships (Teppo & van den Heuvel-Panhuizen, 2014). However, using the NL is often not immediately intuitive and accessible for students. Numbers and distances on NLs can be difficult to understand—especially for students with mathematical learning difficulties (Scherer & Moser Opitz, 2010). Gaining insights into how students use NLs is therefore crucial. Existing studies showed that students with mathematical learning difficulties often show inaccuracies in locating whole numbers on the NL and difficulties in flexible strategy use (van't Noordende et al., 2016). They often rely predominantly on counting strategies and rarely make use of nearby reference points on the NL (Simon et al., 2025). The ability to use these reference points depends largely on students' understanding of numbers and numerical relationships (Peeters et al., 2017), as well as on their interpretation and understanding of distances on the NL (Gaidoschik, 2024).

While several studies already examined how students locate whole numbers on NLs and investigated their strategy use, little is known about which forms of support help students learn to use this challenging tool and how related learning processes unfold. In this paper, we present a case study of a fourth-grade student, referred to as *Emir*. Based on Emir's verbal utterances, eye movements, gestures and actions, we show his learning process in using important reference points on the NL 0–100 and in locating numbers on NLs with varying start and endpoints.

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, pp. 1371-1378. PME.

THEORETICAL BACKGROUND

Different types and interpretations of the number line

The NL represents numbers as points (marks) on a straight line, where equal geometrical distances between points correspond to equal arithmetical differences between numbers. In this paper, we focus on NLs that represent whole numbers. There are different types of NLs, which can differ in the information displayed about numbers: from fully marked and labeled NLs (with equidistant marks for all whole numbers in the represented number range; also called "filled" NLs; e.g., Teppo & van den Heuvel-Panhuizen, 2014), to partially marked NLs with fewer marks, and "empty" NLs, where only the start and endpoints are marked and labeled (also called "proportional" NLs; e.g., Teppo & van den Heuvel-Panhuizen, 2014). NLs can be interpreted in different ways. When understood as an ordinal model, the focus is on the order of whole numbers. In this view, numbers are represented solely as marks on the NL, each corresponding to its ordinal position (e.g., Gaidoschik, 2024). This interpretation is especially evident in fully marked and labeled NLs, where each mark explicitly represents a single number (e.g., Scherer & Moser Opitz, 2010). Another interpretation of the NL is as a measurement model (Fuson, 1984; Gaidoschik, 2024): In this interpretation, numerical relationships are understood as relations between measured quantities (i.e., lengths), highlighting that equal geometric distances correspond to equal numerical intervals (Gaidoschik, 2024). The measurement model is particularly relevant for partially marked or "empty" NLs, where only a few marks are displayed, and numbers need to be located as relative positions.

Difficulties with number lines

Interpreting positions and distances on the NL is challenging for students, particularly for students with mathematical learning difficulties (Scherer & Moser Opitz, 2010). Such challenges are reflected, for example, in the inaccurate locating of numbers on the NL. Research suggests that these inaccuracies are associated with difficulties in the use of appropriate strategies (van't Noordende et al., 2016): for example, students with mathematical learning difficulties predominantly use counting strategies and rarely use nearby reference points (Simon et al., 2025). Whether and how students make use of reference points on the NL depends on their prior knowledge and understanding of numbers and numerical relationships (Peeters et al., 2017). For example, locating 50 exactly halfway between 0 and 100 may indicate an understanding of proportional relationships of 50 as half of 100 (while this may also be recalled) (Gaidoschik, 2024).

Number line support

While several studies already investigated, for example, differences in NL accuracy between students with and without mathematical learning difficulties, only a few support studies have been conducted. The existing support studies focus on pre-post comparisons of children's performance (e.g., Fitzsimmons et al., 2023) and investigate the effects of NL interventions on other, general numerical skills (e.g., Kucian et al.,

2011). To the best of our knowledge, there are no studies published yet that qualitatively inquired into students' learning processes, specifically regarding the use of reference points and interpretations of distances on NLs. To address this gap, the presented case study offers a fine-grained analysis of the learning process of one student, focusing on reference-point use and distance interpretations, and shows how specific instructional supports are connected to challenges and turning points observed.

THIS STUDY

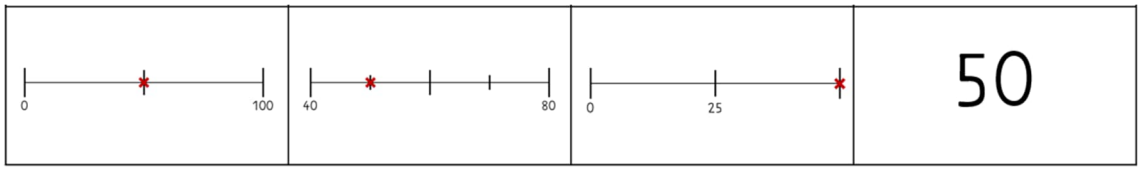
Participant(s). Emir (pseudonym; 9.8 years old) was part of a larger support study with several students in a German primary school. The study took place in the last weeks of fourth grade, i.e., before the end of primary education and the transition to secondary school. The students were chosen by their teachers based on their need of support.

Procedure and data. At the beginning of the study, an individualized diagnostic was conducted through an individual interview with Emir, during which he worked on a variety of NL tasks. Subsequently, he participated in one-on-one support sessions with the first author (support teacher). Every session took place in a quiet room at school and lasted 24 minutes on average. During all sessions, Emir wore eye-tracking glasses *Tobii Pro Glasses 3* (100 Hz, infrared, binocular) to capture the sessions through video and audio, while at the same time tracking his eye movements. This allowed the simultaneous capture of his verbal utterances, gestures, actions, and eye movements.

Analyses. For the analyses, data were triangulated: Emir's utterances were transcribed and multimodally annotated, integrating his eye movements, gestures, and actions. All data were integrated and jointly represented in the analyses (see results section for an example). MAXQDA software was used for data organization. For the analyses of Emir's learning process, we used the qualitative content analysis following Mayring (2014), in an inductive manner. After selecting the units of analysis, we paraphrased the content-bearing elements relevant to Emir's learning processes. On this basis, we developed categories, which were continuously revised and then used to identify the chronological sequence of Emir's learning process across the support sessions.

Contents of the support sessions. In this paper, we focus on two contents addressed during the support sessions with Emir. (1) First, we concentrate on the use of reference points on the NL 0–100. Specifically, we examine how Emir determines the numbers corresponding to the quarter (1/4) and three-quarter (3/4) marks, as well as using these reference points for locating numbers. This content was predominantly addressed during the first two support sessions and taken up in later sessions. (2) Second, we focus on locating numbers corresponding to marked positions on NLs with varying start and endpoints. The emphasis here was on taking the given marks and numbers into account and interpreting distances on the NLs accordingly, which was necessary to determine the numbers correctly. This part of the support was embedded in the card game *Quartets*. The aim was to collect four cards from a series (Fig. 1). Emir worked with NLs with varying start and endpoints over three support sessions.

Figure 1. Example of four matching cards for the number 50



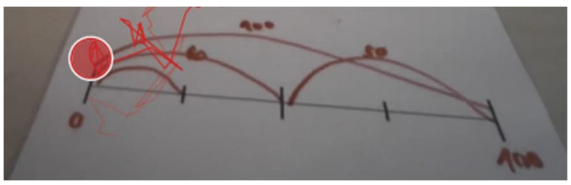
We ask the following research questions: *How does Emir's ability develop to: (1) use the reference points of the one-quarter (25) and three-quarter (75) marks on the NL 0–100, and (2) locate numbers on NLs with varying start and end points, in particular, which challenges and turning points arise?*

RESULTS AND DISCUSSION

In the diagnostic interview prior to the support sessions, Emir showed difficulties in locating numbers on the NL. He often used counting strategies, predominantly from the start point of the NL, and sometimes he even counted in increments of ones on the NL 0–100. He also partially located numbers inaccurately, especially for NLs with varying start and endpoints (e.g., on the NL 60–70, where he needed to locate 67).

Support (1): *Initial approach:* Initially, Emir was asked to name the numbers corresponding to the unlabeled quarter marks (25, 50, 75) on a partially marked NL 0–100. He roughly estimated the values for the quarter marks, assigning 10 to the 1/4 mark and 70 to the 3/4 mark. *Challenge:* This estimation-based approach indicates that he did not yet interpret the quarter marks as equally spaced intervals on the NL. Although he showed numerical knowledge of the midpoint (50) and understood that 100 can be divided into two equal halves of 50 (Fig. 2), he was not able to apply this knowledge to determine the numbers belonging to the 1/4 and 3/4 marks.

Figure 2. Visualization and labeling of the distances from 0 to 100, 0 to 50, 50 to 100 (Screenshot from Emir's eye-tracking video; gaze indicated by red dot)



Support: The support teacher introduced a NL with unlabeled intervals of five and provided Emir with the opportunity to use this NL to determine the number at the 1/4 mark. *Observed change*: Emir determined 25 at the 1/4 mark by counting in increments of five and subsequently integrated 25 into his numerical knowledge (25+25=50): He appeared to recognize—consistent with the concept of (repeated) halving—that equal geometrical distances on the NL correspond to equal numerical intervals (see below).

Teacher: So, we have steps of 5 here (*writes 5 below the corresponding mark*). And what number is here? (*retraces the 1/4 mark with a pen*)

Emir: Um… (*looks at the marks from the start point to the 1/4 mark*) 25.

Teacher: Great. How did you figure that out? (*writes 25 below the 1/4 mark*)

Emir: (*looks back and forth between the start point and the 1/4 mark*) Because … there was a thick mark (*points to and looks at the 1/4 mark*). And thick marks (*looks back and forth between the midpoint and the 1/4 mark*). Um … a thick mark (*looks back and forth between the midpoint and the 1/4 mark*) means that it's, um … (*looks away from the NL*) the middle of the number. (*looks again at the midpoint, the 1/4 mark, the midpoint*) Because 25 plus 25 is 50.

New challenge: To determine the number corresponding to the reference point 3/4 (75), Emir again relied on counting in increments of five from the midpoint to 75. Although he could temporarily use 75 as a reference point (e.g., for locating 80), his understanding was not yet fully developed: at one point, he incorrectly assigned the number 70 to the 3/4 mark previously labeled 75. These difficulties re-emerged at the beginning of the next support session, when he could not name the number at the 3/4 mark—suggesting that the previous activities had not yet led to a sufficient understanding. *Support:* In a subsequent detailed interaction focusing on distances on the NL (100, 50, 25), the numbers 50 and 25 as well as the principle of (repeated) halving were addressed. The concept of equal distances between the marks was made explicit, supported by pointing gestures (Fig. 3, left) to visualize equal geometric and arithmetic intervals from 0 to 25 and from 25 to 50. *Turning point:* Drawing on his knowledge of 25, 50, and the corresponding intervals between 0, 25, and 50, Emir was able to determine the number corresponding to the 3/4 mark, as shown below.

Teacher: From 0 to 25, if I take one large step (*draws an arc with the pen from 0 to 25*), how big does that step have to be?

Emir: Um… (*looks at the beginning, the midpoint, and the 1/4 mark*) plus 25.

Teacher: Great. And from 25 to 50? (*draws an arc with the pen from 25 to 50*)

Emir: (*looks from the 1/4 mark to the midpoint*) 25.

Teacher: Great. Do you know which number should go here? (*points to the 3/4 mark*)

Emir: (*looks from the midpoint to the 3/4 mark*) 75. (*writes 75 at the 3/4 mark*)

From this point on, Emir was able to correctly identify the number 75 at 3/4 throughout all support sessions and to use it confidently and reliably as a reference point. He was also able to explain his approach verbally and through actions (Fig. 3, right).

Figure 3, left: Supporting gestures to visualize geometric distances; right: Emir's representation of 75 by arranging intervals from 0 to 50 and 50 to 75

Support (2): *Initial approach:* For NLs 0–100, Emir was able to draw on his existing knowledge and to locate numbers correctly without difficulty. *Challenge:* However, he was initially unable to transfer his knowledge to NLs with varying start and endpoints. For the NL labeled 0 at the start and 40 at the midpoint (see Fig. 4, left), he misinterpreted the scale and assigned 40 to the endpoint (red cross). In an alternative attempt to solve the task, he drew on his familiarity with the NL 0–100 and labeled the endpoint as 100. This indicates that Emir was initially not yet able to transfer his understanding of equal distances and corresponding numerical intervals from the NL 0–100 to NLs with different start and endpoints.

Figure 4. NLs used in the support sessions

Support: As a wide variety of NLs were used, it was not possible for Emir to rely on memorized reference points without sufficient understanding. Instead, Emir needed to attend to the given marks and numbers on each NL and interpret the spatial intervals accordingly to locate numbers accurately. Across varying NLs, Emir's understanding of NLs was repeatedly scaffolded through explanations, pointing gestures, and visualizations of equal distances and corresponding equal numerical intervals (similar to Fig. 3). *Observed change (with intermittent setbacks):* Through extensive practice in interpreting distances, Emir was able to correctly solve first tasks independently. However, *new challenges* emerged: He sometimes relied on a single reference point for orientation, without taking other provided marks and numbers into account, and located numbers by moving in increments of ten from that point. For example, on the NL labeled with 0 at the start and with 25 at the midpoint (see Fig. 4, middle), Emir assigned the number 35 to the endpoint. *Support*: Extensive practice and explanations of equal distances were continued, as shown in an example in the excerpt below.

Teacher: How long is this distance? (*points to the distance between 0 and 25*)

Emir: Um, there (*looks from 0 to 25*), 25.

Teacher: Exactly. Then this one here (*points to the distance between 25 and X*) must also be 25.

Emir: Ah. (*points and looks at the cross*) That's why it's 50.

Further challenges: Emir occasionally tried to draw on his existing knowledge of numbers on the NL 0–100 and interpreted distances as if 0 and 100 were the start and endpoints. In addition, he showed occasional insecurities in his number knowledge, for example, when halving odd tens, and determining distances between numbers ending in five. Another difficulty that occurred was that while Emir was able to determine the correct distances, he sometimes assumed that the distance he had determined corresponded to the number at the red cross. This was observed only with NLs

containing “difficult” numbers, suggesting that interpreting these NLs placed high demands on Emir’s working memory that may have led to these difficulties. *Turning point / ongoing improvement:* Rather than a single event, Emir’s turning point was a gradual stabilization: The difficulties decreased as Emir practiced with an increasing variety of NLs, engaged with discussed intervals and numerical interpretations. Over time, Emir increasingly succeeded in interpreting distances and numerical intervals to correctly locate numbers. He was also increasingly able to explain distances and numerical intervals on NLs with varying start and endpoints, as shown below for the NL 40–80 (see Fig. 4, right).

Emir: 50.

Teacher: Exactly. Can you explain, why it is 50?

Emir: Because the total distance is 40, and half of that is 20. And this (*points and looks at the cross*) is 10 (*looks back and forth between the start point and the cross*; *points from the start point to the cross and back*). And then, 50.

SUMMARY AND OUTLOOK

The aim of this case study was to investigate Emir’s learning process (1) in using important reference points of the quarter marks on the NL 0–100 and (2) in locating numbers on NLs with varying start and endpoints. The findings highlight that learning processes like Emir’s—of coming to use the NL and its reference points—do not necessarily progress linearly but may be volatile, involving recurring challenges and intermittent setbacks. Across both the NL 0–100 and NLs with varying start and endpoints, a key challenge for Emir was developing a relational interpretation of numbers on the NL. A pivotal turning point lay in interpreting the NL 0–100 in the sense of the measurement model (Gaidoschik, 2024) to use the reference point 75 (3/4 mark). The measurement interpretation enabled Emir to consider the positions of numbers in relation to one another and to understand that equal geometrical distances on the NL correspond to equal numerical intervals. This interpretation continued to support his learning on NLs with varying start and endpoints, albeit more gradually, contributing to an ongoing improvement. It allowed Emir to determine relative positions of numbers more systematically, despite the additional challenges posed by the varying numbers and marks.

The detailed insights from this case study showcase that a *key challenge* in students’ coming to understand and use the NL can lie in issues with a relational understanding of the NL, where introducing the measurement model may provide a *pivotal turning point*. Although a first positive impact of the measurement interpretation may be visible, the learning process appears to be a gradual, stepwise process—shaped both by the characteristics of the NL and by the learner’s interpretation of the NL.

Methodologically, our analyses underscore the importance of data triangulation for examining learning processes. For instance, we were able to infer with confidence that Emir was using the 1/4 mark on the NL as a reference point, as evidenced by his

verbalization of “25,” his gesture toward the 1/4 mark, and his visual fixation on that location as indicated by eye-tracking data. Eye tracking, which to our knowledge has not previously been used to investigate learning processes involving the NL, proved especially valuable in detailing Emir’s difficulties in understanding numerical relationships on the NL: When working with NLs with varying start and endpoints, Emir often did not attend to all relevant marks but attended to single reference points, which in turn led to wrong interpretations. Eye tracking revealed which parts of the NL Emir attended to and which received little or no visual attention, thereby providing fine-grained insight into his relational understanding.

Further analyses with larger samples of students could help identify additional challenges and critical turning points and contribute to the characterization of learning processes. Such insights can inform educational practice by enabling anticipation of potential difficulties and by guiding the design of more targeted and effective support.

References

Fitzsimmons, C. J., Morehead, K., Thompson, C. A., Buerke, M., & Dunlosky, J. (2023). Can feedback, correct, and incorrect worked examples improve numerical magnitude estimation precision?. *The Journal of Experimental Education, 91*(1), 20–45.

Fuson, K. (1984). More complexities in subtraction. *Journal for Research in Mathematics Education, 15*(3), 214–225.

Gaidoschik, M. (2024). Number line estimation tasks: Ein fachdidaktischer Blick auf einen in den kognitiven Neurowissenschaften prominenten Aufgabentyp. *JMD, 45*, 1.

Kucian, K., Grond, U., Rotzer, S., Henzi, B., Schönmann, C., Plangger, F., Gälli, M., Martin, E., & von Aster, M. (2011). Mental number line training in children with developmental dyscalculia. *NeuroImage, 57*(3), 782–795.

Mayring, P. (2014). *Qualitative content analysis: Theoretical foundation, basic procedures and software solution*. Beltz.

Peeters, D., Verschaffel, L., & Luwel, K. (2017). Benchmark-based strategies in whole number line estimation. *British Journal of Psychology, 108*, 668–686.

Scherer, P. & Moser Opitz, E. (2010). *Förderung mathematischer Kompetenzen bei Kindern mit Lernschwierigkeiten*. Kohlhammer.

Simon, A. L., Rott, B., & Schindler, M. (2025). Number line strategies of students with mathematical learning difficulties and students with general learning difficulties: Findings through eye tracking. *Education Sciences, 15*(11), 1461.

Teppo, A., & van den Heuvel-Panhuizen, M. (2014). Visual representations as objects of analysis: The number line as an example. *ZDM Mathematics Education, 46*, 45–58.

van’t Noordende, J. E., van Hoogmoed, A. H., Schot, W. D., & Kroesbergen, E. H. (2016). Number line estimation strategies in children with mathematical learning difficulties measured by eye tracking. *Psychological Research, 80*, 368–378.