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PREFACE

The International Group for the Psychology of Mathematics Education (IGPME) was founded in 1976 at the 3rd International Congress for Mathematics Education in Karlsruhe.

The IGPME organised their first conference in 1977 in the IOWO in Utrecht under the chairmanship of Prof. Freudenthal. The Second International Conference for the Psychology of Mathematics Education took place from 4th-9th September 1978 in Haus Ohrbeck on the Invitation of the Fachgruppe Didaktik der Mathematik der Universität Osnabrück. This conference was attended by 70 participants from 11 countries.

The lectures were divided into 5 sections, each under a separate chairmanship:

1. The Acquisition of Arithmetical Concepts
   (Mrs. R. Rees and G. Vergnaud)
2. The Learning of Generalisation and Proof
   (A.W. Bell)
3. Interpersonal Aspects of Classroom Communication
   (H. Bauersfeld)
4. The Nature of Mathematical Thinking
   (A. Vermandel and E. Cohors-Fresenborg)
5. Intuitive and Reflective Processes in Mathematics
   (E. Fischbein and R.R. Skemp)

These five main topics were chosen with the following points in mind:

For a reflection on the psychological foundations of learning mathematics first an analysis of the special features of mathematical thinking is necessary. Without a certain amount of explanation of the relationship between a philosophical basis of mathematics and central psychological categories of thinking and learning, a more intensive occupation with the learning processes of mathematics is hardly possible. It appeared that among the participants the opinions had not been fully thought out and the discussion remained controversial. That such consideration was necessary at all was seen in different lights by the participants (Theme 4).

Theme 2 and 5 were chosen in order to examine the relevance of certain mathematical thought processes for the learning of mathematics. The emphasis on mathematical methods independent of the content makes possible a general discussion of the psychological problems of learning mathematics. The constant interchange between productive and reflective mathematical activity in the solution of problems, mathematising, proving, and generalising should also be made clear in mathematics education; in order to attain a profounder understanding of mathematical processes and not merely technical facility (Theme 2).

Intuition plays an active part in mathematics. This fact - as in general the phenomenon intuition - was until now considered of little account in psychology, as opposed to philosophy where it is often seen as of fundamental importance in mathematical processes. However, not always is its influence positive. As has frequently been shown both in the forming of new mathematical concepts and in learning mathematics, intuitive perceptions can lead along false paths. Therefore, especially in mathematics education the possible effects of intuition must be brought to attention. On the other hand, mathematics is of general interest in psychology for the following reasons:
Mathematics seems to the psychologist as a particularly concentrated example of the functioning of human intelligence, hardly concealed by falsifying effects. Through the study of the psychology of learning mathematics, in the interplay between intuition and reflection, a more profound understanding can be attained in this respect for "intelligent learning" in general; which in turn allows inferences on the nature and function of intelligence itself (Theme 5).

A concern with central concept-building and scientific methods of mathematics for the improvement of mathematics teaching cannot be separated from a consideration of the social structures and interactions in the classroom. In particular in these problems it can be seen that psycho-social phenomena can be, independent of the content of the instruction, causes of difficulty for the schoolchildren in understanding and of obstacles to learning. Psychologists as well as mathematicians appear indeed to be aware of this problem, but as yet only a few papers have been presented. The discussion in the IGPME has not lead to progress during this conference; but by the way of the lectures the need for further research in this field was made clear (Theme 3).

Because of the significance of arithmetical concepts for the learning and understanding of mathematics, the only topic with actual mathematical content was that in Theme 1. Psychological research has taken up the examination of the acquisition of arithmetical concepts with special attention, because this is of exemplary character for learning mathematical concepts. Furthermore, the reason for the occurence of difficulties in understanding "higher mathematics" - and even more in handling it - often lies in precisely those elementary deficits of knowledge of and facility with calculation procedure.
Insufficient arithmetical skill, on the other hand, plays a not insignificant part in the social discussion on poor mathematical ability in schoolchildren, so this may be why quite a number of papers dealt with this problem.

The present volume contains the final versions of the conference contributions. Where a contribution was presented in a particular section, the number of this section is indicated in brackets in the contents list.

Elmar Cohors-Fresenborg
Ipke Wachsmuth

CONTENTS

EXPLORATORY PROBLEMS IN ELEMENTARY SCHOOL MATHEMATICS
A STUDY OF LEARNING AND RETENTION

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I. DISCOVERING AND PRACTICING

1.1 Discovery Learning

Discovery Learning has played a major role in recent years in research and investigation concerning the improvement of mathematics learning. Among the most known propagators of this approach are G. Polya (1957, 1962) and J. Bruner (1960, 1961). Polya, concentrated mainly on discovery as a mode and means of problem solving. He even developed a detailed model for this approach. Bruner expounded the discovery approach as the major means for learning, which, in his opinion, could do away with all the ills of school learning. Their teaching found a major application in a complete mathematics curriculum, based upon the discovery approach, as developed by the University of Illinois at Urbana (U.I.C.S.M).

Other major studies of this approach were summarized by Shulman and Keisler (1966) and later by Nunnaly and Lemond (1973).

In all of these studies (except that of Polya) the major aim of this mode of learning was the discovery of new concepts and the acquisition of new material. The major finding, supported by most studies, was that while on tests, given immediately after learning took place, expository learning might show better results than discovery learning, the outcome is reversed on retention tests. All studies also emphasize the fact that the motivation of the learner is much stronger in cases of discovery learning than in cases of expository learning.

1.2 Why Exploratory Behavior

Daniel Berlyne (1960) was one of the major researchers to investigate factors which influence exploratory behavior. He considered the need of arousal balance as a major reason for higher motivation induced by discovery learning. Berlyne also coined the term *epistemic curiosity*, a term of great potential for any educator, and particularly for the mathematics educator. According to this theory the organism attains the ideal level of arousal at a certain medium level of excitation by a stimulus. If the level is too low the organism might reject the stimulus altogether and search for other, more exciting sources of arousal. If it is too high, the organism would reject the stimulus. In the classroom setting this would mean that a too low level of stimulation would be rejected as too boring, while too high a level would be rejected as frustrating. Graphically we could describe the relation between the level of arousal and the level of excitation generated by stimulus as one of an inverted U.

1.3 Factors Generating Exploratory Behavior

Most studies dealing with exploratory behavior concentrate on visual exploration in which the major measure is the amount of time spent on observation of certain stimuli. In these studies the major stimuli components which influence the level of arousal and generation of exploratory behavior are *novelty* and *complexity*.

Novelty depends in general upon the amount of change in the stimulus from the time of the previous encounter by the organism. This change may be of different natures such as change of periodicity of appearance, of place of usual appearance, of amount of information usually embodied in the stimulus, etc.

Complexity is a much more involved concept, particularly difficult to define and measure in cases of mathematics learning. Researchers of arousal generated by visual stimuli usually define the level of complexity as depending upon the number of different dimensions which can be discerned in the stimulus. In cases of mathematical problems, complexity can perhaps be described by the remoteness of the associations needed for the solution. This remoteness can perhaps be measured by free association of subjects exposed to the problem[*].

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[*] An attempt by the first author to obtain an equal interval scale for complexity of a set of mathematical problems by the method of paired comparison did not produce any reliable outcome.

It is easy to see that in cases of learning by discovery both of these factors, novelty and complexity, operate at a much stronger level than in expository learning.

1.4 Retention of Material Learned by Discovery

As mentioned before many studies show that the highest difference in performance between material learned by expository methods and material learned by discovery is observed on retention tests. Two different opinions were forwarded as reasons for this phenomenon. Gagné (1966) is of the opinion that in the case of discovery learning the learner has to sort out and organize on his own the various concepts that he encounters in the material, and to embed these in his frame of reference. In this way he eliminates the interference of other concepts. Kendler, on the other hand, sees the advantage of the discoverer in the fact that he organizes the material in his own language, whereas when learning by exposition it often happens that the material is presented in a language unsuited to the particular learner.

Bruner (1961) essentially supports the stand taken by Gagné. He considers the major problem to be the problem of retrieval, and retrieval is facilitated in cases when the learner plays a major role in embedding the material in his own frame of reference.

1.5 Process versus Product

As noted above, previous studies investigated the impact of the discovery approach upon the learning of new material. In this application of the discovery approach one can argue that the theoretical approach, as outlined before, doesn't necessarily imply that the discovery approach will always be superior to the expository one. The reason is that in the learning of new material the factors of novelty and complexity operate, independently of the mode used to expose the student to the new material. One can even argue that essentially *all* learning is discovery learning, as while listening to a "good" exposition the learner is continuously "discovering" what will be "coming next". On the other hand, one can argue that it is quite possible that "learning" in a discovery approach might operate in a rote non - meaningful mode. If this is so, then the amount of learning and retention of new material might not be a function of the mode of expositions, only of the quality of this exposition. This attitude has been taken by the greatest opponent of discovery learning, David Ausubel (1963). It has also been shown by the first author (Finegold-Avital, 1975) that the rote- discovery mode can be embedded in a general model of discovery learning. The authors of this article do not espouse this approach, for the simple reason, that everything we know about mathematics and human learning tells us the chances are much higher that a discovery approach will be meaningful to the learner, than that an expository approach will be "good". This has to be so because the amount of involvement of the individual organism is - by definition - much higher in the first type of learning than in the second, and we know that involvement is the opening to motivation, and motivation is the lubricant of learning.

But what concerns us here is a better analysis of what one can expect a student to learn from a discovery approach. We shall limit our discussion to the learning of mathematics.

In almost all studies of discovery learning the emphasis is on the final product. The learner is supposed to discover something new, a concept, a theorem, a proof, and know it ever after. In some cases (Woerthen 1966), the researchers also tried to find out whether the student learned something about the process of discovery itself - that is whether he learned from the attempts to discover. The authors do not know even one study in which an attempt is made to find out what a student can possibly learn from the exploratory activity itself - that is from the fact that while he searches for a solution to a problem, or for a generalization and creation of a concept from a set of examples, he turns over in his mind a large number of mathematical facts. If the problem for exploration is chosen in such a way that while the student is trying to solve it he has to solve a large number of algorithmic exercises of a certain nature, these solutions can serve as drill and practice for the development of the skill in using the given algorithm.

1.6 Drill and Practice

The usual procedure used in schools to practice the application of an algorithm is to ask the learner to solve many isolated exercises, each of which is an exemplar of the given algorithm. Each of these exercises is an isolated bit of information, and the only property that relates them to one another is the fact that they are exemplars of the same algorithm. This makes this type of practice extremely boring, and the student is ready to follow suit and perform only because of the routine acceptance of school discipline. Our theoretical analysis of arousal due to epistemic curiosity assured as that drill and practice carried out, not for their own sake, but as a means of gathering information for the solutions of an exploratory problem, will increase motivation, and in this way improve the understanding and usage of the said algorithm. This hypothesis is the main subject of this study. The authors think that most human activities are goal directed, and the drive to satisfy a certain need is a strong motivating factor. In this case epistemic curiosity, the desire to see what is behind the set of examples that he is generating, will be the driving force, and the practice will be a side effect. In our statistical analysis we shall compare the improvement in the ability to carry out arithmetical computations, of children who did their practice in an exploratory setting, with the improvement, in the same area, of children who did the same type of practice in the usual classroom way - "teacher assigns - I do".

1.7 Formulating Hypotheses and Reinforcement

Even if, for the teacher, the objective of the exploration might be practice in the application of an important algorithm, the objective for the student will be the search for some generalization which will always take the form of an hypothesis. Reinforcement theory teaches us that to maintain a goal directed activity, some success in the achievement of the goal must be ascertained. In the case of this study such a success would generally be the formulation of some tenable hypothesis. As elementary school classrooms are usually heterogeneous, it is desirable that the exploratory problems should be of such a nature that they permit a sequence of partial hypotheses, so that every child's activity can be reinforced. This is not always either easy, or perhaps even possible, to achieve.

However, the authors think that in most cases the mere collection of data will serve as reinforcement for many children. We can base this assumption on the following considerations. We know that every living organism, that has achieved a certain level of development, tends to explore his environment. We assume that for the human being at school age, intellectual activity is part of his environment, so that exploration thereof is by itself a reinforcing activity. As with all secondary drives, these will usually operate if the organism has not been deprived of some of its basic needs to satisfy primary drives. The authors base their assumption on the often observed fact that most people do become involved in an arithmetical puzzle, and if they have enough leisure time on their hands, and if the puzzle is of such a nature that they think they can contribute to its solution, will devote more time to solving of it. The first author has observed this fact, time and again, while lecturing to soldiers in the field in the various wars he has lived through in Israel. In this case every piece of additional data, which fits the problem, serves as a reinforcement to encourage further activity. Obviously the formulation of a valid generalization would be considered a better accomplishment, but many children will continue with the activity for the sake of gathering more data. This will be particularly true if the data themselves can be considered by the searcher to be of an hierarchical level of complexity, as for instance larger integers, fractions, or more representations of the same number, etc.

1.8 Nature of Exploratory Problems

The analysis, as given until now, shows that if appropriate problems can be found, these can be used to generate exploratory behavior, in the form of collection of data, to serve as a basis for the formulation of hypotheses. While the student is collecting these data he is practicing the application of a useful algorithm. This practice can serve to improve the application of the given algorithm. We expect that the exploratory nature of the problem will serve as an incentive to the student, so that the practice of application of the algorithm will be carried out with higher motivation, and therefore produce better learning, than the usual type of exercises used in school.

We can now formulate a series of requirements which must be satisfied by a mathematical problem, so that it can be considered to be of an exploratory nature to be used as such an incentive to the student. We shall list here these requirements and hope that the reader will see for himself that they are a direct outcome of the mode of suggested usage of these problems in the classroom.

1. The problem must be ameneable to an inductive investigation - that is the student should be immediately convinced that the collection of data is the means to help him reach a solution, or an hypothesis.
2. The solution of the problem must appeal to the student as a goal for the attainment of which it is worthwhile to strive.
3. The collection of data itself can be of various levels of difficulty, so as to give the student a feeling of some accomplishment through the accumulation of more and more data.
4. Subgoals of gradually increasing difficulty can be formulated so that every child can contribute and obtain reinforcement at his level of ability.
5. While investigating the problem through the collection of data the student is practicing an important skill.
6. Some subgoals can be attained in a short time.

We may add one more requirement to ensure that the exploration will deal with a true problem and not with an isolated puzzle.

7. The problem can be expanded to generate new goals.

The analysis, as carried out in the previous sections, also provides us with an understanding of the psychological nature of the factors which will operate when the student is investigating such an exploratory problem. In the following we summarize this analysis in the form of four tenets.

1. The novelty and complexity of the problem will arouse the epistemic

curiosity of the student and increase motivation.

2. The existence of a goal, over and above the immediate task, will produce goal directed behavior.

3. Reinforcement due to the attainment of some subgoals, or due to success with the collection of more complex data, will insure persistence and tenacity on the task.

4. The investigatory nature of the practice will produce better retention than usually achieved with regular practice.

1.9 The Teaching of Exploration

As the usual type of school work is not geared for exploration, one cannot expect the student to become an efficient explorer unless the teacher provides some educational hints about ways of good exploration. Such hints were extensively developed by George Polya (1958, 1963). We shall list them here in the form of instructions, as might be given by a teacher, without any additional elaboration.

1. Investigate inductively by collecting examples which are directly related to the task of your investigation.

2. Vary your examples in some regular way, but here and there, try some of the out-of-way examples - nobody else might have thought of.

3. Look at your examples and search for a pattern. Formulate a conjecture about the pattern. Check that all your examples fit the pattern and the conjecture.

4. If all your data fit the pattern and satisfy your conjecture, check with more data - perhaps of a more unusual nature.

5. Check particularly some special cases such as borderline examples, etc. Do these fit the pattern?

6. Try to discover patterns which fit some subsets of the data, as for instance even (or odd) numbers, points at the vertices, etc.

7. If needed revise your conjecture and check again.

8. Any possible pitfalls? Are your data of a special nature? Did you overlook some exceptional cases for which your conjecture couldn't hold?

9. Can you give intuitive reasons for the general truth of your conjectures? Can you provide a valid proof?

10. Can you expand the problem to generate new questions and open up new roads for investigation?

1.10 Educational Goals Besides Practice

We have concentrated until now mainly on the practice effect and on the reinforcement the learner can obtain from the formulation of valid conjectures. We shall elaborate now, even if shortly, on other desirable educational outcomes one might expect from students' involvement with exploratory problems. We shall discuss only one such outcome, namely the ability to carry out an orderly systematic search of data.

In many exploratory problems the task requires the systematic accounting for a sequence of examples of special cases, or even of all examples that fit a certain rule. Tasks of this nature are rarely practiced in school. Our observations have shown that without special education, students, even in higher grades, after a few systematic steps, tend to abandon the system and skip from case to case without any apparent understanding of the needs for an orderly search of appropriate data. One can hope that an extensive use of exploratory problems in school will educate the student to develop some organized way of systematic exploration, particularly in cases when such an organization might be helpful.

1.11 Proving Conjectures

It is cunderstood that discovering the existence of a pattern in a finite set of data is far from being a general proof of the validity of a conjecture formulated about this pattern. We expect the teacher to emphasize again and again, that the discovered pattern and the related conjecture still need a proof of their general validity. Nevertheless, we consider it expedient and desirable to expose children to exploratory problems even if we know for sure that they will have no means to prove the conjectures they might formulate. We believe that the seeds sown at an early age have a good chance to blossom in later years. Obviously, whenever a proof, even of an heuristic nature, might be at the level of the student, the teacher should encourage the search for its discovery. Nevertheless, no harm will be done if the student who has formulated a valid conjecture is told that only in his later studies will he acquire means which will help him investigate the general validity of his conjectures.

II APPLYING EXPLORATORY PROBLEMS IN ELEMENTARY SCHOOL GRADES

2.1 Theory and Experiment

The theoretical analysis, as carried out in the previous chapter, leads us to believe that practice in arithmetical computetion, obtained as a side effect when the student tries to investigate an exploratory problem, should produce better learning and retention, than the same practice, carried out by the student as a result of an assignment given by the teacher. However, educational experience tells us that theoretical predictions do not always realize in the classroom setting. The number of factors influencing learning and retention is so large that it is impossible to take all of them in account, and outcomes are often very different from what can be predicted on the basis of a theoretical analysis. A major disturbing factor in our case might be age. We don't know to what extent children in elementary school can be attracted by an exploratory problem; or at what age can children see through a pattern and attempt to formulate a conjecture; or whether novelty and complexity in an intellectual task - such as mathematics - can arouse elementary school children and sustain their interest for a lengthy exploration of a problem and collection of numerical data. One might also speculate that young children, with no previous experience in individual exploration, might prefer well defined, teacher assigned, exercises and learn more from such prescribed exercises than from individual exploration. All these questions suggest that an experiment has to be carried out in a field setting to compare the results of practice through exploration with practice through teacher assigned exercises.

2.2 Educational Problems Investigated in this Study

The following questions were formulated for investigation in a controlled experiment in a field setting:

1. In what grades, if any, will elementary school children be attracted by an exploratory problem and sustain interest to collect numerical data and search for a pattern?
2. Will practice through exploration, as carried out by elementary school children, produce better performance on numerical tasks than teacher assigned exercises?
   (i) On tasks given immediately after instruction (investigation) took place?
   (ii) On retention tasks given a few weeks after instruction (investigation) was over?
3. Will elementary school children prefer practice in the form of exploration to the usual classroom mode of teacher assigned exercises?

Another question might concern the optimal age of the elementary school learner during which one should introduce exploratory problems.

2.3 Choosing the Sample

It is very difficult to carry out a well controlled experiment with a randomized sample in an educational setting. The nature of the school prescribes that such a study should be carried out in whole undivided classes. As this is a first study of this nature, it was decided to limit the sample to two classes, one experimental and one control, in each of four grades first, third, fourth and fifth.

To limit as much as possible the influence of the Hawthorn effect it was decided that:

1. The same teacher should teach both classes, experimental and control in all grades.
2. As the regular classes are usually taught by different teachers, it was decided that the second author should take over and substitute for the regular teachers in all classes and all grades.

3. The control classes should solve exactly the same examples as generated by the experimental ones during their investigation.
4. As the hypotheses in this study concern practice, and not original learnings, it was decided to choose topics the children have learned with their regular classroom teacher, and whenever needed apply a pretest which should provide a baseline to measure the differential amount of improvement due to practice.
5. Each grade should explore two problems.

2.4 Choosing the Exploratory Problems

We have to choose the problems so that these satisfy the requirements as listed before. In the following we describe all problems used in this experiment. The terminology here is not the one used in the classroom, where, for obvious reasons, we had to use a more detailed description adapted to the level of the learners.

Grade 1. Problem 1:

The concept of a mathematical tree, with n nodes and $n-1$ branches was introduced. The students were shown how to write the natural numbers 1 to n at the n nodes of a given tree, one number at each node, and then to assign to each branch the difference between the two numbers assigned to its nodes. The exploratory problem was to find an arrangement of the n integers at the n nodes so that all integers from 1 to $n-1$ should appear at the respective branches.

Grade 1. Problem 2:

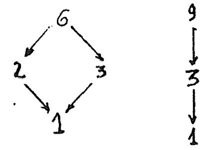
Students were asked to investigate whether the number of possibilities to write a given positive integer n as a sum of k addends (without paying attention to order) is always the same as the number of possibilities of writing the same number n as a sum of addends so that the number k will be the largest number in each sum.

Purpose: Practice of addition and subtraction of integers.

Grades 3 and 4. Problem 1:

The students were exposed to division lattices of various

integers. For instance



Students were asked to separate positive integers into disjoint classes according to the structure of their division lattice.

Grades 3 and 4. Problem 2:

The students were shown a routine in which one assigns to al given integer the sum of 1 and its prime factors, each factor appearing in the sum as many times as it appears as a factor in the integer.
One does then the same to the sum so obtained. For instance starting with 18 one gets: 18→ 1+ 2 + 3 + 3 = 9→ 1+ 3 + 3 = 7→ 1+ 7 = 8→ 1 + 2 + 2 + 2 = 7.
The students were asked to investigate whether each number greater than 6 leads to the loop 7→ 8.

Purpose: Practice in factorization of integers and in location of their divisors.

Grade 5. Problem 1:

Students were introduced to Farey sequences, in which the proper positive factions, in reduced form, are arranged in rows, in order of magnitude, so that the denominators of all fractions in the n-th row are not greater than n. The students were asked to produce sequences up to n-10 and discover interesting properties of this arrangement. Each row begins with 0/1 and ends with 1/1.

Grade 5. Problem 2:

The students were asked to discover shortcuts to the computation of sums of factions:

$1/(1\times2) + 1/(2\times3)+\ldots+1/(9\times10)$; $1(1\times3) + 1/(3\times5)+\ldots+1/(17\times19)$ etc.

Purpose: Practice in the ordering of factions according to size and in the addition of fractions.

One can easily see that all problems satisfy the basic requirements formulated in Section 1.8. There is also no doubt that each of these problems has the potential to excite the exploratory curiosity of students of appropriate grades. One could assume that the problems will be novel to the students, and the heterogeneity of the various steps of the investigation, together with the large number of items in each step, assured their complexity.

The expandibility of each problem wasn't discussed in either of the classes because of lack of time. However, this property, is also satisfied by all problems. To give only one example for the second problem of grade 1. One could ask for partioning into sums of different integers, odd integers, consecutive integers, a prescribed number of integers etc.

As already mentioned it was decided to collect all examples generated by the experimental group and use these as exercises in the control group. Also, because the regular teacher of grades 3 and 4 ascertained that her students have had no practice, either in the factorization of integers, or in accounting for all divisors of a given integer, no pretest was administered to these classes. For technical reasons the school couldn't arrange for us to administer a retention test to grades 3 and 4.

2.5 Other Experimental Conditions.

The school chosen for study is situated in an area of mixed level of income. No grouping system is used in the placement of students, so that each class contained children from various levels of income, with a certain percentage of the students coming from families considered in Israel as disadvantaged. As, so far as we knew, this was the first study of this nature, we concentrated only on the practice effect and tested only the influence of the exploratory activities upon the computational skill in the algorithms students applied in their investigation. We shall not discuss the nature of the various conjectures generated by the students. These require an in-depth analysis, and, in our opinion, a deeper study over a longer period of time would be more appropriate.

2.6. Results and Interpretation.

Means and standard deviation for all grades involved in the study are given in table 1. As a means for a baseline comparison we added teacher scores for each grade. These scores were given by the regular classroom teacher at the end of the trimester

Table 1. Means and Standard Deviations for Experimental and Control Groups in All Grades

	Grade 1				Grade 3				Grade 4				Grade 5			
	Exper.(28)		Control(30)		Exper.(20)		Control(23)		Exper.(28)		Control(33)		Exper.(27)		Control(30)	
Teacher Scores	Max. 10				Max. 10				Max. 10				Max. 10			
	Mean	St.D.	Mean	St.D.	Mean	St.D.	Mean	St.D.	Mean	St.D.	Mean	St. D.	Mean	St.D.	Mean	St.D.
	6.28	1.56	7.36	1.86	7.50	1.24	7.95	1.75	7.03	1.75	7.39	1.56	7.22	1.50	7.07	1.78
Pretest	Max. 22												Max. 20			
	Mean	St.D.	Mean	St.D.									Mean	St.D.	Mean	St.D.
	19.75	3.56	18.63	4.05									14.71	4.51	11.70	4.61
Posttest Immediate	Max. 22				Max 30				Max. 30				Max. 20			
	Mean	St.D.	Mean	St.D.	Mean	St.D.	Mean	St.D.	Mean	St.D.	Mean	St.D.	Mean	St.D.	Mean	St.D.
	21.75	3.56	20.33	3.82	20.75	5.59	17.28	7.88	24.91	3.58	20.69	6.14	16.35	4.40	13.27	5.25
Posttest Retention	Max. 22												Max. 20			
	Mean	St.D.	Mean	St.D.									Mean	St.D.	Mean	St.D.
	21.35	2.84	19.83	6.13									15.82	5.19	12.83	5.71

preceeding the experiment.

1. In all grades in which a pretest was administered the mean score on the immediate test was higher than the mean score on the pretest, for both groups, experimental and control. This tells us that the students improved their performance in either mode of instruction.

2. Even though the mean scores on the retention test dropped in comparison with the scores on the immediate test, these were always higher than the comparable scores on the pretest. This tells us that the improvement due to instruction in either mode was of a more permanent nature.

3. In all grades 1,3 and 4 the mean teacher score was higher for the control group than for the experimental group. Nevertheless, the mean score was higher for the experimental group than for the control group on the immediate and on the retention test. This tells us that in these grades the experimental approach improved immediate, and retention, learning to a greater extent than the traditional approach.

4. The table shows that in grade 5 the mean teacher score and the mean score on the pretest favored the experimental group as compared with the control group. An I.Q. test developed in Israel (Milta) administered to the experimental and control groups showed a similar advantage of the experimental group, a mean of 117 against 110. Nevertheless, an analysis of covariance, with teacher and pretest scores as covariates showed that even when these scores were statistically equaled out the experimental group still showed some slight advantage over the control group (15.05 against 14.45 on the immediate test and 14.30 against 14.20 on the retention test).

5. As mentioned in the description of the design of the experiment, grades 3 and 4 investigated the same problem, and were tested with the same tests. The table shows a definite advantage for grade 4 in the experimental and in the control group, even though the regular teacher of these classes assures us that neither of these classes has ever practiced, before the experiment, the concepts and skills which were used in the study. It is our opinion that the advantage of grade 4 is due to both biological maturation and educational sophistication.

6. A very interesting outcome of the study can be seen in the comparison of the variances of the experimental and control groups on various tests. These variances are always larger for the control group than for the experimental group. This may support the idea that in an investigatory approach it is easier for the learner to embed the material in his own frame of reference and such an embedment diminishes individual differences.

2.7. Measuring Attitudes.

We have mentioned in the design that the second author took over and taught both the experimental and the control groups, in all grades, and solved, as exercises, the examples generated by the experimental group. This approach should have diminished the Hawthorn effect and limited the differences between the groups to the instructional mode. At the end of the experiment a Likert type questionnaire, measuring attitude and preference for the particular mode of instruction, was administered to both groups in grades 3,4 and 5. The response expressed by the experimental group was definitely more favorable than the one expressed by the control group. We shall give details only about one item. Question 5 asked: Did you speak at home about the exercises (in arithmetic) done at school? The options were (1) Yes, with parents, siblings or friends, (2) Only with parents, (3) Only with friends (4) I did not speak with anybody. Response (1) was selected by 41.0% of the experimental group as against 22.6% of the control group in grade 3; by 42.7% against 21.6% in grade 4 and by 40.6% against 17.6% in grade 5. In our opinion these numbers tell a lot in support of the approach.

2.8 Suggestions for Further Study

The authors consider this study as a preliminary step in an area of investigation of significant importance to the teaching of mathematics in elementary school. The limitations of this study stem mainly from two facts:

(i) The investigations were short termed, a few lessons and two problems in each grade.

(ii) The emphasis was only on practice of algorithms and no attention was paid to the ability of elementary school children to formulate hypotheses.

There is a need for a longterm study which would investigate the possibility to construct a course of study in which in every topic taught at school the drill and practice will be carried out through the introduction of an exploratory problem.

While replicating this approach attention should also be paid to children's capability to formulate hypotheses on the basis of patterns, and to the possibility of educating children to improve their performance in this area.

REFERENCES

Avital, S., Practice Through Inquiry in Mathematic Learning, Orbit, 5 1970.

Berlyne, D.E., Conflict, Arousal and Curiosity, N.Y. McGraw-Hill, 1960.

Bruner, J.S., The Act of Discovery, Harvard Ed., Rev. 31, 1961.

Finegold, M. and Avital, S., Ed. Stud. in Maths.

Gagné, R.M., The Learning Requirement for Enquiry, J.Res. Sc. Teaching 1, 1963.

Nunnaly, J.C. and Lemond, L.C., Exploratory Behavior and Human Development; in H.W. Reese (Ed.) Advances in Child Development and Behavior. (Vol. 8), N.Y.: Academic Press, 1973.

Shulman, L.S. and Keisler, E.R., Learning by Discovery, Chicago, Rand McNally Co., 1966.

Polya, C., How to Solve it: A new Aspect of Mathematical Method, Garden City, N.Y. Doubleday, 1956.

--- Mathematical Discovery, N.Y., John Wiley and Sons, 1962.

USE OF GRAPHS FOR CLASSIFICATION OF STUDENT DEMONSTRATIONS

by N. BALACHEFF*

This paper aims to give a methodological contribution to the study of students demonstrations, especially on the two points ;

- analyse of logical structure of a proof,
- for a set of demonstrations, the classification of proofs relatively to their logical structure.

The major part of student solving process remains unapproachable in his mind. We can analyse only what he gets sensible in that process. So, we choose to start our study from the data which are constitued by students proof explanations.

Our work approach is based on two remarks :

- The language used changes, from one author to another, and for one author from a demonstration to another. On various modes it is a mixture of the natural language and the mathematical one.
- The speech which expresses a demonstration is a list of statements. Generally we cannot keep at each step results which could be used again. So, usually a statement is not consequent of the one (those) which is (are) preceding in the speech. Then, we try to give a standard representation of a demonstration, which shows off its logical structure, and such as we can make comparisons.

This we obtain :

- By translating in first predicate calculus language the mathematical statements used in the demonstration. Two statements are synonymous if they have the same translation.
- By associating to the demonstration a directed bipartite graph, with one class of vertices (called *nodes*) represent the mathematical statements and the other class (called *stars*) representing the inferences which have been made clear in the speech.

The graph of demonstration

First we must discern a special kind of statement ; the hypothesis.

Because a statement can be an hypothesis in a part of the proof and obtained from others statements in an other part,

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For example :

In the demonstration of $f^{-1}(A_\cap B) = f^{-1}(A)_\cap f^{-1}(B)$, the statement $x \in f^{-1}(A_\cap B)$ is an hypothesis in the proof of $f^{-1}(A_\cap B) \subset f^{-1}(A)_\cap f^{-1}(B)$, but is obtained with the last inference in which of $f^{-1}(A)_\cap f^{-1}(B) \subset f^{-1}(A_\cap B)$. So, we do not want to merge these two kinds of statements, and we decide to give a name to each hypothesis. We now consider, for a demonstration, the set of mathematical statements and names of statements.

For a graph of nodes and stars, we shall call *input* an edge oriented from a node to a star, and *output* an edge oriented from a star to a node.

We associate to a demonstration a graph as follows :

The nodes of the graph are the mathematical statements and the names of statements, and the stars represent the inferences which have been made clear in the formulation of the demonstration such as :

A statement, or name of statement, E and a statement F are sespectively contiguous to an input and an output of the same star if and only if in the formulation F is explicitely a direct consequent of a list of statements where appears E.

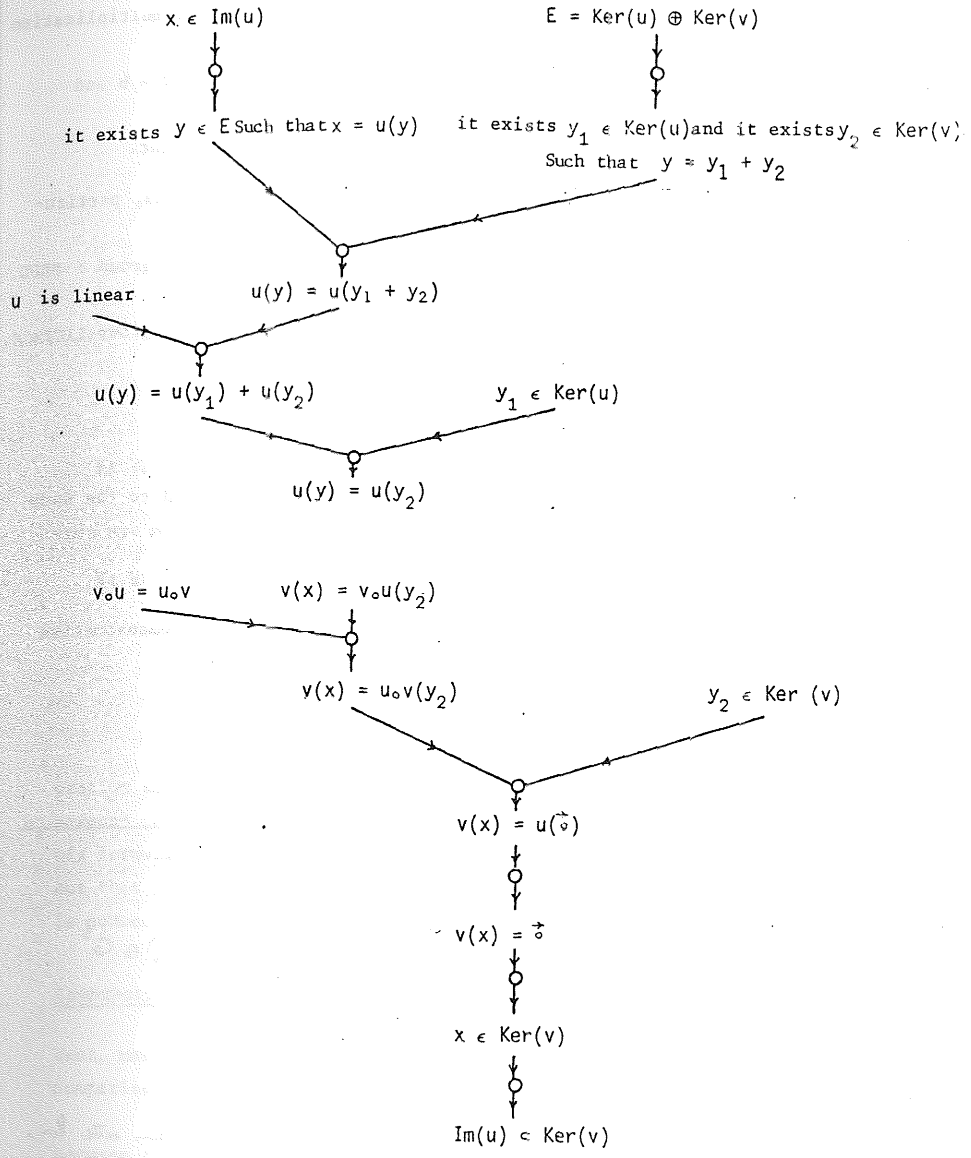
Example :

If **u** *and* **v** *are two endomorphism* $E \rightarrow E$ *such that* $u_\circ v = v_\circ u$ *and* $E = ker\ (v) \oplus ker\ (u)$ *then* $Im\ (u) \subset ker\ (v)$.

Proof :

Let be x any element of Im(u) ; it exists y such that $x = u(y)$. $E = ker(v) \oplus ker\ (u)$ *hence it exists* $y_1 \in ker\ (u)$ *and it exists* $y_2 \in ker\ (v)$ *such as* $y = y_1 + y_2$. *Thus* $u(y) = u(y_1 + y_2)$, *and u is linear hence* $u(y) = u(y_1) + u(y_2)$ *and* $u(y) = u(y_2)$ *because* $y_1 \in ker\ (u)$. *We have* $v(x) = v_\circ u(y_2)$ *then* $v(x) = u_\circ v(y_2)$ *from hypothesis and* $y_2 \in ker(v)$ *hence* $v(x) = u(\vec{o})$ *and* $v(x) = \vec{o}$ *or* $x \in ker(v)$; *and :* $Im(v) \subset ker(v)$.

We obtain the following graph, which shows off the structure of the demonstration.



It is not easy to associate a graph to a demonstration as it is formulated by students. Because in the speech there is not only mathematical statements but also meta-mathematical statements which may be essential to understand the proof. Also, students use many different kinds of process to make clear an inference. We study those problems, and for some of them we propose solutions [1].

To illustrate our approach we use the demonstrations proposed by students of Grenoble University for the next problem :

Let be G a non-void set closed under an associative multiplicati
for which :

For all a $\in$ G and b $\in$ G it exists x $\in$ G such that ax = b and it exists x' $\in$ G such that x'a = b.

Show that G have for this law a right identity element.

It is only an illustration, there is no statistical results, particularly the population is not a representative sample. It is made of :

- 37 students of the Scientific University first year, we name that group : DEU
- 38 third year students who are studying mathematics, we name that group:LICEN

We collect written formulations of the demonstrations.

Structure of the graph and structure of the demonstration

The structures property of a graph are directly connected to the form of the demonstration of which it is the representation. Some of them are characteristic, and others not :

For example :

- There is a directed circuit in the graph if and only if in the demonstration a statement appears among its own antecedents :

Si e, l'élément neutre existant alors a.e = a. $\forall a \in G$

Supposons que e existe, on a alors, Ae = A

Posons A = ax, par hypothèse ax = b. $\forall (a,b,x) \in G^5$.

$\Rightarrow$ (a.x)e = b car e élément neutre.

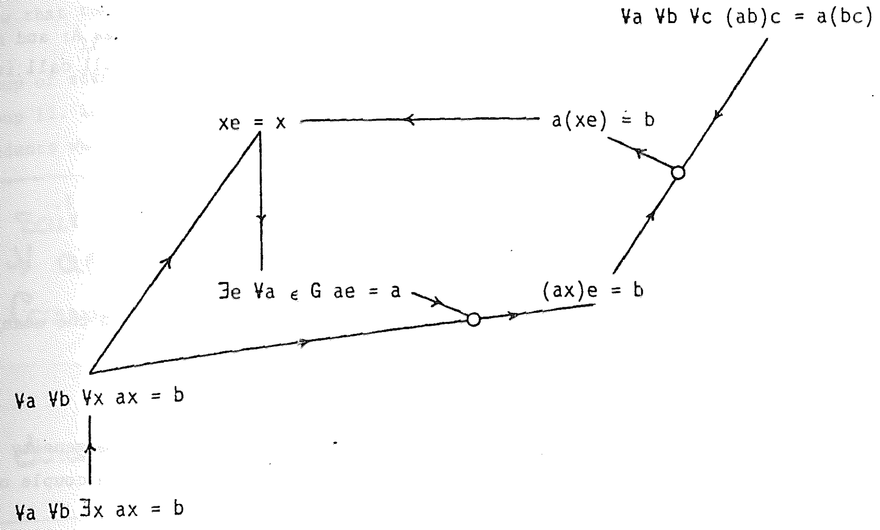
$\Rightarrow$ a(xe) = b car la loi est associative

or on a ax = b d'où x.e = x

donc il existe bien un élément neutre à droite pour cette loi.

(Deug A 9)

We associate to this formulation, the graph :



If the graph is not connected then in the formulation of the demonstration some inferences are not explicit. Either the connected components correspond to some way of solution which the student probes, leaves, but keeps in his formulation. We have an example of such a graph in our previous example. But that property is not characteristic, clearly it could be that the graph is connected and some inferences are not explicit in the formulation.

Comparative analysis of the demonstrations

The representation by a graph, of a demonstration formulated by student, makes clear its logical structure. So, we subsitute for the demonstrations comparison, the comparison of the graphs which represent them.

The demonstrations comparison interest us from the logical relations between the mathematical statements point of view. So, for a graph we consider the set of couple of nodes such as both elements of a couple are respectively contiguous to an input and an output of the same star.

If two demonstrations are identical then the set of couple of nodes of their representations are the same. But if two demonstrations are strangers, the set of couple of nodes of their representations have a void intersection. But other possibilities may occur, and to estimate the proximity of two demonstrations we define an *index resemblance*.

Index ofResemblance

Let be G_1 and G_2 the graphs with set of couple of nodes A_1 and A_2 respectively associated to the demonstrations D_1 and D_2 . We shall call index of resemblance the number noted $R(D_1,D_2)$ and defined by :

$$R(D_1,D_2) = \frac{\text{card } (A_1 \cap A_2)}{\text{card } (A_1 \cup A_2)}$$

That index verifies :

$R(D_1,D_2) = 1$ If and only if D_1 and D_2 are identical

$R(D_1,D_2) = 0$ If and only if D_1 and D_2 are strangers

It is easy to show that R is an index of similarity in the meaning of the statisticians.

Index of homogeneity

For a set of demonstrations we shall call index of homogeneity the arithmetical mean of the index of resemblance associated to each couple of graph of representation.

Demonstration classifying

For a set of demonstrations we can associate to all couple of elements of that set an index of resemblance. We classify the demonstrations by constituting classes having the best homogeneity. Hierarchical analysis gives us tools to do that . An other way to classify demonstrations consists to locate them in a set of reference, using the index of resemblance.

Now we shall illustrate by an example results which we can obtain by such classifications. We begin by a semantic analysis which allows us to value the classifications relevance.

Semantic classification

For a non-void set G which is closed under a single-valued binary operation noted $*$.

A right-identity e is defined by :

$$\forall x \quad x * e = x$$

In fact for that definition students substitute the matrix :

$$\square * \bigcirc = \square$$

Then we note three sorts of strategy :

I : Fix the content of ○, and show that for any content of □ then the equality is verified.

II : Produce an expression consistent with the matrix.

III : Having produced two expressions consistent with the matrix, show that the contents of ○ are identical.

If the quantifiers are implicitly considered, by ones of the first group of strategy, it is not the case for others ; particulary those of the group II. Inside that group we distinguish two subsets : the subset of the students who propose a demonstration as :

On sait par hypothese que :

$\forall a \in G, b \in G, \exists x \in G / ax = b$

Comme a et b quelconques, on peut prendre a = b

$\Rightarrow ax = a$

$\Rightarrow x$ est element neutre à dte

donc il existe $e \in G$ / e = el. neutre à dte.

(LICENCE 21)

---

and we call it II-A ; other students constitue a subset named II-B.

For the group III also, we can distinguish two subsets :

- Those who demonstrate : if $a * e = a$ and $a * e' = a$ then $e = e'$.
- Those who demonstrate : if $a * e = a$ and $b * e' = b$ then $e = e'$.

We name respectively those subsets : III-A, III-B. Note that only two demonstrations of LICENCE, and one of DEUG, are not classified, because calculations are not enough advanced.

We give in the following table the demonstration distribution among the differents categories :

	I	A	B	II	A	B	III
DEUG	9	11	10	21	3	2	5
LICENCE	15	16	3	19	1	1	2

Hierarchical classification ofthe demonstrations

For each population we find a very low index of homogeneity :

DEUG : 0,076

LICENCE : 0,165

That allows us to assert directly, a very large diversity of the demonstrations, particularly for DEUG. To illustrate that point, we give the index of resemblance table of the demonstrations which we have classified as strategy I.

DEUG

	1	13	14	16	22	24
13	0					
14	0	0				
16	0	0	0			
22	0	0	0	0		
24	0	0	0	0	0	
26	0	0	0	0	0	0
27	0	0	0	0	0.09	0

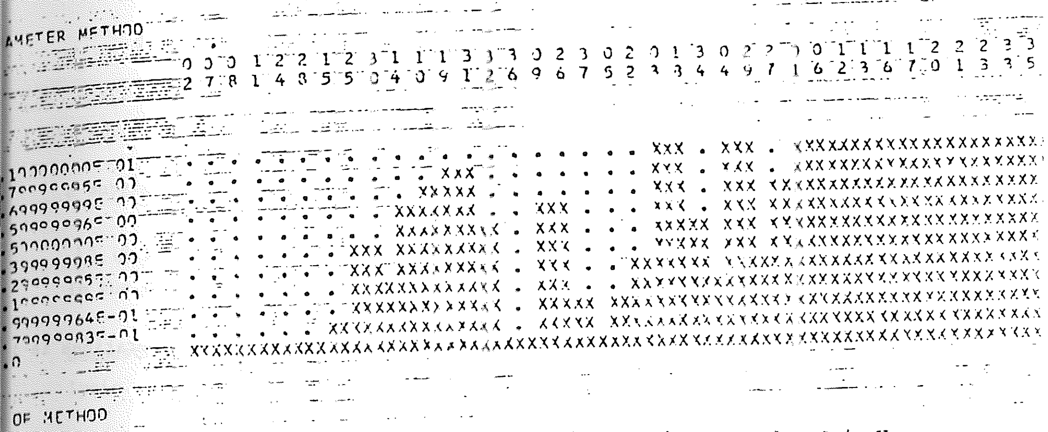
LICENCE

	3	4	5	9	10	11	14	15	19	22	25	30	31
4	0,33												
5	0,28	0,16											
9	0	0	0										
10	0	0	0	0									
11	0	0	0	0	0								
14	0	0	0	0	0,57	0							
15	0	0	0	0	0	0	0,18						
19	0	0	0	0	0,66	0	0,62	0,09					
22	0,33	0,20	0,22	0	0	0	0	0	0				
25	0	0	0,07	0	0,20	0	0,15	0,07	0,27	0			
30	0,25	0,14	0,18	0	0,22	0	0,27	0,18	0,3	0,20	0,36		
31	0	0	0	0	0,80	0	0,71	0,10	0,83	0	0,30	0,33	
	0,22	0,12	0,16	0	0,50	0	0,50	0,07	0,55	0,18	0,20	0,50	0,62
	0,4	0,25	0,25	0	0	0	0	0	0	0,28	0	0,22	0

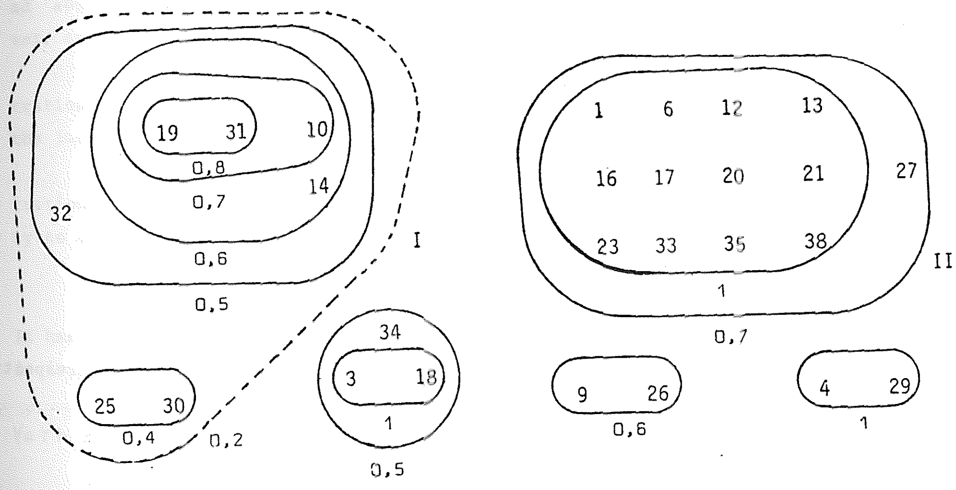
These table shows that for DEUG, we can consider that each student had proposed an original demonstration. For LICENCE there is yet a large diversity, but some demonstrations can be regrouped in classes not reduced to one element. For example the set $\{10, 14, 19, 31\}$ have an index of homogenity of 0,69.

Classification for LICENCE

We use a programm of hierarchical analysis to form the classes with the best homogeneity [2] . It gives us the following result :

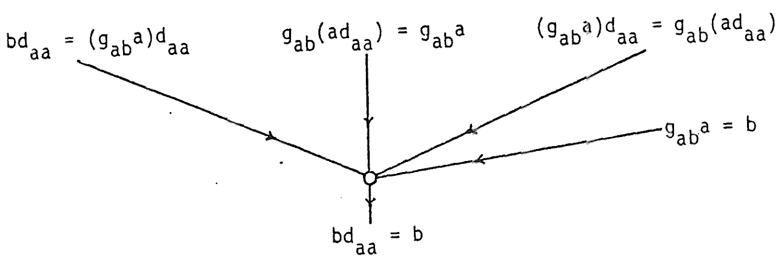


We keep classes which have for homogeneity more than 0,4. We represent them in the following diagramm :

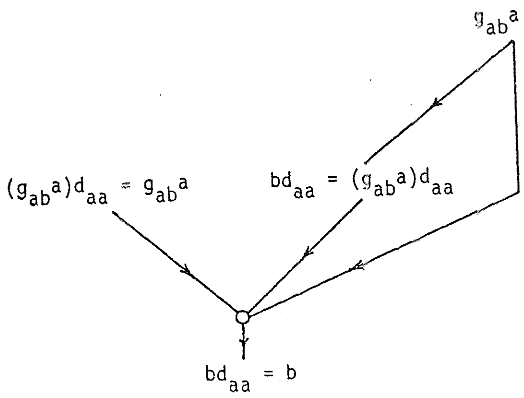


There are two important classes. One we note II on the diagramm and which regroups demonstrations of the class II-A ; and one we note I, which regroups demonstrations of the class I.If we state a more detailed analysis, for those groups we find a common kernel of inferences.

For the class I, the kernel is the subgraph(*) :

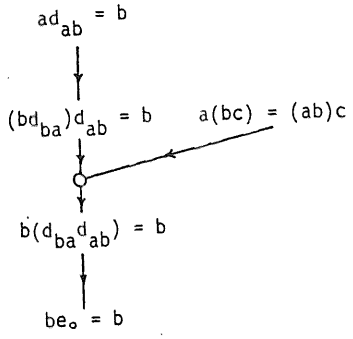


The analyse of $\{25,30\}$ shows that the kernel of their representations i



(*) To remove the quantifiers we use functions of Skolem, so g and d are respectively the functions associated to the predicates $\forall a \forall b \; \exists x \;\; xa = b$, $\forall a \forall b \; \exists x \;\; ax = b$.

We note that the node $(g_{ab}a)d_{aa} = g_{ab}a$ can be obtained from the nodes : $(g_{ab}a)d_{aa} = g_{ab}(ad_{aa})$ and $g_{ab}(ad_{aa}) = g_{ab}a$, of the above kernel. The class $\{9,26\}$ has for common kernel the graph :



In fact this graph is the graph associated to LICENCE 9 and it is a subgraph of the graph associated to LICENCE 26 !

The demonstration formulation whose graphs are in the classes $\{3,18,34\}$ and $\{4,29\}$ are only outlines of a demonstration. We classify them in the semantic classification using meta-mathematical statements.

This analyse allows us to classify in a relevant way 22 among the 38 demonstrations. The classes show off are essentially those of the semantic analysis.

Study relatively to a reference set

As a reference set, we have choosen the one of formal demonstration of that algebra theorem. The algorithm used gives demonstrations intrinsically ad absurdum. It consists to associate the negation of the theorem to the data set, and then demonstrate that we obtain an inconsistent system [3] .

But, as a matter of fact, it is possible to extract a direct demonstration from a demonstration which introduces the negation of the theorem at the last step of calculus.

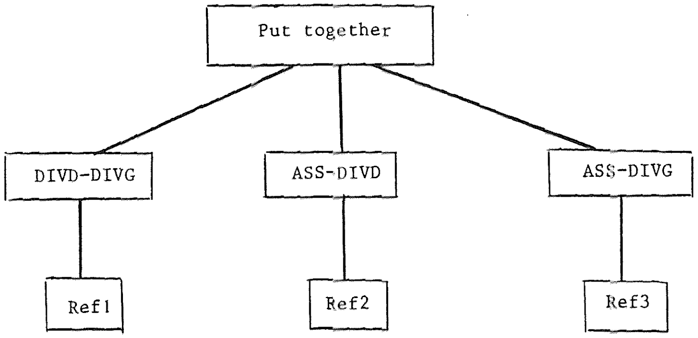
Among the demonstrations formulated by students we do not find demonstrations ad absurdum. The reason is clear, for such demonstrations it is necessary to express the associative law as :

$$\forall x \forall y \forall z \forall u \forall v \forall w \quad (x.y = u \;\&\; y.z = v) \supset (u.z = w \sim x.v = w)$$

There fore we limit the reference set to the direct demonstrations. It has eight elements, which we can regroup in three subsets Ref1, Ref2, Ref3. It is possible to characterize these subsets as follow :

Let be ASS, DIVD, DIVG respectively the associative law, $\forall a \forall b \; \exists x \;\; ax = b$, $\forall a \forall b \; \exists x \;\; xa = b$.

The following diagramm illustrates this classification :

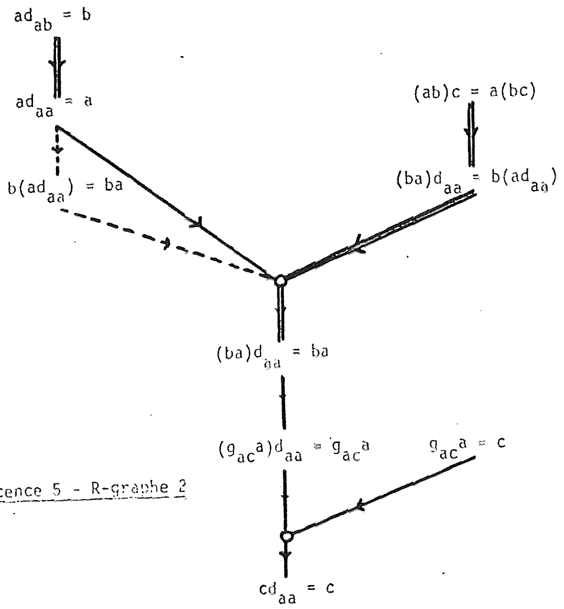


The comparison of DEUG demonstrations with those of the reference s give no results, due to the very low index of resemblance. So the DEUG demonstrations, and the demonstrations of the reference set are strangers.

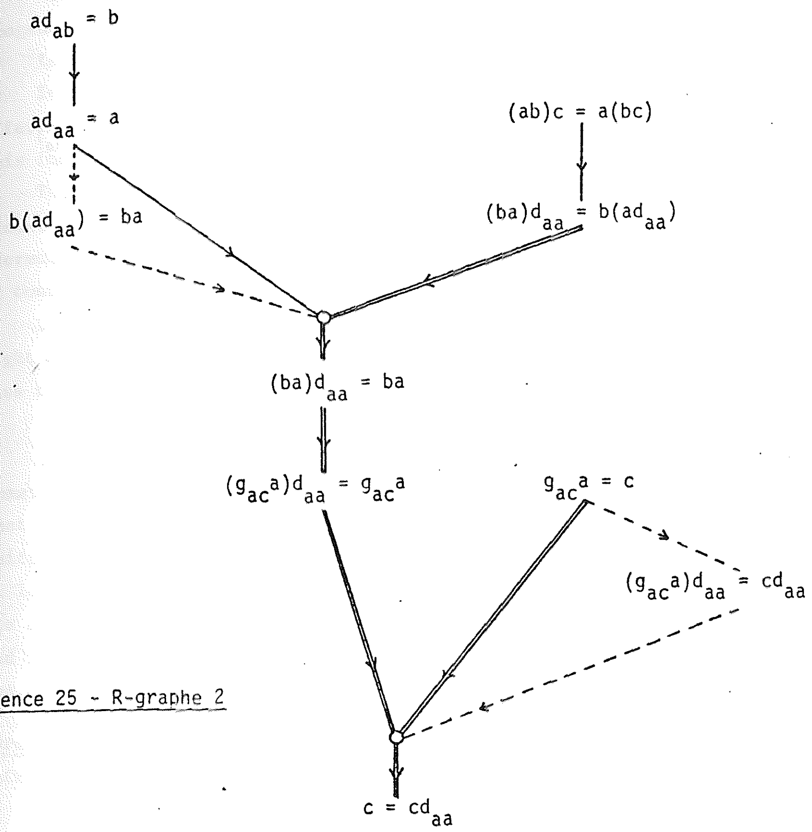
For LICENCE, indexes are still rather low, except for some demonstrations. Those indexes show off a resemblance with the elements of Ref2. To permit a more detailed analysis we represent on the same diagramm, the graph of t student demonstration and the one of the demonstration of reference.

A bow is drawed

- In continuous line, if it appears only in the reference graph.
- In broken line, if it appears only in the student graph.
- In double line, if it appears in both.

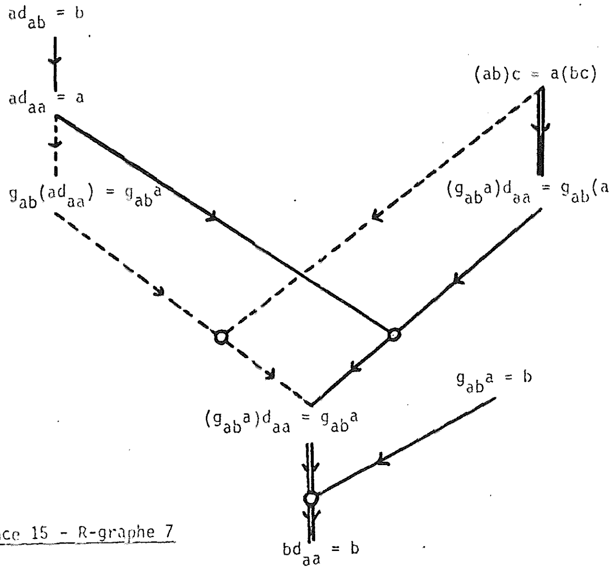


Licence 5 - R-graphe 2

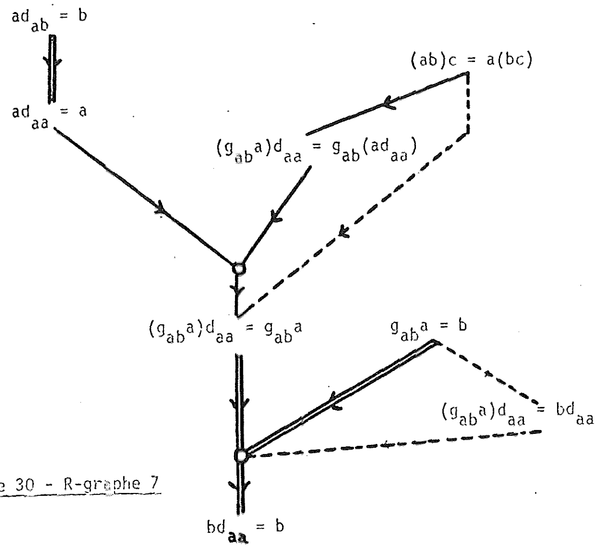


Licence 25 - R-graphe 2

The main difference between the student demonstration and the referen one, is the construction of the statement $b(ad_{aa}) = ba$ from $ad_{aa} = a$ multiplyi on left each member of the equality. Each student explicits $(ba)d_{aa} = b(ad_{aa})$ but does not calculate ad_{aa}.



Licence 15 - R-graphe 7



Licence 30 - R-graphe 7

It seems to be clear in these examples that the students do not consider the associative law as a predicate, but as a syntactical rule on use of the parenthesis in two ways :

- Displacement of parenthesis
- Suppression of parenthesis.

Conclusion

Whith the graph of demonstration we have a tool to give an accurate and homogeneous description of student demonstrations.

In a first way we can use it to study the structure of demonstrations.

In a second way, using an index of resemblance, we can use it to study the differents processes of demonstration. It seems, that in the classification with this index, classes will correspond to a pattern of demonstration.

This classification will be relevant if :

- The index of homogeneity is not lower
- The formulation of the demonstration is not significant only by its meta-mathematical content.

It may be a good tool to study what kind of demonstrations students use, right or not. Is-it possible, for the demonstrations which are false, to characterize the mistakes ? What kind of strategy is associated to right demonstrations ?

In our illustration the reference set is obtained using an algorithm of automatic demonstration. Those algorithms are combinatorial. So, generally, it is not possible to enumerate all the demonstrations. In fact there is a great difference between the deduction rules of the students and these of mathematical logic. Thus it might be interesting to consider as reference set the possible demonstrations as the mathematician would consider it, or those proposed in handbooks. Our approach may allows us to put foreward the demonstrations students remember more easily, or those which show them difficulties.

Bibliography

1. BALACHEFF N. Les graphes de démonstration : outil pour l'étude des démonstrations naturelles.
Thèse de 3ème cycle, Grenoble USMG, 1978.

2. JOHNSON S.C. Hierarchical clustering schemes
Psychometrika . Vol 32 n°3 Sept. 1967.

3. LABORDE J.M. Un développement algébrique de l'algorithme d'exclusion et quelques problèmes géométriques en algèbre de Boole,
Thèse Grenoble USMG 1977.

Andrea Bartal, Sarolta Pálfalvi, János Surányi

The "Work-textbook" for secondary-school mathematics teaching

1. Introduction

A secondary school project going on since 1973 searches for methods by which the students become active participants in the acquisition of mathematics and which turn lessons into more effective opportunities for learning.

Lessons based on the students' independent work require enormous planning by the teacher. A potentially large scope is given to the students' individual work, but as differences will necessarily appear in their progress, we have to plan the lessons on different but parallel levels. Of course, an adequate manual is also necessary for it.

At present, the important learning aids are the textbooks written in a descriptive style, but our experiments show that the students hardly use them: they cannot learn mathematic from their textbooks. We think that one of the main reasons for this is that the textbooks neither direct nor guide the learning process.

In the first stage of the project we prepared worksheets for some parts of the material which were studied by the students individually or in small groups. Accordingly to the experiments done in the first two years /the worksheets were tried out in 10 classes/ nearly the whole teaching process can be built on the students' individual work.

Students are able - though on different levels - to cover a larger amount of material and are able to make individual discoveries.

2. The work-textbook

A so-called "work-textbook" based on those experiments has been prepared. The most important part of it is the Exercise-booklet. As problem-solving ability can only be developed through practice, we have to provide opportunities for students to gain experience which is appropriate for their age. This experience enables them to discover individually the most important parts of knowledge. To this end methods are required which ensure an advance in the cognitive process of any student in his own rhythm. This can only be reached through the student's individual work even during a lesson.

The exercise-booklet gives a possible guidance of the students' individual work. It breaks down theme to be dealt with into a series of problems, the solution of which leads the students to the discovery of the new knowledge. The problem of different student abilities is helped by giving the students tasks, at different levels, for learning one and the same notion.

The complementary illustrations, questions and remarks direct the students' attention to the points of interest of a certain theme. Not everything is worth discovering nor able to be discovered. These ideas are communicated in the exercise-booklet in the appropriate manner. Other texts in the exercise-booklet serve as summaries of the experience of a longer series of tasks, the deepening of the individually discovered ideas, etc.

Sometimes the exercise-booklet compells the student to shape his discoveries: i.e. give definitions on the basis of his experience, formulate his observations, draw conclusions, try to prove them, etc. An empty space is left on the page for students' formulations of answers and on the other part of the page he will find a rectangular shaped frame. The answers will then be discussed in the class until a correct definition arises. This the students write in the rectangle. In this way the material is easy to survey and repeat. The new knowledge is easily distinguished from the questions leading to it.

In the process of discovery learning based on the exercise-booklet, students "fill it up", i.e. they write their answers in the empty spaces, draw the necessary illustrations, formulate definitions, propositions and work through the texts appearing in the exercise-booklet. We may say, with some slight exaggeration, that while they discover the new material they themselves take part in the writing of their own textbook.

3. Other Parts of the Work-textbook

A second part of the work-textbook consists of a series of readings. These are longer texts than those occurring in the exercise-booklet. They help to deepen a theme, give a preview of some field of mathematics connected with the theme under discussion, or present unsolved problems etc. They serve as enrichment material and can be used /if desirable/ again in many different ways.

An essential task of mathematics teaching is to teach the students to read mathematical texts in an efficient manner, which is not easy at all. These readings provide opportunity for this too. Some hints on how to do it can be found in the methodological guide to be described later.

The readings help with the problem of different student abilities, because a part of them require a relatively higher ability for abstraction and a deeper understanding of connections.

A third part of the book is a collection of exercises. This gives exercises suitable to differentiation corresponding to the chain of thoughts of the exercise-booklet.

The small encyclopedia completing the work-textbook for children summs up the important definitions, theses and the most frequently used methods of proving. Examples and illustrations help the understanding. The small encyclopedia is compiled in accordance with the topics in the curriculum and is intended to replace a manual.

The Teacher's Role

The role of the teacher also changes when the teaching is based on the students' individual work. He has to withdraw during the lesson and to guide the students' work indirectly. He has to pay attention to the activity of each group and each student, to observe whether they have understood the exercise or need some further explanation; whether they started by chance, in a completely inappropriate direction; whether they have answered the questions posed, tried to draw appropriate conclusions, etc. If somebody is ready, the teacher has to give him a new exercise.

After the individual work a common discussion of results is essential. The summing up of experiences does not mean the detailed discussion of each question. If for example all students have solved a problem and the difficulties have been cleared up with students individually, it is unnecessary to repeat all these explanations to the whole class. Similarly, when groups are working on different problems the teacher has to decide which of them is of common interest.

The common work, the discussion of experiences and solutions means that the students draw conclusions by debating with each other and correct their former conclusions. In order to have a real debate among students the teacher has to guide it from the background. It is very important that every student draws the conclusion from the debate and understands it clearly. The teacher has to ensure that the students correct themselves, write down the final conclusions into the frames.

Nevertheless the exercise-booklet is not intended to tie the teacher's hand, it attempt to help his work. He can use it unchanged or take some parts of it for the individual work of the students, or for work in small groups, or he can even discuss the exercises with the whole class. He can replace some exercises by other ones. He can also consider the booklet as an example and work out other series of exercises according to the needs of the class etc.

4. The Methodological Guide

For the teacher a methodological guide accompanies the work-textbook. It outlines the work-textbook's aim, structure and the possibilities for its use. It mentions the different kinds of students' independent work, the structure of lessons based on the use of the work-textbook, gives ideas for the organisation of the teaching/learning process of the lessons and for the differentiation between students.

After the general remarks referring to teaching methods the methodological guide gives advice for the teaching of different themes dealt with in the work-textbook. It describes how the different themes are connected with each other, where are their places in the structure of the curriculum and the later development of the themes. It indicates which parts have necessarily to be taught and which can be left out if desired. It shows how difficult exercises are worth solving in classes below the average and in classes above it. It describes what types of answers can be expected from students of different abilities.

Heinrich Bauersfeld: Interpersonal aspects of classroom communication.
More elaborated and in German the paper has appeared in:
H.Bauersfeld (ed.): Fallstudien und Analysen zum Mathematik-unterricht. Auswahl Reihe B, Band 95, H.Schroedel Verlag Hannover 1978,
under the title: Kommunikationsmuster im Mathematikunterricht
pp. 158-170

THE LEARNING OF PROCESS ASPECTS OF MATHEMATICS

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INTRODUCTION

This paper will review recent research, curriculum innovation and current thinking related to problem-solving, proof and mathematisation. Problem-solving is the most heavily researched aspect of mathematical activity. It is here that the possibilities of the learning and use of strategy, and the capability to combine productive and reflective/monitoring states of mind have been studied in most detail. The most recent results suggest that the explicit learning of a structure for the problem-solving process and a fuller awareness of the distinction between these two modes of operation, productive and reflective, is successful. It also appears that this is more possible with older and abler students, though the research results are not entirely consistent in this respect.

The discussion of proof will be concerned not simply with the formal presentation of arguments but with the student's own activity of arriving at conviction, of making verification, and of communicating convictions about results to others. Pupils' developing demands for greater degrees of objectivity are recognised, and their increasing recognition of the need to connect a new result with agreed existing knowledge. It will be seen to be important, for these developments, for mathematical activity to take place in a community of peers (e.g. the classroom) and, for at least some of the time, for the results under discussion to be new ones arrived at in the course of the activity and at other times, to be already known and accepted results but which need to be fitted into a previous framework of knowledge.

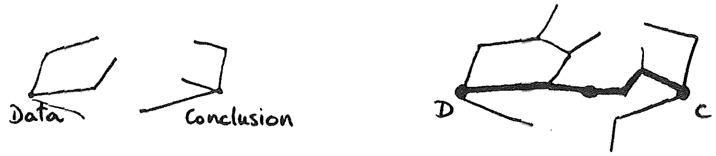
Mathematisation

Freudenthal (1973) describes mathematical activity as the organisation of a field by mathematising, that is by forming spatial or numerical or other relational concepts within it and by structuring it by logical relations. Abstraction, generalisation and representation by symbols and other models all form a part of this activity. Krutetskii has also attempted to characterise mathematical activity. Some curriculum material has been produced embodying these ideas, but little or no formal evaluative work has been done.

2. PROBLEM-SOLVING

Four phases are generally identified in the problem-solving process. These are, first, an input or an assimilation phase, in which the goal and the data are identified; second, an exploration phase involving the production of a number of ideas connected with the data or with the goal and the search for connection points among these; third, a point of illumination, a Eureka experience, when it is felt that a complete chain exists from data to goal, and fourth a verification phase in which the various links and connections are worked through and tested, and the plausibility of the conclusion checked and alternative solutions sought (Poincare, Polanyi 1957). (Recent work by Greeno (1975) has shown that it is possible to program a computer to approximate pupils' problem-solving in geometrical proof problems by providing in the program the capability to perform the main processes described in this outline, that is to recognise patterns or concepts, to make inferences and to alternate production activities with tests of approach towards the goal.)

The essence of problem-solving is the making of some new connection which reaches a desired conclusion from a set of given or already known data. However, the term is usually restricted to cases where there are at least two stages in the connection process, so that some degree of search is required, first for implications of the data and secondly, by working backwards, for pre-implications of the conclusion. The train of connection is completed when some correspondence is found between the first set and the second set of generated alternatives (see Figure 1). (This is the point of illumination.)

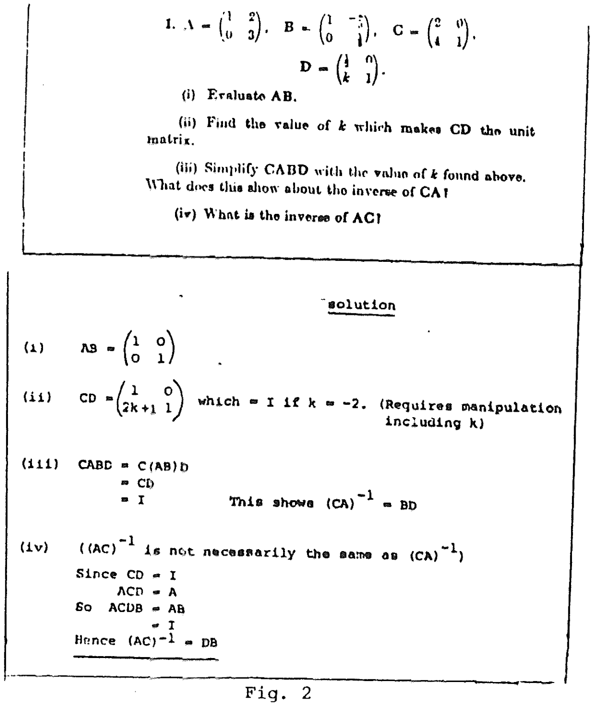


(a) (b)

Fig. 1

Tasks requiring only one step of connection are usually referred to as applications or exercises. The following example (Fig. 2) from an examination paper will illustrate these points (SMP O level, 1969). Here parts (i), (ii) and (iii) each require the recognition, one step at a time, of the relevance of some learned principle and the use of it. In part (iv) the solution demanding the least insight would presumably be to calculate AC from the data given and then to use the standard method of finding the inverse. The better method is to argue that what is required is a matrix X, such that ACX = I. No such matrix has been found in the question so far, but since we know that CD = I it is plausible to consider ACD which equals AI, or A. We now have AC at the beginning of a composite matrix and it only remains to complete it so as to obtain I. Since AB has been shown to equal I in part (i), B is the fourth matrix required and thus we have ACDB = I, giving DB as the required inverse. Here the notions of considering ACD and of subsequently completing it with B both need to be selected from a number of possibilities to make a route to the solution. This is a characteristic two-step problem-solving process.

Ex.



1. $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$, $C = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$,

$D = \begin{pmatrix} \frac{1}{2} & 0 \\ k & 1 \end{pmatrix}$.

(i) Evaluate AB.

(ii) Find the value of k which makes CD the unit matrix.

(iii) Simplify CABD with the value of k found above. What does this show about the inverse of CA?

(iv) What is the inverse of AC?

solution

(i) $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $CD = \begin{pmatrix} 1 & 0 \\ 2k+1 & 1 \end{pmatrix}$ which = I if k = -2. (Requires manipulation including k)

(iii) $CABD = C(AB)D$
$= CD$
$= I$ This shows $(CA)^{-1} = BD$

(iv) ($(AC)^{-1}$ is not necessarily the same as $(CA)^{-1}$)
Since $CD = I$
$ACD = A$
So $ACDB = AB$
$= I$
Hence $(AC)^{-1} = DB$

Fig. 2

The teaching of problem-solving

Problem-solving requires attention to the data and the pursuit of the implications of each of its components. This implies an analytical approach to the data and is therefore connected with the trait of field independence. For example, it could be argued that this is what is required, in the example above, to resist the urge to use a direct but long method, for calculating the inverse of AC, but rather to use the definition AC.X = I. The other highly important factor is the ability to hold and co-ordinate mentally a number of different factors simultaneously. This is related to Pascual-Leone's concept of the mental space or M-space. Both field independence and size of M-space are factors which increase with age and in which there are considerable individual differences, but which are apparently not very susceptible to influence by educational factors (Case, 1975). This suggests caution regarding the traditional assumption that a diet of hard problems is the best training for a mathematician. Any beneficial effect from this would depend on the incidental learning of problem-solving strategies, rather than on a stretching of the information-processing capacity; it would therefore be more advantageous to teach the strategies more directly. This is what we shall discuss next.

Attempts to teach problem-solving have mainly centred on the question of the effectiveness of Polya's heuristic strategies or some variant of them in improving problem-solving performance. In general the most successful of these have been those with students of college or university age, while those with younger school pupils have not all been successful. Schoenfeld (1978) considered that in order to use successfully a set of strategies in a general mathematical problem-solving situation, it was necessary to add to particular heuristics a managerial strategy. He replaces a miscellaneous collection of heuristics by a flow chart intended to guide the problem-solver right through the process; this is based on the four phases identified above, with relevant heuristics attached to each phase. His subjects were first year university students to whom he taught a full three-month course of problem-solving using his heuristic scheme. In a smaller experiment on the explicit and intensive teaching of a set of five strategies, the experimental students performed very considerably better than the control gorup, who had practice in problem-solving without organised instruction in the heuristic scheme.

Lucas (1974) taught a university calculus course with a successful heuristic orientation. On the other hand Post achieved no significant improvement in classes of 12 year olds (1967), or fifteen year olds (1976), when taught by their normal teachers using a predesigned General Problem Solving Program. These results tend to support the conclusion that, while older and abler students can recognise and objectify aspects of problem-solving procedure sufficiently well to exert conscious guidance on their problem-solving processes, younger and less able students are unable to do this. However, Covington and Crutchfield (1965) had success with younger children on general heuristic training, and Resnick and Glaser (1975) have also worked with younger children and report success related to the use of

strategies. The conflicts among these results suggest that successful strategy learning depends strongly on the teacher as well as on the material.

Kantowski (1976) conducted an observational study with fourteen year old pupils of high ability in the solution of geometrical problems. She showed a strong relationship between success and the use of goal oriented heuristics. Thus the ability to come out of the productive mode of thinking, to reflect and to monitor progress towards the goal, led to success. However, the strategy of looking back was not improved during the time of the experiment. Kantowski also remarks that above a basic minimum, the knowledge of heurstics appeared to be more useful than further information on specific principles. Ehrenpreis & Scándura (1974) demonstrate a similar point. Recently, with a group of collaborators, Kantowski has made a more detailed study of problem-solving strategies used by subjects of various ages and abilities. The results reported above appear to be confirmed, but there is little further relevant information yet available.

Most of the studies on problem-solving consist of training over a fairly limited period followed by an immediate post-test. More representative of the educational situation is a study by Scott (1977). This shows a significant positive effect on achievement in geometry and algebra at age 16, related to the experience of at least one year of an inquiry programme at age 11 or 12. This inquiry training was with one of two specific teachers and the method consisted of presenting the students with an event requiring explanation, such as the larger of two blocks of wood floating in liquid while a smaller sized piece sinks to the bottom. The student's task was to ask the teacher questions that were answerable by 'yes' or 'no' until he felt that he could correctly explain why everything in the experiment happened the way it did. Previous studies (Scott, 1973) had shown that such inquiry training had a significant positive effect on the analytical aspects of cognitive style and that these results persisted at least up to the age of 16, and the present study extends this result by showing the positive effect also on mathematics achievement. In answer to a questionnaire, the students attributed their greater success in geometry to the fact that the inquiry training had developed logical thought and had helped them to reason "behind the facts", and that it had helped them in problem-solving and proof. Their success in algebra they attributed partly to the latter and partly to the contribution of the training to helping them to develop a strategy for inquiry. An aspect of the inquiry training which may well be significant for the retention and transfer of the thinking skills is that each problem-solving activity was followed by a reflective strategy session during which the students analysed the questions used and recorded earlier and categorised them according to their information-gathering value. Apparently techniques suggested in these strategy sessions included thinking of a start, middle and end to an experiment, getting all the facts, asking precise questions. These were retained during the five years between the teaching and the study in question, and referred to in the questionnaire by the students. It may also be significant that the inquiry sessions were group activity in which all students were aware of the questions being asked and indeed one of their number himself built up a blackboard display of the information being gained. This makes a reflective discussion of the value of the question possible in a way that it would not be if the pupils had been solving mathematical problems on their own.

Problem-solving strategies and the curriculum

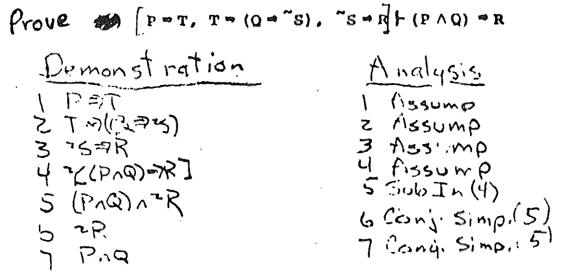
Apart from the USMES program (for 10-11 year olds), and a section of the Open University first year course, the learning of strategies for problem-solving does not appear to form a regular part of any school or college curriculum. There exist a number of printed courses which are discovery oriented, which ask many questions and attempt to generate an attitude of enquiry in mathematical learning. Examples are the primary school series, Investigating School Mathematics, used widely in Canada and the United States, and the School Mathematics Project series in use in many English secondary schools. It is also true that in traditional courses, particularly in geometry, problems were set on a regular basis. However, in the newer courses this tendency has declined and problems which are more than direct applications of immediately learnt material tend to be relegated to the 'puzzle corner'.

Certainly there is little or no serious attempt to develop the capacity to solve problems. The question therefore arises whether the research evidence on training in problem solving strategies is sufficient to recommend that such definite training should form a part of all mathematics courses. There are several reasons why one might be cautious about such a suggestion. First the problem-solving discussed is that of well-formulated, fairly hard mathematical problems. The set of strategies required for solving genuine real life problems differs in a number of respects. For example the set of behaviour categories used for the USMES scheme, aimed at developing elementary school children's skills for solving real life problems (Shann 1975), includes defining the problem, defining sub-tasks, relating sub-tasks to problem, relating the contribution of other people to the problem, acquiring skills and relevant information, planning action, reformulating the plan in response to obstacles, planning the next step, organising, analysing and interpreting data, relating results of the sub-tasks to the total problem, and communicating findings. These might be described in general as maintaining an awareness of the relation of the immediate task to the overall goal, working in communication and co-operation with others, and generally organising the problem-solving process. This represents a more generally useful acquisition than mathematical problem-solving skills; but still there are even more general thinking skills which education might be expected to develop, for example some awareness about how to arrive at well-founded opinions on social or moral questions, including a sense of what is relevant, and the distinction between fact and opinion. Other useful general awarenesses include that of the value of group discussion for generating ideas (brainstorming) and the value of independent approaches to a problem in providing an overall check. The second reason for caution about embracing mathematical problem-solving strategies is that the solving of well-formulated problems is not even the most characteristic <u>mathematical</u> activity. Problem formulation, proof, generalisation, representation and mathematical abstraction have more claim to this distinction than does problem solving as such. However, some attempt to promote the wider training in heuristic strategies as part of the curriculum has been made by Kantowski and her colleagues, who held a problem-solving workshop at the NCTM congress in April 1978. The problems used were of four types: alpha-numeric, as in the well-known CROSS + ROADS = DANGER problem, logical problems ('Mr. Flute does not play the sackbut....'), questions requiring mathematical induction, and arithmetic word problems.

3. PROOF

Historically this was recognised as an aspect of the mathematical process even before problem-solving. However, views about its importance diverge, particularly as between mathematicians and users of mathematics, such as scientists or engineers. In typical English courses, both ancient and modern, proof has appeared only to any great extent in the context of Euclid's geometry, and although in some countries modern mathematical courses have given a much stronger emphasis to proof, this has often been the least successful aspect of the reform. There is also very considerable confusion about the nature of proof, and this we shall commence by discussing.

The CSMP (Carbondale) view is that by beginning with detailed chains of inference using the stated laws of logic, pupils can acquire a firm foundational knowledge of what a proof is without having to induce this knowledge from the ordinary proofs they see presented by the teacher (CSMP, 1972). See extracts from pupils' work (Figure 3).



Prove $[P \Rightarrow T,\ T \Rightarrow (Q \Rightarrow {\sim}S),\ {\sim}S \Rightarrow R] \vdash (P \wedge Q) \Rightarrow R$

Demonstration	Analysis
1 $P \Rightarrow T$	1 Assump
2 $T \Rightarrow (Q \Rightarrow {\sim}S)$	2 Assump
3 ${\sim}S \Rightarrow R$	3 Assump
4 ${\sim}[(P \wedge Q) \Rightarrow R]$	4 Assump
5 $(P \wedge Q) \wedge {\sim}R$	5 Sub In (4)
6 ${\sim}R$	6 Conj. Simp.(5)
7 $P \wedge Q$	7 Conj. Simp. 5)

Fig. 3

Dienes (1973) appears to suggest that the final level of proof in school mathematics is a purely formal system in which strings of symbols are transformed according to stated rules; after sketching a study of totally ordered sets, the following proof that 2 comes after 4 is given:

Rule 1: Rxy $\Rightarrow$ NRyx
Rule 2: (Rxy and Ryx) $\Rightarrow$ Rxz

Theorem: NR SSO SSSSO

Proof: 1. R.See (Axiom)
?. R SSSO SSO (e = SSO)
3. R SSSSO SSSO (e = SSSO)
4. R SSSSO SSO (3, 2, Rule 1)
5. NR SSO SSSSO (4, 1)

Lester (1975), following Suppes, uses a similar but simpler system, as a step towards examining "the development of the ability to write a correct mathematical proof" in pupils aged 9 to 17. Though these may present interesting exercises, they have little or nothing to do with the essence of proof as a means of gaining greater certainty of an obtained result, or of embedding it in accepted mathematical knowledge.

Thom (1973) also dissents:

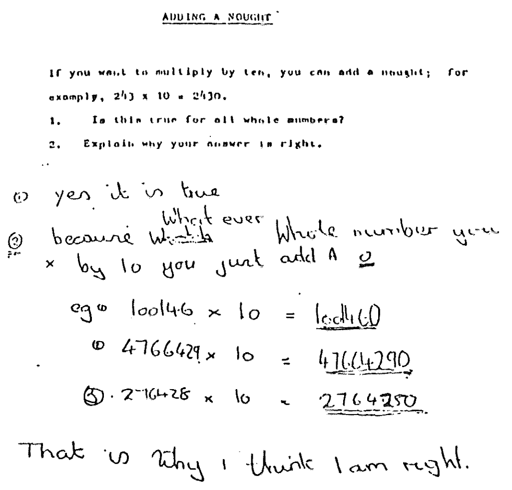
> "The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of 'meaning', of the 'existence' of mathematical objects; .... 'meaning' in mathematics is the fruit of constructive activity, of an apprenticeship......."

Development of the understanding and use of proof

In research of my own (Bell, 1976) I have distinguished three main dimensions of development in the understanding and use of proof. The first concerns the degree of regularity or rationality expected by the pupil. At the lowest level the pupil does not expect regularity in his observations; for example, pupils studying the number of non-crossing diagonals of polygons of different numbers and size, may collect a table of values containing mistakes, particularly in regard to polygons with larger numbers of sides, and may accept this irregularity, not checking their work in expectation that some error has occurred. Another example occurs in responses to a coin-turning problem (given three coins showing tails, obtain three heads by a succession of moves each consisting of turning over two coins).

There is a distinction here between pupils who assume that the continuation of trials must eventually produce success, and those who recognise that this is a rule-governed situation to which there is a definite answer - possible or impossible. The ability to make and use records of trials is important at this point. Later there grows a sense of the value of objectivity and of public acceptability of the knowledge being discovered, and pupils may attempt to verify for themselves a relationship discovered by another. It is then necessary to develop the ability to work with statements, treating them literally, rather than fuzzily; they need to be sufficiently precise to be capable of confirmation or refutation. For example, one fifteen year old boy, investigating the incidence of junctions or orders of 3, 4 and 5 in networks, asserted that "with bigger networks, more 4-junctions were necessary", and confirmed this by reference to his own somewhat idiosyncratic collection of examples. His statement needed to be made more precise and subjected to check on other pupils' examples. Related also to this dimension is the developing awareness of the set of elements for whch a given generalisation holds, with an increasing attention to defining the boundaries of such a set and the properties of special elements such as 0 and 1. For example, in the coin-turning problem, a clear awareness of what is the set of possible moves is the key to recognition of the compellingness of the proof.

A second dimension refers to what I have called the <u>explanatory quality</u> of the proof response. This implies the recognition that a proof or explanation of a particular result must go beyond the restatement of the result itself and must connect it with existing knowledge, avoiding implicit circularities. For example, when asked to explain the property that the addition of a 0 to the end of a number had the effect of multiplying by 10, some pupils were quite unable to penetrate the property and simply gave examples of its use and reasserted its truth (Fig. 4); while others, having more sense of explanation, appealed to the algorithm for long multiplication, and showed that an application of this, using the multiplier 10, produced the quoted result, not realising that the algorithm itself depended on the property being proved.



ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for example, 243 x 10 = 2430.

1. Is this true for all whole numbers?

2. Explain why your answer is right.

(1) yes it is true

(2) because ~~Which~~ What ever Whole number you × by 10 you just add A 0

eg (1) 100146 × 10 = 1001460

(2) 4766429 × 10 = 47664290

(3) · 276428 × 10 = 2764280

That is why I think I am right.

Fig. 4

The <u>third dimension</u> of development observed was the level of sophistication of the proof techniques or logical transformations which are available to the pupil. These range from the awareness of the value of an induction-type argument, seen at the elementary stages as a demonstration of how a case for n + 1 arises from a case for n; other techniques, such as <u>reductio ad absurdum</u>, which involves the assumption of the truth of a hypothesis which one in fact knows to be false, also the distinction between theorem and converse, or implication and equivalence. These demand considerably more of the quality of intellectual detachment which Piaget recognises as one of the important components of formal reasoning. For further discussion and illustration of these developments see Bell, 1976.

Van Hiele (quoted in Wirszup, 1976) has defined five levels of development in geometry which correspond fairly well with the stages observed in my work. The work of Piaget, particularly in The Growth of Logical Thinking, also bears closely on the present discussion and indeed in our study of this question we draw on many facets of Piaget's general description of the development of adolescent thinking, particularly the development from egocentrism to the sense of the public nature of knowledge and the value of public verification.

Another approach to levels of understanding in proof is provided by van Dormolen (1977), following van Hiele. He distinguishes three levels of proof, related to the levels of abstraction at which the pupil is working. At the first level the pupil is concerned, say, with a particular isosceles trapezium or the rotation about O of a particular point of the plane or the sum of a known number of consecutive odd numbers. At this level the pupil is concerned with the single particular object given, so, for example, his proof that the diagonals of the trapezium are equal consists of measuring them with a ruler. At the second level the object is seen as the representative of a class of similar objects and an argument is given which applies in principle to this whole class. For example, the observation that the trapezium can be reflected in a line of symmetry, though expressed in particular terms, would apply immediately to the general case. These arguments have been called quasi-general arguments above. The third level is that in which the particular situation is seen as a facet of a general system of defined constructs and of propositions describing their relationships. The argument at this level is expressed in fully generalised terms using the particular objects merely as illustration. For example, the argument about the trapezium begins by giving a definition of such an object, and that about the rotation deals with the rotation of a general point (a,b).

Studies on the learning of proof

King (1973) reports the development and testing of a unit of instruction on proof for able 11 year olds. The subject matter consisted of six theorems of the kind suggested by the Cambridge (Mass) Conference on School Mathematics (1963), for experiment with pupils of this age: the first theorem was, "if $N|A$ and $N|B$, then $N|(A + B)$", while others extended this to $A - B$, $A + B + C$, and converses. It would appear

that the content of these theorems was easily understood, and that the major teaching effort required was in expressing the proofs in concise symbolic form. This proved possible in 17 instructional sessions with these pupils, but one might question its value. It would justify itself only if such modes of expression were taken up and used elsewhere by the pupils.

An experiment with sixth form pupils (aged 17) was conducted by myself and Edmonds (Bell 1976). The criterion tests in this case required the judgement of the validity from complete or incomplete sets of cases, examples of complete explanatory arguments, of fragmentary explanations and of general restatements of the data containing no explanatory quality. The teaching included some discussion of these points in relation to proofs written by the different pupils and passed around the class for discussion of validity. The results showed that the ability to detect an incomplete set of cases was improved, but the ability to recognise a complete explanation was not significantly affected. For example, one question in the criterion test required the judgements of the validity of the following two arguments in a problem called 'Add and Take'. In this problem it is supposed that a number between 1 and 10 is added to 10 and then taken from 10, and the two results added. The question is whether the result will always be the same and why. The responses proposed are as follows:

Susan:

The result will always be 20. If you chose a number between 1 and 10 and add it to 10, then if you take the first number away from 10 it will be whatever is needed to make 20.

Yvonne:

Always 20. Whatever you add you always take it away so it cancels out. But as you add 10 and take the number from 10, you get double 10 which is 20.

Have these pupils proved their answers?

Susan's: Yes/No   Yvonne's: Yes/No

Give your reasons:

____

____

Although it is clear to us that the second of these arguments contains the essential points of a valid explanation while the first merely restates the problem, this judgement was found difficult by the pupils who tended to demand further justification of the fact that adding and subtracting the same number gives zero. This perhaps highlights the fact that in actual proof activities judgement is always required regarding which aspects of the argument can be assumed as obvious and which require exposure.

The place of proof in the curriculum

Proof may appear in the curriculum in three distinct ways. First, as informal deduction, as when one asks of a newly discovered or displayed principle, is it always true, or why is this true, and one finds some general argument or global insight which establishes conviction. Secondly, as deductive exposition; this ranges from explaining why minus times minus is plus rather than simply asserting it, up to the presentation of an entire course or a major part of it within an axiomatic framework. Thirdly, as the activity of systematisation, when a collection of already known materials in a particular topic area is examined and organised deductively around one or more major theorems, the others being shown to be consequences of it or extentions of it. Of these, at the ages under consideration, that is up to about 16, I think that informal deduction has a big part to play, deductive exposition is not very important, and the activity of systematisation is a desirable element of the course for the ablest pupils.

An example of the abandonment of an axiomatic framework is seen in the following extracts from the Scottish course, 1965 and 1975 editions.

98 GEOMETRY

F. The circle

From the fact that *a circle possesses rotational symmetry about its centre*, we *deduced* that:

(i) equal chords, equal arcs, and equal angles at the centre go together
(ii) the length of an arc is proportional to the size of the angle subtended at the centre $\left(\frac{a}{b} = \frac{\alpha}{\beta}\right)$
(iii) two chords are equal ⇔ the chords are equidistant from the centre.

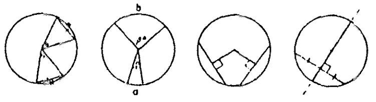


FIG. 6

From the fact that *every diameter of a circle is an axis of bilateral symmetry* we *deduced* that:

(i) a diameter perpendicular to a chord bisects the chord
(ii) a diameter which bisects a chord is perpendicular to it
(iii) the perpendicular bisector of a chord passes through the centre of the circle.

162 GEOMETRY

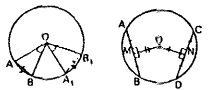
Revision Topic 9 Rotation and Circles

Reminders

1 A *rotation* is a transformation of the plane in which all points are rotated about a given centre through an angle which is fixed in magnitude and sense.
Every line in the plane is rotated through the same angle.
The centre of rotation is an *invariant* point.

2 *Rotational properties of the circle.* In equal circles or in the same circle,

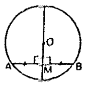
equal angles at the centre
⇔ equal arcs subtending the angles
⇔ equal chords cutting off the arcs
⇔ equal sectors



3 *Equal chords* of a circle are equidistant from the centre. Chords of a circle which are equidistant from the centre are equal.

4 A *regular polygon* of *n* sides can be inscribed in a circle by drawing *n* equally spaced radii and joining their ends.

5 *Properties of the circle by bilateral symmetry.*
a The perpendicular from O to AB bisects AB.
b The join of O to the midpoint of AB is perpendicular to AB.
c The perpendicular bisector of AB passes through O.
We may use Pythagoras' theorem or trigonometrical calculations in triangles like △OAM.



6 The *equation of a circle* with its centre at the origin and radius *r* is $x^2 + y^2 = r^2$.

For further discussion of these questions and an example of systematising activity, see Bell (1971-4); also Sawyer in OISE (1967), and Willson (1978).

A recent booklet for 15-16 year olds, published by the Leapfrogs group (1978) and entitled, 'Conversations about Pythagoras', contains an excellent discussion both of the different kinds of proof of this theorem, embedding it in different aspects of other knowledge, and also of the nature of proof itself.

4. MATHEMATISATION

This term embraces the processes of mathematical abstraction, generalisation and representation. Its use implies an attempt to encapsulate the essence of mathematical activity. Wheeler (1978) has collected papers on this theme from a number of mathematicians and educators. One of his own examples of mathematisation involves activity with a set of coloured prisms and cubes. With these it can be seen that eight small cubes can be put together to form a certain larger cube, and this can in fact be done in two ways with size 1 cubes making up a size 2, and with size 5 cubes making up a size 10. The recognition that the relationship exists independently of the particular size of cube is a simple example of mathematical abstraction. Another is when a set of 10 flat tiles is assembled to make a cube, and then the tiles are each pushed to the side a little until each overlaps the one below by a small amount. The shape is no longer cubical but the volume remains the same. What of the surface area? This has increased; by how much? What is the maximum amount by which it can be increased? Is there a limit? Suppose thinner tiles had been used to make the cube. What would now be the result for the surface area. And so on. In this example a physical transformation has provoked questions about abstractions which have soon led to speculation about possibilities which are beyond those capable of being represented by the concrete material. Although the normally accepted story about abstraction is that it takes place by observing the common elements in a set of particular cases, in fact it seems more often to be the case that one becomes aware while manipulating a single particular situation that some elements in it are arbitrary, and thus the particular case represents a generality. This situation also contains a feature somewhat characteristic of mathematical abstraction, that is of the awareness of an infinite set of possibilities.

An example seen by myself occurred when a student was investigating the problem of dividing a given square into various numbers of smaller squares. She had succeeded in dividing the square into 4, 9, 16 etc. smaller squares, but could not find other ways of division. The breakthrough occurred when she recognised that what had been done to the given square could be done to one of the smaller squares, as shown in Figure 5a. This was a crucial act of mathematisation: it involved abstraction and transfer of the mode of division. Moreover it can be repeated indefinitely. A further breakthrough occurred when it was recognised that the reverse process is possible - a block of small squares can be made a single square (Figure 5b).

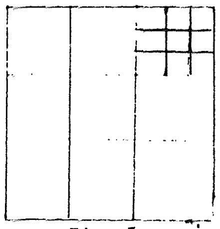


Fig. 5a

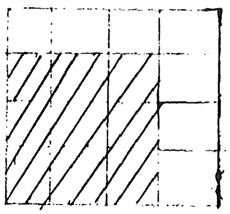
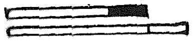


Fig. 5b

The two examples just discussed show aspects of mathematical abstraction, that is the recognition of some structural concept in a situation. An example showing a different aspect of mathematisation, that of representation, is the '23 francs' problem used by L. Johannot (1947). This states, "We both start with the same amount of money. Then I give you 23 francs. How much more than I do you have afterwards?". The problem was posed in this form, and if not solved correctly at first, four variations were offered to see if the subject could use them successfully. These were (i) a concrete form in which the initial piles of money were representated each by about 3 matches, and one or two matches were transferred, (ii) a diagrammatic form in which the initial amounts were shown as rectangles or lines and the transferred amount marked off  (iii) a numerical particularisation, for example, starting with 50 francs, and (iv) an algebraic representation $x + 23$, $x - 23$. The subject was generally first questioned to see if he could supply the alternative form. Only the oldest subjects were able to generate these various representations spontaneously. The most essentially mathematical aspect of the handling of this situation rests in the relating of the various representations one with another. It is the ability easily to make the transition between them which marks mathematical competence.

Abstraction and Generalisation

In the discussion so far mathematisation has not been separated out into the strands of abstraction, generalisation and representation. It may be useful at this point to draw some distinctions. Crudely, generalisation takes place when the set of objects under consideration is extended; in abstraction, a set becomes regarded as an element of a new higher level set. Thus generalisation is related to the inclusion relation, abstraction to the set-element relation. To come a little nearer to reality, generalisation usually involves seeking for the limits of the set over which a certain property is true; the result is a statement of the generalisation with its conditions. The prior identification of the property in question is an act of abstraction. Thus, for example, starting with the observation that 24 x 63 = 36 x 42, an act of concept recognition leads to the question, for what numbers a, b, c, d is it true that 'ab' x 'cd' = 'dc' x 'ba'. Investigation establishes the conditions on the numbers a, b, c, d for which this statement is true and the statement together with the condition then constitutes a generalisation. Processes such as that by which the sine of an angle, initially defined in a right-angled triangle, is redefined in terms of co-ordinates and a rotating arm so as to provide a definition for angles of any magnitude, are better described not as generalisation, but as concept extension, which is an act of abstraction.

Representation, Symbolisation and Modelling

If generalisation is the characteristic pure mathematical process, that of applied mathematics is modelling, that is the representation of some situation via a diagram, a symbolic expression or some other form of analogy. In my view this process has a very great part to play in the mathematical education of the non-specialist student. The existence and nature of this aspect of mathematical process is generally well understood so we shall proceed immediately to consider its place in the curriculum (for a description see Hall, 1972).

Modelling

Ormell has for the last ten years been developing courses with this aspect of mathematics as a basis for pupils aged 17. An example of Ormell's method of "projective modelling" is the following. A possible new method of building high rise flats is considered. Each floor is built on ground level and on completion of each floor the building is raised by hydraulic jacks sufficiently to enable the next floor to be built underneath. Estimates are made of the cost of each floor with the foundations, of other overheads and of the cost of raising each floor to this position. Questions are asked regarding the average cost per flat for buildings of various heights, and whether there is a limiting cost to this average. The model underlying this example is of the form $2.5n + 400 + \frac{3000}{n - 1}$ (Knowles, 1971). One aspect of this course is the learning of modelling itself through a number of exercises, starting from hypothetical proposals, such as the installation of ridges on a road to slow down traffic, or the design of a horizontal transport system from town centre to station in a given town. The other aspect of the course is that the study of functions, such as the linear, quadratic and other polynomial functions, is conducted in the context of such a hypothetical model whose properties are explored. The writing of a modelling essay originally formed part of the examination of this course, but this has recently been discontinued, partly on account of problems of assessment. Ormell, following Peirce, describes mathematics as "the science of possibility", and this displays the purpose of mathematical modelling as the prediction of the likely effect of changes in a given situation. The strategy most often used at the start of a modelling activity is to change something in a situation and to observe the consequences of this change. Classification and comparison can then be made. At this stage is it valuable to have at one's disposal a range of forms of symbol and graph, such as arrow diagrams, Venn diagrams, Cartesian graphs, tree diagrams. A knowledge of the basic structures of sets, relations and functions can also be useful in enabling one to see what kinds of mathematics to apply.

Symbolisation

An example of how different symbolisations are possible in a given situation, each of which has its own distinct advantages, is provided by the game of Frogs. In this, some pegs of two colours are arranged in a row, separated by an empty hole. Red pegs, let us say, move to the right and blue to the left. They may either slide one space, or into the hole if adjacent to it, or jump into it over one peg of the opposite colour. The object is to interchange the colours in the least possible number of moves. To pursue this investigation some representation of the game is necessary. The most direct one is that in which the position of all the pegs is recorded in a row after each move. However, further study makes it clear that to describe a move it is sufficient to state which colour is moved since there is always only one move possible for a peg of a given colour. Thus a whole game can be recorded as, for example, RBRBR. Alternatively, it is possible to record simply whether a slide or a jump is made. This also turns out to describe a move uniquely. These briefer forms of representation enable patterns of moves during a given game to be recognised and to be extended to longer games. Also of interest regarding the potentiality of a good symbolisation is the following diagram which displays all the possible states and moves of the game of Towers of Hanoi with three pegs and three discs. In this the pegs are coded A, B and C, and the discs are coded by position. The first letter denotes the peg on which the largest disc is placed, the next that holding the next largest disc, and so on. On this diagram it is possible to see exactly what sequences of moves constitute minimal successful games (Jullien, 1972).

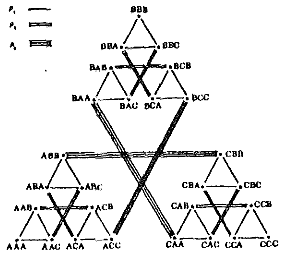
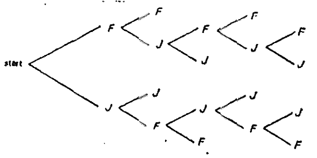


Fig. 7

The question which arises in relation to symbolisation is whether for some persons the use of a symbol to represent the movement or object in itself represents a difficulty. Such slender evidence as we have from a comparison between two groups of children working on reflections and rotations of a hexagon, one using a teacher's symbolism, the other choosing their own, suggests that the difficulty resides not in the symbolism itself but in whether or not the concept which the symbol denotes is itself well understood (Lunzer, Bell and Shiu, 1976). Studies by Collis (1975) and Küchemann (1977) also support this conclusion. It would be interesting however to have further evidence on this question. We display in Figure 9 two questions originally used by the Australian Council for Educational Research which might be used to explore pupils' abilities with regard to the adoption and use of a symbolism.

Jill and Fred arrange to play a tennis tournament under the following rules:

The first person to win 2 games in a row or a total of 3 games wins the tournament. The following diagram shows the various ways in which the tournament could occur. (J represents a win of a game for Jill; F represents a win of a game for Fred.)



4. How many ways can the tournament possibly occur?

A 4
B 10
C 18
D none of the above

Fred has three clean bottles, one large, one medium, and one small, called L, M, S respectively. He also has a large tub of water.

L can hold 1¼ times as much as M.
M can hold 1¼ times as much as S.

An 'f' in front on the name of a bottle means that it is filled from the tub. For example, fS means S from the tub.

Similarly an 'e' in front of the name of a bottle means it is emptied into the tub.

A hyphen between the names of two bottles means that water is poured from the first named bottle into the second until the first one is emptied or the second one filled.

7. Which one of the following would leave S half full?

A fM, M-S, eS, M-S
B fM, M-L, fS, S-L
C It can be done, but not by either of the ways shown above.
D It can't be done with the means available.

Fig. 9

Krutetskii (1976) has collected sets of exercises which expose the difference in capability between able mathematical students and others. Some of these consist of sets of problems where the same mathematical structure is embodied in a number of different contexts. This recognition of common structure is an example of mathematical abstraction but it lacks the unlimited possibility which exists in the examples given above. We include here, for comparison, Krutetskii's own list of mathematical abilities.

1. Obtaining mathematical information
    - A. The ability for formalized perception of mathematical material, for grasping the formal structure of a problem.
2. Processing mathematical information
    - A. The ability for logical thought in the sphere of quantitative and spatial relationships, number and letter symbols; the ability to think in mathematical symbols.
    - B. The ability for rapid and broad generalization of mathematical objects, relations, and operations.
    - C. The ability to curtail the process of mathematical reasoning and the system of corresponding operations; the ability to think in curtailed structures.
    - D. Flexibility of mental processes in mathematical activity.
    - E. Striving for clarity, simplicity, economy, and rationality of solutions.
    - F. The ability for rapid and free reconstruction of the direction of a mental process, switching from a direct to a reverse train of thought (reversibility of the mental process in mathematical reasoning).
3. Retaining mathematical information
    - A. Mathematical memory (generalized memory for mathematical relationships, type characteristics, schemes of arguments and proofs, methods of problem-solving, and principles of approach.
4. General synthetic component
    - A. Mathematical cast of mind

Mathematisation in the curriculum

The ability to mathematise is undoubtedly a universal human capacity, like the ability to talk or to represent by drawing (ATM, 1977). It seems likely that if mathematical curricula were focused more sharply on the development of this key ability, many people could reach higher levels of mathematical competence than they do. Little work of this kind exists at present but we shall assemble such relevant evidence as we can.

A most common type of generalisation is that in which a sequence of situations gives rise to a sequence of numbers. For example, a row of joined triangles may be made and the number of matchsticks required related to the number of triangles formed. Wills (1967), using a number of problems of this general type, taught high-school pupils a strategy of trying small numbers, then somewhat bigger ones and then by seeing what operations needed to be performed with the bigger numbers, the move to the expression of the general formula, both verbally and symbolically. This strategy was successfully learned.

In England a number of schools and groups connected with the Association of Teachers of Mathematics have carried such work further and developed pupils' abilities to formulate questions, to generate examples, to seek generalisations and justify them, and to ask further questions extending the investigation. One school has a special O level examination in which 40% of marks are given for the assessment of four extended investigations of this kind, done in the pupils' own time.

As an example, one pupil following this course began with the following problem:

> I have a number of sweets which divided into 4s leaves remainder 2. If divided into 5s, 1 is left. How many sweets have I?

The sequence of possible solutions for this numerical example was first found, and then the generalisation was formulated and explored for the case of any divisor and any remainder. A partial but not complete conjecture regarding the differences of the obtained series was formulated and justified empirically, and a definite procedure was found for deciding the starting number of the sequence (this piece of work is contained in the Appendix to Bell, 1976).

The Leapfrogs group has produced a series of some 20 small booklets for pupils aged 9 to 13 which contain many situations starting from games or simple structured material or interesting situations, e.g. codes, set up for children's explorations and generalisation. The South Nottinghamshire Project has also based its course for 11 to 13 year olds fairly extensively on such activities (Bell, Rooke and Wigley, 1978).

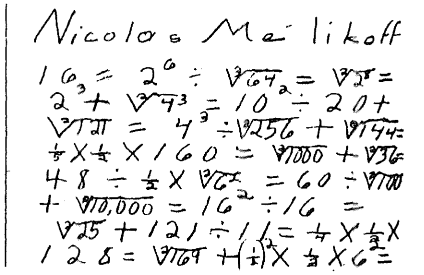
Papert (1972) has exploited the use of the computer with graph plotter and also with a mobile artificial turtle for giving quite young pupils experience of characteristic matheamtical activities. They attempt to make the turtle describe patterns of increasing complexity using sub-routines, iteration and partial solutions and testing for the cause of errors.

A group of the Association of Teachers of Mathematics, in preparing proposals for new mathematics courses for 16 and 17 year olds, based their course on the concepts of generalisation and modelling and tested examination questions designed to evaluate pupils' capacity for generalisation (ATM, 1978).

The Mathematics for the Majority Continuation Project produced activities for less able 14-16 year olds based on games and investigations where simple mathematical strategies were called into play by situations requiring decisions (Kaner, 1974).

At the 1979 IGPME Conference, Vermandel and Cohors-Fresenborg identified the process of abstraction and of interrelating a situation and a symbol system as constituting the essence of mathematical activity. The latter also presented activities with 11 year old children involving the design of automatic (e.g. coin operated) machines from bi-stable units. The pupils had to interrelate descriptions of the machine's operation, the physical bi-stable units and diagrammatic representations of these. Lowenthal described work with 8 year old pupils involving the representation of simple coin-tossing games by graphs, and here again the emphasis was on the translation between situation and representation.

Some of the most striking examples of mathematisation activities by teachers and pupils are given by Goutard (1968, 1970). The following is part of a 'free composition' by a six year old, after six months at school, working with Cuisenaire rods.



Nicolas Mélikoff

$16 = 2^6 \div \sqrt[3]{64} = \sqrt[3]{8} =$

$2^3 + \sqrt[3]{4^3} = 10^2 \div 20 +$

$\sqrt[3]{27} = 4^3 \div \sqrt{256} + \sqrt{144} =$

$\frac{1}{5} \times \frac{1}{2} \times 160 = \sqrt[3]{1000} + \sqrt{36} =$

$48 \div \frac{1}{2} \times \sqrt{6^2} = 60 \div \sqrt{100}$

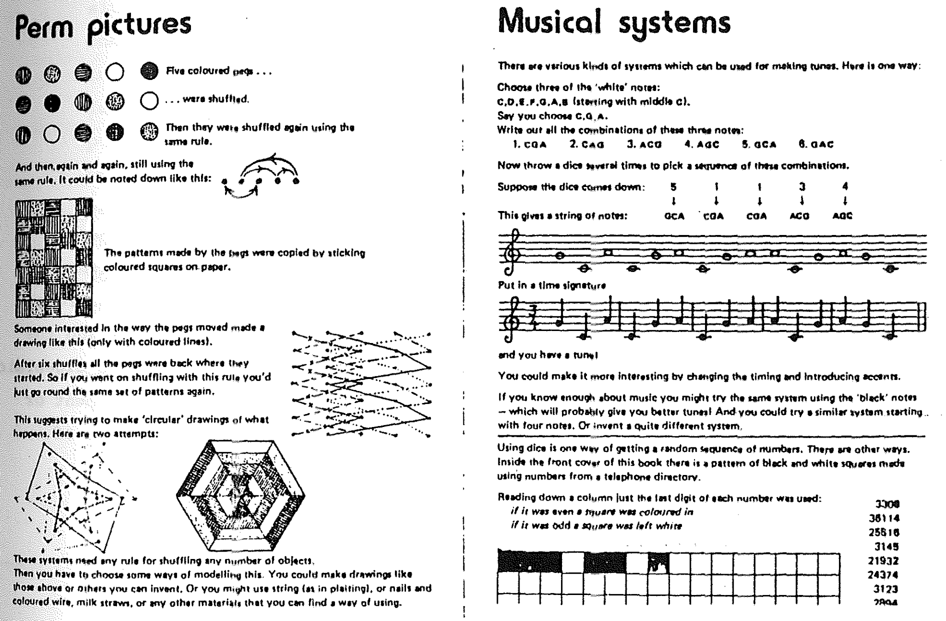
$+ \sqrt{10{,}000} = 16^2 \div 16 =$

$\sqrt{25} + 121 \div 11 = \frac{1}{4} \times \frac{1}{2} \times$

$128 = \sqrt{169} + \left(\frac{1}{2}\right)^2 \times \frac{1}{2} \times 6^2 =$

Fig. 6

Figure 7 is from one of the Leapfrogs booklets. It shows activities involving the controlled creation of patterns and working with representations.



Perm pictures

Five coloured pegs . . .

. . . were shuffled.

Then they were shuffled again using the same rule.

And then, again and again, still using the same rule. It could be noted down like this:

The patterns made by the pegs were copied by sticking coloured squares on paper.

Someone interested in the way the pegs moved made a drawing like this (only with coloured lines).

After six shuffles all the pegs were back where they started. So if you went on shuffling with this rule you'd just go round the same set of patterns again.

This suggests trying to make 'circular' drawings of what happens. Here are two attempts:

These systems need any rule for shuffling any number of objects.
Then you have to choose some ways of modelling this. You could make drawings like those above or others you can invent. Or you might use string (as in plaiting), or nails and coloured wire, milk straws, or any other materials that you can find a way of using.

Musical systems

There are various kinds of systems which can be used for making tunes. Here is one way:

Choose three of the 'white' notes:
C,D,E,F,G,A,B (starting with middle C).
Say you choose C,G,A.
Write out all the combinations of these three notes:
1. CGA 2. CAG 3. ACG 4. AGC 5. GCA 6. GAC

Now throw a dice several times to pick a sequence of these combinations.

Suppose the dice comes down:	5	1	1	3	4
	↓	↓	↓	↓	↓
This gives a string of notes:	GCA	CGA	CGA	ACG	AGC

Put in a time signature

and you have a tune!

You could make it more interesting by changing the timing and introducing accents.

If you know enough about music you might try the same system using the 'black' notes – which will probably give you better tunes! And you could try a similar system starting with four notes. Or invent a quite different system.

Using dice is one way of getting a random sequence of numbers. There are other ways. Inside the front cover of this book there is a pattern of black and white squares made using numbers from a telephone directory.

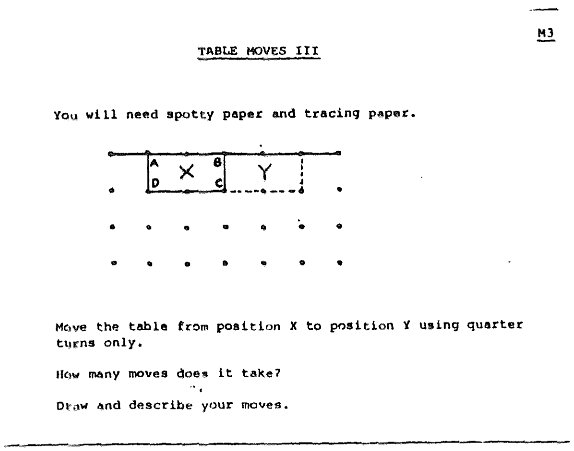
Reading down a column just the last digit of each number was used:
if it was even a square was coloured in
if it was odd a square was left white

3308
38114
25516
3145
21932
24374
3123

Fig. 7

See also the ATM books, Notes on Mathematics in Primary Schools and Notes on Mathematics for Children.

A strategy related to symbolisation which is developed by some of the South Nottinghamshire Project material is shown in the task illustrated below, Figure 8, entitled Table Moves. In this, quarter turns of the table anti-clockwise about the leg A are denoted by the symbol A and thus a word such as ABCDBBD describes a sequence of moves of the table around the plane. Questions such as whether two different words may represent the same displacement of the table arise, including the possibility of identity movements and of how to decide for a given displacement what is the shortest sequence of moves which will achieve it. Throughout this study the interaction between the geometry and the algebra is being explored. Each time the question is whether an algebraic algorithm can be found to correspond to a given geometrical movement or relationship. This principle can be explored also in other areas of the curriculum and made explicit.



M3

TABLE MOVES III

You will need spotty paper and tracing paper.

Move the table from position X to position Y using quarter turns only.

How many moves does it take?

Draw and describe your moves.

Fig. 8

5. CONCLUSIONS

It will be clear that the kinds of work reported here differ greatly as between the three aspects of the mathematics considered. (It should also be admitted that work on the development of algorithmic thinking in relation to computers has been neglected entirely.) In problem-solving there has been substantial research, inspired by Polya's work, on the possibility of teaching Polya's set of heuristic strategies at various ages, but few attempts to study pupils' spontaneous problem-solving activities, or to identify levels of development, which might assist the design of more sympathetic curriculum interventions, to ehhance pupils' problem-solving powers. In proof there was some research to report on developmental dimensions and levels. In mathematisation (representation, generalisation, abstraction) there was experimental curriculum material for various ages to describe, but as yet no evaluation of its use - indeed, little relevant test material exists, though some is being developed currently at Nottingham. My own view is that progress in the development of pupils' capacities in these general processes of mathematics is most likely to come from teaching based on pupils' investigation of problem situation, selected and arranged so as to foster the learning of coherent bodies of mathematical ideas, with interactions among pupils in the group, and with the teachers' intervention mainly taking the form of guiding by general questions which it is intended that the pupil will eventually take over as strategies. For example, 'Make up another similar example', 'What happens if you change it in some way?', 'Is it always true?', 'Look back and note what you have done'. The key concepts in the three areas we have studied are in problem solving the ability to maintain the tension between productive and reflective modes; in proof, the need for a group in which to communicate and defend newly discovered propositions, and in mathematisation, the need for material which promotes work which inter-relates the particular and the general and the situation and its representation. Perhaps the greatest challenge is to produce material which fosters these developments and at the same time develops the pupils' knowledge of important particular mathematical ideas. For, as Gagné has pointed out, effective action requires not only strategies of thought but also the possession of a well-organised set of particular intellectual skills.

REFERENCES

Association of Teachers of Mathematics Notes on Mathematics for Children. Cambridge University Press, 1977.

Association of Teachers of Mathematics Mathematics for Sixth Formers. Nelson, Lancs.: ATM, 1978.

Bell, A.W. Proof in Transformation Geometry, Parts 1-5. Math. Tchg. Nos. 57, 58, 61, 63, 66; 1971-4.

Bell, A.W. The Learning of General Mathematical Strategies, Shell Centre for Mathematical Education, University of Nottingham, 1976.

Bell, A.W., Wigley, A., Rooke, D. Journey into Maths, The South Nottinghamshire Project. Blackie, 1978.

Case, R. Gearing the demands of instruction to the developmental capacities of the learner. Review of Educational Research, 45, no.1, 1975.

Collis, K.F. The Development of Formal Reasoning. University of Newcastle, New South Wales, 1975.

Covington, M.V. & Crutchfield, R.S. Facilitation of Creative Problem Solving. Programmed Instruction 4, 1965.

CSMP Elements of Mathematics Program: Book O. Carbondale, Ill.: Cemrel, 1972

Dienes, Z.P. The Six Stages in the Process of Learning Mathematics. National Foundation for Educational Research, 1973.

Ehrenpreis, W. & Scandura, J.M. Algorithmic approach to curriculum construction. Journal of Educational Psychology, 1974, pp.491-498.

Freudenthal, H. Mathematics as an Educational Task. Dordrecht, Holland: Reidel, 1973.

Goutard, M. Mathematics and Children. Reading: Educational Explorers, 1968.

Goutard, M. Mathematiques sur Mesure. Paris: Hachette, 1970.

Greeno, J.G. Indefinite goals in well structured problems. 1975.

Hall, G.G. Modelling - a philosophy for applied mathematicians. Bull, Inst. Math. Applic., 1972.

Johannot, L. Recherches sur le Raisonnement Mathematique de l'Adolescent. Neuchatel: Delachaux et Niestlé, 1947.

Jullien, P. Trois Jeux, in La Mathematique et ses applications. Paris: Cedic, 1972.

Kaner, P. Back to Square One: a long look at MMCP. Times Educational Supplement, 4 Oct, 1974.

Kantowski, M.G. Processes involved in mathematical problem solving. Journal for Research in Mathematical Education, 1976.

Kilpatrick, J. Problem Solving in Mathematics. Review of Educational Research, 1969.

Knowles, F. An approach to applicable mathematics, Mathematics Teaching, 1971.

Krutetskii, V. The psychology of mathematical abilities in school children, University of Chicago Press, 1976.

Leapfrogs Action Books, Treatments: Coldharbour, Newton St. Cyres, Exeter, 1978.

Küchemann, D. Generalised Arithemtic. Unpublished paper, Chelsea College, London, 1977.

Lester, F.K. Developmental aspects of children's ability to understand mathematical proof. J. Res. Math. Educ. 6, No.1, 1975.

Lucas, J.F. The Teaching of Heuristic Problem-Solving Strategies in Elementary Calculus. J. Res. Math. Educ. 5, No.1, 1974.

Lunzer, E.A., Bell, A.W., & Shiu, C.M. Numbers and the World of Things. University of Nottingham, School of Education, 1976.

OISE Geometry K to 13. Ontario Institute for Studies in Education, 1967.

Ormell, C. New Applicability in the Classroom. Sixth Form Mathematics Curriculum Project, University of Reading, 1976.

Papert, S. Teaching Children to be Mathematicians versus Teaching about Mathematics. Int. J. Math. Educ. Sci. Technol. 3, 1972.

Poincare, H. Science and Method. Nelson.

Polanyi, M. Problem Solving. Brit. J. Phil. Sci. 8, 1957.

Post, T.R. The effects of the presentation of a structure of the problem-solving process upon problem-solving ability in seventh grade mathematics. Ph.D. thesis, Indiana University, Diss. Abstr. 28: 4545A.

Post, T.R. & Brennan An experimental study of the effectivness of the formal versus an informal presentation of a general heuristic process on problem solving in tenth grade geometry. Journal for Research in Mathematics Education, 1976.

Resnick, L., & Glaser, R. Problem solving and intelligence, in Resnick, L.B. (ed.), The Nature of Intelligence. Hillsdale, New Jersey: Lawrence Erlbaum Associates, 1975.

Schoenfeld, A. Problem solving strategies in college level mathematics. Unpublished paper, University of California, Berkeley, 1978.

Shann, M. Measuring problem solving. Boston University, School of Education, 1975.

Silver, E. Students' perception of relatedness in word problems. Paper read at AERA, Toronto, 1978.

Thom, R. Modern Mathematics: Does it exist? in Howson, Developments in Mathematical Education, CUP 1971.

Van Dormolen, J. Learning to understand what giving a proof really means. Educational Studies in Mathematics 8, 1977.

Wheeler, D. Mathematisation - notes for a study group. Unpublished papers, Concordia University, Montreal, 1978.

Willson, W. Wynne The Mathematics Curriculum: Geometry. Blackie, 1978.

Wirszup, I. Breakthroughs in the psychology of learning and teaching geometry, in Space and Geometry, papers from a research workshop. Columbus, Ohio: ERIC, 1976

VISUALISING AND MATHEMATICS IN A PRE-TECHNOLOGICAL CULTURE

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Papua New Guinea is a strange, fascinating country which is at present going through an amazing period of change. All countries experience change, but it is possible that few have ever experienced change so rapid as that in Papua New Guinea. "From stone-age to twentieth century in one lifetime" is no overstatement. Apart from those living in the few small towns the majority of the population have little contact with the technological society and culture which we know so well. And yet, there are two universities, and I was fortunate to be able to spend three months last year working at one of them, the University of Technology at Lae. (The other is at Port Moresby, the capital).

In any culture it is likely that one will find a few people who possess certain skills naturally, one might say, whether or not that culture prizes those skills. For example, we in the U.K. treat as amazing oddities those individuals who have exceptional memories - they are often given entertainer status, and are not awarded the same respect as they would be in Papua New Guinea. The 'big-men' there have, amongst other attributes, exceptional memories.

The inverse is that in Papua New Guinea there will be some individuals who possess those skills necessary for doing mathematical and scientific work. Some of these have been found, and at the University of Technology there are a few 'local' tutors employed to teach. But in a pre-technological culture these skills are rare, their worth is not appreciated and their presence is not even recognised. In these conditions teaching mathematics and science is far more complex than it usually is.

But in this paper I do not want to discuss the strategy of educational development in Papua New Guinea. I want to describe some of the data I obtained in my research there and to encourage you to think what it may imply about children learning mathematics in your own cultures and countries. Jerome Bruner once said that the last animal to discover water is the fish. I think it is most instructive to step outside one's own culture for a while to see what it is about the cultural "water" which helps to support and sustain the mathematical "fish" living in it.

I have another motive. It is all too easy when listening to descriptions of research, to generalise. Particularly so because we, being interested in mathematics learning, know about 'generalising' and its importance. It is almost as difficult for us to stop generalising as it is for some people in certain cultures to start generalising.

In an International body such as this I feel it is very important to recognise that as well as sharing a common interest in mathematics learning there are many differences between us. In particular, what may be the case in one country, or culture, may be quite different from that in another country, or culture. I hope that my data from Papua New Guinea will be a strong reminder of this.

I was testing there, in great detail, twelve first year University students, who were therefore already highly selected. Their ages varied from 16 to 26, and they were from three specifically chosen parts of the country, the Capital, a Highland region and an Island region. They were studying a variety of courses, but all were entering a field of technology, eg. engineering, agriculture, architecture, cartography, accountancy, etc.

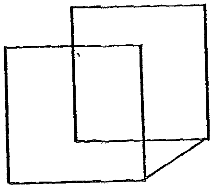
My research was concerned with the visual and spatial aspects of mathematics and I used approximately forty different tasks. (I have written two reports about this work, listed at the end). It would be impossible and inappropriate for me to attempt to give all the details in this short paper. So, what I will do is to summarize the main ideas and give a few specific examples which will illuminate my general comments. The results will be grouped under five main headings: Picture Conventions, Drawing, Visualizing, Language and Cognitive Characteristics.

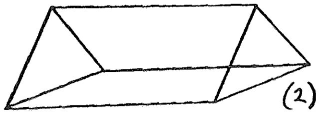
(a) Picture Conventions It was clear from several tasks that there existed a general unfamiliarity with many of the conventions and "vocabulary" of the diagrams commonly used in Western education and which are now entering PNG schools. Some tasks showed this rather dramatically because they focus directly on the convention. For example, this one:

The student was asked to make models using plasticine "corners" and cocktail sticks, based on drawings. The drawings used were similar to those used by Deregowski (1974). They concerned the use of cues such as shaping, dotted lines, etc., to indicate depth.

The representation of a three-dimensional object by means of a two-dimensional diagram demands considerable conventionalising which is by no means immediately recognisable by those from non-Western cultures. Two of the Highlands students produced perfectly flat, 2D objects when shown the following diagrams:

(1)





(2)

It is clear that these students were unfamiliar with the oblique convention where the front (square) of the object (cube) is drawn and the rest displaced from it. Only the Manus students and two POM students produced the same objects that Western students would produce, i.e. part of a cube (1) and a triangular prism for (2). Perhaps the best way to indicate the other students' problem is to say that if (1) is part of a cube then the plan view should look like this 
However, if the plan view is that then the front view will not be as shown on the card.

Several students made shapes which had the "correct" front view as on the card but the plan view was either 

or even 

There is of course no information in the diagram which says how long "a" is, merely our visual experience with foreshortening and our practice with the oblique convention.

Other tasks, which involve understanding conventions plus other skills were even harder. It has often been reported that students from non-Western cultures are poor at spatial skills, but often forgotten is the fact that "pictorial" spatial tests invariably involve conventions. We are so familiar with these that we take their knowledge for granted and assume a universality of understanding which is quite erroneous.

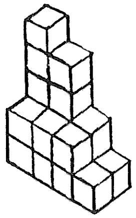
Conventions are of course learnt, as are the reasons for needing them, and the relationship between the pictures and the reality they are conventionalising. The hypothesis is therefore provoked: perhaps much of found difficulty with spatial tasks lies in understanding their conventions, and that if these are known by those people, from both non-Western and Western cultures, who are supposedly weak spatially then perhaps they would not appear to be quite so incapable.

(b) Drawing Several of the tasks required the students to draw and three tasks in particular are illustrative for our purposes here.

In one task, the student was asked to copy the drawings from a specimen set, produced from Plate 1 of Bender (1938).
The drawings use straight and curved lines, dots, closed and open shapes, geometric and irregular shapes.
This task revealed two types of difficulty. Firstly the obvious lack of expertise at drawing and copying. Much erasing, head-scratching and tongue-clicking was in evidence particularly after the student had drawn something and was then comparing his effort with the original.

The other difficulty with this task (and with others) was over the criteria to be satisfied. Once again "copy" implies to us "identical". But "How accurate is accurate?" seemed to be their unasked question. So, scales varies, lines bent, angles varied, and curvatures altered. Of course, if Westerners attempt to draw and copy PNG patterns and designs, they often make similar "mistakes" through ignorance of the criteria to be met. There is nothing obvious nor logical about criteria like these. They must be learnt.

In another task, the student was presented with a small wooden block made from 1 cm wooden cubes. 19 cubes were used and the student viewed the block from across the table - this was his view:



The student was asked to sketch the block as it appeared to him.

In this task, unlike the first, the student must make the decisions of what to include and what to omit and he must imagine what the "ideal" (picture) is that he is trying to reproduce. Several students could not remember ever having been taught how to draw real objects. It was possible to obtain improved drawings by pointing out specific clues like "keep verticals vertical" and "keep parallel lines in the object parallel in the drawing". (These are both adequate hints for drawing small objects.) Advising the student to close one eye helped also, emphasizing that we take the photographer's one-eyed view of the world very much for granted. We don't realise how much it conditions us in our drawings.

Map drawing, whilst by no means appearing simple, did seem to be a a more familiar task. The students could have learned this at school of course or perhaps they find this a more natural and sensible use of visual representation than the drawing of objects from strange angles. They certainly seemed more at ease with the mapping task, and later, when I was asking them about their village gardens or fishing areas they would enthusiastically, and sometimes spontaneously, draw me a sketch map with many details included. I found that the maps the students drew were in the main adequate for the communication purposes they were meant to serve. They were as accurate as they needed to be.

These tasks, then, point to the skills of drawing, and to the criteria to be satisfied, particularly to the recognition of the purpose to be fulfilled by the drawing, by which accuracy is judged. This seems to me to be one of the most important values of drawing - that by doing it one learns about drawing and one is enabled to read other people's drawings.

You can only read this text because you know the conventions employed. In schools reading and writing are usually taught concurrently and the whole complex procedure of forming the letters, writing words, keeping to the line, writing from left to right, leaving certain spaces, etc., is learned by having to be a "user" of conventions, by being a writer, not just a reader.

(c) Visualising This ability is, for me, right at the heart of any spatial work, and I was interested to see the quality of visualising in the students I was working with. Reports of other research (Philp and Kelly 1974) suggested that 'ikonic processing' was likely to be the predominantly used cognitive strategy. Other studies (Lean 1975) suggested that students were weak spatially, largely on the basis of group spatial testing. My first impressions were toward the latter view, but as the work progressed and I understood the difficulties more clearly, particularly those relating to conventions, the following conjecture seemed more likely:

When the object is well known and the convention used in representing it is a familiar one, then imagining and visualising with regard to that representation is well done.

One task which showed this was the sub-test "Matchbox Corners" from Spatial Test 2, produced by the National Foundation for Educational Research. A matchbox was drawn with dotted "hidden lines"and a black dot placed on one corner. Four drawings of the matchbox, rotated in 3D space, were then presented and the student was asked to draw a black dot on the corresponding "same" corner of each. Five different sets were used and the task was presented here untimed.

Very few errors were made and yet the task is known to involve a high degree of spatial ability.

Another task which illustrates their strength is the sub-test "Word Recognition" from the Multi-Aptitude Test, Psychological Corporation, U.S.A. 18 typed English words were presented in varying degrees of obliteration and the student was asked to write the original word. Despite the fact that English was each student's second or third language, they did remarkably well at this task. By contrast, a line diagram counterpart of the previous task was very difficult for the student. It was produced using drawings from Kennedy and Ross (1975). The drawings showed the outlines of 'familiar' (to these students) objects, e.g. people, animals, birds, aeroplanes, house, car. Two forms were presented, one with approximately 80% omitted, the other with approximately 40% omitted. The student was asked what the diagrams showed originally.

The behaviour change from the previous task was fascinating to watch. Whereas for the word completion the students often "drew" letters with their fingers to help them imagine the word, they did not do this with the incomplete line drawings. They merely looked, turned the paper round occasionally and guessed very hesitantly.

Clearly, even if the "objects" were known to them the representations of them were not. Again, the contrast with the word-completion was marked - they had been taught the written representation of English words for several years at school, but not the drawings.

Finally in this section a task which illustrates the strong link (for these students, at least) between visual memory and visualising.

12 small everyday objects (e.g. coin, key, pin, etc.) were set out on a 3 x 4 rectangular board. The student was given 45 secs to look at the arrangement, the objects were then tipped of the board and the student was asked to replace them correctly. Only one student made any error. He had two adjacent objects wrong, and was suffering from malaria at the time!

There was concerned attention given to this task by all the students who in most cases replaced the materials carefully and deliberately. The typical Westerner attempts this quickly, before the memory fades and it was, therefore, interesting to see how long the memory stayed with these students. Certain students were presented with the objects again a day later and were successful at replacing them, a week later (one student 10 out of 12) and the same student 2 weeks after the initial viewing (all correct - he had, therefore, corrected his mistake of a week previously!) This delayed request was clearly not unreasonable, and most of the students attempted the task as if they were confident of success.

Other stimuli were used, with varying degrees of success. Feathers and playing cards were the two most difficult stimuli, and it was clear that no students were using a verbal coding. "I just remember how it looks" was a typical comment. Even some of the Islander students who knew some of the shells by name, didn't name them for this task. They were quite surprised when I suggested that some people might remember the location of a shell by using its name. The colour, the shape, the texture and the size, all were used, but not the name. As one student pointed out, some of the shells had the same name so that wouldn't help - the fact that they almost looked the same didn't worry him!

This last point is important, and supports the reports of other researches concerning "ikonic processing". In several of my tasks which could have used it, there was no verbal mediation used by the students. Very little was said at all in fact, unless it was in answer to a question or because the task sought an oral response. Much looking (pointedly), head turning, paper turning, and moving backwards and forwards (as if to focus properly) was in evidence - all suggestive of a "behavioural support system" for visual strategies. No words though.

(d) Language The problems caused by local languages which are not designed for mathematical and scientific use are becoming well known. One task which will illustrate part of the difficulty is this one. The student was asked to translate a list of 70 English words into his own local language. Many languages have some local equivalents but both 'gaps' and 'overlaps' occurred for all students. There are about 750 different languages in Papua New Guinea, several of which can now be written.

Many of the languages appear to have no easy conditional mood - you cannot say "if.......then". The question this provokes is, if it is not said, is it ever thought? Classification does not appear to be hierarchical as for us, e.g. there can exist several words for different shapes but no word for 'shape'. Another researcher (Jones 1974) asked local interpreters to try to translate some mathematics tests into the local language. Many questions were impossible or very difficult. Some examples of the replies are :-

"There is no comparative construction. You cannot say A runs faster than B. Only, A runs fast, B runs slow".

"The local unit of distance is a day's travel, which is not very precise".

"It could be said (that two gardens are equal in area) but it would always be debated".

For comparing the volume of rock with an equal volume of water, "This kind of comparison doesn't exist, there being no reason for it", and hence you cannot say it!

It is not, of course, merely a matter of teaching the language, because spoken language is only an observable result of some unobservable thinking. Differences in language imply differences in thought. So, if you ask 'deeper' questions as I did of a local anthropologist a different order of difference becomes recognisable. As she said (in a letter to me) :-

"Paiela (a Highland group) space has some unique properties:

i) it is not a container whose contents are objects. It is a necessary dimension of the objects themselves.

ii) coordinate space is two-dimensional. (Here she means things like up or down, left or right, above or below, here or there.) Because it is meaningless to locate difference among three things, three-dimensionality becomes a logical impossibility.

iii) Space is in no sense objective but is a conception, a product of the mind's evaluation of sensory data."

Amongst other things it means that size (for them) would be like value (for us). Value is seen in comparison. Hence she says of pig-exchanges, "so long as the actual pig has not yet been produced, it is impossible to know its size. Once the pig is actually given, and once it is actually placed in proximity to other pigs it is possible to evaluate it large or small... The uncompared pig is attributeless or 'unknown' while the compared pig has at least one attribute that can be 'known'."

With another group, the Kamano-Kafe, in the Eastern Highlands the four "units" of length are 'long', 'like-long', 'like-short', 'short'. Similar adjectival rather than invariant units are also reported from other areas.

So our conceptions of space with its items of objective measurement are not universal, nor are they "natural", "obvious", or "intuitive". They are shaped by our culture. They are taught, they are learnt.

(e) Cognitive Characteristics I have only talked of spatial ideas here because of my particular interest in them, but the statement above need not only apply just to 'space'. Our conceptions of our environment, in particular the mathematical way we view the world, is not a universal view. A colleague in Papua New Guinea has just completed an analysis of 150 different counting systems. How many different counting systems do you know? (not number systems, but counting systems). There are apparently many ways of counting.

So, in general, what were the cognitive characteristics of the students I was working with? The most striking point for me was their concern with the specific as opposed to the general. Their languages have many specific terms, few general ones. The taxonomies used in their cultures have few hierarchies. Generalising is not the obvious mode of operating there as it appears to us to be here. There not only seems to be a difficulty with doing it, there is felt to be no need to do it.

I asked a student "How do you find the area of this (rectangular) piece of paper?" "Multiply the length by the width" "You have gardens in your village. How do your people judge the area of their gardens?" "By adding the length and the width" "Is that difficult to understand?" "No, at home we add, at school we multiply." "But they both refer to area." "Yes, but one is about the area of a piece of paper and the other is about a garden." I drew two (rectangular) gardens on the paper, one bigger than the other. "If these were two gardens which would you rather have?" "It depends on many things, I cannot say. The soil, the shade...."

It is possible, and usual in their case, to hold many ideas, which we would consider in conflict. For example one student had five different versions of how the world began: two from missionaries, two from different village stories and one from his science teacher. He felt no need to reconcile these accounts.

When this type of thinking operates it seems that many of the teaching strategies which I know about become meaningless. The use of analogy, the use of counter-examples, strategies which are designed to foster understanding, or the "meaning" of general principles. All of these assume the acceptance of generalising, abstracting, hierarchical and symbolic processing, as important and worthwhile ways to behave.

I wonder whether we could make a better job of teaching mathematics if we concentrated less on teaching children to generalise (for example) and concentrated more on why generalising is an important and worthwhile way to behave. To do something is different from choosing to do something.

Conclusions

I am not suggesting that the children I study and teach in the UK are like Papua New Guinea native university students. Nor of course am I implying that one must treat these PNG students like children.

What I am suggesting is that:

there is more than one way of viewing the world,
the mathematician's view is a particular one,
it is not an obvious one,
it is shaped by a particular culture,
it assumes many cultural "supports",
and increasing our own awareness of these cultural supports will improve the ways we introduce learners to the mathematician's world.

Bibliography

Bender, L. A visual motor Gestalt test and its clinical use. American Orthopsychiatric Association, New York 1938.

Bishop A.J. Spatial abilities in a Papua New Guinea context. A report to the Department of Education, Konedobu, Papua New Guinea. January 1978.

Bishop A.J. On developing spatial abilities. A report to the Mathematics Education Centre, Lae, Papua New Guinea. November 1977.

Deregowski, J.B. Teaching African children pictorial depth perception: in search of a method. Perception, 1974, 3, 309-312.

Jones, J. Quantitative concepts, vernacular and education in Papua New Guinea. Educational Research Unit Report 12, University of Papua New Guinea, 1974.

Kennedy, J.M. and Ross, A.S. Outline picture perception by the Songe of Papua. Perception, 1975, 4, 391-406.

Lean, G. An investigation of Spatial Ability among Papua New Guinea students, in Progress Report 1975, Mathematics Learning Project, University of Technology, Lae, Papua New Guinea.

Philp, H. and Kelly, M. Product and process in cognitive development: some comparative data on the performance of school age children in different cultures. British J. of Ed. Psych., 1974, 44, 248-265.

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Leen Streefland

Young children (6-8) - Ratio and proportion

0. Introduction.

In this report on young children's comprehension of ratio and proportion we start with two observations of one boy and with a number of observations in a classroom situation. This is why in the title we first mention the children and then the subject matter.

The observational matter is described with a view to didactical-phenomenological analysis of ratio and proportion, combined with a psychological analysis of the children's behaviour of approach. This is followed by a description of methods of mathematical instruction in this field developed for children of the same age.

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1. Observations.

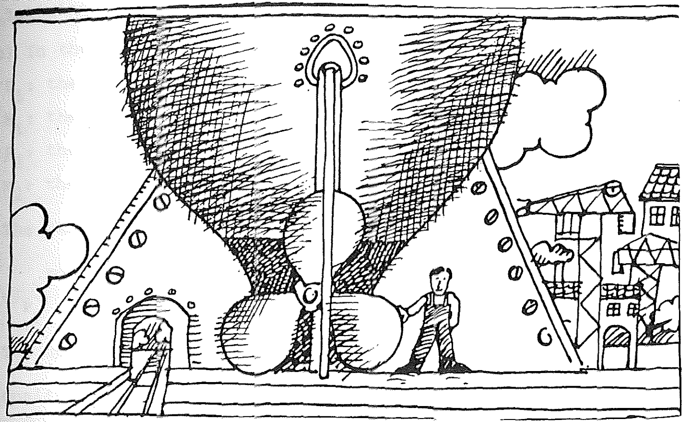
1.1 The ship propeller.

Coen (7;4) wants to know how the propeller of a ship works. We discuss it in many details, viewing

the small electric engine of his boat,
the picture of a sea-going tug on the wall of his room.

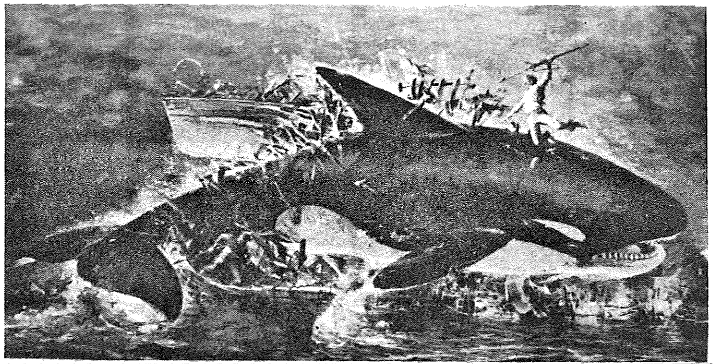
At the end he asks how big is the propeller of a large ship.
His father tells him it would not fit into his room (which is about 3 by 4 meters).

After a moment of silence he jumps to his feet, saying: "It is true. In my book on energy there is a propeller like this (a distance of about 3 cm between his thumb and forefinger) with a little man like that (about 1 cm).



1.2 Orca, the killer whale.

Coen (8;0) and his father pass by a cinema where according to the posters they expect the film "Orca, the killer whale". The spectacular picture shows a little man who on the back of the terrifying, smashing monster, tries to harpoon the animal:



For the sake of sensation the size of the killer whale as compared with that of the man is exaggerated. Coen and his dad look at the picture:

> *What is wrong with it?*

the father asks. The boy replies:

> *That the whale smashes the boat to pieces.*

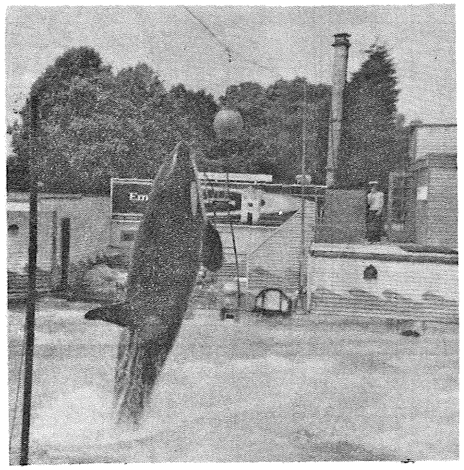
His father does not go further into the matter, although Coen's remark might have aimed at some wrong proportion. The father says:

> *It has something to do with the size of the objects.*

Coen (after a moment of silence):

> *I know what you mean. That whale should be smaller.*
> *When we were in England we saw an orca and it*
> *was only as tall as three men.*

(Here the boy is alluding to an experience of 21 months previously (cf. figure): dolphinarium Windsor Park, England)



Four days later, after just another visit to a dolphinarium with a killer whale (albeit of only a good three meters):

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he explained why he knew the orca in England was as tall as three men:

> *I remembered the orca, jumping out of the water and touching the orange ball.*

(He had not seen the photograph for at least one year; by the way it is a misleading one, because of the different distances of orca and trainer.) After this experience Coen emphatically wished to change 'the three men' into 'three to four men', which indeed corresponds better with the length of both killer whales.

2. Analysis of the observations.

2.1 *Mathematical-didactical.* *)

Both observations involved four magnitudes (or rather rough magnitude values), which were pairwise compared.

(a) In the first example these were

1) - P_1: the propeller of a big ship (bigger than the boy's room),
   M_1: the father,
   P_2: the picture of the propeller of a big ship,
   M_2: the picture of a man beside P_2.

2) - The pairwise comparison can be described by

$$\begin{array}{ccc} \text{room} & & \text{book on energy} \\ P_1 : M_1 & = & P_2 : M_2 \end{array}$$

The comparison is expressed in a qualitative way.

3) - Qualitative comparison and statement of the equality of ratios led the boy to accept his father's suggestion about the size of the propeller: "It is true..."

(b) The second example also involved four magnitudes (or rather rough values of magnitude), pairwise compared:

1) - O_1: the killer whale on the poster,
   M_1: the man on the back of the animal,
   and
   O_2: the orca in Windsor Park,
   M_2: a man.

2) - Now the pairwise comparison, on a qualitative level, may be described by a disproportion

$$\begin{array}{ccc} \text{poster} & & \text{Windsor Park} \\ O_1 : M_1 & \neq & O_2 : M_2 \end{array}$$

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*) cf. [4].

3) - There is, however, a new element involved, since the boy defines a numerical ratio for one of the pairs in order to make the comparison of the pairs easier - "that orca was only as tall as three men".

When comparing the picture on the poster with his experience of two years previously, the boy accepted, as it were axiomatically, a certain ratio orca : man, to wit, an orca = three men. This explains the emphasis with which, after his next experience (Harderwijk), he changed the ratio orca : man into three to four men.

There is a striking resemblance in the boy's approach in both observations. In both cases ratio comes up in an equivalence relation. In both examples the equivalence or non-equivalence of the underlying pairs of rough magnitude values is decided on at a qualitative level, though in the case of the orca a numerical ratio plays a mediating role.

2.2 *Psychological (ratios in cases of similarity).*

Both observations prove that the boy knows about ratios. The way of expressing why in the one case there is equivalence and in the other there is not, shows that similarity as operational equivalence *) is the background pattern. Indeed, these are situations where the boy's criterion is similarity or dissimilarity.

Freudenthal says about this ability: I go even as far as saying that congruences and similarities are built into the part of our central nervous system that processes our visual perceptions. The speed of identification of an object after the object itself or the observer has been rotated, or after its distance from the observer has been changed, presupposes, as it were, a computer programme in the brain which eliminates this kind of transformation. While I do not understand at all what such a programme looks like, its mere existence - which I do not doubt - is an enigma to me.

Other authors make similar statements:

"The main point is, that we have here an elaborate and refined system for coding contour elements which is present in its main essentials at birth and must therefore be 'built in' as a major feature of the visual system." **)

"(...) it seems very likely, that young children can take in and remember size ratios (...)" ***)

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*) Terminology of H.Freudenthal in the last version of his Didactical phenomenology of ratio and proportion. (Internal IOWO publication)

**) cf. [3], p.182.

***) cf. [1], p.96.

Nevertheless, utterances of 5-8 year-olds seem to be aimed at dealing with similarity as an operative equivalence, which is shown by their understanding of, and reasoning about, ratios. These ratios are always object-related within a situation or a system of situations. The (mental) availability of two or more situations of this kind is the precondition for comparing ratios. Various kinds of objects may serve as measures of comparison.

Primarily the child bases his judgment on the ratios under consideration and their equivalence, not on comparing sizes of the same object in different situations but rather on instances of the same situation pictured on different scales.

The child seems to need situations each of which is characterised by the presence of well distinguished objects, in order to come to terms with ratios.

Both in the case of the propeller and the killer whale this phenomenon can be observed:

	book on energy		room
a)	$P_1 : M_1$	$=$	P_2 (= room) : M_2(= father)
	poster		Windsor Park
b)	O_1(rca) : M_1(an)	$\neq$	O_2(rca) : M_2(an)

In his didactical phenomenology of the concept of ratio *) Freudenthal distinguishes internal and external ratios: ratios within a magnitude (or the system of values assumed by this magnitude) on the one hand, and between magnitudes (or the belonging value systems) on the other.

Our examples seem to indicate that the external ratios possess a psychological counterpart. Let us study this question more closely.

The distinction of internal and external ratio has a particular meaning in physics where ratios of magnitudes written as quotients or fractions result in numbers on the one hand, and - very often - in new composite magnitudes on the other.

Psychologically this distinction is also relevant to the process of constitution - or should we say making conscious - of ratio, where internal ratio seems to be preceded by the external one. Here at this stage of development external ratio should be understood as follows: ratio between two or more well distinguished objects in a specified situation which itself is part of a system of such situations.

From a mathematical point of view such ratios can be considered as internal ones because in almost all cases only one kind of magnitude is involved - in the present case, length.

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*) p. 164

However, on viewing magnitude as a physical object or a system of objects of the same kind (including its pictures), then the qualification of these ratios as external ones can easily be justified.

In particular, mapping by similarity appears to be a forceful means of producing and comparing this kind of ratio. Here we mean similarity in a broad sense: producing object related similar situations where the objects themselves even might be displaced with respect to each other - a broader meaning than the usual one of similarities as mappings of the plane or the space.

A further psychological analysis of both observations shows the following details:

(a) Propeller.

The first pair in room book on energy

$$P_1 : M_1 \quad = \quad P_2 : M_2$$

was not explicitly given. The boy had to construct it from the context. Probably this happened after he had produced from memory the second pair, which was neither physically nor visually present.

It appears from the boy's utterance that the mental construction of the similarity between both situations then led him to state a balanced proportion.

The boy's mental activity was aiming at embedding the size of the ship propeller in the field of his experiences on ships. It was not that he doubted the trut of his father's statement.

(b) In the second situation the boy could use the height of a man as a (rough) standard to measure the length of an orca.

Again he had to draw upon his memory to compare the available pair (orca and man in the poster) with the mental pair (orca and man in Windsor Park).

Here the mental activity was initiated through the problem posed by the poster. This led him to state inequivalence, in other words the conclusion that the orca on the poster was too large in proportion: "The whale should be smaller".

3. Conclusion.

Within the domain of congruence and similarity, as described in the preceding sections, 6-8 year-olds understand the meaning of ratio and proportion. This is witnessed by the way they deal with problems in this domain.

As a consequence, teaching this aspect of mathematics should not be directed exclusively at formalising the concept. It is important to make the children conscious of the way they reason about ratios and their characteristics.

A method which can contribute to this aim, is creating situations which are both surprising and conflict provoking, while initial interpretations and judgments are subjected to criticism and correction. In the next sections a few examples from teaching practice will be considered more closely, within and beyond the theoretical frame which has been set up above.

4. Activities concerning ratio and proportion in the lower grades of primary education.

4.1 Introduction.

In the following sections we shall discuss some classroom activities on ratio (and proportion) for 6-8 year-olds.

In a way these classroom activities can be characterised by the term 'reappraisal of the direct perception' in order to focus on ratios in the perceptive field or 'perceptive accentuation of ratio' as Van Parreren calls it. (cf.(6.11)).

It is not only of psychological, but also of mathematical-didactical interest to observe the way in which this 'perceptive emphasis on ratio' might take place and especially 'why' children are capable of this. Moreover it will be of interest to see whether, or in which aspect, the classroom experiences will show the role of internal and external ratios as described on pag. 5.

The examples will display the complicated learning process, which is, on the one hand 'steered by the field' and on the other hand 'steered intentionally', that is to say "by the child's own plans or by those taken from other children", (cf.(6.12)), thanks to the inquiring character of the instruction. The teacher tries to confront the children with surprising problem situations in which the children are motivated to feed back their reactions into their 'intentional steering'.

4.2 Liz Thumb

(maybe a relative of Lewis Caroll's Alice in Wonderland).

The teacher tells the story of Liz Thumb, the girl that once upon a time became as small as a thumb:

Liz Thumb shakes hands with a mouse

Liz Thumb is pictured on a worksheet. The pupils are asked to draw a flower, a stone and one of their own shoes at Liz Thumb's side. The results will show - according to the classroom experiences - quite a lot of wrong proportions, which lead to an interesting discussion:

> *Liz Thumb is not as tall as my thumb ; she is smaller;*
> *look here,*

one of the pupils remarks and he measures Liz Thumb on the sheet by means of his thumb. Apparently she has not been drawn full-sized so two scales are involved, namely the real Liz Thumb and her picture on the worksheet. This is the first spontaneous observation on ratios. But, there will be more problems demanding an explanation. Quite a number of children draw flowers or shoes, which are far too small.

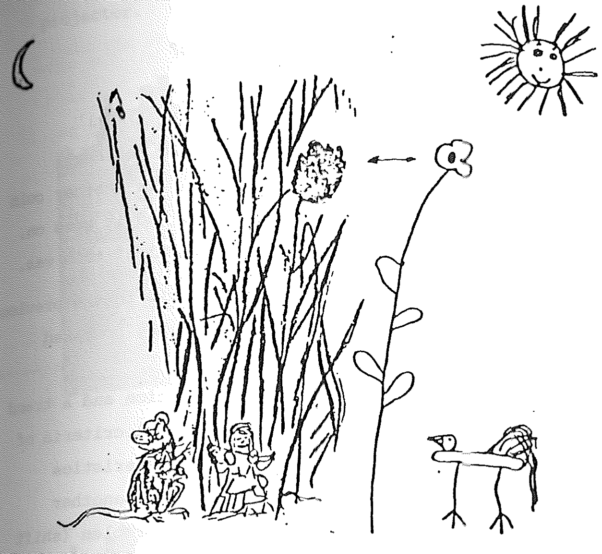
Note: They were supposed to draw one of their own shoes.

By means of a sequence of questions the teacher tries to lead the children:

> *How long is the grass?*
> *How tall will the flower be?*
> *Will the mouse fit into the shoe you have drawn?*

By comparing the drawn objects pair-wise (grass and flower; shoe and mouse) the children are surprised to discover that their previous ideas were not correct.

Sometimes they maintain their opinion concerning the size of objects they have drawn:



> *That sun of yours is too small,* the teacher objects.
> *Uhh ... that is why it is far away anyhow,* is an answer.

Later on in the schoolyear (cf.section 4.5) the influence of perspective on ratios will be taken into account.

4.3 Overhead-projector.

After Liz Thumb an overhead-projector trick is played. The teacher asks the children to close their eyes. She puts a cuisenaire rod (number two) on the projector table. The children are asked to open their eyes. They can only observe the black shadow on the screen. Then the teacher raises an impossible question:

> *What is the colour of this rod?*

One pupil reacts:

> *I don't know. You will have to measure it.*

The image (not the bar itself) is now measured by means of a green rod:

> *It fits, so the rod on the projector table is green, isn't it?*

But the children ask:

> *That green rod should be put on the projector table*

to enable them to compare both images and after that to make a better choice.

In the same lesson the teacher places a coin on the projector table. The screen shows a black circle:

A little round, a counter, the pupils mean.

This is a coin, the teacher tells, *which one is it?*

One guilder, because it is so big.

No, the teacher says, *this one is a guilder,*

and she puts one on the projector table. The pupils then make another guess about the first coin. It is wrong again. The teacher puts another coin on the projector table, namely the one of the pupils' guess and it goes on. By comparing the images the pupils determine what kind the first coin was.

So it is obvious that the children only used external ratio in the previous examples, in the sense that rather than comparing objects with images they compared objects with each other and images with each other.

Moreover, one can recognize a certain shift of interpretation and a trend of abstraction: The children had first to abandon the natural criteria of monetary value and colour of sticks, which are just the characteristics asked for by the teacher. The children were compelled to take another criterion - ratio - into consideration in order to solve the problem (shift of interpretation). On the other hand replacing monetary value and colour by ratio is a special case of abstracting.

The mathematical gist of the previous classroom situation is a mental construction based on visual data. In the course of instruction you see the children develop a strategy by trial and error which leads to a solution.

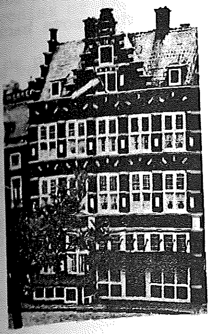
Probably the children are unable to use the scale factor of enlargement of the overhead-projector.

Though they knew the images were bigger than the real objects, they only compared the images. Ratios have been treated by the children in the external way. But how should one organize experiences concerning the internal ratio of an object and its image.

To do this we have the children looking at photographs.

4.4 Reality-turn.

The teacher shows a part of a photograph on the screen of an overhead-projector.



The children are asked to indicate their own height on the picture. First they compare themselves with the classroom-door in order to relate their apparent size to the door on the picture.

Notice that the children based their solution of the problem on external ratios.

It is a big surprise when the remainder of the picture is uncovered, to see a little girl standing next to the house as tall as the height of the house (cf. 7,259-266)):

"I got it, it's a doll's house!" one of the pupils shouts.

First it was a real house with a real door. This view defined the choice of the ratios.

But when the true proportions became clear, the pupils switched to the opinion that the house was a doll's house with a correspondingly little door.

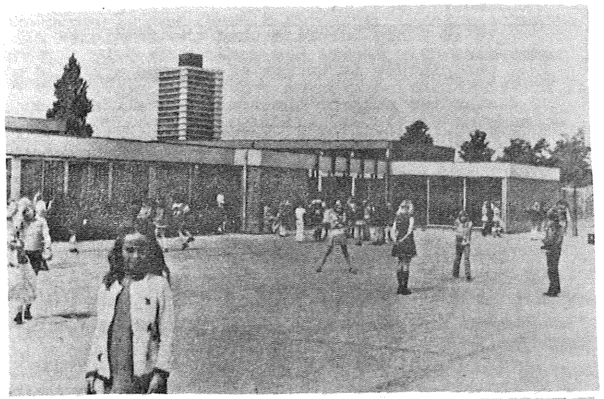
So the children discovered that the internal ratio between the classroom-door and the door on the picture was not a given constant.

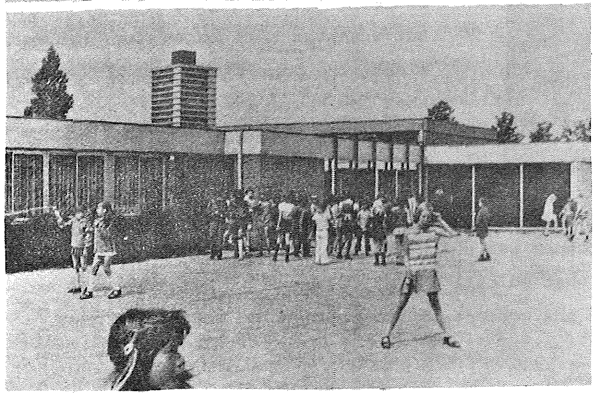
They discovered it by what we call a "reality-turn". Reality-turns are a good preparation for the concept of scale. They can come up in many contexts.

4.5 Perspective.

Another trick that disorders the internal ratios of an object and its image, is perspective.

At school all the pupils will receive a "camera" (that is to say the cover of a matchbox) in order to make a picture of the teacher. They move back, because they want a picture of the teacher from top to toe. By doing so they discover that the cover can also be turned.
The pupils learn to interprete ratios from the point of view of a photographer ..... a little bit to the left, going backwards etc.
Ratio will then have been embedded in the setting of perception.
The pupils then have a look at some photographs of the school:





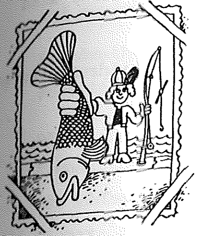
Where did the photographer stand when he took these pictures of the school?

Which classroom is this? If you were on the picture, what would be your size then?

Which of the photographs has been taken from a short distance?
How do you see that?

One pupil reacted to the last question by pointing out the appartment-building, which rises high above the school. He says: the appartment-building will become smaller as you come closer to the school, because the school will get bigger then. After that he counts the little balconies of the appartment-building to be sure.

This early concept of perspective in young children can be used to disorder internal ratio: besides the normal pictures of the school we look at some "mad" photographs.

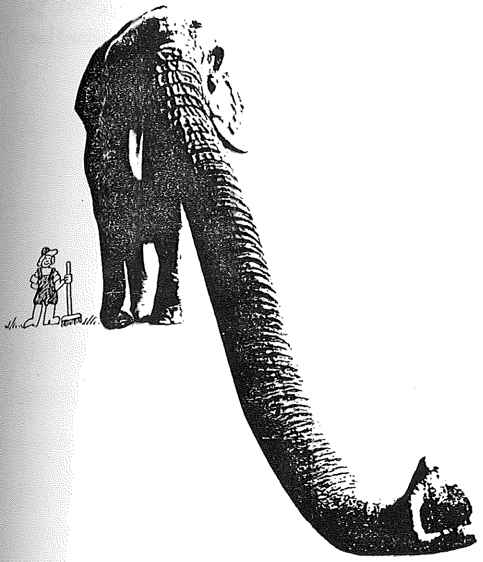


Peter says:
"The fish I caught is twice as tall as me, isn't it?"

A pupil replied:
"Peter is a liar! He cannot carry a fish like this, or can he?"

The child's own experience in fishing is important.

Perspective rather than experience was the clue when the children looked at the picture of the elephant.



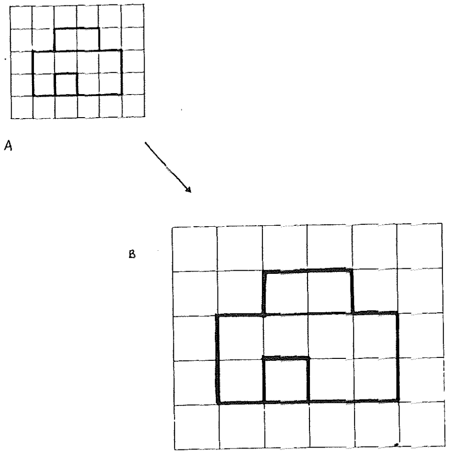
"The elephant is that big, his keeper fits into its trunk".
"How is it possible?"

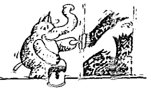
4.6 *In conclusion.*

By means of perspective and reality-turn the children grasp the non-invariance of the internal ratio.

By an open approach, favouring intentional steering, we found a way to develop *all* instructional and research aspects of ratio in agreement with the child's mental development.

This statement is emphasized by the seemingly wrong answers children give to the question on the next worksheet





The shape has been enlarged linearly by a factor 2.
What is the factor of enlargment of the area?

Many pupils answered the area has remained unchanged.
"The grid has grown" they reasoned, or "we have come closer to the house".

In our opinion these "wrong" answers prove the necessity to search for activities which embody all aspects of ratio and proportion and to make them conscious to the pupils.

References.

(0) Brink, F.J.van den, *Strokenaanpak I & II* (An approach of ratios by bars, I,II) (a mathematical-didactical internal IOWO study), IOWO, Utrecht, 1975.

(1) Bryant, P.: *Perception and understanding in young children*, London,1974.

(2) Desjardins, M. and J.C.Hétu: *L'activité mathématique dans l'enseignement des fractions*, Quebec,Canada 1974.

(3) Dudwell,P.: *Children's perception and their understanding of geometrical ideas*, in: "Piagetian cognitive development research and mathematical education". National Council of Teachers of Mathematics 1971.

(4) For an extended and thorough didactical-phenomenological analysis of ratio and proportion, you are referred to the last sections of:

Freudenthal,H.: *Weeding and Sowing - Preface to a science of mathematical education*, Dordrecht,Holland; Boston,USA 1978

or:

Freudenthal,H.: *Lernzielfindung im Mathematikunterricht*, in: Zeitschrift für Pädagogik, Jahrgang 20,Heft 5, Oktober 1974, p.719-739.

(5) Piaget, J.: *Understanding causality*, New York 1971 (especially Ch.12: *Linearity, proportionality and distributivity*).

(6) Parreren,C.F.van: *Niveaus in de ontwikkeling van het abstraheren*, Utrecht 1978.

(7) *Five years IOWO - on H.Freudenthal's retirement from the directorship of IOWO*, Educational Studies in Mathematics, volume 7, n.r 3, Dordrecht, Holland; Boston USA, august 1976.

Elmar Cohors-Fresenborg

Summary

Learning problem solving by developing automata networks.

This is a report on empirical studies on the employment of the automata building-kit "Dynamical labyrinth" in a mathematics course for 10-11 years olds. The kit makes possible by actions the forming of concepts which promote the understanding of automation and programming. The success of the pupils on this course was compared with the results of various intelligence-variables and personality-variables subtests of a school-performance test. On the whole, success in this problemorientatated course correlated well with the results of the results of the intelligence subtests, and slight with the personality-variables subtests. However, a cluster analysis lent significance in part to the personality-variables tests for pupils of average intelligence. On the other hand, sex difference was of no account.

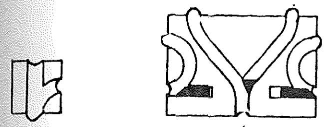
Il s'agit d'observations empiriques concernant l'utilisation des jeux de constructions méchaniques appelés "labyrinths dynamiques" durant les cours de mathématique donnés à des élèves de 10-11 ans. Les jeux de constructions permettent grâce à un procédé concret. l'assemblage de données propres à la comprehension de l'automatisation et de la programmation. Le succès des élèves dans ces séries de cours est comparé aux résultats des differents testes d'intelligence de personnalité et aux aspects personnels du rendement scolaire. Il en résulte une étroite corrélation entre le succès de ces séries de cours concernant la résolution de problèmes et les testes d'intelligence, et par contre un rapport très éloigné aux testes de personnalité dans le groupe d'étude. Il résulte d'après une analyse de cluster que les données moyennes d'intelligence avaient qu'une influence partielle sur les traits de personnalité. Le sexe des individus n'avaient par contre aucune importance.

Learning Problem-Solving by developing networks of automata

1. Introduction

This article reports on the trial of a teaching-course in which children of around 10 years of age are, in a manner of play and with the help of a box of building-bricks (dynamic labyrinth) given insight into the workings of automata and into the development of programmes for computers. With these bricks, networks can be constructed which are comparable with the lines of a model railway as: Each part of the network is a railway-line, on which only one train may run, which itself changes all the points automatically on its run. According to their different uses, the points are represented in the kit by two different bricks; one for use where another rail runs into the main line (junction), and the other for a rail which branches off it. This last is named "points" on the brick. To connect junctions and points there are in the kit straight rails, bends and crossings the same size as the junction. The direction of travel is indicated by tabs on the bricks. All bricks are to be affixed to the appropriate board.

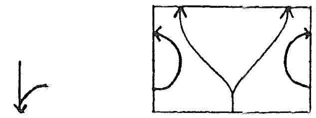
Diagram of bricks



Junction Points

Fig. 1

Symbols for bricks



Junctions Points

Fig. 2

It is possible to construct a network on three conceptual levels: by affixing the bricks (active), by drawing the bricks (iconic), see fig. 1, and by representation of the network by symbols on squared paper (symbolic), see fig. 2. A detailed description of the kit and a suggestion for a course of lessons can be found in COHORS-FRESENBORG 1976 and COHORS-FRESENBORG et al. 1977.

Here a few examples will show how this kit can help to illustrate circuits in machine. The examples are chosen so that at the same time it can be seen how the ability of children to solve problems is encouraged.

2. Plan of lesson course

The course consisted of about 16 lessons. The first few served the purpose of making the pupils familiar with the materials, in particular with the difference between "junction" and "points": the junction has one entrance more than exits and one state; the points have one exit more than entrances an two states. In addition, symbols were worked out for the illustration of automata networks, and the diagrammatic working-out of exercises was practised.

For the organization of the learning-steps and the building of problem-solving strategies, leading to a general "theory" for the solution of certain types of problems, typical is the sequence of exercises taken from the middle of the course, the description of which follows.

Exercise:

To develop a network which describes the switch part of a machine that issues stamps when to 2p pieces are put in. The pupils find that this machine must have one input (the coin slot), two outputs (more money please, issue of stamp), and two states: (ready to operate, 1 x 2p received). The behaviour of the machine is described in the following table:

state	input	output	new state
ready	2p	more coins	2p received
2p reiceved	2p	stamp	ready

Table 1

Network:

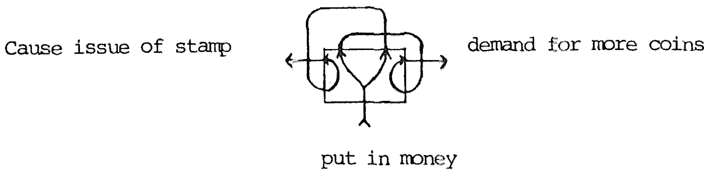


Fig. 3

Following this exercise the pupils are set to constructing a network to describe a machine which does the following:

In a drinks factory beer and cider bottles are on a conveyorbelt in the sequence: B C B C B C ... An automatic sorter must separate the beer from the

cider bottles. The exercise for the pupils is to recognise that this is the same mathematical problem as before, just in different words, i.e. that the automata tables can be interchanged merely by changing the names of the inputs, outputs and states, i.e. they are isomorphic. Only after they have completed this exercise do the pupils learn that there is a brick "flip-flop", fig. 4, and that this can be interpreted as either a 2 x 2p slot-machine or a 2-bottle sorter. The network in fig. 3 and the flip-flop are to be seen as functionally equivalent concepts. As automata they are isomorph.

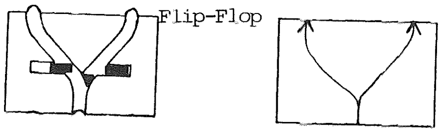


Fig. 4

In the following exercise these two intuitatively different uses are further developed: in the first instance to machines with two outputs which issue goods only after several coins have been put in, in the second instance to sorters (with n outputs) which sort n different bottles. In fig. 5 the example is a 3 x 2p slot-machine, in fig. 6 a 3-bottle sorter.

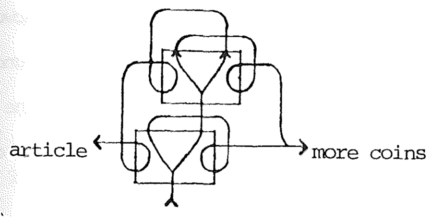


Fig. 5

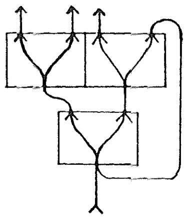


Fig. 6

This time the bottle sorter is constructed entirely of flip-flops. For this construction the concept of "feedback" is needed. The pupils should discover this for themselves and then use it for other related problems.

If numbered bottles are sent through a bottle sorter, it can be seen that the same network can be comprehended as a machine showing to which class modulo 3 a number belongs.

For the pupils, the solution of this problem is related to that of "sorting". This is an example that the different description of a problem leads to a conceptual differentation which cloaks functional equivalents. In this way lines of association are engendered which arrange the exercises to different problem areas although the mathematical automaton is the same. Besides these the pupils are given problems which are motivated by the situation of application and described in the language of application, the pupils are given problems which are directly connected with automata networks: a network is given in a table, and this must be built with the bricks (Synthesis problem): For example see table 2.

Here the network from fig. 5 is described in a table which clearly indicates its functions (r: right, l: left).

state		input	output	next state	
W1	W2			W1	W2
l	l	I	more coins	r	l
l	r	I	more coins	r	r
r	r	I	article	l	l

Table 2

In addition to this there is the opposite type of exercise: a network is given on squared paper and the table to describe it as automaton be found (analysis problem). Here the difficulty is in recognising the changes of the points and flip-flops (especially by feedback, e.g. fig.6) in the static diagrammatic drawing.

3. Empirical studies

The course of lessons here briefly described was tried out on about 540 pupils of 10-11 years in two large schools during the school year 1976-77. All pupils were taught according to the same plan of lessons. They had to solve certain problems either in pairs with the bricks or on their own on squared paper. In the discussion of the answers afterwards, especial value was laid on the following: the pupils should recognise that the solution of the given problem was possible by developing the answers to problems they had already solved.

e.g. 3 x 2p machine as development of 2 x 2p machine) or by appropiate interpretation of the problem and consequently a change of terms and presentation (e.g. slot-machine/bottle sorter/class-counter). What was imporant was recognition of the unique features of a concept which are invariable when the language and presentation are changed.

As the course material sketched here is strongly differentiated from the normal content of an arithmetic course, it was in our own interest to compare the performance of the pupils who took part, with that of others and to take into consideration the variables of personality. To this purpose a combined school-performance test KLPI (ROLLETT/BARTRAM 1973) and AVT (ROLLETT/BARTRAM 1977), was set at the beginning of the course, which is related to verbal and mathematical intelligence-variables and some personality-variables.

The individual subtests were correlated with two tests, given here as T1 and T2, on the material of the course, set after 10 and 16 lessons respectively.

The following are short reports of some of the most interesting of the results.

1. Contrary to expectations (based on the usual sex-specific differences recorded in performance in mathematics, e.g. STARK 1971, VIET 1977, or because of differences in approach to buildingblocks and varied experience of them), there was found to be no difference in the averages of T1 and T2 between boys and girls, either in the random sample of the whole or in the performance quartiles.

2. T1 and T2 correlate well with the total value of the intelligence variables of KLPI, in particular with that of subtests "word understandig" (SV), "numbersequence completion" (RE), "understanding and following instructions" (VBA), but less well with arithmetical facility and geometry subtests. Surprisingly slight was the correlation of the random sample with "hope of success, (HE), "fear of failure" (FM), "avoidance of exertion, (AV), and "zeal", (PE).

3. After all the subtests of KLPI and AVT a cluster-analysis (see STEINHAUSEN/LANGER 1977) was worked out. Hereby a division into six clusters was found to be useful. For these six clusters a distribution of the test results from T1 and T2 was calculated. This yielded results as follows: 2 clusters were defined respectively by high and low values in the intelligence variables, the performance on the course was corresponding. 2 clusters with average intelligence quotient were clearly differentiated by the results of AV and FM. In the cluster with low results in AV and FM, few pupils showed low performance in T1 and T2 and more pupils in the upper third of the sample improved between T1 and T2. Both of the other clusters with just below average IQ performance differently in two geometry subtests and in their working-pace. Here the cluster with the higher results in the geometry subtests and slower pace was better than the other in all quartiles of T1 and T2.
   To summarize, it can be said of the four personality subtests that these had the strongest influence on performance in the course where the pupils were of average and lower intelligence.
4. Individual observations made by school psychologists with this material in another school lead to the assumption that through the training with this kit the readiness to and capability of trying out and organizing in the solving of problems was promoted. This was also shown by an improvement in the performance and willingness in the mathematics course which followed. In addition to which, a second trial of the subtests SV, RE and VBA after our course in one of the schools resulted in an improvement in all quartiles which was better than was to be expected from a mere repetion of a test. Experiments in this direction are intended.

4. Summary

This kit makes possible by actions the construction of concepts which are suited to the understanding of automation. By the generation of symbols through actions the likehood of remembering them is better and therefore also the ability to use them as the basis of further methodical deliberations concerning other network constructions. If we differentiate between levels of representation (p.e. enactive, iconic, symbolic in the sense of G. BRUNER)

it is impossible to decide to which level a given network of building-blocks belongs if we do not know the nature of the task. If the network shown in fig. 3 arose from the intention to construct the connection in a stamp-machine, then it can be seen to be a symbolic representation of the machine in spite of the fact, that it is made from building-blocks. But if it is a realisation of table 2 it should be attributed to the enactive level, and the drawing of the same net to the iconic level.

The interdependence and peneration of these three levels furthers the acquisition and structuralization of concepts.

As the paths of the networks constructed with building-bricks are actively traced further research may indicate that the described instruments are suitable for nonverbal acquisition of concepts. So it could be helpful for the instruction of deaf-mutes, blind people or children of immigrants.

References

Cohors-Fresenborg, E.	Dynamische Laybrinthe in: Didaktik der Mathematik, Heft 1, 1976
Cohors-Fresenborg, E. Finke, D. Schütte, S.	Dynamische Labyrinthe, Unterrichtsprogramm Osnabrück 1977
Rollett, B. Bartram	KLPI-Schulleistungstest, Universität Osnabrück 1973
Rollett, B. Bartram	AVT Anstrengungsvermeidungstest, Verlag Westermann, Braunschweig 1977
Stark, G:	Bringen Leistungskurse "Chancengleichheit"? Eine empirische Querschnittsuntersuchung mit Hilfe von lernzielübergreifenden Tests. Deutsches Institut für Internationale Pädagogische Forschung, Frankfurt 1971
Steinhausen, D. Langer	Clusteranalyse, Berlin 1977
Viet, U.	TOR: Test für operatives Rechnen, Beltz-Verlag Weinheim 1977

The box of building-brix "Dynamische Labyrinthe" is produced by Beschützende Werkstatt, Industriestr. 7, 4500 Osnabrück-Sutthausen

SELF APPRAISAL IN THE LEARNING OF MATHEMATICS

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When you work on a piece of mathematics, how much checking do you do? The amount probably depends on the routineness of the work, who is going to use it and for what purpose, your character, and feelings of the moment, the time available and so on. In addition to whatever deliberate checking is done, I suggest that you will automatically be on the watch for anything which seems to be in conflict with what you already know and expect. We, teachers and professional mathematicians, take for granted that our reasoning and results can and should be checked with appropriate thoroughness. Experience suggests that many children learning mathematics do not perceive self appraisal to have any place in what they are doing. This attitude, where it exists, is likely to have a pervasive and damaging effect, as I shall try to show by considering four benefits which tend to accrue only when self appraisal is taken to be an essential part of doing mathematics.

1. Anyone having to use mathematics after they have left the classroom must be able to rely on their results if wasteful mistakes are to be avoided. The situation in which answer book and teacher are ever present authorities is not a very good preparation for the tackling of real problems; nor is the notion that 45% correct is a 'pass'. Whilst 100% accuracy is unattainable for most of us, it is vital that we can distinguish what is certainly correct from what is doubtful, and in the case of doubt, seek a second opinion. To learn and to apply independent checks is an aid towards achieving this sort of reliability.

2. At a different level, the making of deliberate checks and being alert for contradictions helps to focus attention on the nature of mathematics, that it is a consistent system, meaningful rather than arbitrary. Moreover it emphasizes that a person learning mathematics is expected to think, to make and to use the connections that exist. In expounding the importance of the child's own perception of his main task in mathematics lessons, Buxton[1] points out that attention is too readily focussed on the judgement of his final output which is made by the teacher. Cultivating his own sense of judgement can be for a child both a symptom and a cause of a more balanced view of the task.

3. There is a sense of satisfaction in the certainty that one is correct, which provides motivation to continue working in the field with maximum concentration. Many teachers advocate getting students to speculate about the result of a problem before working it out in detail, the intention being similar, to stimulate personal involvement with the work and the pleasure of achievement. Failure remains a possibility, but a less damaging one when the learner has opportunity to notice and correct mistakes or to seek help for himself. In an experiment with calculators in primary schools, Bell, Burkhardt, McIntosh and Moore[2] found it significant that children "could use them as a non-threatening check on their own work", and this seemed of particular benefit to the less confident children. Feelings of confidence and satisfaction are powerful internal motivators.

4. For long term retention of what is learned these feelings also have importance. Memories inevitably become overburdened, and competence in the solution of particular types of problem tends to be lost. If mathematics which was learned hours, weeks, or years ago is to be applied, then an ability to reconstruct methods from partial recollections becomes essential. The recall and reconstruction process can be blocked by feelings of inadequacy leading to panic, but supported if a sense of mastery was originally associated with the topic. Self appraisal has some bearing on the sense of mastery. Furthermore an ability to find interrelationships, developed for the purpose of checking, also applies to the reconstruction of forgotten parts. The extra practice and concentrated attention entailed in checking should also make for thoroughness of initial learning and thus ease of recall.

How can the habit of self appraisal be fostered?

Self appraisal is not a battery of checking procedures, although these are obviously useful; it is both an attitude and an arduous mental activity. As such, an environment conducive to the exercise of a student's own judgement is likely to be more influential than a period of concentrated coaching or exhortation taking place in an environment which otherwise exerts contrary pressures. Willingness of the teacher to make every judgement is a contrary pressure, and so is the presentation of work too difficult or too fast to be thoroughly mastered. Both can be hard to avoid in practice, but if their effect is recognised as damaging, it should be possible to make some moves away from teacher dependence and a predominant concern with syllabus coverage towards an emphasis on self reliance.

Actions which positively encourage self appraisal are difficult to isolate because they are embedded in the teacher's personal way of working with the group and with individuals within the ethos of each class. However it is worth discussing some possibilities.

1. Much communication of attitudes and styles of working occurs by example. In this context, the teacher thinking aloud could be the example, perhaps a dramatised dialogue with himself, That seems odd .. .. but if ... it could ... why isn't ... we ought ... . This requires more than a deliberate mistake strategy; it centres on a display of puzzlement rather than a rapid correction. The critical look at the answer, the casting around that occurs when something seems to be wrong, the trial of different methods, can all be demonstrated regularly, including what is ultimately accepted as satisfactory confirming evidence. To see a teacher make errors and then cope with them can be reassuring. It also emphasises the importance of monitoring your own work.

2. There are many useful techniques, such as testing a general result in a particular, simple, case or the solution to an equation by substitution or checking a multiplication by dividing into the product, and more of these could be taught. Those mentioned come into the general category of checks by working back from the answer to see if it fits known data. Buxton[1] suggests going further, "The practice of letting students have the answer is sensible, and using the answers to correct one's thinking should not be regarded as cheating." In some problems, an alternative strategy could be to check by solving in two different ways, or by making a rough estimate to compare with the detailed solution. This requires versatility, but is of especial value in arithmetic where people do devise short cuts and methods more congenial to themselves than the standard pencil-and-paper-algorithms[3]. This sort of flexibility can be recognised and encouraged. Some of the widespread reluctance to check seems to arise from a fear of not knowing what to do should an error be detected. Again flexibility of method must be part of the remedy.

3. Depending on the circumstances, most teachers give marks for method as well as answers, in effect emphasising that having some idea of how to tackle a problem is worth credit even if you cannot carry the solution through accurately to the end. On the same basis, why not give marks for self appraisal? Sound judgement is normally evidenced in a number of ways, including requests for help when an error is suspected, which can be encouraged in class but perhaps not accepted in a test. It is possible to demand some more artificial indications, such as the explicit writing out of checks and a commitment by the candidate to an opinion about his answers, which of them he thinks are definitely right and which uncertain. To be wrong and know it, is at least some progress and worth a mark? There are potential benefits in that what is examined is somehow made legitimate. There are also dangers. Unless all the children have developed some genuine basis for exercising judgement and hopefully begun to enjoy it, the demand for an opinion is pointless, adding excessive strain, and directing attention away from the tasks they can reasonably attempt.

If children have acquired a tacit assumption that one cannot expect to understand mathematics, then self appraisal is a difficult goal. Yet young children have a very positive drive to make sense of everything for themselves. Some of the above actions may help to restore or maintain this drive in the learning of mathematics.

1. L.Buxton. 'What goes on in the mind?' Times Ed. Supp. 7.7.78. pp18,19.
2. Bell et al. A calculator experiment in a primary school. Shell Centre, University of Nottingham, 1978.
3. e.g. McIntosh 'Some Subtractions.' Maths. Teaching, no.83 June 1978

Toward Acculturation between Traditional, Creative, and Technological Approaches to Mathematics Education

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Introduction

There is a growing recognition that teachers and the various kinds of experts who are supposed to be resources for teachers live in rather separate worlds. Goodlad et al. (1974) and Goodlad (1975), for example, write of the "culture of the schools" and the "culture of the university" as quite distinct entities. Paulo Freire (1973) writes of the gnosiology of extension, challenging the model of the agricultural extension agent which educational specialists have often championed. During the site visits we made to widely scattered and diverse schools (Stake and Easley, 1978), teachers spoke to us quite freely about the disappointment they felt in their college or university training.*

While I do not want to claim that traditional teachers are in a separate culture from the "experts," I do want to appeal to the metaphor of a cultural difference in order to show that differences of this sort in values, rituals, and sanctions can be transcended bit-by-bit if one pays attention to the life style of the group in which an import from another group can be incorporated--hence I'll say acculturated.

The experts in mathematics education are divided into so many perspectives (see Easley, 1975, 1977) that it may seem gratuitous to

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*A handful of Texas teachers who disagreed turned out all to be from one, fundamentalistic church college.

pick on just two. However, in some sense, there is a more profound division between the specialists and teachers who treat mathematics classes as an opportunity for children to create ideas and systems of measurement and calculation and those who see mathematics teaching and even mathematics itself as ideally embodied in technological devices such as computers In the first group, the teacher is a humanist who seizes the opportunity, or helps create the opportunities, to encourage children's autonomous creative thinking and provides them with language and forms to use as they need them in their work. In a sense, the re-invention of algorithms, fundamental measurement, theorems and proofs are celebrated as, in a small way, making oneself one with the historical development of mathematics. In this sense, it is backward-looking in its inspiration for the development of children. The technologist, and technologically oriented teachers, are then forward-looking, intrigued with the power of those diverse solid-state devices that serve increasingly to carry out our calculations for us in work and in play.

There is no doubt that we are going to see a very large involvement of such devices of all sizes in our daily lives. Linking computers with television sets, as in many popular games, is surely a realization of Marshall McLuhan's media revolution. While some use computer technology to teach traditional mathematics and logic (e.g., Suppes), others see the new form of mathematics given in the logic of computer programming (e.g., Papert). There is little doubt, however, that the traditional school teacher is pushed aside in the designed learning environment, in which the student interacts directly with sophisticated computers and thus learns to speak to them in their language, the otherwise strange language of mathematics. There is little doubt that computers and their languages are increasingly used by mathematicians, but there is a humanistic streak in many mathematicians that refuses to let mathematical thought be identified with computer programming. This is the dividing line between these two camps.

The traditional mathematics teachers, the creative mathematics teachers and their theoreticians, and the mathematical technologists are well aware of each others' existence, and come in contact at workshops and professional meetings and yet close examination reveals that little real communication occurs between these three groups about the problems of teaching mathematics. They are like the moieties Wolcott identifies in his case study (1977) of the failure of a systems-oriented group of technocrats to have any lasting impact on the functioning of a school. Professional courtesy covers a great deal of contempt in all six of the directions in which that relation is possible among these three groups.

Teachers repeatedly affirmed to the data gatherers in Stake and Easley's study that they could find no experts who understood their problems and could help them solve them. Experts of all persuasions point to the evidence that their programs or devices do work in situations like those of teachers who claim they are unworkable. Therefore, they conclude, the reason they are not adopted or the reason for failures when teachers try them lies with the teachers and not with their innovations.

Yet the public schools of America are committed to an in-service training program based on experts (in residence in the Superintendent's office or imported from a university or R. and D. center). It is clearly a part of American culture to look to expertise, and schools would lack

professional pride if they did not employ experts in one way or another. The experts of whatever persuasion are in fact capable of winning a handful of converts in most encounters, many of whom eventually leave the school to find a higher degree of support for their non-traditional ideas in some other setting than the classroom. Because of a time delay, experts can sustain the illusion of infusing their ideas into teachers at a steady rate while the net impact is zero, converted teachers leaving the schools as rapidly as new converts are being made. Fortunately the insiduous character of these illusions (the teachers' illusion of keeping up to date and the experts' illusion of influencing teachers) are beginning to be recognized more and more.

This situation does not mean that nothing is being done to improve mathematics teaching in schools. Traditional teachers are continually changing their methods and materials, although from the perspective of the creative or technological cliques the changes may seem negligible. It is too early to say what effect teacher centers (which often rely on the resources of the teachers themselves rather than on experts) will have on the rate and effectiveness of changes. However, the problems that teachers are facing in the United States, and in many other countries, are growing faster than they can solve by their own innovations. Primarily there seem to be two sources of these problems. One is the increasing diversity of social cultural background of children in each classroom, which handicaps the traditional teacher who must draw on some common pool of values and ideas developed at home in order to keep school work moving along. The second is an increase in the skepticism of youth about traditional values which affects middle class youth with respect to their own cultural values perhaps even more than it does some minorities with respect to theirs.

The presumption of the governmental agencies mandating racial balancing, heterogeneous classes, mainstreaming the handicapped, and similar programs to guarantee due educational process to every child, seems to be that schools are competent to adjust to the diversity of backgrounds of the children. But if there are no experts who understand the problems teachers face in their terms,* and teachers reject the drastic recommendations of experts as unworkable with the students they have, there must be a reason for it arising from their perspective of their job. Furthermore, there is too little being done, even when teachers do adopt a handful of the hundreds of innovations they are offered, to solve the problems they face. Therefore, the school condition for all children is worsening with no remedy in sight.

The social foundations of mathematics education cliques

It seems highly probable that the three cliques we have described are not ephemeral, transient groups but rather have stable roles to play in the larger educational scene. I shall not attempt here to elaborate what these roles might be. The tension between traditional mores and innovations is a general topic of considerable interest, but a more satisfactory treatment would require looking into such issues as how school mathematics functions in maintaining or establishing social classes** and what threat would be posed to the American economy and social tensions by uncontrolled

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* Soviet studies in mathematics teaching and learning suggest that the psychologists involved took their problems as traditional teachers defined them.
** See Mehan, 1978, for a strong suggestion of the mechanism of social class definition in teacher-pupil interactions.

upward mobility in mathematics. There may be a delicate balance here that teachers try to maintain in a largely intuitive way. Clearly, not enough is known about the mechanisms of classism, racism, and sexism in mathematics although the evidence that these biases exist is becoming increasingly clear.

We can, however, point out a number of stabilizing factors in the relationships between the three groups we are discussing. Traditional mathematics teaching is highly defensible in any particular teacher's area of responsibility, given the stability of the other parts of the curriculum. Many parents and students will complain if a teacher institutes a change that appears to weaken his/ her students' preparation for subsequent mathematics classes. Empirical evidence is not needed to argue for stability or even, as we say, a return to the basics. The public defense of traditional teaching of mathematics is growing and schools are establishing criteria for graduation which depend on mastery of skills at fixed, or population-normed, criteria. As long as the universities teach mathematics to undergraduates in an authoritarian, demanding way (and only a few colleges or individual professors have adopted the creative proposals of Polya or Moore, or adopted the textbooks of Rothbart and Singer (1976, 1977) based on their ideas), public schools cannot be expected to change their orientation. Likewise, the technological innovations in university-level mathematics teaching are extremely limited, although a few colleges have made computer programming a required course for everyone. It doesn't matter apparently whether universities require specific kinds of math courses for college entrance or not, parental pressure will be high to see that college bond students get as close as they can to what is going to be taught in university math courses. College placement courses are thriving. If they succeed, college courses may stiffen, which leaves the two expert groups the elementary and junior high schools for their field. However, the effects of the college prep and back to basics movement are felt there too.

Another reason for the stability of these three groups in mathematics education is that mathematics is widely celebrated as, at once, a status symbol, an entertaining hobby, and a highly useful skill. However, concentration on the attainment of one of these three aspects is very likely to make the other two appear more difficult of attainment. Thus, if one seeks high status, advancement along the traditional pathway toward the calculus is required. The fun of finite mathematics and the applications of computer languages and accounting are seen as distractions. However, if one takes pleasure in following Martin Gardner's column in Scientific American called "Mathematical Games," the utility of it all and the progress along the status track are bound to receive lesser consideration. Mathematics, despite being usually organized as a single university department, has more to offer the high school student potentially than several of the science departments combined. So, as long as teachers are busy teaching one aspect of mathematics, there are opportunities for mathematics educators to pick up equally valid pieces of mathematics and advocate their inclusion in the curriculum. In short, the field of mathematics is diverse enough to support these three groups.

The social structure of the three groups is such as to make them highly self-sustaining. It seems clear to most observers of the mathematics teacher education scene I know that the majority of persons in the

U.S. preparing to teach mathematics in high school or junior high school look upon mathematics as a conventional set of definitions and procedures that solve a conventional set of problems. Undoubtedly persons are more often motivated to go into teaching because of a positive image of a teacher they liked than because they think teaching should be done differently. This seems especially true of the high school teacher. The specialists in creative approaches to mathematics teaching actively recruit adherents of their approach in universities and in schools. They are identified as the better students because of their sensitivity to children's thinking and/ or the possibilities for development of mathematical ideas in many different situations. Those students who worry about getting pupils to follow standard procedures and obtain standard answers are seen by these teacher educators as having limited vision. (The feeling may be mutual.) Likewise in courses or workshops on the technological approaches to mathematics education, it is clear that those who take naturally to this approach will receive highest grades and recommendations for advanced study. It appears, in fact, that teachers faced with growing socialization difficulties in the classroom may increasingly seek to leave the school in favor of graduate study or research programs. It is easy to convince one who wants to leave that a whole new set of goals and procedures would do wonders for the schools. Those who are committed to stay are much less likely to want to try such a major overhaul, especially single-handedly.

There are undoubtedly some mutal benefits to a three-way polarization of the field of mathematics education. Not only can the mis-fits of each pole have another place to go within the field, but there is a ready explanation by those left behind of why they left: they weren't able to cope with the demands of that group. Neither schools, nor centers of research and advanced study have an easy time coping with diversity of opinion. Individual opinions are likely to change as one gives up on one perspective and tries another. This is perhaps made easier by a physical move than by trying to get others to accommodate to a person's changed perspective.

The problem and a solution

A field like mathematics education which seems to be primarily organized around three poles where the majority of participants are trying to achieve a pure development of the ideals of that pole does not lend itself to helping large numbers of children who have difficulty learning mathematics. While some of my colleagues have argued that every field of work suffers from tension between the experts and the practitioners, I do not know of any outside of education where the actual utilization of the best scholarship is so low. Neither do I know of any where the teachers of professional workers are so blatantly disinterested in the problems as defined by practitioners in the field. Clearly, those in this field who are concerned about the mathematical learning of the majority of children should find a way to pull together in spite of their ideological differences, instead of pulling away from each other. Those who are only concerned to develop their personal status in one group may be ignored in this organized effort. A few dozen persons, in fact, could mount an organization to accomplish something, if they could work together in one place for a time. The question naturally arises what kind of program or policies could they advocate? This is the problem to which the rest of this paper is devoted.

One thing they might agree, at the outset, not to undertake is a research, development, and dissemination program. At least, this would be the advice of Les McLean, John Goodlad, J. Myron Atkin, and Lou Rubin. Several projects of the Centre for Applied Research in Education (CARE) place the responsibility on school staffs for research decisions.

One plan that has been proposed to alleviate this kind of polarization is a rotation of situations. School people come into the research and teacher training situations for a time and experts rotate into the classroom for a time. Another plan is sometimes referred to as the "clinical professor," a person who works in both situations all the time. There was more enthusiasm for these plans a decade ago when student criticism of the lack of relevance of the University to society's problems was high. Perhaps the most serious difficulty with rotational approaches to the problem is that the nature of the barriers are such that mere situational involvement won't create much of a break-through.

One teacher, reported in Stake and Easley (1978) said:

> They must have tricked us. In demonstration teaching they did, the kids looked like my kids, and they seemed to learn, but when I tried it, my students didn't learn.

My own experience in learning Max Beberman's method of teaching ninth grade algebra, which had the attractive feature that no child was left out and lost for more than a few minutes, was that it took a full year of daily observation in Beberman's two classrooms, interspersed with daily conversations with him or with Gertrude Hendrix, who was writing ethnographic field notes on every tactic he employed and their results. That one year followed a few months of intensive study of the design of his textbook, and was followed by a semester in which I had the opportunity to do a major share of the teaching of a ninth grade algebra class. The teacher I worked with then, having had an intensive training with text and films, was doing her apprenticeship with me. She foundered with the class of borderline students. I took over, and she watched. The class recovered miraculously; she did a little follow-up teaching, and eventually, at the end of the semester, she took over full-time teaching. I know that Beberman and Vaughan can be taught to border-line students. But when teachers tell me it can't be, I know that their apprenticeship was nowhere near sufficient to make it possible for them. Deep involvement is needed not just contact to make converts.

Acculturation on a massive scale, in which millions of Americans gradually incorporate into their life-style some features of other cultures, is going on all the time with some assistance from the advertising and manufacturing industries. In Japan, it is even more striking because the Western culture they are incorporating is so different from traditional Japanese culture. The cultural purists are not obliterated in the process; in fact, they are scarcely involved at all, initially. A good example is the California wine industry. When I first tasted Chablis, twenty years ago, it was a sweet wine. Having grown up in a puritanically dry community, I never acquired a taste for dry wines until quite recently. Now, that we have wine-tasting clubs all over our country, California Chablis is made closer to the European version, but initially it was sweetened for American tastes.

The lesson I take for mathematics education is that someone needs to develop and market popular versions of what the purists acclaim as

their true knowledge. To be popular they must be fool-proof, like the Japanese noodles packed in a plastic cup. Just open the plastic package of seasoning, dump it in, and fill to the brim with boiling water. Let sit five minutes and eat. This won't put Japanese restaurants out of business, because there you get the waitress in the kimono. In fact, the fool-proof noodles may increase their business.

Fool-proof is a positive concept which is quite different from teacher-proof, which implies that the teacher is not really interested. Fool-proof food preparations are inventions designed for busy persons who don't have the time and patience to learn gourmet cooking but want to enjoy a taste of another culture. When we visited classrooms in the Case Studies in Science Education (Stake and Easley, 1978) we found boxes of science and mathematics materials piled up in closets or corridors, unopened. They had been designed by purists supported by the National Science Foundation who had discovered striking new ways of teaching children. Hands-on mathematics materials like Cuisenaire rods and Dienes' base ten blocks sat on shelves in the classroom, having been used once or twice at the beginning of the year, while children filled in answers on worksheets or to sets of type problems in books.

It was extremely difficult to find mathematics teaching devices other than textbooks that were being used by large numbers of teachers frequently enough to justify their manufacture. There was no lack of awareness of innovations, but they were not available in fool-proof models, only in the form which required extensive teacher involvement.

Publishers have included a few challenging problems in textbooks, but teachers often don't assign them. They are doing traditional teaching, which is a different thing altogether, and difficult problems upset the routine of teaching procedures and providing lots of practice in following them. In fact, given the gross individual differences in learning of a teacher's class, practice could be individualized and each pupil can copy the appropriate procedures from a worked-out example. This was a common teacher-made solution to the problem of finding oneself out of one's culture and mores. IPI was a less common version of the same solution to the same problem. IPI is almost fool-proof. (See Erlwanger, 1973.) I propose that acculturation be promoted between the three poles of math education via fool-proof devices. To claim this as a plausible solution to the tri-polarized paralysis of mathematics education, I must either find some such fool-proof devices for exhibit or else invent them myself. Furthermore, it must be made plausible that such devices could be developed as carriers of the "culture" from each of the three poles I'm talking about to each of the other two. I would not suppose that a single device could be designed that could move in more than one of the six possible directions. For each device must represent something recognizable from one pole and carry it with least disturbance into the life style of persons in another pole.

In the remainder of the paper, I shall present those ideas I have been able to collect or invent and try to indicate roughly the kind of market research and development that would be necessary before launching an advertising campaign and manufacturing. In each case, I shall try to identify the demand that I believe can be cultivated, the nature of the product to be imported, and the probable effect on mathematics educators and pupils if it is successful.

1. From the creative to the traditional mathematics educator.

Traditional teachers have heard of mathematics games, but worry about the possibility that children will become so involved in the social or competitive aspects of games that they will not really learn any mathematics skills. They object to disguising skills so that people learn effortlessly, since effort is something that has to be learned in order to succeed in later years of schooling. However, if a game could be found or devised that produced some learning not ordinarily covered in traditional middle school curricula, I believe a lot of teachers would give it a try. But it has to be packaged in a fool-proof way if it is going to have any sizeable use by teachers.

Creative mathematics teachers have found children fascinated by trying to measure the areas of all kinds of things from the prints of their feet or bicycle tires to the area of the floor or walls of the school building. Traditional teachers, however, are not likely to use more than once any activity that involves children going out to the bicycle rack or creeping all over the school building because they could not resolve and teach correctly the procedural problems that would arise in remote places. However, if the estimation of areas of strange shapes could be incorporated into a game that could be played by groups of children by pulling their desks together, it might be used a lot because it introduces a concept traditional teachers have found trouble with, area measurement. Imagine a box full of strange plastic shapes that suggest fantastic creatures, a pad of graph paper on which each shape could be traced, a set of rules that require each child to guess the area in square millimeters (or square centimeters for younger children) trace the figure and count the squares, and compute the points earned as some function of the difference between an individual's guess and the group average of counts of the graph paper outlines. Bad errors in counting would affect scores of good guessers negatively so helping each other with the counting (or estimating) process would be an advantage. Designing the plastic shapes would be most important. Short cuts should be possible by suitable alignment with the graph paper, but there should be look-alikes with different short cuts so the game would have staying power and gradually invite children's attention to parallel lines, right triangles, parallelograms, etc. embedded in the shapes. Many traditional teachers would notice children learning these concepts and formulate rules for them to memorize.

The traditional teacher feels responsible for proper procedures--they are the essence of what teaching and learning are about. How to get teachers in such a perspective to recognize and tolerate a spontaneous discovery of a mathematical principle that is obtained by deviating from standard procedures and which is not yet codified into a standard procedure--that is the problem. A game format loosens the rigor of teaching, and the substantial practice in arithmetic is the justification appealed to. However, it is still a delicate question--requiring trial and skillful design--whether, when they see pupils taking advantage of hidden right angles in the pieces, traditional teachers will want the rules formulated and memorized, which might kill the motivation, or whether they'll let these go as harmless adumbrations of later formulas for area.

2. From the traditional to the creative mathematics educator.

It is very difficult to think of what the traditional teacher can offer the convert to creative mathematics, but it is very important to find

something. If acculturation is promoted only in one direction, there will be no lessening of the tension and barriers between them. This point is well made by Freire with respect to the extension agent and the peasant farmer.

Although criterion reference tests arose from an earlier form of educational technology, they are now growing in popularity with traditional mathematics teachers. I believe that they could be designed in such a way that creative mathematics educators would show some interest and even use them frequently. If this happened, it would provide an important link with traditional teachers where now there are none. Creative teachers have a great interest in children's misconceptions. (Such misconceptions have been investigated by Davis, 1975, Ginsburg, 1975, and Erlwanger, 1973.) Other studies of this sort go back to Buswell, 1926, Brownell, 1928, and others. For example, young children often miss the ordinal meaning of the last number in counting, thinking that it is simply attached to the last object. Older children often miss the general principle for combining multiples of ten with digits in decimal notation for integers, writing 10007 for one thousand and 7, while knowing perfectly well that twenty seven is written 27 and not 207.

The link I propose to exploit is that multiple choice test items require distractors that children who understand something incorrectly will choose because they are consistent with their mis-understanding. While traditional math teachers would show little interest in the particular wrong answers that may be chosen, prefering to concentrate on correct ideas, creative approach mathematics teachers are very likely to be interested in information such tests could give them about mis-conceptions in their class. They could then hold a discussion about them or develop inquiries into the extent or origin of such misconceptions. Because they are committed to using children's thought, to assume more of the social responsibility of the traditional teacher, they need to know more about their pupils' thought than they can find out just by conversation with a few at a time. If the children themselves are given the results of criterion reference tests in terms of the misconceptions which their wrong answers suggest, they could initiate discussions with their fellow pupils about the reasons they chose one answer rather than another and whether or not they really believe in the misconceptions suggested. While these uses would, of course, differ partly from the uses made by the traditional teacher, there would be an importation of the idea of testing the whole class against the same criterion, which otherwise would be neglected, and creative teachers and other mathematics educators would soften their critical attitude toward test-based teaching. There is another approach that could be used on a much smaller scale. Called communication among multiple perspectives (CAMP), it proposes week-long workshops in which each participant learns to explain a different perspective from his own to the satisfaction of an expert.

Four more directions require inventions that will fit comfortably into one culture and yet transport an important idea from another culture--and must be developed and marketed by persons more dedicated to acculturation than to refinement of any one "culture." I shall briefly sketch four possibilities, more to indicate the need than as firm proposals.

3. From the technological to the traditional mathematics educator.

While very unsophisticated mathematically, the mini-calculator is becoming cheap enough to consider it as an opening wedge of technology into the traditional classroom. The problem is to introduce it in such a way that it does not replace practice in computing algorithms. Checking one's answers by calculator is too close to just doing them on the machine, and checking by hand is good discipline. However, it might be used if it were introduced with a problem that causes great difficulty, such as solving heterogeneous collections of story problems of the sort that form mathematics applications tests, where the difficulty is not knowing which operations to perform on which numbers. Research is needed to determine whether a game format or a more serious problem-solving format would go over better. However, the following procedure could be tried: Have pupils guess the answer to a story problem and write it down. Then they should try various calculations on a mini-calculator, recording the procedure and results. Finally, they compare the results to the guessed answer to see if some procedures could be eliminated. Now a new guess is made and the process is repeated until convergence on a procedure and an answer is obtained.

The problems for this game or exercise should be carefully designed not to fall into type problems of the sort found in text books, but to be as different from each other as possible, while still being comprehensible enough to permit some reasonable guess.

While purists in the technological approach would not recognize this as a serious application of technology, it might have a good chance of being adopted by many teachers as the least objectionable modernization available. Furthermore, if the scores on the application sections of standardized tests are low, and the prime causes are identified as bad guessing and ignorance of which operation to use when, a package with calculator, work sheets, and problems that could be usea once-a-week with noticeable improvement on standardized tests could make an impression.

4. From the traditional to the technological mathematics educator.

In some respects the technological (computer) educator and the traditional educator have something in common. Both have a faith that correct procedures will keep one out of trouble and incorrect procedures will sooner or later lead to serious trouble. Another concomitant principle is the belief that rules, definitions and procedures are man-made and hence merely conventions. Perhaps the philosophy of conventionalism could be expounded in cartoons in such a way as to develop more respect for conventional teaching by the technologist.

5. From the creative to the technological mathematics educator.

Many technological mathematics educators are interested in helping children write, debug, and run computer programs. Some, like Papert, have developed heuristics for debugging, and most computer languages have automatic aids to debugging. What seems not to have been studied are the patterns of errors in writing computer programs. Just as the creative mathematics teacher finds help in studying children's errors and misconceptions, there should be a way in which educators working with computers could find help in studying children's errors and misconceptions related to programming bugs. Some research would be needed to discover how to package this information for maximum probability of use. More

knowledge of the computer language and the organization of the computer is quite likely to make such specialists insensitive to specific misconceptions, and student difficulties may be treated as merely requiring careful going over the principles, definitions or terms, and making systematic flow charts. Errors caught will be corrected, of course, but the possibility of systematic errors due to misconceptions that could be corrected if recognized seems out of place. For one who knows precisely what the system really is, the question is: How can there be any system in an error?

Several possibilities suggest themselves. Perhaps the simplest is to write a debugging package which would print out interview questions to the programmer on each stop the program makes. For example, it would ask what the programmer intended the computer to do and storing his reply, along with the program and line number on which the reply was made, then asking how the computer was supposed to know to do that, etc. When enough replies are obtained, careful study of them may reveal more pointed questions that could be asked in particular identifiable situations, like: Did you want the computer to do this? If so, you should realize that it doesn't have the information needed, etc. After some years of development, a debugging package that "psyches out" the novice programmer may be developed. Another approach might be to develop text material that explains what the common mis-conceptions are and gives counter examples and arguments against them, as well as explaining what appear to be ambiguous terms in the theory. This, of course, would require more research.

6. From the technological to the creative mathematics educator. Here, there is a great deal that can and already is being done. The problems Papert poses to students working with LOGO and the turtle are creative problems. However, they are not specifically designed to solve some of the teaching difficulties the creative teacher has. For example, children are very prone to question such seemingly arbitrary decisions as $n X 0 = 0$, $n^0 = 1$, if $a = b$, then $b = a$, etc. Perhaps there are simple computer models in which these decisions make a great deal of sense? If so, perhaps without mastery of a computer language, students could be encouraged to work these problems out on a simple mechanical or electrical computer. Some computers, on the other hand employ conventions that don't make much sense, e.g., that $-0 \neq +0$. Perhaps there could be an expansion of conventions in mathematics and computer languages through use of a relatively simple piece of circuitry, a circuit board of some sort, perhaps.

Summary and conclusion

The picture we have painted in this paper needs a great deal of refinement, and that can come if it is sharply criticized and if research is focussed on the social structure of mathematics education, both in the classroom and out. I think, however, that the criticism and research needed to translate my sketch into more reliable knowledge will leave one point unchanged. It will remain clear that most present efforts aimed at the improvement of mathematics learning for large numbers of students in the U.S. are not producing any noticeable effect, and unless changed drastically current programs will not have any noticeable effect in the next ten years. Traditional teaching is strongly entrenched and supported by public opinion even as its special problems grow. The out-of-classroom sector of mathematics education is not oriented at all to helping traditional teachers solve the problem of socializing children of many different backgrounds and an increasingly skeptical attitude toward school through mathematics teaching. The schemes for changing this purpose to another one that is more creative or more technological in focus (or any thing else) are not at all likely to be well received by a teaching force under the kind of pressure and duress our teachers now experience. Only by joining forces with them could the "experts" find a common goal with them more suited to expert taste. That is, extension and dissemination have to change to dialog and finding new common goals, as Freire correctly points out for agriculture.

However, the faculties of education and the research and development laboratories, as well as the superintendents' offices, do not contain many mathematics educators who would be able to give up their commitment to the learner and to learner thinking and begin to try to understand teachers and teacher thinking. Consequently, the job of softening up the rigid barriers that now separate mathematics educators into three or more camps has to be undertaken by those few who see the disaster of millions of very badly educated children and are willing to drop their own ideal approach in order to bore holes in the dams that stop the flow of thinking and concern from reaching masses of children.

List of References

Brownell, W. A. (1928) The development of children's number ideas in the primary grades. Supplementary Educational Monographs, No. 35. Chicago: University of Chicago.

Buswell, Guy T. (1926) Diagnostic studies in arithmetic. Supplementary Educational Monographs, No. 30. Chicago: University of Chicago.

Davis, Robert B. (1975) Cognitive processes involved in solving simple algebraic equations. Journal of Children's Mathematical Behavior, 1 (3): 7-35.

Easley, J. A., Jr. (1975) Communication between educators taking different perspectives. Schriftenreihe des IDM, 5:21-48.

Easley, J. R., Jr. (1977) Seven modelling perspectives on teaching and learning--Some interrelations and cognitive effects. Instructional Science, 6:319-367.

Erlwanger, Stanley H. (1975) Case studies of children's conceptions of mathematics--Part I. Journal of Children's Mathematical Behavior, 1(3):157-283.

Freire, Paulo (1973) Education: The Practice of Freedom. London: Writers and Readers Publishing Cooperative.

Ginsburg, Herbert (1975) Young children's informal knowledge of mathematics. Journal of Children's Mathematical Behavior, 1(3):63-156.

Goodlad, John I. (1975) The Dynamics of Educational Change: Toward Responsive Schools. New York: McGraw-Hill.

Goodlad, John I., Klein, Frances M. and Associates (1974) Looking Behind the Classroom Door. Second Edition. Washington, Ohio: Charles A. Jones Publishing Co.

Mehan, Hugh (1978) Structuring school structure. Harvard Educational Review, 48:32-64.

Rothbart, Andrea M. and Singer, Richard (1976) Male Chauvinist Chess and Other Problems in Number Theory (with an introduction by George Polya) unpublished text, Webster College, St. Louis, Mo.

Stake, Robert E. and Easley, J. A., Jr. (1978) Case Studies in Science Education, Design, Overview and General Findings. (National Science Foundation) Washington: Government Printing Office (GPO Stock No. 038-000-0376-3).

Wolcott, Harry F. (1977) Teachers vs. Technocrats: An Educational Innovation in Anthropological Perspective. Eugene, OR: Center for Educational Policy Management.

ANALYSIS OF THE CONCEPT OF PROBABILITY IN YOUNG CHILDREN

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Various investigators used a decision making technique in order to explore the onset of children's understanding of probability. The basic experimental unit presented two sets of elements, each of which was subdivided into two categories, i.e., different colors. At each trial, one of the two colors was pointed out as the payoff color (POC). The child had to choose the set from which he would try to draw at random an element of the required color. In order to be maximally rewarded, his problem was to identify the set with the higher probability of the desired color. Although the different studies all presented paired comparisons of binary sets, serious disparities were obtained in authors' conclusions concerning the age of attainment of the probability concept. It seems feasible to hypothesize that the apparently contradictory results stemmed from differences in the mathematical features of the problems presented. A more profound analysis of the concept of probability indicates that some of the studies included certain components in their probability problems while the others overlooked the same elements.

The aim of the present experiments was to analyse the concept of probability and thus to find out how children's responses relate to each of the components of probability.

The Experiments

Experiment 1. 36 children within the age limits 5-11 (6 Ss in each year of age) were tested. Each child was confronted with two out of three kinds of instruments: a. Pairs of transparent urns with different compositions of blue and yellow beads in each. b. Pairs of roulettes of different radii, each with its own distribution into blue and yellow sectors. c. Pairs of spinning tops of different volumes, likewise subdivided into the two colors. The probability of the POC in a given set is the ratio of the number of elements of the POC to the total number of elements in the set. These two absolute quantities are the two main dimensions composing the proportion of the target color. Each of these quantities can be either greater or smaller in the correct choice than in the other set; the quantities could also be equal in both sets. The experiment consisted of 22 trials, 11 on each half with a given instrument. The design of the experiment is presented in Table 1. Crossed cells represent impossible combinations. The arrow in each of the examples, points at the subset of the POC in the correct choice. The trial described at the upper right cell of the table, for example, consisted of 4 elements in the left hand set: 3 of the POC and one of the non-payoff color (NPOC), the right hand set included 2 elements of the POC and 2 of the NPOC. The correct choice is the left set since 3/4>2/4 = 1/2. The results are, likewise, given in Table 1. The percentages written lowest in each cell are percentages of correct responses to trials of that category.

Experiment 2. 25 children, within the age limits 4-7, were tested with 32 trials each. In this experiment only roulettes were used. The design of the experiment controlled a third quantitative dimension of probability, i.e., the number of elements of the NPOC. All the possible cross-combinations of the three dimensions are presented in Table 1.

The conjunction of the results of the two experiments points to the "number of elements of the payoff color" as the variable that accounts for most of the variation in the children's responses. The effect of that dimension was significant in both experiments, but more salient in the second one. The level of correct responses when the number of elements of the POC was smaller on the correct choice approached that of pure guesswork for the younger children (52% in Expt. 2). Indeed, the effect of this variable is highly dependent on age, as one can see in Table 2, where the results are broken down according to years of age. This analysis included only problems in which the correct choice contained a smaller total number of elements and a smaller number of elements of the NPOC. The results of Expts 1 & 2 were pooled together for overlapping ages.

Conclusions

The experiments showed that starting from age 6 the children tested performed significantly above chance level. Since according to the design of the experiments, no other principle of choice, except selecting the higher probability, would systematically result in correct choices, this means that at about the age of schoolstart children become capable of discriminating between probabilities. The dominant error among preschoolers was choosing the alternative with the largest number of target elements.

Current Research

The following problems are being studied:

a. Will children who choose according to an absolute principle, persist in applying the same principle also when going over to complementary events? Consider the following comparison: (5:3) versus (3:1). The number of elements of both colors is higher on the

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(1) Parts of this study were carried out in cooperation with Iris Levin from the University of Tel-Aviv and Raphael Falk from the Hebrew University of Jerusalem. The paper is based partly on the author's M.A. thesis which had been supervised by Rivka Eifermann, the Hebrew University of Jerusalem. Current research is supported by a research fund of the Faculty of Social Sciences at the Hebrew University of Jerusalem.

Table 1. Examples of Problems of Probability-Comparisons, According to the Composition of Elements on the Correct Side in Relation to the Incorrect Side.

(Number of Trials)

Percentage of Correct Responses

Experiment 1. Age Interval: 5-11; No. of Ss: 36

Number of elements of the payoff color	Total number of elements		
	Greater	Smaller	Equal
Greater	(2:6)>(1:5) ↑ (4) 80%	(4:4)>(3:6) ↑ (2) 89%	(3:1)>(2:2) ↑ (6) 90%
Smaller		(2:2)>(3:9) ↑ (4) 79%	
Equal		(2:1)>(2:3) ↑ (6) 87%	

Experiment 2. Age Interval: 4-7; No. of Ss: 25

Number of elements of the payoff color	Total number of elements							
	Greater			Smaller			Equal	
	No. of elements of the NPOC			No. of elements of the NPOC			No. of elem of the NPOC	
	G	S	E	G	S	E	G	S
Greater	(2:4)>(1:3) ↑ (4) 88%	(5:3)>(3:4) ↑ (4) 84%	(3:2)>(2:2) ↑ (6) 75%		(4:1)>(2:4) ↑ (2) 86%			(2:6)>(1:7) ↑ (6) 77%
Smaller					(2:2)>(4:6) ↑ (4) 52%			
Equal					(1:2)>(1:5) ↑ (6) 63%			

left hand set. Let us imagine a child who is choosing consistently the set with a higher number of elements of the POC. The child would *always* choose the left hand set. The choice would be correct when the POC is the one whose number is written on the right side in both sets (3/8>1/4), and incorrect when the POC is the one written on the left (5/8<3/4). A consistent choice of the same set, independently of the POC, violates the rule stating that if a>b then 1-a<1-b.

b. To what extent are children integrating the numbers of elements of the two colors by computing the *difference*, rather than the *ratio*, between them? The following is an example of a comparison problem where the choice according to the difference in favor of the POC does not coincide with the correct choice: (4:1)>(10:5); one should choose the left hand set since 4/5>10/15; however 4-1<10-5 and choice according to the difference would point at the right hand set.

Table 2. Percentage of Correct Responses According to Age and to Number of Elements of the Payoff Color on the Correct Side in Relation to the Other Side.

(No. of Trials - in parenthesis)

Age limits	Number of elements of the POC on the correct side	
	Greater	Smaller
4-5	78% (18)	33% (36)
5-6	78% (28)	48% (56)
6-7	93% (28)	76% (56)
7-8	88% (12)	88% (24)
8-9	92% (12)	77% (24)
9-10	96% (12)	90% (24)
10-11	100% (12)	96% (24)
Total	88% (122)	68% (244)

Intuition and Mathematical Education

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The concept of intuition

Mathematicians, especially in the last two centuries, have been very concerned with the problem of intuition. Striving to formalize the different branches of mathematics, they have been faced with the problem of liberating their definitions, theorems and proofs from ideas which were not completely controlled by logic. As a consequence, they became aware of the enormous part played by intuitions in mathematical invention (and mathematical errors)--i.e., of the role played by mental attitudes which, by their very nature, are not explicitly justified.

Great mathematicians such as Felix Klein and Henry Poincaré --though affirming the necessity to always verify intuitive ideas --have, at the same time, accepted that intuitions play an active role in mathematical invention and mathematical education.

Unfortunately, psychology has paid little attention to intuition, in contrast to philosophers who frequently have seen in intuition a fundamental source of knowledge (Descartes, Spinoza, Kant, Bergson, etc.)

In this paper, an attempt will be made to analyze intuition from a psychological viewpoint and to reveal some of its implications for mathematical education.

So, what is intuition? A great number of definitions and interpretations have been offered, each of them generally expressing the author's own philosophy. But all these interpretations mention a common feature: Intuition is immediate knowledge.

An intuitively accepted truth is self-evident--no arguments appear to be necessary to convince us that the given statement is true. For instance, a statement such as: "If A>B and B>C then A>C" would be such an intuitively accepted statement. In the same manner, we accept as being self-evident statements such as: "The shortest way between two points is the straight line;" "the whole is greater than each of its parts;" and "only one line parallel to a given line can pass through a point outside that line."

This fundamental feature of intuitions--i,e., their being self-evident--explains their great impact on the course of thinking. They have constituted the basic source of mathematical axioms through their long acceptance as primitive nonanalyzable concepts, e.g., the concepts of straight line, number, distance, continuity, etc. The efforts to find a unique, rigorously deductive, that is axiomatic, setting for an ensemble of related mathematical concepts and operations have generally followed an earlier, less rigorous and more intuitive approach.

On the other hand, some great discoveries have been made in mathematics in a time when rigorous proofs were impossible, taking into account the state of mathematical knowledge in that time. Hadamard, in his famous "Essay on the Psychology of Invention"

(1949) quotes some striking examples. One example is Fermat's well-known theorem: "The relation $x^m + Y^m = z^m$ is impossible in integral numbers (x, y, z different from 0; m is greater than 2)." In fact, as Hadamard says, no rigorous proof was possible in the time of Fermat, owing to the lack of adequate mathematical tools.

A second example mentioned by Hadamard (1949, pp. 119-120) is that of a discovery made by Galois concerning a theory on periods "of a certain kind of integral." This theory could not have been understood by scientists who lived at the time of Galois since these periods had no meaning in the state of science of that day. The conclusion is that Galois had guessed his theory without being able to give a full, rigorous demonstration of it.

On the other hand, the efforts to axiomatize the various mathematical domains have revealed the limitations and dangers of intuitive interpretations. Some of the authors simply concluded that intuitions had to be banished from mathematics. As Hans Hahn has stated: "Repeatedly, we have found that in geometry questions, even in very simple and elementary ones, intuition is a wholly unreliable guide, and it is of course impossible to adopt this discredited aid as the basis of mathematical discipline (1968, p. 187).

Hahn also describes a number of beautiful examples: the discovery by Weierstrass (1861) of continuous curves which possess no tangent at any point; and the discovery made by Peano (1890)

that moving points can generate (in a finite time) not only curves but also entire plane surfaces (H. Hahn, 1968, pp. 184-186).

A list of other examples can be given concerning mathematical truths which contradict intuition: the logical possibility of non-Euclidian and multidimensional geometries, the Cantorian concept of actual infinity (generating anti-intuitive concepts such as that of a set being equivalent with some of its subsets) etc. One conclusion seems to be clear: Intuition has an important impact on mathematical thinking, whether positive or negative. So it cannot be neglected by mathematical education. Both in trying to use its beneficial qualities and in striving to eliminate intuition from the pupils' mathematical reasoning, we have to take into account its possible effects.

The features of intuition

First of all, we need a better understanding of what intuition is, of its features, sources, and mechanisms. In what follows, we shall try to describe the common features of intuitions.

1. The fundamental characteristic already mentioned is of course that of self-evidence. An intuition is a self-evident representation of facts. Not all mathematical statements, formulas, etc., which are accepted as being completely clear, have the status of intuition. For instance, the well-known formula for solving quadratic equations:

$$x_{1,2} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

does not possess the inner credibility of an intuitive truth; it is not self-evident. If, instead of -4ac, it would have been +4ac, nothing would have been striking or surprising. But the statement "there are as many even numbers as natural numbers" sounds, at least at a first glance, absurd. An intuition is, consequently, more than a pure cognition: it is a belief. This explains the role attributed to intuition in aesthetic judgments, in moral convictions, and attitudes in religious revelations (see Wild, 1938).

From the educational standpoint, some additional remarks have to be made. The obviousness of various mathematical statements may help children understand and memorize them. Elementary geometry makes constant use of such intuitions.

At the same time, however, the self-evidence of some mathematical truths can become an obstacle in understanding the logical structure of mathematics. For most pupils, proofs of apparently evident statements seem to be arbitrarily and pointlessly invented.

Why is it necessary to prove that in a rectangle the diagonals are equal if it is absolutely evident that it is so? The intuitive obviousness of such a statement simply blocks the acceptance of the utility of the corresponding logical proof. To the pupil, the proof appears to be superfluous, that is, arbitrarily

invented, strange to the normal course of sound thinking.

Here some practical teaching recommendations can be made: a) The first mathematical proofs must be introduced in connection with non-evident statements; and b) it is useful to analyze elementary examples of statements which seem to be intuitively evident, but are not correct. Pupils have to learn the necessity of defining rigorously and explicitly the terms they use. Let us take the following example: xy and x_1y_1 are two parallel axes and AB and CD are two perpendiculars on them. The distance between AB and CD is a. Let us draw two curved lines, EF and HG, in such a manner that the distance between two corresponding points (on parallels to xy) remains constant and equal to a. Let us ask the pupils to compare the areas ABCD and EFGH. Generally, the conclusion (intuitively drawn) is that the area EFGH is greater than the area ABCD. But that answer is of course not correct, and it is easy to prove that the areas are equivalent (Figure 1).

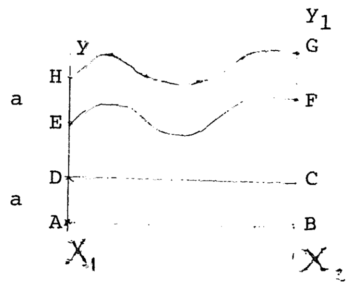


Figure 1

2. As a consequence of their self-evidence, intuitions exert a coercive effect on the processes of conjecturing, explaining, and interpreting various facts.

When some facts or some scientific interpretations appear to be contrary to our natural intuitions, we have a strong feeling of uneasiness. We tend to reject such facts or interpretations or we look for a scheme of explanation which would be able to conciliate the contradictory aspects. Concerning Weierstrass's discovery that there are continuous functions which have no derivatives (and that this is the general case!) Hermite, the French mathematician, has written: "Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions continues qui n'ont pas de derivées" (cf. LeRoy, 1960, 329). Relevant also, is Tannery's statement that: "Je suis très scandalisé par ces points d'abscisse rationnelles qui sont aussi rapprochés que l'on veut de n'importe quel point de l'intervalle (0, 1) et qu'on peut entourer chacun d'un petit segment sans que l'ensemble de tous ces segments recouvrent l'intervalle" (cf. LeRoy, 1960, 329).

In the history of science and mathematics, this coercive nature of intuitions has frequently contributed to a perpetuation of wrong interpretations and to a reluctance to accept the correct ones even when they have been logically proved.

For the school teaching process, the coercive nature of intuition is of essential importance. The following situations may occur when the learned truth contradicts intuitive tendencies:

a. Two opposite interpretations may coexist for a long time without the subject being aware of the contradiction. This becomes especially evident in conflictive situations. For instance, we asked pupils in grade 6 (age 12) to compare the number of points of two segments of different lengths. Some pupils were confused; some answered that in both segments there is an infinity of points. One girl added: "Yes, but the infinity of the longer segment is bigger than the infinity of the shorter one."

A more striking example illustrating, in an experimental manner, the "scandalous behavior of the continuum" (see the quotation from Tannery) is the following: Pupils from grade 5 to 9 (aged 12 to 15) had been presented with the following: "C is an arbitrary point somewhere on a segment AB. We divide and subdivide the segment AB by two, by four, by eight. We continue dividing in the same manner. Question: Will we arrive at a situation such that one of the points of division will coincide with point C?" The great majority of the subjects (about 80%) affirmed that the point would be reached. Starting from at least grade 7, the pupils have learned about irrational numbers; but only a few of them have been really influenced by that fact. One of my university students (who has a B.A. in mathematics), expressed his feelings in the following manner: "Intuitively, I feel that the point C will be reached because the process of division is infinite. On the other hand, I know, from mathematics, that it is possible that the point should not be reached. Surely it will not be reached if the point is an irrational one."

Intuitively, infinity is equivalent to inexhaustible. The intuitive infinity is roughly equivalent to the potential infinity of the mathematicians. In accordance with that primitive meaning, the endless process of dividing the segment will reach, sooner or later every point of the segment. The concept of an hierarchically organized world of infinities has no intuitive background. So, in the above-quoted example, the primitive understanding of infinity coexists with the more sophisticated concept of continuum (learned from mathematics). The pupils know that in addition to rational numbers, there are also irrational numbers. When contrasted with their intuition of infinity, this superstructural knowledge tends to be forgotten.

b. Our above example refers to intuitions coexisting with opposing scientific concepts. A second case is that of intuitions which are so strong that they generally obscure or destroy the corresponding conceptual knowledge. Most pupils asked about the notion of weight would not refer to the force of gravitation. For them, "weight is weight," though they have been taught the correct interpretation. For the naive intuition, the weight of a body is an intrinsic, permanent feature, like hardness or taste and an external factor seems to be superfluous. The intuition of weight, as an intrinsic property is so strong, so completely obvious by itself, that the idea of an external factor as the cause of weight will be eliminated sooner or later.

Another relatively similar example is the following: Fifth and sixth grade pupils were asked to mention the factors which they thought would influence the forces of friction in a certain experimental situation. In front of the subject, there was a wooden rectangular board, horizontally oriented. To the board was attached a pulley. On the board, a parallel piece of wood was placed. By the aid of a string, to which a pan with weights could be attached (and using the pulley), the piece of wood could be forced to move.

Ten fifth-graders, ten eixth-graders (both groups not having studied the laws of friction yet), and 20 sixth-graders (who have studied the laws of friction) were tested. The first question was the following: "We want to keep the object moving with a constant speed on the wooden board. The weight attached to the string must be greater, smaller, or equal to the weight of the object?" The results (in percentages) appear in Table 1. (Ft = the attached weight; Fa = the weight of object).

Table 1

Grades	Ft<Fa (correct)	Ft>Fa	Ft=Fa
V	30	40	30
VI (before)	10	55	35
VI (after)	25	60	15

As can be seen, the effect of teaching on that intuition is practically null. The majority of the subjects consider that in order to keep the object moving constantly we need a force greater than the wieght of the object itself or at least equal (reported in Fischbein, 1975c).

A question relevant to our present discussion was the following: The Ss have been asked if the magnitude of the contact surface will influence the force of friction. 80% in grade 5, 25% in grade 6 (before studying) and 55% in grade 6 (after studying) answered affirmatively. This time, a positive effect of teaching may be remarked, but, still in this case, 55% of the pupils maintained their belief that the magnitude of the contact surface affects the force of friction.

3. A third essential feature of intuition is the capacity to extrapolate. Intuition, like thinking, transcends direct, empirical information. But while logical thinking makes predictions on the basis of rigorously and explicitly established arguments, intuition represents a leap which cannot be completely justified. Westcott used that aspect in order to define the act of intuition: "The event which occurs when an individual reaches a conclusion on the basis of less explicit information than is ordinarily required to reach that conclusion" Westcott (1968, p. 100).

This aspect is not always evident, simply because the apparent obviousness of some intuitions themselves hide the incompleteness of the information on which they are based.. For instance, consider the famous Euclide's postulate: "Through a given point outside a straight line one and only one parallel line can be drawn." The statement is intuitively evident; it does not seem to request additional information or any logical proof. In fact, the statement extrapolates to infinity, in a non-legitimate manner, a very narrow, limited experience. But the apparent obviousness of the statement hides the need (and of course, the impossibility) of further proof.

A second similar example concerns the concept of infinity. Here is a fragment of a protocol: Hor Syb (age 13) "A line can be extended to infinity." How do you know this? "I think so." What is the meaning of "to extend to infinity?" "It never ends." How do you know? The child laughs. No answer (Fischbein, 1963, 328).

In fact, the natural concept of infinity is the concept of potential infinity, the only one which Aristotle admitted as meaningful and which was predominant till Cantor. Mathematicians did not wonder about the possibility of "larger" and "smaller" infinities. Intuitively, there is only one kind of infinity. The apparent obviousness of infinity hides, of course, a lot of inner contradictions and a variety of complex and difficult problems. The fact that the intuitive meaning of infinity includes non-legitimate conjectures is hdden by its own natural strong obviousness. It may be said that the natural intuition of infinity is the direct expression of the extrapolative capacity of intuition. As LeRoy says: "L'intuition simple engendre un infini logique" (1960, 337).

4. Intuition is also described as a global, synthetic view as opposed to analytical thinking, which is discursive in its very nature. LeRoy writes in connection with intuition:

> Il s'agira toujours d'une vision directe et rapide, d'une vue synthétique sans analyse préalable....il est clair que le discours qui chemine pas à pas, qui joint les notions une à une, qui va lentement d'un point à un autre, ne saurait suffire a l'establissement de la science. Constamment s'y ajoute pour l'orienter, pour la vivifier une démarche differente: l'éclair d'illumination subite; la lecture condensée d'un vaste ensemble virtuel dans une brève image (pp. 327-328).

Being a condensed view, intuition is frequently expressed by a visual symbolization. The related visual image may be a primitive non-elaborated one or, on the contrary, the result of a highly elaborated technique of representation. In both cases, it is not the image itself which constitutes the intuition but the synthetic intellectual view which it symbolizes.

To summarize then, the following are the essential features of intuitions: *self-evidence*, *coercive effect*, *extrapolative capacity*, and *globality*.

The classification of intuitions

1. *Affirmatory* and *anticipatory* intuitions. An *affirmatory intuition* is the kind of representation or interpretation which appears to be self-evident. The statement that "A straight line is the shortest way between two given points" is an example of an affirmatory intuition. Most of the examples described above are also of this kind. An *anticipatory intuition* is the preliminary, global view which precedes the analytical, fully developed solution of a problem. LeRoy, without being so explicit concerning that dychotomy, writes about the role of intuition in *understanding* and *invention*:

> D'abord elle est necessaire pour comprendre parce que elle est perception des ensembles dans leur unité', dans leur cohérence intrinseque....A plus forte raison en est-il de même quant il s'agit non plus seulement de comprendre mais de construire et d'inventer (p. 337).

2. A second dychotomy refers to what we have called *primary* and *secondary* intuitions. This classification concerns mainly affirmatory intuitions. *Primary intuitions* refer to those cognitive beliefs which develop themselves in human beings, in a natural way, before and independently of systematic instruction. They are common to all those belonging to a given culture. Elementary spatial and temporal intuitions are of this kind. There is a natural intuition of infinity, of chance, etc. It has been found for instance that subjects aged 12, possess a correct, natural intuitive understanding for the following probabilistic concepts: the concept of chance and of the quantification of chances as the relationship between the number of favorable and

and of all possible equally likely outcomes; the fact that the probability of a compound event is obtained by summing up the probability of the components; the fact that increasing the number of imposed conditions to an expected event diminishes its chances (which corresponds to the multiplication of probabilities). By contrast, there is no natural understanding of the compound character of some categories of events nor of the necessity to inventory the different situations which can constitute the same event (for instance, when throwing a pair of dice, there is no intuitive understanding of the fact that there is a difference between the probabilities of getting the pair 5-5 and the pair 5-6). (Fischbein, 1975b, pp. 138-155)

Secondary intuitions are those which are developed as a result of systematic intellectual training (generally in the school setting). For instance, for Aristotle, a body will keep moving with a constant direction and velocity as an effect of a force acting upon it. This is a primary intuition. The same ideas are accepted by common sense nowadays too. By contrast, for a physicist it seems natural to affirm that a body keeps moving with constant direction and velocity if no force intervenes. In the same meaning, Felix Klein (1898) used the term "refined intuition" and F. Severy wrote about "second degree intuition" (1951).

This classification implies the following fundamental hypothesis: intuitions—though appearing as given (i.e., as produced by some a priori intellectual mechanism)--are, in fact, changeable.

They may be built, transformed, corrected, or eliminated as a result of an adequate training. Nobody has checked that statement yet, and nobody has systematically followed the ontogenesis of intuitions. But it is a belief (albeit an intuitive belief) among at least some outstanding mathematicians and mathematical educationists that intuition may be--and must be--developed in connection with mathematical education.

Feller, the author of the well-known Introduction to Probability Theory and its Applications (1968), has expressed his conviction that in teaching probabilities, the basic problem is to develop adequate intuitions.

Patrick Suppes writes about the importance of developing intuitions for finding and giving mathematical proofs: "Put in another way, what I am saying is that I consider it just as necessary to train the intuition for finding and writing mathematical proofs as to teach intuitive knowledge of geometry in the real number system." (1966, 70).

Hans Hahn, who has sharply criticized the use of intuition in mathematics, writes:

> If the use of multidimensional and non-Euclidian geometries for the ordering of our experience continues to prove itself so that we become more and more accustomed to dealing with these logical constructs; if they penetrate into the curriculum of the schools; if we, so to speak, learn them at our mother's knee as we now learn three-dimensional

Euclidian geometry, then, it will no longer occur to anyone to say that these geometries are contrary to intuition. They will be considered as deserving of intuition status as three-dimensional Euclidian geometry today. (1968,p.188).

Hahn implicitly admits, in the above-quoted lines, that intuition can change, that higher order intuitions can be formed by adequate instruction and, finally, that correct, scientifically validated intuitions are an essential complement to the conceptual framework in science and mathematics education.

It is also important to note that Suppes does not write on "teaching the axiomatic method" but rather on"training the intuition for finding and writing mathematical proofs, etc." He thus implicitly accepts that, in order to build an axiomatic theory (i.e., a formal, non-intuitive and sometimes anti-intuitive one), *what we firstly need is some adequate efficient intuitions.* It is a pity that Suppes did not further develop that idea. He is, himself, an experimented builder of axiomatic systems.

In short, what Feller, Hahn, and Suppes describe in the above-quoted lines is the kind of refined intuition termed here as *secondary.intuition*. It is the belief of these authors that new, specific intuitions can be developed--and must be developed--in order to complement the conceptual part of science and mathematical education. It must be emphasized that the terms *primary* and *secondary* intutions have only a relative meaning. Namely, they are related to a given cultural environment; the distinction is useful and

valuable only in the realm of a certain cultural environment. Our primary space intuitions, for instance, are different from those of people belonging to a different culture.

In this respect, Alan Bishop quotes some striking examples in his paper devoted to "Visualization and Mathematics in a Pre-technological Culture" (1978). For instance, he writes that for the Paiela (an African highland group) "space is not a container whose contents are objects. It is a necessary dimension of the objects themselves." For another group, the Kamano-Kafe of the Eastern Highlands, the four units of length are "long," "like-long," "like-short," and "short." Bishop concludes that "our conceptions of space with its items of objective measurement are not universal nor are they 'natural', 'obvious', or 'intuitive'. They are shaped by our culture. They are taught, they are learnt" (pp. 77). We fully agree with Bishop's affirmation that space conceptions are shaped by the cultural environment—that they are taught, they are learnt. But we disagree with the first part of his statement (that space representations are not"obvious,"are not "intuitive"). In fact, in the context of a certain culture, these "space conceptions" do appear as being "natural," "obvious," and "intuitive". In the educational process, we have to take into account the "obviousness" of these intuitions. We cannot say that we may neglect the existence of such intuitions because they are not natural *in an absolute manner*. When I buy 500 g cheese, the weight of the cheese, is, for me, a reality despite the fact that wieght is only relative. The same cheese has no weight in an artificial satellite! From an epistemological standpoint, it is, of course, a fact of fundamental importance that intuitions are not, a priori, genetically built-in truths. But from a psychological, educational point of view, it is also of fundamental importance to identify categories of interpretations (correct or incorrect) which appear as self-evident and imperative (dispite the fact that they are so only in the realm of a certain culture).

A second remark concerning the relativeness of the dichotomy of primary/secondary intuitions refers to the role of age. Primary intuitions are, generally, themselves developmental phenomena. The intuitive understanding of such ideas as number, distance, proportionality, infinity, etc. is the result of the mental evolution in children. (A lot of information can be found in the Piagetian descriptions). But at a certain age—which depends on the nature and the complexity of the concept—the corresponding representation becomes stabilized, and internally structured. What is then obtained is not a pure concept or a pure, formal statement. The result is <u>an interpretation</u> of a certain aspect of the reality which gets the appearance of reality itself. Instead of a learnt conception (and intuition is a "learnt" conception) the respective interpretation appears to be absolute, internally and fully accepted, the only interpretation which makes sense. So, the mental development of the child is not merely a history of his/her concepts and conceptual structure. It includes also the generation and development of his/her intellectual beliefs. What distinguishes primary from secondary intuitions (in the context of a given culture) is the fact that primary intuitions are developed themselves in a more or less spontaneous, natural way, as a result of the everyday experiences of the child, while secondary intuitions are the result of a systematic, long-run, teaching process. In other terms, secondary intuitions are necessarily constructed in connection with a highly elaborated conceptual system which normally cannot be the result of common, non-systematic, everyday experience. It is evident that nowadays a person cannot arrive normally, as a result of personal experience, at such a concept as multidimensional spaces or a statement such as: "The infinite set of points of a straight line is equivalent to the infinite set of points of a square". What we are stressing here is that such concepts or statements can also be regarded as having the property of obviousness, that is the status of intuitively-accepted truths. They also are intuitions, but they are built on the base of a scientifically elaborated framework. Consequently, both categories—primary <u>and</u> secondary intuitions—are interpretations which are acquired by learning and by personal experience. There is no difference between them as regards their nature and mechanisms. Hence, any distinction might seem to be superfluous. In our opinion, however, it is not superfluous because this distinction emphasizes the following fundamental idea: Scientific conceptions, even those which are very far from an intuitive understanding, can aspire to be intuitively accepted as obvious, self-evident truths. If such a transformation can take place, this may constitute an important argument in favor of the empirical origin of intuitions *in general*.

Of course, one can raise the following question: Why is it useful to attach to scientific ideas a feeling of *belief*? Belief is a religious matter. Intuition is needed when reason must give up. Why, then, should scientific knowledge be associated with this emotional rather obscure, dimension of belief, of self-evidence? It is certainly a legitimate question. Essentially the answer is that given by Poincaré. "C'est par la logique qu'on démontre, c'est par l'intuition qu'on invente" (1914, 137). *Scientific concepts and statements will actually be able to participate in a productive thinking process only if they have been deeply integrated in the person's own mental structure as intuitive accepted truths.* In our opinion, Poincaré's statement that "we invent by intuition..." has to refer to both anticipatory and affirmatory types of intuition. Of course, when inventing, when creating, when solving a problem, we guess before we are able to get a complete, logically valid presentation. But it may also be supposed *that a mathematical concept or statement will actually participate in a productive process of thinking only if we get some direct, sympathetic, intuitive understanding of it.* The great revolution in the concept of infinity took place when Cantor broke with tradition and started thinking of infinity not only as a potentiality or as a pure, abstract construct, but as a reality, something which we can deal

with *as such*, i.e., *as an actual set of elements.* When reading Cantor, one's impression is that he genuinely had a feeling of infinity, of the world of infinities, as a real, hierarchically organized world of sets of elements. *The conclusion of this discussion would then be, that adding a direct, intuitive understanding of concepts, statements and proofs to a formally acquired knowledge will improve the productive capacity of mathematical thinking.*

3. A third classification refers to infra-operational, operational, and post-operational intuitions.

a) *Infra-operational* (or practical) intuitions are those intuitions which synthesize a certain amount of practical experience. As a result of such intuitions, we gain the possibility of evaluating or interpreting a given situation, globally and efficiently. Spatial intuitions and basic probabilistic intuitions are of this kind. They generally do not suppose a previous, explicit, analytical search. It is this category of intuitions which characterizes the intuitive period in Piaget's theory. In fact, such infra-operational intuitions normally survive even after the emergence of operational thinking.

b) *Operational intuitions* play a fundamental role in mathematical thinking and consequently, in mathematical education. They are intrinsically involved in reasoning and can be either primary or secondary intuitions.

In a syllogism, the conclusion is determined by the premises. *But the validity of the syllogism as a method of deducing a truth* from previously accepted premises, cannot be proved. We must accept it by intuition (Ewing, 1941, cf. Westcott, 1968, pp. 17-19). This is an example of operational intuition.

It is by intuition that we accept the universality of inductive inferences. If, in a number of trials, it has been found that iron is electrically conductive, we tend to generalize this finding: Iron is, in general, electrically conductive. What is the basis for such a generalization? No explicit proof can be found for the validity of this operation. The only thing which can be said is that, frequently, a generalization from a finite number of findings to a universal statement has not been contradicted by facts. The universality of an inductive inference has no formal, absolute validity. The right to generalize is accepted by belief, by intuition.

All that has been said about logical inferences with reference to empirical facts is also valid for mathematical reasoning. For instance, what has been called "mathematical induction" is based on an intuitively accepted conviction that, on some mathematical grounds, extrapolation is legitimate. There is also some information available concerning the development of what may be called "logical (i.e., operational) intuition." It has been proved that concrete operational children are able to identify the conclusion which follows from given premises in a categorical syllogism (of the forms AAA and EAE). It is more difficult for them to formulate,

by themselves, that conclusion. For instance: On the table in front of the child, there are four red squares, a yellow triangle, a blue triangle, a yellow ractangle and a blue circle (all of them of plastic). The experimenter asks: "What is the shape of the red figures? The normal answer is, of course: "All the red figures are squares" (the first premise). The experimenter hides the figures from the subject's eyes, removes the non-red figures, and covers the remaining ones (red) with a grating. Through the grating, the shapes of the figures are no longer visible, but the color can be identified. The experimenter asks: "What color are the figures underneath?" The normal answer is: "All the figures under the grating are red." This is the second premise of the syllogism. Then the experimenter asks: "What is the shape of the figures under the grating?" The subject is asked to answer in two different manners: a) to verbally formulate the conclusion, and b) to choose, from four written sentences, the correct one (i.e., the figures under the grating are squares). In the above example, we referred to the mood AAA of the first syllogistic figure. In the same manner, we investigated all four moods of the first and of the second syllogistic figure. The subjects were pupils from grades 2, 3, 4, 6, and 8 (20 pupils for each age).

We found that 80% of children of ages 7-8 (grade 2) were able to identify the correct conclusions in an AAA type: In syllogisms of the form EAE and AII, there were 65% correct answers, while on the mood EIO, for both the first and the second figures, there was only one (5%) correct answer (which may well have been by chance). It may be concluded from these data *that there is a natural, logical competence concerning categorical syllogisms of the form AAA, in concrete operational children* (Fischbein *et al*. 1875a). The child concludes that under the grating there are squares, even though he does not see them. It is this kind of belief which is representative of an operational intuition.

Our point here is that a syllogism, as well as any other logical inference, is not a pure conceptual construction. *It always expresses a more basic extra-logical attitude which is the belief of the validity of that inference*. This will be clearer if we engage in the following hypothetical reasoning: 1) If object A is a metal than it will conduct electricity; and 2) Object A is a metal (it has been identified for instance, as being sodium); then,: 3) Object A conducts electricity. *Nothing here is intuitively evident except the validity of the inference.* It may be argued that this just represents a mental habit. Maybe, but it is not only a mental habit: *it is a form of knowledge accompanied by a feeling of intrinsic conviction*. The algorithms for multiplying or dividing two numbers are also learned and they finally become mental habits--but they do not have the intrinsic evidence of an intuition.

Generally speaking, we may safely say that the axioms of logical thinking are, in fact, based on such fundamental beliefs. They constitute the domain of operational intuitions. Mathematical education (and generally intellectual education) should not be satisfied with

training blind automatic intellectual skills corresponding to the formal laws of logical thinking. Such blind rules do not work by themselves in an actual problem-solving process. They may work, in solving blind exercises. We can teach a pupil the truth table of implication. It does not follow that he will use it naturally in a thinking process if the corresponding intuitions have not been built. Let us take an example: 1) If quadrilateral A is a square then its diagonals are equal; and 2) A is *not* a square. Pupils tend, naturally to conclude that A does not have equal diagonals. On the other hand, from the statement: "It has been proved that the diagonals in figure A are equal," pupils tend to conclude that A is a square. When using an implication $p \to q$, children aged 12-13 do not distinguish naturally between the *uncertain* conclusion which can be drawn by affirming q and the *certain* rejection of p when negating q. (See also Carrol, 1975).

Again, the educational problem is not only that of building a set of mental skills for logical thinking. New intellectual beliefs, i.e., intuitions, have to be built. Their role is not only to suggest or to confirm inferences. *Their function is also to measure, logically as well as subjectively, the fullness or, on the contrary, the incompleteness of an argumentation:* "I *feel* that something is wrong in my argumentation"; "I *feel* that my argumentation is missing some elements." Such feelings are possible only if sound, active, operational intuitions have been built. The intuitions for finding and giving mathematical proofs mentioned by

Suppes are (in our classification) <u>secondary</u> operational intuitions.

c. <u>Post-operational intuitions.</u> It is also possible to describe some intuitions as post-operational. After having solved a difficult problem, we are generally able to grasp the entirety in a synthetic, unique, meaningful view. That final step, expressing a feeling of completeness, has all the features of intuition. What had been previously a laborious, difficult, hesitating, step-by-step search-process resulted in a structured, fully accomplished self-evident entity. This final intuition also represents a kind of verification. In this stage, the role of intuition is to validate the intellectual construction <u>as a whole</u>. What a fascinating thing: A global, preliminary intuitive solution must always be followed by a detailed analytical, logically structured argumentation, but we only feel safe with this explicit, logically valid construction if it is in turn completed by such a synthetic, global, intuitive view of the whole argumentation. As Hadamard writes: "Any mathematical argument, however complicated, must appear to me as a unique thing. I do not feel that I have understood it as long as I do not succeed in grasping it in one global idea and...this often requires a more or less painful exertion of thought" (1949, 65-66).

I would strongly support that in mathematics education special attention should be given to training in pupils that capacity for synthetizing in a unique, global view, a long and complex argumentation.

References

Bishop, A.J. Visualising and Mathematics in a Pre-Technological Culture. Paper presented at the Second International Conference of the IGPME, Osnabrück, 1978.

Carrol, C.A. Low Achievers' Understanding of Logical Inference Forms, in "Children's Mathematical Concepts", M.F. Rosskopf (Ed.), Teachers College Press, New York, 1975.

Feller, W. An Introduction to Probability Theory and its Applications. New York: John Wiley, 1968.

Fischbein, E. Conceptele Figurale. Editura Academiei Republicii Socialiste România, Bucuresti, 1963.

Fischbein, E., I. Barbat, I. Minzat. Syllogistic Reasoning in Children and Adolescents. Revue Roumaine des Sciences Sociales, Série de Psychologe, <u>19</u>, 1, 1975a, pp. 21-33.

Fischbein, E. The Intuitive Sources of Probabilistic Thinking in Children. Reidel, Dordrecht, 1975b.

Fischbein, E. , I. Minzat, I. Barbat. Dezvoltarea capacitătii de investigare experimentală la levi. Revista de Psihologie, 4, 21, 1975c, pp. 409-423.

Hadamard, J. An Essay on the Psychology of Invention in the Mathematical Field, Princeton Univer. Press, Princeton, 1949.

Hahn, H. Geometry and Intuition, reproduced in "Scientific American", San Francisco: W.H. Freeman and Comp., 1948, 1968, pp 184-188.

Klein, F. Conférences sur les mathématiques. Conference VI, A. Hermann, Librairie Scientifique, Paris, 1898.

LeRoy, Ed. La pensée mathématique pure, Paris, P.U.F., 1960.

Poincaré, H. Science et méthode, P.U.F., Paris, 1914.

Severi, F. Intuizionismo e astratismo nella matematica contemporare, in Atti del terzo Congresso dell UMI, Ediz. Cremonese, Perella, Roma.

Suppes, P. The Axiomatic Method in High School Mathematics, in "The Role of Axiomatics and Problem Solving in Mathematics," The Conference Board of the Mathematical Sciences, Washington, D.C.: Gin and Co., 1966.

Westcott, M.R. Toward a Contemporary Psychology of Intuition, New York: Holt, Rinehart and Winston, 1968.

Wild, K.W. Intuition. Cambridge University Press, 1938.

THE UNDERSTANDING OF FRACTIONS IN THE SECONDARY SCHOOL

A report of the research carried out by the CSMS team at Chelsea College, London University.

Kathleen Hart

The Project "Concepts in Secondary Mathematics and Science" has been investigating secondary school pupils' understanding of various mathematical topics. In the summer of 1977 a thousand children aged 12+ to 15+ were tested on the topic of Fractions. Two test papers were given, the older children being required to deal with multiplication and division of fractions in problems while the younger ones were given problems which could all be done by addition and subtraction. The initial ideas for items on the test papers came from a group of teachers who interviewed the children they taught to find out their reasoning when dealing with the problems. The two papers each had two parts, one section presented problems and diagrams, the second section was simply a list of computational questions. The problems in the first section had parallel computation examples in the second section. The sample consisted of approximately three hundred children from each year group. Four schools were used for each of the four age ranges, the four schools together reflected the IQ distribution in the British child population. Thus the sample was taken from fourteen schools and was representative, both by year and in total, of the normal IQ range.

The papers of the 12+ and 13+ children were analysed separately from those of the older children and then later those items which were common to the two papers were compared. (In this paper the results for the 12+ and 13+ group will be reported). The analysis was designed to try to form a hierarchy of problem types matched against facility. Problems were deemed to be of the same type if the same children successfully completed them. Two homogeneity coefficients

$$\emptyset = \frac{bc - ad}{\sqrt{(a+b)(c+d)(a+c)(b+d)}} \qquad (1)$$

and

$$H_{Loevinger} = \frac{bc - ad}{(b+d)(c+d)} \qquad (2)$$

were used, the letters abcd are obtained from the pass/fail matrix of any two items

a,b,c,d number of children

		Item 1	
		Fail	Pass
Item 2	Pass	a	b
	Fail	c	d

Maximum homogeneity occurs when $\emptyset$ or H is 1.

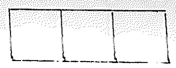
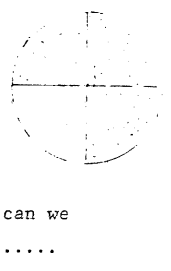
Where $d < a$

For items to be grouped within a certain facility band they had to possess a ∅ value of a particular level with each other. In order to form a chain of groups there also had to be links with items in other facility bands. The "cut-off" point for any group was based predominantly on the different facilities exhibited by the children's responses to the paper, with some interpretation as to mathematical demand. Any items which had low values of ∅ with other items of similar facility were for the moment discarded. Finally the technique of Guttman Scalogram Analysis was applied (taking a 2/3 pass mark for each group) to see how many children did not perform according to the hierarchy of stages being postulated.

The hierarchy for the problems given the children aged 12+ to 13+ appeared to be as follows:-

[each group is illustrated by a typical item]

n = 555

% facility		Fraction Problems Age Range 12+, 13+	
90	A	Labelling parts of a whole	Shade in two thirds of this shape 
80 70	B	Fraction of a number Equivalent fractions with 2 multiplier Addition $\frac{3}{8} + \frac{2}{8}$	In a baker's shop $\frac{3}{\text{[illegible]}}$ of the flour is used for bread and $\frac{2}{\text{[illegible]}}$ is used for cakes. What fraction of the flour has been used?
67 [illegible]	C	Comparison of 2 fractions Finding 2/3 of whole when whole not in thirds.	Put these fractions in order of size, starting with the smallest: $\frac{1}{4}$ $\frac{1}{2}$ $\frac{1}{100}$ $\frac{1}{3}$ ....
55 45	D	Operations on fractions	Shade in $\frac{1}{6}$ of the dotted section of the disc. What fraction of the whole disc have you shaded? .... 
30	E	Two operations on fractions	How many pieces of wood $1\frac{1}{4}$cm long can we get from a piece $8\frac{3}{4}$cm long? ....

The facility gap between groups is narrow, further analysis is being carried out to test other demarcations.
The hardest item (2%) does not appear.

Six per cent of the children obtained a 2/3 mark on a harder group of items without obtaining a similar pass mark on all easier groups. The Guttman Scalogram Analysis gave a coefficient of Reproducibility of .976 and a Coefficient of Scalability of .915. When one assigned to a child the hardest level in which he obtained a 2/3 mark we have:-

Not even at Level A	9.4%
Level A	14.6%
Level B	8.8%
Level C	11.5%
Level D	29.2%
Level E	20.5%

The computational items were subjected to the same analysis and produced the following:-

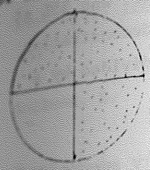
Computation n=555

Facility %

70 – 60	Addition with same denominator	$\frac{2}{8} + \frac{3}{8}$
	Using 'OF' with whole numbers	2/3 of 12
55 – 45	Addition with different denominators	$\frac{1}{3} + \frac{1}{4}$
	Multiplication of a fraction by a whole number	$1\frac{1}{4} \times 7$
36 – 32	Addition with mixed numbers or with different denominators and numerators $\neq$ 1	$32\frac{2}{3} + 5\frac{1}{4}$
	Division of 2 whole numbers or 2 proper fractions	$3 \div 5$
10½	$40 \div 10\frac{1}{2} = 3\frac{17}{21}$	

There was only one computation item solved by over 72% of the children, that was 3 x 10½ (facility 79%).

It had been hypothesised that the computations would be easier but this was certainly not the case particularly when a diagram was the alternative. For example:



Shade in $\frac{1}{6}$ of the dotted section of the disc.

What fraction of the whole disc have you shaded? ....

Facility 52.8%

Computation $\frac{1}{6}$ of $\frac{3}{4}$ Facility 22.9%

The 12+ group was rather better at computation than the 13+ group, whereas the problems were completed at about the same facility level. It was obvious that division was avoided if another method was available. For example:

How many bicycle spokes 10½cm long can be cut from a piece of wire 40cm long?	Facility 72.1%
What length of wire is left over?	Facility 54.8%
Computation 40 ÷ 10½	8.8% Facility

OR

A relay race is run in stages of $\frac{1}{8}$Km each. Each runner runs one stage. How many runners would be required to run a total distance of $\frac{3}{4}$Km? ....

Facility 45.8%

Computation $\frac{3}{4} \div \frac{1}{8}$ Facility 31.5%

Both are of course solvable by repeated addition, the method which is presumably used in the problem form but not in the computation form. The symbolisation in the computation appeared to have prevented the transfer of the method. The inability to transfer was starkly displayed in two adjacent questions:-

3 x 10½ (facility 79%)

40 ÷ 10½ (Answers $3\frac{17}{21}$ or 3 remainder 8½ facility 21.8%).

The children showed a lack of ease in handling fractions in that 37% at 12+ and 30% at 13+ preferred the answer 4cm remainder 1cm for the question "A piece of ribbon 17cm long has to be cut into 4 equal pieces. Tick the answer you think is most accurate for the length of each piece.

a) 4cm remainder 1 piece

b) 4cm remainder 1cm

c) $4\frac{1}{4}$cm

d) $\frac{4\text{cm}}{17}$ "

Similarly 24% of those aged 12+ and 17% of those aged 13+ declared that the answer to 15 ÷ 4 was 3 remainder 3. The question "What is 3 ÷ 5?" was solved by about 30-35% in each year whether it was in problem or computation form. The computation form of the question produced 30% at 12+ and 35% at 13+ who read the question as 5 ÷ 3 or gave $1\frac{2}{5}$ as the answer, demonstrating possibly the lack of acceptance that a small number could be divided by a larger.

By far the most difficult question was:

> Mary and John both have pocket money. Mary spends $\frac{1}{4}$ of hers, and John spends ½ of his.
>
> Is it possible for Mary to have spent more than John? ......
>
> Why do you think this? ....
>
> ....

The facilities were

12+	13+	14+	15+
1.6%	2.3%	1.9%	3.7%

A common mistake was to state that Mary could have spent more than John if she had more to start with, without adding that she should have at least twice as much. The percentage of children doing this was:-

12+	13+	14+	15+
32.1%	38.8%	36.7%	46.0%

However a substantial number of children in each year stated that ½ was greater than $\frac{1}{4}$:-

12+	13+	14+	15+
41.5%	34.3%	27.6%	19.1%

It seems likely that the younger children had absorbed a rule "$\frac{1}{2} > \frac{1}{4}$" without taking into account the circumstances in which it could be applied. It was also apparent that "rules" had been misremembered or in desperation invented. One question required the number which replaced △ in $\frac{2}{7} = \frac{\Box}{14} = \frac{10}{\triangle}$. The answer △ = 21 or 28 occurred in 20% of the responses. The child seems to have been searching for a number pattern occurring in the denominators.

A popular misconception when adding fractions was that the "rule" was "add numerators, add denominators" e.g.

$\frac{3}{8}+\frac{2}{8}= \frac{5}{16}$ (8.5% at 12+, 19.7% at 13+)

$\frac{2}{7}+\frac{3}{4}= \frac{5}{11}$ (18.7% at 12+, 30% at 13+)

Note that the younger children who were nearest the teaching of addition of fractions committed this particular error less frequently. In the word probl form of 3/8 + 2/8, the answer $\frac{5}{16}$ occurred less frequently, 4.9% at 12+, 7.4% at 13+.

References

① Horst, P. "Psychological Measurement and Prediction" Wadsworth Pub. Co. 1966 pp.93-95

② Loevinger, Jane "A systematic approach to the construction and evaluation of tests of ability." Psychol. Monogr. 1947, 61, No.4, pp.36-37.

③ Guttman, L. "The basis for scalogram analysis". Measurement and and Prediction. Princeton Univ. Press 1950

Green, B.F. "A method of scalogram analysis using summary statistics" Psychometrika 1956, 21, 77-78.

POINTS AND ROUNDS
(Abstract)

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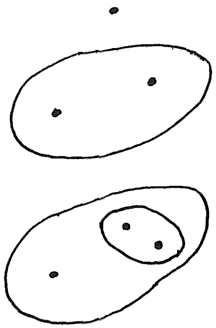
This research has been carried out to help school-masters about teaching operations in a mathematical way beside the usual, practical, manner. How could they diversify representations for their pupils ? For instance, what is possible to introduce brackets ? This experiment has been conducted at three levels : last class of nursery school (age : 5 - 6), first (age 6 - 7) and second (age 7 - 8) classes of elementary school.

With the youngest children, the rule has been presented like that : " You are allowed to rub one point out. As soon as you do so, you must replace it by a round in which you put two new points"

Ex :

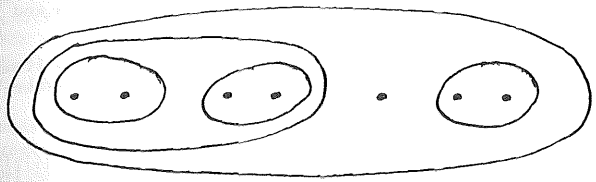


Then, pupils have been asked to run this rule. If necessary, we gave the following precision : if the point we rub out is inside a round, the new round we draw must be itself inside this round. Various exercises have been proposed, from suitable diagrams :

- to put one round or several rounds
- to place points
- to put rounds
- to correct diagrams, etc...

In the first class of elementary school, a constraint is added to the rule : "All the points must be on the same line" (horizontal line). Many exercises are possible.

Ex : Here is a diagram :



Is it correct ? Children reply : "No !". Why ? etc...

With older children, the rule becomes, for instance :

Ex :

a may be replaced by (b a)
b may be replaced by (a a)
or (b b)
or (a b)

and the exercises, from suitable diagrams :

- to lengthen what is written
- to shorten it, etc...

This experiment brought an interesting possibility to study operations and introduce the use of brackets. But the main interest concerns pupils' mathematical formation.

Secondary School Children's Understanding of Reflection and Rotation.

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This paper reports the results of a test, entitled "Reflection and Rotation", that was given in the summer term 1977 to a standardised sample of about 1000 children: 293 2nd year, 449 3rd year and 284 4th year secondary school pupils, whose mean ages would have been about 13:04, 14:04 and 15:04 respectively.

The test consists of 16 questions containing altogether 56 items, with an additional 6 trial items, and takes about 1 hour to administer.

REFLECTION items.

Figure 1

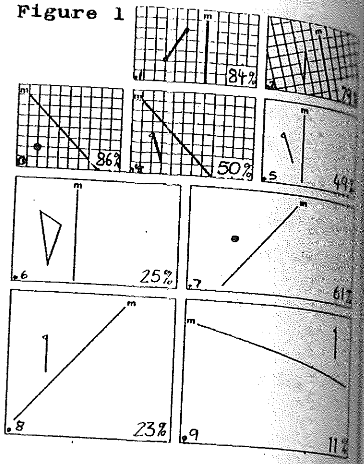
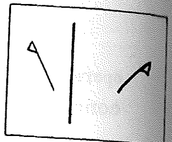
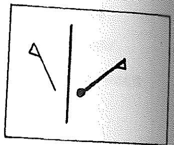


Figure 1 shows the items from question A1 which asks the children to "reflect in each mirror line m, and draw your answers free hand; do not use a ruler". The facilities shown are the percentages of 3rd year pupils answering each item correctly. (On the whole, the 2nd and 4th year facilities rarely differed from these by more than 5%.)

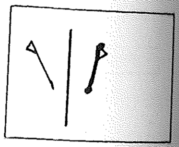
From interviews with individual children and from answers to the written paper, it would seem possible to divide pupils' approaches into 4 types, which might be called
GLOBAL,
SEMI-ANALYTIC,
FULLY-ANALYTIC, and
ANALYTIC-SYNTHETIC.



At the GLOBAL level, the object that is to be reflected is considered as a whole, and the child seems to regard the transformation as some kind of single, all-in-one action involving an intuitive notion of moving "across" the mirror and of producing some kind of "balance" or symmetry between object and image. No reference is made to specific parts of the object or to specific angles or distances.



The SEMI-ANALYTIC approach differs from the global in that the child first reflects part of the object (the base-point of the flag, say) but then immediately draws in the rest of the image (the stem of the flag), concentrating more on its slope or its length than on the location of the other end-point.



With the FULLY-ANALYTIC approach, the child reduces the object to a set of points (2 for a flag, 3 for a triangle) and then reflects each of these before drawing any image-lines. This is basically a much more successful strategy but its limitations can be seen in the adjacent diagram: both end-points of the flag have been estimated quite well but the slope of the stem looks wrong, being the result of simply joining the image-points without reference to the slope of the object. It is the coordination of these two aspects, namely locating a series of image-points but at the same time assessing the slopes, lengths, the total image that derives from joining these points, that is demonstrated by the ANALYTIC-SYNTHETIC approach.

The choice of which of these 4 approaches the child uses seems to vary not only with cognitive level but with the complexity (or the strength of the "distractors") of the items. There seem to be 3 important distractors:
the NUMBER OF POINTS to which the object can be reduced,
the SLOPE OF THE OBJECT, and
the SLOPE OF THE MIRROR LINE.

Thus, while the child may successfully reflect the single point in A1.3 he may revert to a purely global approach, with no part of the object reflected precisely, when confronted with a flag (2 points) as in A1.4 .

Also, there is a tendency to draw images parallel to the object, particularly if the slope of the object is horizontal or vertical as in A1.8 and .9 . This condition also reinforces the tendency to reflect the whole object horizontally or vertically regardless of the slope of the mirror.

A marking scheme was devised for the test that allowed for the use of up to 10 categories for classifying the different responses to each item. (The response-categories for item A1.8 are shown in Figure 2a.)

For each item, the categories were ordered by evaluating the mean score, on the test as a whole, of all those pupils producing responses in a given category. Categories with similar means were then grouped together, thus producing for each item a limited number of clearly distinguishable levels of response. (Thus the 10 categories for item A1.8 were grouped into 4 levels, as shown in Figure 2b.)

The mean scores were then compared across items, and in this way it proved possible to classify the responses to the reflection items as a whole into 5 levels (with an additional level for the rotation items and the items involving combinations of reflections and rotations).

These levels of response are illustrated in Figure 3 below, using the items from question A1. A description of the levels follows.

Figure 2a

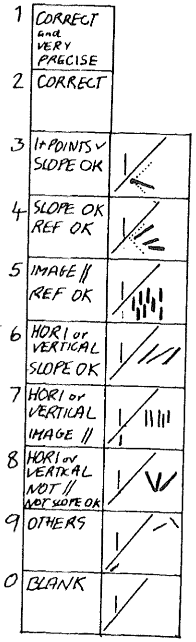


Fig 2b

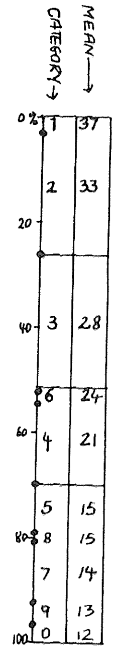
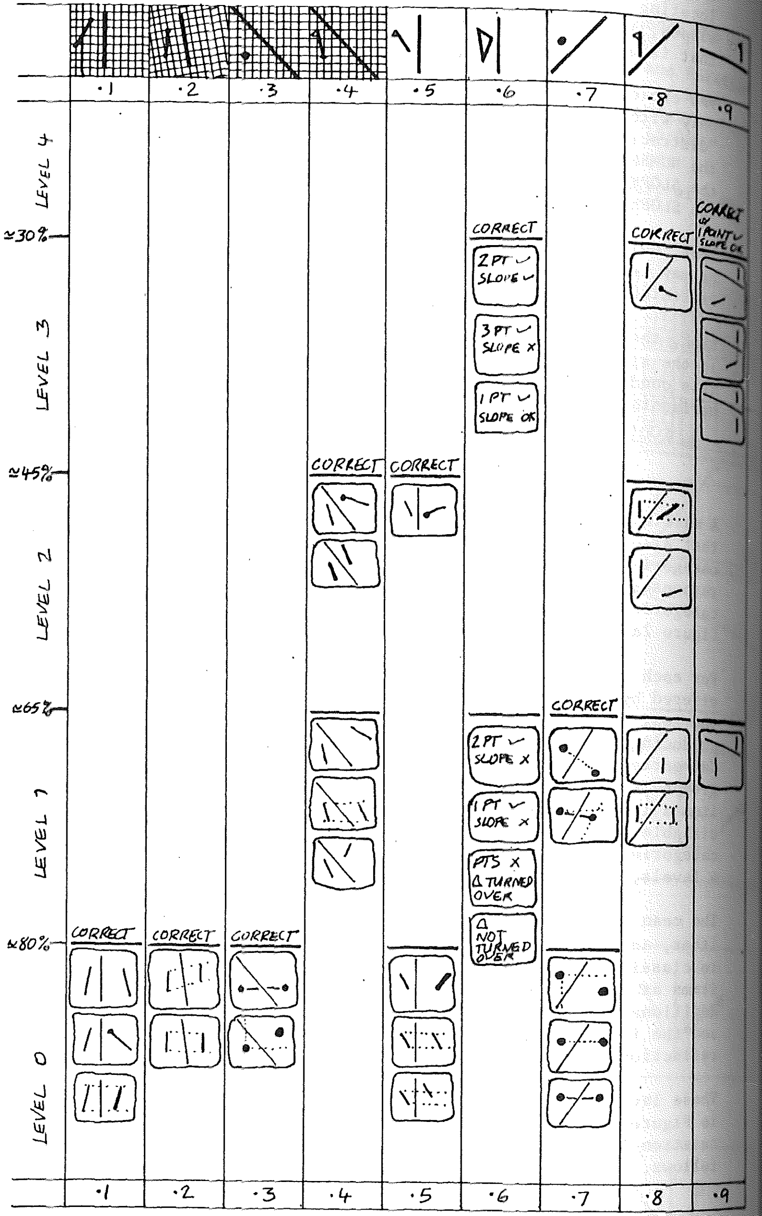


Figure 3.



LEVEL 0. Responses at this level show at best adequate control over just one aspect of any given problem. Thus in item .1 the slope of the image may be drawn correctly OR one of the end-points may be correct; in .3 and .7 the single image-point may be at the correct distance from the mirror (but reflected horizontally) OR the tendency to reflect horizontally may be partially overcome (but with no control of distance). Responses to .5 are at best globally correct.

LEVEL 1. Here the responses show the beginnings of control over two aspects of a problem. Thus,with the help of a grid, children cope successfully with a flag (2 points) when the mirror line is vertical (.1) and with a single point when the mirror line is at 45° (.3); however, the introduction of a second point may reduce the responses to virtually random efforts (.4). Without a grid children still cannot cope fully with a slanting mirror, even when reflecting a single point (.7), nor can they overcome the distraction of a vertical flag (.8 and .9).

LEVEL 2. Responses are now successful with a single point and a slanting mirror (.7); reflections of a flag are likely to be semi-analytical (slanting mirror and grid, or vertical mirror without grid) or global if the flag is vertical (.8).

LEVEL 3. Children can now reflect both points of a flag correctly on a grid, or if the mirror is vertical. They can also cope analytically with the "off the page" reflections that occur in questions A6 and A7. However, their approach may still be only semi-analytical in .8 and even global in .9 . With the triangle (.6) children tend to focus on the points or on the slopes of the sides, without fully coordinating both aspects. Thus they may establish 3 points correctly but not one of the slopes, or the slopes correctly but not one of the points.

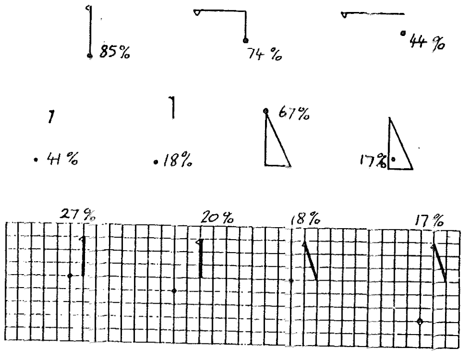
LEVEL 4. At this level, responses show a successful coordination of position and slope for the triangle (.6) and for the vertical flag in .8, though still not necessarily in .9 .

ROTATION items.

All rotation items involved rotations through 90° degrees anticlockwise. The items of question B1, together with the 3rd year facilities, are shown in Figure 4.

Where children tended to reflect horizontally or vertically, the tendency with rotation is to move the object directly "across" the centre of rotation, (by reflecting or by rotating through 180°). Or the line joining the image to the centre is drawn vertically or horizontally regardless of its original slope.

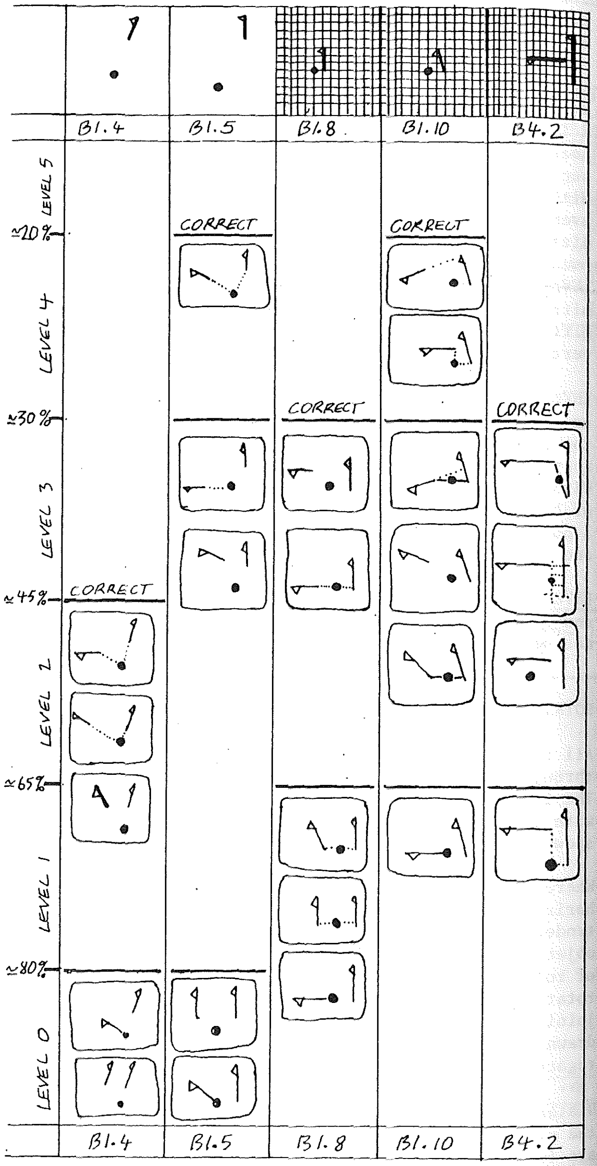
Figure 4.



Distances are more difficult to estimate (the lines joining object and image to the centre are at right angles, whereas the corresponding lines to the mirror are in the same direction), and the grid, instead of being of help, may mislead children into placing points "symmetrically" with respect to the centre, rather than at a right angle.

The different levels of response, for a selection of items from question B1 (and one item from B4, which involves locating a centre of rotation) are illustrated below, in Figure 5.

Figure 5.



Further analysis of the test.

To get an indication of the degree of consistency of levels of response across items, it was felt necessary to go beyond the comparison of means refered to earlier. To this end, the correlations between all item-pairs (assessed on a right-wrong basis) were calculated, using the PHI coefficient. As had previously been found on other CSMS tests, the resulting PHI values tended to be higher between items of very similar content (being parts of the same question or from similar questions) than between items from different questions. However, generally the links between reflection and rotation items did not appear noticeably weaker than the links within reflection and rotation items. This seemed rather surprising and it was decided to test for the possible existence of separate reflection and rotation "factors".

Items of a similar content were grouped together into 3 reflection groups and 3 rotation groups (some of the harder rotation and combinations items were initially left out in order to produce groups with comparable score distributions). These 6 groups were then subjected to a factor analysis (using the Principal Factor program, PA2, from SPSS).

In accordance with what had been suggested earlier, (from the inspection of the PHI links across reflection and rotation items), only one factor with eigenvalue greater than 1 was extracted (Figure 6). (The eigenvalue of the second factor still remained below 1, when the harder items were added to some of the rotation groups or were used to form new groups -despite the resulting disparity in score distribution between some of the groups.) This one-dimensionality of the 6 groups is illustrated by the basic uniformity of the correlations (Pearson's r) in Figure 7.

Figure 6.

Factor	Eigenvalue	% of Variance
1	3.53	58.8
2	.76	12.6
3	.53	8.9
4	.47	7.8
5	.39	6.6
6	.32	5.3

Figure 7.

	ref2	ref3	rot1	rot2	rot3
ref1	.56	.48	.49	.31	.36
ref2		.59	.65	.46	.42
ref3			.63	.49	.50
rot1				.53	.56
rot2					.48

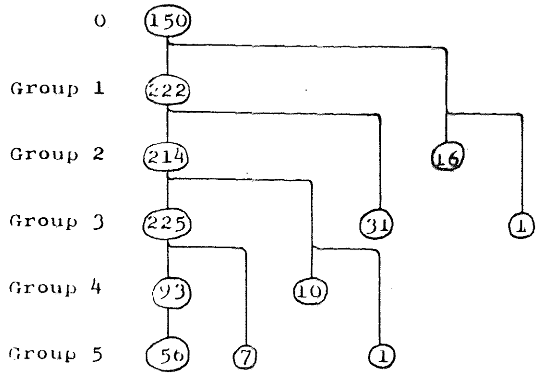
The results of the factor analysis suggested that it would be justifiable to group items from across the whole test; it was thus decided to form 5 groups, with items being chosen to conform to the levels 1 to 5 refered to earlier. Items with excessively low PHI values were left out, although it was felt desirable (from the point of view of reliability) to keep the groups as large as possible. The resulting groups each had a mean PHI between all item-pairs of above .30 . Whether this value can be judged to be sufficiently high is difficult to assess (or perhaps it is a meaningless question); certainly the value is lower than values obtained with some of the other CSMS tests, but this may be due in part to the relative unreliability of items that involve drawing, as opposed to "discrete" right or wrong answers.

The 5 groups were subjected to Guttman scalogram analysis, the resul of which are shown below. (The criterion for success on each group w set as near to 2/3 of items correct as was possible.)

The proportion of pupils' responses that do not form a perfect scale is about 6.4%. Though this is higher than on some other CSMS tests, it would seem nonetheless that the description of pupils' performanc in terms of the levels outlined above is reasonably coherent and meaningful.

		Group 5	Group 4	Group 3	Group 2	Group 1	total
pass rate		9/13	5/7	5/8	4/6	7/11	
mean PHI		.34	.33	.38	.31	.32	
Number of groups passed	5	56	56	56	56	56	56
	4	7	93	100	100	100	100
	3	1	10	225	236	236	236
	2	.	.	31	214	245	245
	1	.	.	1	16	222	239
	0	.	.	.	.	.	150
total		54	159	413	622	895	1026
%		6	15	40	61	84	
errors		8	10	32	16	0	66

REPRODUCIBILITY = .97 COEFFICIENT OF SCALABILITY = .89



Some Trends in Research and the Acquisition and Use of Space and Geometry Concepts

Richard Lesh

Northwestern University

This paper consists of two parts. The first part outlines the results of a multi-university research program dealing with space and geometry concepts which began in 1975. Papers from the research conference which initiated this effort are published in a monograph, Space and Geometry: Papers from a Research Workshop, edited by Martin (1977). A second monograph, Recent Cooperative Research Concerning the Acquisition of Spatial and Geometric Concepts, (Lesh and Mierkiewicz, 1978) contains fourteen research reports from projects associated with the above research effort. This later monograph will be available for the first time in September, 1978--just in time for the Osnabrück Conference.

The second part of the paper will describe a major new cooperative research program that grew out of the space and geometry research described above. A research conference to initate this new effort was held at Northwestern University on January 5-7, 1978. Papers from this conference will appear in a monograph titled, Applied Mathematical Problem Solving (Lesh, Kantowski and Mierkiewicz). This monograph is scheduled to go to the printers in September, 1978. All three of the above monographs are available from the ERIC center at Ohio State University.

Part I: A Cooperative Research Project on Space and Geometry Concepts

In the spring of 1975, the Georgia Center for the Study of Learning and Teaching in Mathematics (GCSLTM) sponsored a series of five research workshops involving (a) teaching strategies in mathematics, (b) number and measurement concepts, (c) space and geometry concepts, (d) models for learning mathematical concepts, and (e) problem solving. This first part of this paper focuses on the activities of the working group that developed from the space and geometry workshop.

Some Assumptions Underlying the Work of the Space and Geometry Group:

The existence of a successful, nonfunded, multi-institutional, cooperative research effort is in itself a significant research innovation in mathematics education. Because of the complexity of most of the important issues in mathematics education, most issues will require intensive study by various individuals and long-term commitments and coordinated research efforts from groups of people. Furthermore, the optimum time to establish connections among individual research efforts is while project plans are in the formative stages--not a year or two after projects have been completed and the reports are appearing in journals or at conferences. When research projects are in the planning stages, it is helpful for groups of individuals working together to (a) identify basic issues that are important to as many people as possible, (b) formulate answerable questions related to the basic issues, (c) coordinate individual projects so simplistic conclusions can be avoided, and (d) establish better bases of communication so individuals can profit by (and build on) the work of others. If research is restricted to answering questions through isolated studies, then asking important questions seems almost irreconcilable with asking an answerable question; and if the language and underlying theoretical constructs are not consistent across studies, the results are difficult to interpret or use.

Papers Presented at the 1975 Space and Geometry Workshop

To review a broad survey of past research, to develop a language for dealing with these problems, and to describe some of the most important directions for future research, the following papers were presented at the 1975 space and geometry workshop.

Mathematical Foundations of the Development of Spatial and Geometrical Concepts.................................... Edith Robinson

Piaget's Thinking about the Development of Space Concepts and Geometry..Charles D. Smock

Breakthroughs in the Psychology of Learning and Teaching Geometry..Izaak Wirszup

Recent Research on the Child's Conception of Space and Geometry in Geneva: Research Work on Spatial Concepts at the International Center for Genetic Epistemology..........Jacques Montangero

Needed Research on Space in the Context of the Geneva Group ....Jacques Montangero and Charles D. Smock

Cross-Cultural Research on Concepts of Space and Geometry ....Michael C. Mitchelmore

Transformation Geometry in Elementary School: Some Research Issues ....Richard Lesh

The proceedings of the space and geometry workshop were published in a book, Space and Geometry: Papers from a Research Workshop (Martin, 1976). In addition, several other Georgia workshops included papers pertaining to the acquisition of spatial and geometric concepts. For example, the "models" workshop (Osborne, 1976) included Martin's article, "The Erlanger Program as a Model of the Child's Conception of Space," and the "number and measurement" workshop (Lesh, 1976) included several papers emphasizing the close links between the development of spatial/geometric concepts and the development of number/measurement concepts.

From the beginning, the space and geometry group considered geometry to be a context to investigate general questions about concept acquisition. That is, research concerning the development of spatial concepts should not be an area of interest only to those who want to teach geometry. For example, many of the models and diagrams teachers use to introduce arithmetic and number concepts presuppose an understanding of certain spatial/geometric concepts. Consequently, misunderstandings about number concepts are often closely linked to misunderstandings about the models used to illustrate them. For this reason, one of the long-range goals of the space and geometry working group was to furnish information to help educators devise "better" instructional models for teaching basic arithmetic and number concepts--and to study some of the "modeling" processes students use to link abstract mathematical concepts to real world phenomena.

After the initial GCSLTM workshops, the space and geometry group maintained communication through a series of meetings at Northwestern University, at the University of Georgia, and at professional meetings. As a result of these planning sessions, the following set of 14 coordinated research projects were completed and published as the monograph, Recent Cooperative Research Concerning the Acquisition of Spatial and Geometric Concepts (Lesh and Mierkewicz, 1978). The following table of contents gives a fairly clear idea about the major themes considered in these 14 studies. A description of these themes will be given in the next section of this paper.

"I. Studies Concerning Pre-Operational Concepts

Perception, Imagery, and Conception........R. Lesh & D. Mierkiewicz
Northwestern University

Apparent Memory Improvement Over Time?!!...R. Lesh
Northwestern University

Haptic Perception, Construction, and Drawing of Geometric Shapes by Children Aged Two to Five..................K. Fuson & C. Murray
Northwestern University

The Role of Motor Activity in Young Children's Understanding of Spatial Concepts..........................J. Musick
Northwestern University

Mathematical Foundations for the Development of Spatial Concepts in Children.....................................I. Weinzweig
University of Illinois
Circle Campus

II. Studies Concerning Transitional Stages from Concrete to Formal Operations

(a) Studies about Rigid Motions

Understanding of Selected Transformation Geometry Concepts...........
D. Thomas
Ohio State University

Variables Influencing the Difficulty of Rigid Transformations........
K. Schultz
Georgia State University

Conservation of Length: A Function of the Mental Operation Involved...
R. Kidder
Longwood College

An Investigation into the Effect of Instruction on the Acquisition of Transformation Geometry Concepts in First Grade Children and Subsequent Transfer to General Spatial Ability.........................
F. Perham
University of Illinois
Circle Campus

(b) Studies about the "Middle" Geometries: Affine Transformations, Similarities, and Projections

An Analysis of Research Needs in Projective, Affine, and Similarity Geometries, Including an Evaluation of Piaget's Results..............
K. Fuson
Northwestern University

The Child's Conception of Rational Distances--An Affine Invariant.....
L. Martin
Missouri Southern
State College

III. Studies Concerning Older Subjects or Formal Operational Concepts

Cognitive Studies Using Euclidean Transformations....................
J. Moyer
Marquette University, an
H. Johnson
Syracuse University

Understanding of Frame of Reference in Preservice Teacher Education Students.......................................C. Dietz & J. Barnett
Northern Illinois
University

Visual Influences of Figure Orientation on Concept Formation in Geometry.......................................N. Fisher
Research Department
Chicago Board of Education

IV. An Analysis of Past Efforts and Directions for Future Research

....A. Coxford
University of Michigan

Issues Considered by the Space and Geometry Group

The following issues were central themes in the 1978 research monograph:

1. What general principles can be found for anticipating the relative difficulty of mathematical ideas? For example, if a child is operational (in the Piagetian sense) with regard to one concept, what does this imply about the child's ability to learn other "related" ideas? Van Hiele, Freudenthal, and several Soviet psychologists (e.g., Yakimanskaya, 1970 and Sergeevich, 1971) have described the way they believe geometric concepts evolve in children. Yet basic controversies and ambiguities occur in each of these theoretical descriptions; and the controversies strike at the heart of many of the most basic issues in developmental psychology. If it is possible to find techniques to anticipate the relative difficulty of geometric concepts, then similar techniques may be useful to organize the sequential presentation of arithmetic and number ideas--or instructional models leading to arithmetic and number concepts.

2. Even within the category of "concrete materials," some materials are more concrete than others. Nonetheless, little has been done to investigate how the figurative content of a problem affects the difficulty of the task. Piaget has focused on the operational aspects of tasks and concepts and has deemphasized the figurative aspects. The influence of figurative content on operational ability is important information for teachers who must devise models to illustrate mathematical concepts.

3. One of the ingenious aspects of Piaget's theory is that he explicitly confronts the fact that ideas (as well as children) develop. That is, (a) a given idea can exist at many different levels of sophistication, (b) this evolution can be traced, and (c) the more primitive levels of a concept have seldom been investigated or accurately described. Yet, little is known about the nature of children's early conceptions of many mathematical ideas.

The first mathematical judgements children learn to make are highly specialized, closely tied to specific content, and involve highly restricted, "messy" primitive concepts that do not give rise to neat, tidv, elegant theories. For this reason, mathematicians have not taken the trouble to describe most of the structures children use when they first come to master a given idea. Nonetheless, to develop effective instructional materials, it i important to present ideas in a form that will be most understandable to children. Therefore, it is useful to formulate acceptable mathematical descriptions of children's primitive conceptions of important mathematical id

4. Some of the best resources for describing the nature of children's early number concepts have come from Piagetian studies. Nonetheless, because Piagetian research has focused on the cognitive processes used by first-grade (i.e., concrete operational groupings) and by sixth-graders (i.e., INCR groups children at intermediate levels of development have been neglected. Furthermore, because psychologists in general (and Piagetian psychologists in particular) have avoided mathematical ideas that are typically taught in elementary school, it is usually possible to make only relatively crude inferences about how children's mathematical thinking gradually changes from concrete operational concepts to formal operational concepts. It is time for mathematics educators to apply Piagetian techniques and theory to concepts that exist at intermediate levels of development--as well as at adult or preschool levels

5. In the past, the space and geometry group focused on "Piagetian" operational analyses of various concepts and on analyses of the figurative contexts in which these concepts were used. However, the availability of certain general problem-solving strategies, as well as certain affective characteristics, also seem to relate to the solvability of a problem. Future research efforts may tend to focus more heavily on basic types of applied problem-solving processes--that is, on processes students use to apply mathematical ideas to real world situations. These processes will be described in the second part of this paper.

Several Lines of Research Not Represented in the 1978 Monograph:

Several lines of research were considered to be important by members of the space and geometry working group--but were not represented in the 1978 monograph. Examples of such research areas include the following:

1. Computer simulation studies developing out of the work of Newell and Simon (e.g.:). One of the best examples of research in this area is represented by the work of Greeno and some of his associates (e.g.:). Computers have been used to model the geometric construction and problem solving procedures used by students. In this way it has been possible to demonstrate that a given set of procedures is internally consistent and sufficient to perform a specified domain of tasks. Computer simulation research is of course preceded by a great deal of clinical observation and experimentation

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with students in real problem situations, and it has served to greatly extend and refine traditional "information processing" models of human learning. Current work in this area includes (a) the representation for visual imagery of abstract concepts (e.g., Schwarts,) and (b) the influence of attentional demands and memory load on task performance capabilities(e.g., Pascual-Leone and Wollmon,).

2. Concept development studies based on the work of the Van Hieles.

Two studies in this area were attempted by various members of the space and geometry group. But, prohibitive problems arose in obtaining manuscripts of the Van Hieles' work. Neither of the studies was completed in spite of the fact that several interesting and well designed studies were considered. It is to be hoped that the kind of international cooperation this present conference is intended to encourage should make this a ripe area of future research.

3. Problem solving studies which focus on geometric content.

Examples of current work in this area include (a) studies based on the work of Krutetskii and other Soviet theorists (e.g., Kantowski and Landa), (b) studies based on the work of Polya, (e.g., Schoenfeld), (c) studies focusing on "modeling" processes and applied problem solving processes (e.g., Lesh). References to this work will be given in the second part of this paper.

Part II: Research on Applied Problem Solving Processes

The original space and geometry research group has now divided into two overlapping research groups--one focusing on the development of space and geometry concepts and the other focusing on applied problem solving processes.

As the first part of this paper suggested, the distinctions were not always clear between research aimed at studying (a) the development of spatial and geometric concepts, (b) "embodiments" or "concrete models" for illustrating arithmetic and number concepts, or (c) problem solving processes. For example, Kieren's ( ) recent work analyzing various mathematical interpretations (and corresponding concrete models) for rational numbers is closely related to research on "modeling" processes in space and geometry. Or, Carpenter's ( ) work with measurement concepts deals with figural models of basic number ideas. Similarly, regarding problem solving processes, a number of space and geometry studies were unavoidably forced to confront the issue of what it is, beyond having a concept, that allows a normally intelligent person to use the idea to deal with math-related problems in everyday situations. Piagetian studies have shown that the operational structure of an idea is an important factor determining the difficulty of the idea. Geometry research has shown that the figurative content of the task situation can also radically influence the difficulty of a given problem situation. And, a student's proficiency with certain problem solving processes also influenced the difficulty of tasks. However, the relevant processes do not necessarily correspond to the typical kinds of problems solving processes discussed by Polya, Krutetskii and others.

To focus on the kinds of processes that seemed to be involved in earlier space and geometry studies, and to review and synthesize relevant research from a variety of related research perspectives, a conference on "Applied Problem Solving" was held at Northwestern University on January 5-7, 1978. Papers from this conference are currently being edited and should be available from the Ohio State ERIC center by November 1978. A tentative list of papers and authors includes:

1. "Problem Solving" Versus "Applied Problem Solving": What is the Distinction? Lesh, Northwestern University.
2. A Survey of Several Psychological Perspectives: What Kind of Problems and Processes Do They Consider? Bell, University of Chicago.
3. Problem Solving Projects or Mathematical Abilities Projects Affiliated with the Georgia Center for Learning and Teaching Mathematics. Hatfield, University of Georgia.
4. Soviet-Style Research on Problem Solving. Kantowski, University of Florida.
5. Polya-Based Resarch on Problem Solving: Some New Directions. Schoenfeld, University of California - Berkeley.
6. Social/Affective Factors Influencing Problem Solving Ability. Higgins and Trimble, Ohio State University.
7. Computer Simulation Research on Problem Solving. Heller and Greeno, University of Pittsburgh
8. Mathematical Learning Disabilities. Lesh and Jones, Northwestern University.
9. Modeling Processes. Kieren, University of Alberta.
10. Information Processing Models of Problem Solving. Carpenter, University of Wisconsin
11. Routine Problem Solving. Sowder, Barnett, and Vos, University of Northern Illinois
12. An Analysis of Problem Solving Research Methodology. Nelson, University of Alberta.

In the remainder of this paper, I would like to identify some themes that I believe will be important in future research efforts on applied problem solving processes.

What is Applied Problem Solving?

What it is, beyond having an idea, that allows a normally intelligent person to use the idea to deal with math-related problems in everyday situations? The upcoming monograph is an attempt to clarify a productive scope and focus for research on applied problem solving--as well as to identify an agenda of priority research issues. Because the papers for this monograph are still in the final formative stages, it would be premature to summarize their conclusions in this paper. However, the following sections taken from my own papers may serve to indicate some of my biases. I hope they will serve as points to stimulate discussion at the Osnabrück Conference.

1. Focus on Processes Needed to Use Mathematical Ideas:

In general, "being able to use a concept" involves something more than simply "having the concept." Getting an idea into a student's head does not guarantee the new idea will be integrated with other ideas that are already understood, nor does it guarantee the student will be able to use the library-type "look-up" skills that allow the idea to be retrieved when it is needed. "Being able to use an idea" may also involve problem solving processes that are not necessary in order to demonstrate the simple attainment of a concept. But what are these processes? Past research on problem solving seems to have made little progress. Perhaps the field is in need of reanalysis. Some promising new directions are described below.

2. Focus on Average Ability Students:

Most information about problem solving processes has come from situations involving older students, exceptionally bright students, individual students working in isolation (often in artificial laboratory situations), or situations involving highly contrived word problems, mathematical puzzles, or proofs. Elementary school children, average (or below average) ability students, and applied problem solving processes have been neglected. For this reason, the "problem solving" processes educators discuss often seem inaccessible to younger children or less gifted students, and applied problem solving processes, like modeling, have been ignored.

Traditionally, a great deal of the research in problem solving has focused on identifying and analyzing the abilities and processes used by bright students and good problem solvers. The assumption is that average and below average ability youngsters can be taught to use the processes used by good problem solvers. Krutetskii (1966), however, has shown that this assumption may be unwarranted. That is, many of the abilities and processes used by gifted youngsters may be inaccessible to average ability youngsters. If this conclusion is correct, another fruitful tack for research to take may be to analyze the disabilities of poor problem solvers or the difficulty-causing processes "learning disabilities" students--and to investigate the extent to which these processeses and disabilities also cause difficulties for average ability problem solvers. Many assumptions about learning and problem solving that seem quite sensible when they are used to describe average ability youngsters become obviously absurd when they are applied to exceptional children--either "gifted" or "learning disabled."

3. Focus on Processes Related to Concept Formation:

The definition of "problems" must take into account the prior experience cognitive capabilities of the student. A problem for one student may be an exercise--or a bore--for another. But, more than this, some of the most important problem solving processes probably also serve to increase the meaningfulness of available concepts. If applications are only considered to be appropriate *after* learning has occurred, then they will (and perhaps should) be neglected. If applications are to find a place in the curriculum, they must play an important role *on the way* to helping students understand the important ideas we want them to learn.

It seems questionable to claim that a person can apply a concept before the concept has been learned, but it seems equally questionable to claim that a concept is *known* if it cannot be used in simple applications. Perhaps, applied problem solving processes play an important part in the formation of most mathematical concepts. Or, perhaps it is naive to think of ideas as being either understood completely or else not understood at all. Perhaps it is more accurate to think of ideas as gradually becoming more and more meaning as they become more complex and are connected to more and more other ideas and events.

There is a variety of ways to make an idea meaningful--some of which involve "translation" processes corresponding to the arrows in Figure 1. These processesses not only play an important role in the development of mathematical concepts, they are also among the most important "modeling" processes students use when they try to apply the concepts in real life situations.

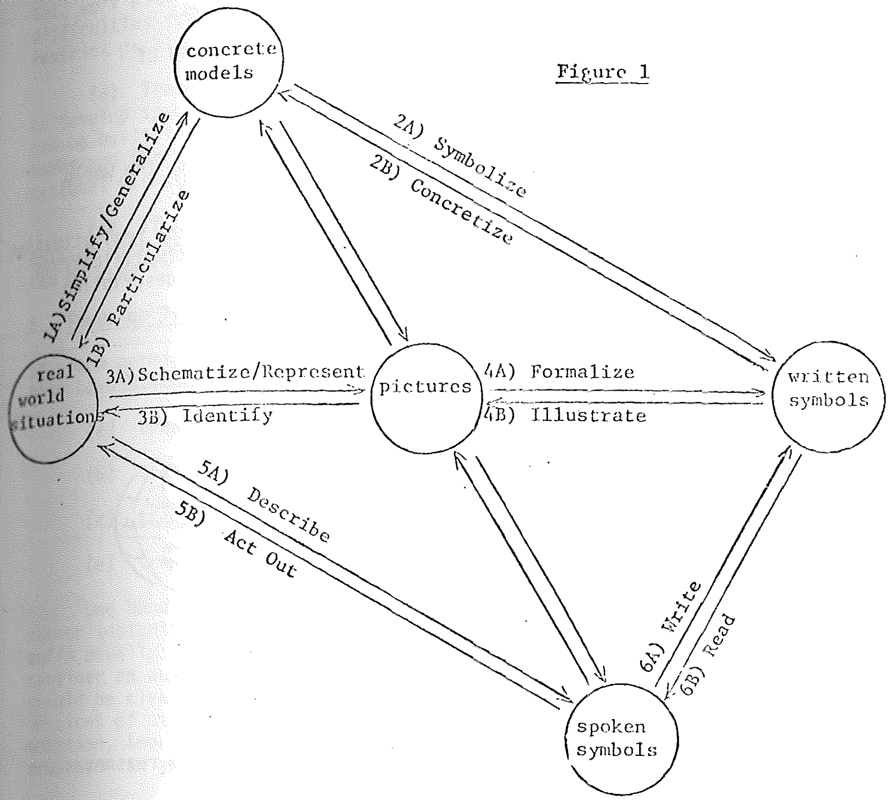


Figure 1

4. Focus on Modeling Processes

Using mathematics to solve real problems usually involves:

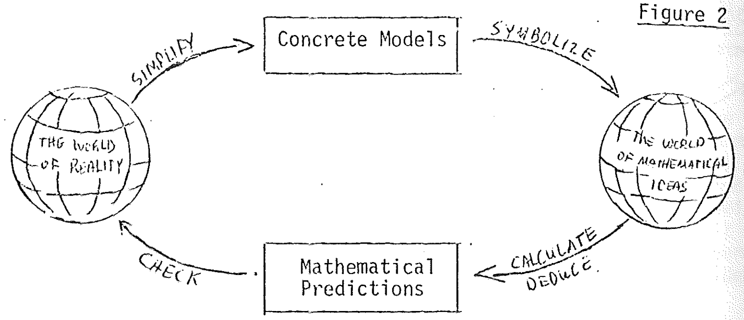
(a) a dive from the world of reality into the world of mathematics;

(b) a swim in the world of mathematics;

(c) a climb from the world of mathematics back into the world of reality, carrying a prediction in our teeth.

But, the "dive" often requires us to ignore some things about the real situati in order to focus on others. That is, a series of successive simplifications are often needed before a situation "fits" a nice mathematical description.

Figure 2



The processes in Figures 1 and 2 are important for a variety of reasons.

(a) They are simplified versions of the "modeling" processes used by gifted applied problems solvers. They represent some of the most important processes students need when they try to use basic geometric, algebraic, or number concepts.

(b) When we say a student "understands" a mathematical concept, part of what we mean is that he/she can use the kind of processes listed in Figures 1 and Yet, students are given few instructional activities that focus directly on these processes--in spite of the fact they are the kind of processes that give meaning to the ideas teachers are trying to teach.

(c) Average or below average students can learn to use these processes. Yet, work with special education students indicates these problems cause difficulties for many students--and that these difficulties can severely restrict problem solving (or even concept formation) capabilities.

(d) Teachers do not need to wait for large scale curriculum projects to develop special instructional activities to teach these processes. They can be built into the kind of lessons that are already included in many textbooks and the kind of problems that are included in the "applied" sections of national assessment tests.

(e) If diagnostic questions indicate a student is having unusual difficulties with one of the processes in Figure 1, other processes in the diagram can be used to strengthen or bypass the difficulty.

5. Focus on Translation/Representation/Notation Difficulties:

If space permitted, several other important "translation" processes could also have been discussed in connection with Figure 1. These include:

(a) translating from one type of model to another,

(b) translating from one type of picture to another,

(c) translating from one verbal description to another, and

(e) translating from one written statement to another.

One handicap with the above translation processes is that they seem rather insignificant compared with the more grandiose processes problem solving buffs usually like to discuss--e.g., reason by induction, reason by analogy, consider an auxiliary problem, or work backwards. On the other hand, it should be clear that the translation processes are really only simplified versions of other familiar problem solving processes--e.g., introduce suitable notation, look for a similar problem, simplify the problem, or restate the probablem in your own words.

The more grandiose processes may be quite appropriate for older students or for exceptionally bright youngsters, but Krutetskii's work suggests they are often inaccessible to average or below average students.

6. Focus on Processes Needed to Use "Social/Technological" Amplifiers:

When educators talk about problem solving situations, they often ignore the fact that most people work on real world problems when other people and other resources are available. People seldom work in isolation using only the power of their own minds to solve problems. Instead, good problem solvers learn to amplify their own powers through effective use of outside resources. For example, when real people solve real problems, one of the most often used problem solving strategies is to "ask someone who can give the needed information." This is not to say that good problem solvers solve most problems by asking someone else to do their work for them. Formulating a problem in wuch a way that a specific bit of information can be requested is not a trivial skill. In fact, one of the most obvious characteristics of good problem solvers is that they are good question askers. Once a question is formulated in a nice way, answer giving is often quite easy.

Applied problem solving usually involves:

(a) formulating the problem into an answerable question,

(b) devising an appropriate procedure for getting the needed information,

(c) carrying out the procedure, and

(d) interpreting and evaluating the answer; and "answer giving is only involved at stage "c."

When a psychologist gathers information about students solving artificial problems in isolated laboratory situations, the data are usually analyzed using a computer. The psychologist does not use the computer because he is too ignorant to calculate the relevant statistics with a pencil and paper; he uses the computer because the important aspects of his own problem solving situation are to select and interpret the appropriate calculations--not to actually go through the drudgery of computation. Furthermore, to decide upon the most appropriate computations to make, the psychologist probably also consulted a statistician before he carried out his analysis. Again, this does not mean that the psychologist is ignorant about statistics; to formulate the problem in such a way that a statistician can help requires "translation" skills like the ones in Figure 1, and a firm understanding of statistics.

7. Focus on Information Retrieval Processes:

After a problem is interpreted in a meaningful way, relevant information must be retrieved from memory or from available resources. Often these "library-type" retrieval skills represent important aspects related to the meaningfulness of the ideas related to a problem. That is, as an idea is related to other ideas; its meaningfulness increases. Yet these processes of forming relationships among ideas are seldom taught either as concept formation skills or as problem solving skills. For example, even in the most situations--say length concepts and area concepts--preestablished concepts are seldom related to new ideas being taught, and problem solving situations are seldom designed to help students form related networks of ideas.

8. Focus on Individual Stages in the Overall Problem Solving Process:

Many people believe that the best way to learn to solve problems is simply by solving problems--and, afterwards, looking back to reconsider processes that were most helpful. Unfortunately, if the problem solving process is not broken into stages, and if students are forced to work entire problems in isolation, the "looking back" technique seldom works. Average problem solvers seldom solve problems quickly enough to look back on more than a few problems. So, the strategies they notice are often useful for only a small number of problems of a particular type, and they do not learn cues to determine when each strategy might be useful. Furthermore, average problem solvers often have difficulty focusing on processes they use to solve problems. Even when they are able to give correct answers, they are notoriously unable to give accurate descriptions of steps in the process they used.

To practice these skills, a teacher can give students a whole list of problems and ask them to postpone trying to solve any of the problems until they have first gone through the entire list--trying to determine which technique would be "best" for each problem. For example, for computational problems, students can "set up" each problem before attempting to actually carry out the procedures they selected; or the teacher can simply hand out the requested information. Finally, the group can work together to estimate the reasonableness of the information they receive. Schoenfeld (   ) has shown that learning a system of strategies or heuristics involves a good deal more than simply learning each of the individual processes in isolation. A "managerial strategy" for selecting and evaluating processes is also needed.

By working in a group, students can investigate critical parts of more problems, and by working on whole sets of problems, one stage at a time, they can focus on "non-answer giving" stages in problem solving. Furthermore, by working with other students, average problem solvers often find it easier to focus on processes that would otherwise have been internal and more difficult to notice.

By focusing almost exclusively on the answer-giving stage of problem solving, and by treating problem solving as though it is always necessary to do all stages of the problem without any kind of assistance, students are often misled about what most real problem solving is really like. Furthermore, when problem solving is carried on in isolation, many useful problem solving strategies become almost impossible to teach to average or below average students. Some examples of these useful but difficult to teach strategies will be given in the next section of this paper.

9. Focus on "Group" Problem Solving Processes

Many individual problem solving strategies are quite difficult for average or below average ability youngsters. But, when these internal process are externalized in the context of small group activities, they are often easier to describe in a form that is understandable to lower ability problem solvers. For example, problem solving strategies like "consider a similar problem," "consider an auxiliary problem," or "consider a special case," can be summarized with the simple advice, "look for a related problem." Yet, to poor problem solvers, this advice often seems quite foolish because, "I already have one problem I cannot do, I do not need another." To poor problem solvers, a more sensible suggestion is, "Look at the same problem from a different point of view."

In several recent research studies, where groups of four students were supposed to work together on problems, individuals often worked independently --each conceptualizing the problem in quite different ways, and each unconscio of misleading biases inherent in their own point of view. This is one reason why "brainstorming" is often a useful problem solving technique.

In group, "brainstorming sessions, students can be bombarded with a variety of different ideas and approaches, and can simultaneously become more self-critical about their own points of view. They can also be made to notice: (a) some people are good talkers while others are good listeners, (b) some people are good generalizers while others are better at working out details, and in general (c) a variety of different roles are beneficial to good problem solving. Good problem solvers must be flexible enough to switch quickly from one role to another while solving a problem.

Many other problem solving strategies are greatly simplified in group situations. Problem solving strategies like "identify the givens," "identify the unknowns," and "eliminate irrelevant information" all having to do with the general recommendation, "understand the problem." However, this advice again seems rather useless to poor problem solvers whose superficial understanding of the problem often leads to selecting or eliminating information on rather artificial bases. On the other hand, except for specific recommendations about identifying knowns and unknowns, it is difficult for poor problem solvers to understand what it means to "understand the problem." It is much easier to say, "Use your own words to describe the problem to a friend," or "Describe some other problems like this problem." Poor problem solvers sometimes flounder with a problem for a long time before noticing (if asked) that they are unable to give a clear description of the problem to a friend. So, once again, group activities can force students to "understand the problem" and "organize the information given." Eventually, they may become self-critical enough to work on problems and no longer need group work to overcome their subjectivity and egocentrism.

When students work in groups to solve problems, they often see that many problems can be solved in a variety of ways--some of which are "better" than others. In fact, in research with gifted youngsters (e.g., Krutetskii, 1976), the hallmark of outstanding problem solvers is not so much whether answers are right or wrong but whether "clever" procedures were used. "Good" problem solvers are flexible thinkers who are capable of solving problems in several

different ways; so, when one path is blocked, another route can be taken.

Speed alone is not always an accurate measure of problem solving ability. For instance, because of the impulsiveness and inflexibility of their thinking "learning disabilities" subjects may actually solve problems (if they are able to solve them at all) faster than their normal peers. However, impulsivity and inflexibility are hinderances to good problem solving that can be compensated in group problem solving sessions.

The above points are not intended to imply that we should explicitly <u>teach</u> group problem solving techniques. Rather, group problem solving situations furnish an effective context to teach individual problem solving procedures--especially when the problem solver is at a relatively primitive level of skill acquisition. In addition to previously mentioned cognitive justifications for small group activities, a number of affective justifications are also apparent.

10. Focus on Social/Affective Factors:

In modern psychology it has become more and more clear that cognitive adaptation exists in an ecological system with other adaptation-seeking mechanisms, and is influenced by them. Learning, socializing, and adjusting cannot be completely separated. Much that we now call learning is social learning.

Many aspects of human learning that we have traditionally regarded as "cognitive" development are specializations of "social" development. The central issues are not so much about how the child develops knowledge, but rather, about how he develops shared or cooperative knowledge.

If we look at the moment-to-moment activities of children, we never find learning as a separate or isolated human activity. We have begun thinking about dialectical processes, about sharings of reality among people that create not only the objective (i.e., intersubjective) reality of science, but also the individual's conception of the self (Mead 1934) and his moral ideology (Kohlbert 1963, 1969). Observed changes in cognitive performance are first and foremost, assimilated into hypotheses about intellectual competence.

In part, learning to be a problem solver means acquiring a problem solving personality. One of the first characteristics of a good problem solver is that he interprets an unusual number of daily situations as problems --that is, as situations where his problem solving skills may be relevant. One of the first steps a good problem solver takes is to identify a given problem as "do-able" or "un-do-able," and next as "easy" or "difficult." Then appropriate solution strategies are selected to fit the initial appraisal. However, it is quite obvious that people who are good problem solvers in one context, in one type of situation, or in one discipline, may be average or below average problem solvers in another.

Research has shown that children may well form severe personae for different situations--family, peer groups, school. The child's "personality" is different at different times, in different situations, in different mental and physiological states, as a function of cognitive, social, and emotional load, and as a function of his particular agenda of the moment. Cognitive performance is moved upward and downward by load factors in the child and in the situations--e.g., noise, emotionality, distraction, confusion, shyness, anxiety. So children fluctuate in their apparent ability depending upon time and place. Psychological descriptions of people have for a long time centered around the premise that people have traits, personality traits and cognitive traits, that endure and that are manifest in all situations. This notion is currently under strong attack, particularly in personality theory.

The above sorts of factors suggest another set of benefits associated with group problem solving activities. For example, Triandis (1976) lists the following points:

(a) Imitation: There is a good deal of learning that takes place when humans see connections between what other people do and rewards they receive. (One of the characteristics related to this factor is the viewpoint that behavior is not viewed as being reliably followed by particular outcomes good or bad. Triandis calls this "ecosystem distrust." Seligman (1975) calls it "learned helplessness." Rowe (19) refers to the factor as "locus of control."

(b) Groups create satisfactions with certain activities. An activity which has low intrinsic value, when the learner is alone, may acquire a much higher value when the learner is in a group. Group norms, or roles adopted by the learner in a given social setting, or interpersonal agreements between the learner and important others, can change the desirability of activities.

(c) Self-esteem is important. The learner's view of the self, particularly beliefs that certain kinds of learning are possible and desirable, is important in learning. Among learners who think that they are not capable of learning some material, or who do not see the desirability of learning it, this factor severely reduces learning. A person's self-concept is in part formed through interaction with other people.

11. Focus on "Real" Problems:

What makes a problem real as opposed to artificial? In this paper, space limitations do not allow a thorough answer to this question. Some attributes which should be considered are apparent from the factors that were mentioned in the preceding sections.

Real problems do not have to be job oriented, and they probably do not even have to involve real world objects--at least not if the real world is defined from an adult perspective. A fanciful or imaginary problem having to do with "Star Wars" may be meaningful and interesting to youngsters. Or problems using poker chips, cuisenaire rods, geoboards or other materials can be quite appropriate for young students. But the problems should not ignore all of the types of modeling processes, information retrieval processes, resource utilization process, etc., that were mentioned in the preceding sections.

As scientists we know that real problems seldom fall neatly into disciplinary categories. A problem which begins as a measurement problem may turn into a probability or statistics problem which may in turn become a social issue or a legal problem.... The same is true at elementary levels. For example, when a sixth grader uses arithmetic to solve problems in everyday situations, problems are seldom simply addition or multiplication or division problems. They often involve combinations or sequences of basic arithmetic operations. In real problem solving one of the first and most important steps is to identify the appropriate computational structures to deal with a given situation.

In real problem solving, the exact nature of the answer (if one exists) may not be clear. Many levels of correctness may be possible, and many paths to solution may be appropriate. Or, in many situations, the goal may be to understand a problem, not to solve it. Or, in other situations, what is involved is more like a project than a problem. That is, a sequence of problems may arise, with each temporary solution leading to new, more sophisticated problems.

Time constraints may or may not be a factor. Often, good problem solvers, just like good runners, know how to pace themselves. They know when to relax and when to exhaust themselves. Many other factors could be mentioned. I hope some of these will be topics of discussion at the Osnabrueck Conference. The above factors and others are discussed in greater detail in the upcoming monograph on Applied Problem Solving.

Summary

Originally, I became interested in space and geometry research because I was interested in finding ways to concretize mathematical concepts--any mathematical concept, not just spatial or geometric ideas. I was interested in Piaget-inspired research analyzing the development of mathematical ideas, but I was convinced that (especially for formal operational youngsters) Piaget's focus on the "structure" of concepts did not give an adequate description of the "dimensions of meaningfulness" for most mathematical concepts. The operational or relational structure of a concept did indeed seem to be important characteristics determining the difficulty of the concept but figurative characteristics also seemed important. Furthermore, certain "applied problem solving processes," such as the ones described in this paper, were clearly important--both when children tried to use mathematical ideas in real situations, and when concepts were in the early stages of development. Geometry seemed to be an ideal context to study modeling processes and the influence of figurative characteristics of ideas.

I was also interested in "mathematics laboratory" methods of instruction. I was impressed by the fact that so many laboratory theorists claimed that mathematical concepts developed through interactions with the environment--and that these interactions were of two basic types: (a) interactions with concrete materials (or lower order "concrete" ideas) and (b) interactions with other people.

Some theorists, like Dienes, have given reasonable explanations of what students are supposed to get out of interactions with concrete materials. But, no one seems to offer a comparably convincing explanation of what students are supposed to get out of interactions with other people. What kind of interactions are supposed to be beneficial in laboratory situations? How are these personal interactions related to the interactions with materials?

I am pleased to say that some of my recent work in group problem solving settings has helped clarify my thinking on several of these issues. However, it is premature to report the results of this work. For the purposes of stimulating discussion at this conference, it seemed preferable to describe some of the factors I believe are important in applied problem solving.

I would like to close this article with a few brief comments about a theoretical model I have been using--and one which I believe is particularly relevant to the kind of research variables described in earlier sections of this paper.

One of my colleagues, Don Saari, an applied mathematician, has developed a mathematical model for the theoretical perspective I am taking. The model was originally based on some relatively new ideas in a branch of mathematics called catastrophe theory. Specifics concerning the mathematical model are probably not relevant to mention here. It is sufficient to mention that catastrophe models are useful:

(a) When several continuously changing variables give rise to abrupt changes (e.g., insights, etc.) in the behavior of an organisism or system,

(b) when behavior at any given moment depends not only on the strengths of the control variables but also on the previous relative strength of the control variables,

Saari's model has the additional characteristic that

(c) problem solving ability (or concept acquisition ability) at any given moment depends not only on available cognitive structures but also on certain social/affective structures.

The above three conditions are typically present in problem solving situations.

The model is especially suited to describe the evolution of adaptive systems--or structures. Therefore, because the characteristic feature of mathematics is its "structure," the model is particularly relevant to the development of mathematical concepts. To put it very simply: learning a system of things is usually quite different from learning each of the things in isolation. That is, the whole is usually more than the sum of its parts. For example, Schoenfeld confronts precisely this issue when he points out that mastering a <u>system</u> of heuristics in isolation. My research suggests that this idea applies equally well to the acquisition of <u>any</u> mathematical system. Mathematical judgements or mathematical ideas almost always require a student to master a coordinated system of relations, operations, or transformations. This is why "structured learning theorists" as diverse as Piaget (who focuses on written-concept" structures), and Ausubel, Bruner and Gagne (all of whom focus on different aspects of "between-concepts" structures) have all been particularly relevant to mathematics learning.

Saari's model is useful because it gives a precise mathematical description of previously ambiguous terms like assimilation, accommodation, intuition, etc. In this way it draws together and coordinates useful "structured learning variables" from theories like Piaget's and Ausubel's--as well as a number of useful "process" variables from computer simulation/information processing theories like Greeno's or Krutetskii's. The model also defines "problems" and "problem solving processes" in such a way that they are closely related to concept formation, to modeling processes, to information retrieval processes, and to other factors that were mentioned in this paper.

UNIVERSITE DE L'ETAT A MONS

LOGIC AND LANGUAGE IN GAME SETTING

- FOR CHILDREN AGED 8 TO 10 -

⁂

F. LOWENTHAL

Summary

Adults and children, teachers and pupils use words to express verbally their thoughts; nevertheless they don't always interprete these words in the same fashion and they do not always understand each other nor do they understand why !

For this reason we tried a non-verbal (graphical) technique of communication which allows the children to represent and express what they are doing. This method introduces constraints in the children's communication but these constraints are so obviously connected to the graphical support that the children are conscious of their existence. This in turn compells them to use only terms they know perfectly well, or to discuss the new terms till these are mastered.

⁂

1. Introduction

A child can only reach the stage of formal thought, of social and hierarchical organization if he masters a certain logic and a formalism through which logical operations can be expressed. When studying the development of logical abilities in children Piaget (Piaget and Inhelder, 1966) considered mainly the case of an isolated child describing verbally the way he manipulates "things". Freudenthal (1973) objected to Piaget's use of verbal language because it introduced a bias. Bruner (1966a; 1966b) believes that the cognitive development of a child is function of the child's language : Bruner considered mainly the case of a child in interaction with other people and representing "things".

We believe that the cognitive development is function of the way the child, in interaction with others, manipulates representations. This in turn depends on the type of representations which are available, i.e. it depends upon the support which is used to communicate with the child.

Our thesis is as follows : "Children (between 8 and 10 years) will have at their disposal a formalism which supports a logic in a natural way. This enables the child to communicate and explore the univers to formulate certain logical conclusions and at the same time to de lop his critical powers and evaluate both himself and others. The child who has this formalism at his disposal and who also assimila it, manages to use this system to guide his own verbal (natural) la guage. Such a formalism, therefore enables the child to develop na rally without stopping at certain stages described by Piaget".

2. Methodology.

a) The children

We present in this paper the results of our work in 4 classes normal children. When the work started in January 1977, we worked w one 3rd grade (children aged 8 to 9 year) and one 4 th grade (9 to year). This study was continued after September 77; work was also carried out in two other classes after January 78 (3rd grades, 8 to year). All children were of similar background. All received one 50 minute lesson each week. We used the same programme for the 4 class

b) The chosen formalism : graphs and automata.

Among all formalisms we know, the most suitable seems to be th of new maths. The fact that this formalism has been used fruitfully does not mean that it is the only possible one (Cohors-Fresenborg, but it offers some attractive advantages :

1) it is strong enough to develop classical logic (propositional calculus) and bears a logic in a natural way;
2) it is flexible, apparently abstract but can be used as concrete support of the child's thought;
3) it is nearly non-verbal and will thus not be in conflict with th child's natural language.

Generally we used multicoloured graphs (diagrams, flow charts) dots represent the objects under discussion and the arrows of the graph represent the relationships between these objects, each colou symbolising a relationship, very clearly defined and fixed by the agreement binding the author and the reader of the graph; the defini tions can be altered but the logical structure of the message has to be respected.

An interesting property of these graphs is the possibility to use them to completely describe a finite automaton (Trakhtenbrot and Barzdin, 1973). We want to use them slightly differently : each dot will code an inner state of the automaton and an output at the same time, the colour of the arrow will code the input and the arrows will serve as "next inner state and next output" function. This will also lead to a complete description of a finite automaton.

c) The importance of a game setting.

Kohl (1974), among many others, pointed out the importance of the child becoming aware of the arbitrariness of "the rules of a game". He also insisted upon the part played by "coding" the rules of a game. This together with the stimulating and reassuring aspect of the game (Bruner, 1972), appears fundamental in the development of the critical sense.

By definition we call a "play", a succession of moves according to a given set of rules of a game, such that these successive moves lead to a conclusion. We do not want actually to code the rules of a game. We want to use finite automata, presented by graphs, to code all possible plays of the game under consideration.

d) The technique : how to use games and automata.

This has already been described in other papers (Lowenthal, 1976, 1977, 1978). We will only give here a brief summary.

We did not intend to create what Bruner (1966c) calls "an aid to teaching", but rather a precise structure, a framework within which children speak, think and reason (1). We wanted to use the full power of the chosen formalism in order to let children represent graphically a game by a finite automaton and observe how they would use this representation. The following steps were used for this purpose :

1st. Learn a game and play it;
2nd. Give a verbal description of what happened;
3rd. Describe all possible plays of the game with a multicoloured graph;
4th. Forget the initial game and modify the graph arbitrarily;
5th. Describe and play a game, illustrated by the new graph (or tell a logical story which corresponds with the new graph).

Let us show what be the connection between this procedure and Higginson's "mathetic capacity" (Higginson, 1978).

We define a finite automaton as follows : it is a quintuple (Q,A,B,§,a) where Q,A and B are finite alphabets (Q is the set of tions for inner states of the automaton, A is the set of notations outputs and B is the set of notations for inputs), § is a mapping domain (Q x A) x B and range Q x A, a is an arbitrarily chosen ele of Q x A. Whenever we verbally present a game to the children (thi of a game of "heads or tails"), we introduce notions like :

1) who has to play (this is our set Q),
2) scores (this is our set A),
3) moves of the players (this is our set B);

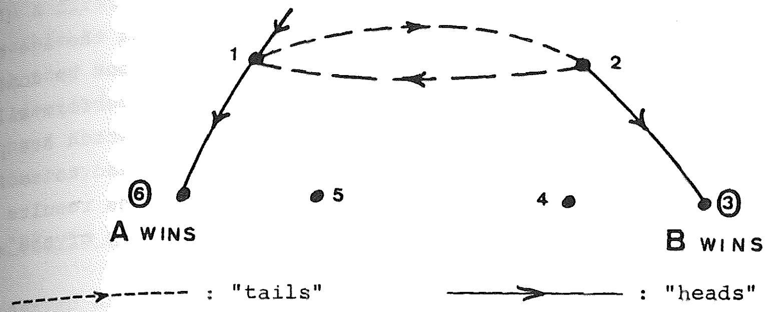
we must also specify the starting situation (our a) and precise the elements of Q x A for which we have in fact a winner. The correspon automaton can now be described by means of a multicoloured graph. Th elements of the set Q x A constitute our universe of discourse. Each them receives a name. An arrow of a fixed colour is associated to ea possible move (e.g. red for "tails" and green for "heads"). Using th dots and colours we draw the graph (diagram) of the function §; we a a special "start" arrow showing where is a and for each dot correspo ding to a winning situation, we need a label mentioning the name of winner.

We show in table 1 how children construct the set Q x A for a g of "heads or tails" and in figure 1, how the corresponding automato can be described.

Table 1.

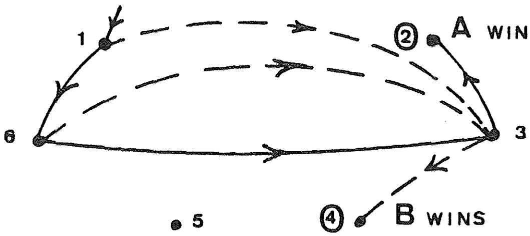
Score \ Player	A must play	B must play
(0,0)	1	2
(0,1)	3	4
(1,0)	5	6

Figure 1.



We then ask the children to produce a new graph (step 4 of our procedure). One could for instance get the graph shown in figure 2.

Figure 2.



Finally we ask the children to describe verbally a game corresponding to the "picture" they invented (step 5). In our case, rule 1 of this new game would be : "If the score is (0,0) and A has to play (thus state 1), then either A gets 'heads', the score becomes (1,0) and B must play, or A gets 'tails', the score becomes (0,1) and A must play again". We let the reader try to figure out the other rules of this game !

3. Facts observed.

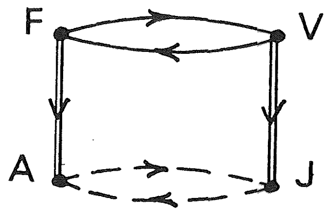
a) "Tell a story".

When we ask the pupils to tell a story corresponding to a graph we notice that there are (at least) two different languages, each corresponding to a logic such that none of them is an extension of the other : we have the language of graphs, which is not ambiguous and t verbal language, which is ambiguous.

A child does not dominate the verbal language at the age of 8 a cannot express in words all the logical operations; but he dominates the non-verbal language of graphs and can use it to perform all the classical operations, to solve the logical problems which are presented in a non-verbal fashion. This support may even lead to social in actions : children might even attempt to formulate the results of th mental and collective reasoning in their approximation of the adult verbal language.

To illustrate this we wish to describe what actually happened in a classroom. Two months after we started our work a child presented t graph shown in figure 3.

Figure 3.



——→—— : throws the ball to
═══→═══ : throws the ball to
- - - -→- - - - : invites

(F, V, A and J represent pupils).

The author told then his story : "F throws the ball to V, then F throws the ball to A. A invites J, while V throws the ball to J".

The other pupils reacted immediately and drew the attention of th author to the discrepancies between the graph and the story:
1) there must be two balls,
2) arrows of different colours must have different meanings.

After 30 minutes of intensive collective work they reached a solution : in a first step they added a simple arrow from A to V, it was then possible to tell a story using only one ball; one of them noticed then that they had solved the first difficulty, but not the second one; they all decided to come back to the original picture; after a vote, they changed the meaning of the double arrow into "pays a visit to". It was now easy to read the whole story.

b) "Play a game".

1) We observed unexpected reactions when we played a game of "addition modulo 3" with children aged 8 : both players have to throw a dice with 3 green sides and 3 red sides, the winner is the first player who reaches a multiple of 3 (other than 0), starting from 0 and using only the operations "+ 1" (green side) and "+ 2" (red side).
- The pupils were able to describe easily plays where, for example, the first player wins : they were free to choose either a graphic description or a description using words and sentences only; most of them chose the graphic support. Those who prefered the graphs were generally able to give at least one correct description of the required play as a succession of red and/or green arrows, although they made many mistakes as far as the names of the dots were concerned; the other pupils had more difficulties and forgot often to use the words describing the last move.
- These children found out, using the graphic support, that there is an automorphism bound to this game (and to its graph) : one could replace everywhere the function "+ 1" by "+ 2" and conversely. The children expressed this as follows : "As soon as we have one solution, we have another one (i.e. another play leading to the same winner) by replacing all red arrows by green ones and all green arrows by red ones".

2) The children discovered, by looking at the graphs, that some games are obviously unfair : one player has "more possibilities" to win than the other one. Later on they built in an arbitrary way new automata. These corresponded to equally unfair games but in favor of the other player. The children very quickly realized this (15 minutes) and asked to be allowed to play the part of the player who is advantaged. Sometimes they decided to "be fair" and collectively changed the graph, or simply changed the labels : plays winning for A

became winning for B and conversely !

3) Three months after we started this programme, all our pupils, one or two in each class, were able to built a representation the plays of a game : they used easily a multicoloured graph ponding to a finite automaton. Using these representations, they discovered that different plays, for the same set of rules, may lead to the same winner.

4) The children willingly and collectively modified the graph they derived from the verbal rules of a game : they created a new fi automaton and used it to play according to the rules introduced the new graph, they played "by following the arrows" 3 months af the start of our programme.

5) At the beginning of our work, they cared a lot for "the way they followed" (i.e. the succession of arrows). At that stage partial scores seemed important : the pupils were also manipulating pawns on tracks adapted to the game. Five months later, they cared only for the last dot of "a path" (i.e. a play) and ignored the manip lations of objects which were implied by the scores. They cared for the end points : "to know who is the winner". While playins were satisfied with following the arrows, thus manipulating the representation. Their way of creating a new finite automaton was thus an abstract creation, easily achieved by children aged 8 to

6) During a long process the children played and learned to express verbally certain notions : after 3 months work, they were able to discover traps and to specify details in the new games they creat they also accepted the responsability for the oddities they produ One month later they realised that the game of the new automaton globally different from the original one. It is at this moment th they attempted for the first time to formulate verbally the new r the best pupils succeeded only in doing so 11 months after the pro gramme started. After 18 months all were able to express verbally the new rules. It seems important to note that these successive progresses were usually first made by one or two pupils (not alway the best ones) who then explained the situation to their companions

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Two remarks made by children spontaneously translating the rules of their new graph into words (after 11 months) should be quoted : 1st in a 5th grade (10 year) : "I know, it is the same" and 2nd in a 4 th grade (9 year) : "I do not understand : it is the same".

4. <u>Discussion</u>.

Our pupils used a non-verbal formalism as support for their thought : the graphic representation showed mistakes, impossibilities, misinterpretations that they were unable to detect when using the verbal language only. To set things right, they discussed the problems with one another and manipulated ("trial and error") the graphic representations.

Special notions (like "isomorphism") were discovered through the graphic support, <u>in particular situations</u>.

The children were actually reasoning in a purely abstract way. They were able to express this through the graphic support only (age : 8 to 9). They became able to express this verbally between 11 and 18 months after we started our programme.

The two remarks quoted (§ 3.6°) suggest another approach of the problems raised by the cognitive development of children. It is here that the stumbling block of non-identity arises : the older child understands that there can be two different representations of the same thing; the younger child does not. This is not a problem concerning the action of the child : all children know how to play with the graphs they have invented for themselves. Neither is it a problem with the logical structure of that part of language used only to describe manipulations of objects : the youngest of our two children can do the transformation from the formal representation to usual language but cannot explain verbally the identity of the two systems; but the older child can. <u>What is developing is a metalanguage</u>, a language for describing language. The development of a metalanguage might start before the age of 8 but the <u>manipulation of representation</u> can only hasten its completion, and thus facilitate the cognitive development of a child.

5. Conclusions.

a) The children realize that a verbal label can cover, in a vague many different concepts and they learn to reason by comparing bal statements and graphs.

b) On the one hand, we believe that the following is an explanatio what is actually happening : one of the essential factors of co tive development is the mastery of a metalanguage. In the prese school system, such a metalanguage develops naturally as the chi grows and the verbal language becomes its own metalanguage. We believe that the confrontation between verbal and non-verbal language, in children aged 8, makes a better development of the metalanguage possible.

c) On the other hand, we no longer believe in the "stage of concret operations", as described by Piaget. Such a stage exists in chil who live in the present "system". Does it still exist as soon as take into account non-verbal teaching techniques ?

Note.

(1) According to Dienes (1975) "what mathematics a child learns is little importance ... What matters is the kind of mental discipline he acquires and the kind of mental habits and techniques which he learns". The aim of our method was certainly not teaching new techniques. We wonder now whether this method might lead to teaching strategies and provoke an interest in generalization, abstraction, structuration and coding system in a broa domain than mathematic education. We are presently working wit the same pupils and attempt to test these ideas concerning generalization, ...

Bibliography.

- BRUNER J.S. : 1966a. Toward a theory of instruction. Cambridge, Mass.: Harvard University Press.
- BRUNER J.S. : 1966c. The process of education. Cambridge, Mass. : Harvard University Press.
- BRUNER J.S. : 1972. Nature and uses of immaturity, American Psychologist, vol. 27, n° 8, 1-28.
- BRUNER J.S., GREENFIELD P.M. and OLVER R.R. : 1966 b. Studies in cognitive growth. New York : John Wiley.
- COHORS-FRESENBORG E. : 1978. Learning problem solving by developing automata networks, Revue de Phonétique Appliquée, vol. 46-47, (Proceedings of the 1st Mons Conference on Language and Language Acquisition).
- DIENES Z.P. : 1975. Mathematical games II : Finite geometries, Journal of Structural Learning, vol. 5, n° 1-2, 73-95.
- FREUDENTHAL H. : 1973. Mathematics as an educational task. Dordrecht : D. Reidel.
- HIGGINSON W. : 1978. Language, Logic and Mathematics, Revue de Phonétique Appliquée, vol. 46-47, (Proceedings of the 1st Mons Conference on Language and Language Acquisition).
- KOHL H. : 1974. Math, Writing and Games in the Open Classroom. New York : New York Review Book (NYR 109).
- LOWENTHAL F. : 1976. Jeux mathématiques comme base du langage. (In "Langage et Pensée Mathématiques"), Luxembourg : Centre Universitaire de Luxembourg, 439-441.
- LOWENTHAL F. : 1977. Games, graphs and the logic of language acquisition, Communication and Cognition, vol. 10, n° 2, 47-52.
- LOWENTHAL F. : 1978. Logic of natural language and Games at Primary School, Revue de Phonétique Appliquée, vol. 46-47, (Proceedings of the 1st Mons Conference on Language and Language Acquisition).
- PIAGET J. and INHELDER B. : 1966. La psychologie de l'enfant. Paris : Presses Universitaires de France.
- TRAKHTENBROT B.A. and BARZDIN' Y.M. : 1973. Finite Automata - Behavior and Synthesis. Amsterdam : North-Holland.

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TWO COGNITIVE MODES IN ARITHMETIC WORD PROBLEM SOLVING

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1. INTRODUCTION

This study is part of a more extensive research effort designed to formulate the textual structure of arithmetic word problems so as to enable us to reach a well-motivated grading of these problems that would predict the actual degree of difficulty encountered by students in their attempt to solve problems of this kind.

In a previous study [6] we dealt with the underlying semantic structure of word-problem texts and suggested that it is only through its global, though rigeraus, analysis of the text that the role of isolated linguistic elements in it can be properly appreciated.

The present study consists of several extensions of the above framework with a particular emphasis on some additional properties of predicate structure, and on the textual approach suggested in our previous analysis. Our analysis has lead us to some practical predictions that have been tested in a computor-based arithmetic program. The empirical findings will be discussed against the background of our theoretical considerations.

2. DYNAMIC VS. STATIC DESCRIPTIONS

Following Lyons [5] let us introduce some useful semantic distinctions in terms of which we can later discuss the textual organization of word problem texts :

> *We will draw a high level distinction between static and dynamic situations. A static situation (or state of affairs, or state) is one that is conceived of as existing, rather than happening and as being homogeneous, continuous and unchanging throughout its duration. A dynamic situation, on the other hand, is something that happens (or occurs, or takes place) : it may be momentary or enduring; it is not necessarily either homogeneous or continuous, but may have any of several temporal contours .*
>
> (Lyons, 1977, p. 483)

The two major categories of word problem text we will distinguish are : dynamic and static problem texts. We will further distinguish between a dynamic 'narrative' text which describes a causal link and a dynamic sequential text in which the events are chronologically but not causally linked.

We will follow Prince's definition of a minimal story as reported by Lipsky [4] by way of a starting point for the characterization of what we consider to be a minimal narrative word problem text. A minimal story, according to him has the following properties.

> *a. It consists of 3 and only 3 events. of which the first 2 are conjoined by] feature and the second and third by 2 features, 1 of which is identical to the first;*
>
> *b. The events must occur in chronological order 1-2-3.*
>
> *c. The first and third events must be stative, and the second active, and the third event must be caused by the second and be the inverse of the first .*
>
> (Lipsky, p. 194)

Along these lines we can say that a minimal narrative[1] problem text will be composed of 3 underlying strings: one of them designates an

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[1] Our previous treatment analyzed descriptive (static) texts as consisting of three underlying strings which can be schematically represented as:

Information component: (1) $(\underset{n}{\exists} x) Fx$

(2) $(\underset{m}{\exists} x) Px$

Question component : (3) $(\underset{?}{\exists} x) Rx$

where F, P and R are well-defined in terms of semantic dependencies.

initial state of affairs (hence, Initial-State, or Si); another designates some event (E), whose function is to introduce a change in the initial state; and a third string which designates the state-of-affairs subsequent to the change, the final-state (Sf).

In arithmetic word problem texts the change from Si to Sf is specifically concerned with the numerical characteristics of some of the objects mentioned. Accordingly, the initial and final states, too, are defined in terms of the relative numerical characterization of the objects.

As minimal narrative problems refer to a single event, their structure is intrinsically dependent on the temporal dimension. Hence, the three problem strings are temporally distinct, so that Si is coupled with t_1, (t - denoting time), Sf with t_3 and t_2 (the temporal location of the changing event)(E) is placed between them ($t_1 < t_2 < t_3$).

Let us use the term simple narrative problem text for a word-problem text in which the sequential textual order of the strings reflects the natural temporal order of the occurences ($t_1 - t_2 - t_3$), so that string 1 refers to the initial state, string 2 refers to the event which brings about a change in it, and string 3 refers to the final state.

e.g. Example 1
1. Dan had 3 marbles — Si in t_1
2. He found 2 marbles — E in t_2
3. How many marbles does he have now? — Sf in t_3

The term transformed narrative problem text will be used to refer to a problem in which the sequential textual order of the strings does not reflect the natural temporal position of the event (i.e., not t_1-t_2-t_3 but some variation of it).

e.g. Example 2[2]:
1. How many marbles did John gain[3] if — E in t_2
2. he had 2 marbles before , and — Si in t_1
3. now has 5 marbles, altogether. — Sf in t_3

A static problem text refers to a single state-of affairs: two of its underlying strings refer to sub-parts of it and the third to its entirety.

The predicates employed in describing it may all be 'static' so that the distinct sets and their union are linguistically encoded in terms of different NP's (2 boys and 3 girls...); different locations of objects (2 books on the upper shelf and 3 on the lower one) or even different points in time (2 candies before lunch and 4 candies after the lunch...).

The predicates may denote an activity, which according to Lyons' distinction is a kind of durative, agent-based process, (2 girls were dancing and 3 were playing the clarinet) or even events (Dan bought 3 pens and 3 pencils).

The fact, however, that the individual predicates denote events does not make the whole situation a dynamic one.

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2) Note in Ex.2 that a 'transformed' narrative problem involves a syntactical change, as well as the use of specific time adverbials (e.g. previously, before, after, etc.).

3) The reader should not confuse the relative place of the question with the natural order of the occurences. For example: "How many marbles did David have in the morning, if he gained 3 marbles and now he has 5 marbles altogether". (which keeps the natural sequence of $t_1 - t_2 - t_3$, but like example 2, has the question as string 1.
Changing the place of the question does not transform the temporal order in the text.

Thus, we distinguish three types of word problem texts containing predicates denoting events: in two cases the situation referred to is a dynamic one and in one case it is static. A dynamic situation is described by a narrative problem text when the single event mentioned forms the causal link between two states of affairs (as in example 1, above) and a sequential problem text when no causal relationship obtains but the two events mentioned are inherently temporally ordered (as in the following example: Ruth bought 5 candies and gave two of them to Dan...).

A descriptive problem text refers to a static situation which combines unordered events. Thus, the two events are not inherently sequential, but rather juxtaposed.

3. THE COGNITIVE ASPECT: TEMPORAL SEQUENTIALITY AND THE FLOW OF INFORMATION.

In the previous section we discussed the temporal dimension of dynamic word-problem texts, disregarding the specific textual function, of each string i.e., the fact that two of them constitute the information component of the problem and one constitutes its question component [6]. The task to be performed in solving word-problems requires the adequate processing of given information and its subsequent use for the acquisition of new information. We must note that this very task itself has its own temporal structure according to which a solution can be reached only on the basis of previously acquired information.

Thus we have two independent temporal parameters each of which monitors a different temporal sequence: one of these is related to the set of events described in the text and the other is related to the set of cognitive moves the solver has to take. (see also, Paige et al.[8]

The main question which arises concerns the interaction between these two independent temporal parameters, as both are present in every dynamic problem solving situation, and may interfere with each other.

From ontological point of view the textual distinction we have made corresponds to two different types of occurences in 'reality': dynamic descriptions characteristically refer to events distinguished along the time axis, while static descriptions refer to a state-of affairs whose temporal dimension is either constant or irrelevant. Accordingly, the division between the sets of objects in a dynamic situation is produced by some manipulation on one of the sets in question, while the corresponding division in a static situation is not due to a change in the objects but is due to various observers'-induced conceptual and linguistic mechanisms.

We think that the distinction between dynamic and static texts, which encode chronologically presented events, vs. simultaneously presented objects has a bearing on children's understanding of mathematical notions and their applications.

Let us consider the notion of 'structure' and some adjacent notions such as :'transformation', 'inverse', 'invariant under transformation', which are central in mathematics [10]. These notions require a simultaneous (spatial) conceptualization of the component structure. It seems, then, within our framework, that the closest verbal counterpart of a spatial structure will be a static word problem text. On the other hand, it has been noted that the intrinsic sequentiality of the linguistic mode of expression creates a bias towards a temporal interpretation of sequentially presented events.

In Grice's terms (1975) such textual juxtaposition of two events generates a coversational implicature4) concerning a causal relationship. The fact that there is a preference for a causal over a sequential and a sequential over a descriptive (static) interpretation of chained linguistic strings has already been noted by linguists, particularly with reference to studies of the connectives [3]. The dynamic type of word problem texts most naturally demonstrates this type of preference for sequentially ordered texts. A math word problem solving task combines the 'structural' and 'sequential' cognitive modes, and it seems to us that this very fact may be problematic for young learners.5)

The following empirical results were analyzed in the light of this distinction, hoping that we could clarify some of the processes involved in math word problem solving.

4. EMPIRICAL DATA: STATIC VS. DYNAMIC WORD-PROBLEM TEXTS.

The data presented herein is taken from a computor-based program [12] in arithmetic which includes a strand for word-problems with a sample of about 300 problems. Each of the above 300 problems was solved by 400 to 1000 children (grades 2-6) during the last year, and the proportion of success in the solution of each problem was recorded.

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4) A conversational implicatures is roughly, a cencellable implication generated by some deductive inference from the given text and very general rules of conversation. It is cancellable because it is not part of the meaning of the utterance strictu sensa. Thus, the implicature concerning temporal ordering can be cancelled as in "Tom spoke to his parents and phoned his aunt, but not necessarily in that order".

5) Just as a reminder consider how difficult it is for the child at certain age to attend properly to one object (bead) as once being wooden and once being brown (Piaget's experiment, Piaget, 1952 (1941) p. 165), Vs. the task of removing few beads from a pile of beads.

From the above sample we examined all and only the problems that required the subtraction operation (1 binary operation) for their solution. There were 34 such word problems in our sample. These were categorized as follows:

Type of problem	*No. of W.P in sample*	*A sample problem*
1. static description	*13*	*There are 10 chairs in the room. 4 of them are white and the rest are brown. How many brown chairs are there in the room?*
2. dynamic description	*14*	*How many crayons does Dan have, if he recieved 10 crayons and lost 4 of them?*
*3. comparison*7)	*7*	*Ruth has 10 crayons and Dina has 4 crayons. How many crayons more than Dina, does Ruth have?*

The questions we concentrated on were the following :

a. What effect does the global type of text (dynamic vs. static text) have on the level of difficulty of word problems? (variable 1)

b. In a dynamic word problem text the question component (i.e., the information sought) may refer to the final state (t_3), or to a previous point in time (i.e., do we ask about t_3 in the sequence $t_1 < t_2 < t_3$, or about t_1 or t_2 ?). What effect does such variation have on the level of difficulty of word problems? (variable 2).

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7) At present we do not have within our framework an adequate analysis of 'comparison'-word-problems, therefore we will mainly deal with the other two types.

c. Is a 'transformed' narrative text (where the surface order of the strings deviates from the order of events) more difficult than a simple narrative word-problem text? (variable 3).

d. What is the effect of lexical factors such as: verbal cues or distractors,[8] on the level of difficulty of word problems in the presence of the above variables? (variable 4).

The data given is not based on a pre-planned designed experiment, therefore we could not answer all the questions through one statistical analysis, but through a number of steps. However, the data reported is based on a sample of 400-1000 Ss' on each word problem.

5. FINDINGS

On the basis of the above theoretical considerations we sought to examine which of the two cognitive modes will appear dominant on the assumption that greater success in solving dynamic word problems will demonstrate a sequential preference while greater success in solving static word problems will demonstrate a structural grasp.

Table 1 will present means and S.D for each type of text :

Table 1

Type of text	*Mean*	*S.D.*	*No. of Questions*
Dynamic	*75.8*	*10.7*	*13*
Static	*52.0*	*7.9*	*14*
Total	*63.5*	*15.2*	*27*

---

[8] Which were already examined empirieally, [2],[7],[11].

One-way analysis of variance yielded F=43.56; p<0.0001), thus our prediction concerning the effect of type of the text (dynamic vs. static) was found to be statistically significant, and the sequential mode was found to be more dominant than the structural one.

In order to check relative effect and mutual interaction we have examined together variable 1 and variable 4.

Table 2 will present a summary table of 2-way Analysis of Variance of the Proportion of Success (as the criterion) by variable 1: Type of Text (dynamic Vs. static) and variable 4: Verbal Cue. Table 3 will present a Multiple Classification Analysis of the above variables.

Table 2

2-WAY ANALYSIS OF VARIANCE

Summary Table

Source of Variation	*Sum of Squares*	*d.f.*	*Mean Square*	*F*	*Signif. of F*
Main effects	*3858.504*	*2*	*1929.254*	*21.492*	*0.000*
Verbal cue	*48.388*	*1*	*48.388*	*0.539*	*0.470*
Type of Text (static vs. dynamic)	*3128.167*	*1*	*3128.167*	*34.849*	*0.000*
2-way interactions					
V. Cue X Type of Text	*73.633*	*1*	*73.633*	*0.820*	*0.374*
Explained	*3932.142*	*3*	*1310.714*	*14.602*	*0.000*
Residual	*2064.586*	*23*	*89.765*		
TOTAL	*5996.727*	*26*	*230.643*		

Table 3

MULTIPLE CLASSIFICATION ANALYSIS

(Proportion of Success by Verbal Cue and Type of Text)

GRAND OF MEAN : 63.52					
Variable + category	*N*	*Unadjusted*		*Adjusted for independents*	
		dev'n	*eta*	*dev'n*	*beta*
Verbal Cue (Var. 4)					
distractor	*15*	*-4.65*		*-1.27*	
aiding cue	*12*	*5.81*		*1.59*	
			0.35		*0.10*
Type of Text (var. 1)					
dynamic	*13*	*12.33*		*11.84*	
static	*14*	*-11.45*		*-10.99*	
			0.80		*0.77*
Multiple R square					*0.643*
Multiple R					*0.802*

As one can note from tables 2 and 3, the effect of the dynamic vs. static type of text is highly sinificant even in the presence of a well established variable such as Verbal Cue ([2],[7],[11]). The Type of Text exlains most of the 64% of the variance, and has no interaction with V. Cue.

Our prediction with regard to question C concerning the surface order of the strings (i.e., transformed vs. simple narrative text) was that a simple narrative text will be easier than a transformed one. Table 4 will present means and S.D. for groups of word-problems which differ along this dimension: (other variables were controlled).

Table 4

MEANS AND S.D. FOR SIMPLE AND TRANSFORMED NARRATIVE TEXTS

Category	*N*	*mean*	*S.D.*	*F*	*P*
Simple text	*8*	*76.3*	*8.66*	*2.84*	*N.S*
Transformed text	*5*	*75.2*	*14.60*		

Our prediction was not justified. A change in the surface of the strings by itself, has no effect on the degree of difficulty of the word-problem texts.

With regard to question b, concerning the order of information presented, our prediction was that if the task consists of finding information related to t_3 it will be easier than otherwise (i.e., the information sought is related to t_1 or t_2).

Table 5 will present Means and S.D. for word-problems which we could differentiate along this dimension in our sample.

Table 5

Category of Variable 2	*Mean*	*S.D.*	*F*	*P*
question about t_3	*84.7*	*9.29*	*11.161*	*0.044*
question about t_2 [9]	*61.0*	*2.82*		

---

[9] Our sample included just 5 problems of this kind, in none of which the question referred to t_1, controlling other variables.

Thus, it was found that when the question component does not refer to the final point in time (t_3) the solving task is more difficult.

How could this be accounted for? It appears that when t_1 or t_2 are referred to in the question, there is an interference between the temporal parameter of the text (see section 3) and the order of the solving activity. In this type of text different points in time have to be considered simultaneously, which makes it according to our hypothesis similar, in this respect, to the static texts discussed earlier. To check this hypothesis we compared the proportion of success of problems where information about t_2 is sought to static problems. The results are presented in table 6.

Table 6

MEAN AND S.D. FOR STATIC PROBLEMS AND DYNAMIC PROBLEMS, ASKING ABOUT t_2.

Category of Word Problems	*Mean*	*S.D.*	*F*	*P*
Static Problems	*52.07*	*7.86*	*2.40*	*N.S*
Dynamic Problems (asking of t_2)	*61.00*	*2.82*		

Thus, the significant difference between static and dynamic word problems is greatly reduced when the child has to re-order the events along the time axis and simultaneously cope with all the information available to him, despite the fact that the text is a dynamic one. For this point we have found convincing support from empirical data presented by G. Vergnaud [12] which contained

questions referring to t_1 and t_2 with results in the same line.[10]

6. FINAL REMARKS.

The main conclusion which emerges from the above findings is that word-problems designating dynamic situations are cognitively easier than word problems designating static ones.

We can at this point offer only some tentative suggestions by way of accounting for the observed results: it seems to us that dynamic problem texts differ radically from static texts in the cognitive requirements they pose to the solver. In dealing with a dynamic text the reader can go step by step processing each unit of information at the time: the causal or temporal link guarantees the potential 'coherence' of the text as a whole, so that it can be readily interpreted as referring to a fully constructible situation.

In a static text, on the other hand, the states referred to are simply placed in juxtaposition: the reader must first be able to construct for himself a state-of-affairs as a whole before he can see the link between its subparts. The same is true when the requirement to re-order the events is due to the fact that the question is asked about t_2 or t_1.

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[10] The distinction between state and events seems to correspond Vergnaud's distinction between states and transformations. In spite the fact that he considered a different set of variables, when we compare pairs of problems which differ only with respect to the t_i in the question component, we get support to our data: (numbers in parantheses are proprtion of success).

Question is about	t_1	t_3
	Bertrand (.69)	Pierre (.85)
	Bruno (.22)	Paul (.64)

In sum, the relation between the subparts of a dynamic situation depends on a dimension inherent to them (causality, temporality) while the corresponding relation between the subparts of a static situation is not intrinsic to them but derives from their status vis-a-vis the global structure of the situation.

This difference between the two types of text seems to us to account for their different cognitive modes.

If our observations are correct, they seem to point to a more general cognitive phenomenon which goes beyond the specific mathematical context in which we have so far studied it. Natural extensions of this investigation would be in the area of reading comprehension and in the area of non-verbal information processing.

REFERENCES

[1] Grice, Logic and Conversation in Cole & Morgan (eds.) Speech Acts, 1976.

[2] Jerman, M. and Rees, R., 'Prediciting the Relative Difficulty of Verbal Arithmetic Problems', Educational Studies in Mathematics 4 (1972), 306-323.

[3] Lakoff, Robin, 'Ifs Ands and Buts about Conjunction' in Fillmore & Langendoen (ed.) Studies in Linguistic Semantics, 1971.

[4] Lipsky, J.M. 'From Text to Narrative'.

[5] Lyons, J. Semantics, Cambridge Press (1977), Vol.2.

[6] Nesher, P. Katriel, T. 'A Semantic Analysis of Addition and Subtraction Word Problems in Arithmetic'. Educational Studies in Mathematics, 8 (1977), 251-269.

[7] Nesher, P. and Teubal, E., 'Verbal Cues as an Interfering Factor in Verbal Problems Solving', Educational Studies in Mathematics 6 (1974), 41-51.

[8] Paige, J.M. and Simon, H.A., 'Cognitive Processes in Solving Algebra Word Problems' in Problem Solving, Kleinmuntz, B. (ed.), John Wiley and Sons Inc., 1966.

[9] Piaget, J. The Child Conception of Number, Routledge and Kegan Paul, London, 1952 (1941).

[10] Piaget, J., Structuralism, Basic books, N.Y., (1970).

[11] Searle, B., Lorton, P. Jr., P., 'Structural Variables Affecting CAI Performance on Arithmetic Word Problems of Disadvantaged and Deaf Students', Educational Studies in Mathematics 5 (1974), 371-384.

[12] Suppes, P. Loftus, E.F., and Jerman, M., 'Problem Solving on a Computer Based Teletype', Educational Studies in Mathematics 2 (1969), 1 15.

[13] Vergnaud, G., 'Structures Additives et Complexité Psychogénétique', La Revue Française de Pédagogie, 1976.

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THE DEVELOPMENT OF PROPORTIONAL REASONING IN THE CHILD AND ADOLESCENT THROUGH COMBINATION OF LOGIC AND ARITHMETIC[1]

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INTRODUCTION

Two different types of information can be brought to mathematical education from experimentation with children and adolescents in a cognitive developmental setting:

(1) What is the natural *hierarchy* in building a higher-order concept such as proportional reasoning, involving the integration of many lower-order concepts acquired earlier ? Such information will be useful in establishing a *learning hierarchy* for the particular concept.

(2) What is the type of *meaningful situation* that can be presented to a child at a particular level of thinking that brings about a reaction of response and recognition and also arouses interest ?

The concept of ratio and proportion is a typical mathematical concept where information is useful. This concept has been quite widely studied with initial research undertaken by Piaget and Inhelder in the field of probability (Piaget and Inhelder, 1951) and in physical laws (Inhelder and Piaget, 1955). Further studies confirmed that the concept is acquired quite late during adolescence (Lovell and Butterworth, 1966; Lunzer and Pumphrey, 1966; Fishbein, Pampu and Manzat, 1970; Strauss, 1977). Karplus devised a test to determine levels of reasoning in ratio and proportion tasks, isolating different strategies classified according to levels of operational thinking (Karplus and Peterson, 1970; Karplus and Karplus, 1972).

Steffe and Parr (1968) have found that correlation at the Elementary School level is low between graphic and symbolic embodiments of the concept of proportion, showing that content is important besides structure in the problem. Kieren (1976) has furthermore isolated the multiple aspects of the concept of proportion.

Related to the general problem of stages and their hierarchy, Laurendeau and Pinard (1962 and 1968) have devised particularly useful methods for the chronological differentiation of stages. Some have been taken up in this study. Stages have also been reinterpreted with new ideas by Pascual-Leone (1970, 1976) and by Case (1972a, 1972b, 1978). The relationship between psychology and the learning of mathematics has been dealt with extensively by Skemp (1971).

The relation between mathematics and education has been given a new impetus with Freudenthal (1973, 1978) and the whole body of research undertaken by the IOWO group. Piaget's general influence on mathematics learning has also been given considerable attention (Lovell, 1966,1972; Sullivan, 1967; Beilin, 1971; Weaver, 1972). The teaching and learning of ratio and proportion, in particular, has given rise to quite a number of studies (Romberg and de Vault, 1967; Duquette, 1972; Wollman and Karplus, 1973; Novillis, 1973; Streefland, 1978).

A particular interesting aspect of instruction refers to the theory of *advanced organizers*, where instructional material is established to be meaningful and related to what the learner already knows (Ausubel, 1963; Lesh, 1976). Work undertaken in this perspective shows how complex the problem is, but how promising is still the relation between hierarchical construction and education.

The piece of research presented here, with an example fully described, is a summary of work which has been going on for some years at Laval University on the development of proportional reasoning. Various replications of the study and many variations in experimental conditions led to the results described here. Extensive discussions with students, in the framework of a Doctoral thesis (Bellemare, 1966,1967) and various Masters'thesis (Scallon, 1966; Rousseau-Savard, 1969; Cloutier, 1970; Bégin, 1973; Cardinal, 1973) allowed the working out of the various stages in the development of the concept and the interpretation of strategies as related to problem structure at each level (Noelting et Cloutier, 1970;

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Noelting, Cloutier et Cardinal, 1975).

A non-school problem was chosen to embody the concept of proportion: preparing a mixture with varying numbers of glasses of Orange Juice and Water. Items were built by considering all possible variations of the situation. Thus the universe of content was fixed at the beginning. Perceptual variations in the situation, or difficulties arising from a lesser or greater familiarity with the problem, both leading to the phenomenon of *horizontal decalage*, were eliminated at the outset. This unity in content allowed scalogram analysis to be undertaken which led to high coefficients in scalability. Parallel research with the proportion concept embodied in different experimental situations led to low scalability of items. Unity of content is necessary in order to *determine* a hierarchy. Once the levels are established, learning hierarchies should *vary* content, in order to induce generalization of the concept at each level.

The problem of *advanced organizers* in the educational setting is particularly significant in the framework of a hierarchical theory of development. As will be seen, the order of successive stages has not only be found constant, but a rationale is found for the passage of one stage to the other. Thus at each stage only *certain* problems are really meaningful and lead to a change in the scheme applied to solve them. An understanding of the relationship between meaningful problems and the schemes applied at each level could be an interesting outcome of this work.

Piaget's chronology was adopted for the various levels found, as the author had directed various pieces of research which constituted parts of Piaget's work, had participated during twelve years in the working out of the Piagetian system (Inhelder and Piaget, 1955, 1959), and still considers himself as Genevan, though having evolved through contact with the North American empirical tradition. In this work, an attempt was made to combine structural analysis with standarized procedures pertinent to empirical research. Thus presentation of the problem was the same for all subjects, criteria for attaining a stage was given as success or failure at distinct items, stages were differentiated through non-parametric tests, and strategies are described mathematically at each level. In this presentation (which is a summary of a forthcoming publication) only the outline of the work is given.

Once the general ground work of a succession of integrative stages had been established, the problem of mechanisms of passage from one stage to the next had to be tackled. The publication of Piaget (1975) come as a great help in confirming certain outcomes of the research, such as the existence of "periods" of development with similar "phases" of centration and decentration. Two theoretical points come out: i) the existence of various *periods* of equilibration or reconstruction, where similar processes of *centration-decentration* were taken up at ever higher levels of complexity; ii) the existence of *phases* at each period, with similar processes coming about, such as *centration* opening up a period and *hierarchical organization* closing it. It turned out that four phases were found when Piaget had theoretically postulated three, which is a minor change in the general perspective of phase-development. The distinction between a β or *categorial* phase of differentiation, followed of a ρ or *relational* phase (connecting the two categories produced) gives a theoretical basis for the new model.

THE EXPERIMENT: METHOD

Instrument: the Orange Juice Experiment

A test was devised comprising 23 items. Each item consisted of the comparison of the relative orange taste of two drinks, made up of a certain number of glasses of orange juice and a certain number of glasses of water (see Table 1). Each mixture was expressed as an ordered pair e.g. (4,1)vs.(1,4), with the first term corresponding to the number of glasses of orange juice and the second the number of glasses of water.

The items were the outcome of a certain number of previous experiments. Items 24 and 25 were later added to make up the final stage IIIB.

Procedure

Two items are first discussed with the whole group, with explanations given (items I and II, see Figure 1).

Then each child or adolescent is asked to answer the experimental items 1 to 23, first choosing among three possible choices, then explaining why he made his choice.

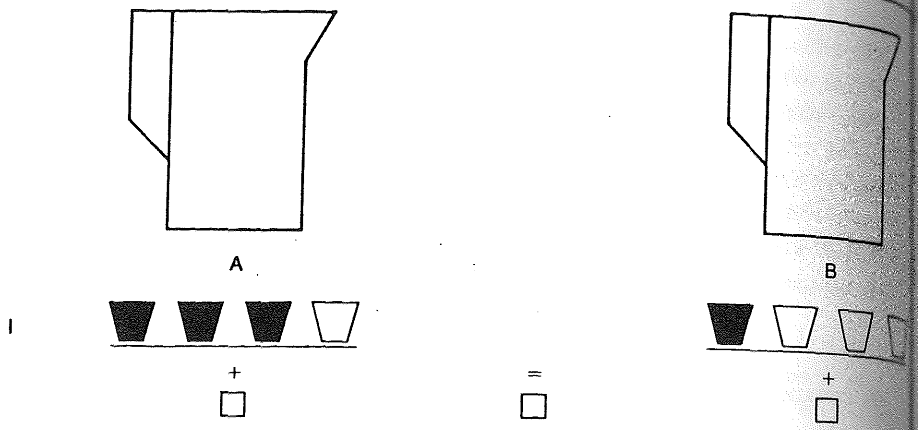
ORANGE JUICE (FORM A)

Date ____________________

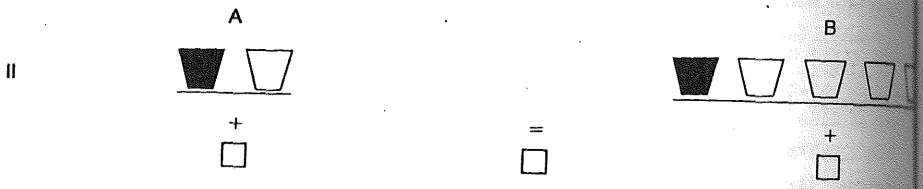
Name ____________________

Age ____________________ Date of birth ____________________

School ____________________ Class ____________________



Why ? ____________________



Why ? ____________________

FIG. 1. — **First page of Orange Juice** Experiment (Group Form A).

TABLE 1

ITEMS OF ORANGE JUICE EXPERIMENT (GROUP FORM A) WITH CORRECT ANSWER AND STAGE

Items	Composition	Correct answer	Stage
I	(3,1) vs. (1,3)	A	IA
II	(1,1) vs. (1,4)	A	IB
1	(1,0) vs. (1,1)	A	IB
2	(4,1) vs. (1,4)	A	IA
3	(1,2) vs. (1,5)	A	IB
4	(1,2) vs. (2,1)	B	IA
5	(1,1) vs. (1,2)	A	IB
6	(3,1) vs. (2,2)	A	IA
7	(1,1) vs. (2,2)	E	IIA
8	(2,3) vs. (1,1)	B	IC
9	(2,2) vs. (3,3)	E	IIA
10	(2,2) vs. (3,4)	A	IC
11	(1,1) vs. (3,3)	E	IIA
12	(1,2) vs. (2,4)	E	IIB
13	(2,1) vs. (3,3)	A	IC
14	(2,3) vs. (1,2)	A	IIIA
15	(4,2) vs. (2,1)	E	IIB
16	(2,1) vs. (4,3)	A	IIIA1
17	(1,3) vs. (2,5)	B	IIIA1
18	(2,1) vs. (3,2)	A	IIIA1
19	(2,3) vs. (3,4)	B	IIIA2
20	(6,3) vs. (5,2)	B	IIIA2
21	(3,2) vs. (4,3)	A	IIIA2
22	(4,2) vs. (5,3)	A	IIIA2
23	(5,2) vs. (7,3)	A	IIIB
24	(3,5) vs. (5,8)	B	IIIB
25	(5,7) vs. (3,5)	A	IIIB

NOTE. — Items 24 and 25, of Stage IIIB, have been added after further experimentation, in order to complete the stage of Higher Formal Operations.

Correction

Items 1 to 15 are corrected as given by the subject. Items 14 and 16 to 25 need the examination of explanations, in order to eliminate accidentally correct answers due to sole centration effects and no operations being put into use. Centration exists when just two terms are compared additively.

Examples:

Item 22 : (4,2)vs.(5,3)	Chooses A (correct)	*"Because there is less water in A than in B"* (centration on water: item failed).
Item 22 : (4,2)vs.(5,3)	Chooses A (correct)	*"A gives two wholes and B:1* (operation: item passed).
Item 22 : (4,2)vs.(5,3)	Chooses A (correct)	*"4/6 = 16/24, 5/8 = 15/24. The first is larger"* (operation: item passed).

Sample

A sample of 321 subjects were tested, from 6 to 16 years of age (see Table 3). This corresponded to one class per grade level of Elementary and Secondary Schools. Mathematically advanced classes were chosen at each level, from the same socio-economic level (upper-middle class) of a suburb of Quebec City.

Treatment of results

A first-order analysis of the choice made was first undertaken. This consisted in two parts: 1) an analysis of the order of items according to Guttman; 2) a content analysis of items in order to determine categories of items of same structure, making up a stage.

A second-order analysis was then performed to determine: 1) the structure of items common to a category; 2) the strategy put into use to solve items of each category; 3) *imbedding* of one structure into the next inside an hierarchical order; 4) *processes* determining changes in strategy from a stage to the next.

TABLE 2

ITEMS OF ORANGE JUICE EXPERIMENT (GROUP FORM A)

ORDERED ACCORDING TO DEGREE OF SUCCESS, THEN CATEGORIZED TO FORM STAGES

Stage	Item	Composition	Frequency of success	Characteristics
0	0	(1,0)vs.(0,1)	-	Differentiation of terms.
IA	2	(4,1)vs.(1,4)	319	Difference between first terms of ordered pairs.
	6	(3,1)vs.(2,2)	319	
	4	(1,2)vs.(2,1)	319	
IB	1	(1,0)vs.(1,1)	311	Like first term, difference between second terms of ordered pairs.
	3	(1,2)vs.(1,5)	307	
	5	(1,1)vs.(1,2)	305	
IC	8	(2,3)vs.(1,1)	295	Equality vs. difference between terms of ordered pairs.
	13	(2,1)vs.(3,3)	291	
	10	(2,2)vs.(3,4)	297	
IIA	9	(2,2)vs.(3,3)	251	(1,1) equivalence class.
	11	(1,1)vs.(3,3)	244	
	7	(1,1)vs.(2,2)	231	
IIB	12	(1,2)vs.(2,4)	186	Any equivalence class.
	15	(4,2)vs.(2,1)	156	
IIIA1	16	(2,1)vs.(4,3)	141	Ordered pairs with two corresponding terms multiple of one another.
	17	(1,3)vs.(2,5)	131	
	14	(2,3)vs.(1,2)	107	
	18	(2,1)vs.(3,2)	88	
IIIA2	20	(6,3)vs.(5,2)	87	Same after simplyfing one pair or extracting (1,1) unit.
	22	(4,2)vs.(5,3)	71	
	19	(2,3)vs.(3,4)	65	
	21	(3,2)vs.(4,3)	59	
IIIB	23	(5,2)vs.(7,3)	51	Pairs without corresponding terms multiple of one another.
	24	(3,5)vs.(5,8)	-	
	25	(5,7)vs.(3,5)	-	

TABLE 3

COMPARISON OF AGE DISTRIBUTION AT EACH STAGE
OF ORANGE JUICE EXPERIMENT (GROUP FORM A)

Age	N	Stage							
		0	IA	IB	IC	IIA	IIB	IIIA	III
6	14	0	1	2	8	3	0	0	0
7	26	1	1	7	14	2	1	0	0
8	35	1	0	4	12	10	6	2	0
9	43	0	1	2	9	12	13	6	0
10	32	0	0	1	3	13	8	6	0
11	38	0	0	1	5	12	7	9	1
12	34	0	3	1	0	9	5	14	4
13	31	0	2	0	0	2	9	17	2
14	20	0	0	0	I	1	2	10	1
15	29	0	0	0	0	0	8	16	6
16	19	0	0	0	0	1	2	8	5
Total	321	2	8	18	52	65	61	88	27
p^a		–	–	–	–	<.01	<.01	<.01	<.01
Age of accession[b]		–	–	–	–	8;1	10;5	12;2	(17;0)

NOTES. — [a] Probability level of difference between age distribution of the stage, compared with preceding one, assessed by Kolmogorov-Smirnov Test.

[b] Age of accession to a stage is the age where 50% of Ss solve at least one item of the stage.

RESULTS

Items are ordered according to difficulty, then submitted to a Guttman-type scalogram analysis with the help of a computer program (Dixon, 1971). Coefficients obtained were CR = 0.942 and MMR = 0.781.

$$PPR = \frac{CR - MMR}{1 - MMR} = 0.735.$$

This shows that items formed a so-called "perfect" hierarchical scale.

Adjacent items on the scale were then grouped through a process of categorization (Table 2). Subjects passing items of one level, but failing at the next, were considerd to make up a "stage". These stages were compared, as to the age distribution of subjects, with the Kolmogorov-Smirnov Test (Siegel, 1956). This is a non-parametric test, as the scale involved is ordinal. Adjacent-stage comparison gave significant differences for the last five stages (IC to IIIB). Earlier stages had to be differentiated in individual experiments. Orange Juice Experiment Form B was worked out with 18 Ss per age between 4 and 9 years. Stages (A, (IB + IC) and IIA were differentiated at the .05 level. An infant form of the test: Form D, with true orange juice being used, revealed the existence of a symbolic stage, where "collections" are not yet constituted and only "elements" are considered (Table 4).

Examination of problems involved at each stage, and strategies used to solve them, led to assign operational levels to these stages, following the Piagetian chronology of development. It was decided that strategies involving comparisons between terms only were of the *preoperational* level (Table 5). Strategies involving joint multiplication or division of terms, and yielding equivalence classes, were considered as *concrete operations*. While strategies, where comparisons took place after previous equivalences had been mentally reconstructed, were considered as an *operation on an operation*, and assigned to *formal operations*. Typical items of each stage are given in Table 4.

Explanations subjects give at each stage, for solving the particular problem of the stage, were set in mathematical form (Table 5). Symbols used are described in the section titled Symbolism.

Finally, the succession of stages was analyzed in terms of equilibration processes (Tables 7 and 8).

TABLE 4

STAGES IN THE DEVELOPMENT OF THE CONCEPT OF RATIO
(ORANGE JUICE EXPERIMENT, GROUP FORM A)

Stage	Name	Age of accession (50% Ss)	Typical item	Characteristi of stage
0	Symbolic	(2;0)	■ □ (1,0) vs. (0,1)	Identificatio of elements.
IA	Lower Intuitive	(3;6)	■■■■□ ■□□□□ (4,1) vs. (1,4)	Comparison of first terms
IB	Middle Intuitive	6;4	■□□ ■□□□□□ (1,2) vs. (1,5)	Like first ter comparison of second terms.
IC	Higher Intuitive	7;0	■■■□□□□ ■■□ (3,4) vs. (2,1)	Inverse relati between terms both ordered
IIA	Lower Concrete Operational	8;1	■□ ■■□□ (1,1) vs. (2,2)	Equivalence cl of (1,1) ratio.
IIB	Higher Concrete Operational	10;5	■■□□□ ■■■■□□□□□□ (2,3) vs. (4,6)	Equivalence cl of any ratio.
IIIA	Lower Formal Operational	12;2	■□□□ ■■□□□□□ (1,3) vs. (2,5)	Ratios with tw corresponding multiple of o another.
IIIB	Higher Formal Operational	15;10	■■■□□□□□ ■■■■■□□□□□□□□ (3,5) vs. (5,8)	Any ratio.

TABLE 5

PROBLEM-SOLVING STRATEGIES AT DIFFERENT STAGES IN THE DEVELOPMENT OF PROPORTIONAL REASONING

Stage	Name	Age of accession (50% *S*s)	Typical item (a,b) vs. (c,d)	Strategy
0	Symbolic	(2;0)	(1,0) vs. (0,1)	$a_1 \in A,\ d_1 \in D$
IA	Lower Intuitive	(3;6)	(1,4) vs. (4,1)	$c > a$ Therefore $(c,d) > (a,b)$
IB	Middle Intuitive	6;4	(1,5) vs. (1,2)	$a = c,\ b > d$ Therefore $(c,d) > (a,b)$
IC	Higher Intuitive	7;0	(2,1) vs. (3,4)	$a > b,\ c < d$ Therefore $(a,b) > (c,d)$ even though $a < c$
IIA	Lower Concrete Operation	8;1	(1,1) vs. (2,2)	$m(1,1) = (m,m)$
IIB	Higher Concrete Operation	10;5	(2,3) vs. (4,6)	$m(a,b) = (ma,mb)$ with $a \neq b$
IIIA	Lower Formal Operation	12;2	(1,3) vs. (2,5)	$ma = c$ $m(a,b) = (ma,mb)$ $ma = c,\ mb > d$ $(c,d) > (ma,mb)$ Therefore $(c,d) > (a,b)$
IIIB	Higher Formal Operation	15;10	(3,5) vs. (5,8)	$a+b = g$ $(a,b)g = (a/g,b/g)$ $c+d = h$ $(c,d)h = (c/h,d/h)$ $hg = gh$ $h(a,g) = (ha,hg)$ $g(c,h) = (gc,gh)$ $(gc,gh) > (ha,hg)$ Therefore $(c,d) > (a,b)$

SYMBOLISM

Symbolism of items in algebraic form was introduced to express the strategy common to a stage. A uniform method of placing the two ordered pairs in each item was found necessary, in order to make items comparable. The following rule was applied: the ordered pair with the smaller first term is put first, e.g. (1,4)vs.(4,1). When first terms are equal, the ordered pair with the greater second term is put first, e.g. (1,5)vs.(1,2). With equivalence classes, the pair with the lowest terms thus comes first, allowing multiplicative covariation: 3(1,1) = (3,3). This rearrangement when symbolizing an item will be called the *standardized form*. In the test, the order of pairs in an item is put at random.

The following symbols were adopted to express the sets and subsets of each item. The sets, when placed in the standardized order are called G and H, with g and h expressing their number. The respective subsets of orange juice and water of set G are A and B, with a and b their number. The subsets of set H are C and D, with c and d their number. Thus an item in standardized form remains (a,b) vs. (c,d). Individual elements of a subset such as A are called a_1, a_2, etc. Various operators are introduced by subjects to modify the terms of the ordered pairs, i.e. a, b, c and d. The symbols f, j, m and n will be used to denote integers (excluding zero and 1).

In the examples given, item numbers preceded by the letter A come from the group test A. Those preceded by the letters B and D come from the individual and infant forms respectively.

QUALITATIVE DESCRIPTION OF STAGES

Examples of behavior of some characteristic stages are given.

In the choice made: A means first pair is greater, B second, E equivalence between pairs. Space does not allow to give examples of each stage. However principal stages are given (beginning and end of periods of equilibration).

Two *modes of behavior* are distinguished and will be found at *each stage:*

Mode E or *covariation*, with two *external ratios* being compared.

Mode I or *internal division*, with two *internal ratios* being compared.

Internal ratios are those between terms *within a state*; external ratios are *between states*. Here we find again a most important distinction made already by Freudenthal et al. 1976 (p. 313). The distinction is here made somewhat differently, as "composition" between terms is involved. The important thing is to make the distinction.

Stage IA: Lower intuitive. Centration on the first terms of the ordered pairs.

Success at items such as (1,4)vs.(4,1) and (1,2)vs.(2,1).

The child compares the number of glasses of orange juice in both pairs, or opposes predominance of juice in one pair and water in the other.

Examples of success:

Diane, 4;0	Item B4: (1,2)vs.(2,1) Chooses B (success).	*"Here there is more orange".*
Nathalie, 5;0	Item B3: (4,1)vs.(1,4) Chooses A (success).	*"Because there is a lot of orange juice and only one glass of water".*
Gilles, 4;0	Item B4: (1,2)vs.(2,1) Chooses B (success).	*"There is a lot of orange juice* (B)*. There is a lot of water* (A)*".*

Examples of failure:

i) Globalism

France, 5;0	Item B6: (1,0)vs.(1,1) Chooses B (failure).	*"Because there are many".*

ii) Centration

Louis, 4;7	Item D6: (1,1)vs.(1,0) Chooses E (failure).	*"It will taste the same because there is one glass of orangeade there* (B) *and one glass of orangeade there* (A)*".*

Stage IIA: Lower concrete operation. Equivalence class of ratio (1,1).

Success at items such as: (1,1)vs.(2,2) and (2,2)vs.(3,3).

Examples of success:

i) Covariation (Mode E)

Johanne, 11;0	Item A7: (1,1)vs.(2,2) Chooses E (success).	*"Each glass dilutes one glass. So A has one glass of juice and B has two; A has one glass of water and B two. They are equal only there is more liquide mixed in B".*

ii) Division (Mode I)

Martine, 8;0	Item B12: (2,2)vs.(3,3) Chooses E (success)	*"Two for two, here (A); three for three, here (B)".*

Subjects differentiate between state and transformation. The relation between complementary terms in the pair is stabilized as an "invariant". The relation between corresponding terms between pairs is mobilized as a transformation (either co-multiplication or co-division). This yields the simplest equivalence class, the 1:1 ratio. Strategy for the internal mode is $m/m = n/n$, corresponding to transposition of a ratio. Strategy for the external mode is $m(1,1) = (m,m)$; $(m,m)/m = (1,1)$, corresponding to complexifying or simplifying a ratio.

However ratios, where terms are not equal, are failed.

Examples of failure:

i) Centration on the residue after (1,1) covariation (thus using strategy of the stage)

Louise, 11;0	Item A12: (1,2)vs.(2,4) Chooses A (failure).	*"Because the left side has one glass of water more, while the right side has two of them more".*

ii) Centration on either juice or water (regression to earlier strategy)

Diane, 8;0	Item A12: (1,2)vs.(2,4) Chooses A (failure).	*"It is that there are less glasses of water".*

Stage IIIB: Higher formal operation. Common Denominator and Percentage algorithms.

Examples of success:

i) Common Denominator (Mode E)

Sylvie, 14;0	Item A19: (2,3)vs.(3,4) Chooses B (success).	*"At the right, there is 3/7 of juice for 4/7 of water, that is 15/35 of juice; at the left there is only 14/35".*

ii) Percentage (Mode I)

Réjean, 13;0	Item A23: (5,2)vs.(7,3) Chooses A (success).	*"A = 71 3/7% because 5/7 orange juice. B = 70% because 7/10 orange juice".*

INTERPRETATION

Characteristics of stage IIIB: Differentiation between logical and algebraic operations, with hierarchical integration.

At stage IIIB, a combinatorial system is formed, where algebraic and logical transformations are differentiated and integrated. These are defined as follows:

Algebraic:

A *binary operation on terms* is an operation in set which combines two elements of the set into a third element of the set.

The *addition* of two terms a and b of a ratio to find their sum g is a binary operation on terms. The terms of a ratio as considered here are natural numbers, with their sum a natural number.

The *multiplication* of two denominators to find their product is a binary operation on terms. The operation of *join* or *meet* on two denominators to find their LCM or HCF is also a binary operation on terms.

Logical:

A *binary operation on statements* is a connective introduced between two statements. Let us consider a binary operation on terms (an algebraic operation) as a *statement*.

We shall define a new operation, of a logical nature, consisting in *combining* or *separating* such *statements* in order to constitute an algorithm. Thus we *operate on operations*, which is Piaget's definition of a *formal operation*.

Let us give two examples:

a) *Equivalence class of a ratio*.

The equivalence class of a ratio (a,b), reduced to its lowest terms, is obtained by the joint multiplication of its two terms a and b by an integer m. Let us define the following statements, which assert that the defined operations have been performed, on the first and second terms respectively, of an ordered pair (a,b) expressing a ratio:

$$p \ : (a,b) \longmapsto (ma,b)$$
$$q \ : (a,b) \longmapsto (a,mb)$$

where m is an integer (excluding 0 and 1). Statements corresponding to nil transformations will be rendered as follows:

$$\bar{p} \ : (a,b) \longmapsto (a,b)$$
$$\bar{q} \ : (a,b) \longmapsto (a,b)$$

where m thus becomes the nil element of a multiplicative group. The operations p and q when performed, can be reversed (reversibility). The connection between the two statements defining the logical operation pertaining to the equivalence class is a *biconditional* with the following truth table:

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

We remain inside the equivalence class when the two terms of the ratio are jointly multiplied (or divided), defining the connection $p \cdot q$,

or when no transformation is performed, giving $\bar{p} \cdot \bar{q}$. The two connections $p \cdot \bar{q}$ and $\bar{p} \cdot q$ are outside the system.

b) *Strict order between ratios.*

Let us define the following statements, which assert that the described additive operations on elements have been performed on the first and second term respectively of an ordered pair expressing a ratio:

$$r \ : (a,b) \longmapsto (a \pm f,b)$$
$$s \ : (a,b) \longmapsto (a,b \pm f)$$

where f is an integer, excluding 0. Statements corresponding to nil transformations will be rendered as follows:

$$\bar{r} \ : (a,b) \longmapsto (a,b)$$
$$\bar{s} \ : (a,b) \longmapsto (a,b)$$

where f is thus the nil element of an additive group. The connection between the two statements defining the logical operation necessary to determine a strict order between ratios is an *exclusive disjunction* with the following truth table:

r	s	$r \underline{\vee} s$
1	1	0
1	0	1
0	1	1
0	0	0

Strict order between ratios is determined by the *agreement and difference* method (as defined by John Stuart Mill), where only *one* difference is examined at a time when many variables are concerned.

The following connections are thus relevant: $r \cdot \bar{s}$ or $\bar{r} \cdot s$. The connection $r \cdot s$ yields no information and $\bar{r} \cdot \bar{s}$ gives identity, with no change occurring and the connection included in equivalence and not strict order.

The formal operation as an "operation on operations" (a logical operation on arithmetical operations).

When the development of proportional reasoning is examined (Table 5), one finds that the intuitive and concrete levels of thinking are characterized by the use of *one* logical connection between arithmetical operations *at a time*. Thus the *agreement and difference* method based on *exclusive disjunction* is applied at stage IB, while the equivalence class transformation based on the *biconditional* is applied at stages IIA and IIB. Ratios being composed of *two* terms, equivalence and order between ratios must be defined in terms of *binary* transformation, best formalized with the help of logic.

On the other hand, arithmetical operations on ratios at the same levels (arithmetical addition or multiplication) are performed on *terms* quite independently from logical transformation.

The distinction between logical collection and arithmetical operation is unnecessary to establish formally at these levels, as these transformations are either independent or closely related.

When the IIIA stage is reached, the strategy applied to compare two ratios is a *conjunctive* multiplication of terms followed by a *disjunctive* additive comparison. We find here a combination of logical connectives, the biconditional connective being *reversed* into the exclusive disjunction, which is its exact complement with respect to their truth tables. However, in the instance mentioned, one finds that one connection is between multiplication, the other between addition. The change between logical connection and arithmetical operation is made simultaneously. This stage is still marked by non-differentiation (and non-integration) between logic and arithmetic.

At stage IIIB, differentiation between logical transformations and arithmetical operations (multiplicative and additive) occurs. A combinatorial system is formed where the subject passes easily from an arithmetic operation on terms (finding a common denominator) to a logical transformation (finding equivalences). Thus we have the following:

Arithmetical operation on *terms*:

$a+b = g$, finding the *denominator* of a fraction when two terms of a ratio are given;

$g \cdot h = gh$, finding a *common denominator* when two fractions are given.

Logical operation on statements:

Biconditional $(a,b) \mapsto (ma,mb)$ *equivalence class*

Exclusive disjunction $(a,b) \mapsto (a \pm f,b)$ *strict order*

These distinctions are best summarized in the following Table:

TABLE 6

LOGICAL AND ARITHMETICAL OPERATIONS
INVOLVED IN PROPORTIONAL REASONING AT THE FORMAL STAGE

Nature of operations	Operations and objects on which they are performed	
	Arithmetical operations on terms	Logical operations on ordered pairs
Additive or disjunctive	I $a+b = g$	III $(ha,hg) \gtrless (gc,gh)$
Multiplicative or conjunctive	II $h \times g = hg$	IV $h(a,g) = (ha,hg)$

The two reversibilities at the level of formal operations.

One particularly important difference between logic and arithmetic is the type of *reversibility* or *opposite* involved. *Arithmetical operations* have as opposite an *inverse*: multiplication is inversed in division, addition in substraction.

Logical statements have as opposite a *negation*. When a statement bears on the assertion of a transformation having taken place, the negation of the statement corresponds to a nil transformation. Nil transformations are extremely widespread , as they correspond to constants or to combinatorial systems where variables are made to remain *constant* for definite periods.

This study on proportional reasoning shows that the final stage of organization of this concept corresponds to the possibility of combining operations having an *inverse* with operations having a *negation*, i.e. of algebra and logic.

The process aspect of development.

The concept of equilibration.

Equilibration theory is based on the process of adjusting existing schemes to fit more complex objects in the outer world. It is the outer world which unbalances an existing scheme and forces it to evolve. But the process of change and the reconstruction of a new, "magnified" scheme, is the subject's business, and has to do with what is ordinarily called "understanding". When a subject says: *"I do not understand"*, he means that the new object in front of him is too complex for him to adjust, or is presented to him in such a manner as not to enable him to proceed easily to an adjustment. While the expression: *"Now, I understand"* means that the appropriate modifications have been made and the subject is able to integrate the unfamiliar object and play with it adequately.

An example is a subject, familiar with natural numbers, who is placed in front of a rational number. He will interpret it in the light of what he "knows". The number 4/9 will be considered "large", while 2/3 is "small". A certain number of "modifications" have to be made to the concept of natural number, in order to fit it (or "equilibrate" it) to the new "object" which is the rational number.

The main emphasis of equilibration is on the dual aspect of subject and object intervening in a process of interaction and reciprocal construction.

At equilibrium, the subject grasps the object exhaustively and bears a judgment which is adequate to the whole object.

However, this same object, which is autonomous in the environment, can become more complex. The consequence will be that one part of the object will be grasped by the subject, while the other part is either ignored or interpreted erroneously. This leads to *centration* or *confusion*.

Piaget (1975) has called "équilibration majorante" the process through which an "increased" or "magnified" scheme is constructed, adapted to the new object in the environment.

The phases of magnifying equilibration.

Magnifying equilibration comes about in a step-wise process. Let us examine its premises and consider these phases.

1. - Data in the environment are interpreted by the subject, or *assimilated* through identifiable patterns of behavior or *schemes*.
2. - Unfamiliar data in the environment cause a *disturbance* in the functioning of the scheme.
3. - The subject reacts to a disturbance in the environment through a process of *compensation*.
4. - The mechanism of compensation is not a one-step process, but consists of a succession of identifiable *phases*.
5. - Three phases have been described by Piaget, which are the following:
   - i) At the α-*phase*, the subject "neglects" the disturbance or simply "avoids" it.
   - ii) At the β-*phase*, the subject "modifies" his scheme in order to "assimilate" the new datum.
   - iii) At the γ-*phase*, the subject integrates the new datum in an hierarchical system.
6. - Finally each *period* of development, i.e. *sensori-motor, preoperational* and *operational*, is the seat of a complete process of *magnifying equilibration*, each period beginning with a phase of *nonbalance* and terminating with the construction of *an hierarchical system.*

Construction of proportional reasoning.

We shall examine the data, obtained on the development of the concept of ratio in the child and adolescent, to test:

(1) whether equilibration theory holds, and if so,
(2) what is the nature of its "phases",
(3) whether we find these at each of the "periods" of development.

The development of proportional reasoning was studied here. Two distinct findings are made:

a) Development of the ratio concept occurs in stages which can be both chronologically and structurally differentiated.
b) These stages can be seen as resulting from changes which can be reorganized into two distinct "periods".

The first preoperational period bears on *terms*: a *natural number* is equilibrated with its *inverse* generating, in a four-stage process, the concept of *1:1 ratio* varying inside its equivalence class.

The second, operational period, bears on *ordered pairs*: the *1:1 ratio* is differentiated in an *a:b* ratio, where terms are independent in magnitude, both in their state and their transformations, generating in a four-stage process, the Common Denominator and Common Factor algorithms.

The passage from a *stage* interpretation to a *phase* interpretation can be made if certain definite *processes* can be described, explaining the passage from one stage to another. This can be undertaken by examining the strategy put to work at a particular stage: how it *succeeds* with items characteristic to the stage, and how it *fails* with items of the next stage of difficulty. As the scalogram shows that all subjects pass through the stages in the same order, some rationale should come out from this examination of a strategy, which is the last step in failure for a particular group of items, to a strategy, which is the first step in success for the same group of items. This leads to a study of *errors* made by subjects at the end of a stage of development, when treating a problem of the next stage. Focus was laid on how these errors can be corrected by a process of *differentiation* of the existing scheme, when working upon an *object* of the immediately superior level of difficulty, with ensuing *integration.*

This study was applied at two different *periods* of equilibration, one corresponding to *preoperational processes* leading to the construction of the concept of *ratio* (Table 7), the other to the *operational processes* leading to the construction of the Common Denominator and Common Factor algorithms (Table 8).

CONCLUSION

This study reveals that the *phases* of equilibration, described by Piaget (1975), are in fact distinct *stages*, structurally defined and imbedded in one another.

Equilibration, inside a *period*, takes place in *four phases*, Piaget's β-phase being followed by a *phase* we shall name ρ, giving the following succession:

α-phase : *centration* on known part of an object, ignoring unknown part.

β-phase : *assimilation* of unknown part as complement of known part (*categorial differentiation*), with comparison of *states*.

ρ-phase : *relation* between complementary parts or states inside each object, (*relational differentiation*), with comparison of *relations*.

γ-phase : *differentiation between internal and external relations* with their *hierarchical integration in operations*.

Development *between phases* involves *restructuring* of a scheme in order to fit a more complex object. It is a *qualitative change*. Development *within a phase* involves extension of the scheme to quantitative variations of the object. It is a *quantitative change*. Development thus takes place along two dimensions: *structure* and *quantity*. This is related to the *comprehension* of the scheme, on the one hand, and its *extension* to objects, on the other.

As will be noted, the scalogram analysis was made on individual items and not on categories making up stages. Items in a category vary in difficulty according to *order* of presentation (the first time an item of a stage is presented, it is often failed), *quantitative load* (larger items being more difficult) and *contrast* (difference between subsets favoring seizure of data). This bidimensional aspect of hierarchical development gives it the appearance of a linear development, which it is not. Corresponding to the dual aspect of development: in comprehension and extension, is found the dual aspect of instruction: *understanding* and *exercice*.

TABLE 7

THE FOUR STAGES OF EQUILIBRATION OR ADAPTATIVE RECONSTRUCTION IN THE GENESIS OF EQUIVALENCE CLASSES OF RATIOS

PROBLEM: Working out the relation between terms of a ratio takes place here.
The difficulty in understanding the equivalence class of a ratio is differentiating between complementation of terms in the state and covariation of terms in the transformation.
Math. object involved: ordered pair.- Part known: 1st term. - New part: 2nd term.

ilibration stage	Typical item with strategy	Operatory mechanism	Comments
stage ration scheme familiar of (1st).	IA. - (1,4)vs.(4,1) $a < c$ therefore $(a,b)<(c,d)$	Centration on familiar part of object (1st term). New part rejected or treated by same scheme (confusion).	Direct relation between familiar part of object (1st term) and whole (ratio) in all items passed with success.
stage part seen wing in- effect whole.	IB. - (1,5)vs.(1,2) $a = c, b > d$ therefore $(a,b)<(c,d)$	Assimilation of unknown part of object through inversion of scheme.	Oscillation between centration on first or second term of ratios.
stage rnal ensation terms.	IC. - (2,1)vs.(3,4) $a > b, c < d$ therefore $(c,d)<(a,b)$ even if $c > a$	Compensation of parts in each object with internal comparison, then conclusion.	Comparison between 1st and 2nd terms in each pair, and conclusion when possible.
stage archical ization states in formation.	IIA.- (1,1)vs.(2,2) Mode E $m(1,1) = (m,m)$ Mode I $1/1 = m/m$	Differentiation between complementation of parts and inversion of scheme. Parts compensate each other (reciprocals) but covary in same direction (complexifying or simplifying ratio).	Complementary parts covary both in direct and inverse directions. Equivalence class of 1:1 ratio is constructed.

TABLE 8

THE FOUR STAGES OF EQUILIBRATION OR ADAPTATIVE RECONSTRUCTION
IN THE GENESIS OF THE COMMON DENOMINATOR OR PERCENT ALGORITHM

PROBLEM: At the end of last period, the child has grasped to inverse relationsh
between terms in the 1:1 ratio.
At this period, he must understand their independence as to size (stat
and variation (transformation).

Equilibration stage	Typical item with strategy	Operatory mechanism	Comments
α-stage Fragmentation between multiplicative and additive parts of (a,b) ratio.	IIA.- i) (1,1)vs.(2,2) $m(1,1)=(2,2)$ (success) ii) (2,3)vs.(4,6) → 2(1,1)+(0,1) vs. 4(1,1)+(0,2)(failure)	Application of (1,1) scheme and centration on excess when present.	Adequate trea of 1:1 ratio. $a:b$ ratio is d into multipli aspect (1:1 r and excess c additively.
β-stage Terms of ratio seen to be independent in state.	IIB.- i) (2,3)vs.(4,6) $m(a,b)=(ma,mb)$ (success) ii) (3,1)vs.(5,2) → (2,1)+(1,0) vs. 2(2,1)+(1,0) (failure)	Differentiation of terms in the state. Application of (a,b) scheme with centration on excess.	Equivalence cl of any ratio i grasped. But c conjunctive va ation possible. Excess with re to equivalence treated additi
ρ-stage First coordination between conjunction and disjunction.	IIIA. - (3,1)vs.(5,2) $mb = d$ $m(a,b) = (ma,d)$ $ma > c,\ mb = d$ therefore $(ma,mb) > (c,d)$ whence $(a;b) > (c,d)$	Differentiation of terms in the transformation. Multiplicative conjunction combined with additive disjunction.	Only ratios wh part of corresp terms are multi one of another treated, otherw fragmentation.
γ-stage Hierarchical organization of logical connectives and algebraic operations.	IIIB. - (3,5)vs.(5,8) i) Algebraic addition $a+b = g$ $(a,b) \rightarrow (a,g)$ $c+d = h$ $(c,d) \rightarrow (c,h)$ ii) Algebraic multiplication $hg = gh$ iii) Logical comultiplication $h(a,g) = (ha,hg)$ $g(c,h) = (gc,gh)$ iv) Logical disaddition $ha < gc,\ hg = gh$ $(ha,hg) < (gc,gh)$	Differentiation between logical and algebraic aspects of system. Combinational system along two dichotomies, logical: conjunction vs. disjunction; algebraic: multiplicative vs. additive.	Equivalence clas of each ratio i grated by comm inator or comm factor. Then additive treatment of num ators. Algorith of rational num addition is esta lished.

EDUCATIONAL COMMENTS

Specific to the proportion concept.

(1) The concepts of *ratio* between quantities (e.g. 2 glasses of orange juice for 3 glasses of water), *partition of a set* (e.g. 2 glasses of orange juice in 5 glasses of liquid) and *fraction of a unit* (e.g. 2/5 juice in each glass) should be carefully distinguished. These logical distinctions are being worked upon presently.

(2) *Proper* (< 1) and *improper* (> 1) fractions should be treated simultaneously, e.g. 1/2, 2/4 ...; 2/1, 4/2 ... .
Improper fractions should be considered as rational numbers, without immediate retrieval of the unit, where two distinct notations confuse the child.

(3) From level IIA onwards, any fraction should be envisaged under both its *internal aspect* (*mode I*) and *external aspect* in the equivalence class (*mode E*). This ensures mobility and prepares the future construction of algorithms.

(4) The passage from equivalence of *unit fractions*, to equivalence of *any fraction*, should be considered a difficult step, and taking up many years. The child must here differentiate between *disjunction of terms in the state* (in 2/3, the terms are different) and *conjunction in transformation* (2/3 varies conjunctively when transformed into 4/6). The difference between state and transformation with respect to fractions is the proper problem of the elementary school. Equivalences of 3/5 or 5/7, for instance, in concrete situations, are still considered difficult at the end of the elementary school.

(5) A lot of time should be devoted to problems like 1/3 + 2/9, which make the passage from concrete to formal operations. The *multiple relation between denominators*, in such problems, should be discovered by the pre-adolescents themselves. They must here make a difference between i) *equivalent* fractions (such as 1/3 ≃ 2/6 ≃ 3/9) and ii) fractions with *like denominators and different numerators* (such as 1/9, 2/9,

3/9 ...). Two different systems are here involved, which the subject must differentiate and then integrate. We find here a combination of the logical operation in the equivalence class and arithmetical operation in the addition of fractions with like denominators. This is a coordination of *logic* and *arithmetic* applied to the same content. It is characteristic of the *formal level* of thinking. It is *abstract thinking*, an *operation* (combination) on *two concrete operations* (equivalence and numerator addition).

(6) *Common denominator (mode E) and reduction to unit (mode I)* should be seen as *inverse strategies* used to liken different denominators. Herron and Wheatley (1978) have aptly suggested the unit factor method as an alternate way for solving proportion problems. *Percentage* itself should be seen as a better way of expressing the ratio to a unit, when the first term is smaller than 1 (½/1 or .5/1 becoming 50%).

Relative to the concepts of periods and phases of equilibration.

(1) Respect of periods of equilibration i.e. *concrete* and *formal* modes of thinking.
A sharp distinction should be made between *concrete operations* and *formal operations*. It is a distinction between *operation on terms* and *operation on operations*. In period I, the operation is performed on data themselves (i.e. *a*, *b*, *c*, *d*). In period II, the operation is performed on internal data, previously constructed from the external data (e.g. *ma*, *a/b*, etc.). The difference is especially important to make for teachers in Grade 6, when children are at the frontier of concrete and formal thinking. Some operations, though lengthy, are easy because they involve *successive* manipulation of concrete data. Others, though apparently much simpler, are difficult for the pre-adolescents, because they involve *simultaneously* simplification of fractions (equivalence of fractions) and operations. This seems automatic to an adult, but involves retention of constructed data, then operation on these.

(2) Equilibration should be made of what the child knows, to the unknown which is brought to him. In particular the new variable introduced

at each period should be clearly identified by the teacher and related to existing schemes in the child. This point has been particularly stressed by Ausubel (1963) and Lesh (1976).

Conclusions of a general nature bearing on mathematical thinking.

(1) Emphasis should be put on laying strong foundations rather than rapid but evanescent techniques (this has been said many times). Thus notions should be constructed in their *hierarchical order*. Motivation is kept up by the process of discovery and construction, and the field is widened and strenghtened by varying the content for a same structure.

(2) *Equilibration* - The novel aspect introduced by magnifying equilibration is the constant interplay between the two processes of *interaction with data* and *organization of data*. The dialectical process of uncovering new data (interaction) is constantly related to the process of structuring the data inside a coherent whole (organization). In this whole the new and the old are interrelated in a specific manner. This study on the development of proportional reasoning shows that at the first preoperational period of equilibration, *inversion* of a scheme plays a fundamental part in establishing this relation, while at the second operational period, *separation* of a scheme into autonomous and integrated sub-schemes plays this part. Thus equilibration to novelty is always related to reorganization of the subjet's internal structure. Growth consists in being open to the world, but proceeding with system. This opens up a new field of study: the epistemology of mathematical construction.

(3) At all levels concrete problems (with concrete or graphic data) should be worked out in parallel to symbolic problems (with numbers). The absence of correlation between these two types of activities has been stressed by Steffe and Parr (1968).

(4) Stage IIIB, in proportional reasoning, is characterized by the combination of *logical reasoning* and *algebraic operations*. The difference between the *nil transformation*, characteristic of logic (e.g. $p\bar{q}$), and

the *inverse transformation*, characteristic of operations (e.g. 2+3 = 5, 5-3 = 2), should be made much earlier. Usually a nil transformation is introduced in a combinatorial setting at the formal level: e.g. $pq \vee p\bar{q} \vee \bar{p}q \vee \bar{p}\bar{q}$. This should be prepared at the concrete level (elementary school) by introducing the difference between *constancy* and *variation*. Their combination in the constancy-variation scheme is already put into use, at a practical level, at stage IB of proportional reasoning (middle intuitive: 5-6 years of age). The very important role of the *agreement and difference* principle (basis of scientific reasoning as set forth by Roger Bacon and John Stuart Mill) renders the early introduction of the constancy-variation scheme imperative. The *all other things being equal* principle is the basis of organized thinking at all levels. It should be introduced early in the curriculum in simple problems.

(5) *Axiomatics vs. constructivism.* - Disinterest for mathematics on the part of the layman and the child is due to a certain compulsive character of axiomatics, in contrast with the more creative quality of constructivism. Mathematics as a science is first constructive, with emphasis on process, then results are consolidated, with emphasis on structure. As recent literature shows, two successive moments can be distinguished in mathematical thinking: *construction* and *consolidation*. *Mathematical education* is primarily interested in the first moment: *construction*, while formalized *mathematics* puts its main interest in the second: *consolidation*. Thus two distinct disciplines emerge: *mathematics* and *mathematical education*, with different methods and aims.

(6) *Intuition* plays a large part in mathematical thinking as Fishbein (1978) shows, and belongs to the first moment of mathematical thinking: *construction*. Cognitive-developmental theory regards *intuition* as the result of *actions* of subjects upon the environment in view of finding a coherent structure. Coherent organization of actions exists before formalization. These coherent actions upon defined objects, and the resulting intuitions, should be given a central part in mathematical education. Genetic epistemology will have to work out the passage from coherent actions to formalization (second moment of

mathematical thinking). Thus a relation will be established between *mathematical education*, with emphasis on intuition and organized thinking, and *mathematics* itself, with the emphasis on proof and formalization.

(7) Controversy about the double nature of a mathematical theorem: 1) *derivability*, 2) *applicability*, finds here some solution. Not only is mathematical thinking "applicable". But application to physical objects, when unsatisfactory, leads to "reorganisation" of the strategy applied and establishment of a new mathematical coherence or of derivability.

REFERENCES

AUSUBEL, D.P. The psychology of meaningful verbal learning. New York: Grune and Stratton. 1963.

BEGIN, Louis-Réal. Genèse de la notion de rapport chez un groupe d'enfants de 4 à 9 ans. Thèse de maîtrise, Université Laval, Québec, 1973.

BEILIN, H. The training and Acquisition of Logical Operations. In Piagetian Cognitive-Development Research and Mathematical Education, edited by M.F. Rosskopf, L.P. Steffe and S. Taback, pp. 81-124. Washington, D.C.: The National Council of Teachers of Mathematics, 1971.

BELLEMARE, Thérèse. La méthode Cuisenaire et le développement opératoire de la pensée. Thèse de doctorat, Université Laval, Québec. 1966.

BELLEMARE, Thérèse. La méthode Cuisenaire-Gattegno et le développement opératoire de la pensée. Neuchâtel: Delachaux et Niestlé. 1967.

CASE, R. Learning and development: A neo-Piagetian interpretation. Human Development, 1972, 15, 339-358 (a).

CASE, R. Validation of a neo-Piagetian mental capacity construct. Journal of Experimental Psychology, 1972, 14, 287-302 (b).

CASE, R. Intellectual Development from Birth to Adulthood: a Neo-Piagetian Interpretation. To appear in R. Siegler (Ed.) Children's Thinking: What Develops? Proceedings of the Thirteenth Annual Carnegie Symposium on Cognition. 1978.

CARDINAL, Gilbert. Méthodologie de construction expérimentale d'épreuves opératoires. Thèse de maîtrise, Université Laval, Québec. 1973.

CLOUTIER, Richard. Standardisation de l'épreuve des Concentrations. Thèse de maîtrise, Université Laval, Québec. 1970.

DIXON, W.G. Biomedical Computer Programs. Berkeley: University of California Press. 1971.

DUQUETTE, Raymond J. Some thoughts on Piaget's findings and the teaching of fractions. In The Arithmetic Teacher, Volume 19, Number 4, 273-275. Washington D.C.: The National Council of Teachers of Mathematics. 1972.

FISHBEIN, E. Intuition.
Paper presented at the 2nd International Conference for the Psychology of Mathematics Education. Osnabrück. 1978.

FISHBEIN, E., PAMPU, E. and MANZAT, I. Comparison of ratios and the chance concept in children. Child Development, 41, 365-376. 1970.

FREUDENTHAL, H. Mathematics as An Educational Task. D. Riedel Publishing Co., Dordrecht, Holland. 1973.

FREUDENTHAL, H. Weeding and Sowing. Preface to a science of mathematical education. Dordrecht: Holland. Boston: U.S.A., D. Reidel Publishing Company (Eds.). 1978.

FREUDENTHAL, H., JANSSEN, G.M. and SWEERS, W.J. (Eds.). Five years IOWO. Educational Studies in Mathematics, 1976, 1-2, 189-367.

HERRON, J.D. and WHEATLEY, G.H. A Unit Factor Method for solving Proportion Problems. The Mathematics Teacher. 1978, 71, 1, 18-21.

INHELDER, B. et PIAGET, J. De la logique de l'enfant à la logique de l'adolescent. Paris: Presses Universitaires de France, 1955. Deuxième édition 1970. Translated as: The growth of logical thinking from childhood to adolescence. An essay on the construction of formal operational structures. New York: Basic Books. 1958.

INHELDER, B. et PIAGET, J. La genèse des structures logiques élémentaires. Neuchâtel: Delachaux & Niestlé, 1959. Deuxième édition 1967. Translated as The Early Growth of Logic in the Child: Classification and Seriation. London: Routledge & Kegan Paul. 1964.

KARPLUS, R. and PETERSON, R.W. Intellectual development beyond elementary school II; Ratio, a survey. School-Science and Mathematics. 1970, 70(9), 813-820.

KARPLUS, R. and KARPLUS, E.F. Intellectual development beyond elementary school III: Ratio, a longitudinal survey. Berkeley: Lawrence Hall of Science, University of California. 1972.

KIEREN, T.E. On the Mathematical, Cognitive and Instructional Foundations of Rational Numbers. In R. Lesh (Ed.) Number and measurement: Papers from a Research Workshop. Columbus: ERIC/SMEAC, Ohio State University. 1976.

LAURENDEAU, M. et PINARD, A. Les premières notions spatiales de l'enfant. Neuchâtel: Delachaux et Niestlé. 1968. English edition New York: Basic Books.

LAURENDEAU, M. et PINARD, A. La pensée causale. Montréal: Institut de Recherches Psychologiques et Paris: Presses Universitaires de France. 1962. Translated as: Causal Thinking in the Child. A Genetic and Experimental Approach. New York: International Universities Press, Inc. 1962. Second edition 1968.

LESH, R.A. An interpretation of advanced Organizers. Journal of Research in Mathematics Education, 1976, 7, 69-74.

LOVELL, K. Mathematical Concepts. In Analyses of Concept Learning, edited by H.J. Klausmeier and C.W. Harris. New York: Academic Press. 1966.

LOVELL, K. Intellectual growth and understanding mathematics implications for teaching. In The Arithmetic Teacher. Washington D.C.: The National Council of Teachers of Mathematics. April 1972, 19, 277-282.

LOVELL, K. and BUTTERWORTH. Abilities underlying the understanding of proportionality. Mathematics Teaching. 1966, 37, 5-9.

LUNZER, E.A. and PUMPHREY, W. Understanding proportionality. Mathematics Teaching. 1966, 34, 7-12.

NOELTING, G. et CLOUTIER, R. Le développement de la notion de proportion chez l'enfant et l'adolescent. Rapport de recherche, Laboratoire de Psychologie du développement, Université Laval, Québec. 1970.

NOELTING, G., CLOUTIER, R. et CARDINAL, G. Stades et mécanismes dans le développement de la notion de proportion chez l'enfant et l'adolescent. (L'Épreuve des Concentrations). Rapport de recherche. Université Laval, Québec. 1975.

NOVILLIS, C. An Analysis of the Fraction Concept into a Hierarchy of Selected Subconcepts and Testing of Hierarchical Dependencies at Grade Levels 4, 5, 6. Ph.D. Thesis, University of Texas. 1973.

PASCUAL-LEONE, J. A mathematical model for the transition rule in Piaget's developmental stages. Acta Psychologica. 1970, 32, 301-345.

PASCUAL-LEONE, J. Metasubjective problems of constructive cognition: Forms of knowing and their psychological mecanisms. Canadian Psychological Review. 1976, 17, 110-125.

PIAGET, J. L'équilibration des structures cognitives, problème central du développement. Paris: Presses Universitaires de France. 1975. Translated as: The Development of Thought. New York: Viking Press. 1977.

PIAGET, J. et INHELDER, B. La genèse de l'idée de hasard chez l'enfant. Paris: Presses Universitaires de France. 1951. Translated as: The Origin of the Idea of Chance in Children. New York: Norton. 1975.

ROMBERG, T. and de Vault, V. Mathematics Curriculum: Needed Research. In J. Hooten (Ed.), Journal of Research and Development in Education, Vol. 1, No. 1. Athens, Georgia. 1967.

ROUSSEAU-SAVARD, C. Epreuve des Concentrations: étude de l'instrument et de ses résultats. Thèse de licence. Université Laval, Québec. 1969.

SCALLON, G. Etude de l'habilité à résoudre des problèmes nouveaux avec le développement génétique et le degré de connaissance chez des élèves du cours primaire. Thèse de licence. Université Laval, Québec. 1966.

SKEMP, R.R. The Psychology of Learning Mathematics. Penguin Books, 1971.

SIEGEL, S. Non Parametric Statistics for the Behavioral Sciences. New York: McGraw Hill. 1956.

STEFFE, L.P. and PARR, R.B. The Development of the Concepts of Ratio and Fraction in the Fourth, Fifth, and Sixth Years of the Elementary School. Technical Report No. 49, Research and Development Center for Cognitive Learning. Madison: University of Wisconsin. March 1968.

STRAUSS, S. Curvilinear Development in Proportional Reasoning or That's Funny, I wouldn't Have Thought you Were U-ish. Tel Aviv University. Unpublished manuscript. 1977.

STREEFLAND, L. Some Observational Results Concerning the Mental Constitution of the Concept of Fraction. Educational Studies in Mathematics. Dordrecht, Holland: D. Reidel Publishing Company. 1978, 9, 51-73.

SULLIVAN, E.V. Piaget and the School Curriculum: A Critical Appraisal. The Ontario Institute for Studies in Education, bulletin no. 2. Toronto: The Institute, 1967.

WEAVER, J.F. Some concerns about the application of Piaget's theory and research to mathematical learning and instruction. In The Arithmetic Teacher. Washington D.C.: The National Council of Teachers of Mathematics. April 1972, 19, 263-270.

WOLLMAN, W. and KARPLUS, R. Intellectual development beyond elementary school V: Using ratio in different tasks. Berkeley: Lawrence Hall of Science, University of California, June 1973.

INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF
MATHEMATICS EDUCATION

OSNABRUECK SEPTEMBER

BRUNEL UNIVERSITY DEPARTMENT OF EDUCATION

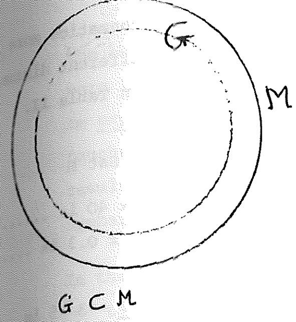
THE STRUCTURE OF MATHEMATICAL ABILITIES: SOME EXAMPLES ILLUSTRATING DIMENSIONS OF DIFFICULTY - RUTH REES

Our earlier work at Brunel was concerned with studies of the performance of various groups of students on specific mathematical topics. The work has now developed so that a structure of mathematical performance is emerging. Investigations, involving both psychometric and case-study methods, of some basic mathematical difficulties of students in Further Education, Secondary Schools, a University and Teacher Training Institutions have show (1,2,3,4) that there appear to be two major determinants of mathematical performance. The first of these is associated with the general ability 'g' as measured by standard psychological tests. The second is an ability which appears to relate to the capacity for 'mathematical inference' one of its characteristics being its relative independence of the 'g' measures.

We have found that when students solve the problems aloud the kinds of difficulty they find with items demanding 'mathematical inference' are recognisably different from those associated with the 'g' type problems. The essential difference appears to be that the appropriate mathematical operati has to be recognised for the latter and deduced for the former.

The basic mathematics under investigation has been concerned essentially with concepts and operations applied to number and to a few algebraic and geometrical problems. The mathematics involved possibly represents a minim basis of mathematical competence for the average school leaver.

The characteristics of the two sets of mathematical problems are shown in Table I. Those problems which correlate with the measures of general ability 'g' are called set G. Those problems which are relatively independe of the 'g' measures and appear to be associated with a more specific mathem ability are called set M.



Set G: 'General Ability' problems.
Mathematical processes are 'digestable' for the majority of students.

Set M: More specific Mathematical Ability problems.
Mathematical processes are 'digestable' for fewer students.

CHARACTERISTICS OF PROBLEMS

SET G	SET M
1. Students perform reasonably well provided they recognise the problem.	1. Performance is relatively worse for all kinds of students. Solutions have to be deduced.
2. Teachers' estimates of students' performance are reasonably accurate.	2. (i) Teachers often do not appreciate the nature of the difficulties and underestimate these difficulties. (ii) A learning-teaching cycle appears to be generated by these problems.
3. Students appear to solve algorithmically.	3. Students appear to need more mathematical 'inference' for solution.

TABLE I

It is found that some students may perform well on both sets G and M, some perform well on G and poorly on M, some perform poorly on both sets. What does not appear to be possible is that students perform badly on G and better on M. This may explain why so many students of good general ability sometimes do relatively badly in mathematical work. The composition of a test i.e., the number of problems of the 'M' kind compared with the number of the 'G' kind will be critical in determining the student's total score.

The findings also show that not only are some topics in mathematics more difficult than others but that within topics there exist different dimensions of difficulty. These can be illustrated by the exemplars in Table II.

Topic	Set G	Set M
1. ⓧ of integers	1. 41 x 43 is	1. 41 x 40 is .....
2. ⓧ of numbers less than 1	2. 0.4 x 0.4 is	2. 0.3 x 0.3 is ....
3. Fractions	3. $\frac{9}{16} + \frac{5}{64}$ is	3. $5\frac{7}{16} - 3\frac{5}{8}$ is ...
4. Equivalent fractions/ ratio/equations/ reciprocal	4. If $\frac{2}{3} = \frac{1}{3}$ x is...	4. If $\frac{1}{x} = \frac{3}{4}$, x is ...
5. Proportion		✓

TABLE II

Probably the majority of pupils and older students will solve the first example in both sets using the algorithm of 'long multiplication'; many of these will become confused with the role of the zero in set M. The more mathematically able students will provide a variety of ways for the solution of 41 x 40. These ways will include $40^2 + 1 \times 40$ and "adding" a zero to 41 x 4.

The solutions to the second pair of examples again highlights the 'blockage' created by operating on numbers less than unity. Examples such as 0.4 x 0.4 (0.5 x 0.5 etc) will generate the following solution. "Four fours are sixteen; point sixteen". The student has the right answer and therefore appears to cope quite well!

However, examples such as 0.3 x 0.3 (0.2 x 0.2 etc.) will generate the following response from the majority of pupils. "Three threes are nine; point nine". The more mathematically able student may also be 'triggered' into such a response but then usually checks his solution and sees his error.

The majority of pupils appear to treat numbers less than unity as if they were integers. The effect of operating on such numbers does not appear to be explicitly taught and emphasised. Many pupils have said "we have been taught the effects of operating on integers e.g., multiplication makes the result larger, but we are then expected to know what happens with small numbers."

In the third pair of examples many students will confidently solve the 'G' type even though the answer may be wrong e.g., adding the numerators and denominators respectively. They will be less confident about a method for the 'M' type fractions..

In solving the fourth pair of examples students who recognise 'equivalent fractions' or the trick of 'cross-multiplication' will solve the example in set G. Many of these students will get stuck on the solution of the example in set M because it is a non-integer. This latter example generates many different "perceptions" by various kinds of students as indicated under the 'Topic' heading.

The recordings show that most students try to solve mathematical problems algorithmically. The set G kind of problems are more amenable to the application of straightforward algorithms. The set M kind of problems require more mathematical appreciation and demand more inferential thinking although ultimately they may be solved algorithmically.

Algorithmic solutions do not necessarily imply a lack of understanding by pupils. An algorithm engenders a measure of confidence and removes some of the anxiety apparently inherent for most pupils in mathematical problem solving. Perhaps a good teacher should seek to provide understanding of a topic and plenty of practice in the application of correct algorithms. The solutions may then appear rote but will nevertheless conceal understanding for many pupils. Those pupils who escape the net of understanding will at least acquire a rote skill of practical application to work and living.

It is therefore important for teachers to know where, within topics, the blockages occur and why they occur.

Discussion with students reveals that they think that many of the set 'M' type problems are not taught explicitly.

The implications of the differentiation in performance between the sets is that most students could be taught the 'set G' type of mathematics and their success will be largely determined by their general ability. The learning of the 'set M' kind of mathematics, however, may well require teaching of a special quality. It is part of this research study to investigate teaching strategies for some of these concepts and skills.

The following examples have been selected to illustrate the two methods used in the research i.e., students' written solutions and students' verbalising their way to solutions.

I PROBLEM TYPE: SET M

Written (1) If $\frac{1}{R} = \frac{1}{2} + \frac{1}{6}$ then R is

a. $\frac{1}{8}$ b. 8 c. $\frac{2}{3}$ d. $1\frac{1}{2}$

Response	F%	Main Wrong Response %	
Further Education	15.8	51.7	(c)
Schools C.S.E.	7.4	45.4	(c)
Schools 'O'	36.7	45.5	(c)
Primary Teacher Trainees	34.3	57.4	(c)
University Undergraduates	64.4		(c)

(2) If $\frac{1}{x} = \frac{3}{4}$ then x is

a. $\frac{3}{4}$ b. 3 c. 4 d. $\frac{4}{3}$

Response	F%	Main Wrong Response %	
Further Education	34.5	37.3	(a)
Schools ('O' & C.S.E)	43.0	31.0	(a)

Verbal

The language laboratory and tape-recordings of students solving this kind of problem reveal the following "conceptual models" used by students

(i) 'Equivalent fractions' (13 year old school pupils).
(ii) 'Confusion' : " x on the bottom" (F.E. craft students).
(iii) 'Cross-multiplication ' (F.E. technician students).
(iv) 'Equation' (University undergraduates).
(v) 'Turn both sides upside down' (Few students).

The set 'M' problems generate more searching around for a conceptual model on which to operate effectively.

II PROBLEM TYPE: SET G

Written $\frac{9}{16} + \frac{5}{64}$ is

a. $\frac{45}{64}$ b. $\frac{41}{64}$ c. $\frac{14}{80}$ d. $\frac{14}{64}$

Response	F%	Major Wrong Response %	
Schools (mostly C.S.E.)	34	40	(c)
Schools ('O')	85	13	(c)
F.E. (craft)	63.9		
F.E. (technician)	88.0		

Verbal

Students are more confident when they try to solve set G problems - even although their conceptual models and answers may be wrong. A problem of this kind tends to "trigger" them into a solution.

It would be interesting to explore the relationship between these kinds of problems and the "instrumental" and "relational" kinds of learning and teaching discussed by Professor Richard Skemp.

Discussion in recent years of the mathematical performance of students has tended to project a depressive picture. We must not forget the enthusiasm and better attitude generated amongst many primary school pupils by schemes such as, for example, the Nuffield primary project in the U.K.

Many children at both primary and secondary school levels are able to tackle cheerfully many of the non-computational aspects of mathematics. The computational tasks of mathematics (and these are considerable) involve techniques that are conceptually difficult. The expression 'simple arithmetic' is a misnomer that does an injustice to many of the mathematical tasks which we are concerned to teach to our pupils.

I believe that the computational message is, however, one of hope. For nearly all pupils a substantial core of 'G' mathematical computational problems is possible, substantial that is with respect to most ways of earning one's living. In addition for these pupils there are many non-computational aspects which they can appreciate. As teachers we need to teach 'concepts' clearly and the application of correct algorithms resulting from these concepts, It is therefore essential for effective teaching to appreciate where and why the learning blockages are likely to occur. In the present situation it appears that it is the more mathematically able pupils who can solve the set M kind of problem. Perhaps as teachers we should aim to create a more mathematically able society by seeking to transform all 'M' type problems to 'G'!

REFERENCES

1. Monograph
"Mathematics in Further Education. Difficulties experienced by Craft and Technician Students", Hutchinson Educational Ltd., February 1973.

2. "An Investigation of Some Common Mathematical Difficulties experienced by Students", 'Mathematics in School' (Mathematics Association), Vol.3. January 1974.

3. "The Dimensions of Mathematical Difficulties". A set of problems proving more difficult than teachers expect at all levels in the educational system. W.D. Furneaux and Ruth Rees, Brunel University Department of Education, Occasional Publications Series No.1 September 1976.

4. "The Structure of Mathematical Ability" W.D. Furneaux and Ruth Rees. 1977. Brit.J. Psychology, in the press.

A REVISED MODEL FOR REFLECTIVE INTELLIGENCE

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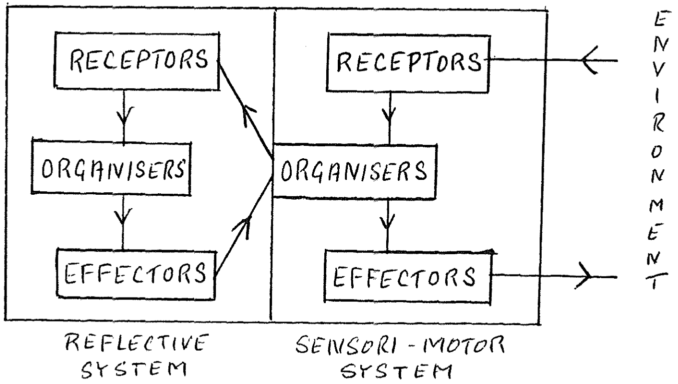
Some earlier work.

The model which I have been using until recently was first published seventeen years ago (Skemp, 1961), where the following formulation was given:

> "Reflective intelligence is the functioning of a second order system which
> (i) can perceive and act on the concepts and operations of the sensori-motor system;
>
> (ii) can perceive relationships between these concepts and operations, and
>
> (iii) can act on them in ways which take account of these relationships and of other information from memory and from the external environment."

In diagram form:-



That model was first formulated, and tested, in the context of mathematics: the main hypothesis being (ibid, page 50):

> "For mathematical achievement a necessary, though not sufficient, condition is the presence of reflective intelligence as well as sensori-motor intelligence."

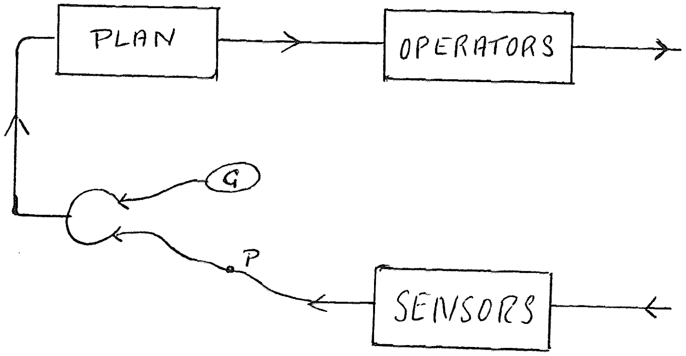
To test this hypothesis I devised a set of tests for evaluating pupils' ability at reflective activity on class-concepts and on operations. Although these tests were non-verbal and non-mathematical, correlations with a mathematical criterion were obtained of .42 (maths with reflective activity on concepts) and .72 (maths with reflective activity on operations). These results were closely replicated a year later, when the correlations obtained were respectively .48 and .73. In an extensive investigation using a slightly revised version of the tests of reflective activity, and a different mathematical criterion, Harrison (1967) obtained corresponding correlations of .41 and .49. He also obtained a multiple correlation between the mathematical criterion on the one hand, and a weighted sum of scores from two of my tests on the other, of .64. This was higher than the correlation obtained between a test of numerical ability taken alone and the maths criterion (.57); but lower than a multiple correlation obtained from a combination of numerical ability and verbal reasoning (.75). Since I would certainly regard verbal reasoning as a reflective activity, these results are consistent with the basic hypothesis stated above, and also with my own experimental results.

Generalising the model.

My interest in this area has long had two distinct, though complementary, interests: first, that of a mathematician and teacher of mathematics; and second, that of a psychologist, to whom mathematics appears as a particularly concentrated and low-noise example of the functioning of human intelligence. From the latter viewpoint, it seemed that by studying the psychology of learning mathematics, the improved understanding of intelligent learning which can be gained by working in this area should be generalisable to give a better understanding of the nature and function of intelligence itself: with a potential for applications extending over a very wide range of human activities.

Such has proved to be the case, and this generalisation will be published in 1979. Mathematics now re-appears as an important special case, as might have been expected. There are, however, new features which I do not think I would have arrived at without having moved into the more general field. (Especially, it is becoming possible to answer better the question 'Why do we learn mathematics?' by seeing it as a special case of 'Why do we learn anything?')

Those parts of the new model which are needed for the present discussion are shown in the diagram below, and the next.



First order system)
Sensori-motor system) called "Delta-one".

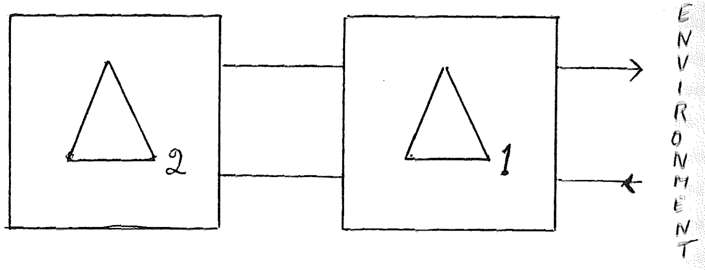
This corresponds to the right-hand part of the earlier model. The boxes for 'receptors' and 'effectors' are unchanged, except for being re-named (sensors = receptors, operators = effectors). (They are also changed in position, but this has no importance.) The important difference is the expansion of the former 'organisers' into four parts. This makes the model into a cybernetic one, and sharpens the divergence from 'stimulus-response' ways of thinking. The dot marked P is an internal representation of the present state of the operand (whatever it is in the environment that the sensori-motor system is acting on). The loop marked G is an internal representation of the goal state (that state to which the operand is to be taken). The large letter C to the left of these is the comparator; that which compares present state and goal state, and transmits this difference to the box marked PLAN. This last determines how the energy available to the operators is applied to the operand, so as to take it to the goal state (and if necessary keep it there). The whole system, excluding the operand, forms a director system.

The clearest connection with mathematics arises from P and G with the emphasis that these are internal representations of a present state and a goal state of the operand. Some of these internal representations are innate, but these with which we are here concerned are learnt. Now, as was said by Heraclitus, "We cannot step twice into the same river". The present experiences from which (sometimes) we learn become part of our past, and will never again be encountered in exactly the same form. But the situations in which we use what we have learnt, to direct our actions, lie in the future as it becomes present, or as by anticipation we bring it into our present thinking.

It follows that if our mental representations are to be of any use to us, they must represent, not singletons from among the infinite variety of our possible encounters with the environment, but common properties of past experiences which we are able to recognise on future occasions. That is to say, these mental representations must be concepts. So the development of the model into a director system further emphasises the importance of concept formation. Primary concepts represent regularities abstracted directly from the environment; secondary concepts are abstracted from other concepts; and mathematics is concerned with increasingly abstract concepts - with the higher order regularities which can be found among the regularities we have already abstracted.

The successful direction of our sensori-motor activities is thus closely dependent on our building up appropriate conceptual structures (schemas), by which can be represented a wide assortment of present states and goal states within a particular category. It is the building up, testing, and up-dating of these schemas which I now conceive as one of the major functions of intelligence.

This brings us to the second part of the new model. For these changes in the schemas available to delta-one (the sensori-motor system) can hardly be thought of as random. Quite the opposite. The purely mental activities involved in learning are as purposeful as the sensori-motor activities involved in observable action: with this difference, that the goals of learning are new states of delta-one in which it can do its job better. To account for this, we need a second order director system which acts on delta-one itself, and whose function is the improvement of delta-one in various ways. The diagram below shows this combination, in the simplest possible way.



This looks very much like the original (1961) model, but there is this important difference: that each part now represents a director system. The job of delta-one is to direct action on the physical environment. The job of delta-two is to direct learning, and other mental activities which will be mentioned later.

Intuitive and reflective modes of functioning.

This conception of intelligence, as a superior form of learning which can be observed at its most advanced form in man, represents a sharp breakaway from the psychometric approach. Since intelligent learning may be inferred from observations of babies from very early years (see for example Bower, 1974), the new model assumes that delta-two as well as delta-one is present from birth. The difference between intuitive and reflective intelligence are now explained, not as the development of a new system which was not there before, but as two modes of functioning of the same system. In the intuitive mode, consciousness is centred in delta-one, the objects of consciousness being in the physical environment. When intelligence functions in the reflective mode, consciousness is centred in delta-two, and the objects of consciousness are in delta-one: particularly its concepts, schemas, and plans. What develops with age, from infancy through adolescence, and which we may continue to call for short 'reflective intelligence', is not the delta-two system itself, but the ability at will to move the centre of consciousness from one level to another in this way.

This ability to make our own mental processes the object of conscious observation introduces a number of new powers and dimensions to our abilities. Those which are particularly relevant to mathematics include:

(i) Inferential processes, including logical thinking and proof.
(ii) Examining our concepts and schemas for inconsistencies and false inferences, and trying to put these right.
(iii) Checking work which has been done, and correcting errors.
(iv) Improving and systematising the knowledge we already have, e.g. by perceiving particular cases as examples of higher order regularities.
(v) Reflective extrapolation.
(vi) Reflective planning.
(vii) Problem solving.

All of these reflective activities are of central importance in mathematics; and as already has been said, mathematics offers particularly clear examples of all these reflective activities. For in pure mathematics, the object of both attention and manipulation are concepts and schemas of a high order of abstraction.

Symbols and models.

When the second order system delta-two was introduced, there was a tacit assumption that the same model of a director system can be sued for delta-two as for delta-one. This assumption must now be examined, since it raises an important and difficult question which is central to our understanding of the reflective process.

For delta-one, the objects whose present states have to be perceived and represented within an appropriate schema are physical objects, accessible to sense organs with which we are familiar such as our eyes and ears. But for delta-two, the objects to be perceived and manipulated are concepts and schemas, mental objects which are certainly inaccessible to our ordinary senses. Nor have any researches, anatomical or neurophysiological, given us any suggestion of internal sense organs, analagous to our eyes and ears, by which we can perceive our own concepts and schemas. So we should be surprised, not at the difficulties of reflecting on our own thought processes, but at the fact that we can do so at all. And beyond this, to know how we do it involves yet another degree of difficulty: that of reflecting on the reflective process itself. In trying to answer both of these questions, an examination of some of the functions of symbols will enable us to make a beginning.

A symbol is itself a concept, of a kind. A particular word will not be pronounced exactly the same by any two persons. Handwriting and the types used for printing also differ. That we can recognise different utterances as examples of the same symbols therefore implies a simple kind of conceptualisation. They are primary concepts, since the examples from which they are formed are sounds, marks on paper, etc... They have in addition the following special properties:-

(i) A fresh example can be produced when we want, by speaking, writing, drawing, projecting on a screen, or otherwise uttering. This implies that the utterance of a particular symbol can be set as a goal state for delta-one.

(ii) It is mentally connected with another concept (or schema), which is its meaning.

(iii) If it is to be used for communication it is also necessary that the symbol activates matching concepts (so far as one can tell) in both utterer and the receiver.

The process of making a concept accessible to reflective consciousness seems to be closely associated with associating them with symbols. Since symbols are primary concepts, they are the least abstract kinds of concepts we have; and this offers a possible explanation of why they are more accessible than others. It may indeed be the case that secondary concepts are not directly accessible to reflection at all, but only indirectly by having become connected with symbols. The symbols then act as both handles and labels, by which concepts are retrieved from memory store, identified, and manipulated.

Symbols may thus be thought of as an interface between delta-one and delta-two, by which delta-two is able to act on the concepts and schemas belonging to delta-one in the ways which have been listed. The use of symbols for discussion is also related to their use for reflection. In each case, various aspects of delta-one are the objects of attention and manipulation. In mathematics these are mathematical concepts and schemas, and also algorithms, proofs, operations, problems, and possible plans for solving these. The major difference between discussion and silent reflection is that in discussion the utterance of the symbols evokes (approximately) matching concepts (etc.) in the delta-ones of both utterer and receiver. A major similarity is that a certain degree of distancing is involved. When consciousness is centred in delta-two, the concepts, schemas, and processes of delta-one are treated somewhat objectively even by their owner. Discussion, including disagreement of a constructive kind, can be a valuable partner for reflection; sometimes requiring it (when one's own arguments are called into question), sometimes stimulating it (by setting into motion thought processes which otherwise might not have taken place). A necessary condition is a reasonably good match between the schemas used by the participants, and their use of the same symbol-system.

Theories and models.

These are both particular kinds of schema; and though there is not a sharp division between them, it is convenient to think of a theory as more general as a model. If we regard a theory as a schema within which can be conceptualised all possible states within a particular universe of discourse, together with the various connections between these states, then a model may be thought of as being something constructed for a particular task (or class of tasks) by making use of some of the concepts already available from the schema, and putting these together in a particular way. (For example, the simple interest formula $I = \frac{PTR}{100}$ uses concepts taken from the system of rational numbers, and from elementary algebra.)

Even for the functioning of delta-one without any reflection, some kind of model - a mental representation of the particular situation within which the action takes place - is essential for the successful direction of this action. For this kind of use, it is not necessary to be conscious of the model in use, particularly if the task is a routine one and a suitable model is already available. But if the task is problematic, i.e. of a kind which we have not done before (nor perhaps any one else), considerable reflection may be needed both for constructing a suitable model, and in deciding how it should be manipulated so as to yield a solution of the problem. This requires that the concepts from which the model is constructed shall be associated with a suitable structure of symbols, again usually drawn from an available pool; such as words, mathematical symbols, lines and dots. This process of making a model accessible to conscious reflection by connecting it to a symbol structure may conveniently be called <u>formulating</u> it. It is clearly necessary also before we can communicate it, though in friendly company the two activities may sometimes be combined as 'thinking aloud'. The process of formulating a model, by making it accessible to reflection, also exposes it to the tidying-up, self-critical activities of delta-two. Internal inconsistencies, gaps in the inferential process, imperfect correspondence between the model and whatever it is meant to be a model of, can no longer be overlooked. This is not only a good thing in itself, since it improves the quality of the model for use in directing action; it is also a good thing as a preliminary to communication.

We may also note that models may be approached with two different emphases: model using, and model making. In the former the model is brought into use, and put to work as a tool by which to direct action, with relatively little attention to the model itself: in much the same way as we might pick up a hammer or screwdriver almost without thinking about them, our attention being on the job for which we are using them. In other circumstances, attention may shift to the model itself; to choice, possibly its construction, or critical examination in relation to the job to be done and in relation to alternative models. Using the same analogy, suppose that we want to knock a nail into a thin panel without breaking the latter. For this, the heaviest hammer we can find may be put to a different use, by being held to the other side of the panel to provide inertial resistance while the nail is tapped in. Here we are devising what is functionally a new tool

from our existing set of tools: and this involves a temporary shift of emphasis from the job to be done, to finding a tool by which we can do it.

<u>A higher order model</u>.

This brings us full circle, to the same subject matter as has been under discussion from the beginning of this paper, but viewed now in a different way. We can now apply the argument of the previous section at the level of the model making process itself.

Making and using models is something which we all do all the time. Much of the time we do it unconsciously and intuitively. What I have been doing in this paper is a second order process, that of formulating and offering for discussion a model <u>of</u> the model making and model using processes. And what I hope may be achieved thereby is that by making these processes accessible to the activity of reflective intelligence, the additional powers to our thinking which we know that this brings about in other areas may also take place in the interesting but difficult area now under disucssion.

<u>REFERENCES</u>

BOWER, T.G.R. (1974). <u>Development in Infancy</u>, Freeman, San Francisco.

HARRISON, D.B. (1967). '<u>Reflective Intelligence and Mathematics Learning</u>', Unpublished Doctoral Thesis, University of Edmonton, Alberta.

SKEMP, R.R. (1961). 'Reflective Intelligence and Mathematics', <u>Brit. J. Educ. Psychol</u>., XXX1, pp45-55.

SKEMP, R.R. (1979). '<u>Intelligence, Learning, and Action: A Foundation for Educational Theory and Practice</u>', to be published by Wiley in 1979.

Review of Recent Research
Related to the Concepts of Fractions and of Ratio

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Discussions about and reviews of research on fractions or rational numbers have been relatively extensive during the past several years. Kieren (1976a) elaborated the mathematical, cognitive, and instructional foundations of rational numbers in one of a set of papers from a research workshop sponsored by the Georgia Center for the Study of Learning and Teaching Mathematics (GCSLTM). For the same meeting, Payne (1976) described the studies which he, and graduate students working with him at The University of Michigan, had conducted over a period of years. For the Third International Congress on Mathematical Education, Kieren (1976b) briefly summarized both previous studies and ongoing research being conducted by the Rational Number Working Group of the GCSLTM. Other papers provide insights on selected aspects of research or assessment efforts.

Reviews by Suydam and Dessart (1976) and Suydam and Weaver (1975) have summarized the results of a wide range of studies on fractions, with a focus on presenting findings which might have relevance for teachers. Instructional procedures, many concerned with computational algorithms for fractions, have characterized such studies.

In another approach, procedures which have been developed through research at The University of Michigan are carefully detailed for teachers. Thus Coxford and Ellerbruch (1975) describe a sequence for developing the fraction concepts; this is refined, with even more detail, by Ellerbruch and Payne (1978).

Far less attention has been devoted to ratio concepts aside from their involvement in interpretations of fractions. While some attention has been given to the curricular aspects of teaching ratio and proportion, most research has explored the topic in relation to statements about proportionality made by Piaget.

This paper will not attempt to describe myriad studies in detail. Instead, highlights from previous documents will be cited to indicate the status of our knowledge about fraction and ratio concepts and operations, to draw attention to comments from researchers on their findings, and to note points to consider as research proceeds.

Assessment Results on Fractions

Data from the first mathematics assessment of the National Assessment of Educational Progress indicate that performance with fractions is at a much lower level than performance with whole numbers (Carpenter et al., 1978). In most American schools, some intuitive ideas about fractions are introduced in the primary grades; however, it is "not too likely that they will generally think of fractions as quantities" (p. 34). In-depth study of fraction concepts, operations with fractions, and emphasis on fractions as quantities occurs in the last half of grade 4, and continues into grades 5 and 6, with review and reinforcement continuing in grades 7 and 8. Consequently,

> 9-year-olds (grades 3 and 4) assessed in January and February would not be expected to demonstrate much formal knowledge of fraction concepts and algorithms, although 13-year-olds should be thoroughly operational with fractions. (Carpenter et al., 1978, p. 34)

In their overview of the findings on the data on fraction items, Carpenter et al. noted:

- Only 20 to 37 percent of the 9-year-olds could name a fractional part of a whole, and only 9 to 21 percent were correct on naming a fractional part of a set of discrete elements. There appears to be an absence of fraction concepts for nine-year-olds, rather than any erroneous knowledge for the majority of this age group.
- While 65 percent of the 13-year-olds were correct in naming the fractional part of a set of discrete elements, they and adults performed at about the same level (20 and 25 percent respectively) when naming the fractional part of a group, with 17-year-olds slightly better (36 percent). Thus, the overall results were low.
- None of the age groups appear to possess adequate knowledge of properties of fractions, especially ordering and comparing. Fifty-six percent of the 13-year-olds and 83 percent of the 17-year-olds could select a fraction between two given fractions. No age group could order adequately six fractions from smallest to largest; the respective percentages correct were 0.2, 3.3, 14, and 20. The three older groups were asked to select a number closest to 3/16; the percentages correct were 19, 39, and 36, respectively.
- A majority of 13- and 17-year-olds could successfully multiply 1/2 x 1/4 (62 and 74 percent respectively), although fewer (42 and 66 percent respectively) were able to add 1/2 and 1/3 correctly. On an addition problem problem, 30 percent of the 13-year-olds and 16 percent of the 17-year-olds added numerators and added denominators. This indicates a lack of understanding of fractions as quantities and a severe failure to develop computational skill or even knowledge of computation algorithms.
- On an exercise involving a minor translation to a number sequence, selection of the multiplication operation, and multiplication of a fraction by a whole number, 78 percent of the 17-year-olds, 57 percent of the 13-year-olds, and 17 percent of the 9-year-olds could do the task.

- Another exercise called for the manipulation of an algebraic fraction, x/y. While it was administered to 9-year-olds, it is difficult to see any reasonable expectation for them to cope with the symbolism and understand the problem; nevertheless, 15 percent of them hit upon the correct answer. Eighteen percent of the 13-year-olds and 41 percent of the 17-year-olds were correct.

In a section of implications for instruction, Carpenter et al (1978) noted:

> Emphasis should be placed on providing a sound initial development of fraction concepts using concrete objects. Results indicate that many students have little computational skill with fractions and probably little conceptual understanding. An increase in the amount of time spent on operations with fractions is not necessarily an appropriate remedy. The development of algorithms should be paced so as to connect firmly with the main ideas in the initial development. (p. 54)

Bright (1978) synthesized results from NAEP, NLSMA, and several other assessments. He reported that

> Performance seems to stabilize at about grade 8 (age thirteen), and there is noticeable improvement in performance from grade 6 to grade 8. As might be expected, when the lowest common denominator does not appear in the problem, students have considerably more trouble than when it does appear. Their skills with multiplying fractions seem to lie somewhere between these two levels. It is impossible to determine from the data, however, whether their difficulties result from a lack of understanding of the processes, an inability or inattention to "reducing" the answers to lowest terms, or a lack of practice. (p. 158)
>
> It should be noted in passing that the multiplication algorithm, because it can be performed without writing the fractions with a common denominator, is mechanically an easy one. Conceptually, however, multiplying fractions is more complex than adding them . . . this hierarchy of difficulty is supported empirically. Apparently the mechanical difficulties of adding fractions without a common denominator outweigh the conceptual difficulties of multiplication. (p. 159)

> . . . stabilization of performance for whole-number computation occurs earlier and at a higher level than for fractional-number computation. Perhaps this is a practice effect that reflects the introduction of whole-number computation before fractional-number computation. (p. 160)

Suydam and Osborne (1977) also summarized data from various assessments, concluding that "the topics with which difficulty (or weakness) were reported can be ranked in this order of frequency: first, fractions . . . " (p. 212).

From interviews with 176 seventh graders, Lankford (1972) reported errors made on fraction examples. Of the possible answers for computation with fractions, 35 percent were right, 33 percent were wrong, and 32 percent were omitted. It was observed that performance with fractions was much below that with whole numbers. He noted the most prevalent incorrect practices:

Addition

1. Adding numerators and placing the sum over one of the denominators or over a common denominator.
2. Adding numerators for the numerator of the sum and the same for the denominators.
3. Writing equivalent fractions with common denominators.
4. Adding the numerator and denominator of one fraction for the numerator of the sum, and the same with the second fraction for the denominator of the sum.

Subtraction

1. Subtracting numerators for the numerator of the difference and the same with denominators.
2. Using borrowing, or the borrowed number, incorrectly.
3. Writing equivalent fractions by dividing a denominator into the common denominator or adding this quotient to the numerator of the original fraction for the numerator of the equivalent fractions.
4. Writing equivalent fractions by choosing a common denominator and using it for the denominator of the new fraction but retaining the numerator of the old fraction.

Multiplication

1. First writing equivalent fractions, unnecessarily, then incorrectly multiplying numerators and placing the product over the common denominator.

2. Introducing errors with conversions to simpler form.
3. Writing the reciprocal of the second factor before multiplying.
4. In a mixed number times a fraction, multiplying the fractions and affixing the whole number.
5. In a mixed number times a whole number, multiplying the whole numbers and affixing the fraction.

Division

1. Dividing numerators and placing the product over the common denominator.
2. Making errors in dividing numerators.
3. In dividing a mixed number by a whole number, dividing the whole numbers and affixing the fraction.
4. In dividing a mixed number by a fraction, dividing the fractions and affixing the whole number.
5. Multiplying numerators of like fractions instead of dividing.
6. Multiplying numerators and denominators without writing the reciprocal of the divisor.

Of the responses on eight comparison exercises with fractions, 35 percent were right answers, 17 percent were wrong, and 48 percent were omitted. The errors included:

1. The fraction and the 1 were incorrectly compared.
2. Whichever divisor was thought to be larger made the corresponding sum, difference, product, or quotient larger.
3. Attempting to perform the operations before making the comparison.
4. Comparing the sums, etc., incorrectly.
5. Thinking of all whole numbers as greater than fractions.

Lankford includes all of the responses made by the students, providing a rich source of data which can serve for researchers (and teachers) as a guide. The sometimes tortuous explanations which the students use to describe their manipulations of numbers indicate clearly weaknesses in their understanding of the rational number construct.

As Ginther et al. (1976) noted,

> Our present instructional program does not provide our students, even in the best of schools, with the fractional skills, in computation and in problem solving, that they should have. (p. 10)

They conducted a study to investigate whether deficiencies in fraction skills are due to current instructional programs. A 78-item battery of fraction tests was developed; tests were classified according to cognitive level as computation, comprehension, or applications. More than 1500 average eight-grade students in programs considered to be "modern" took 10-item parts of the battery. Results indicated that

> students do not have the fractional skills needed to compute or to solve problems. Responses to number line and diagrammatic items indicate they understand fractional concepts; however, performance on other items indicates a poor understanding of the structure of the rational number system.

Thus, the data appear to indicate that even instruction purportedly aimed at developing understanding of the rational number construct does not necessarily result in better performance. Such assessment results have spurred research. They have led, on the one hand, to recommendations that fractions be de-emphasized or deleted from the curriculum, with arguments frequently pointing to increased use of the metric system or calculators and thus decreased need for fractions. Others, noting the long history of poor results with achievement on fractions, have concluded that instruction on rational numbers should be postponed until the learner is more mature. For the past two decades, the appropriate time for rational number instruction has seemed to be when the child reaches the Piagetian stage of formal operations (ages 11-14). Earlier, the investigations of the Committee of Seven (Washburne, 1931; Gillet, 1931) led to delineation of the mental ages at which success could be expected (according to the Committee's standard of 80 percent of the total number of examples in

the retention test solved correctly by 75 percent of the group); they also listed "optimum mental ages", "at which the curve definitely flattened, indicating that there was little to be gained by postponement":

	Minimum M.A.	Optimum M. A.
Meaning of fractions	9-0	10-9
Addition and subtraction of similar fractions and mixed numbers, with no carrying	9-10	11-1
Multiplication of fractions	12-3	14-2
Division of fractions	12-3	

On a very different basis, Freudenthal (1973) argued that teaching of the addition algorithm for rational numbers should be delayed until it can be developed as a consequence of the algebraic ideas from which it arises. The student should be given opportunities to deduce rational number ideas from more general ones.

As researchers have considered the process of learning fractions and rational number ideas, however, the emphasis has been not on postponement, but on sequencing and on clarification of the constuct itself.

The Psychological Foundation of Rational Numbers

It has long been recognized that

> The fraction concept is complex and cannot be grasped all at once. It must be acquired through a fairly long process of sequential development. (Hartung, 1958, p. 377)

In commenting on the child's difficulty with rational numbers, Kieren (1976b) noted:

> A child may through natural experience attain certain "fractional" and part-whole notions. Yet, attaining a conceptualization of rational numbers requires a sophisticated application of the proportionality schema, a general notion of inverses, the ability to do arithmetic

> with two distinct abstractly defined operations, and some concept of one dimensional vections (addition) and function composition (multiplication). (pp. 1-2)

Compounding the difficulty is the fact that rational numbers have many interpretations:

> 1. Rational numbers are fractions which can be compared, added, subtracted, etc.
> 2. Rational numbers are decimal fractions and form a natural extension of the whole numbers.
> 3. Rational numbers are equivalence classes of rational numbers.
> 4. Rational numbers are numbers of the form p/q, where p and q are integers, that is, "ratio" numbers
> 5. Rational numbers are multiplicative operators, states related by such operators, or dilatations.
> 6. Rational numbers are elements of a quotient field.
> 7. Rational numbers are measures or points on a number line.
>
> (Kieren, 1976b, p. 2)

Kieren noted that the acquisition of whole-number concepts is "aided by everyday experiences which both precede and accompany formal instruction" (p. 2). Such experiences with rational numbers, however, are not particularly common. He also stated that

> While there seems an obvious relationship between the acquisition of the proportionality schema and rational number acquisition, the nature of this has only been explored in a limited fashion. Relationships to other fundamental intellectual structures such as the INRC group structures are hypothetical at this stage. Suffice it to say then that a "natural" developmental sequence for the concept of rational numbers is not known at this stage. (p. 3)

Thus, in working with rational numbers, "children are dealing with mathematical structures which do not have an obvious basis in natural thought" (Kieren, 1976a, p. 103).

He also discussed the little that formal instruction has done to "enrich the experience base of children". Instruction has focused on the

the first two of his interpretations, ignoring the remainder. He believes that

> Because each interpretation of rational relates to particular cognitive structures, ignoring a conglomerate picture or failing to identify particular necessary structures in developing instruction can lead to a lack of understanding on the part of the child. (Kieren, 1976a, p. 127)

He makes the point that

> the many interpretations of rational numbers have themselves many related instructional strategies. These in turn employ numerous physical and symbolic models, Without a conglomerate view, it is easy to design instructional settings which contain contradictory elements or models, or which do not easily lead to the development of some rational number concepts. For example, if one interprets rationals as measures and uses a number line model, multiplication of rationals is not naturally generated. The number line model may conflict cognitively with an area model or an exchange model for generating multiplicative ideas. (Kieren, 1976a, p. 127)

Quite obvious should be the fact that he proposes that curriculum developers must consider both short- and long-term objectives for instruction on rational numbers, relate these objectives with the seven interpretations of rational numbers, and then select appropriate interpretations to develop certain objectives.

> Given these, they can then ascertain the necessary cognitive structures for meeting the objectives and develop sequences of instructional activities which contribute to the growth of these structures. (Kieren, 1976a, p. 128)

Researchers must use a similar process to study selected interpretations, identify the most important cognitive structures, develop settings to ascertain the extent to which a child has such structures, and study the growth of such structures developmentally.

Kieren is, obviously, attempting to devise a schema for improving research and development efforts. Thus he noted:

> The seven interpretations noted above each relate to or are suggestive of schema or cognitive structures which

> at least theoretically appear to underlie rational number concept development. Further, these mathematical structures and related cognitive structures then induce a set of related instructional structures. While our theory is yet lacking in formal propositions, an example of the relationship between mathematical, cognitive and instructional structures is given below:
>
> Rational numbers notions conceptualized as operators have as some of their underlying cognitive structures the proportionality schema, a generalized notion of reversibility, conservation of quantity under equi-sized decomposition, composition of transformation and the conservation of results under such a composition and some or all of the INRC group structures. (From a developmental point of view it is clear that such rational number notions could have several levels or stages.)
>
> The mathematical and cognitive structures above could be developed using the following instructional structures: measurement division problems, proportional exchange settings, or similarity mapping settings. (pp. 3-4)

In another analysis of the fraction concept, Wagner (1976) underlined the definitional problem in another way, indicating

> the need for looking at the fraction concept as a complex "megaconcept" made up of a variety of components or subconcepts . . . not sequential in nature. Instead, the development of the fraction concept can be visualized as strands of subconcepts that become increasingly more intertwined, each enhancing the understanding of the other, until they become so interwoven that they are one. In addition, strands from other subject areas (mathematical and otherwise) become intermeshed, contributing understanding to and deriving understanding from the developing fraction concept. . . . This development was viewed as a gradual process of internal construction. The culmination, which involves an understanding of the isomorphic character of all the models for fractions, is the concept of the system of rational numbers. (pp. 7920-7921)

Identification of the attributes of fractional parts has been of some concern to researchers. Piaget, Inhelder, and Szeminska (1960) indicated that a child's concept of fractional parts is based on seven attributes:

(1) a whole is composed of separable elements

(2) separation can occur into a determinate number of parts

(3) subdivision exhausts the whole

(4) a fixed relationship exists between dividing cuts and subdivisions

(5) subdivisions are equal

(6) these parts are wholes in their own right

(7) the whole is conserved

They found that children well into elementary school had not attained these attributes. Payne (1976) verified this and added other components which he sees as essential to early rational number learning:

(8) symbolic control over fractions

(9) continuous and discrete part-whole relationships

(10) fractions larger than one

(11) equivalent subdivisions

Novillis (1976), working with number lines, added:

(12) ability to identify the unit

Kieren (1976a) suggested:

(13) arbitrary subdivision of the unit

(14) proportionality

(15) composition of transformations

(16) control over two-dimensional symbols

(17) control of primitive forms of vector addition and function composition

(18) functional capability with equivalence classes and quotient fields

(19) connections with natural and real numbers

It has also been apparent to these researchers that children must have experiences in a variety of contexts. Some of the specific results of their research are included in the next section.

Research on Instruction on Fractions

As was noted in the introduction, Suydam and Dessart (1976) and Suydam and Weaver (1975) have provided reviews of the research on fractions with a particular emphasis on studies which may have direct implications for classroom practice. These reviews provide citations to studies which have tended to be concerned with computational skills with fractions. Instead of duplicating these syntheses, attention here will focus largely on studies in which the researchers were concerned with the processes children use as they learn fractional concepts and develop skills with the operations on fractions.

Payne (1976) reviewed the studies completed in two periods of work at The University of Michigan (with additional comments on other related research).

Period 1: Emphasis on comparing and analyzing the strengths and weaknesses of algorithms for a given fraction operation

Bidwell	1968	division of fractions	comparing three approaches (common denominator, complex fraction, and inverse operation)
Green	1969	multiplication of fractions	comparing two approaches (area and finding fractional parts of) and two modes (diagrams and manipulative materials)
Bohan	1970	equivalent fractions	comparing three sequences using diagrams or paper-folding
Coburn	1973	equivalent ratios	comparing two approaches (ratio and regions)

Period 2: More intensive examination of what children learn while being taught a carefully developed sequence (study of the cognitive structures resulting from different combinations of rules and concrete models)

Muangnapoe	1975	initial fraction concept and symbols
Williams	1975	initial fraction concepts
Galloway	1975	initial fraction sequence in grades 1-5
Choate	1975	cognitive structures of children taught a rule for comparing fractions or a conceptual approach to comparison
Ellerbruch	1975	effects of changing order of rules and concrete models for equivalent fractions, addition, and subtraction

Among the comments made by Payne about the various studies, some have particular significance in indicating specific points learned:

- . . . it is important to do an analysis of the various components of the learning structure for an algorithm. Performance on some major subskills needed in the development of an algorithm showed spots where more effective instructional material was needed. (p. 149)
- The Common Denominator method lacks internal structure compared with the Complex Fraction or Inverse Operation methods. (p. 149)
- It appears in retrospect that the use of visual materials in developing algorithms has a more important effect on retention than does a purely logical mathematical development. (p. 155)
- It is striking that the cognitive structure for pupils in the Area approach were superior to those in the approach Finding Fractional Parts. Since there are more applications and practical uses of FFP, what may be needed is an investigation of how this approach can be taught so that the cognitive structure has a better fit. (p. 155)
- Evidently it is much more complicated to relate a child's thought to his use of concrete materials and/or diagrams of the materials than is usually assumed. (p. 156)
- Paper folding helped make a logical connection between the concrete models and the generalization for getting higher terms. (pp. 162-163)
- Taking Bidwell, Green, and Bohan together, it appears algorithms were learned best when they were logical, and retention seemed best when there was a heavy visual component to instruction. (p. 163)
- Bohan's study demonstrated that instruction on operations on fractions can begin with multiplication. (p. 163)
- In choosing an initial model to be given main emphasis, one must look ahead to subsequent work. (p. 167)
- These data and other evidence point to a more consistent and sound cognitive structure being built using Region as the initial model. (p. 167)

- It seems likely that improvement . . . in symbolic form might come by spending more time initially with concrete materials, or else more time is needed to relate the symbolic form to the concrete materials. (p. 167)

- Logical developments of algorithms [and] concrete objects that seemed to fit well with steps in the algorithms appeared to help achievement, and this was more pronounced on retention tests. (p. 167)

These comments, all reflections on work during the first period of studies, affected the research questions and procedures used in the later studies. These are characterized by extensive work with children in small groups to explore processes and refine sequences of instruction. Payne noted additional perceptions and conclusions from these studies:

- Teaching word names before fraction symbols eliminated the "reversal problem". Children were having perceptual difficulties with diagrams. (p. 169)

- The set model was dropped from the sequence because it seemed to interfere with the measurement ideas associated with regions and because achievement was low on the set model. Difficulties persisted with fractions associated with number lines . . . (p. 171)

- All investigators found that the set model for fractions was difficult to teach. After teaching the set model . . ., the region model with emphasis on measurement seemed to disintegrate. The set model is very closely related to ratio; in fact, they may be the same thing. Since ratio is a topic . . . considered important, a way is needed to teach the initial set model and yet maintain the measurement model. (p. 179)

- What seems clearest from Choate's work is that steps in a rule developed side by side with visual diagrams is the least effective way to teach an algorithm . . . there may be difficulty in teaching two seemingly different things at the same time. (p. 176)

- There is evidence that relating a spatial representation and a verbal algorithm helps in learning equivalent fractions. (p. 180)

- . . . suggests that a ratio-proportion treatment for the initial work with equivalent fractions be delayed. (p. 180)

- This writer is convinced that there should be some spatial models, probably because they give better structures and also better retrieval from semantic memory. The way the rules and spatial models can and should be related is still a major question. (p. 181)

- Overall, the Michigan studies show that it takes a very much longer period of time to teach any part of the fraction work than has generally been allowed in instructional materials or curriculum guides. (p. 182)

It is vital to note that conflicts apparently arise when instruction based on different interpretations (such as ratio and measurement) are used together. Sambo (1975) found that textbooks for nine-to-eleven-year olds typically present fraction concepts in a variety of modes simultaneously, probably leading to confusion on the part of the children.

Owen (1975) tested children aged 8 and 9 on such pre-measurement attributes as conservation of length and area, and then instructed them using the Initial Fraction Sequence developed at Michigan.

> He found that for the 9-year-olds there was a significant difference in achievement of IFS objectives between students with high pre-measurement scores . . . and those with low scores. This finding is suggestive of an hypothesis that attainment of the structures of the full stage of concrete operations may be a necessary condition for learning initial rational number concepts. (Kieren, 1976b, p. 9)

Kieren also notes that several studies are attempting to establish further the connection between measurement attributes and number learning (e.g., Babcock, 1976; Sambo, 1976).

In outlining related and continuing work, Kieren (1976b) noted:

> Recent theoretical and empirical work have focused on the importance of the notion of unit in rational number concept acquisition. Novillis (1973) in a study of a hierarchy of rational number sub-concepts with 10- and 11-year-olds observed that these children had difficulty identifying the unit in region and number line representations of rational numbers. In a more detailed study of this phenomena, Novillis (1976) found that while 12-year-olds could successfully locate rationals less than one on a number line from zero to one, they had great difficulty doing so if the number line was extended. It is apparent from these results, as well as observations from Owen and Babcock in their work, that children at least up to the age of 12 or 13 have difficulty disassociating the notion of unit from "everything else". Thus a child, when given a picture of 4 cakes and asked to shade ½ cake, tends to shade 2 cakes and a similar phenomenon exists with more frequency in number line settings. (p. 10)

In the first conference of the IGMPE at Utrecht, Streefland (1978) discussed some observational results concerning fraction concepts. He noted that

> the subdivision of continuous or discrete quantities in equivalent parts is almost always the only way used to introduce fractions in primary education, . . . [Moreover] after a few illustrations . . . the equivalence of fractions is almost exclusively dealt with in an algorithmic way. (p. 51)

Freudenthal (1973) has similarly suggested that instruction has put a premature emphasis on learning algorithms.

Streefland (1978) cited the need

> of emphasizing the relationship between equivalent parts and quantities resulting from subdivision of an arbitrary unit and the fractional names, which enable us to describe those parts and quantities. (p. 53)

Payne (1976) also emphasizes this point, and, as will be apparent in the next section, delineates it clearly in the Initial Fraction Sequence.

Organizing phenomena and situations which have something essential in common and using concrete models on the logical level of the child are also points confirmed by Streefland. He concluded with these comments:

> The mental constitution of a concept is determined by the wealth of phenomena which discharge this concept, or by the ways education utilizes this wealth. As far as the concept of fraction is concerned, the didactical approach is based on a defective phenomenological analysis. As the observations showed its mental development involves many aspects. Besides, the different aspects of the concept are at different states of development, at a particular moment.
>
> It seems possible to identify situations in which a concept is concretized and which are so convincing to the child that they can serve as paradigms. . . .
>
> The materials on which the mental constitution of the concept of fraction is based will then have a definitive meaning, even when the concept is tending to operate at an algorithmic level. (pp. 72-73)

It should be clearly evident from the citations and comments that there is much agreement among researchers as they consider both components of the process of learning and teaching fraction ideas, and as they consider needed research.

Instructional Sequences on Fractions

Some of the results of the research efforts at The University of Michigan were presented by Coxford and Ellerbruch (1975), in a chapter in the Thirty-seventh NCTM Yearbook. They noted that when the child comes to school, it is likely that the "part of something" idea is his or her concept of fraction. This concept must be extended, with attention focused on three questions:

(1) What is the unit?

(2) How many pieces are in the unit?

(3) Are the pieces the same size?

They stressed that "much emphasis must be placed on equal size pieces", and that the child must be made aware that fractions are used to answer the question, "How much?" The child must be able to make six connections, between the concrete model form, the oral form, and the written form. Children should have a great deal of experience working with concrete materials, which should be used to establish all six connections before using diagrams to represent regions or lengths, but delaying the use of sets of objects until the children's understanding of fractions used with regions and segments is very firm.

In the most recent NCTM Yearbook, Ellerbruch and Payne (1978) present in detail the Initial Fraction Sequence developed in the Michigan Studies. As the title of their article suggests, the teaching sequence provides a guide to instruction of (1) initial concepts of fractions, (2) the addition of like fractions, (3) equivalent fractions, and (4) the addition of unlike fractions. They begin by noting the importance of ascertaining prerequisite knowledge and teaching any lacking prerequisites:

> Before beginning computation with fractions, pupils must be able to represent fractions using concrete objects and

> diagrams, to recognize and use both oral and written words for fractions, and to recognize and use the symbols for fractions. Consequently, this teaching sequence begins with a thorough introduction to fraction concepts. Next, the addition of like fractions is developed, using the single mathematical model of the mesure of rectangular regions. The same model is then used to demonstrate that equivalent fractions name the same part of a given region. Rectangular regions are used to develop the generalization for expressing equivalent fractions in higher terms and then used in the addition of unlike fractions. (p. 129)

It is of importance that they based the introduction to fractions on a single mathematical model, a point both their experiences with children and their research confirmed.

The major steps in the sequence for developing initial fraction concepts, language, and symbols are then given:

> The child must learn
>
> 1. to use concrete objects and make equal-size partitions;
> 2. to recognize and use the oral names for the various-size parts;
> 3. to draw diagrams of the concrete objects and attach the oral names to the parts;
> 4. to use the concrete objects and diagrams together with the oral names to write the fraction symbols. (p. 131)

Of particular importance is the placement of fraction names and symbols:

> Research has indicated that pupils make a substantial number of "reversals when writing the numerator and denominator . . . When the oral names were taught before the written symbols, however, practically no reversal errors were made. This is why stress is placed on teaching the oral names before the written symbols. (p. 135)

Major steps in the sequence for generating equivalent fractions in higher terms are delineated:

> The student must
>
> 1. recognize that <u>two equivalent fractions name the same amount</u>, or show the same measure;
> 2. develop and state the <u>generalization</u> that multiplying the numerator and the denominator by the same number generates an equivalent fraction;

> 3. find the "new" numerator, given a fraction and a "new" denominator;
> 4. when given a pair of fractions and a common denominator, find a "new" pair of fractions using the common denominator;
> 5. when given a pair of fractions, determine a common denominator and use this common denominator to find two equivalent fractions. (pp. 140-141)

Ellerbruch and Payne point out that

> The first two steps are easily related to the spatial models pupils learned earlier with rectangular regions. Steps 3, 4, and 5 are perceptually more difficult, at least when initially taught. Consequently, these last three steps are taught first using a set of rules and subsequently by relating them to the spatial models. Ellerbruch (1975) found that with this sequence pupils attained better skill with equivalent fractions and that their understanding was comparable. This is in contrast to a sequence where the spatial models are used to teach steps 3 through 5 prior to teaching rules. (In a similar study, Choate [1975] found that using concrete models simultaneously with rules for comparing fractions was not effective.) (p. 141)

It is to be expected that continuing efforts on the part of Payne and others will result in delineation of steps to extend the child's developmental of rational number ideas and skills with the operations.

<u>Ratio and Proportionality</u>

Ratio is a topic which has long been recognized as difficult by both mathematics and science teachers. Comparatively little research has been conducted specifically on ratio or the teaching of ratio concepts, however. Rather, researchers' attention has largely been turned toward proportionality, largely as a reaction to the work of Piaget, with numerous attempts to confirm or extend or refute his hypotheses and findings. Inhelder and Piaget (1958) postulated that the learner does not have the cognitive skills necessary for an understanding of proportionality until the age of 12 or 13; ratio is a task requiring formal reasoning.

Piaget's theory proposes that four cognitive structures called logical operations (identify, negation, reciprocity, and correlation) unite to form a single structure, the INRC group. The group composition then allows operations on operations, supporting the development of proportionality.

Piaget investigated the child's acquisition of proportionality by examining children's reactions to such situations as equilibrium on a balance and shadow size. He found that younger children (aged 7 to 12) dealt with these problems arithmetically. On enlargement tasks, they assumed that any enlargement is sufficient, or they decided on a fixed enlargement no matter what the question asked. Older children demonstrated understanding of proportional increase and decrease, as well as reciprocity between various relations. They overtly indicated the use of mathematical relationships between variables. Children demonstrated an intuitive understanding of proportionality before they could deal with it quantitatively, however.

Affirmation of Piaget's results on proportion reasoning has come from studies by Lovell (1961) and Lunzer (1965) among many others; both concluded that only rarely have average to bright junior high school children attained the formal reasoning stage. In studies with older secondary school students (e.g., Lawson and Blake, 1976) or with college students (e.g., McKinnon, 1976), it has been found that at least half the students had not attained the formal level. This is a larger percentage than expected, and the majority of those who attain formal reasoning attain it at a later age than Piaget predicted.

A few studies investigated the possiblity of speeding up the transition from concrete to formal operations. The results of most of these studies have been less than successful. Lawson and Wollman (1976), for instance, reported specific transfer taking place, but not non-specific transfer. It may be that the subjects judged to be transitional made the most growth on tasks in which no training was given. This result seems somewhat consistent with the conservation training studies in which children's progress depended on their initial developmental level.

Successful results have been reported by several researchers. McKinnon (1970) found that a group which took an inquiry course (in science) made a significantly larger gain on the posttest than the group which took the regular science class. Fischbein et al. (1970) found that

> 9- to 10-year-old Ss, after a brief instruction, become able to perform chance estimates by comparing numerical ratios and to understand the concept of proportionality . . . (p. 388)

They also hypothesized:

> the data now available in genetic psychology with regard to the difficulty of accelerating the transition to a higher developmental stage refer mostly to the transition from the intuitive to the concrete operational stage. Would it not be reasonable to assume that during the process of mental development the differences between the successive stages are gradually reduced? If so, instruction might be able to set up structures corresponding to formal operations already at the concrete operational stage with much greater ease and more stability than would be the case for the transition from the preoperational to the operational stage. (p. 388)

Karplus et al. (1970a, 1970b, 1972, 1974a, 1974b) have isolated various strategies that children use to explain their answers to a proportion problem. In a series of five studies, they classified responses from a group-administered, paper-and-pencil test according to a hierarchical schema, with categories ranging from concrete to abstract levels of thought. The studies were intended to indicate how well ratio concepts, in particular, function at various ages and to explore conditions under which learning these concepts might be facilitated.

The tasks included the Island Puzzle, two forms of the Paper Clips task, the Candy task, the Ruler task, the Pulley task, one geometrical task intended to assess ability to recognize a fraction of a whole, and one numerical task requiring the application of proportional reasoning. These were administered to groups from grade 4 through adult level.

Easley and Travers (1976) summarized the interpretation of the five studies by noting that all five studies revealed a tendency for the median frequency of tasks to move from lower-level to higher-level categoreies as grade level increased.

> The researchers, overall, were disturbed by the implications of their findings. In study I, intellectual development, as assessed by their taxonomy, reached a "disappointingly low level" in the high school age group and did not progress much further. In study II, it was found that successful proportional reasoning was not reached until the last years in high school. This concern was reiterated in III, where evidence was found that many students did not advance to more abstract categories of thought during the intervening two years of that longitudinal study. Another disturbing implication, prompted by the data of studies II and III, was evidence of apparent obstables to learning which may be inadvertently set up by "mathematics courses, by teachers, and by the children's cultural environment". Of particular notice was the dramatic contrast between the responses of urban and suburban 11th and 12th grade students. It was found that 80% of the suburban students, but only 9% of the urban students, were classified at the highest level P. (p. 39)

Karplus et al. (1974a) placed emphasis on the context in which the problem was presented. Form B of the Paper Clips Task was more abstract than Form A, forcing students to make use of the data. They concluded that perhaps students' attitude toward handling of data was reflected in their responses. It was also apparent in Wollman and Karplus (1974b) that tasks tending toward concreteness (Ruler and Paper Clips tasks) led to more correct responses than did the abstract tasks. Easley and Travers summarize:

> The lack of applicability of proportional reasoning to physical relationships raised questions about the appropriateness of many instructional strategies, particularly at the junior high school level, where only about 15% of the subjects were found to have reached the highest level. The researchers speculated that one source of the problem may be that ratios are introduced as fractions and proportions as equivalent fractions. (p. 39)

Another outcome of the analysis is the finding that younger children tended to make predictions by guessing on the basis of appearances and used the numbers they were given haphazardly. The children in the middle age range used two different kinds of strategies. They either used all the data by computing the difference between two of the measurements and adding the difference to the third measurement, or they used a scaling factor which was not related to the ratio inherent in the measurements. The older children tended to use either a combination of the latter two strategies or were able to solve the problem using a proportion.

In their critical commentary on the set of articles, Easley and Travers (1976) state:

> It should be clearly noted that Karplus and his colleagues are investigating a different problem from that studied by Piaget. . . . [They] are careful not to identify what they call proportional reasoning . . . with Piaget's Formal Operations. At least, they are quite open to the possibility that these may turn out to be different things. . . . [Thus] we have evidence from high school seniors (and even many adults) who do not employ "abstract logic" or "proportional reasoning" in these problems, but we are not entitled to infer that they have not achieved the stage of formal operations as Piaget defines it, even though Inhelder and Piaget state that proportional reasoning is only attained at the stage of formal operations. (pp. 39-40)

They also point out that Piaget used interviews, with a probing technique, while Karplus used paper-and-pencil tests, with interviews only to check the levels of performance attained from the written responses.

> This difference in purpose and procedure explains in part the differences in age distribution found between the two groups of studies. A second major contribution to these differences, which the research to date cannot isolate from the first, is the phenomenon Piaget calls <u>decalage</u> . . .

> Another difference is that Karplus et al. (study V) employ a concept of types of reasoning which depends on the external static form of arguments. This contrasts sharply with Piaget's interest in internal dynamic processes. (p. 41)

Another difficulty was cited by Nelson et al. (1969) who demonstrated that, in checking proportionality, as in so many other problems in cognitive development,

> verbal deficiencies or ambiguities in task presentation often mask the actual capacity of the child to perform in the required manner. . . . Once the instructions were clear there was no difficulty in comparing two proportions and there was no interference from the perceptual cues that otherwise presumably controlled their responses. (p. 261)

In Karplus et al. (1977), proportional reasoning and control of variables were studied with samples of 13- to 18-year-olds in seven countries (Denmark, Sweden, Italy, United States, Austria, Germany, and Great Britain). They noted that

> Previous studies of these areas have helped to identify the gradual progress in adolescent reasoning for population samples in the United States, Great Britain, and Switzerland (Karplus & Peterson, 1970; Levine & Linn, 1977; Lovell & Butterworth, 1966; Lunzer & Pumfrey, 1966; Suarez, 1974; Wollman, 1975). (p. 411)

The task for assessing proportional reasoning was similar to the Paper Clips Task, Form B. The task for assessing control of variables had students separate the effects of sphere weight and release height on the dependent variable, indicating how far a target sphere was pushed by a sphere rolling down the plane. They found that, on the proportional reasoning task, about 1/4 of the students gave ratio responses, 1/6 used additive reasoning, and the remainder were divided between intuitive and transitional categories. They concluded that

> The effects of gender were not associated systematically with a particular form of school organization, nor with classes segregated by gender. Significant differences that did occur always favored the boys. Girls tended to use additive reasoning more commonly than did boys.

> Differences in achievement among countries were much smaller than differences among groups within a country. Austrian boys distinguished themselves on proportional reasoning, but Austrian students as a whole did not perform comparably well on control of variables. In two countries, Denmrk and Germany, there was a much lower frequency of additive reasoning than in other countries.
>
> Both socioeconomic status and selectivity of school affected student performance significantly, though the magnitude of the effect depended on the task and on the country. (p. 416)

In a recent study with older students, Parete (1978) developed a paper-and-pencil test to assess formal reasoning level, validating it with a group of community college freshmen. The test included four items on proportional reasoning (an enlarging machine problem, a salesman's commission problem, a sales tax problem, and a magnifying glass problem). He found that 23.4 percent of the students were at the early formal stage and 37.2 percent were at the late formal stage; the remainder were at early (22.9%) or late (16.5%) concrete stages. He noted that

> The distribution on the Proportionality Task seems to be shifted more toward advanced substages than are the distributions for the other tasks. This may be due to the fact that mathematics problems which require proportional reasoning for a solution are introduced quite early in the public school mathematics curriculum. Although Piaget has claimed many times that his evaluation of subjects' reasoning levels is content free, recently he has admitted that a subject's familiarity with the content of any of his tasks may in some degree influence the subject's performance on the task, especially in older subjects. (p. 41, manuscript chapter 4)

The addition strategy, the scaling strategy, and finally, consistently forming equivalent ratios were noted.

Thus, the results on many studies indicate the wide variance in the ages at which students can perform on proportional reasoning tasks using mathematical relationships. In fact, it may be that some learners never attain the formal reasoning level. This, obviously, has implications for instruction.

Learning and Instruction on Ratio and Proportion

In this section are cited studies which considered specifically factors related to learning and teaching ratio and proportion ideas in school programs. They emphasize some of the questions apparent previously.

Hart (1978) described aspects considered as basic to the development of ratio concepts by the Concepts in Secondary School Mathematics project in Great Britain:

(1) doubling or halving

(2) multiplication by an integer

(3) given a rate per unit, apply this rate

(4) finding a rate per unit and then applying it

(4) enlarging a drawing in ratio 2:1, 3:2, 5:3, etc.

(5) finding a ratio a:b using an intermediate quantity, c

(6) using a fractional multiplier

(7) simple percentages

She reported that administration of the test to about 200 children indicated that

> Doubling ability is no indicator of the ability to tackle other ratio questions. Children have tortuous methods of obtaining larger numbers, the addition strategy being fairly consistent on harder questions. Techniques for dealing with fractions or percentages are remembered incorrectly and indeed answers which are outside any sensible range are not recognized as such. This was summed up perhaps by one child who said "I was doing it mathematically not logically". (p. 6)

Steffe and Parr (1968) tried to partition out the potential cause of difficulty due to the mode of presentation of a proportion problem. They reported that:

> 1. There is little correlation between the ability of children at the fourth, fifth, and sixth grades to perform successfully in proportionality situations at a symbolic level, such as $\frac{6}{15} = \frac{\square}{5}$, and their ability to perform

> successfully on proportionality situations based on ratio or fractional pictorial data.
>
> 2. Children solve many proportionalities presented to them in the form of pictorial data by visual inspection both in the case of ratio and fractional situations.
>
> 3. Whenever the pictorial data, which display the proportionalities, are not conducive to solution by visual inspection, the proportionalities become exceedingly difficult for fourth, fifth, and sixth grade children to solve, except for the high ability sixth graders.
>
> 4. For the denominator test, the proportionalities represented pictorially by a ratio situation were easier for the children to solve than the proportionalities represented pictorially by a fractional situation.
>
> 5. The children of high intelligence are much more adept at solving proportionalities for both a symbolc and pictoral representation than are children of low intelligence.
>
> 6. The fifth and sixth graders performed significantly better than the fourth graders on all tests and subtests involved. (p. 26)

As implications, Steffe and Parr noted:

> 1. Much more care must be taken in the fifth and sixth grades to develop a sequence of lessons which are designed to enhance children's ability to represent visual data mathematically in the case of ratio or fractions, indeed if that ability can be enhanced.
>
> 2. Special attention must be devoted to the lower ability children . . . (p. 26)

Other researchers have noted other attributes necessary in the development of ratio and proportion ideas. Abramowitz (1975b), for instance, noted that it is necessary to have the ability to make a distinction between "bigger than" and "times as much", as well as the ability to understand inverse relations between unit size and the number of units used in a measuring task. As she considered the results of previous research and her own study, she indicated that

> These results suggest that those investigating the developmental acquisition of proportionality must be careful not to generalize too quickly from performance on any one proportion task to the concept of proportionality in general.

> It seems possible for Ss to have an intuitive understanding of proportion without concurrently having the mathematical facility to solve proportion problems.
>
> A surprising result from the factor analysis is that skill tests of facility with fractions load on a different factor than tasks involving proportionality. . . . proportion tasks demand a knowledge of this facility but also an understanding of how to and when to use it in an appropriate situation. . . . It suggests that drill alone may be insufficient in teaching proportion. The teaching of fractions must be supplemented with tasks which help students conceptualize what they are doing with these numbers.

Karplus et al. (1977) believe that the findings from their study have implications for teachers:

> First of all, it is clear that a substantial fraction of students between 13 and 15 years of age lack the ability to articulate proportional reasoning and/or control of variables. Science and mathematics programs in all but the top level of selective schools should take this diversity of student reasoning into account in so far as content selection, laboratory activities, and textbook choices are concerned.
>
> Second, it appears from the small but significant country-to-country differences that teaching can have some influence on the development of reasoning by the students in the age range being investigated here. Applying proportional or control-of-variables reasoning is not the result of a process exclusively internal to the young people.
>
> Third, the development of the reasoning patterns considered in this paper should be an important objective of teaching programs for 13- to 15-year-olds, an objective that is not achieved comprehensively by present school practice in any of the countries included in our study. Other investigations (Kurtz, 1976; Lawson & Wollman, 1976) have identified certain promising teaching approaches that have been presented elsewhere (Karplus et al., 1977). A key component of these approaches is to direct the attention of students at their own reasoning, the thought processes by which they have arrived at valid or invalid conclusions, correct or incorrect answers. Other aspects are based on (1) identification of the reasoning required for the understanding of science concepts, (2) arranging for active participation in learning by the students, and (3) providing concrete experiences at the beginning of new topics (pp. 416-417)

Those who have developed teaching materials on ratio (e.g., Higgins, 1967 ; Geisert, 1972; Kurtz, 1977) have reported success. Furthermore,

Ward (1974) found that pupils who had used a textbook emphasizing ratio and proportion in grades 5 and 6 scored higher in grades 6, 7, and 9 than those who had had a program in which proportion was not introduced until grade 7.

Further study of ratio and proportion ideas is obviously needed, with clearer delineation of what is entailed and how the ideas are acquired. Instructional sequences, such as those being developed for fractions, may be needed to help to fulfill the promise implied by the last-cited study.

Summary

This paper presents highlights from several sources, in order to indicate the status of knowledge about fraction and ratio concepts and operations, to draw attention to comments from researchers about their findings, and to note points to consider as research proceeds.

Data indicate that many students have little conceptual understanding and computational skill with fractions; performance on fractions is far lower than with whole numbers. Errors students made in solving fraction items indicate how students manipulate numbers without understanding. Questions have arisen about the appropriate age level at which to provide instruction on fractions, since achievement results are so poor with the current placement.

- Researchers have focused not on postponement, but on sequencing and on clarification of the construct itself.
- Emphasis should be placed on providing a sound initial development of fraction concepts using concrete materials.
- The development of fraction ideas should be paced so as to connect firmly with the main points in the initial development.
- Instruction has put a premature emphasis on learning algorithms.

Seven interpretations of rational numbers have been proposed and discussed in depth in terms of implications for curriculum development and research. The fraction concept is complex, made up of a variety of components or subconcepts and attributes.

- Both short- and long-range objectives for instruction must be considered in relation to the interpretations of rational numbers. Then appropriate interpretations should be selected to develop certain objectives, necessary cognitive structures

to meet the objectives ascertained, and sequences of instructional activities to contribute to the growth of these structures determined.

A comprehensive review of studies conducted at the University of Michigan includes such reflections as:

- It is important to do an analysis of the various components of the learning structure for an algorithm.
- It appears that algorithms were learned best when they were logical, and retained best when there was a heavy visual component to instruction.
- Evidently it is much more complicated to relate a child's thought to use of concrete materials or diagrams than is usally assumed.
- Word names should be taught before fraction symbols.
- The set model seemed to interfere with the measurement model.
- Relating a spatial representation and a verbal algorithm helps in learning equivalent fractions.
- It takes much longer to teach fractions than has generally been allocated in the curriculum or instructional materials.
- In the presentation of the Initial Fraction Sequence developed at Michigan, rectangular regions are used as the single mathematical model for the introductory work.

Comparatively little research has been conducted specifically on ratio concepts. Research attention has largely been turned toward proportionality, largely as a reaction to the work of Piaget. It appears that no more than half the students have attained proportional reasoning even by age 18. Certainly there is wide variance in the ages at which students can perform on proportional reasoning tasks by using mathematical relationships. Additive and scaling strategies are used by many.

- Pictures aid children in solving proportionalities. Concrete experiences and active participation appear necessary.
- Much care must be taken to develop a sequence of lessons designed to enhance ability to represent data mathematically for ratios and fractions.
- A key component of instruction is to direct the attention of students to their own reasoning, to the thought processes by which they arrived at valid or invalid solutions.

It appears that instruction can influence the attainment of ratio and proportion ideas, just as it can influence the attainment of fractional ideas. Research and curriculum development must be synchronized to provide instructional programs appropriate for children's psychological and mathematical status and needs.

References

Abramowitz, Susan. Proportionality: As Seen by Psychologists and Teachers. ERIC: ED 111 689. 1975 (a).

Abramowitz, Susan. Adolescent Understanding of Proportionality: Skill Necessary for Its Understanding. ERIC: ED 111 690. 1975 (b).

Babcock, G. The Relationship Between Success with Linear and Area Measure Concepts and Achievement on Fractional Number Concepts. Research proposal, University of Alberta, 1976. Cited by Kieren, 1976 (b).

Bright, George. Assessing the Development of Computation Skills. In Developing Computational Skills (Marilyn N. Suydam, editor). 1978 Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1978.

Carpenter, Thomas; Coburn, Terrence G.; Reys, Robert E.; and Wilson, James W. Results from the First Mathematics Assessment of the National Assessment of Educational Progress. Reston, Virginia: National Council of Teachers of Mathematics, 1978.

Coxford, Arthur and Ellerbruch, Lawrence. Fractional Numbers. In Mathematics Learning in Early Childhood (Joseph N. Payne, editor). Thirty-seventh Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1975.

Easley, John A. and Travers, Kenneth J. Expanded Abstract and Analysis of "The Karplus Studies". Investigations in Mathematics Education 9: 34-42; Spring 1976.

Ellerbruch, Larry W. and Payne, Joseph N. A Teaching Sequence from Initial Fraction Concepts through the Addition of Unlike Fractions. In Developing Computational Skills (Marilyn N. Suydam, editor). 1978 Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1978.

Fischbein, Efraim; Pampu, Ileana; and Manzat, Ion. Comparison of Ratios and the Chance Concept in Children. Child Development 41: 377-389; June 1970.

Ford, Elinor Rita. The Spectrum of Proportion Mastery Among New York Parochial School Children. (Columbia University, 1974.) Dissertation Abstracts International 35A: 3573; December 1974.

Freudenthal, H. Mathematics as An Educational Task. Dordrecht, Holland: D. Riedel Publishing Co., 1973.

Geisert, Paul George. A Study of the Hierarchical Competencies Underlying the Problem Solving Use of Proportion. (The Florida State University, 1971.) Dissertation Abstracts International 32A: 5103-5104; March 1972.

Gillet, Harry O. Placement of Arithmetic Topics. Journal of the National Education Association 20: 199-200; June 1931.

Ginther, Joan; Ng, Katie; and Begle, E. G. A Survey of Student Achievement with Fractions. SMESG Working Paper No. 20. Stanford, California: Stanford Mathematics Education Study Group, October 1976. ERIC: ED 142 409.

Hart, Kathleen. The Understanding of Ratio in the Secondary School. Mathematics in School 7: 4-7; January 1978.

Hartung, Maurice L. Fractions and Related Symbolism in Elementary-School Instruction. Elementary School Journal 58: 377-384; April 1958.

Higgins, Jon Lyle. The Development and Evaluation of Mathematics Curriculum Materials for Use in a Junior High School Physical Science Program. (The University of Texas, 1967.) Dissertation Abstracts 28A: 1620; November 1967.

Inhelder, B. and Piaget, J. The Growth of Logical Thinking from Childhood to Adolescence. New York: Basic Books, 1958.

Karplus, Elizabeth F. and Karplus, Robert. Intellectual Development Beyond Elementary School I: Deductive Logic. School Science and Mathematics 70: 398-406; May 1970 (a).

Karplus, Robert and Peterson, Rita W. Intellectual Development Beyond Elementary School II: Ratio, A Survey. School Science and Mathematics 70: 813-820; December 1970 (b).

Karplus, Robert and Karplus, Elizabeth F. Intellectual Development Beyond Elementary School III--Ratio: A Longitudinal Study. School Science and Mathematics 72: 735-742; November 1972.

Karplus, Elizabeth F.; Karplus, Robert; and Wollman, Warren. Intellectual Development Beyond Elementary School IV: Ratio, the Influence of Cognitive Style. School Science and Mathematics 74: 476-482; October 1974 (a).

Karplus, 1974 (b): See Wollman and Karplus, 1974 (b).

Karplus, Robert; Karplus, Elizabeth; Formisano, Marina; and Paulsen, Albert-Christian. A Survey of Proportional Reasoning and Control of Variables in Seven Countries. Journal of Research in Science Teaching 14: 411-417; 1977.

Kieren, Thomas E. On the Mathematical, Cognitive, and Instructional Foundations of Rational Numbers. In Number and Measurement (Richard A. Lesh and David A. Bradbard, editors). Columbus, Ohio: ERIC/SMEAC, 1976 (a).

Kieren, Thomas E. Research on Rational Number Learning. Athens: Georgia Center for the Study of Learning and Teaching Mathematics, 1976 (b).

Kurtz, Barry Lloyd. A Study of Teaching for Proportional Reasoning. (University of California, Berkeley, 1976.) Dissertation Abstracts International 38A: 680; August 1977.

Lankford, Francis G., Jr. Some Computational Strategies of Seventh Grade Pupils. Charlottesville, Virginia: The University of Virginia, October 1972. ERIC: ED 069 496.

Lawson, Anton E. and Blake, A. J. D. Concrete and Formal Thinking in High School Biology Students as Measured by Three Separate Instruments. Journal of Research in Science Teaching 13: 227-235; 1976.

Lawson, Anton E. and Wollman, Warren T. Encouraging the Transition from Concrete to Formal Cognitive Functioning -- An Experiment. Journal of Research in Science Teaching 13: 413-430; 1976.

Lovell, Kenneth. A Follow-Up Study of Inhelder and Piaget's "The Growth of Logical Thinking". British Journal of Psychology 52: 149+; 1961.

Lunzer, Eric E. Problems of Formal Reasoning in Test Situations. In European Research in Cognitive Development (P. H. Mussen, editor). Volume 30, Monographs of the Society for Research in Child Development. Chicago: University of Chicago Press, 1965.

McKinnon, Joe W. The College Student and Formal Operations. In Research, Teaching and Learning with the Piaget Model (John W. Renner et al., editors). Norman, Oklahoma: University of Oklahoma Press, 1976.

McKinnon, Joe W. The Influence of College Inquiry-Centered Course in Science on Students Entry into the Formal Operational Stage. (University of Oklahoma, 1970.) Dissertation Abstracts International

Nelson, Katherine J.; Zelniker, Tamar; and Jeffrey, Wendell E. The Child's Concept of Proportionality: A Re-examination. Journal of Experimental Child Psychology 8: 256-262; October 1969.

Novillis, C. The Effects of the Length of the Number Line and the Number of Segments in Each Unit Segment on Seventh Grade Students' Ability to Locate Proper Fractions on the Number Line. 1976. Cited in Kieren, 1976 (b).

Owen, D. Pre-Measurement Abilities and Initial Fraction Learning in Children. 1976. Cited in Kieren, 1976 (b).

Parete, Jesse David, Jr. Formal Reasoning Abilities of College Age Students: An Investigation of the Concrete and Formal Reasoning Stages Formulated by Jean Piaget. Unpublished Docatoral Dissertation, The Ohio State University, 1978.

Payne, Joseph N. Review of Research on Fractions. In Number and Measurement (Richard A. Lesh and David A. Bradbard, editors). Columbus, Ohio: ERIC/SMEAC, 1976.

Piaget, J.; Inhelder, B.; and Szeminska, A. The Child's Conception of Geometry. New York: Basic Books, 1960.

Sambo, A. A. A Psycho-Mathematical Development of the Concept of Rational Numbers. A colloquium paper at the University of Alberta, Fall 1975. Cited in Kieren, 1976 (b).

Sambo, A. A. Transfer Effects of Measure Concepts on Fractional Number Learning. Research Proposal, University of Alberta, 1976. Cited in Kieren, 1976 (b).

Steffe, Leslie P. and Parr, Robert B. The Development of the Concepts of Ratio and Fraction in the Fourth, Fifth, and Sixth Years of the Elementary School. Technical Report No. 49. Madison: Wisconsin Research and Development Center for Cognitive Learning, March 1968.

Streefland, Leen. Some Observational Results Concerning the Mental Constitution of the Concept of Fraction. Educational Studies in Mathematics 9: 51-73; 1978.

Suydam, Marilyn N. and Dessart, Donald J. Classroom Ideas from Research on Computational Skills. Reston, Virginia: National Council of Teachers of Mathematics, 1976.

Suydam, Marilyn N. and Osborne, Alan. The Status of Pre-College Science, Mathematics, and Social Science Education: 1955-1975. Volume II: Mathematics Education. Final Report, National Science Foundation. Columbus, Ohio: The Ohio State University, 1977. Available from ERIC/SMEAC.

Suydam, Marilyn N. and Weaver, J. Fred. Using Research: A Key to Elementary School Mathematics. Columbus, Ohio: ERIC/SMEAC, 1975.

Wagner, Sigrid. Conservation of Equation and Function and Its Relationship to Formal Operational Thought. 1976. ERIC: ED 141 117.

Washburne, Carleton. Mental Age and the Arithmetic Curriculum: A Summary of the Committee of Seven Grade Placement Investigations to Date. Journal of Educational Research 23: 210-231; March 1931.

Wollman, Warren and Karplus, Robert. Intellectual Development Beyond Elementary School V: Using Ratio in Differing Tasks. School Science and Mathematics 74: 593-613; November 1974 (b).

Mathematical Thinking & the Brain

David Tall

§1. Introduction

Though our knowledge of brain activity is still rudimentary, it is becoming increasingly clear that our partial understanding of the phenomenon can give significant insight into mental processes; in particular our growing understanding of the brain itself highlights certain aspects of mathematical thinking. In considering the brain we must first decide which level (or levels) of structure we wish to study; we may concentrate on the molecular level, investigating the chemical changes involving DNA, RNA and proteins, we may look at the neuronal level and consider their function, or we may desire a more global model of activity. Investigations into the chemistry show hopeful signs that may lead to a better understanding of the nature of memory [4], [10]. At the cellular level a great deal is known. For instance the neurons always fire in a specific direction [10, p.93], so that electrical mental activity is an ordered structure. Thoughts cannot be reversed in the simple way we reverse a film by running it backwards, the best we can do is to key into certain parts of the ordered schema to re-run a portion of it. Is it any wonder therefore that children have problems with reversible properties in mathematics ? Within the brain the reversal of a property cannot be a mirror image, it requires the construction of an alternative route.

At these levels of brain structure we can therefore already see signs of useful information developing, but it seems that a more global pattern is essential to have much hope of encapsulating the more complex nature of mathematical thinking. Biological theories at this level are limited, but a mathematical model has been suggested by Zeeman [13], regarding the brain in terms of a dynamical system. This of course is a theoretical model based not on specific measurements, but rather on the nature of the electrical activity of the brain. It can be viewed in terms of resonance [8] which we shall outline in this paper. As with any model, it does not describe the actuality of the brain, however it suggests a plausible manner in which the brain may work at a global level. It concentrates on the qualitative nature of the thinking processes: the resonances and the abrupt changes in resonance (the latter being described in terms of catastrophe theory).

In this paper we shall concentrate on the notion of resonance and look at some specific examples of this phenomenon in mathematical thinking. In particular we shall be interested in the kind of thinking used in creating a mathematical proof. The mathematician does not arrive at his proof by a process of inexorable logic. At first he may be confused by the data, then out of a partial understanding of the problem may come strategies of attack which may at first only go part of the way, to be later refined into a more complete and acceptable version. The final polished form of proof is rarely a logical structure either, it concentrates on certain novel turning points and suppresses routine detail. So if the final edifice built by the mathematical mind is not an absolute logical structure, nor is the tortuous route by which the proof was obtained, it is stretching the imagination to think that the brain itself can be described in terms of purely logical circuitry.

In this paper the term *generic thought* will be used to describe the notion of thinking within a resonance framework. Certain aspects of a problem may fit together and cause a mental resonance leading to an ordered mental schema in action (perhaps, but not necessarily, in terms of a mathematical algorithm); then if the result is incomplete, subsequent thought may lead to suitable refinements, and so on. The chief problem in describing the process is that certain fundamental mental actions are so fast that the individual cannot say how the total chain of activity happened in his mind, only that certain aspects have occurred. The examples given will exhibit this limitation. They consider student's answers to mathematical problems and show how a resonance interpretation of thought may be considered. After discussing the notion of generic thought we shall compare this with the idea of 'generalisation' and consider the appropriateness of basing teaching schemes on mathematical hierarchies.

§2. Resonance

René Thom has outlined the beginning of a theory of resonance in brain activity in his essay "Topologie et Signification" [9,p.201]:

> *Si nous considérons la totalité de nos activités cérébrales comme un système dynamique (selon le modèle de C. Zeeman [12], nous serons amenés à supposer qu'a tout champ moteur codifié en verbe correspond un mode propre, un attracteur A de la dynamique cérébrale subit un stimulus spécifique s, qui la met dans un état instable d'excitation; cet état évolue ensuite vers la stabilité par sa capture par l'attracteur A, dont l'excitation engendre par couplage aux motoneurones l'exécution motrice d l'ordre. ... Lorsque, sous l'effet d'un message, la dynamique mentale ne présente pas d'attracteur qui la capture de manière solide, c'est qu'alors le message est sans grande signification. Comme le montre bien l'assimilation du phénomène de comprendre à une résonance dynamique, l'absence de signification d'un texte n'est jamais totale: il se forme toujours des résonances plus ou moins fluctuantes, mais qui ne peuvent attacher l'esprit. ...*

Thus an external stimulus presented to the brain via its receptors sends it into an unstable state of excitation which evolves very quickly to a state of stability; this new state of stability causes the effectors to carry out the appropriate physical or mental actions.

The biological nature of this activity is not fully understood, short term it may be an electrical process, but long term chemical changes in the brain are constantly modifying the possible configurations of the electrical activity. It may be, for instance, that short term memory is an electrical phenomenon and long term memory is caused by chemical change [4, p.240]. This means that as the brain is enriched by sensory experience and mental activity its available configurations for electrical resonance change and respond in different ways to a given stimulus. Thus a stimulus which at first causes minimal resonance may through repetition cause a chemical change in the brain which eventually leads to resonance, remembering and response. Such conditioning lies behind stimulus-response theory, but it does not explain higher mathematical reasoning.

In mathematics we are concerned with more complex phenomena; at this level we may obtain certain insights into mental activity by considering it in terms of resonance. For instance it is possible in dynamical systems theory to superimpose two stable resonances leading to an unstable state which evolves to a third stable resonance [9]. Thus resonance allows for a kind of creative response which is not just the result of learned stimuli.

R. R. Skemp [6] has given a different analogy for resonance by likening it to the resonating of the undamped strings of a piano when a sound is made. The piano strings respond with a given sound. In this analogy the resonance is limited by the physical configuration of the piano. The mind is more subtle because its resonance configuration changes with experience and time. Unlike a passive piano which only responds to the actions upon it, the mind is active, with much of this activity being subconscious. The process of long-term memory storage takes several hours, as can be shown by the common occurrence of loss of memory after concussion. Experimental evidence has indicated that memory fixation occurs over a period of thirty minutes to three hours [4, p.239]. Problems which are attempted sometimes prove intractible at the time, yet a solution can leap to mind after a considerable period, indicating possible subconscious activity during at least part of the intervening time. University lecturers teaching difficult subjects such as analysis consider the need of a long period of study before the basic concepts are understood, again understanding can occur after a period of relaxation away from the topic (given some work on it in the first place!).

In considering the long-term activity of the brain, new sensory input, or brain activity itself may change the possible resonance configurations. Given an established resonance, an alternative may develop, and during the period when two (or more) alternatives are possible, a slight change in sensory input may cause a dramatic change from one resonance to the other. The passage of time may lead to the dominance of a new resonance and the decay of the old. Such a model is a possible description of Piaget's observed transitions from one stage of development to another [7].

In this paper we shall not concern ourselves with transitions and catastrophe theory, concentrating more on the resonance phenomenon. Resonance has certain simple implications; for instance with one or more factors resonating strongly and others not at all, attention is centred on resonating factors. With limited conscious capacity the brain may not consciously register non-resonating details. From this viewpoint we discriminate by virtue of the design of the brain, only noting the essential elements which resonate. (The musical analogy may be inappropriate here, since differences may cause brain resonance as well as similarities.)

Resonance may be viewed as an essential factor of human thought. It may allow us to recognise an object from a slightly different viewpoint. It may also explain how communication between two individuals is possible; similar, but not identical, experiences may be communicated as the two brains resonate on the important essentials and neglect the differences.

§3 Mathematical Examples

Consider the following part of a university examination question:

Show that the matrix $\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$ *has two linearly independent eigenvectors.*

The solution technique is straightforward, with one technical problem. When $\alpha \neq 0$ there are precisely two eigenvalues, $1+\alpha$, $1-\alpha$, and two eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, but for $\alpha = 0$ there is one eigenvalue 1 and all (non-zero) vectors are eigenvectors. For the eigenvalue $1+\alpha$ one finds the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ by solving

$$\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (1+\alpha)\begin{pmatrix} x \\ y \end{pmatrix}$$

which simplifies to

$$\alpha y = \alpha x$$
$$\alpha x = \alpha y.$$

Virtually all students reaching this point deduced from the equations that the solution was $x = y$, or $\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, giving the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Only three out of 170 attempted to consider the case $\alpha = 0$, which can have any solution for x,y, (and this despite the emphasis in the course of not dividing by zero in an equation such as $\alpha x = \alpha y$).

One could give various interpretations as to why this occurred, the most likely being that students consider α as a general scalar and did not think of it being a specific number, for instance $\alpha = 0$.

Regarding the thought processes as a resonance phenomenon, one can see the average student being drawn through the problem by the inexorable resonance of the technique, following an ordered mental schema, and failing to perceive the non-resonating singularity of $\alpha = 0$.

Example 2

In an 'A-level' question (for British students ending their school years aged 18), the following problem was posed:

> *The inclined faces of a fixed triangular wedge make angles* α *and* β *with the horizontal. Two particles of equal weight connected by a light inextensible string which passes over the vertex of the wedge, rest on two faces of the wedge. If both particles are in limiting equilibrium, show that*
>
> $$\mu = \tan \tfrac{1}{2}|\alpha-\beta|$$
>
> *where* μ *is the coefficient of friction between each particle and the face of the wedge.*

Such a question, involving a modulus sign (as in $\tan \frac{1}{2}|\alpha-\beta|$) had not been set in recent years, though the modulus sign is part of the syllabus. A complete solution depends on considering the cases $\alpha \geq \beta$ and $\beta \geq \alpha$ separately. Many students drew a picture with α,β similar in size and failed to see this subtlety. Out of a sample of 100 students, seventeen managed to proceed correctly with what was essentially the case $\alpha \geq \beta$ and obtained $\tan \frac{1}{2}(\alpha-\beta)$ from their diagram. This looks similar to the answer $\tan \frac{1}{2}|\alpha-\beta|$ and is the same for $\alpha \geq \beta$. Only one of these seventeen went on to consider the case $\alpha \leq \beta$ to obtain a complete solution. However five others drew a different diagram which led to the solution $\tan \frac{1}{2}(\beta-\alpha)$. This is also correct (for $\beta \geq \alpha$), but β,α now occur in a different order from that in $\tan \frac{1}{2}|\alpha-\beta|$ and the solution looks visibly wrong. Four of these five students immediately used an ad hoc argument to introduce the modulus and reconcile their solution - a much higher proportion than in the other case where the solution looks more acceptable. Apart from the one correct solution mentioned earlier and these four, the remaining 95 made no reference to the modulus sign whatsoever, almost as if it was not there. Many, of course, were doubtless worried by it, but knew they would only lose a few marks if the rest of the question were correctly handled; the time factor truncated their attempts and they concentrated on the aspect of the question that they could handle.

As an amusing additional remark, the examiner failed to consider the singular case $\alpha = \beta$!

Example 3

To show that mental resonance need not be purely in terms of mathematical algorithms, consider the individual case of an eight year old child faced with the following question:

"Guessing Answers": 19+12 is nearly 20+10 which equals 30.
A good guess for the answer of 19+12 would be 30.
Try to make a good guess for the answer of these.
The first is done for you.
1. 21+8 is nearly 20+10 which equals 30.
2. 31+7 3. 19+21 4. 27+11 5. 42+9 6. 11+12
7. 17+27 8. 32-11 9. 43-16 10. 38-19

The child's solutions were as follows:

2.	*31+7*	*is nearly 29+10*	*which equals 39*
3.	*19+21*	*31+10*	*41*
4.	*27+11*	*29+10*	*39*
5.	*42+9*	*51+1*	*52*
6.	*11+12*	*23+1*	*24*
7.	*17+27*	*44+1*	*45*
8.	*32-11*	*10+10*	*20*
9.	*43-16*	*18+10*	*28*
10.	*38-19*	*10+10*	*20*

The first comment to make is that this is rather a stupid question. Thirty seven pages earlier in the book was the last reference to "guessing to the nearest number ending in 0", and this child had clearly forgotten. So he approached the question in his own way. Having admitted this, can we diagnose what mental activity caused him to write down the answers he did?

Within the terms of Thom's theory we must be prepared for two different mental resonances causing instability and settling down as an entirely new resonance. Not only may the child not know why he came to a certain conclusion, but the observer should not expect to be able to logically deduce his reasoning from his observations. Even had we been able to get the child to "think out loud" as he did the problems, this might have interfered with his thinking, causing possibly a different outcome, it would have also been a very rudimentary analysis in his own terms. Discussing his solution afterwards, taking each question in turn, the first light dawned on the questioner in the solution of question 5 where the child mentioned that he knew that 42+9 was 51, so he added on 1 "to make it a guess". It was only realised much later by the observer that all the answers with one exception were "the correct answer plus 1". Questions 8,9,10 were interesting in that the child kept to the format "?+? is nearly something plus something..." instead of subtracting. Perhaps his strategy was to keep to the rhythm of the worked solution and change the answer by one to make sure it was a "guess" (after all a "guess" can't be correct, otherwise it isn't a guess!)

4. Generic Thought

It is difficult to give a precise definition of a high level concept like that of 'generic thought'. Perhaps we could begin by saying that all thought is generic, but that the term 'generic' is used when it is viewed as a resonance activity. Sensory input causes an excitation of the brain which quickly stabilises through resonance activity and produces a chain of effective mental or physical action. It is the chain of mental action, a mental schema in the sense of Piaget, that we call generic thought. Of course the chain reaction leads to further mental excitation which itself may be temporarily unstable and lead to a totally different stable resonance. This may manifest intself as an observable discontinuity in thought. To give more precision I would prefer to confine the term "generic" to the connected chains of mental activity rather than the sudden discontinuities, but in practice such a distinction may not be possible in view of our inability to monitor brain activity accurately. If such measurement were possible we would almost certainly find that connected chains of thought contained many discontinuities anyway, but that we were so accustomed to them that we no longer sensed the dramatic nature of the discontinuities in the same manner that occurred the first time. For the moment then we shall use the term 'generic thought' to describe the whole process including the connected schemas and discontinuities.

The concept of generic thought helps to give a rational insight into how mathematicians often get the general grasp of the principles involved in a mathematical argument whilst (initially) omitting details. At this stage the essential features resonate. Logically the missing details may render a proof invalid, but in practice the details can often be tidied up to give an acceptable proof. Such a style of thought is often found in Lakatos [3]. It is a human activity often at variance with the formal idea of a step-by-step logical proof.

Generic thought is a natural consequence of the resonance of mental activity, which causes certain resonating ideas to come to the fore, whilst others are ignored. Having followed a chain of mental activity (together with possible discontinuities), the preliminary, or partial proof may be considered unsatisfactory and the resultant instability can lead to further chains of thought which may (or may not) lead to improvements.

In the first example of the last section, considering α to be any real number is generic in a mathematical sense, whilst $\alpha = 0$ is a singular case. This ties up with the use of the term 'generic' in algebraic geometry. The other examples are not generic in the same mathematical sense, but the phenomena are clearly analogous.

§5. Generalisation and Learning Hierarchies

Much attention is focused in mathematical education these days on the topic of generalisation and the ability to abstract mathematical properties from particular examples. The notion of generalisation in this sense is quite different from that of generic thought, though the latter often involves thinking in 'general terms'. The difference is that generic thinking is construed as mental resonance in the electrochemical activity of the brain whereas generalisation is a mathematical classification of concept hierarchy. The distinction between these is absolutely fundamental in mathematical education, for learning hierarchies are sometimes falsely equated with mathematical hierarchies when the mathematical hierarchy may not correspond with the actual order of difficulty for a particular individual. It may not be, for example that a particular individual needs to abstract a generalisation from a number of different exemplars, though another individual may find this helpful.

The work of Dienes [2], establishes various principles of teaching in Piaget's concrete operational stage. For instance in his mathematical and perceptual variability principles he advocates the use of a number of exemplars from which to abstract a higher order concept. Skemp [5] also talks of the use of several exemplars from which to abstract a higher order concept, but he also speaks of 'noise' (properties in the exemplars different from those to be abstracted). With many examples and much noise, abstraction may be inhibited. Noise may be interpreted as interference in the electrical circuits which inhibits resonance. Thus there is a delicate balance between the exemplars used and the noise involved.

Krutetskii has observed [3, page 335] that more capable older pupils are able to generalise from a single example. He also notes an interesting case (p.296) which is worth citing:

> *A capable pupil would do a problem of a definite type and in two or three months he would be given a problem of the same type (but not the one he had done earlier) .... Often a "feeling of familiarity" would come to the pupil: he would believe he had done this problem already (not one of the same type, but the same one).*

The implications of this observation in terms of resonance and generic

thought need not be dwelt upon.

If we pass to the research mathematician, we may find that in a particular case he is far happier with a high level generalisation than with a lower order example containing excessive distracting factors, (for instance he may well prefer linear maps in Banach spaces to 3 × 3 real matrices). The advocates of an "example - generalisation" hierarchy would say that this high level generalisation is now his 'example', but that simply does not hold water when it is clear that he finds it easier manipulating the curtailed simplicity of the generalisation than the excessive detail of a supposedly lower level example.

What is common to the child and the research mathematician will not be seen through a mathematical classification in terms of example - generalisation, but through a mental classification: what electrochemical configurations are available in the brain to be able to process the information and to solve the problem at hand. What is common between the child and the researcher is the nature of brain activity, which may be interepreted as resonances, continuous schemas, instabilities and discontinuities of thought.

We may also briefly mention the thesis that in a highly structured subject like mathematics one needs to master subordinate tasks before preceding to higher order tasks, or, in other words, to understand all the parts before preceding to the whole. This arises through equating logical structure with learning hierarchy. Practice tells us this is not always so (see also [9] p.98); indeed a grasp of many of the parts may lead to sufficient resonance to put the other parts into context.

§6. Conclusions

In this paper we have suggested a way in which mathematical thinking may be related at a qualitative level with brain activity in terms of resonance and generic thought. This allows for the strengths and weaknesses of brain activity in a model theoretic sense which is not found in logical or computer-based simulations. The paper is itself an example of generic thought; it yields a possible approximation to a theory with all the consequent weaknesses of mental activity which may need radical correction with the passage of time. Nevertheless, mathematics is an activity of the human brain, so to understand the nature of mathematical thinking we must eventually have an insight into the workings of the brain itself. As yet we lack a complete understanding, but in the meantime a qualitative model in terms of resonance gives an alternative paradigm within which we may be able to perceive some of the subtle qualities of thought which seem to elude us at present.

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REFERENCES

1. Z. P. Dienes — Building Up Mathematics Hutchinson 1960

2. V. A. Krutetskii — The Psychology of Mathematical Abilities in School Children Chicago University Press 1976

3. I. Lakatos — Proofs and Refutations Cambridge University Press 1976

4. S. Rose — The Conscious Brain Pelican 1976

5. R. R. Skemp — The Psychology of Learning Mathematics Pelican 1971

6. R. R. Skemp — Relational Mathematics and Instrumental Mathematics - some Further Thoughts Paper presented to IGPME (British Section) Warwick University May 18, 1977

7. D. O. Tall — Conflicts and Catastrophes in the Learning of Mathematics. Mathematical Education for Teaching 2, 4 1977

8. R. Thom — Topologie et Signification, Modeles Mathematiques de la Morphogenese Union Generale D'Editions 1974

9. R. F. Thompson — Introduction to Physiological Psychology Harper & Row 1975

10. J. D. Williams — Teaching Techniques in Primary Maths NFER 1971

11. E. C. Zeeman — Topology of the Brain in Mathematics & Computer Science in Biology & Medicine Medical Research Council Publication, 1965

12. E. C. Zeeman — Brain Modelling in Catastrophe Theory, Selected Papers 1972-1977 Addison Wesley 1977

13. E. C. Zeeman — Duffing's Equation in Brain Modelling in Catastrophe Theory, Selected Papers 1972-1977

THE ACQUISITION OF ARITHMETICAL CONCEPTS

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Arithmetic is usually considered a rather vulgar part of mathematics comparatively with geometry and algebra. Some historical research both in the history of education and in the history of mathematics would be necessary to understand why this is so. Probably the leap that the child is supposed to make from elementary arithmetic at the primary level to the noble mathematical concepts of equation, function, point, line and plane at the secondary level has something to do with this prejudice against arithmetic.

The current trend of modern mathematics has made the situation for arithmetic rather worse than it was before. And the silly reaction against modern mathematics that has sprung up in the last few years is very often presented as a return to good, useful and commonsense arithmetic.

The title chosen for this paper: "the acquisition of arithmetical concepts" intends to give a better cognitive status to arithmetic, too easily associated with boring, out of date calculation, and to relate it to modern mathematics instead of opposing one to the other.

As a matter of fact arithmetic, even in its elementary aspects, deals with very important mathematical concepts. To establish the link between ordinary mathematical situations and the relevant mathematical concepts is probably the most challenging question in mathematical education. This link is important both for a better ability to solve ordinary arithmetical problems and for a better understanding of some more sophisticated concepts.

It is well known that the capacity in calculation of secondary-school pupils is not very good, but it would be misleading to think of these difficulties only in terms of "calculation capacity". Most difficulties met by pupils are difficulties in the concepts involved, and not in the calculations. Of course, this does not mean that calculation by itself raises no problem.

It is this problem of concept that some groups are trying to tackle in studying the complexity of different capacities and in organizing conditions making possible the emergence of these capacities. Solving a problem by choosing the right calculation is a very strong criterion of the acquisition of concepts. Operational thinking in mathematics has a lot to win in being tested and analyzed through problem solving situations, because a concept that is not operational is not really a concept. Of course, a solving procedure is not a concept either. It must also be thought over, or explained, in terms of properties and relationships.

But there is so little stress on physical and natural situations that psychologists feel the need (especially in France) to call mathematicians back to the relationships that mathematics have with reality. The concept of number would not even exist if man had not met problems of measurement, and the concepts of function and equation refer to very elementary aspects of the ordinary activity of the child in the world.

Summarizing this short epistemological introduction, I would say that teachers and psychologists must be firmly aware of two kinds of considerations :

- on the one hand : start from the analysis of situations implying quantities, transformations of quantities, and relationships between quantities; and test mathematical knowledge against problem-solving in such situations.

- on the other hand : make successive steps to abstract different sorts of mathematical objects and different sorts of symbolic representations of these objects.

When a pupil uses a certain procedure to solve a problem, he is using what I call a "theorem in action" which is not yet a theorem. When a pupil is able to repeat verbally a theorem, it is not necessarily a theorem in action. There is no operational thinking without the coordination of these two criteria.

Psychologists and psychologically-minded teachers can be very helpful in studying the problem of developmental complexity, because developmental complexity cannot be inferred directly from mathematical complexity. For instance, two independent axioms such as commutativity and inversion of additive transformations are not mastered at the same level by the child, because they do not refer to the same task in problem-solving and because the complexity of these tasks is not the same. Another example on which different researchers (Noelting, Kathleen Hart, Karplus and others) have found very important results is proportion: different properties, almost equivalent for the mathematician, are not at all equivalent for the child.

But the problem of complexity should not be seen in too general terms. Of course it always refers to the complexity of concepts but, as a psychologist, I suggest a task oriented, behavior-oriented and symbolic-representation-oriented analysis of complexity.

Complexity of tasks :

The same concept can be involved in a variety of situations and tasks that do not need the same "relational calculus". By "relational calculus" I mean the transformation and composition of relationships given in the situation. For instance, the same relational calculus is not required to substract 6 from 24 in the three following situations :

A I had 24 francs and spent 6 francs. How much do I have now?

B My grand-mother has just given me 6 francs and I now have 24 francs. How much did I have before?

C I spent 24 francs this morning and my mother gave me some money in the afternoon. I now have 6 francs less than I had at breakfast. How much did my mother give me?

Of course, the analysis of tasks is still more useful when you have much numerical data, including non-relevant data, or when you must find intermediary results, or when you need some measurement or counting to complete your data. But as I cannot go into details in this paper, and only want to suggest a few questions, I will mention two important notions, which the mathematician usually ignores, and which must be reintroduced into our reference-frame if we want to analyse the relationships that the pupils take into account in arithmetical tasks. These notions are:

1) TIME :

Situations usually happen in time with a very blatant order-relationship : initial state, transformation, state, transformation, ... final state. Especially for additive transformations (plus or minus something) it is necessary to keep time in mind to analyse the tasks.

2) DIMENSION :

Numerical situations usually deal with quantities that are not pure numbers but magnitudes of various kinds. These magnitudes can be elementary magnitudes, or they can be products or quotients of elementary magnitudes. It is necessary to take these dimensions into account not only to analyse tasks but also to lead pupils to some important theories such as dimensional analysis, linear mapping and vector space theory.

you cannot make proper classifications of tasks if you only refer to the notion of number. Numbers are undoubtedly central in mathematics but it is impossible to understand what difficulties children meet with numbers if you do not look at them as magnitudes of different sorts, transformations, or relationships.

Moreover there are specific obstacles coming from the specific nature of the physical concepts involved. These obstacles must be studied too: the ratio of distance to time, the ratio of mass to volume raise more problems for children than the ratio of money to a quantity of sweets. The analysis of these specific problems is part of the analysis of tasks.
So is the analysis of problems raised by the choice of numerical data in certain subsets of values: decimals, rationals, values under 1 ...

All this brings us to the idea that it is hopeless to try to understand the acquisition of arithmetical concepts and to propose better conditions for the child to understand them if one does not make the effort to analyse the tasks through which these concepts are made meaningful and useful to the child. This analysis, if we want it to be systematic, must include many aspects: relational aspects, number of data, subsets of chosen values, physical or other concepts that are involved. The possible tasks may appear as an unnecessarily wide variety and it may happen that some distinctions are useless when we know more about them. But, for the time being, because not enough mathematicians and teachers are aware of it, it seems more profitable to accept that variety of tasks as a richness and a source of new interesting experimentations.

Complexity of behavior and procedures :

For the same task, the same problem or the same situation, you get from pupils a variety of procedures. There is not only one way of getting the right answer, there is not only one wrong answer and there is not only one way of obtaining the same wrong answer. These different procedures, right or wrong, are not equivalent from a cognitive point of view.

More generally speaking, the behavior of pupils in a new situation can be very different and the observation of that behavior is very useful to comprehend what makes sense to them. I will only give two examples.

The first one refers to a rather restricted notion of procedure. In the very classical problem

> given a,b,c, such that $b = f(a)$, find $x = f(c)$ when f is a direct proportion (consumption of something in function of time for instance)

one finds more than twenty different sorts of procedures that imply different ways of looking at the relationships in the task and different ways of dealing with them. Amongst the successful ones, there are differences which show different degrees of generality. The properties taken into account are not the same.

As concerns the unsuccessful procedures, it is possible to classify them into different subclasses which do not have the same meaning from a cognitive point of view.

That example is very important because many calculation capacities rely upon it.

I will turn to a second example which does not refer to a strict meaning of procedure but rather to a wider notion of behavior. If I take the metaphor of the rat in the maze, when no aim is given to him, and of the "orientation responses" that he usually makes, I will say that very often, situations in the class-room are at first mazes to the pupils. It is most interesting to observe their "orientation responses". Of course there are all sorts of such responses but one can restrict oneself to some subsets of them. You can study for instance the way they tackle a new set of materials (blocks for instance) or a new situation including relationships and data which they have never met before. Our research-group* is now studying the questions that pupils invent and find meaningful when they face data with no question. Of course some questions are stereotypes but it is interesting to observe the choice of questions, the formulation of them and the discussions about what is interesting or not, what is answerable or not.

I will conclude this part on procedures by saying that the study of the difficulty of tasks (of items in a test for instance) is undoubtedly the first priority. But it would be misleading to think of the problem of complexity only in terms of difficulty of tasks. The complexity of procedures and behavior is also very important. The easiest way to tackle a difficult problem may be easier than a more sophisticated way of tackling a simpler one. Theorems in action, as seen through the analyses of procedures, are supposed to lead to explicit mathematical theorems. It is important to know which ones are more natural and which ones more complex. For instance, the study of procedures may lead to the conclusion that it is better to explicit the homomorphic property of linear mapping

$$f\ (\lambda x) = \lambda f\ (x)$$

than the usual expression

$$f\ (x) = a\ x$$

but this also raises the problem of the complexity of representations.

The complexity of representations :

It is well known that mathematicians use different sorts of representations according to the problem they have to solve or to the theorem they have to prove. But this idea has not been systematically studied in education. The problem of representation is not so simple after all, because there are at least two different meanings of the word.

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* This group studies the acquisition and teaching of multiplicative structures on 11-15 year-old-pupils. It is sponsored by D.G.R.S.T. n° 76-7-1093. It comprises mathematicians (André Rouchier, Patrick Marthe et René Mètregiste) and psychologists (Graciela Ricco and myself).

1) The most widely accepted one is "symbolic representation" according to which different sorts of symbols stand for objects of different mathematical status: elements, operations, relationships, classes, functions ... Of course there may be different sets of symbols referring to different (or to the same) sets of objects.

2) The other important meaning of representation is "thinking" (conceptual or preconceptual thinking) when no reference is made to any explicit system of symbols, even to ordinary verbal language. Representation is then that sort of thing that you can infer from the behavior and explanations of a subject.

The linguistic distinction between "signifier" ("signifiant" in French) and "signified" ("signifié" in French) does refer to these two meanings of representation. But for the psychologist, the "signified" should be analyzed through all aspects of behavior, and especially action in problem-solving and not only through the symbols by which the subject tries to represent things.

In simpler words, you can learn more about the meaning of substraction for a child if you study the way he deals with problems that need the use of a substraction, than if you only study his use of words and signs referring to substraction. For a psychologist, the "signified" refers more to action and behavior than to explicit symbolization although both references should be taken into account and not be opposed to each other. Concepts and preconcepts are better analyzed through the actual capacities of pupils in problem-solving situations than through verbal definitions and theorems.

But this does not solve the problem of the role played by symbolic representations. As a matter of fact, mathematicians agree that symbolic representation is very important in mathematics: they spend a lot of time to come to an agreement on the best and canonical ways of designating and writing mathematical objects.

How can we transfer that problem into didactics?

My first point will be that we must try to make operational the problem of symbolic representation by finding tasks in which a symbolic representation helps pupils to solve the task. This is not so easy after all: most teachers know that pupils very often solve the problem first, and write its representation afterwards. We must try to find, for all levels of education, classes of problems and symbolic systems such that the former ones could not be easily solved without the help of the latter ones: classes of problems that could not be solved without being "put into equation". Of course I use the word "equation" as a generic metaphor, because equations are not the only way of representing a problem. One of the aims of teaching should be, for all symbolic systems, the discovery of problems implying a variety of data and relationships that makes it necessary for the child to use a symbolic representation to make the right interpretation of relationships and the right choice of data.

If you test, against that criterion, the arithmetical equations that children are supposed to use at primary school, they are very seldom operational: not only because they come after the solution has been found, but also because they do not usually represent the relationships in the problem, but rather the operations the child has made to solve it. Venn-Euler circles, trees, arrow diagrams should be tested against the same criteria. It is probably the case that a system is efficient at certain levels and for certain classes of problems and not for the other levels or for other problems. I do not have many results, only some data* on the use of arrow-diagrams in additive problems when a succession of states and transformations occurs. These data seem to show a better ability of pupils to use diagrams than equations at the end of primary school and beginning of secondary school. I also have some observations concerning the use of diagrams and tables for multiplicative problems.

My second point concerning the problem of symbolic representations is that it is sometimes better to differentiate between situations that are not the same for the child although they do come to the same numerical equation. For instance a + x = b refers to many different classes of problems which are not at all equivalent for the child. So how can he find it natural to represent them by the same equation? Of course it is justified, at a certain level, to deal only with numbers and work on the general equation a + x = b, but that work can hardly be meaningful to the pupil if he is still trying to understand how a + x = b can represent such different problems as addition of cardinals, additive or negative transformation of a state into a state, composition of transformations or transformation of a relationship.

The same sort of problem arises for ax = b.

So, a symbolic system can be too abstract if different situations have the same representation in it. It may then be heplful for the pupil to use more differentiated systems for

a while, and to prepare the introduction of the most abstract system by these intermediary steps. It is often good to use several symbolic systems at the same level and link one to another.

I will stop here the list of problems which I meant to raise under the theme of complexity: tasks, procedures, representations. They all refer to the order relationship (total or partial order, genetic or not) in acquisition. I will turn now, very briefly, to an example that can illustrate the ideas which I have tried to develop.

Let be two problems A and B that concern primary school children :

A - John has played marbles with friends. In the morning, he won 14 marbles. In the afternoon, he lost 31 of them. He now has 23 marbles. How many did he have before playing?

B - John has played marbles with friends. He had 14 marbles before he started. He played in the morning and in the afternoon. In the afternoon he lost 31 marbles. He now has 23 marbles. What happened in the morning?

And here are four systems representing these problems (there are others of course).

I A $\boxed{x} \xrightarrow{+14} \square \xrightarrow{-31} \boxed{23}$

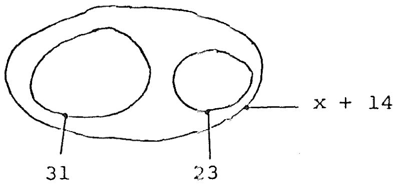
B $\boxed{14} \xrightarrow{x} \square \xrightarrow{-31} \boxed{23}$

II A $x + 14 - 31 = 23$

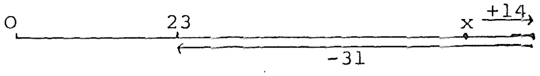
B $14 + x - 31 = 23$

equivalent

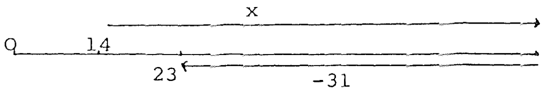
III A or B



IV A



B



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* These data have been collected by a group that studies the coordination of maths-teaching between primary and secondary schools. It is sponsored by the Ministry of Education, and is led by Paule Errecalde (INRP).

These four systems are not so obvious to use.

System II for instance designates by the same x a state (positive number) in A and a transformation (positive or negative number) in B. The sign plus and minus do not have the same meaning either, as appears in the solution

$$x = 23 + 31 - 14$$

sign minus means inversion of a positive transformation in A and substraction of a natural number in B.

System III is not very natural in that case and needs some thinking to be used correctly, especially for B because one does not know the sign of x (x would have a different sign if you had 74 instead of 14).

System I is obvious. This is not amazing because the example has been chosen on purpose. Of course, there are situations for which system III or system II are obvious.

System IV is similar to system I as regards the difference between a state (position) and a transformation (move) but it is not very easy to use it, because the direction of moves should be known: this is not the case in B.

Let us turn now to the procedures: for example problem A can induce different procedures:

a add 31 to 23; then substract 14

b substract 14 from 31; then add this result to 23

c make an hypothesis on x; apply + 14 then - 31 and change the hypothesis according to the gap between this result and 23.

There are some other good procedures and many bad ones. But let us limit ourselves to a, b and c.

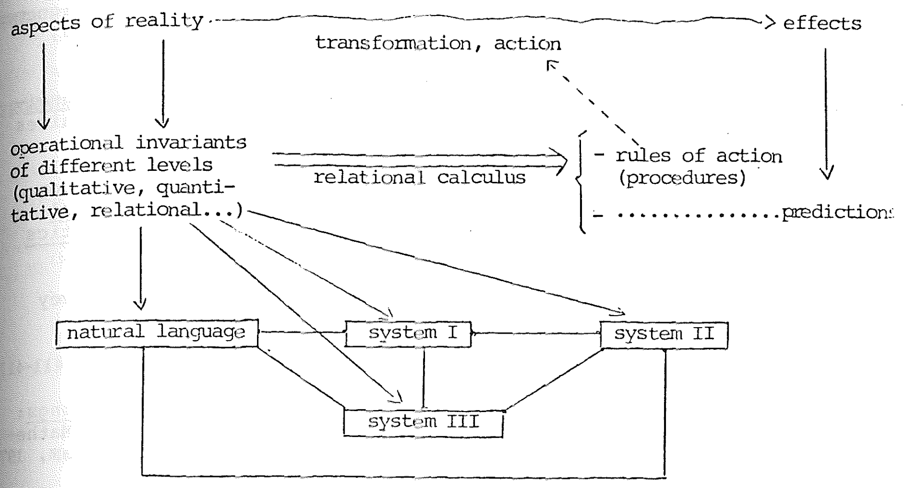
b consists of the composition of two transformations in the first step: it is a high level procedure for primary school children.

a is not so difficult but it involves the inversion of the direct transformations - 31 and + 14 into the symmetrical ones. This is not at all obvious for 9 or 10 year-old-children.

c does not require any composition or inversion of transformations. It is attempted (if not successfully) at a younger age, especially with small numbers.

a, b and c do not represent the same "relational calculus". The properties taken into account, or the "theorems in action" working at the "signified level" are not the same.

All this can be summed up in the following schema :



This schema is an oversimplification of the problems which I have tried to raise. It only traces the most important aspects and criteria of operational and symbolic thinking in building up both references of concepts with action in reality and handling of symbolic systems.

The first two levels illustrate the idea that operational concepts are built up through action and are tested against effectiveness of action and prediction in reality. The last two levels show the link between conceptualization and explicit symbolization and lay the stress on the plurality of symbolic systems and on the special part played by natural language.

Jean Piaget has done very important work on the building up of invariants through action. It seems to me that more work than what has been done up to now should be pursued on relational invariants and on the use that children can make of symbolic systems. This work can hardly be done by pure psychological studies. It requires didactic studies for which both mathematicians and psychologists are needed.

REFERENCES

BROUSSEAU, G Les obstacles épistémologiques et les problèmes en mathématiques. *La problématique et l'enseignement de la mathématique. Some questions related to the use of problems in the teaching of mathematics*. Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques. Louvain-la-Neuve, 1976, 101-117.

DESJARDINS, M., HETU, J.C. *L'activité mathématique dans l'enseignement des fractions*, Presses Universitaires du Québec, 1974.

HART, K. The Understanding of Fractions in the Secondary School. *Acts of Second International Conference for the Psychology of Mathematics Education*, Osnabrück, 1978 (this volume).

KARPLUS, R., KARPLUS, E., FORMISANO, M., PAULSEN, A. A Survey of Proportional Reasoning and Control of Variables in Seven Countries, *Journal of Research in Science Teaching*, 14 (1977), 411-417.

LUNZER, E., BELL, A., SHIU, C. *Numbers and the World of Things: a Developmental Study*. Shell Centre for Mathematics Education, University of Nottingham, 1976.

NESHER, P., KATRIEL, T. A Semantic Analysis of Addition and Substraction Word Problems in Arithmetic. *Educational Studies in Mathematics*, *8*, 1977, 251-269.

NOELTING, G. Constructivism as a Model for Cognitive Development and (eventually) Learning. *Acts of Second International Conference for the Psychology of Mathematics Education*, Osnabrück, 1978 (this volume).

REES, R. *Mathematics in further Education*. Hutchinson Educational, 1973.

SCHULMAN, M. Review of Recent Research Related to the Concepts of Fractions and Ratios. *Acts of Second International Conference for the Psychology of Mathematics Education*, Osnabrück, 1978, (this volume), (contains a good bibliography of English Litterature).

SKEMP, R. *The Psychology of Learning Mathematics*, Pelican, 1971.

VERGNAUD, G. Calcul relationnel et représentation calculable. *Bulletin de Psychologie*, *315*, 1974-1975, 378-387.

VERGNAUD, G., DURAND, C. Structures additives et complexité psychogénétique. *La Revue Française de Pédagogie*, *36*, 1976, 28-43.

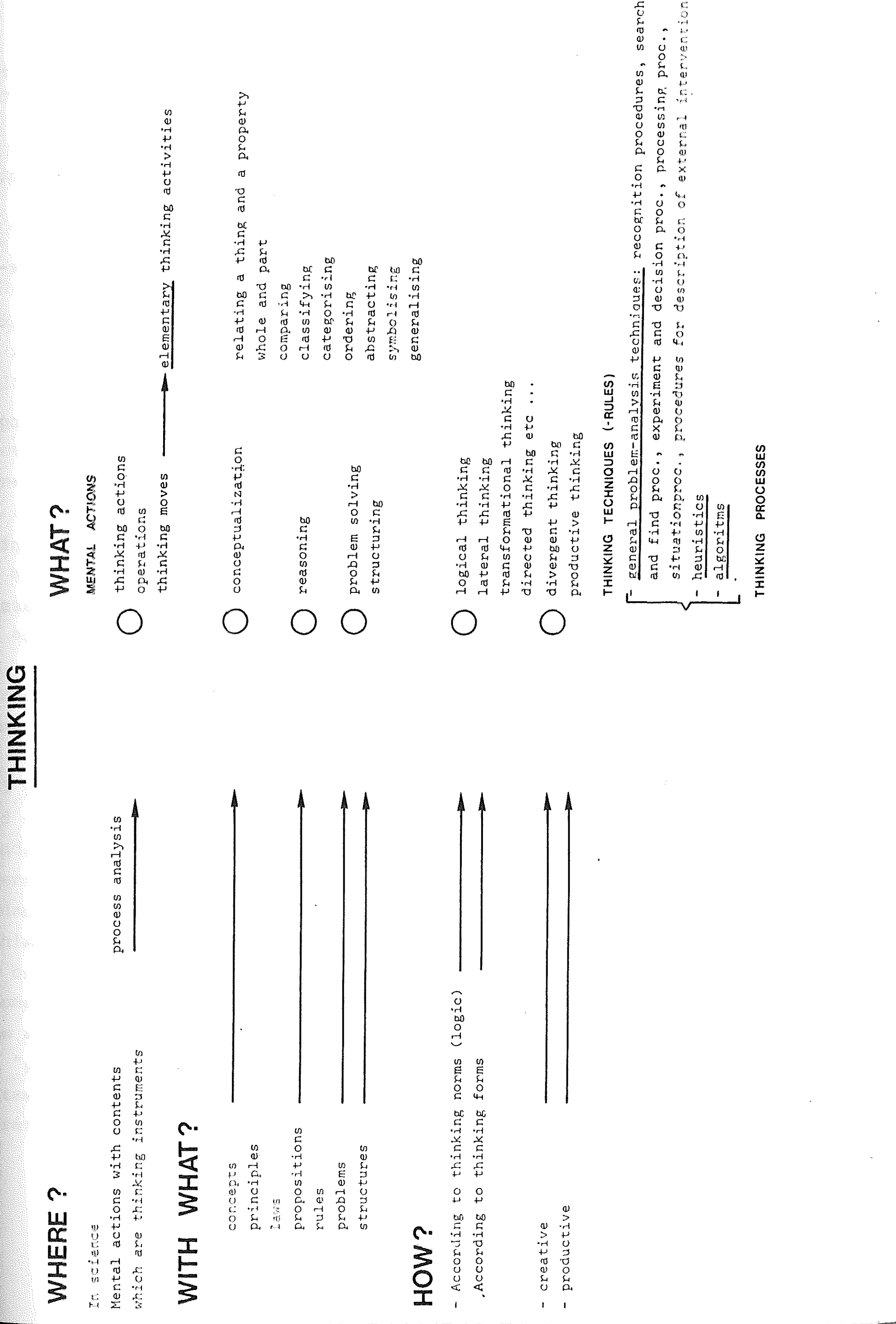
VERGNAUD, G., RICCO, G. Psychogenèse et programmes d'enseignement: différents aspects de la notion de hiérarchie. *Bulletin de Psychologie*, *330*, 1976-77, 877-882.

VERGNAUD, G., RICCO, G., ROUCHIER, A., MARTHE, P., METREGISTE, R. Quelle connaissance les enfants de sixième ont-ils des "structures multiplicatives" élémentaires? Un sondage, *Bulletin de l'APMEP*, *313*, 1978, 323-357.

A. Vermandel and E. Cohors-Fresenborg

The nature of mathematical thinking

If we want to study thinking, we are confronted with a great variety of possible methods and points of interests.
An analysis of the range of thinking instruments which every science, mathematics included, offers us, gives us as constituting elements : concepts, principles, definitions, laws, propositions, rules, problems, structures.
Regarding learning of thinking we can transform them in mental activities of Man, like, in different formulations, thinking operations, thinking moves, thinking processes, inter-psychic transformations between stimulus entrance and reaction ouput, etc.
This gives us a.o. studies on conceptualisation, reasoning, recognising of rules, problemsolving, structuration (restructuration) etc.
We can also focuss on more elementary thinking activities which seem fundamental for every science-practice, like the relating of a thing and a property, a part and a whole, comparing, ordening, abstrahating, symbolising, generalising, categorising, classifying.
These strictly formal categories one encounters on every level of thinking and they can then also show all gradations of difficulty.
We can study how thinking has to be carried out to lead to (correct) thinking results.
We then define thinkingnorms and -forms.
In this way we arrive to notions like logical thinking, vertical thinking, lateral thinking, directed thinking, relational thinking, transformational thinking.
If we look more to the result, then certainly important are divergent (versus convergent) thinking, creative and (re) productive thinking, and also the notions discovery and invention.
Organised wholes of more elementary thinking activities form the thinkin techniques which one can distinguish in a) general problem-analysis techniques like recognition procedures, search- and find procedures, checking and decision procedures, processing procedures, transformation procedures, situation procedures, procedures of the processing of external interventions.
b) heuristic rules in the more narrow sense
c) algorithms
Also thinking processes like the concatenation of information processes and thinking strategies which make thinking more efficient are important.



It is clear that the whole of these studies is an important base for the research and promoting of mathematical thinking.

What seemed us on occasion of this conference a necessary point of discussion, is to define more precisely the expression MATHEMATICAL thinking, more exactly the term mathematical mental action.

We prefer an action-centered terminology like that of Gal'perin because it meets with our position that in mathematics teaching, mathematics should be learned as a conscious activity, as a proceedings rather than as a whole of prefabricated knowledge-units and automatisms.

We accept with Gal'perin that an <u>action</u> is a working on the objects in an purposeful way. When an action is carried out on material objects (ev. their symbolic representation) we are dealing with <u>material</u> (ev-materialised actions). A <u>mental action</u> on the contrary is carried out on non material, notional objects or representations f.e. the adding up of two numbers in the head.

According to Gal'perin mental actions develop gradually through interio sation of external behavior.

An important function thereby are <u>verbal actions</u>, i.e. actions carried out on words representing notions, they are loudly spoken actions without external objects, f.e. a child counts aloud in making a simple addition.

These verbal actions are seen as transition between material and verbal acting.

Mental actions make it possible for man to anticipate mentally the result of material actions.

To be able to function optimally, a complete mental action has to give insight, has to be general, consciously executed and thus controlable, and be readity to apply.

The acquiring of new mental actions best happens through a stagewise building up of them (1).

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hematical actions

t is a mathematical action ? One has immediately the inclina-
n to call mathematical action, the action carried out with
hematical objects, like numbers, vectors, lines, functions geometrical
lles etc.
sides the fact that the notion "mathematical object" is not clear
her, this point of view can not satisfy, because it is a vicious
cle. Neither are we convinced that pupils in factorising alge-
ic expressions or multiplicating fractions do a lot of mathe-
ics. Are they mathematicians because one works with symbols
skillfully combines and transforms them according to pre-set
les?

e acceptable is to give those actions the status of mathematics
ch one has to carry out to <u>construct</u> a mathematical (deductive)
ory, like for example the act of defining, creating structures,
proof theorems, to extent.
find mathematical actions which are generally accepted as such
can also start from psychological analysis of a method which
hout any doubt belongs to mathematics, namely the axiomatic
thod...

axiomatic method has indeed in contemporary mathematics grown
o something much more than a technical procedure of the mathematician
rving the internal organisation of his field.
still plays an organising role but thanks to the notion of struc-
e and the accompagnying morphism thinking it goes a lot further
n traditional deductivity. It appears to be an excellent research
strument advancing the unification of mathematics and indispensable
fundamental research. It offers the model itself of an irrepla-
ble thinking process sui generis which makes it possible, without
lling on the world of experience to know properties of reality.
axiomatic method is according to DIEUDONNE (2) nothing else
n the application of a principle at the base of every science,
, the "principle of consciously incomplete knowledge" in which
systematically neglects certain aspects of the objects under
nsideration.

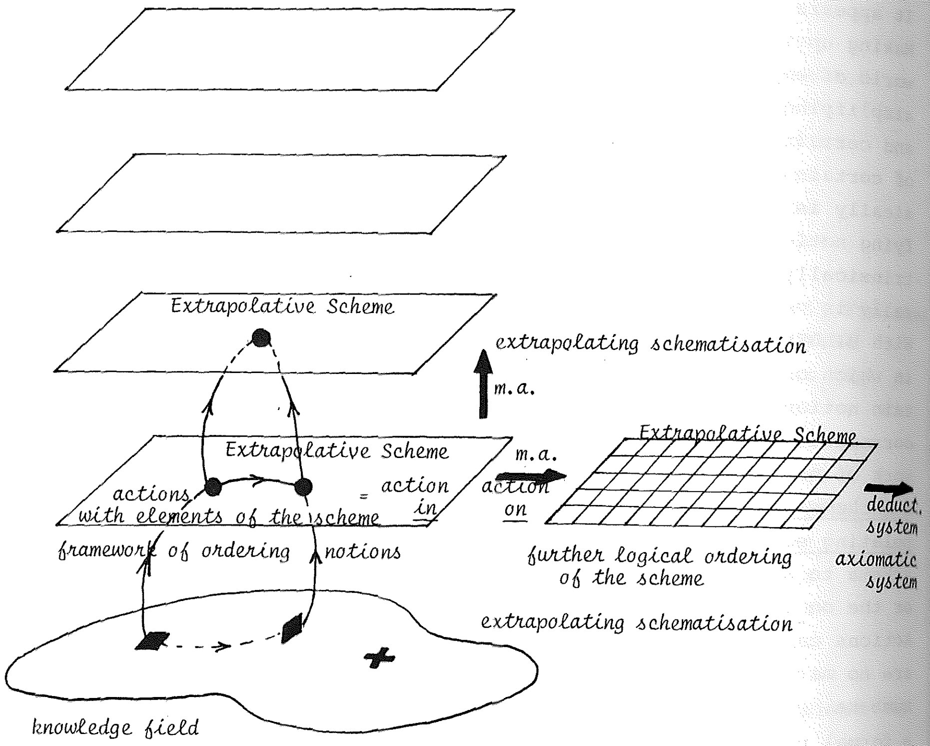
In mathematics however one takes care to sum up exhaustively the properties of the studied objects (axioms) and one forbids oneself to use anything else but those properties and the rules of logic. This action which is psychologically identical to isolating the mathematical features,that characterise a problem in applied mathematics,we will indicate provisionaly with the term schematising.

Extrapolating schematising

The use of the term "schematisation" for the workmethod which seems to be fundamental for the axiomatic method, - namely, the rendering of objects which one studies in such a way that consciously certain aspects of these objects are left out of consideration, can be justified by giving an example of a simple scheme: a geographical map.

Its properties are the following ones : it gives no completely exact representation of reality. Neither the trees in the forest, nor the depth of the rivers is indicated. The map not only does not render everything, but the description it gives is simplified and shortened. The cities, the streets, the mountainpeaks are only indicated by conventional signs, symbols which do not take into account all peculiarities of the things indicated and which one has to interpret. The map doesn't give everything, but nothing stops us from reworking one or other aspect, she is only finished in so far as we have decided. As essential properties of a scheme we already can give : incomplete, simplified, symbolised. What the map wants to describe, this or that country we can call its "external meaning".
On it self, as autonomous object it has however a reality of its own, we can speak of an internal structure. Nothing stops us from considering this last one on itself without taking into account the meaning which it can have, if one consults it regarding its external meaning. Even considered like this, the map poses lot of problems : geometrical, topological, colorationproblems etc. it can be usefull to solve.
In reasoning intrinsically in a scheme we do turn from its original

meaning, but the results of the reasoning make it better comprehensible for us.
Since the meaning for an axiomatic lies exactly in showing properties of objects without referring to their external meaning (experience) we will make in connection with axiomatization the following demands to schematising. The resulting scheme must make intrinsic reasoning possible. To indicate this we'll speak of extrapolating schematisation.
In this way elementary geometry can be considered as a mental scheme of which "the external meaning" can be found in a certain natural structure of the surrounding fysical world. The intrinsic structure of this scheme is its logical structure. It appears in the possibility to build it up axiomatically.
Making up the scheme "geometry" consists of, - starting from the world of empirical data -, the thinking, of a certain number of simplifying and ordening notions which are sufficiently precise and certain relations, the axioms which are a schematic form of certain natural and necessary connections. To reason intrinsically is then simply to explicate what is proper to these simplifying notions, we could say the rolling off of the chain of "intrinsically necessary" consequences. Shortly, to reason intrinsically is very simply to reason logically as we are used to do.
With HILBERT (3) we find this action defined as a sort of mapping in which chosen objects of a knowledge field are reflected on certain notions of a framework of ordening notions and the chosen facts correspond to a logical relation between these notions. The framework of notions is then nothing else than an extrapolating scheme.
A mathematical action is then every action carried out on an extrapolating scheme like for example further logical ordening of this scheme to a deductive system, transforming, deducing, axiomatizing, or the on its turn again extrapolating schematising.
Actions carried out in the scheme, i.e. with elements of the scheme, are no mathematical actions, like for example calculating with numbers, executing of algorithms, applying algebraic formula checking a formal proof etc. .



mathematical action: m.a.

An example of a mathematical action

Essential for mathematics is that also the actions carried out on the elements of an extrapolating scheme like the bringing together to a whole, the composing, mapping, transforming, ordering etc. are object of schematising.

Here we find a mathematical action clearly coming to the foreground in modern mathematics and which is certainly an elementary thinking move of mathematical thinking.

Where activities like bringing together, mapping, composing, transforming, ordening in their "external meaning" are experienced as actions carried out on objects, they get after schematising "thing character".

The thinking process which leads to this characterchange is very specific so that we would like to give it a name. Let's speak provisionally of "reification" (4).

After the reification of actions these are themselves regarded as objects, i.e. able to be an element of sets and inturn actions can be carried out on them.

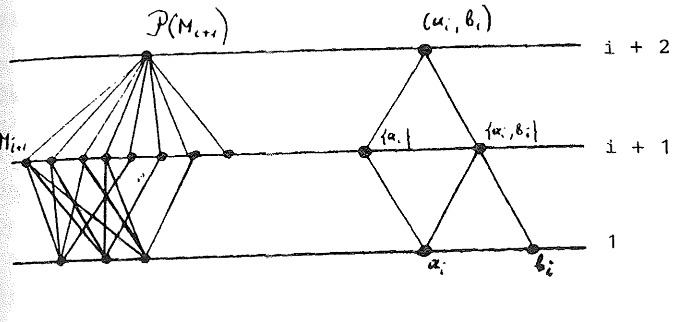
In the building up of mathematical theories such cycli follow each other in a reasonably quick tempo.

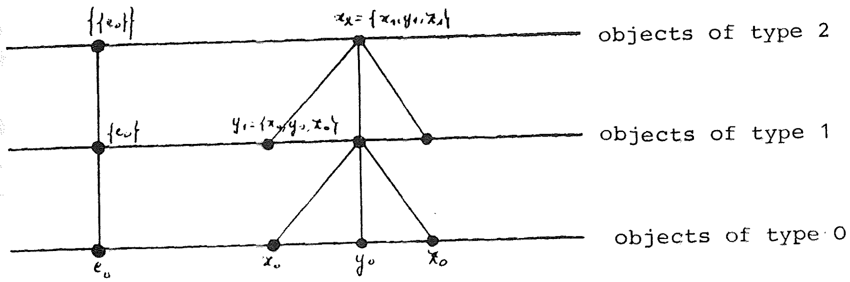
An example of reification is the following: the action of comparing two collections of objects A and B to see if they have each as much elements as the other can happen by taking for every thing in A one in B. This situation can be schematized by making abstraction of the nature of the elements of A und B by introducing the term bijection of a set A on a set B. One says of two sets A and B that they are equipotent if one can map A one to one anto B. This action is reificated by using as a synonym of "A is equipotent with B", the expression "A has the same cardinal number as B". Next, one simply talks of the cardinal number of a set and one handles cardinal numbers as wellknown objects. A next step is to determine the set ω of all natural numbers. ω in turn will get a thingcharacter, so that expressions like $\omega + \omega$ get meaning.

Another example is the construction of a topology: the predicate "open" is true for a subset O of a given set S, if O is an element of a special set of subsets of S, which statisfies the axioms of a topological space.

In these examples we also see that relations are reificated, another essential part of mathematical schematizing. Where relations in their "external meaning" are experienced as a connection, a relation between to objects, they get after schematizing "thing-character". The ultimate mathematical object R can be taken as an element of a set, it can be transformed, mapped composed with another relation S, etc... Let us consider a set of people.
Many relations can be noticed in it (in the undefined meaning of the word) :... is greater than ..., ... is older than ..., ... lives in the same neighbourhood as ... a is b's father.
Let's mark the last relation V. In first instance, V indicates the relation between a and b. We can safely say "a and b are in relation V". c and d too are in relation V, e and f are not in relation V. If a and b are in relation V, b and a are not.
A first reification exists in expressing this as: the ordered pair (a, b) is in relation V. In a next step all such ordered pairs are grouped.
The set of all ordered pairs in relation V is called "relation V".
This immediately leads to the mathematical definition:
"Relation = set of ordered pairs".
The reification in modern mathematics thought find an example in the typ theory by RUSSELL (1908). An intuitive model of type theory is the following type-hierarchy. Let's suppose a defined not-empty domain of objects. We call the objects of the domain individuals. RUSSELL considered the individuals for atomic objects. The individuals are situated on the lowest level of the hierarchy, they are being named "of the type 0". Next the properties of the individuals (collections of individuals) and the relations between individuals (sets of n-tupels of individuals) are being considered. The whole of all these sets forms the whole of the thinking of the next level above the individuals, namely that of the objects of type 1. Then certain (not necessary all) sets and relations on the thus considered hierarchy are considered. This gives a next step in the typehierarchy.
Certain prescriptions are valid for this. If we indicate the variables by x_i, Y_i, z_i, where the index indicates the type in the sense that x_{12} varies on the type of level 12 and we shall call (atomatic) formulas the signconnections, formed by variables and the signs = and $\in$, then is f.e. the formula $x_1 \in x_{i+1}$ admitted

ut not $x_i \in x_{i+2}$; $x_i = Y_i$ but not $x_5 = x_7$. Since $x_i \in x_i$ is ot admitted, it isn't possible to asign a property to itself.
This method can be repeated a finite number of times. Each time one becomes a next layer in the typehierarchie by taking certain relations and sets in the till the acquired hierarchy. In this anner we get the typehierarchie of which the next scheme gives an idea:





Each set of the type i+1 is defined by the elements of a lower type. This typical, always repeating hierarchical thinking move of the reification in the axiomatic schematizing, together with the related class and relation thought once more clearly indicates that the mathematicians now base themselves on an elaborated intuition which is no longer factual, which is no longer bounded on the actual perception, but on the repetition of a few well defined mathematical actions.

The further schematizing (see the first figure) does not necessarily mean the transition to a higher type stage, because in a higher schema the lower must not be an object.

In the terminilogy of type theory the expression "mathematical thinking" can be explained as follows: mathematical thinking is mental action which includes at elast three type stages. The measure is refered to a description of this process in a formal system. The aim of this explanation of "mathematical thinking" with type stages is to seperate exactly the *mathematical* concepts. An introduction with concepts of concepts does not seperate, as we show in our example below. The measure of type is relative, as the respective basis point 0 changes with its mathematical usage. If someone uses number theory, the numbers are for him objects of type 0; if, however, he constructs the numbers in set theory, then they are of a higher type.

In the explanation of mathematical thinking it is important that at least two interrelated type stages are present. It is not enough that a mental action is carried on with objects which have in the framework of some theory a higher type. This can be illustrated by the following examples:

Calculation with numbers which are considered by the user as objects of type 0 is not mathematical thinking. In the same way, the usage of formulas in the (type-theoretical) higher mathematics cannot be counted as mathematical thinking.

The classification of plants by biologists represents only a simple mental action. Neither does the formation of wider concepts in these category systems raise the type, as it corresponds with the uniting of sets in set theory.

The abstraction of the regularities of a natural language to compile a grammar is only a mental action; similarly the sorting of various sorts of grammar into grammar types still does not belong to mathematical thinking, as the resulting sets of grammars are subsets of the whole set of grammars. If, however, the concept of a formal grammar is constructed from these examples, then this act belongs to mathematical thinking. However, the *use* of the rules of a formal grammar is just as little a part of mathematical thinking as is the *use* of calculation rules.

The analysis of various electric switches in order to realise a problem in only a mental action. The formulation of axioms and comparision within the framework of the theory of automata belongs to mathematical thinking.

In general it can be said that, for example, classification, arranging in order, reasoning and calculation are not mathematical thinking, but formalisation, formulation of axioms and proofs are. In the examples above (grammars, automata) it has also become clear that the objects of mathematical thinking can also be non-mathematical materials. But one can also with mathematical objects (for example numbers, axiom systems, calculation rules, proofs) perform actions with are not mathematical thinking.

This has for didactics of mathematics the following consequences: classification, arrangement in order, reasoning, calculation, etc., must certainly be encouraged as they are mental actions which can be component parts of mathematical thinking without themselves being this thinking. The actual emphasis must however be laid on such mental activities in which the bridging of more than one type stage is done consciously, and by which the achievement of object results, which lie more than one stage ahead of the relative startingpoint. The distinctive feature in the personality of the mathematician is, hat these after reification become objects, of which the generation process and thereby the jump form one type to the next are often not needed for further mental actions. In this sense the mathematicians make themselves by their actions partially superfluous; the further calculation with the objects and formulas he has made can be the job of non-mathematicians. The didactics of mathematics should educate to mathematical personalities for whom non-mathematical objects exist everywhere in the world, about which mathematical thinking is possible.

(1) P.A. GAL'PERIN, An experimental study in the formation of mental actions in E. Stones, *Readings in education psychology*, London, Methuen & Co Ltd., 1970, p. 142-154.

(2) J. DIEUDONNE, *Algèbre lineaire et geométrie élémentaire*, Paris, 1964, p. 21

(3) D. HILBERT, Axiomatisches Denken, *Mathematische Annalen*, Bd. 78, 1918, p. 405

(4) After the Latin "res" : thing, "facere": make.

(5) Sec. E. ARTIN, Geometric Algebra, New York, 1957
F. BACHMANN, Aufbau der Geometrie aus dem Spiegelungsbegriff, Berlin, 1959

(6) PAPY, Mathématique Moderne 6, Bruxelles, 1966, p. 43.

Shlomo Vinner

The Concept of Number Via Numeration Systems

This paper deals with the concept of number at the levels of high school and
ollege. We examine the "nature of number" aspect, an aspect that expresses itself
plicitly and explicitly in School Mathemetics. Some students approaches are dis-
ssed together with some consequences about School Mathematics.

§1. Two Contradictory Tendencies in the Field of Math Education.

One can point at two main tendencies in Math Education (which are a special
e of tendencies in Science Education). The first one is to emphasize the struc-
re of the mathematical systems which are taught (or meant to be taught) in
hools. (The most distinguished representatives of this tendency are perhaps
S. Bruner and Schwab.) The second tendency is to study the students' intellectu-
l development and to establish teaching on the findings of this study. (The re-
resentatives of this tendency are the disciples of Piaget. Piaget himself speaks
ry little about teaching in specific terms.) These two tendencies might be con-
radictory since there might be cases where there is a big gap between the struc-
re one wants to teach and the students' stage of intellectual development (for
stance, speaking in Piagetian terminology, it is pointless to teach a child
dition and subtraction of natural numbers if he does not master the "conser-
tion of numbers.") Some attempts to explain the failure of certain new Math.
apters used the above argument, namely, the chapters failed because the stu-
nts had not reached the intellectual level needed for the concepts upon which
e chapters were established.

§2. The Psychological Analysis of Mathematical Concepts Versus their Mathematical Analysis as a Starting Point.

As a result of the possible contradiction mentioned in §1 and of the tendency to see the student at the center of the education process there is a tendency now among Math. Educators to ignore the structure of Mathematics both in research and in teacher training. The claim is that the main interest should be in the study of the students' mathematical concepts and for this there is no need to bother with the structure of Mathematics in a serious manner as done by mathematicians and philosophers of Mathematics. However, a little consideration of this argumentation shows us immediately that is is wrong. This is so because it is impossible to analyze the students' mathematical concepts without having a mathematical analysis as a starting point. Even in relatively simple concepts as the concept of number in the ages 4-6, Piaget (5) uses a very special analysis of the concept of number. (This analysis can be related to G. Frege (3) since he was the first one to formulate it in such a monumental way. Probably, he was not the first one to think about it.) According to this analysis a natural number represents a class of (finite) sets such that between each two there exists a one to one correspondence. In some of the experiments in (5) Piaget tries to examine whether the child, when making one to one correspondences, sees the connection between them and numbers. Not having Frege's analysis at his background it is doubtful whether Piaget would come up with the same experiments and analysis he does have in (5). Moreover, in his investigations about the development of space concepts (6) Piaget points out that the order of geometrical distinctions the child makes is from the projective to the Euclidian distinctions. So again, Piaget had to master mathematical notions such

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projective" and "Euclidian" before starting his research. This particular know-
e undoubtedly determined the direction of the whole research.

Finally (to end with a more recent illustration) in a research conducted by
ischbein (2) about the intuition of infinity at the ages 11-15, the analysis
precedes the research mentions Cantor and his transfinite cardinal numbers
$\aleph_1$,... .Some basic facts of set theory determined the questions that were posed
he students, although it was clear that these facts were far beyond the reach
these students.

The point we want to make here is that the <u>mathematical analysis</u> of the Mathe-
ics structure and the mathematical concepts is crucial for psychological research
the development of mathematical concepts) and for teaching. As we said, this
t had been overlooked both in research and in teacher training.

Another point worth mentioning in this context is the following: in many ex-
iments by Piaget and his successors, a failure of the student to give the "right"
er is an evidence of intellectual immaturity or a lack of readiness for the
ject. The impression thus made is that intellectual maturity is "context-free."
this impression is wrong. One has to specify about intellectual maturity and
say according to what scientific or mathematical analysis it is considered. In
e there exists more than one analysis of the concept (and in the histroy and
hematics this is almost always the case) it turns out immediately that intel-
tual maturity is a <u>relative notion.</u>

§3. The Concept of Number as an Abstract Object and its Implication on School Mathematics.

This paper deals with the concept of number at the high school and college levels. As we mentioned above there is no way of dealing with it without relating to a certain mathematical analysis. The one we chose is the one made by Frege (3). This analysis was used later on by B. Russel as a basis for his work (see, for instance, (9)). Probably it is the prevailing approach to numbers in modern mathematics and the philosophy of mathematics.

The aspect that we want to examine is that of the number being an abstract object. Frege claimed that "each individual number is an independent object" (this is the title of one of the chapters in (3)). Frege admits that numbers are not spatial objects but claims that not every object is somewhere. This view had had a lot of influence on the foundation of mathematics and also on the first movement of the New Math. A number (according to Frege) is a class that contains all sets such that between each two of them there exists a one to one correspondence. Thus the notion of a set and the ntion of one to one correspondence were introduced to the elementary math curriculum and got strong emphasis in the first grade. But not only this. There are two other topics in the school curriculum that result from the above idea. The first one is the numeral and the distinction between numerals and numbers (imposed sometimes on very young children). The second is the numeration systems (different from the decimal system).

It is clear about the first topic that the whole distinction between numbers and numerals is pointless unless one considers numbers as abstract objects. This distinction can be exemplified again (either at the elementary or at the junior

igh level) by numeration systems. The number which is an "independent object" has any representations.(1)

We will try here to examine to what extent the idea of the number as an ab- tract object has become an active part of the student mathematical thinking and lso to examine to what extent it is really needed for the mathematical activities ill a certain college level.

§4. The Questionnaire and its Analysis.

There is one direct way of finding out whether one considers numbers as ab- tract objects and that is just asking him. But this is, of course, an ineffective ay to get reliable information. First, one might answer the question according to e expectations he thinks you have. Second, since the whole subject is very often eyond one's verbal awareness a direct question might have a crucial influence on e formation of the verbal answer. Such an influence might distort the non verbal ncept in an unwanted direction. Therefore, there is left the indirect way to find e students' approach. Here we used a questionnaire that (so we believe) might (at east roughly) reveal in an indirect way the students' point of view. We will present it first and then make some comments. Its exact formulation was the following:

We would like to remind you that the expression 31_4, for example, denotes a mber written in the base 4 system (the number is thirteen in this case). 1101_2

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(1) Numeration systems are presented sometimes in such a way that their relation to the idea that a number is an abstract object becomes invisible. This point will be discussed in §7.

denotes a number written in the base 2 system (also in this case the number is thirteen).

Please read carefully the following section and answer the questions.

In one homework exercise about numeration systems that grade eight students got was written: "The natural numbers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and so on and so on. It is possible to write these numbers by means of different bases. Thus, for example 31_4, 1101_2 are different forms of writing the number 13."

In another homework exercise was written: "The natural numbers can be written in many different ways. One of them is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and so on and so on. This is the position method in base 10. Different forms are obtained by using other bases. Thus, for instance, 13, 1101_2, 31_4 are different forms of writing the same number."

Question 1: According to your opinion, is there any difference between the two homework exercises from the content point of view?

A. Yes. B. No. C. I don't know.

Question 2: Which formulation do you prefer?

A. The first formulation. B. The second formulation. C. I do not have any preference.

Question 3: According to your opinion which formulation is correct?

A. The first formulation. B. The second formulation. C. Both formulations. D. None.

It was explained to the students that all the arithmetical details mentioned in the questionnaire were correct and no calculations are needed to answer the questions.

Now, some comments about the questionnaire; in our opinion, the second homework exercise represents the idea of numbers as abstract objects. It does not explain what numbers are. It deals only with the aspect of number representation. Our ssumption was that if a student really has the idea of numbers as abstract objects (and this idea is <u>active</u>) then the chance is quite good that he will prefer the second homework exercise to the first one. So his answer profile will be either 1A2B3B or 1A2B3C (the second one in the case he is tolerant about other views. Of course, to confirm this we had to look at the answer in 1A about the difference etween the two homework exercises.

The way the questionnaire is formulated it suggests also an alternate approach to numbers. According to this approach (the first homework) the natural numbers are identical with base ten numeration system. This approach is probably the very first approach of children. When you write 13 on a sheet of paper and ask the child what number you have written his answer will be "thirteen." Here is (probably in an implicit way) the idea that the number thirteen is the symbol "13." Of course, this approach might look <u>inconsistent</u> under some sophisticated attacks.(2) (We will deal with this problem in §7.) But a person might have an inconsistent point of view (to him it seems consistent). Our purpose here is only

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(2) We are also aware of some other highly sophisticated attacks on the interpretations of the homework exercises in the questionnaire. We won't get into it here because we think they are not relevant to the students in our sample. We say that after being well acquainted with many of the students and after a careful reading of all the answers to the questionnaire.

to discover such a point of view, not to judge it.

Students who identify numbers with the decimal system might have the following answer profile: 1A2A3A or 1A2A3C.

We discussed the above questionnaire with some members of the math department (Hebrew University, Jerusalem) and most of them accepted the suggested interpretations to the questionnaire. Interestingly enough, some of them had the answer profile 1A2A3C. This might show that in everyday thinking they do identify numbers with the decimal system although as mathematicians they are familiar with the idea that numbers are abstract objects. This might show as well that one can acquire some theoretical principles but they remain <u>inactive</u>. In practice some other principles (sometimes even implicit) take over.

Finally, our main interest was to examine the above approaches in adults; however for the sake of comparison we also distributed the questionnaire to a younger population.

<u>§5. The Sample and the Classification of the Answers.</u>

Our sample of adults consisted of two different groups. The first one included math major students in their first, second and third years of their studies and also mathematical track students in their 12th grade in a high school (169 students altogether). The second group included 127 non-math students in their first year of college (part of this group took a math course for science and another part took an Introduction to Logic course at the department of philosophy). The younger population sample consisted of high school students in grades 9-11 (228 students altogether). The questionnaire was distributed in the academic year 1977/78.

Our first work on the data was to look at the verbal answers in 1A and to classify them. We divided them into four different groups:

1. Answers that express (or might express) the "numbers are abstract objects" approach. We will call it from now on the abstract object approach and denote it by AO.

2. Answers that express (or might express) the approach implicitly suggested by the first homework exercise in the questionnaire, namely, an approach according to which the natural numbers are identified as the symbols: "1," "2," "3," and so on. In other words, numbers are identified as the decimal system. We will call this approach from now on the decimal system approach and denote it by DS.

3. Answers that, although they do not express explicitly the AO approach, point out that the decimal system and other numeration systems "have equal status" and the decimal system has no priority to other systems. We will call this approach the equal status approach and denote it by ES.(3)

4. Answers that, as far as we could see, do not express any approach to numbers. They will be deonted by NI (no information).

Here are some classified examples. Our classification is very often a result of interpretation considering both the answer profile and the verbal answer. In case the answer is indirect a greater amount of interpretation is needed for mak-

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(3) It is hard to say about students belonging to this group whether they explicitly have the AO approach. Probably they do not have the DS approach since a necessary condition for belonging to this group was the preference of the first homework exercise. The ES approach is much closer to the AO than to the DS approach since it does not identify the natural numbers with one specific representation.

ing the classification and we do not claim that our judgement is absolute. The text is open for further interpretations. Moreover, since sometimes it was hard to decide whether an answer is AO or ES we counted these two groups together.

We bring the answers exactly as they were written (after translation from the Hebrew), sometimes in an imperfect style and even in non-grammatical forms. All the emphases are ours. In parantheses we state the grade, course or year of studies (in the case of math majors) to which the student belongs and also his answer profile.

Categories ES and EO

1. (grade 9,1A2B3C). In the first homework the assumption is that the natural numbers have no base. In the second answer, on the other hand it is said that one of the forms of writing the natural numbers is base 10. In general, the starting point is the natural numbers but this does not mean that they are not written in the a base form. This is why the second answer is better.

2. (grade 9, 1A2B3B). The second answer says that also 1, 2, 3,... are a base, not the natural numbers themselves.

3. (grade 10, 1A2B3C). The difference is that the first homework relates to the decimal system as common and only afterwards how it is possible to write the same numbers by other methods. In the second homework the decimal system is only one of the methods to express numbers.

4. (grade 11, 1A2B3B). The difference is that in the first answer the natural numbers are the base 10 system whereas in the second homework it is only one out of many forms to express the natural numbers.

5. (grade 12, 1A2B3B). In the first homework they define the natural numbers whereas in the second homework they define the ways of expressing numbers.

6. (Math for Science, 1A2B3C). The natural numbers are not 1, 2, 3,... This is only one way of writing them. Therefore 31_4, 1101_2 are not different forms of writing the number 13 but they are three different forms of expressing the same number.

7. (Math for Science, 1A2B3B). Base 10 system is one of the methods to write the natural numbers. This is written in the second homework. In the first homework they relate to this system not as if it was one method among many others but as if it were the only truth and all other methods (of different bases) are translations from it to other languages. It is like a child who is sure that Hebrew is the only language that exists in the world and all other languages are only translations. He does not relate to it as one language among many others that are all equal.

8. (Math for Science, 1A2B3B). In the second homework the natural numbers have an independent existence whereas in the first homework they are described by the decimal system which is preferable.

9. (Logic, 1A2B3C). In the first homework they identify what it is called by "natural numbers" with the base 10 system. In the second homework the base 10 system is one possibility of writing natural numbers.

10. (Math major, 1st year, 1A2B3B). In the second homework it is emphasized that the natural numbers are not 1, 2, 3,..., 10,... but abstract concepts that can be denoted by 1, 2, 3,...10,...

11. (Math major, 1st year, 1A2B3B). The first homework identifies the natural numbers with 1, 2, 3,... It is not true. This is only a way of writing them.

12. (Math major, 1st year, 1A2B3B). The first homework determines an identity between the concept of the natural number and the symbol that represents it (in the decimal system). The second homework does not define the concept of the natural number. It only emphasizes it in its different forms of representation.

13. (Math major, 3rd year, 1A2B3C). A natural number is an abstract object

that can be represented as we want. The motivation for defining the natural numbers is the intuitive idea of counting. The first homework expresses the motivation. The second expresses the abstraction.

Category DS

1. (grade 9, 1A2A3A). The natural numbers cannot be written in different forms. They can be written only in one form and that is 1, 2, 3,..., 10 and so on.

2. (grade 10, 1A2A3A). In the first homework they relate to the bases as a method of writing the natural numbers of the decimal system. In the second homework the decimal system is one of the systems of writing natural numbers.

3. (Math for Science,1A2A3A). The natural numbers are 1, 2, 3, 4,... as it is mentioned in the first homework. It is incorrect that one of the forms of writing the natural numbers is 1, 2, 3,... as it is said in the second homework.

4. (Math for Science, 1A2A3A). In the first homework they say what the natural numbers are and later on they show some ways of writing them. In the second homework they don't say what the natural numbers are but they show right away how to write them.

5. (Logic, 1A2A3A). The first homework mentions the fact that the natural numbers are 1, 2, 3,..., 10,... In the second homework nothing is said about what natural numbers are but only how to write them.

6. (Math major, 1st year, 1A2A3A). The natural numbers are 1, 2, 3,... and it is not true that this is only one way of writing them.

7. (Math major, 2nd year, 1A2A3A). The first homework says that the natural numbers are 1, 2, 3,...,10,... and only later on speaks about their forms. The second homework only says that they can be written in different forms but does not say the decimal system is the main one.

8. (Math major, 3rd year, 1A2A3C). In the first homework it is clear what natural numbers are and their writing in other bases is only different notation for the same natural numbers. In the second homework, one can think there is more than one kind of natural numbers.

Category NI

1. (grade 9, 1A2B3B) In the first homework it is written in base 2, 4, etc. In the second homework, on the other hand, it is said that this is the position method.

2. (grade 10, 1A2A3A). The first answer is correct and the second is incorrect.

3. (grade 11, 1A2B3C). In the first homework examples were given. The second is more general.

4. (grade 12, 1A2B3C). The second homework explains better the base system.

5. (Math for Science, 1A2B3C). According to the first homework in all bases there are 10 digits and according to the second homework the position method has 10 digits.

6. (Logic, 1A2B3B). The second homework explains better and gives the feeling that the student better understands the explanation.

7. (Math major, 1st year, 1A2A3C). The first homework is simple. The second is complicated.

8. (Math major, 1st year, 1A2B3C). The second homework is more accurate.

9. (Math major, 2nd year, 1A2B3C). The second homework is more general, more detailed and more instructive.

10. (Math major, 3rd year, 1A2A3C). The second homework is too general.

§6. The Results.

Before presenting the tables we have to say that a necessary condition for meaningful answers to the questionnaire is the knowledge of numeration systems (bases). This topic is officially included in the Israeli junior high math curriculum. However, we told the students that in case they do not know the subject they should avoid filling the questionnaire. So the information we are giving relates only to the 523 students who have the required knowledge for answering the questionnaire.

In Table I we give the percentages fo students (out of the whole groups) that belong to categories AO,ES and DS. Group 1 consists of 228 grade 9-11 students. Group 2 consists of 127 university freshmen. Part of them took the course Math for Science and the other part took Introduction to Logic. Group 3 consists of 168 math major students in their 1st, 2nd and 3rd years of study and also of some 12th grade high school students in the mathematical track.

Table I

	Categories AO and ES	Category DS
Group 1 N=228	6.73%	0.87%
Group 2 N=127	27.56%	6.29%
Group 3 N=168	39.04%	4.13%

In most cases the results in the subgroups were not significantly different from the results in the whole groups therefore we do not specify about the subgroups. However, there were two exceptions in two subgroups of extremely good students. The first one was the 12th grade students in the mathematical track. The second one consisted of 1st year math students in an honorary calculus course. We give their results in Table II.

Table II

	Categories AO and ES	Category DS
12th grade students in mathematical track N=27	56.63%	0%
Honorary calculus students N=13	46.15%	0%

Now, one might claim that the questionnaire we used is not an efficient tool to discover the AO, ES or DS approaches. There might be students who have the above approaches and still their answers to the questionnaire won't show it. Our first reaction to this is that perhaps these approaches are not <u>active</u> and therefore the questionnaire did not <u>evoke</u> them. Our second reaction is to try to get an <u>upper bound</u> to the percentages of students that do have one of the above approaches. So we made another attempt to deal with the data and this was to use only the answer profile as an indication of the students' approach. We did it because we do believe in <u>non-verbal (implicit) knowledge</u>. This notion relates to situations where people lack the ability to formulate concepts, principles or ideas but they behave as if they were guided by certain concepts, principles or ideas. We cannot get into it here because of space problems. This was done in a detailed way in (10), (11), and (12). Here we will only show how the statistical picture looks if we take the answer profiles 1A2B3B, 1A2B3C as an indication of having the AO or the ES approach and the answer profiles 1A2A3A, 1A2A3C as an indication of the DS approach. At first sight this move is not justified, especially if we consider the NI answers in §5. At a second thought, it might be justified since a choice of a profile answer might be an indication of some hidden implicit approach (exactly as a choice of a picture might be an indication of some hidden emotions in an affective test). If a student (No. 2 in NI) claims that "the first answer is correct and the second is incorrect" and cannot say why, we might assume that there is some implicit principle that determines his behavior. The same holds true for the student (No. 8 in NI) who claims "the second homework is more accurate." So we suggest that the implicit pronciples of the above students are DS and AO or ES, respectively. Of course (and this is explained in 10, 11, 12) when making such suggestions we need later on more evidence to support them and this we do not have here. Therefore, we will have only a very rough estimate, but still worth looking at becuase it will probably give us the upper bound we mentioned above.

Table III shows the percentages of students in the three groups having answer profiles 1A2B3B, 1A2B3C and 1A2A3A, 1A2A3C.

Table III

	Answer profiles 1A2B3B, 1A2B3C	Answer profiles 1A2A3A, 1A2A3C
Group 1	29.05%	9.64%
Group 2	32.29%	11.81%
Group 3	55.95%	11.90%

For the sake of analogy we give also Table IV which is analogous to Table II.

Table IV

	Answer profiles 1A2B3B, 1A2B3C	Answer profiles 1A2A3A, 1A2A3C
12th grade students in mathematical track	62.97%	7.41%
Honorary calculus students	53.84%	15.38%

Tables I-IV give us a rough estimate (a kind of "first order approximation") to students' approaches to numbers. They show us that a very low percentage of students in grades 9-11 explicitly have either the AO or the ES approach to numbers. Among non-math major students at the undergraduate level about 40% explicitly have it. If we consider also implicit approaches (Table III) the situation is little better.

However, the percentages of students who do not explicitly have either the AO, ES or the DS approaches to numbers (and probably have no explicit idea about the nature of numbers) are the following:

In the first group - 92.40%

In the second group - 66.15%

In the third group - 56. 83%

The percentages of students not having even an implicit approach are:

In the first group - 61.31%

In the second group - 55.90%

In the third group - 32.15%

The difference between group 1 and group 2 can be explained by orientation to mathematics. In group 3, the better the subgroup is the more it is oriented toward the AO or ES approaches.

We cannot claim that similar results will be found in similar groups in other places since we know very little about our students' backgrounds, but our guess is that the picutre won't be essentially different.

§7. The Practical Aspects.

I. The AO Approach and other Approaches to Numbers

In our opinion there is not only one correct approach to numbers (namely, the AO approach). This is not the right place to develop other approaches but we can point at some references which are sources for different views. For instance, Benaceraff (1) ("numbers are not objects at all" it is written there on p. 70), Wittgenstein (13) ("Calculating in the decimal notation must have its own life... The life of the decimal notation would have to be independent" it is written there on p. II-53), and also Quine (7) (The section: numbers, mind and body, pp. 262-266). Even the "naive" DS approach can be elaborated in such a way that it will be satisfactory (and by the way, in many mathematical systems in the theory of models symbols are the elements of the systems). If this is the case then the AO approach is not an inseparable part of mathematics. Therefore, when teaching it we teach our students the philosophy of mathematics. But if so, why teaching only one approach What about the others?

It seems to us that there is no justification for doing it anyway. Table I shows us that what we do in schools in order to teach the AO approach (and we do it a lot altogether) is not enough. The high school students do not have it explicitly. We do not claim that the failure is because of developmental obstacles. But it is clear that in order to teach it (in case it is possible) we need much more time and effort which perhaps we want to spare for other topics in mathematics.

We ourselves do not think that it is so important for non-math students to have an explicit view about the nature of numbers. Most people can function very well without it exactly as they function without having an explicit moral philosophy. The most impressing detail in Table I (§6) is the difference between group 1 and group 2. The development of a "philosophical" approach to numbers did not occur because of any direct teaching in this direction. It is probably the result of additional few years of mathematical experience and life experience. The question whether this development would occur without teaching numerals and numeration systems at previous levels is an open one but it is unimportant. The moment you admit that several different philosophical approaches to numbers are legitimate you have to give up the idea of teaching one specific approach. Since it is not possible to teach all of them it is better to give up the idea of teaching any of them. As Table I shows us, there is a stage in the intellectual development at which some people (not all of them) develop a "philosophical approach" to numbers. This might be the AO approach (as in our case(4)) or any other approach.

II. The Spontaneous Development of a Cognitive Style

Our suggestion for the average and the weak students (who are altogether at least 90% of the student population) is to avoid teaching about the nature of numbers. It is quite possible to teach school math until the 12th grade without getting into the question of what numbers are. Through manipulations with numbers dur-

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(4) As we mentioned above, we do not know if this fact is a result of the math curriculum or that the AO approach is more natural and this is the reason why the majority of studnets (having an explicit or implicit approach to numbers) chose it.

ing several years the students might develop an (implicit or explicit) approach to the nature of numbers.(5) It does not even have to be consistent from our point of view. It is enough that the student considers it consistent and that it enables him to understand and organize his knowledge about numbers. By avoiding teaching about the nature of numbers we let our students avoid questions that do not bother them and avoid answers that they cannot understnad. By this we encourage plurality of spontaneous approaches to numbers and save unnecessary conflicts and misunderstandings. By this we give a chance to our students to develop their own individual cognitive styles. It is a mistake to compare this individual cognitive style to the AO approach and to claim that it is immature only because it is not identical with it (this is often done,as we mentioned in §2, in developmental theories when there is a difference between the students' point of view and the common view, which is, as a matter of fact, the scientific point of view. When such a difference is met the student is declared as one who has not yet reached a certain stage in the intellectual development). As we mentioned above, well known mathematicians also may have implicitly or explicitly different approaches to numbers and this, of course, does not mean that they lack mathematical maturity. It shows only a diffrnece in the cognitive style.

It is true, perhaps, that the AO approach is more preferable for the math major student since in higher mathematics he has to deal with general mathematical systems (abstract groups, rings, fields, linear spaces, topological spaces, etc.). These general systems can be briefly characterized as sets of abstracts objects

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(5) And again, this does not have to be the AO approach. It can be the DS approach or even other original unknown approaches.

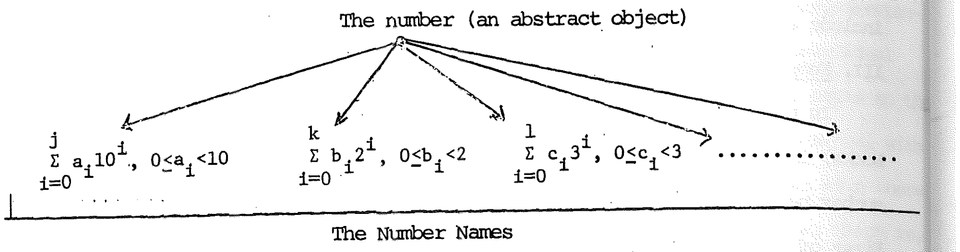
having certain relations (described by the axioms of the system). It seems reasonable to think that a student having the AO approach will have better readiness for higher mathematics (but reasonable does not necessarily mean true). However, it is not justified to teach the AO approach to all students just because of the one percent of them that will become math majors at the university. This one percent will have enough occasions to get it until they reach the university level, and if not we can include a special chapter about it in the 12th grade in the mathematical track.

III. Numerals

In new elementary math books (and of course in junior high and senior high math books) a lot of space and effort is devoted to the distinctions between numbers and number names (numerals). Especially in the elementary level this is very confusing. First of all, it is confusing for the elementary teachers. In most cases they lack the mathematical background to understand the AO approach and it is meaningless for them. Second, it is confusing becuase both they and their students face only number names but never face numbers (how can you face an abstract object?). This might lead to a mysterious approach to numbers in the best case or to frustration, denial and rejection in the worst case. And above all, this is totally unnecessary. It is possible to teach, in an intelligent way, all the school math without the distinction between numbers and numerals. (This will be shown in a detailed way elsewhere.) The decisive majority of school students is not capable of understanding the highly sophisticated arguments that require the distinction between numbers and numerals.

IV. Numeration System

In §1 we said that one of the rationales for including Numeration Systems in the curriculum was to give the students another chance to meet the AO approach. The including of this topic in school curriculum became quite controversial. For instance, see Lackner (4) (for it) and Rappaport (8) (against it). Without getting into this controversy here, we would like to make one comment about the AO approach and Numeration Systems. There is a way to teach Numeration Systems without getting at all into the problem of the AO approach. This can be explained by the following diagrams. The AO model for teaching Numeration Systems is:



In case our <u>mental image</u> for numbers will be within the DS approach (namely, the number is identical with its decimal representation) then the model for teaching Numeration Systems will be:

(a) The number $= \sum_{i=0}^{j} a_i 10^i$, $0 \le a_i < 10$

(b) In addition to (a) also : $\sum_{i=0}^{j} a_i 10^i = \sum_{i=0}^{k} b_i 2^i = \sum_{i=0}^{l} c_i 3^i = \ldots$

$0 \le b_i < 2 \quad 0 \le c_i < 3$

The mathematical theorem behind this is of course the following:

For every two natural numbers n and g, $g>1$, there exist natural numbers m and d_i, $i=0,\ldots,l$, $0 \le d_i < g$, such that $n = \sum_{i=0}^{m} d_i g^i$.

This theorem can be discussed and proved in a very detailed way in the senior high level in at most two hours (including drill and exercies). At this stage, a

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high level of understanding can be achieved. The question is (and it concerns also other topics in the elementary and the junior high level) whether we should spend in lower grades a whole trimester for topics we can teach (in a better way) in upper grades in two hours?

References:

1. P. Benaceraff, What Numbers Could Not Be, Philosophical Review, Vol. 74, 1965, pp. 47-73.
2. E. Fischbein, D. Tirosh and P. Hess, The Intuition of Infinity. To appear.
3. G. Frege, The Foundation of Arithmetic, Northwestern University Press, Evanston Illinois, 1968.
4. L. M. Lackner, What about Numeration Systems at the Primary Level, School Science and Math, Vol. 74, No. 2, 1974, pp. 152-156.
5. J. Piaget, The Child's Conception of Number, Routledge and Kegan Paul, 1971.
6. J. Piaget and B. Inhelder, The Child's Concept of Space, Routledge and Kegan Paul, 1956.
7. W. V. O. Quine, Word and Object, MIT Press and John Wiley and Sons, 1960.
8. D. Rappaport, Numeration Systems - A White Elephant, School Science and Math, Vol. 77, No. 1, 1977, pp. 27-30.
9. B. Russel, Selections from Introduction to Mathematical Philosophy, in The Philosophy of Math, Edited by P. Benaceraff and H. Putnam, Prentice-Hall, 1964 pp. 113-133.
10. S. Vinner, The Naive Concept of Definition in Math, Educational Studies in Mathematics, Vol. 7, 1976, pp. 413-429.

11. S. Vinner, The Elimination of Some Mathematical Confusions through the Analys of Language, Mathematical Education for Teaching, Vol. 3, No. 2, 1978, pp. 2 35.

12. S. Vinner, Conscious and Unconscious Processes in Mathematics Learning. To appear.

13. L. Wittgenstein, Remarks on the Foundations of Mathematics, Basil Blackwell, 1956.

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